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Desigualdades Geométricas e Interpolación de Operadores p-Schatten.

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires
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Título: Desigualdades Geométricas e Interpolación de operadores p -Schatten.

Resumen: En el conjunto de las perturbaciones de la identidad por operadores de la clase p-Schatten que resultan positivos e inversibles (denotado por Δ_p con $1 \leq p < \infty$) intoducimos la métrica $d_p(a,b)$ definida por el ínfimo de las longitudes de las curvas uniendo a con b, medidas con una métrica de Finsler. Tal espacio métrico comparte propiedades análogas a la de un espacio métrico de curvatura no positiva en el sentido de Alexandrov a pesar de no serlo. Entre ellas podemos citar la existencia de curvas minizantes (geodésicas), la convexidad de la función distancia entre dos geodésicas, la proyección a todo conjunto cerrado y convexo. Por otro lado vemos que las geodésicas resultan ser la curva interpolante que se obtiene al aplicar el método de interpolación de Calderón a los espacios p-Schatten, finalmente obtenemos como corolario directo de este resultado una serie de desigualdades de tipo Clarkson.

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Palabras claves: operadores p-Schatten, operador positivo, grupo de Banach-Lie, convexidad geodésica, espacio métrico de curvatura no positiva, espacio homogéneo.

Title: Geometric Inequalities and Interpolation of p-Schatten Operators.

Abstract: In the set of perturbations of the identity by p-Schatten class operators which result positive and invertible (that we denotes by Δ_p with $1 \leq p < \infty$) we introduce a metric $d_p(a, b)$ defined by the infimum of the lengths of the curves joining a with b, measured with a Finsler metric. Such metric space shares properties with a non-positive curvature metric space in the sense of Alexandrov despite not being such space. Among them we can cite the existence of minimazer curves (geodesics), the convexity of the function distance, the projection to a closed and convex set. On the other hand we notice that the geodesics are the interpolating curve obtained when applying the Calderon's interpolation method, finally as a by-product of this fact, we obtain Clarkson's type inequalities.

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Keywords: p-Schatten class, positive operator, geodesic convexity, space of non-positive curvature, Banach-Lie group, homogeneous space.

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quien me demostró que la vida tiene otro significado.

Las matemáticas pueden ser definidas
como aquel tema del cual
no sabemos nunca lo que decimos
ni si lo que decimos es verdadero.

Bertrand Russell (1872-1970).
Filósofo, matemático y escritor inglés.

Introducción

En los años 40 Alexandrov [1] y Busemann [17] mostraron que la noción de espacio con curvatura acotada superior o inferiormente tenía sentido para una clase más general de espacios métricos que las variedades Riemannianas, denominados "espacios geodésicos" (espacios métricos donde alguna desigualdad de comparación de triángulos es válida). Los trabajos clásicos sobre este tema son [41], [16] y [8].
La geometría de los espacios métricos de curvatura no positiva es ciertamente rica y tiene aplicaciones en muchos campos de la matemática, como teoría geométrica de grupos, topología, sistemas dinámicos y teoría de probabilidades.
Recordemos los principales resultados válidos para los espacios métrico (X, d) de curvatura no positiva en el sentido de Alexandrov (ANPC):

1. *Existencia de curvas cortas:* Dados dos puntos $p, q \in X$, X simplemente conexo, existe una única geodésica que conecta $p$con q.

2. *Convexidad de la función distancia:* Para cualquier par de geodesicas γ, δ en X, la función
$$f(t) = d(\gamma(t); \delta(t))$$
es convexa.

3. *Proyección a subconjuntos cerrados y convexos:* Sea C un subconjunto cerrado y convexo de X y $q \in X$. Entonces existe un único $q_0 \in C$ tal que
$$d(q_0, q) \leq d(p, q),$$
para todo $p \in C$.

En este trabajo probamos resultados análogos a los mencionados anteriormente en una familia de subvariedades diferenciables las cuales no resultan ser espacios de curvatura no positiva en el sentido de Alexandrov (sin embargo son espacios de curvatura no positiva en el sentido de Busemann para $1 < p < \infty$ y para $p = 1$ satisface una desigualdad similar pero para una familia distinguida de curvas cortas). Más precisamente:
Sean H un espacio de Hilbert separable y $B_p = B_p(H)$ la clase de operadores p-Schatten de H, $1 \leq p < \infty$. Denotamos con
$$\Delta_p = \{1 + X : X \in B_p,\ 1 + X > 0\},$$

donde 1 es el operador identidad y X es un operador autoadjunto perteneciente a la clase p-Schatten (conjunto que denotamos B_p^{sa}).
El conjunto Δ_p es una variedad diferenciable, con una carta natural dada por $exp : B_p^{sa} \to \Delta_p$. Asimismo, es fácil ver que Δ_p es simplemente conexa.
Dado $Y \in B_p^{sa}$ y $1 + X \in \Delta_p$ consideramos la métrica de Finsler

$$\|Y\|_{p,1+X} = \|(1+X)^{-1/2} Y (1+X)^{-1/2}\|_p.$$

A partir de dicha estructura métrica sobre los espacios tangentes podemos calcular la longitud de una curva $\alpha : [0,1] \to \Delta_p$ de la manera natural

$$L_p(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_{p,\alpha(t)} dt.$$

Sea d_p la métrica dada por el ínfimo de las longitudes de las curvas uniendo dos puntos cualesquiera, medidas en la métrica de Finsler. La variedad Δ_p es completa (tanto en sentido geodésico como en la métrica d_p). Cabe aclarar que esta métrica no es Riemanniana.
En esta tesis estudiamos las propiedades de (Δ_p, d_p) poniendo énfasis en aquellas que comparte con los espacios métricos de curvatura no positiva. A continuación comentamos la organización y los principales resultados obtenidos.
En el capítulo 2, introducimos la notación y los resultados preliminares necesarios para todo el trabajo.
En el capítulo 3, analizamos las propiedades topológicas y la estructura diferenciable de Δ_p.
En el capítulo 4, para estudiar la geometría en el caso $p = 1$, correspondiente a perturbaciones de operadores nucleares, nuestro enfoque, similar al de [49], consiste en utilizar herramientas de la geometría diferencial, más precisamente métricas de Finsler, y tratar tal conjunto como un espacio homogéneo.
Los casos restantes, $1 < p < \infty$, se estudiaron en el capítulo 5 utilizando fuertemente la convexidad uniforme de los espacios tangentes B_p^{sa} (propiedad no disponible en el caso p =1).
Los resultados obtenidos son los siguientes:

Teorema I. Sean $a, b \in \Delta_p$ con $1 \leq p < \infty$. La curva $\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^t a^{\frac{1}{2}}$ es la más corta que los une y por lo tanto

$$L_p(\gamma_{a,b}) = d_p(a,b) = \|\log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_p.$$

Además, si $1 < p < \infty$, $\gamma_{a,b}$ es la única curva con tal propiedad.

La existencia de curvas cortas nos permite introducir la noción de convexidad en Δ_p de una forma natural. Decimos que $K \subset \Delta_p$ es convexo si para todo $a, b \in K$ la geodésica $\gamma_{a,b}(t) \in K$ para todo $t \in [0,1]$.
Otro hecho interesante que satisfacen las subvariedades estudiadas es que la distancia antes definida resulta ser convexa sobre geodésicas, es decir:

Teorema II. Sean $a, b, c, d \in \Delta_p$, $1 \leq p < \infty$, $\gamma_{a,b}$ y $\gamma_{c,d}$ las respectivas geodésicas que los une. Luego, para $t \in [0,1]$

$$d_p(\gamma_{a,b}(t); \gamma_{c,d}(t)) \leq (1-t)d_p(a,c) + td_p(b,d).$$

Para $1 \leq p < \infty$, las subvariedades Δ_p satisfacen que la función exponencial aumenta distancias (propiedad que verifica toda variedad Riemanniana de curvatura seccional no positiva) es decir:

Teorema III. Sean $a \in \Delta_p$ y $X, Y \in B_p^{sa}$. Entonces

$$\|X - Y\|_{p,a} \leq d_p(exp_a(X), exp_a(Y)).$$

En [7], Ball et al. introdujeron la noción de convexidad p uniforme en espacios de Banach. Recordemos que un espacio de Banach $(X, \|.\|)$ es p-uniformemente convexo para $2 \leq p < \infty$ si existe una constante $C \geq 1$ tal que

$$\|\frac{v+w}{2}\|^p \leq \frac{1}{2}\|v\|^p + \frac{1}{2}\|w\|^p - C^{-p}\|\frac{v-w}{2}\|^p,$$

para todo $v, w \in X$.
Por ejemplo, los espacios B_p con $1 < p < 2$, son 2-uniformemente convexos con $C = 1/\sqrt{p-1}$ (ver [7]), y p-uniformemente convexos con $C = 1$ si $2 \leq p < \infty$ (desigualdades de Clarkson, ver Proposición 2.2.3). Una generalización natural de la convexidad p uniforme a un espacio métrico geodésico (X, d) es la siguiente:
Para todo $x \in X$ y cualquier curva geodésica minimal $\eta : [0,1] \to X$, tenemos

$$d(x, \eta(1/2))^p \leq \frac{1}{2}d(x, \eta(0))^p + \frac{1}{2}d(x, \eta(1))^p - \frac{1}{2^p C^p}d(\eta(0), \eta(1))^p.$$

En la Proposición 5.3.11 establecemos que Δ_p es r-uniformemente convexo con $r = \max\{p, 2\}$.

Teorema IV. Sean $X \in B_p^{sa}$ y $\gamma : [0,1] \to \Delta_p$ una geodésica. Entonces para $1 < p < \infty$, existe una constante $c_r > 0$ tal que

$$d_p(e^X, \gamma_{1/2})^r \leq \frac{1}{2}(d_p(e^X, \gamma_0)^r + d_p(e^X, \gamma_1)^r) - \frac{1}{4}c_r d_p(\gamma_0, \gamma_1)^r.$$

En la seccion 5.3.3 enunciamos y demostramos el teorema sobre la existencia y unicidad de una geodésica minimizante entre un punto y un subconjunto convexo y cerrado:

Teorema V. Sea K un subconjunto cerrado y convexo de Δ_p con $1 < p < \infty$. Entonces para cada punto $a \in \Delta_p$, existe un punto $q_0 \in K$ tal que

$$L_p(\gamma_{a,q_o}) = d_p(a, K).$$

Para $1 < p \leq 2$, las subvariedades satisfacen una condición del tipo de curvatura no positiva en el sentido de Alexandrov (ver Teorema 5.3.15).

Teorema VI. Dados $X \in B_p^{sa}$, $\gamma_t : [0,1] \to \Delta_p$ una geodésica y $1 < p \leq 2$. Entonces para todo $t \in [0,1]$

$$d_p(e^X, \gamma_t)^r \leq (1-t)d_p(e^X, \gamma_0)^r + t d_p(e^X, \gamma_1)^r - t(1-t)c_r d_p(\gamma_0, \gamma_1)^r.$$

En el capítulo 6, al aplicar el método de interpolación complejo de Calderón a los espacios B_p con las métricas de Finsler antes definidas, obtenemos que las geodésicas minimizantes resultan ser la curva de interpolación. Más precisamente:

Teorema VII. Sean $a, b \in B(H)$ positivos e inversibles, $1 \leq p, s < \infty$, $n \in \mathbb{N}$ y $t \in (0,1)$. Entonces

$$(B_{p,a;s}^{(n)}, B_{p,b;s}^{(n)})_{[t]} = B_{p,\gamma_{a,b}(t);s}^{(n)},$$

donde $B_{p,b;s}^{(n)}$ denota el espacio de n-uplas de operadores de la clase p-Schatten dotado con la norma $\|(X_0, \dots, X_{n-1})\|_{p,b;s} = (\|X_0\|_{p,b}^s + \dots + \|X_{n-1}\|_{p,b}^s)^{1/s}$ e indicamos con $(B_{p,a;s}^{(n)}, B_{p,b;s}^{(n)})_{[t]}$ el espacio de interpolación asociado al par $B_{p,a;s}^{(n)}$ y $B_{p,b;s}^{(n)}$.

Una aplicación directa de este teorema nos brinda una serie de desigualdades de tipo Clarkson para las normas $\|.\|_{p,a}$.

Teorema VIII. Dados $a, b \in B(H)$ positivos e inversibles, $n \in \mathbb{N}$, $X_0, ..., X_{n-1} \in B_p, 1 \leq p < \infty$ y $t \in [0,1]$, entonces

$$\tilde{k} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p \leq \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \theta_j^k X_j \|_{p,\gamma_{a,b}(t)}^p \leq \tilde{K} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p,$$

donde $\theta_0, ..., \theta_{n-1}$ son las n raíces de la unidad,

$$\tilde{k} = \tilde{k}(p,a,b,t) = \begin{cases} n^{p-1} \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-pt} & \text{si } 1 \leq p \leq 2, \\ n \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-pt} & \text{si } 2 \leq p < \infty, \end{cases}$$

y

$$\tilde{K} = \tilde{K}(p,a,b,t) = \begin{cases} n \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{pt} & \text{si } 1 \leq p \leq 2, \\ n^{p-1} \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{pt} & \text{si } 2 \leq p < \infty. \end{cases}$$

Teorema IX. Dados $a, b \in B(H)$ positivos e inversibles, $n \in \mathbb{N}$, $X_0, ..., X_{n-1} \in B_p, 1 \leq p < \infty$ y $t \in [0,1]$, entonces

$$k \sum_{j=0}^{n-1} \|X_j\|_{p,a}^2 \leq \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \theta_j^k X_j \|_{p,\gamma_{a,b}(t)}^2 \leq K \sum_{j=0}^{n-1} \|X_j\|_{p,a}^2,$$

donde $\theta_0, ..., \theta_{n-1}$ son las n raíces de la unidad,

$$k = k(p,a,b,t) = \begin{cases} n^{2-2/p} \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{si } 1 \leq p \leq 2, \\ n^{2/p} \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{si } 2 \leq p < \infty, \end{cases}$$

y

$$K = K(p,a,b,t) = \begin{cases} n^{2/p} \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{si } 1 \leq p \leq 2, \\ n^{2-2/p} \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{si } 2 \leq p < \infty. \end{cases}$$

Los resultados obtenidos en esta tesis tienen como precedentes a los siguientes trabajos:

1. En 1955, Mostow [52] dotó de una estructura Riemanniana al conjunto M_n^+ de matrices positivas e inversibles; la métrica inducida transforma a dicho conjunto en un espacio simétrico y de curvatura no positiva.

2. Corach et al. [22], estudiaron las propiedades geométricas de la variedad G^s, donde G^s es el conjunto de elementos autoadjuntos e inversibles de una álgebra C* A con identidad.

3. Mata Lorenzo y Recht en [49] dieron un marco general a la geometría de espacios homogéneos reductivos de dimensión finita en un álgebra de Banach. Más recientemente, Beltiţă [11] publicó un libro sobre espacios homogéneos de operadores que recopila la resultados anteriores.

4. Andruchow et al. mostraron en [3] que si $A \subset B(H)$ es una álgebra C^*, a, b elementos positivos e inversibles en A, y $\|.\|_a$ y $\|.\|_b$ las correspondientes normas cuadráticas en H inducidas por dichos elementos, i.e. $\|x\|_a = \langle ax, x\rangle$, entonces el método de interpolación complejo está determinado por $\gamma_{a,b}$. Tal curva es la única geodésica de la variedad de elementos positivos e inversibles de A, que une a con b.

5. En [46], Larotonda dió una estructura Riemanniana al conjunto Σ de operadores de Hilbert-Schmidt unitizados, positivos e inversibles, a través del producto interno definido por la traza. Esta métrica convierte a Σ en una variedad de Hilbert métricamente completa, simplemente conexa y de curvatura no positiva.

Geometric Inequalities and Interpolation of p-Schatten Operators.

Contents

Chapter 1

Introduction

In the 1940s Alexandrov [1] and Busemann [17] showed that the notions of upper and lower curvatures bounds make sense for a more general class of metric spaces than Riemannian manifolds, namely for "geodesic spaces". For more details on such metric spaces we refer to [41], [16] and [8]
La geometría de los espacios métricos de curvatura no positiva es ciertamente rica y tiene aplicaciones en muchos campos de la matemática, como teoría geométrica de grupos, topología, sistemas dinámicos, teoría de probabilidades, etc.
We recall the basic properties of these spaces:

1. *Existence of short curves :* Any two points in a simply connected Alexandrov non positive curvature space (ANPC) can be connected by a unique geodesic.

2. *Projection to closed and convex subsets:* Let (X, d) be a global Busemann non positive curvature space (BNPC), C a closed, convex subset of X and $q \in X$. Then there exists a unique $q_0 \in C$ with

$$d(q_0, q) \leq d(p, q),$$

   for all $p \in C$.

3. *Convexity of the distance function:* If γ, δ be geodesics in X, with X a BNPC, then the

$$f(t) = d(\gamma(t); \delta(t))$$

   is convex.

Let H be a separable Hilbert space and $B_p = B_p(H)$ the p-Schatten class with $1 \leq p < \infty$. We denote by

$$\Delta_p = \{1 + X : X \in B_p, 1 + X > 0\},$$

where 1 is the identity operator and X is a selfadjoint operator belongs to B_p (set which we denote by B_p^{sa}).
The set Δ_p is a differentiable manifold, with a natural chart given by $exp : B_p^{sa} \to \Delta_p$. It is easy to see that is simply connected.
Let $Y \in B_p^{sa}$ and $1 + X \in \Delta_p$ we consider the Finsler metric

$$\|Y\|_{p,1+X} = \|(1+X)^{-1/2} Y (1+X)^{-1/2}\|_p.$$

From the metric structure on the tangent spaces we can measured the length of a curve $\alpha : [0,1] \to \Delta_p$ de la manera siguiente

$$L_p(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_{p,\alpha(t)} dt.$$

We consider the metric space (Δ_p, d_p) where d_p is given by the infima of the lengths of curves joining two given points in Δ_p, measured with the Finsler metric. The manifold is complete (in the sense geodesic and metric)
In this thesis we study the properties of (Δ_p, d_p) focusing on those that share with the non positive curvature metric spaces. Few words about the structure of this work.
In the Chapter 2, we introduce the notation and the necessary preliminaries.
In el Chapter 3, we analyze the topological and differential properties of Δ_p.
In the Chapter 4, we consider the geometry in the case $p = 1$. We treat the set as a homogeneous space (i.e. $\Delta_1 = M/G$)- with a connection and Finsler metric, in a way that for arbitrary elements $a, b \in \Delta_1$, there exists a unique geodesic with endpoints a and b. We investigate the basic facts of Finsler geometry on orbit spaces M/G for isometric proper actions of classical Banach-Lie groups. The geodesics are minimal curves in the metric space M/G.
For the cases $1 < p < \infty$, we use strongly the uniformly convexity of the tangent spaces .
The principal results obtained are the following:

Theorem I. Let $a, b \in \Delta_p$, the geodesic $\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^t a^{\frac{1}{2}}$ is the shortest curve joining them. So

$$L_p(\gamma_{a,b}) = d_p(a,b) = \|\log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_p.$$

Furthermore, if $1 < p < \infty$, $\gamma_{a,b}$ the unique curve with such property.

Another fact that the submanifolds Δ_p hold is the convexity of the distance function (property that verifies any Riemannian manifold of non positive sectional curvature), i.e.

Theorem II. Let $a, b, c, d \in \Delta_p$, $1 \leq p < \infty$, $\gamma_{a,b}$ and $\gamma_{c,d}$ the respectively geodesics joining them. Then, for all $t \in [0, 1]$

$$d_p(\gamma_{a,b}(t); \gamma_{c,d}(t)) \leq (1-t)d_p(a,c) + td_p(b,d).$$

For $1 \leq p < \infty$, the submanifolds Δ_p verifies that the exponential map increases distances.

Theorem III. For all $a \in \Delta_p$ and $X, Y \in B_p^{sa}$we have

$$\|X - Y\|_{p,a} \leq d_p(exp_a(X), exp_a(Y)).$$

In [7], Ball et al. introduced the notion of p-uniform convexity in a Banach space. A Banach space $(X, \|.\|)$ is said to be p-uniformly convex for $2 \leq p < \infty$ if there is a constant $C \geq 1$ such that

$$\|\frac{v+w}{2}\|^p \leq \frac{1}{2}\|v\|^p + \frac{1}{2}\|w\|^p - C^{-p}\|\frac{v-w}{2}\|^p$$

holds for any $v, w \in X$.
These inequalities turned out to be useful instruments in Banach space theory and the geometry of Banach spaces. For instance, the B_p spaces are 2-uniformly convex with $C = 1/\sqrt{p-1}$ if $1 < p < 2$ (see [7], Proposition 3), and p-uniformly convex with $C = 1$ if $2 \leq p < \infty$ (Clarkson´s inequality, see Proposition 2.2.3).
One natural generalization of the p-uniform convexity to a geodesic metric space (X, d) is the following:
For any $x \in X$ and any minimal geodesic $\eta : [0, 1] \to X$, we have

$$d(x, \eta(1/2))^p \leq \frac{1}{2}d(x, \eta(0))^p + \frac{1}{2}d(x, \eta(1))^p - \frac{1}{2^pC^p}d(\eta(0), \eta(1))^p \qquad (1.1)$$

If $C = 1$ and $p = 2$, then the inequality (1.1) corresponds to the CAT(0)-property.
In Proposition 5.3.11 we establish that Δ_p is r-uniform convex with con $r = \max\{p, 2\}$.

Theorem IV. Let $X \in B_p^{sa}$ and $\gamma : [0,1] \to \Delta_p$ be a geodesic. Then for $1 < p < \infty$ there exist a constant $c_r > 0$ such

$$d_p(e^X, \gamma_{1/2})^r \leq \frac{1}{2}(d_p(e^X, \gamma_0)^r + d_p(e^X, \gamma_1)^r) - \frac{1}{4} c_r d_p(\gamma_0, \gamma_1)^r,$$

where $r = \max\{p, 2\}$.

In Section 5.3.3 we prove the existence and uniqueness of geodesics that realizes the distance betwen a point and a convex and closed subset of Δ_p:

Theorem V. Let K be a convex and closed subset of Δ_p with $1 < p < \infty$. Then for any $a \in \Delta_p$, there is a unique $q_0 \in K$ such that:

$$L_p(\gamma_{a,q_o}) = d_p(a, K).$$

For $1 < p \leq 2$, the submanifolds satisfies a type of non positive curvature in the sense of Alexandrov (ver Teorema 5.3.15).

Theorem VI. Let $X \in B_p^{sa}$, $\gamma_t : [0,1] \to \Delta_p$ a geodesic and $1 < p \leq 2$. Then for all $t \in [0,1]$

$$d_p(e^X, \gamma_t)^r \leq (1-t) d_p(e^X, \gamma_0)^r + t d_p(e^X, \gamma_1)^r - t(1-t) c_r d_p(\gamma_0, \gamma_1)^r.$$

Finally, in Chapter 6, al aplicar el complex interpolation methos to the spaces B_p with the con Finsler metrics before defined, we obtain that the geodesics curves are the interpolating curve. More precisely:

Theorem VII. Let $a, b \in B(H)$ positive and invertible operators, $1 \leq p, s < \infty$, $n \in \mathbb{N}$ and $t \in (0,1)$. Then

$$(B^{(n)}_{p,a;s}, B^{(n)}_{p,b;s})_{[t]} = B^{(n)}_{p,\gamma_{a,b}(t);s},$$

where $B^{(n)}_{p,b;s}$ denotes the space of n-tuples of p-Schatten operators endowed with the norm $\|(X_0, \ldots, X_{n-1})\|_{p,b;s} = (\|X_0\|^s_{p,b} + \ldots + \|X_{n-1}\|^s_{p,b})^{1/s}$ and one indicates with $(B^{(n)}_{p,a;s}, B^{(n)}_{p,b;s})_{[t]}$ the complex interpolation space, associated with the couple $B^{(n)}_{p,a;s}$ and $B^{(n)}_{p,b;s}$.

As a direct consequence of this theorem, we obtain Clarkson's type inequalities **for** the norms $\|.\|_{p,a}$.

Theorem VIII. Let $a, b \in B(H)$ positive and invertible operators, $X_0, ..., X_{n-1} \in B_p$, $n \in \mathbb{N}$,$1 \leq p < \infty$ and $t \in [0,1]$, then

$$\tilde{k} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p \leq \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \theta_j^k X_j\|_{p,\gamma_{a,b}(t)}^p \leq \tilde{K} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p,$$

where $\theta_0, ..., \theta_{n-1}$ are the n roots of the unity,

$$\tilde{k} = \tilde{k}(p,a,b,t) = \begin{cases} n^{p-1} \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-pt} & \text{if } 1 \leq p \leq 2, \\ n \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-pt} & \text{if } 2 \leq p < \infty, \end{cases}$$

y

$$\tilde{K} = \tilde{K}(p,a,b,t) = \begin{cases} n \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{pt} & \text{if } 1 \leq p \leq 2, \\ n^{p-1} \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{pt} & \text{if } 2 \leq p < \infty. \end{cases}$$

Theorem IX. Let $a, b \in B(H)$ positive and invertible operators, $X_0, ..., X_{n-1} \in B_p$, $n \in \mathbb{N}$, $1 \leq p < \infty$ and $t \in [0,1]$, then

$$k \sum_{j=0}^{n-1} \|X_j\|_{p,a}^2 \leq \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \theta_j^k X_j\|_{p,\gamma_{a,b}(t)}^2 \leq K \sum_{j=0}^{n-1} \|X_j\|_{p,a}^2,$$

where $\theta_0, ..., \theta_{n-1}$ are the n roots of the unity,

$$k = k(p,a,b,t) = \begin{cases} n^{2-2/p} \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{if } 1 \leq p \leq 2, \\ n^{2/p} \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{if } 2 \leq p < \infty, \end{cases}$$

y

$$K = K(p,a,b,t) = \begin{cases} n^{2/p} \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{if } 1 \leq p \leq 2, \\ n^{2-2/p} \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{if } 2 \leq p < \infty. \end{cases}$$

The results obtain in this thesis have as precedents the following works:

1. In 1955, Mostow [52] gave a Riemannian structure to the set M_n^+ of positive invertible matrices; the metric induced makes a symmetric and non positive curvature space.

2. Corach et al. [22], studied the geometric properties of G^s, where G^s is the set of invertible and selfadjoint elements of a C*-álgebra A with unity.

3. Mata Lorenzo y Recht en [49] gave a framewok to the geometry of reductive homogeneous spaces of infinite dimension. Recently, Beltiţă ([11]) published a book about .

4. Andruchow et al. proved in [3] that if $A \subset B(H)$ is a C^* algebra, a, b two invertible positive elements in A, and $\| \, \|_a$ and $\| \, \|_b$ the corresponding quadratic norms on H induced by them, i.e. $\|x\|_a = \langle ax, x \rangle$, then the complex interpolation method, is also determined by $\gamma_{a,b}$. This curve is the unique geodesic of the manifold of positive invertible elements of A, which joins a and b.

5. In [46], Larotonda gave a Riemannian structure to the set of positive, invertible and unitized Hilbert- Schmidt operators Σ induced by the trace.

Chapter 2

Preliminaries

2.1 Linear Operators in Hilbert Spaces

Definition 2.1.1. *A complex linear space H is called a Hilbert space if there is a complex valued function $\langle .,. \rangle : H \times H \to \mathbb{C}$ with the properties*

1. $\langle \eta, \eta \rangle \geq 0$ *and* $\langle \eta, \eta \rangle = 0$ *if and only if* $\eta = 0$;
2. $\langle \eta + \xi, \sigma \rangle = \langle \eta, \sigma \rangle + \langle \xi, \sigma \rangle$ *for all* $\eta, \xi, \sigma \in H$;
3. $\langle \beta\eta, \xi \rangle = \beta \langle \eta, \xi \rangle$ *for all* $\eta; \xi \in H$ *and* $\beta \in \mathbb{C}$;
4. $\overline{\langle \xi, \eta \rangle} = \langle \eta, \xi \rangle$ *for all* $\eta; \xi \in H$;
5. *H is complete with the norm defined by* $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$.

Troughout, we will suppose that H is a separable Hilbert space.
A linear map (operator) $T : H \to H$ is said to be bounded if there is a number K with

$$\|T\xi\| \leq K \|\xi\| \qquad \forall \xi \in H.$$

The infimum of all such K is called the norm of T, written $\|T\|$. Boundedness of an operator is equivalent to continuity. Let $B(H)$ denote the algebra of bounded operators acting on H.
To every bounded operator $T \in B(H)$ there is another $T^* \in B(H)$, called the adjoint of T, which is defined by the formula

$$\langle T\eta, \xi \rangle = \langle \eta, T^*\xi \rangle,$$

for all $\eta, \xi \in H$.

$$\|T\| = \sup_{\|\eta\|\leq 1, \|\xi\|\leq 1} |\langle T\eta, \xi\rangle| = \|T^*\| = \|T^*T\|^{1/2}.$$

Definition 2.1.2. *The identity map on H is a bounded operator denoted* 1.
An operator $T \in B(H)$ is called self-adjoint *if $T = T^*$.*
An operator $T \in B(H)$ is called positive *($T \geq 0$) if $\langle T\eta, \eta\rangle \geq 0$ $\forall \eta \in H$. We say $T \geq S$ if $T - S$ is positive.*
An operator $T \in B(H)$ is called unitary *if $TT^* = T^*T = 1$. We denote by $U(H)$ the set of all unitary operators in H.*

We denote by $Gl(H)$ the general linear group of all invertible bounded operators on H and by $Gl(H)^+$ the subset of the positive and invertible operators.
Given T be a positive operator in $B(H)$. Then, exist a unique positive operator B such that $T = B^2$ which we denote $T^{1/2}$.

Definition 2.1.3. $|A| = (A^*A)^{1/2}$.

Every invertible operator T admits a representation in the form

$$T = PU,$$

where $P = |T^*|$ and $U \in U(H)$. Such decomposition is called a polar decomposition of T.

Definition 2.1.4. *Let $T \in B(H)$.*
The resolvent set of T, denoted $\rho(T)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $\lambda 1 - T$ is bijective with a bounded inverse.
If $\lambda \in \rho(T)$, then $R_\lambda(T) = (\lambda - T)^{-1}$ is called the resolvent of T (at λ).
If $\lambda \notin \rho(T)$, then λ is in the "spectrum of T" = $\sigma(T)$.
Note: From the Open Mapping Theorem, if $\lambda - T$ is bijective, then its inverse is continuous

1. *$\lambda \in \sigma(T)$ is said to be an eigenvalue of T if $ker(\lambda - T) \neq 0$. If $0 \neq x \in ker(\lambda - T)$ we say that x is an eigenvector. The set of eigenvalues is called the point spectrum of T.*

2. *A scalar $\lambda \in \sigma(T)$ which is not an eigenvalue and for which $ran(\lambda - T)$ is not dense is said to be in the residual spectrum of T.*

Definition 2.1.5. *$T \in B(H)$ is said to be* compact *if $T(ball\ H)$ has compact closure in H, where ball $H = \{\eta \in H : \|\eta\| \leq 1\}$. The set of compact operators in H is denoted by $B_0(H)$.*

Remark 2.1.6. In this remark we recall some basic properties of compact operators.

1. $B_0(H)$ is a linear space and if $\{T_n\} \subset B_0(H)$ and $T \in B(H)$ such that $\|T_n - T\| \to 0$, then $T \in B_0(H)$,
2. If $T \in B_0(H)$ and $S \in B(H)$, then TS and ST are compact,

In addition, we consider an important characterization of compact operators. The following conditions are equivalent:

1. $T \in B_0(H)$.
2. T maps bounded sets into precompact sets (i.e. sets with compact closure).
3. T maps bounded sequences into sequences which have convergent subsequences.
4. $T^* \in B_0(H)$.
5. There is a sequence $\{T_n\}$ of operators of finite rank (i.e. $ran(T_n)$ is finite dimensional) such that $\|T_n - T\| \to 0$.

Theorem 2.1.7. *(Riesz-Schauder) Let $T \in B_0(H)$. Then, the following hold:*

1. $0 \in \sigma(T)$,
2. $\sigma(T) - \{0\}$ *consists of eigenvalues of finite multiplicity (i.e. the dimension of the λ-eigenspace ($ker(T - \lambda)$) has finite dimension $\forall \lambda \in \sigma(T) - \{0\}$),*
3. $\sigma(T) - \{0\}$ *is either empty, finite or a sequence converging to* 0 *(i.e. it is a discrete set with no limits other than* 0*).*

Now, we recall a result that essentially says that: "compact operators on a Hilbert space, can be 'diagonalized' over an orthonormal basis".

Theorem 2.1.8. *(Canonical form for Compact Operators, [60], Th. 1.4) Let $T \in B_0(H)$. Then T has the norm convergent expansion,*

$$T = \sum_n s_n(T) \langle \phi_n, . \rangle \psi_n.$$

(where he sum may be finite or infinite), each $s_n(T) > 0$, decreasingly ordered with $s_n \to 0$ and ϕ_n, ψ_n are orthonormal sets (not necessarily complete). Moreover, the $s_n(T)$ are uniquely determined. The $s_n := s_n(T)$ are eigenvalues of $|T| = (T^*T)^{1/2}$ and are called singular values of T.

2.1.1 The p-Schatten class

If $T \in B_0(H)$ we denote by $\{s_n(T)\}$ the sequence of singular values of T (decreasingly ordered). For $1 \leq p < \infty$, let

$$\|T\|_p = (\sum_n s_n(T)^p)^{1/p}$$

and

$$B_p(H) = \{T \in B(H) : \|T\|_p < \infty\},$$

called the $p-$ *Schatten class* of $B(H)$ (to simplify notation we use B_p). That is the subset of compact operators with singular values in l_p.
By convention $\|X\| = \|X\|_\infty = s_1(X)$. A reference for this subject is [36].
Now, let us recall some properties of the classes B_p

Theorem 2.1.9. *Let* $1 \leq p < \infty$.

1. B_p *is a* $*$*-ideal of* $B(H)$,
2. $\|X\|_p = \|UXV\|_p$ *for all* $X \in B_p$ *and* $U, V \in U(H)$,
3. *The set of all finite-dimensional operators is dense in* B_p,
4. *If* $p_1 < p_2$ *and* $T \in B_{p_1}$, *then* $T \in B_{p_2}$ *and* $\|T\|_{p_2} \leq \|T\|_{p_1}$,
5. *If the operators* T_j *with* $j = 1, 2, ..., n$ *belong respectively to the spaces* B_{p_j} *and* $\sum_{j=1}^{n} p_j^{-1} \leq 1$, *then the operator* $T = T_1T_2...T_n$ *belongs to the space* B_p, *where* $p^{-1} = \sum_{j=1}^{n} p_j^{-1}$, *and*

$$\|T\|_p \leq \|T_1\|_{p_1} \|T_2\|_{p_2} \cdots \|T_n\|_{p_n}.$$

In particular, if $T \in B_p$ *and* $S \in B_q$, *with* $p^{-1} + q^{-1} = 1$, *then* $TS, ST \in B_1$ *and*

$$\|TS\|_1 \leq \|T\|_p \|S\|_q, \qquad \|ST\|_1 \leq \|T\|_p \|S\|_q.$$

Remark 2.1.10. A norm $|||.|||$ defined on $I \subset B(H)$ that satisfies

$$|||UXV||| = |||X|||,$$

for all $X \in I$ and for pair of unitary operators U, V is called a norm unitarily invariant.

From now on an operator T will be called nuclear if it belongs to B_1, i.e. if

$$\|T\|_1 = \sum_n s_n(T) < \infty.$$

Another characterization of a nuclear operator will be given below, which makes it possible to introduce the notion of a trace for such operators.

Lemma 2.1.11. *Let T be a positive operator. Then the sum*

$$tr(T) := \sum_{n=1}^{\infty} \langle T\eta_n, \eta_n \rangle$$

has the same value (finite or infinite) for any orthonormal basis $\{\eta_n\}$ of H. The number $tr(T)$ is called the trace of T and it has the following properties:

1. $tr(T + S) = tr(T) + tr(S)$,
2. $tr(\lambda T) = \lambda tr(T)$ *for all* $\lambda \geq 0$,
3. $tr(UTU^{-1}) = tr(T)$ *for any unitary operator* U,
4. *If* $0 \leq T \leq S$, *then* $tr(T) \leq tr(S)$.

The connection between the p-Schatten operators and the trace is simple.

Theorem 2.1.12. $T \in B_p$ *if and only if* $\|T\|_p = (tr\,|T|^p)^{1/p} < \infty$.

Theorem 2.1.13. *If* $1 < p < \infty$ *and*

$$\phi : B_q \to B_p^*,\ \phi(T)(S) := tr(ST),$$

then ϕ is an isometric isomorphism:

$$\|T\|_q = \sup\{|tr(VT)| : V \in B_p, \|V\|_p \leq 1\} = \|\phi(T)\|.$$

2.2 Uniform Convexity

We begin by recalling the definition and some of the properties of uniformly convex Banach spaces which can be found in [10], [15], [29] and [30].

Definition 2.2.1. *A Banach space X is called uniformly convex if and only if for all $\epsilon \in (0, 2]$, the modulus of convexity*

$$\delta_{\|.\|}(\epsilon) := \inf\{1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1; \|x - y\| \geq \epsilon\}$$

satisfies $\delta_{\|.\|}(\epsilon) > 0$

Theorem 2.2.2. *For a Banach space X the following are equivalent:*

1. *X is uniformly convex*
2. *X has an equivalent uniformly convex norm $\|.\|$ with modulus of convexity of power type q; i.e. for some $k > 0$ one has $\delta_{\|.\|}(\epsilon) \geq k\epsilon^q$ for all $\epsilon \in (0, 2]$.*
3. *X has an equivalent uniformly smooth norm $\|.\|$, i.e. such that its modulus of smoothness*
$$\rho_{\|.\|}(\tau) := \frac{1}{2}sup\{\|x+y\| + \|x-y\| - 2 : \|x\| = 1; \|y\| \leq \tau\}$$
satisfies $\lim_{\tau\downarrow 0} \frac{\rho_{\|.\|}(\tau)}{\tau} = 0$.
4. *X has an equivalent uniformly smooth norm $\|.\|$ with modulus of smoothness of power type s, i.e. such that for some $c > 0$ one has $\rho_{\|.\|}(\tau) \leq c\tau^s$ for all $\tau \geq 0$.*
5. *X has an equivalent norm which is both uniformly convex and uniformly smooth and which has moduli of convexity and smoothness of power type.*

Let us recall classical inequalities for the p-Schatten class B_p.

Proposition 2.2.3. 1. *For* $1 \leq p \leq 2$

$$2^{p-1}(\|A\|_p^p + \|B\|_p^p) \leq \|A-B\|_p^p + \|A+B\|_p^p \leq 2(\|A\|_p^p + \|B\|_p^p), \tag{2.1}$$

$$2^{2-2/p}(\|A\|_p^2 + \|B\|_p^2) \leq \|A-B\|_p^2 + \|A+B\|_p^2 \leq 2^{2/p}(\|A\|_p^2 + \|B\|_p^2). \tag{2.2}$$

2. *For* $2 \leq p < \infty$

$$2(\|A\|_p^p + \|B\|_p^p) \leq \|A-B\|_p^p + \|A+B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p), \tag{2.3}$$

$$2^{2/p}(\|A\|_p^2 + \|B\|_p^2) \leq \|A-B\|_p^2 + \|A+B\|_p^2 \leq 2^{2-2/p}(\|A\|_p^2 + \|B\|_p^2). \tag{2.4}$$

The inequalities (2.1) and (2.3) are called Clarkson inequalities. The proofs of these inequalities can be found in [14].
These inequalities have useful applications, in particular they imply the uniform convexity of B_p.

Theorem 2.2.4. *For $1 < p < \infty$, B_p is uniformly convex.*

Proof. See [31] for $p \geq 2$ and [50] for general p. □

2.3 Different notions of convexity in metric spaces

Following [33], one can define two different notions of convexity of metric spaces. A midpoint map for a metric space (X, d) is a map $m : X \times X \to X$ satisfying

$$d(m(x,y),x) = \frac{1}{2}d(x,y) = d(m(x,y),y) \qquad \forall x,y \in X.$$

Definition 2.3.1. *([33]) Let (X, d) be a metric space admitting a midpoint map. (X, d) is called*

1. *ball convex if for all $x, y, z \in X$*

$$d(m(x,y),z) \leq \max\{d(x,z), d(y,z)\}. \tag{2.5}$$

   for any midpoint m. It is called strictly ball convex if the inequality is strict whenever $x \neq y$.

2. *distance convex if for all $x, y, z \in X$*

$$d(m(x,y),z) \leq \frac{1}{2}[d(x,z) + d(y,z)], \tag{2.6}$$

   for any midpoint map m.

Note that the condition (2.6) implies condition (2.5), and also that strictly ball convexity implies the uniqueness of a midpoint map.
Now, we give the definition of uniform ball convexity of metric spaces:

Definition 2.3.2. *Let (X, d) be a metric space admitting a midpoint map. (X, d) is called uniformly ball convex if for all $\epsilon > 0$ there exists a $\rho(\epsilon) > 0$ such that for all $x, y, z \in X$ satisfying $d(x,y) > \epsilon \max\{d(x,z), d(y,z)\}$, it holds that*

$$d(m(x,y),z) \leq (1 - \rho(\epsilon)) \max\{d(x,z), d(y,z)\}$$

for the (unique) midpoint map m.

2.4 Non positive curvature metric spaces

For the development of the theory of nonpositively curved metric spaces, we shall consider works that have been carried out in two different directions: the works of H. Busemann and the works of A. D. Alexandrov and his collaborators. Both Busemann and Alexandrov started their works in the 1940s, and the two approaches

gave rise to rich and fruitful developments, with no real interaction between them. The ramifications of these two theories continue to grow today, especially since the rekindling of interest that was given to nonpositive curvature by M. Gromov in the 1970s.
Let us briefly describe the basic underlying ideas of these works. First we need to recall a few definitions. For more details on metric spaces with non-positive curvature we refer to [41].
We now introduce the notion of geodesic space (X, d).

Definition 2.4.1. *A complete metric space (X, d) is called a* geodesic length space*, or simply a* geodesic space*, if for any two points $x, y \in X$, there exists a shortest geodesic joining them, i.e. a continuous curve such that $\gamma : [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$, and*

$$d(x, y) = L^d(\gamma).$$

Here, $L^d(\gamma)$ denotes the length *of γ (respect to the metric d) and it is defined as*

$$L^d(\gamma) := \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\}.$$

A curve $\gamma : [0,1] \to X$ is called a geodesic *if there exists $\epsilon > 0$ such that*

$$L^d(\gamma|_{[t,t']}) = d(\gamma(t), \gamma(t')) \quad \textit{whenever} \left|t - t'\right| < \epsilon.$$

Finally, a geodesic $\gamma : [0,1] \to X$ is called a shortest geodesic *if*

$$L^d(\gamma) = d(\gamma(0), \gamma(1)).$$

For a geodesic metric space the condition (2.6) can be phrased as follows: A geodesic metric space is distance convex if and only if for all $x \in X$ the distance function $d_x := d(x, .)$ is convex, where the convexity of d_x means that the restriction of d_x to every geodesic is a convex function.
We recall Busemann's definition of nonpositive curvature, which has the advantage of being the simplest to describe.

Definition 2.4.2. *A geodesic space (X, d) is said to be an Busemann nonpositive curvature space if for every $p \in X$ there exists $\delta_p > 0$ such that for any $x, y, z \in B(p, \delta_p)$ and any shortest geodesic $\gamma_1, \gamma_2 : [0,1] \to X$ with $\gamma_1(0) = \gamma_2(0) = x \in B(p, \delta_p)$ and with endpoints $\gamma_1(1), \gamma_2(1) \in B(p, \delta_p)$, we have*

$$d(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2} d(\gamma_1(1), \gamma_2(1)).$$

Now we consider the point of view of Alexandrov.

Definition 2.4.3. *A geodesic space* (X, d) *is said to be an Alexandrov nonpositive curvature space if for every* $p \in X$ *there exists* $\rho_p > 0$ *such that for any* $x, y, z \in B(p, \rho_p)$ *and any shortest geodesic* $\gamma : [0, 1] \to X$ *with* $\gamma(0) = x$, $\gamma(1) = z$, *we have for* $0 \leq t \leq 1$

$$d^2(y, \gamma(t)) \leq (1-t)d^2(y, \gamma(0)) + td^2(y, \gamma(1)) - t(1-t)L^d(\gamma)^2.$$

It should be noted that a metric space which is nonpositively curved in the sense of Alexandrov is also nonpositively curved in the sense of Busemann, but that the converse is not true. For instance, any finite-dimensional normed vector space whose unit ball is strictly convex is nonpositively curved in the sense of Busemann, but if the norm of such a space is not associated to an inner product, then this space is not nonpositively curved in the sense of Alexandrov. Alexandrov mentions this example in [2], p. 197.

2.5 Manifolds

In this thesis we work with smooth manifolds modeled in Banach spaces or usually called Banach manifolds, thus it is a topological space in which each point has a neighborhood homeomorphic to an open set in a Banach space (a more involved and formal definition is given below). We refer to Lang's book ([45]) for the basic differential geometry of this type of manifolds.
We recall the definition of this object.

Definition 2.5.1. *Let* X *be a set. An atlas of class* C^r *on* X *is a collection of pairs (called charts)* (Ui, ϕ_i) *satisfying the following conditions:*

1. *Each* U_i *is a subset of* X *and the* U_i *cover* X.
2. *Each* ϕ_i *is a bijection of* U_i *onto an open subset* $\phi_i(U_i)$ *of some Banach space* E_i *and for any* i, j, $\phi_i(U_i \cap U_j)$ *is open in* E_i.
3. *The map*
$$\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$
is a C^r*-isomorphism for each pair of indices* i, j.

One can then show that there is a unique topology on X such that each U_i is open and each ϕ_i is a homeomorphism. Very often, this topological space is assumed to

be a Hausdorff space, but this is not necessary from the point of view of the formal definition.
If all the Banach spaces E_i are equal to the same space E, the atlas is called an E-atlas. However, it is not a priori necessary that the Banach spaces E_i be the same space, or even isomorphic as topological vector spaces. However, if two charts (U_i, ϕ_i) and (U_j, ϕ_j) are such that U_i and U_j have a non-empty intersection, a quick examination of the derivative of the crossover map
shows that E_i and E_j must indeed be isomorphic as topological vector spaces. Furthermore, the set of points $x \in X$ for which there is a chart (U_i, ϕ_i) with $x \in U_i$ and E_i isomorphic to a given Banach space E is both open and closed. Hence, one can without loss of generality assume that, on each connected component of X, the atlas is an E-atlas for some fixed E.
A new chart (U, ϕ) is called compatible with a given atlas $\{(U_i, \phi_i) : i \in I\}$ if the map

$$\phi_{\phi}^{-1} : \phi(U \cap U_i) \to \phi_i(U \cap U_i)$$

is an r-times continuously differentiable function for every $i \in I$.
Two atlases are called compatible if every chart in one is compatible with the other atlas. Compatibility defines an equivalence relation on the class of all possible atlases on X.
A C^r-manifold structure on X is then defined to be a choice of equivalence class of atlases on X of class C^r. If all the Banach spaces E_i are isomorphic as topological vector spaces (which is guaranteed to be the case if X is connected), then an equivalent atlas can be found for which they are all equal to some Banach space E. X is then called an E-manifold, or one says that X is modeled on E.
A morphism $f : X \to Y$ will be called a submersion at a point $x \in X$ if there exists a chart (U, ϕ) at x and a chart (V, ψ) at $f(x)$ such that ϕ is an isomorphism of U onto a product $U_1 \times U_2$, and such that the map

$$\psi f \phi^{-1} = f_{V,U} : U_1 \times U_2 \to V$$

is a projection. We say that f is a submersion if it is a submersion at every point.
For manifolds modeled on Banach spaces, we have the usual criterion for submersion in terms of the differential.

Proposition 2.5.2. *Let X, Y be manifolds of class C^p modeled on a Banach spaces. Let $f : X \to Y$ be a C^p-morphism. Let $x \in X$. Then:*
f is a submersion at x if and only if exists a chart (U, ϕ) at x and (V, ψ) at $f(x)$ such that $f'_{V,U}(\phi(x))$ is surjective and its kernel splits.

Recall that a Finsler metric on a manifold M (infinite dimensional) is a function $F : TM \to \mathbb{R}$ satisfying the following conditions:

1. The function F is continuous on the complement of the zero section,
2. It defines a norm on each tangent space T_aM, with $a \in M$.

   This means that $F(X) > 0$ for $X \neq 0$, $F(cX) = |c|X$ for $c \in \mathbb{R}$, and $F(X+Y) \leq F(X) + F(Y)$.

Remark 2.5.3. In the finite dimensional theory of Finsler manifold [9], one defines Finsler structures by functions $F : TM \to \mathbb{R}$ which are smooth on the on the complement of the zero section and positively homogeneous and strong convex on each tangent space. Since, we work with not necessarily smooth norms, we have to give up this requirement.

Finally, we recall that a set G endowed with a group structure and an analytic Banach manifold structure is called a *Banach-Lie group*; if these two structures are compatible in the following sense: the mapping

$$G \times G \to G,\ (g,h) \to gh^{-1},$$

is analytic. A reference for this subject is [38].

Chapter 3

The set Δ_p

3.1 Topological and differentiable structure of Δ_p with $1 \leq p < \infty$.

Consider for $1 \leq p < \infty$ the following set of Fredholm operators,

$$\mathcal{L}_p = \{\beta + X \in B(H) : \beta \in \mathbb{C},\ X \in B_p\}.$$

$\mathcal{L}_p$ is a complex linear subalgebra consisting of the p-Schatten class perturbations of multiples of the identity. There is a natural norm for this subspace

$$\|\beta + X\|_{(p)} = |\beta| + \|X\|_p.$$

Lemma 3.1.1. *$(\mathcal{L}_p, \|\ \|_{(p)})$ is a complex Banach space.*

Note that if $\beta + X, \mu + Y \in \mathcal{L}_p$, then

1. $\|\beta + X\| \leq \|\beta + X\|_{(p)}$,
2. $\|(\beta + X)(\mu + Y)\|_{(p)} \leq \|\beta + X\|_{(p)} \|\mu + Y\|_{(p)}$.

In particular, $(\mathcal{L}_p, +, .)$ is a Banach algebra and $(\mathcal{L}_p, \|.\|_{(p)})$ is the unitization of $(B_p, \|.\|_p)$.
The selfadjoint part of $\mathcal{L}_p$ is

$$\mathcal{L}_p^{sa} = \{\beta + X \in \mathcal{L}_p : (\beta + X)^* = \beta + X\},$$

Remark 3.1.2. 1. Note that since $\dim H = \infty$, the multiples of the identity $\beta 1$ and the operators $X \in B_p$ are linearly independent. Therefore

$$\beta + X \in \mathcal{L}_p^{sa} \text{ if and only if } \beta \in \mathbb{R}\ ,\ X^* = X.$$

Formally,

$$\mathcal{L}_p = \mathbb{C} \oplus B_p \qquad \text{and} \qquad \mathcal{L}_p^{sa} = \mathbb{R} \oplus B_p^{sa}.$$

Inside $\mathcal{L}_p^{sa}$, we consider

$$\Delta_{p,\mathbb{C}} = \{\beta + X \in \mathcal{L}_p :\ \beta + X \geq 0\} \subset Gl(H)^+.$$

and, in particular if $\beta = 1$ we denote by

$$\Delta_p = \{1 + X \in \mathcal{L}_p :\ 1 + X \geq 0\}.$$

We begin proving some elementary facts abouts the topology of $\Delta_{p,\mathbb{C}}$.

Proposition 3.1.3. *$\Delta_{p,\mathbb{C}}$ is open and convex subset of $\mathcal{L}_p^{sa}$*

Proof. Convexity
It is apparent that $\Delta_{p,\mathbb{C}}$ is convex. Let us prove that it is open.
Let $x_0 = \beta_0 + X_0 \in \Delta_{p,\mathbb{C}}$, since $x_0 > 0$ we get that

$$-\beta_0 < \beta \qquad \text{for all } \beta \in \sigma(X_0).$$

Set $r > 0$ such that $-\beta_0 + r < X_0$ or equivalently $X_0 + \beta_0 - r > 0$.
Consider

$$B(x_0, \frac{r}{2}) = \{x = \mu + X \in \mathcal{L}_p^{sa} / \|x - x_0\|_{(p)} < \frac{r}{2}\}$$

Given $b = \mu + X \in B(x_0, \frac{r}{2})$, we have:

1. $\mu > \beta_0 - \frac{r}{2}$ since $-\frac{r}{2} < \mu - \beta_0 < \frac{r}{2}$ and
2. $X - X_0 \geq -\frac{r}{2}$, from the inequality $-\|X - X_0\|_p \leq -\|X - X_0\| \leq X - X_0$.

Hence,

$$\begin{aligned} \mu + X &> \beta_0 - \frac{r}{2} + X = \beta_0 - \frac{r}{2} + (X - X_0) + X_0 \geq \beta_0 - \frac{r}{2} - \frac{r}{2} + X_0 \\ &= \beta_0 - r + X_0 > 0 \end{aligned}$$

therefore $\mu + X \in \Delta_{p,\mathbb{C}}$ □

In particular $\Delta_{p,\mathbb{C}}$ is differentiable (analytic) submanifold of $\mathcal{L}_p^{sa}$.
The next step is to prove that $\Delta_p \subset \Delta_{p,\mathbb{C}}$ is a submanifold. For this purpose, we consider

$$\theta : \beta + X \to \beta$$

Lemma 3.1.4. *θ is a submersion.*

Proof. It is sufficient to show that $d\theta_{\beta+X}$ is surjective and $ker(d\theta_{\beta+X})$ is complemented .
Since $\mathcal{L}_p^{sa}$ and $\mathbb{R}$ are Banach spaces and θ is a continuous linear map we get that $d\theta_{\beta+X} = \theta$. It is evident that $d\theta_{\beta+X}$ is surjective.
Finally the kernel of $d\theta_{\beta+X}$ has codimension 1, and hence is complemented.

□

It follows that Δ_p is a submanifold, since $\Delta_p = \theta^{-1}(\{1\})$. For $a = 1 + Y \in \Delta_p$, we identify the tangent space $T_a\Delta_p$ with B_p^{sa}, and endow this manifold with a complete Finsler metric by means of the formula

$$\|X\|_{p,a} = \|a^{-\frac{1}{2}} X a^{-\frac{1}{2}}\|_p, \tag{3.1}$$

for $X \in B_p^{sa}$.
If $\mathcal{I}(H)$ is an ideal in the algebra $B(H)$, we denote by $Gl(H, \mathcal{I}(H))$ the subset of $Gl(H)$ consisting of those operators of the form $1 + a$ with $a \in \mathcal{I}(H)$, i.e.

$$Gl(H, \mathcal{I}(H)) = \{1 + a \in Gl(H) : a \in \mathcal{I}(H)\} = \{b \in Gl(H) : b - 1 \in \mathcal{I}(H)\}$$

The standard examples occur when $\mathcal{I}(H)$ is the ideal of compact operators, in which case $Gl(H, B_0(H))$ is the so-called *Fredholm group* of H, or when $\mathcal{I}(H)$ is the ideal of Hilbert-Schmidt operators and when $\mathcal{I}(H)$ is the ideal of trace class operators B_1. Under the conditions stated above, $Gl(H, \mathcal{I}(H))$ is a group. In particular, $Gl(H, B_p)$ is a group for $1 \leq p < \infty$.
There is a natural action of $Gl(H, B_p)$ on Δ_p, given by

$$l : Gl(H, B_p) \times \Delta_p \longrightarrow \Delta_p,\ l_g(1 + X) = g(1 + X)g^*.$$

This action is clearly differentiable and transitive, since if $1 + X, 1 + Y \in \Delta_p$ then

$$l_g(1 + X) = (1 + Y),$$

for $g = (1 + Y)^{\frac{1}{2}}(1 + X)^{-\frac{1}{2}} \in Gl(H, B_p)$.
We denote

$$B_{p,a} = (B_p^{sa}, \|.\|_{p,a}).$$

The next proposition justifies the definition of the Finsler metric (3.1):

Proposition 3.1.5. *The norm $\|.\|_{p,a}$ is invariant for the action of the group $Gl(H, B_p)$. That is: for each $X \in B_p$, $a \in Gl(H)^+$ and $g \in Gl(H, B_p)$, we have*

$$\|X\|_{p,a} = \|l_g(X)\|_{p,gag^*}.$$

Proof. Let $a \in Gl(H)^+, g \in Gl(H, B_p)$ and $X \in B_p$, observe that

$$gXg^* = ga^{1/2}a^{-1/2}Xa^{-1/2}a^{1/2}g^*.$$

Denote by $z = ga^{\frac{1}{2}}$ then

$$(gag^*)^{-\frac{1}{2}} = (ga^{\frac{1}{2}}a^{\frac{1}{2}}g^*)^{-\frac{1}{2}} = (zz^*)^{-\frac{1}{2}} = |z^*|^{-1},$$

therefore

$$(gag^*)^{-\frac{1}{2}}gXg^*(gag^*)^{-\frac{1}{2}} = |z^*|^{-1} za^{-\frac{1}{2}}Xa^{-\frac{1}{2}}z^* |z^*|^{-1}.$$

From the polar decomposition applied to $z \in Gl(H)$, $z = |z^*| \rho_z$ with ρ_z unitary, we have

$$(gag^*)^{-\frac{1}{2}}gXg^*(gag^*)^{-\frac{1}{2}} = \rho_z a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\rho_z^*.$$

Since $|srs^*| = s|r|s^*$ for all unitary s, we get

$$\begin{aligned} \|gXg^*\|_{p,gag^*}^p &= tr|\rho_z a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\rho_z^*|^p = tr(\rho_z|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}|^p\rho_z^*) \\ &= tr|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}|^p = \|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\|_p^p = \|X\|_{p,a}^p. \end{aligned}$$

□

Let e^X be the usual exponential map, i.e. $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$. The restriction of this map to B_p^{sa} is injective and takes values in $Gl(H, B_p)$, in fact, if $X \in B_p^{sa}$, then

$$e^X = 1 + X + \frac{1}{2}X^2 + ... = 1 + X(1 + \frac{1}{2}X + ...) \in \Delta_p.$$

Moreover, any positive element a in Δ_p is of the form $a = e^X$ for the unique $X \in B_p^{sa}$. Let us denote $X = \log(a)$. Therefore,

$$e^{B_p^{sa}} = \Delta_p.$$

One can compute the *length* of the curve α in Δ_p by

$$L_p(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_{p,\alpha(t)}dt.$$

Definition 3.1.6. *Let $a, b \in \Delta_p$. We denote by*

$$\Omega_{a,b} = \{\alpha : [0,1] \to \Delta_p : \alpha \text{ is a } \mathcal{C}^1 \text{ curve, } \alpha(0) = a \text{ and } \alpha(1) = b\},$$

the set of smooth curves in Δ_p joining a to b.
As in classical differential geometry, we consider the geodesic distance between a and b (in the Finsler metric) defined by

$$d_p(a,b) = \inf\{L_p(\alpha) : \alpha \in \Omega_{a,b}\}.$$

If $K \subseteq \Delta_p$, let

$$d_p(a,K) = \inf\{d_p(a,k) : k \in K\}.$$

Note that by Proposition 3.1.5, the geodesic distance d_p is invariant under the action of $Gl(H, B_p)$, i.e. $d_p(a,b) = d_p(gag^*, gbg*)$ for all $g \in Gl(H, B_p)$.
From now on, we denote by

$$\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}},$$

with $a, b \in \Delta_p$.

Remark 3.1.7.

1. Through these notes, we use alternatively the following notation for the curve $\gamma_{a,b}$

$$a \natural_t b = \gamma_{a,b}(t) = exp_a(t\ exp_a^{-1}(b)),$$

which is called the t-power mean between a and b in the literature (see [53]), and the *relative operator entropy*

$$S(a/b) = a^{\frac{1}{2}} \log(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})a^{\frac{1}{2}} = \dot{\gamma}_{a,b}(0),$$

defined in [35].

Lemma 2 in [34] shows that for $a, b \in \Delta_p$ and $t \in \mathbb{R}$

$$a \natural_t b = b \natural_{1-t} a.$$

2. Note that this curve looks formally equal to the geodesic (or shortest curve) between positive definitive matrices (regarded as a symmetric space, see [52]) and positive invertible elements of a C*-algebra [22].

3. Note that $\gamma_{a,b} \in \Omega_{a,b}$ (because pqp is positive and invertible whenever p, q are positive and invertible).

Proposition 3.1.8. *Given a, b in Δ_p, the curve $\gamma_{a,b}$ has length $\| \log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) \|_p$.*

Proof. Since the group $Gl(H, B_p)$ acts isometrically and transitively on Δ_p, it suffices to prove the theorem for $a = 1$. Then

$$\|\dot{\gamma}_{1,b}(t)\|_{p,\gamma_{1,b}(t)} = \| \log(b) b^t \|_{p,b^t} = \|b^{t/2} \log(b) b^{-t/2}\|_p = \| \log(b)\|_p,$$

because $\log(b)$ and b^t conmute for every $t \in \mathbb{R}$. $\square$

3.1.1 The Exponential Metric Increasing (EMI) property

The inequality (3.3) will be useful in the next chapters; its proof for matrices can be found in [52]. This inequality, in the context of matrices, is called by Bhatia the *exponential metric increasing property* and our proof is based on a similar argument used in [13].
We begin with the following inequality

Theorem 3.1.9. *(see [39]): Let A, B, X be Hilbert space operators with $A, B \geq 0$. For any unitarily invariant norm $|||.|||$ we have*

$$|||A^{1/2} X B^{1/2}||| \leq ||| \int_0^1 A^t X B^{1-t} dt||| \leq \frac{1}{2}|||AX + XB|||. \tag{3.2}$$

Proposition 3.1.10. *For all $X, Y \in B_p^{sa}$*

$$\|Y\|_p \leq \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}\|_p, \tag{3.3}$$

where $dexp_X$ denotes the differential of the exponential map at X.

Proof. The proof is based on the inequality (3.2) and the formula below:

Claim 3.1.11. *$dexp_X(Y) = \int_0^1 e^{tX} Y e^{(1-t)X} dt$.*

Here is a simple proof of this equality. Since

$$\frac{d}{dt}(e^{tX} e^{(1-t)Y}) = e^{tX}(X - Y)e^{(1-t)Y},$$

we have

$$e^X - e^Y = \int_0^1 e^{tX}(X - Y)e^{(1-t)Y} dt,$$

and hence

$$\lim_{h\to 0} \frac{e^{X+hY}-e^X}{h} = \int_0^1 e^{tX} Y e^{(1-t)X} dt.$$

Let $X, Y \in B_p^{sa}$. Write $Y = e^{\frac{X}{2}}(e^{-\frac{X}{2}} Y e^{-\frac{X}{2}}) e^{\frac{X}{2}}$. Then using the inequalities (3.2) we obtain

$$\begin{aligned} \|Y\|_p &\leq \| \int_0^1 e^{tX} (e^{-\frac{X}{2}} Y e^{-\frac{X}{2}}) e^{(1-t)X} dt \|_p = \|e^{-\frac{X}{2}} (\int_0^1 e^{tX} Y e^{(1-t)X} dt) e^{-\frac{X}{2}} \|_p \\ &= \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}} \|_p. \end{aligned}$$

This proves the proposition. □

The inequality (3.3) can be rephrased in terms of the the linear map $dexp_X(Y)$ as follows

Corolary 3.1.12. *For any $X \in B_p^{sa}$, the map $T_X : Y \to e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}$ is bounded and invertible. The inverse is contractive, that is, $\|T_Y^{-1}(Z)\|_p \leq \|Z\|_p$.*

Proof. This map is clearly bounded and invertible and the bound for the inverse is a consequence of the previous proposition . □

Chapter 4

The geometry of Δ_1

4.1 Introduction

In this chapter we focus on the case $p = 1$;

$$\Delta_1 = \{1 + a \in \mathcal{L}_1 : \ 1 + a > 0\},$$

The main reason to consider this specific case $p = 1$ is that positive trace class operators usually appear in Physics [62] and Probability, as densities of non negative functionals.
We consider the homogeneous space $Gl(H, B_1)/\mathcal{U}_1$, where $\mathcal{U}_1$ is the subgroup of $Gl(H, B_1)$ of unitary operators. This space can be identified with Δ_1 . This space is a reductive homogeneous space, such structure we allow construct a covariant derivate and via the covariant derivate introduce the notion of a geodesic as a solution to a ordinary differential equation. We then show that the geodesics are solutions of a variational problem (Prop. 4.3.7). They are critical points to the so called energy functional and furthermore shortest paths between their endpoints.

4.2 Reductive structure of Δ_1 .

For $1 + a \in \Delta_1$, let

$$\mathcal{I}_{1+a} = \{g \in Gl(H, B_1) : l_g(1 + a) = 1 + a\},$$

the *isotropy group* of $1 + a$. In particular, for $1 \in \Delta_1$

$$\mathcal{I}_1 = \{g \in Gl(H, B_1) : p(g) = 1\} = U(H) \cap Gl(H, B_1) = \mathcal{U}_1.$$

where $p(g) = gg^* = l_g(1)$.
Let us recall the definition of homogeneous reductive space

Definition 4.2.1. *A homogeneous space G/F is reductive (RHS) if there exists a vector space decomposition $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G, such that $\mathfrak{m}$ is invariant under the action of F*

In order to give an RHS structure to $Gl(H, B_1)/\mathcal{U}_1$, under the action of $Gl(H, B_1)$, we must find a decomposition

$$\underbrace{T_1 Gl(H, B_1)}_{\mathfrak{g}} = \underbrace{T_1 \mathcal{U}_1}_{\mathfrak{f}} \oplus \mathfrak{m}.$$

Recall that $\mathfrak{g} = T_1 Gl(H, B_1)$ and $\mathfrak{f} = T_1 \mathcal{U}_1$ can be identified with B_1 and iB_1^{sa}, respectively. Then, we have

$$B_1 = iB_1^{sa} \oplus \mathfrak{m}.$$

The most natural choice is $\mathfrak{m} = B_1^{sa}$. Note that B_1^{sa} is $\mathcal{U}_1$-invariant:

$$l_g(B_1^{sa}) = \{gXg^* : X \in B_1^{sa}\} = B_1^{sa}.$$

Now, we observe that $p(Gl(H, B_1)) = \Delta_1$, so it is clear that there exists an analytic isomorphism given by the polar decomposition such that

$$\Delta_1 \backsimeq Gl(H, B_1)/\mathcal{U}_1.$$

From the above remarks, we get

Proposition 4.2.2. *Δ_1 has an RHS structure under the action of $Gl(H, B_1)$.*

There is a natural connection on Δ_1, i.e. a smooth distribution of subspaces of B_1, $g \to H_g$ such that

1. $B_1 = H_g \oplus V_g$, if $g \in Gl(H, B_1)$;
2. $l_g(H_1) = H_1$ if $g \in \mathcal{U}_1$;
3. $H_g u = H_{gu}$ if $g \in Gl(H, B_1), u \in \mathcal{U}_1$.

In fact, $H_g = gB_1^{sa}$ and $V_g = \{X \in B_1 : Xg^* + gX^* = 0\}$ satisfies these properties The spaces H_g are called horizontal.
As the fibre bundle has a connection, a smooth curve γ admits a unique horizontal lift Γ in $Gl(H, B_1)$, with the following properties:

1. The curve Γ lifts γ: $l_{\Gamma(t)}(\gamma(0)) = \gamma(t)$.

2. $\Gamma(0) = 1$.

3. $\dot{\Gamma}(t) = \frac{d}{dt}\Gamma(t) \in H_{\Gamma(t)}, \qquad t \in [0,1]$.

This curve Γ is called the horizontal lifting of γ, and is also characterized as the unique solution of the following linear differential equation

$$\begin{cases} \dot{\Gamma} = \frac{1}{2}\dot{\gamma}\gamma^{-1}\Gamma \\ \\ \Gamma(0) = 1 \end{cases}$$

These are standard facts from the theory of homogeneous reductive spaces [49]. Now, in order to construct a covariant derivative in Δ_1 , we use its reductive structure.

Definition 4.2.3. *The differential equation*

$$\dot{\Gamma} = \frac{1}{2}\dot{\gamma}\gamma^{-1}\Gamma,$$

is called the transport equation for γ.

The transport equation induces a covariant derivative of a tangent field X along γ, namely

$$\frac{DX}{dt} = \Gamma(t)\frac{d}{dt}((Tl_{\Gamma(t)^{-1}})_{\gamma(t)}X(t))\Gamma(t)^* = \dot{X} - \frac{1}{2}(X\gamma^{-1}\dot{\gamma} + \dot{\gamma}\gamma^{-1}X).$$

From now on, we denote with $a, b, ..$ etc. the elements of Δ_1 .
As with HRS in finite dimension, the invariant of the induced connection can be computed. We shall be concerned with the curvature tensor, wich is given by:

$$R(X,Y)Z = -\frac{1}{4}a[[a^{-1}X, a^{-1}Y], a^{-1}Z],$$

for $X, Y, Z \in T_a\Delta_1$.
The exponential mapping of this connection can be also computed: given $a \in \Delta_1$ and $X \in B_1^{sa}$ the exponential mapping is

$$exp_a : B_1^{sa} \to \Delta_1,\ exp_a(X) = a^{\frac{1}{2}}e^{a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}}a^{\frac{1}{2}}.$$

Rearranging the exponential series we obtain a simpler expression

$$exp_a(X) = ae^{a^{-1}X}.$$

Notice that exp_a is a diffeomorphism and its inverse map is

$$\log_a = exp_a^{-1} : \Delta_1 \to B_1^{sa},\ \log_a(b) = a^{\frac{1}{2}} \log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) a^{\frac{1}{2}}.$$

Definition 4.2.4. *A curve γ is a geodesic if $\dot{\gamma}$ is parallel, i.e.*

$$\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}. \tag{4.1}$$

The basics properties of the geodesics can be summarized in the following statement.

Proposition 4.2.5. *Let $a \in \Delta_1$, $X \in T_a\Delta_1$ and γ a geodesic. Then*

1. *The curve $g\gamma g^*$ is also a geodesic for all $g \in Gl(H, B_1)$,*
2. *The unique geodesic γ such that $\gamma(0) = a$ and $\dot{\gamma}(0) = X$, is*

$$\gamma(t) = a^{\frac{1}{2}} e^{ta^{-\frac{1}{2}} X a^{-\frac{1}{2}}} a^{\frac{1}{2}} = exp_a(tX) \quad t \in \mathbb{R},$$

3. *Let $b \in \Delta_1$. There is one and only one geodesic $\gamma_{a,b}$ such that $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$, namely*

$$\gamma_{a,b}(t) = a^{\frac{1}{2}} (a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^t a^{\frac{1}{2}} \quad t \in \mathbb{R}.$$

Proof. The proof is straightforward. □

4.3 Minimality of geodesics

In this section we prove that the unique geodesic joining two points is the minimun of the p-energy functional for $p \geq 1$.
Note that given $a, b \in \Delta_1$, if $\gamma_{a,b}$ is the unique geodesic joining them, then by Proposition 3.1.8 we get that $L_1(\gamma_{a,b}) = \| \log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) \|_1$.
The next step consists in showing that geodesics are short curves, i.e. if δ is another curve joining a to b then

$$L_1(\gamma_{a,b}) \leq L_1(\delta).$$

and hence

$$d_1(a,b) = \| \log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_1.$$

We are now ready to prove the main result in this section.

Theorem 4.3.1. *Let $a, b \in \Delta_1$, the geodesic $\gamma_{a,b}$ is the shortest curve joining them. So*

$$d_1(a,b) = \| \log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_1.$$

Proof. Since the group $Gl(H, B_1)$ acts isometrically and transitively on Δ_1 , it suffices to prove the theorem for $a = 1$. Then

$$\gamma_{1,b} = b^t = e^{t\log(b)} \qquad \text{and} \qquad L_1(\gamma_{1,b}) = \| \log(b)\|_1.$$

Let $\gamma \in \Omega_{1,b}$; so write $\gamma(t) = e^{\alpha(t)}$ we get from the EMI property that

$$\|\gamma(t)^{-\frac{1}{2}}\dot{\gamma}(t)\gamma(t)^{-\frac{1}{2}}\|_1 = \|e^{-\frac{\alpha(t)}{2}} dexp_{\alpha(t)}(\dot{\alpha}(t))e^{-\frac{\alpha(t)}{2}}\|_1 \geq \|\dot{\alpha}(t)\|_1.$$

Finally,

$$\begin{aligned} L_1(\gamma) &= \int_0^1 \|\dot{\gamma}(t)\|_{1,\gamma(t)}dt = \int_0^1 \|\gamma(t)^{-\frac{1}{2}}\dot{\gamma}(t)\gamma(t)^{-\frac{1}{2}}\|_1 dt \geq \int_0^1 \|\dot{\alpha}(t)\|_1 dt \\ &\geq \| \int_0^1 \dot{\alpha}(t)dt\|_1 = \|\alpha(t)|_0^1\|_1 = \|\alpha(1) - \alpha(0)\|_1 = \| \log(b)\|_1. \end{aligned}$$

□

Remark 4.3.2. 1. The geometrical result described above can be translated to the language of the relative entropy

$$d_1(a,b) = \|a^{-\frac{1}{2}} S(a/b) a^{-\frac{1}{2}}\|_1 = \|S(a/b)\|_{1,a}\,.$$

2. For each $a \in \Delta_1$ and $\alpha > 0$ the exponential map $exp_a : T_a\Delta_1 \to \Delta_1$ maps the ball $\{X \in T_a\Delta_1 : \|X\|_{1,a} \leq \alpha\}$ onto the geodesic ball $\{x \in \Delta_1 : d_1(a,x) \leq \alpha\}$, since

$$d_1(a, exp_a(X)) = d_1(a, a^{\frac{1}{2}} e^{a^{-\frac{1}{2}} X a^{-\frac{1}{2}}} a^{\frac{1}{2}}) = \|X\|_{1,a}\,.$$

3. If we can decompose a tangent vector V in other two commuting vectors X, Y such that $V = X + Y$ and $\|V\|_1 = \|X\|_1 + \|Y\|_1$, then the curve

$$\delta(t) = \begin{cases} e^{2tX} & t \in [0, \frac{1}{2}] \\ e^{X} e^{(2t-1)Y} & t \in [\frac{1}{2}, 1] \end{cases}$$

is piecewise smooth and joins 1 to e^V in Δ_1; moreover

$$\|\dot{\delta}\|_{1,\delta} = \begin{cases} 2\|X\|_1 & t \in [0, \frac{1}{2}] \\ \\ 2\|Y\|_1 & t \in [\frac{1}{2}, 1]. \end{cases}$$

Hence $L_1(\delta) = \|X\|_1 + \|Y\|_1 = \|V\|_1$, which proves that the curve δ is a minimizing piecewise smooth curve joining 1 to e^V, and it is not one of the smooth geodesics $\gamma_{a,b}$.

For instance, take $V = V^* \in B_1^{sa}$ of the form $V = p_1 + p_2$ with p_i one dimensional mutually orthogonal projections. Then the p_i commute, and $\|V\|_1 = 2 = \|p_1\|_1 + \|p_2\|_1$.

Corolary 4.3.3. *If $X, Y \in B_1^{sa}$ commute, then for all $a \in \Delta_1$ we get*

$$\|X - Y\|_{1,a} = d_1(exp_a(X), exp_a(Y)).$$

In particular on each line $\mathbb{R}X \subseteq B_1^{sa}$ the exponential map preserves distances.

Note that the previous Corollary tells us that when X and Y commute the natural parametrization of γ_{e^X,e^Y} is given by $\gamma(t) = e^{(1-t)X}e^{tY}$. In this case

$$d_1(e^X, \gamma_{e^X,e^Y}(t)) = td_1(e^X, e^Y).$$

Corolary 4.3.4. *If, for some $a, b \in \Delta_1$, 1 lies on the geodesic $\gamma_{a,b}(t)$, then a and b commute and*

$$\log(b) = -\frac{1-t}{t}\log(a), \tag{4.2}$$

where $t = d_1(a, 1)/d_1(a, b)$.

Proof. We know that, for some t , $1 = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$. Thus $b = a^{-(1-t)/t}$ so that a and b commute and (4.2) holds. □

Definition 4.3.5. *For every $s \in \mathbb{R} - \{0\}$ we define the s-energy functional*

$$E_s : \Omega_{a,b} \to \mathbb{R}^+, \qquad E_s(\beta) := \int_0^1 (\|\dot{\beta}(t)\|_{1,\beta(t)})^s dt.$$

Remark 4.3.6. 1. For $s = 1$, one has the *length functional*

$$L_1(\beta) := \int_0^1 \|\dot{\beta}(t)\|_{1,\beta(t)} dt,$$

and for $s = 2$, one has the *energy functional*

$$E(\beta) := \int_0^1 (\|\dot{\beta}(t)\|_{1,\beta(t)})^2 dt,$$

2. For any curve β such that $\|\dot{\beta}(t)\|_{1,\beta(t)}$ is constant we have

$$E_s(\beta) = (L_1(\beta))^s = (E(\beta))^{\frac{s}{2}}.$$

In Theorem 4.3.1 we proved that the geodesic between a and b minimizes the length functional. This fact is valid also for the s-energy functional for $s \in (1, \infty)$.

Proposition 4.3.7. *Let $a, b \in \Delta_1$ and $s \in [1, \infty)$. Then the s-energy functional*

$$E_s : \Omega_{a,b} \to \mathbb{R}^+ \qquad E_s(\beta) := \int_0^1 (\|\dot{\beta}(t)\|_{1,\beta(t)})^s dt,$$

takes its global minumum $d_1^s(a, b)$ at $\gamma_{a,b}$.

Proof. Let $\beta \in \Omega_{a,b}$ and $s \in (1, \infty)$. By Hölder´s inequality

$$(L_1(\beta))^s = (\int_0^1 \|\dot{\beta}(t)\|_{1,\beta(t)} dt)^s \leq \int_0^1 (\|\dot{\beta}(t)\|_{1,\beta(t)})^s dt = E_s(\beta).$$

On the other hand, $(L_1(\gamma_{a,b}))^s = E_s(\gamma_{a,b})$. This implies

$$E_s(\gamma_{a,b}) = (L_1(\gamma_{a,b}))^s \leq (L_1(\beta))^s \leq E_s(\beta).$$

□

Proposition 4.3.8. *Given $a, b \in \Delta_1$ we get*

1.

$$d_1(a, b) = d_1(a^{-1}, b^{-1}).$$

2. For all $t_0, t_1 \in \mathbb{R}$

$$d_1(a\sharp_{t_0} b, a\sharp_{t_1} b) = |t_0 - t_1| \, d_1(a, b),$$

and in particular

$$d_1(a, a \sharp_t b) = |t| \, d_1(a, b).$$

Proof. 1. It is easy to see that $S(a/b) = -a^{\frac{1}{2}}\log(b^{-1}/a^{-1})a^{\frac{1}{2}}$, as a consequence from $\log(1/t) = -\log(t)$. Then

$$\begin{aligned} d_1(a,b) &= \|\log(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|_1 = \|a^{-\frac{1}{2}}S(a/b)a^{-\frac{1}{2}}\|_1 \\ &= \|-a^{-\frac{1}{2}}a^{\frac{1}{2}}\log(a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}})a^{-\frac{1}{2}}a^{\frac{1}{2}}\|_1 \\ &= \|-\log(a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}})\|_1 = d_1(a^{-1},b^{-1}). \end{aligned}$$

2. It is apparent that $L_1(\gamma_{a,b}|_{[t_0,t_1]}) = d_1(\gamma_{a,b}(t_0),\gamma_{a,b}(t_1))$. Then

$$d_1(a\sharp_{t_0}b, a\sharp_{t_1}b) = \int_{t_0}^{t_1}\|\dot\gamma(t)\|_{1,\gamma(t)}dt = |t_0 - t_1|d_1(a,b).$$

□

By Proposition 4.3.8 and remark 3.1.7 we have that

1. $a\sharp_{\frac{1}{2}}b = b\sharp_{\frac{1}{2}}a$.
2. $d_1(a, a\sharp_{\frac{1}{2}}b) = \frac{1}{2}d_1(a,b) = \frac{1}{2}d_1(b,a) = d_1(b, b\sharp_{\frac{1}{2}}a)$.

The previous equalities justifies the following definition

Definition 4.3.9. *For $a,b \in \Delta_1$, we denote by $m(a,b)$ the midpoint of a and b (following the notation used in [41]),*

$$m(a,b) := a\sharp_{\frac{1}{2}}b.$$

Definition 4.3.10. *Let $K \subseteq \Delta_1$. K is called* convex *if for all $a,b \in K$ the geodesic $\gamma_{a,b}(t) \in K$ for any $t \in [0,1]$.*

4.4 Convexity of the geodesic distance

The purpose of this section is to show that the norm of the Jacobi field along a geodesic γ is a convex function.

Definition 4.4.1. *J is a vector field along to a geodesic γ, if $J(t) \in T_{\gamma(t)}\Delta_1$ for all t.*
A vector field J along to a geodesic γ is a **Jacobi field** *if*

$$\frac{D^2J}{dt^2} + R(J,V)V = 0, \tag{4.3}$$

where $V(t) = \dot\gamma(t)$ and $R(X,Y)Z$ the curvature tensor.

Theorem 4.4.2. *If $J(t)$ is a Jacobi field along the geodesic $\gamma(t)$, then $\|J(t)\|_{1,\gamma(t)}$ is a convex map of $t \in \mathbb{R}$.*

The method of the following proof is based on a similar argument used in [23].

Proof. Note that by the invariance of the connection and the metric under the action of $Gl(H, B_1)$, we may supose that $\gamma(t) = e^{tX}$ is a geodesic starting at $\gamma(0) = 1$, with $X \in B_1^{sa}$.
Then for the field $K(t) = e^{-\frac{tX}{2}} J(t) e^{-\frac{tX}{2}}$ the differential equation (4.3) changes to

$$4\ddot{K} = KX^2 + X^2K - 2XKX. \tag{4.4}$$

Since the group $Gl(H, B_1)$ acts by isometries, we have

$$\|J(t)\|_{1,\gamma(t)} = \|\gamma(t)^{-\frac{1}{2}} J(t) \gamma(t)^{-\frac{1}{2}}\|_1 = \|K(t)\|_1,$$

thus the proof reduces to show that for any solution $K(t)$ of (4.4), the map $t \to \|K(t)\|_1$ is convex for $t \in \mathbb{R}$.
Fix $u < v \in \mathbb{R}$ and let $t \in [u, v]$. We shall prove that

$$\|K(t)\|_1 \leq \frac{v-t}{v-u}\|K(u)\|_1 + \frac{t-u}{v-u}\|K(v)\|_1.$$

Let $X = \sum_{i \in \mathbb{N}} \lambda_i \langle ., e_i \rangle e_i$ be the spectral decomposition of $X \in B_1^{sa}$ where $\{e_i : i \in \mathbb{N}\}$ is an orthonormal basis of H.
Consider the matrix valued map

$$k(t) = (k_{ij}(t))_{i,j \in \mathbb{N}},$$

where $k_{ij}(t) = \langle K(t)e_i, e_j \rangle$ for all $t \in \mathbb{R}$.
The differential equation (4.4) is equivalent to the equations

$$\ddot{k_{ij}}(t) = \delta_{ij}^2 k_{ij}(t),$$

where $\delta_{ij} = \frac{\lambda_i - \lambda j}{2}$.
A simple verification shows that all solutions of $\ddot{f}(t) = c^2 f(t)$ satisfy

$$f(t) = \phi(u, v, c; t) f(u) + \psi(u, v, c; t) f(v),$$

where

$$\phi(u, v, c; t) = \begin{cases} \frac{\operatorname{Sinh} c(v-t)}{\operatorname{Sinh} c(v-u)} & \text{if c} \neq 0; \\ \frac{(v-t)}{(v-u)}, & \text{if c=0.} \end{cases}$$

$$\psi(u,v,c;t) = \begin{cases} \frac{\operatorname{Sinh} c(t-u)}{\operatorname{Sinh} c(v-u)} & \text{if } c \neq 0; \\ \frac{(t-u)}{(v-u)}, & \text{if } c=0. \end{cases}$$

Then each $k_{ij}(t)$ satisfies

$$k_{ij}(t) = \phi_{ij}(t)k_{ij}(u) + \psi_{ij}(t)k_{ij}(v),$$

where $\phi_{ij}(t) = \phi(u,v,\delta_{ij};t)$ and $\psi_{ij}(t) = \psi(u,v,\delta_{ij};t)$. In matrix form

$$k(t) = \Phi(t) \circ k(u) + \Psi(t) \circ k(v),$$

where $\Phi(t) = \{\phi_{ij}(t)\}$, $\Psi(t) = \{\psi_{ij}(t)\}$ and $\circ$ denotes the Schur product of matrices, i.e. $\{a_{ij}\} \circ \{b_{ij}\} = \{a_{ij}b_{ij}\}$. Thus we have that

$$\|k(t)\|_1 \leq \|\Phi(t) \circ k(u)\|_1 + \|\Psi(t) \circ k(v)\|_1. \tag{4.5}$$

We make the following claim:

Claim 4.4.3. *Let $\Psi(t), \Phi(t)$ and $k(t)$ as above, then*

1. $\|\Phi(t) \circ k(u)\|_1 \leq \frac{v-t}{v-u}\|k(u)\|_1$,
2. $\|\Psi(t) \circ k(v)\|_1 \leq \frac{t-u}{v-u}\|k(v)\|_1$.

Proof. We only prove the first inequality, the second is analogous. Define for each $n \in \mathbb{N}$ and $A = \{a_{ij}\}_{i,j\in\mathbb{N}}$

$$A_n = \begin{cases} a_{ij} & \text{if } 1 \leq i,j \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $n \to \infty$,

$$\Phi(t) \circ k(u)_n \xrightarrow{\|\cdot\|_1} \Phi(t) \circ k(u), \tag{4.6}$$

since

$$\begin{aligned} \|\Phi(t) \circ k(u)_n - \Phi(t) \circ k(u)\|_1 &\leq \max_{i>n}|\phi_{ii}(t)| \sum_{i>n} |k_{ii}(u)| \\ &= \frac{(v-t)}{(v-u)} \sum_{i>n} |\langle K(u)e_i, e_i\rangle| \\ &\leq \sum_{i>n} |\langle K(u)e_i, e_i\rangle| \to 0. \end{aligned} \tag{4.7}$$

Next we use a theorem by Ando, Horn and Johnson ([5]), according to which if A and P are $n \times n$ matrices, with P positive semidefinite, then

$$\|A \circ P\|_1 \leq (\max_{1\leq i\leq n} p_{ii})\, \|A\|_1 \,.$$

Thus

$$\|\Phi(t) \circ k(u)_n\|_1 = \|\Phi(t)_n \circ k(u)_n\|_1 \leq (\max_{1\leq i\leq n} \phi_{ii}(t))\|k(u)_n\|_1. \tag{4.8}$$

We conclude from (4.6) and (4.8) that

$$\|\Phi(t) \circ k(u)\|_1 \leq \frac{v-t}{v-u}\|k(u)\|_1.$$

□

Consequently we get

$$\|k(t)\|_1 \leq \frac{v-t}{v-u}\|k(u)\|_1 + \frac{t-u}{v-u}\|k(v)\|_1.$$

□

Remark 4.4.4. For each $n \in \mathbb{N}$ both matrices $\Phi(t)_n$ and $\Psi(t)_n$, are positive definite. This follows from Bochner's Theorem applied to $\Phi(t)_n$ and $\Psi(t)_n$ considered as functions of c. In both cases the matrix is of the form $\{F(\lambda_i - \lambda_j)\}_n$ where $F(c)$ is the Fourier transform of a positive function (see [32], formula 1.9.14, page 31).

A consequence of this result follows:

Theorem 4.4.5. *Let $\gamma(t), \rho(t)$ be geodesics in Δ_1 , then $t \to d_1(\gamma(t), \rho(t))$ is a convex map in $\mathbb{R}$.*

Proof. Suppose the $\gamma(t)$ and $\rho(t)$ are defined in $[u, v]$. We consider $h(s,t)$ defined as follows:

1. the map $s \to h(s,u), 0 \leq s \leq 1$ is the geodesic joining $\gamma(u)$ with $\rho(u)$;
2. the map $s \to h(s,v), 0 \leq s \leq 1$ is the geodesic joining $\gamma(v)$ with $\rho(v)$;
3. for each s, the function $t \to h(s,t), u \leq s \leq v$ is the geodesic joining $h(s,u)$ with $h(s,v)$.

Let $J(s,t) = \frac{\partial h(s,t)}{\partial s}$. Hence, for each fixed $s, t \to J(s,t)$ is Jacobi field along the geodesic $t \to h(s,t)$. Finally, we define

$$f(t) = \int_0^1 \|J(s,t)\|_{1,h(s,t)} ds.$$

From Theorem 4.4.2, $t \to \|J(s,t)\|_{1,h(s,t)}$ is a convex function for each s. Hence, $t \to f(t)$ is also convex for $t \in [u,v]$. But $f(u) = \int_0^1 \|J(s,u)\|_{1,h(s,u)} ds$ is the lenght of $s \to h(s,u)$ and therefore $f(u) = d_1(\gamma(u), \rho(u))$. Similarly, $f(v) = d_1(\gamma(v), \rho(v))$. Now, for $u \leq t \leq v$, $f(t) = \int_0^1 \|J(s,t)\|_{1,h(s,t)} ds$ is the lenght of the curve $s \to h(s,t)$ which joins $\gamma(t)$ with $\rho(t)$ and then we get $d_1(\gamma(t), \rho(t)) \leq f(t)$. Convexity of $d_1(\gamma(t), \rho(t))$ follows and the Theorem is proved. □

Remark 4.4.6. A particular consequence of the above theorem is that there are no closed nonconstant geodesics in Δ_1. Indeed if $\beta : [0,1] \to \Delta_1$ is a nonconstant geodesic such that $\beta(0) = \beta(1) = a$, then for all $t \in (0,1)$

$$d_1(a, \beta(t)) \leq t d_1(a, \beta(0)) + (1-t) d_1(a, \beta(1)) = 0.$$

Definition 4.4.7. *A subset K of Δ_1 is called* convex *if for all $a, b \in K$ the geodesic $\gamma_{a,b}$, joining a and b, is contained in K.*

Corolary 4.4.8. *Let $a, b, c \in \Delta_1$. Then for all $t \in$ [0,1]*

$$d_1(a \sharp_t b, a \sharp_t c) \leq t d_1(b,c). \tag{4.9}$$

In particular,

$$d_1(b^t, c^t) \leq t d_1(b,c).$$

Remark 4.4.9. The relation (4.9) is known as "convexity of the metric" in the Riemannian context.
There is a clear interpretation of the corollary above. If M is a Riemannian manifold, the sectional curvature is nonpositve if and only if

$$d(\rho_s(x), \rho_s(y)) \leq s d(x,y),$$

for all $x, y \in M$ and all $s \in [0,1]$, where $\rho_s(x) = exp_p(s\ exp_p^{-1}(x))$ and $p \in M$ is fixed (see [8]).
This expression reduces, in our (non Riemannian) case to

$$d_1(p \sharp_s x, p \sharp_s y) \leq s d_1(x,y),$$

which is (4.9).

Corolary 4.4.10. *Let $a \in \Delta_1$, a fixed. Then*

$$f(\gamma(t)) \leq (1-t)f(\gamma(0)) + tf(\gamma(1)),$$

where $f(x) = d_1(a,x)$ and $\gamma(t)$ is a geodesic. In particular, if $r > 0$ the ball centered at a with radius r, i.e $\{x \in \Delta_1 : d_1(a,x) \leq r\}$, is a convex set.

4.4.1 The Metric Increasing Property of the Exponential Map

In this section we provide a proof of the metric increasing property (MIP) of the exponential map (Theorem 4.4.12) which is based on the exposition in Corach, Porta and Recht [23]. We begin with a lemma of approximation.

Lemma 4.4.11. *Let $\gamma(t)$ be a curve in Δ_1 , then $\log(\gamma(t))$ can be approximated uniformly by polynomials for $t \in [t_0, t_1]$.*

Proof. Throughout the proof $Hol(U)$ and S^2 denote the set of all complex analytic functions defined in U, with U an open set of complex plane and the Riemann sphere, respectively. Let $\sigma(t)$ be the spectrum of $\gamma(t)$, $\sigma(\gamma) = \bigcup_{t\in[t_0,t_1]} \sigma(t)$ be the spectrum of γ in the algebra $C([0,1], \mathcal{L}_1)$, and $G \subseteq \mathbb{C} - \{z : Im(z) \leq 0\}$ an open neighbourhood of $\sigma(\gamma)$.
Since $\sigma(\gamma)$ is compact, $S^2 - \sigma(\gamma)$ is connected and $\log(z) \in Hol(G)$. Then there is a sequence P_n of polynomials such that $P_n(z) \to \log(z)$ uniformly on $\sigma(\gamma)$ ([56], Theorem 13.7).
Since $P_n(z)$ are analytic on G, $\sigma(\gamma) \subseteq G$, and $P_n(z) \to \log(z)$ uniformly on compact subsets of G, then $\|P_n(\gamma(t)) - \log(\gamma(t))\|_1 \to 0$ as $n \to \infty$. □

The Finsler structure of Δ_1 is not Riemannian. However Δ_1 shares some properties with Riemannian manifolds of non-positive sectional curvature. For instance, the following:

Theorem 4.4.12. *The exponential map in Δ_1 increases distances, i.e. for all $a \in \Delta_1$, $X, Y \in B_1^{sa}$ we have*

$$d_1(exp_a(X), exp_a(Y)) \geq \|X - Y\|_{1,a}. \tag{4.10}$$

Proof. Let $\gamma_1(t) = e^{tX}$, $\gamma_2(t) = e^{tY}$, $a = 1$ and $f(t) = d_1(\gamma_1(t), \gamma_2(t))$. By Theorem 4.4.5, f is convex, with $f(0) = 0$. Hence $\frac{f(t)}{t} \leq f(1)$ for all $t \in (0,1]$.

Note that
$$\frac{f(t)}{t}=\frac{1}{t}\|\log(e^{tX/2}e^{-tY}e^{tX/2})\|_1=tr|\frac{1}{t}\log(e^{tX/2}e^{-tY}e^{tX/2})|.$$
Taking limits we have
$$\lim_{t\to 0^+}\frac{f(t)}{t}\leq f(1).$$
Observe next that by the previous lemma, $\log(x)$ can be approximated on any interval $[x_0,x_1]$ with $0<x_0<x_1$ uniformly by polinomials $P_n(x)$. In particular
$$\lim_{n\to\infty}P_n(x)=\log(x)\qquad\text{and}\qquad\lim_{n\to\infty}\dot{P_n}(x)=\frac{1}{x}.$$
(in morm $\|.\|_1$).
Then
$$\lim_{t\to 0^+}|\frac{1}{t}\log(e^{tX/2}e^{-tY}e^{tX/2})|=|X-Y|.$$
From this equality and convexity we conclude that
$$f(1)\geq\|X-Y\|_1,$$
this completes the proof for $a=1$.
By the invariance of the distance under the action the previous inequality implies that
$d_1(exp_a(X),exp_a(Y))\geq\|X-Y\|_{1,a}$ for all $a\in\Delta_1$, and all $X,Y\in B_1^{sa}$. □

Remark 4.4.13. For $a=1$, from the theorem above we get
$$\|X-Y\|_1\leq\|\log(e^{-\frac{X}{2}}e^{Y}e^{-\frac{X}{2}})\|_1,$$
for $X,Y\in B_1^{sa}$, which can also be written
$$\|\log(x)-\log(y)\|_1\leq\|\log(x^{-\frac{1}{2}}yx^{-\frac{1}{2}})\|_1,\tag{4.11}$$
with $x,y\in\Delta_1$.

Proposition 4.4.14. Δ_1 *is a complete metric space with the geodesic distance.*

Proof. Consider a Cauchy sequence $\{a_n\}\subseteq\Delta_1$. By (4.11) $X_n=\log(a_n)$ is a Cauchy sequence in B_1^{sa}. Then there exists an operator $X\in B_1^{sa}$ such that $X_n\xrightarrow{\|.\|_1}X$. Hence
$$d_1(a_n,e^X)=\|\log(e^{\frac{X}{2}}e^{-X_n}e^{\frac{X}{2}})\|_1\to 0,$$
when $n\to\infty$. □

4.5 Non-positive Curvature

It would be very interesting to understand the relations between the geodesic distance and general metric spaces with non-positive curvature.
In particular, the geodesics $\gamma_{a,b}$ (in the sense of the equation (4.1)) are also short curve in the metric space (Δ_1, d_1), since

$$\begin{aligned} L^d(\gamma_{a,b}) &= \sup\{\sum_{i=1}^{n} d(\gamma_{a,b}(t_{i-1}), \gamma_{a,b}(t_i)) : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\} \\ &= \sup\{\sum_{i=1}^{n} L_1(\gamma_{a,b}\Big|_{[t_{i-1},t_i]}) : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\} \\ &= \sup\{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\dot{\gamma}_{a,b}(t)\|_{1,\gamma_{a,b}(t)} dt : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\} \\ &= L_1(\gamma_{a,b}). \end{aligned}$$

In other words

$$L^d(\gamma_{a,b}|_{[t,t']}) = L_1(\gamma_{a,b}|_{[t,t']}) = d_1(\gamma_{a,b}(t), \gamma_{a,b}(t')).$$

By the above argument, we have the following statement

Proposition 4.5.1. *The metric space (Δ_1, d_1) is a geodesic space and $m(.,.)$ is a midpoint map corresponding to the shortest geodesic $\gamma_{a,b}$ for all $a, b \in \Delta_1$.*

Busemann has defined non-positive curvature for chord spaces [18]. These are metric spaces in which there is a distinguished set of geodesics, satisfying certain axioms. In such a space, denote by $m(x, y)$ the midpoint along the distinguished geodesic connecting the pair of points x and y. Then the chord space is non-positively curved if, for all points x, y and z in the space,

$$d(m(x,y), m(x,z)) \leq \frac{1}{2} d(y,z),$$

where d is the metric.
Consequently, if in the metric space Δ_1 we consider the following distinguished set of geodesics

$$\mathcal{G} = \{\gamma_{a,b} : a, b \in \Delta_1\}.$$

we get by (4.9) the following statement.

Theorem 4.5.2. *The metric space (Δ_1, d_1) is a chord space non-positive curved.*

4.5.1 An alternative definition of sectional curvature

In this section, we shall see that it is possible to give a definition of sectional curvature in Δ_1. Recall that in [51] Milnor observes in his optical interpretation of curvature, that the sectional curvature, $s_a(X,Y)$, can be obtained by the following limit

$$s_a(X,Y) = 6 \lim_{r\to 0^+} \frac{r\,\|X-Y\|_{1,a} - d(exp_a(rX), exp_a(rY))}{r^2 d(exp_a(rX), exp_a(rY))}.$$

where X, Y are tangent vectors at a point a. We will see that this limit makes sense in our context.
Suppose that $r > 0$ is close to 0 such that $e^{-rX/2}e^{rY}e^{-rX/2}$ lies within the radius of convergence of the series $\log(u)$. Then by a straightforward computation, we get the following equality (see [4])

$$\log(e^{-rX/2}e^{rY}e^{-rX/2}) = r(Y-X) + r^3\kappa(X,Y) + o(r^3),$$

where

$$\kappa(X,Y) = \frac{1}{6}YXY + \frac{1}{12}XYX - \frac{1}{12}(XY^2 + Y^2X) - \frac{1}{24}(X^2Y + YX^2).$$

Before stating the existence of the limit above we need the following definition and lemma.

Definition 4.5.3. *Let V a vector space and f be a function from V to $\mathbb{R}\cup\{+\infty\}$. We shall say that $Df(x_0)(v)$ is the* right derivate of f at x_0 in the direction *v if the limit*

$$Df(x_0)(v) = \lim_{t\to 0^+} \frac{f(x_0+tv) - f(x_0)}{t}.$$

exists. In this case, we call $v \to Df(x_0)(v)$ the right derivate of f at x_0.

Remark 4.5.4. ([6], Proposition 4.1) Let V a vector space and f be a nontrivial convex function from V to $\mathbb{R}\cup\{+\infty\}$. Suppose $x_0 \in Dom(f)$ and $v \in V$. Then the limit $Df(x_0)(v)$ exists in $\overline{\mathbb{R}}$ and satisfies

$$f(x_0) - f(x_0 - v) \le Df(x_0)(v) \le f(x_0+v) - f(x_0).$$

For $a \in \Delta_1$, we denote by

$$P_a : T_a\Delta_1 \to \mathbb{R}^+, \qquad P_a(X) = \|X\|_{1,a}$$

Lemma 4.5.5. *For $a \in \Delta_1$, P_a is a convex function. Moreover, P_a is right differentiable on B_1^{sa} and satisfies*

$$\|X\|_{1,a} - \|X-Y\|_{1,a} \le DP_a(X)(Y) \le \|X+Y\|_{1,a} - \|X\|_{1,a}.$$

Proof. By the remark 4.5.4 it suffices to prove that P_a is convex. Clearly this is obvious from the usual properties of a norm, since for all $\lambda \in (0,1)$

$$P_a(\lambda X + (1-\lambda)Y) \le \lambda P_a(X) + (1-\lambda)P_a(Y).$$

□

Theorem 4.5.6. *Let $a \in \Delta_1$ and $X, Y \in B_1^{sa}$. The limit*

$$s_a(X,Y) = \lim_{r\to 0^+} \frac{r\,\|X-Y\|_a - d_1(exp_a(rX), exp_a(rY))}{r^2 d_1(exp_a(rX), exp_a(rY))}$$

exists and verifies

$$1 - \frac{\|Y - X + \kappa(X,Y)\|_a}{\|X-Y\|_a} \le s_a(X,Y) \le 0.$$

Proof. Since the metric on B_1^{sa} and the geodesic distance are invariant by the action of $Gl(H, B_1^{sa})$, it suffices to consider the case $a = 1$. Note that

$$\lim_{r\to 0^+} \frac{1}{r} d_1(e^{rX}, e^{rY}) = \lim_{r\to 0^+} \|Y - X + r^2\kappa(X,Y) + o(r^2)\|_1 = \|Y-X\|_1.$$

Then it is sufficient to show the existence of the following limit

$$\lim_{r\to 0^+} \frac{1}{r^3}(r\,\|X-Y\|_1 - \|r(Y-X) + r^3\kappa(X,Y) + o(r^3)\|_1),$$

which is equivalent to the existence of the limit

$$\lim_{r\to 0^+} \frac{1}{r^2}(\|X-Y\|_1 - \|(Y-X) + r^2\kappa(X,Y)\|_1).$$

This limit exists and is equal to $-DP_1(Y-X)(\kappa(X,Y))$, therefore

$$s_1(X,Y) = \frac{-DP_1(Y-X)(\kappa(X,Y))}{\|Y-X\|_1}.$$

By the MIP property, this limit is non positive. On the other hand,

$$-DP_1(Y-X)(\kappa(X,Y)) \ge \|Y-X\|_1 - \|Y - X + \kappa(X,Y)\|_1$$

and therefore

$$s_1(X,Y) \ge 1 - \frac{\|Y - X + \kappa(X,Y)\|_1}{\|X-Y\|_1}.$$

□

Chapter 5

The geometry of Δ_p with $1 < p < \infty$

5.1 Introduction

In this chapter we consider the geometrical structure of the manifold Δ_p $(1 < p < \infty)$, with the Finsler metric defined before. For each $a \in \Delta_p$, the tangent space $T_a\Delta_p$ can be identified with B_p^{sa} which is a Banach space uniformly convex. This fact is consequence of the Clarkson´s inequality (see (2.1) and (2.3)). The uniform convexity of the norm has a rich geometrical structure, for example uniqueness the short curve connecting two points.
The concept of uniform convexity for Banach spaces was introduced in [20]. In this work, Clarkson showed that the classical Banach spaces L^p $(1 < p < \infty)$ are uniformly convex. For $p \geq 2$, the proof is based on the inequality

$$\|f - g\|_p^p + \|f + g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p) \tag{5.1}$$

for the L^p-norm.
This inequality, valid on the tangent spaces, has geometric consequences in Δ_p, via the exponential map.

5.2 Clarkson´s inequalities and Uniform Convexity

It is well-known that in a Hilbert space H one has

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \tag{5.2}$$

for all $x, y \in H$ and $0 \leq t \leq 1$.
In a Banach space E, there are inequalities analogous to (5.2). In this section we recall some of these inequalities, from now on we denote by $W_p(t)$ the function $t(1-t)^p + t^p(1-t)$.
Let $f : E \to \bar{\mathbb{R}}$ be a proper functional and D a nonempty convex subset of E, then f is said convex on D if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $0 \leq t \leq 1$ and $x, y \in D$. Recall that f is said uniformly convex on D if there exists a function $\mu : [0, \infty) \to [0, \infty)$ with $\mu(s) = 0$ if and only of $s = 0$ such that

$$f(tx + (1-t)ty) \leq tf(x) + (1-t)f(y) - t(1-t)\mu(\|x - y\|) \tag{5.3}$$

for all $x, y \in D$ and $t \in [0, 1]$. If the inequality (5.3) is valid for all $x, y \in D$ when $t = \frac{1}{2}$, then f is said to be uniformly convex at center, Zălinescu [64] remarked that for a convex function f, f is uniformly convex on D if and only if f is uniformly convex at center on D.
We consider for $r > 1$ the functional

$$f_r : B_p \to [0, \infty), \qquad f_r(X) = \|X\|_p^r.$$

Now, we recall some inequalities which allow to prove the uniformly convexity of f_r.
Ball, Carlen and Lieb in [7] proved the following inequalities:

Proposition 5.2.1. *For $A, B \in B_p$ and $1 \leq p \leq 2$ it hold*

$$\|A\|_p^2 + (p-1)\|B\|_p^2 \leq \frac{1}{2}(\|A + B\|_p^2 + \|A - B\|_p^2). \tag{5.4}$$

Proposition 5.2.2. *For $1 < p < \infty$, f_r is uniformly convex where $r = \max\{p, 2\}$.*

Proof. By the previous remark as f_r is a convex function, it is sufficient to see that f_r uniformly convex at center on B_p.
First, we consider the case $2 < p < \infty$. In (2.3), setting $A = X/2$ and $B = Y/2$, one has

$$\begin{aligned} f_p(\frac{X+Y}{2}) = \|\frac{X+Y}{2}\|_p^p &\leq \frac{1}{2}(\|X\|_p^p + \|Y\|_p^p) - \frac{1}{2^p}\|X - Y\|_p^p \\ &= \frac{1}{2}(\|X\|_p^p + \|Y\|_p^p) - \frac{1}{4}\mu(\|X - Y\|_p). \end{aligned}$$

where $\mu(s) = \frac{1}{2^{p-2}} s^p$.
If $1 < p \leq 2$, then using (5.4) setting $A = (X+Y)/2$ and $B = (X-Y)/2$, we get

$$\begin{aligned} f_2(\frac{X+Y}{2}) = \|\frac{X+Y}{2}\|_p^2 &\leq \frac{1}{2}(\|X\|_p^2 + \|Y\|_p^2) - \frac{1}{4}(p-1)\|X-Y\|_p^2 \\ &= \frac{1}{2}(\|X\|_p^2 + \|Y\|_p^2) - \frac{1}{4}\mu(\|X-Y\|_p), \end{aligned} \tag{5.5}$$

with $\mu(s) = (p-1)s^2$. □

Recall the following result of Xu ([63], Th. 1)

Theorem 5.2.3. *Let $s > 1$ be a fixed real number. Then the functional $\|.\|^s$ is uniformly convex on the whole Banach space $(X, \|.\|)$ if and only if X is s-uniformly convex, i.e. there exists a constant $c > 0$ that $\delta_X(\epsilon) \geq c\epsilon^s$ for all $0 < \epsilon \leq 2$.*

Therefore one obtains, that B_p is r-uniformly convex with r as above. For the sake of simplicity, we denote

$$c_r = \begin{cases} p-1 & \text{if } r = 2, \\ \frac{1}{2^{p-2}} & \text{if } r \neq 2. \end{cases}$$

By the definition of uniform convexity of f_r, we have that for each $t > 0$

$$\begin{aligned} \mu_0(t) = \inf \quad \{ & \frac{sf_r(X) + (1-s)f_r(Y) - f_r(sX + (1-s)Y)}{W_r(s)} : \\ 0 & < s < 1, X, Y \in B_p(H), \|X-Y\|_p = t\}. \end{aligned}$$

It is easy to check that $\mu_0(ct) = c^r\mu_0(t)$ for all $c, t > 0$. This leads to $\mu_0(t) = \mu_0(1)t^r$ for $t > 0$ and hence the following inequality.

Proposition 5.2.4. *For all $X, Y \in B_p$, $t \in [0,1], 1 < p < \infty$ and $\hat{c}_r$ as above, we have:*

$$\|tX + (1-t)Y\|_p^r \leq t\|X\|_p^r + (1-t)\|Y\|_p^r - W_r(t)\hat{c}_r\|X-Y\|_p^r. \tag{5.6}$$

where $\hat{c}_r = \mu_0(1)$.

Remark 5.2.5. For the L^p spaces ([56], Ch. 3), it is well known the best possible function $h(t)$ such that

$$\|tf + (1-t)g\|_p^r \leq t\|f\|_p^r + (1-t)\|g\|_p^r - h_r(t)\|f-g\|_p^r, \tag{5.7}$$

with $f, g \in L^p$ is $h_r(t) = W_r(t)\alpha_r$, with

$$\alpha_p = \frac{1 + t_p^{p-1}}{(1+t_p)^{p-1}},$$

where $t_p \in [0,1]$ is the unique solution of the equation

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0.$$

if $p > 2$, and $\alpha_2 = p - 1$ if $1 < p < 2$.
With a similar proof to Propostion 5.3.15, from the inequality (5.5), we obtain that if $X, Y \in B_p$ and $1 < p \leq 2$, then for all $t \in [0,1]$

$$\|tX + (1-t)Y\|_p^2 \leq t\|X\|_p^2 + (1-t)\|Y\|_p^2 - t(1-t)(p-1)\|X-Y\|_p^2, \tag{5.8}$$

and in consequence we get $\hat{c}_2 \leq (p-1)$.

5.3 The geometry of Δ_p

5.3.1 Minimal Curves

In this section we investigate the existence of minimal curves for the Finsler metric just defined.
We prove that the curve $\gamma_{a,b}$ joining a with b is the minimun of the s-energy functional for $s \geq 1$.
For a piecewise differentiable curve $\beta : [0,1] \to B_p^{sa}$, one computes the *length* of the curve β by

$$L(\beta) = \int_0^1 \|\dot{\beta}(t)\|_p dt.$$

The proof, that we present here, involves the uniformly convexity of the tangent spaces.
Let δ be a piecewise smooth curve $\delta \subseteq \Delta_p$. Then $\delta = e^{\beta}$ for a uniquely determined piecewise smooth curve $\beta = \log(\delta)$ such that $\beta \subseteq B_p^{sa}$. By Claim 3.1.11

$$\dot{\delta} = dexp_{\beta}(\dot{\beta}) = \int_0^1 e^{(1-t)\beta}\dot{\beta}e^{t\beta}dt.$$

We begin comparing the lengths of the curves δ and β.

Theorem 5.3.1. *Let $\delta = e^{\beta} \subseteq \Delta_p$ be a piecewise smooth curve. Then $L(\beta) \leq L_p(\delta)$.*

Proof. Let us compute the speed of δ using the Proposition 3.1.10:

$$\|\dot{\beta}\|_p \leq \|e^{-\frac{\beta}{2}} dexp_{\beta}(\dot{\beta}) e^{-\frac{\beta}{2}}\|_p = \|\dot{\delta}\|_{p,\delta}.$$

□

Corolary 5.3.2. *Let $X, Y \in B_p^{sa}$ and $a \in \Delta_p$. Then*

$$\|X - Y\|_{p,a} \leq d_p(exp_a(X), exp_a(Y)).$$

Proof. Let $\delta \in \Omega_{e^X, e^Y}$, put $\delta = e^\beta$ as before and $a = 1 \in \Delta_p$. Then

$$\|Y - X\|_p = \|\beta(1) - \beta(0)\|_p = \| \int_0^1 \dot{\beta} dt\|_p \leq \int_0^1 \|\dot{\beta}\|_p dt = L(\beta) \leq L_p(\delta).$$

Hence

$$\|Y - X\|_p \leq d_p(e^X, e^Y). \tag{5.9}$$

□

Theorem 5.3.3. *Assume that, for the geometry induced by the norm $\| \cdot \|_{p,a}$, the unique short curve joining 0 to X in B_p^{sa} is the straight segment $\gamma(t) = tX$. Then $\gamma_{a,b}$ is the unique piecewise smooth curve joining a to b in Δ_p.*

Proof. Let $\delta = e^\beta$ be a short, piecewise smooth curve joining 1 and e^X in Δ_p. Now $L(\beta) = \|X\|_p$. Since $L(\beta) \leq L_p(\delta)$, then β is a piecewise smooth curve in B_p^{sa}, joining 0 to X, with length less or equal than $\|X\|_p$, which is the length of the straight segment tX ($t \in [0,1]$) in B_p^{sa}. Then $\beta(t) = tX$, and $\delta(t) = e^{tX}$. Then general case follows by the homogeneity of the metric of Δ_p. □

Remark 5.3.4. The hypothesis of Theorem 5.3.3 holds for any $p \in (1, \infty)$, since its is a simple consequence of the uniform convexity of these spaces (see Theorem 2.2.4).

As a simple consequence of the Corolary 5.3.2 we obtain the completeness of the (Δ_p, d_p).

Proposition 5.3.5. *Δ_p is a complete metric space with the geodesic distance d_p.*

Proof. Consider a Cauchy sequence $\{a_n\} \subseteq \Delta_p$. By (5.9), $X_n = \log(a_n)$ is a Cauchy sequence in B_p^{sa}. Then there exists an operator $X \in B_p^{sa}$ such that $X_n \xrightarrow{\|\cdot\|_p} X$. Hence

$$d_p(a_n, e^X) = \| \log(e^{\frac{X}{2}} e^{-a_n} e^{\frac{X}{2}})\|_p \to 0,$$

when $n \to \infty$. □

We summarize in the following proposition the basic properties of the metric space Δ_p.

Proposition 5.3.6. *Given $a, b \in \Delta_p$ and $g \in Gl(H, B_p)$ we get*

1. $$d_p(a, b) = d_p(a^{-1}, b^{-1}).$$

2. *For all $t_0, t_1 \in \mathbb{R}$*
$$d_p(a\sharp_{t_0}b, a\sharp_{t_1}b) = |t_0 - t_1|\, d_p(a, b),$$
and in particular
$$d_p(a, a\sharp_t b) = |t|\, d_p(a, b).$$

3. *If $X, Y \in B_p^{sa}$ commute, we have*
$$\|X - Y\|_{p,a} = d_p(exp_a(X), exp_a(Y)).$$
In particular on each line $\mathbb{R}X \subseteq B_p^{sa}$ the exponential map preserves distances.

4. *If 1 lies on the geodesic $a \sharp_t b$, then a and b commute*
$$\log(b) = -\frac{1-t}{t}\log(a).$$
where $t = d_p(a, 1)/d_p(a, b)$.

5. *Let $s \in [1, \infty)$. Then the s-energy functional*
$$E_s : \Omega_{a,b} \to \mathbb{R}^+, \qquad E_s(\beta) := \int_0^1 \|\dot{\beta}(t)\|_{p,\beta(t)}^s dt,$$
has its global minimun $d_p^s(a, b)$ precisely at $\gamma_{a,b}$.

Proof. The proof of this proposition is similar to the proof of Corolary 4.3.3 and 4.3.4 and Proposition 4.3.7 and 4.3.8. □

Given $a, b \in \Delta_p$ and $\gamma_{a,b}$ the shortest curve joining them, we can define the following midpoint map
$$m : \Delta_p \times \Delta_p \to \Delta_p,\ m(a, b) := \gamma_{a,b}(\frac{1}{2}).$$

Definition 5.3.7. *Let $K \subseteq \Delta_p$. We say that K is convex if, for any $a, b \in K$, $\gamma_{a,b}(t) \in K$ for any $t \in [0, 1]$.*

5.3.2 Weak Semi Paralelogram Law

Let $(V, \langle .,. \rangle)$ be an euclidean space, i.e. V is a real vector space (finite or infinite dimensional) and $\langle .,. \rangle$ is a positive definite symmetric bilinear form on V. Then $\|v\| = \sqrt{\langle v, v\rangle}$ defines a norm on V and the parallelogram law states that for $u, v \in V$ we have

$$\|v - w\|^2 + \|v + w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

If we consider a parallelogram with vertices x, x_1, x_2 and $x_3 = x_1 + x_2 - x$, then the parallelogram law reads

$$d(x_1, x_2)^2 + d(x, x_3)^2 = 2d(x, x_1)^2 + 2d(x, x_2)^2$$

where $d(a, b) := \|a - b\|$. If $z = \frac{x_1+x_2}{2}$ is the midpoint of x_1 and x_2, then we get

$$d(x_1, x_2)^2 + 4d(x, z)^2 = 2d(x, x_1)^2 + 2d(x, x_2)^2$$

Definition 5.3.8. *Let (X, d) be a metric space. We say that (X, d) satisfies the* semi parallelogram law *(SPL) if for $x_1, x_2 \in X$ there exists a point $z \in X$ such that for each $x \in X$ we have*

$$d(x_1, x_2)^2 + 4d(x, z)^2 \leq 2d(x, x_1)^2 + 2d(x, x_2)^2.$$

Note that the point z occuring in the preceding definition plays the role of a mid-point between x_1 and x_2.
The above condition can be rephrased as follows in a geodesic length space:
For any $x \in X$ and any minimal curve $\eta : [0, 1] \to X$ with $\eta(0) = x_1, \eta(1) = x_2$, we have

$$d(x, \eta(\frac{1}{2}))^2 \leq \frac{1}{2}d(x, \eta(0))^2 + \frac{1}{2}d(x, \eta(1))^2 - \frac{1}{4}d(\eta(0), \eta(1))^2. \tag{5.10}$$

One natural generalization of the p-uniform convexity to a metric space is the following:

Definition 5.3.9. *Let (X, d) be a geodesic length space and $p \geq 2$. We say that (X, d) is p-uniformly convex if for any $x \in X$ and any minimal curve $\eta : [0, 1] \to X$ with $\eta(0) = x_1, \eta(1) = x_2$, there exists a constant $c_p > 0$ such that*

$$d(x, \eta(\frac{1}{2}))^p \leq \frac{1}{2}d(x, \eta(0))^p + \frac{1}{2}d(x, \eta(1))^p - \frac{1}{4}c_p d(\eta(0), \eta(1))^p. \tag{5.11}$$

If $p = 2$ and $c_2 = 1$, then the inequality (5.11) corresponds to the SPL.

At this point we can easily prove the r-uniform convexity “at the origin ” of Δ_p.

Lemma 5.3.10. *Let $X, B \in B_p^{sa}$ and γ_t be the geodesic joining e^B with e^{-B}. Then for $1 < p < \infty$*

$$d_p(e^X, \gamma_{1/2})^r \leq \frac{1}{2}(d_p(e^X, \gamma_0)^r + d_p(e^X, \gamma_1)^r) - \frac{1}{4}c_r d_p(\gamma_0, \gamma_1)^r, \tag{5.12}$$

with r and c_r as above.

Proof. By (2.3), if $2 < p < \infty$

$$\begin{aligned} 2(\|X\|_p^p + \|B\|_p^p) &= 2(d_p(e^X, 1)^p + d_p(e^B, 1)^p) \\ &\leq \|X - B\|_p^p + \|X + B\|_p^p \\ &\leq d_p(e^X, e^B)^p + d_p(e^X, e^{-B})^p. \end{aligned}$$

Since $d(e^B, 1) = \frac{1}{2}d(e^B, e^{-B}) = \frac{1}{2}L_p(\gamma_t)$ we have

$$\frac{1}{2^p}L_p(\gamma_t)^p \leq \frac{1}{2}(d_p(e^X, \gamma_0)^p + d_p(e^X, \gamma_1)^p) - d_p(e^X, \gamma_{1/2})^p.$$

Now, we consider the case $1 < p \leq 2$. Applying the exponential map and using the EMI property in the inequality (5.4) we obtain

$$\begin{aligned} \|X\|_p^2 + (p-1)\|B\|_p^2 &= d_p(e^X, 1)^p + (p-1)d_p(e^B, 1)^2 \\ &\leq \frac{1}{2}(\|X + B\|_p^2 + \|X - B\|_p^2) \\ &\leq \frac{1}{2}(d_p(e^X, e^B)^2 + d_p(e^X, e^{-B})^2). \end{aligned}$$

Since $d(e^B, 1) = \frac{1}{2}d(e^B, e^{-B}) = \frac{1}{2}L_p(\gamma_t)$, we have

$$(p-1)\frac{1}{2^2}L_p(\gamma_t)^2 \leq \frac{1}{2}(d_p(e^X, \gamma_0)^2 + d_p(e^X, \gamma_1)^2) - d_p(e^X, \gamma_{1/2})^2.$$

□

The following proposition establishes the r-uniform convexity for Δ_p.

Theorem 5.3.11. *Let $X \in B_p^{sa}$ and $\gamma_t : [0, 1] \to \Delta_p$ be a geodesic. Then for $1 < p < \infty$*

$$d_p(e^X, \gamma_{1/2})^r \leq \frac{1}{2}(d_p(e^X, \gamma_0)^r + d_p(e^X, \gamma_1)^r) - \frac{1}{4}c_r d_p(\gamma_0, \gamma_1)^r, \tag{5.13}$$

with r and c_r as above.

Proof. Given $a = \gamma_0, b = \gamma_1 \in \Delta_p$, let $m = m(a,b)$ be the midpoint of a and b. We claim that there exist $g \in GL(H, B_p)$ and $X \in B_p^{sa}$ with

$$l_g(a) = e^X;\ l_g(b) = e^{-X}.$$

First; observe that $h_1 = b^{-\frac{1}{2}}$ satisfies $l_{h_1}(b) = 1$. Let $x := l_{h_1}(a)$ and we define $h_2 : x^{-\frac{1}{4}}$ and $g := h_2 h_1$.
Then

$$l_g(a) = h_2 h_1 a h_1 h_2 = l_{h_2}(x) = x^{\frac{1}{2}}.$$

$$l_g(b) = h_2 h_1 b h_1 h_2 = l_{h_2}(1) = x^{-\frac{1}{2}}.$$

Now the claim above follows with $x = e^{2X}$.
By the invariance of the distance under the action of $GL(H, B_p)$ and the claim, it suffices to verify the inequality for pairs a, b with $b = a^{-1}$. This case follows from the previous lemma. □

Corolary 5.3.12. *The metric space (Δ_p, d_p) is strictly ball convex for $1 < p < \infty$.*

Proof. Let $a, b, c \in \Delta_p$ with $a \neq b$, then

$$\begin{aligned} d_p(c, \gamma_{a,b}(\tfrac{1}{2}))^r &\leq \frac{1}{2}(d_p(c,a)^r + d_p(c,b)^r) - \frac{1}{4}c_r d_p(a,b)^r \\ &< (\max\{d_p(c,a), d_p(c,b)\})^r. \end{aligned}$$

□

Corolary 5.3.13. *If $a, b, c \in \Delta_p$ $(1 < p < \infty)$ are three arbitrary points then abc_Δ will denote the geodesic triangle of vertices a, b, c (which by the uniqueness the geodesics is uniquely defined), then we have*

$$2^r (L_A)^r \leq 2^{r-1} B^r + 2^{r-1} C^r - c_r A^r$$

with $A = d_p(b,c), B = d_p(c,a), C = d_p(a,b)$ and L_A the length of the geodesic joining a to $m(b,c)$.

Another interesting consequence of Theorem 5.3.11 is the uniformly convexity of (Δ_p, d_p).

Corolary 5.3.14. *For $1 < p < \infty$, the metric space (Δ_p, d_p) is uniformly ball convex.*

One can to compute the modulus of uniform convexity explicitly in these cases. Consider $\epsilon > 0$, set $d_p(a,b), d_p(a,c) \leq s$ and $d_p(b,c) \geq \epsilon s$. Then,

$$d_p(a, m(b,c))^r \leq s^r - \frac{1}{4}c_r d_p(b,c)^r \leq s^r - \frac{1}{4}c_r(\epsilon s)^r = [1 - c_r(\frac{\epsilon}{2^{2/r}})^r]s^r.$$

Then an admissible value for the modulus of uniform convexity is $\rho_{\Delta_p}(\epsilon) = 1 - [1 - c_r(\frac{\epsilon}{2^{2/r}})^r]^{1/r}$.
Note that the formula of the modulus is similar with the one obtained by Clarkson for the space L^p.

Proposition 5.3.15. *Let $X \in B_p^{sa}$, $\gamma_t : [0,1] \to \Delta_p$ a geodesic and $1 < p \leq 2$. Then for all $t \in [0,1]$*

$$d_p(e^X, \gamma_t)^r \leq (1-t)d_p(e^X, \gamma_0)^r + td_p(e^X, \gamma_1)^r - t(1-t)c_r d_p(\gamma_0, \gamma_1)^r. \tag{5.14}$$

Proof. Let us denote $W_2(t) = t(1-t)$.
Given any geodesic $\gamma_t : [0,1] \to \Delta_p$, it suffices to prove the previous inequality for all dyadic $t \in [0,1]$. It obviously holds for $t = 0$ and $t = 1$. Assume that it holds for all $t = k2^{-n}$ with $k = 0, 1, ..., 2^n$. We want to prove that (5.14) also holds for all $t = k2^{-(n+1)}$ with $k = 0, 1, ..., 2^{n+1}$. For k even this is clear. Fix $t = k2^{-(n+1)}$ with k odd; and put $\Delta t = 2^{-(n+1)}$. Then by (5.13)

$$d_p(e^X, \gamma_t)^r \leq \frac{1}{2}(d_p(e^X, \gamma_{t-\Delta t})^r + d_p(e^X, \gamma_{t+\Delta t})^r) - \frac{1}{4}c_r d_p(\gamma_{t-\Delta t}, \gamma_{t+\Delta t})^r. \tag{5.15}$$

By the assumption for multiples of 2^{-n}

$$\begin{aligned} d_p(e^X, \gamma_{t\pm\Delta t})^r &\leq (1 - t \mp \Delta t)d_p(e^X, \gamma_0)^r + (t \pm \Delta t)d_p(e^X, \gamma_1)^r \\ &- W_2(t \pm \Delta t)c_r d_p(\gamma_0, \gamma_1)^r. \end{aligned}$$

Thus, by (5.15)

$$d_p(e^X, \gamma_t)^r \leq (1-t)d_p(e^X, \gamma_0)^r + td_p(e^X, \gamma_1)^r - [g(t, \Delta t)]c_r d_p(\gamma_0, \gamma_1)^r,$$

where $g(t, \Delta t) = (\Delta t)^2 + \frac{1}{2}W_2(t - \Delta t) + \frac{1}{2}W_2(t + \Delta t)$.
Since,

$$W_2(t) = (\Delta t)^2 + \frac{1}{2}W_2(t - \Delta t) + \frac{1}{2}W_2(t + \Delta t) = g(t, \Delta t). \tag{5.16}$$

then

$$d_p(e^X, \gamma_t)^r \leq (1-t)d_p(e^X, \gamma_0)^r + td_p(e^X, \gamma_1)^r - W_2(t)c_r d_p(\gamma_0, \gamma_1)^r.$$

□

Corolary 5.3.16. *Let* $\gamma_t, \eta_t : [0,1] \to \Delta_p, 1 < p \leq 2$ *and* $t \in [0,1]$*, then*

$$\begin{aligned} d_p(\eta_t, \gamma_t)^r &\leq (1-t)^2 d_p(\eta_0, \gamma_0)^r + t^2 d_p(\eta_1, \gamma_1)^r - t(1-t)c_r(L(\eta)^r + L(\gamma)^r) \\ &+ t(1-t)[d_p(\eta_0, \gamma_1)^r + d_p(\eta_1, \gamma_0)^r]. \end{aligned}$$

Proof. Applying (5.14) twice, we obtain that

$$\begin{aligned} d_p(\eta_t, \gamma_t)^r &\leq (1-t)d_p(\eta_0, \gamma_t)^r + t d_p(\eta_1, \gamma_t)^r - t(1-t)c_r L(\eta)^r \\ &\leq (1-t)[(1-t)d_p(\eta_0, \gamma_0)^r + t d_p(\eta_0, \gamma_1)^r - t(1-t)c_r L(\gamma)^r] \\ &+ t[(1-t)d_p(\eta_1, \gamma_0)^r + t d_p(\eta_1, \gamma_1)^r - t(1-t)c_r L(\gamma)^r] \\ &- t(1-t)c_r L(\eta)^r \\ &= (1-t)^2 d_p(\eta_0, \gamma_0)^r + t^2 d_p(\eta_1, \gamma_1)^r - t(1-t)c_r(L(\eta)^r + L(\gamma)^r) \\ &+ t(1-t)[d_p(\eta_0, \gamma_1)^r + d_p(\eta_1, \gamma_0)^r]. \end{aligned}$$

□

In particular if $p = r = 2$, the metric space (Δ_2, d_2) results an Alexandrov nonpositive curvature space (ANPC) and we get

$$d_2(\eta_t, \gamma_t) \leq (1-t)d_2(\eta_0, \gamma_0) + t d_2(\eta_1, \gamma_1)$$

or equivalently, d_2 is strictly convex on geodesics.
Now, we try to extend this result for $p > 1$ with $p \neq 2$.
Recently, Larotonda [47] using the theory of dissipative operators and the theory of entire functions, derived several operator inequalities for unitarily invariant norms. Among then, if $X, Y \in B_p^{sa}$

$$\| \log(e^{\frac{-t}{2}X} e^{tY} e^{\frac{-t}{2}X}) \|_p \leq t \| \log(e^{\frac{-X}{2}} e^{Y} e^{\frac{-X}{2}}) \|_p. \tag{5.17}$$

This inequality establishes the convexity of the geodesic distance d_p in the Finsler manifold Δ_p, that is

Proposition 5.3.17. *Let* $a, b, c, d \in \Delta_p$*, then*

$$d_p(\gamma_{a,b}(t), \gamma_{c,d}(t)) \leq t d_p(a,c) + (1-t) d_p(b,d). \tag{5.18}$$

Proof. Consider the geodesic rectangle with vertices a, b, c, d. Let $\gamma_{c,b}$ be the short curve joining c to b in Δ_p, and consider the triangle with sides c, b, d, and the geodesic triangle with sides b, a, c. Note that $\gamma_{c,b}(t) = \gamma_{b,c}(1-t)$ and the same holds for $\gamma_{a,b}$. Then, by the triangle inequality

$$d_p(\gamma_{a,b}(t), \gamma_{c,d}(t)) \leq d_p(\gamma_{a,b}(t), \gamma_{c,b}(t)) + d_p(\gamma_{c,b}(t), \gamma_{c,d}(t)),$$

and by (5.17)

$$d_p(\gamma_{c,b}(t), \gamma_{c,d}(t)) \leq t\, d_p(b, d).$$

Also

$$d_p(\gamma_{b,c}(1-t), \gamma_{b,a}(1-t)) \leq (1-t)\, d_p(a, b).$$

Adding these two inequalities yields the convexity of d_p. □

Remark 5.3.18. From the uniqueness of short curves and the inequality (5.18), we obtain that for $1 < p < \infty$, (Δ_p, d_p) is a Busemann non positive curvature space.

Now, we investigate when $d_p(exp_a(X), exp_a(Y)) = \|X - Y\|_{p,a}$.
The method of the proof that follows is based on a similar argument used in [4]. In this work the authors studied the occurrence of the equality for a C*-algebra A with trace, in the 2-norm and the operator norm.

Theorem 5.3.19. *Let $a \in \Delta_p$, $X, Y \in B_p^{sa}$. Then we have*

$$d_p(exp_a(X), exp_a(Y)) = \|X - Y\|_{p,a}, \tag{5.19}$$

if and only if $a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}$ and $a^{-\frac{1}{2}}Ya^{-\frac{1}{2}}$commute.

Proof. Suppose that $a = 1$. Clearly, $d_p(e^X, e^Y) = \|X - Y\|_{p,1}$ if X and Y commute. Let $\gamma_{b,c}$ be the geodesic joining $b = e^X$ with $c = e^Y$, and let $\alpha \subseteq B_p^{sa}$ such that $\gamma_{b,c} = e^{\alpha}$. Then α, that joins X with Y, satisfies the inequality

$$\|X - Y\|_p \leq L(\alpha) \leq d_p(e^X, e^Y) = \|X - Y\|_p.$$

This implies, by the uniqueness of short curve connecting two points, that

$$X + t(Y - X) = \alpha(t) = \log(\gamma_{b,c}(t)) = \log(b\sharp_t c).$$

Since the map $f(t) = d_p(e^{tX}, e^{tY})$ is convex with $f(0) = 0$ and $f(1) = d_p(e^X, e^Y) = \|X - Y\|_p$, and sX, sY satisfy the hypothesis of this theorem for $s \in [0, 1]$, we have that

$$sX + st(Y - X) = log(b^s \sharp_t c^s),$$

for $t, s \in [0,1]$. If one computes $\frac{\partial^3}{\partial t^3}$ at $s = 0$on both sides of this equality, one obtains

$$0 = \frac{1}{2}(t^2 + t)[[X, Y], X - Y].$$

Then, $[[X, Y], X] = [[X, Y], Y]$. Therefore, by the properties of the trace,

$$tr([[X, Y], X]X) = tr([[X, Y], Y]X) = 0,$$

this implies that

$$tr(X^2 Y^2) = tr(XYXY).$$

This means that we have equality in the Cauchy-Schwarz inequality

$$\begin{aligned} tr(XYXY) = tr((YX)^* XY) &\leq [tr((YX)^* YX)]^{\frac{1}{2}} [tr((XY)^* XY)]^{\frac{1}{2}} \\ &= tr(X^2 Y^2) = tr(XYXY). \end{aligned}$$

So XY is a multiple of YX, i.e. $XY = \beta YX$, $\beta \in \mathbb{C}$. Replacing this equality above equality we get,

$$tr(X^2 Y^2) = tr(XYXY) = \beta tr(XYYX) = \beta tr(X^2 Y^2).$$

This implies that $\beta \geq 0$. If $\beta = 0$, then X and Y commute. Otherwise $\beta = 1$. This completes the proof for $a = 1$. For the general case, note that by the invariance under the action

$$d_p(exp_a(X), exp_a(Y)) = \|X - Y\|_{p,a}, \tag{5.20}$$

is equivalent to

$$d_p(e^{a^{-\frac{1}{2}} X a^{-\frac{1}{2}}}, e^{a^{-\frac{1}{2}} Y a^{-\frac{1}{2}}}) = \|a^{-\frac{1}{2}} X a^{-\frac{1}{2}} - a^{-\frac{1}{2}} Y a^{-\frac{1}{2}}\|_p.$$

Hence, it follows from the case $a = 1$, that the equality (5.20) holds if and only if $a^{-\frac{1}{2}} X a^{-\frac{1}{2}}$ and $a^{-\frac{1}{2}} Y a^{-\frac{1}{2}}$ commute. □

5.3.3 Best approximation

Given a subset $K \subseteq \Delta_p$ and an element $a \in \Delta_p$ put

$$d_p(a, K) = \inf\{d_p(a, k) : k \in K\}.$$

We shall prove (Theorem 5.3.20) that, as in a Hilbert space, one can define a metric projection onto convex closed subsets of Δ_p. In other words given K a convex closed subset of Δ_p and $a \in \Delta_p$, there is a unique $k_0 \in K$ such that the length of the geodesic joining a with k_0 is the distance between a and K. That is, there is a unique solution to the minimization problem

$$\begin{cases} k_0 \in K \\ d_p(a,k_0) \leq d_p(a,k) \qquad \forall\, k \in K \end{cases} \tag{5.21}$$

Theorem 5.3.20. *(Best Approximation) Let $K \subseteq \Delta_p$ be a convex closed set, $1 < p < \infty$ and $a \in \Delta_p$. Then the problem (5.21) has a unique solution. In other words, there is a unique $\pi_K(a) = k_0 \in K$ such that $d_p(a,q_0) = d_p(a,K)$.*
In addition, if $\widetilde{a}$ belongs to the geodesic segment joining a with $\pi_K(a)$, then $\pi_K(\widetilde{a}) = \pi_K(a)$.

Proof. Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence in K, such that

$$d_p(a,k_n) \to d_p(a,K),$$

by Theorem 5.3.11 we obtain that

$$\begin{aligned} \frac{1}{4}c_r d_p(k_n,k_m)^r &\leq \frac{1}{2}(d_p(k_n,a)^r + d_p(a,k_m)^r) - d_p(a,k_{n,m})^r \\ &\leq \frac{1}{2}(d_p(k_n,a)^r + d_p(a,k_m)^r) - d_p(a,K)^r, \end{aligned} \tag{5.22}$$

where $k_{n,m} = \gamma_{1/2} \in K$, with γ_t is the geodesic joining k_n and k_m.
This implies that $\{k_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in K, hence convergent to some $k_0 \in N$. Since K is closed, $k_0 \in K$. By the continuity of the distance we have

$$d_p(k_0,a) = \lim_{n\to\infty} d_p(k_n,a) = d_p(a,K),$$

For the uniqueness part, let $k_1, k_2 \in K$ such that

$$d_p(k_1,a) = d_p(a,K) = d_p(k_2,a).$$

Replacing k_n and k_m by k_1 and k_2 respectively in (5.22), we obtain

$$\begin{aligned} d_p(a,k_{1,2})^r &\leq \frac{1}{2}(d_p(k_1,a)^r + d_p(a,k_2)^r) - \frac{1}{4}c_r d_p(k_1,k_2)^r \\ &= d_p(a,K)^r - \frac{1}{4}c_r d_p(k_1,k_2)^r, \end{aligned}$$

since $k_{1,2} \in K$, the above inequality proves that $d_p(k_1,k_2) = 0$. ☐

Definition 5.3.21. *Let $K \subseteq \Delta_p$ be a convex closed set, $1 < p < \infty$ and $a \in \Delta_p$. By Theorem 5.3.20 there is exactly one point $\pi_K(a) \in K$ such that*

$$d_p(a, \pi_K(a)) = d_p(a, K).$$

Then $\pi_K(a)$ is called the projection of a to K. The map $\pi_K : \Delta_p \to K$ is called the projection map to K.

Remark 5.3.22.

1. One important fact here is the existence of a unique projection without assuming any kind of compactness of K.
2. It clearly that $\pi_K^2 = \pi_K$.

Theorem 5.3.23. *Let $K \subseteq \Delta_p$ be a convex closed set, $1 < p < \infty$ and π_K the projection map onto K. Then π_K is continuous.*

Proof. Let the sequence $\{c_n\}$ converge to c in Δ_p. For simplicity, denote $\pi_K(c_n)$ by u_n. Now $\{u_n\}$ is a Cauchy sequence in K, otherwise there are positive numbers ϵ and subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ such that $n_k < m_k$ and $d_p(u_{n_k}, u_{m_k}) \geq \epsilon$ for all k. Put $a_k = u_{n_k}, b_k = u_{m_k}$ and $M_k = \max\{d_p(c, a_k), d_p(c, b_k)\}$.
Note that $M_k \to d_p(c, K)$ as $k \to \infty$. Now $d_p(c, a_k) \leq M_k, d_p(c, b_k) \leq M_k$ and $d_p(a_k, b_k) \geq (\frac{\epsilon}{M_k})M_k$. This implies

$$d_p(c, m(a_k, b_k)) \leq M_k\left(1 - \rho_{\Delta_p}\left(\frac{\epsilon}{M_k}\right)\right) \leq M_k\left(1 - \rho_{\Delta_p}\left(\frac{d(a_k, b_k)}{M_k}\right)\right).$$

Also $\rho_{\Delta_p}(\frac{\epsilon}{M_k}) \leq 1 - \frac{d_p(c,K)}{M_k}$, letting $k \to \infty$, one has $\delta_{\Delta_p}(\frac{\epsilon}{M_k}) \to 0$ and ϵ can not be positive. Thus $\{\pi_K(c_n)\}$ is a Cauchy sequence in K and therefore converges to a point z in K, as $d_p(c, z) = d_p(c, K)$, then $z = \pi_K(c)$. □

Another useful property of the Δ_p spaces with $1 < p \leq 2$ is the following Pythagoras type inequality.

Corolary 5.3.24. *Under the same conditions stated above, we have that for all $k \in K$ with K a closed and convex set and $t \in (0, 1]$*

$$d_p(a, \pi_K(a))^2 + (1-t)(p-1)d_p(\pi_K(a), k)^2 \leq d_p(a, k)^2, \tag{5.23}$$

in particular

$$d_p(a, \pi_K(a))^2 + (p-1)d_p(\pi_K(a), k)^2 \leq d_p(a, k)^2. \tag{5.24}$$

Proof. Let $\gamma_t : [0,1] \to \Delta_p$ be the geodesic joining $\gamma_0 = \pi_K(a)$ and $\gamma_1 = k$, then $\gamma_t \in K$ by the convexity of K. Hence, by (5.3.15)

$$\begin{aligned} d_p(a, \pi_K(a))^2 &\le d_p(a, \gamma_t)^2 \\ &\le (1-t)d_p(a, \pi_K(a))^2 + td_p(a,k)^2 - W_2(t)(p-1)d_p(\pi_K(a), k)^2, \end{aligned}$$

and therefore

$$td_p(a, \pi_K(a))^2 \le td_p(a,k)^2 - W_2(t)(p-1)d_p(\pi_K(a), k)^2.$$

Now if $t \in (0,1]$, this is the desired inequality . □

We shall prove now that the inequality (5.24) is a characterizes solutions of the minimization problem.

Theorem 5.3.25. *Let $K \subseteq \Delta_p$ be a convex closed subset and $a \in \Delta_p$ with $1 < p \le 2$. Suppose that $q_0 \in K$ verifies* (5.24)*, then q_0 is the unique solution of* (5.21)*.*

Proof. For all $k \in K$ and $t \in (0,1]$ we have

$$d(a, q_0)^2 + (p-1)d(q_0, k)^2 \le d(a,k)^2.$$

Then $d(a, q_0) \le d(a, k)$. For the uniqueness part, let $q_0, q_1 \in K$ satisfying (5.24), then

$$(p-1)d(q_1, q_0)^2 \le d(a, q_1)^2 - d(a, q_0)^2 = d(a, K)^2 - d(a, K)^2.$$

□

Chapter 6

Geometry and Interpolation

6.1 Introduction

In [40], the Clarkson's inequalities were generalized to a larger class of functions including the power functions. Bhatia and Kittaneh [14] proved similar inequalities for trace norms on linear combinations of n operators with roots of unity as coefficients.
On B_p, we have defined the following weighted norm :

$$\|X\|_{p,a} := \|a^{-1/2}Xa^{-1/2}\|_p,$$

associated with $a \in Gl(H)^+$.
The material of this chapter is organized as follows. Section 6.2 contains a brief summary of the Complex interpolation method. In Section 6.3 we apply this method and obtain that the curve of interpolation coincides with the curve of weighted norms determined by the positive invertible elements

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}.$$

In Section 6.5, we present an elementary interpolation argument to obtain new Clarkson's type inequalities.

6.2 The Complex Interpolation Method

We recall the construction of interpolation spaces, usually called the complex interpolation method. We follow the notation used in [12] and refer to [44] and [19] for details on this construction.

A compatible couple of Banach spaces is a pair $\bar{X} = (X_0, X_1)$ of Banach spaces X_0, X_1, both continuously embedded in some Hausdorff topological vector space $\mathcal{U}$.
Observe that for all $a, b \in Gl(H)^+$ and $1 \leq p < \infty$, the Banach spaces $(B_p, \| \,.\, \|_{p,a})$ and $(B_p, \| \,.\, \|_{p,b})$ are compatible considering $\mathcal{U} = (B_p, \|.\|_p)$. We will simply write this pair of spaces $\overline{B_p}$ when no confusion can arise.
If X_0 and X_1 are compatible, then one can form their sum $X_0 + X_1$ and their intersection $X_0 \cap X_1$. The sum consists of all $x \in \mathcal{U}$ such that one can write $x = y + z$ for some $y \in X_0$ and $z \in X_1$.
Suppose that X_0 and X_1 are compatible Banach spaces. Then $X_0 \cap X_1$ is a Banach space with norm

$$\|x\|_{X_0 \cap X_1} = max(\|x\|_{X_0}, \|x\|_{X_1}).$$

Moreover, $X_0 + X_1$ is also a Banach space with norm

$$\|x\|_{X_0 + X_1} = inf\{\|y\|_{X_0} + \|z\|_{X_1} : x = y + z\}.$$

A Banach space X is said to be an intermediate space with respect to $\bar{X}$ if

$$X_0 \cap X_1 \subset X \subset X_0 + X_1,$$

and both inclusions are continuous.
Given a compatible pair $\bar{X} = (X_0, X_1)$, one considers the space of all functions f defined in the strip $S = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1\}$ with values in $X_0 + X_1$, and having the following properties:

1. $f(z)$ is continuous and bounded in norm of $X_0 + X_1$ on the strip S.

2. $f(z)$ is analytic relative to the norm of $X_0 + X_1$ on $\overset{\circ}{S}$.

3. $f(j + iy)$ assumes values in the space X_j $(j = 0, 1)$ and is continuous and bounded in the norm of these spaces.

This space of functions is denoted by $\mathcal{F}(\bar{X}) = \mathcal{F}(X_0, X_1)$.
One equips the vector space $\mathcal{F}(\bar{X})$ with the norm

$$\|f\|_{\mathcal{F}(\bar{X})} = max \,\{\sup_{y \in \mathbb{R}} \|f(iy)\|_{X_0}, \sup_{y \in \mathbb{R}} \|f(1 + iy)\|_{X_1}\}.$$

The space $(\mathcal{F}(\bar{X}), \| \,\|_{\mathcal{F}(\bar{X})})$ is a Banach space.

For each $0 < t < 1$ the complex interpolation space, associated to the couple $\bar{X}$, which we denote $\bar{X}_{[t]} = (X_0, X_1)_{[t]}$, is the set of all elements $x \in X_0 + X_1$ representable in the form $x = f(t)$ for some function $f \in \mathcal{F}(\bar{X})$, equipped with the complex interpolation norm

$$\|x\|_{[t]} = inf\{\|f\|_{\mathcal{F}(\bar{X})}\,;\, f \in \mathcal{F}(\bar{X}),\ f(t) = x\}.$$

The two main results of the theory are:

Theorem A. The space $\bar{X}_{[t]}$ is a Banach space and an intermediate space with respect to $\bar{X}$.

Theorem B. Let $\bar{X}$ and $\bar{Y}$ two compatible couples. Assume that T is a linear operator from X_j to Y_j bounded by M_j, $j = 0, 1$. Then for $t \in [0, 1]$

$$\|T\|_{\bar{X}_{[t]} \to \bar{Y}_{[t]}} \leq M_0^{1-t} M_1^t.$$

6.3 Geometric Interpolation

In this section, we state the main result of this chapter. First, we introduce the notation.

For $1 \leq p < \infty, n \in \mathbb{N}, s \geq 1$ and $a \in Gl(H)^+$, let

$$B_p^{(n)} = \{(X_0, \dots, X_{n-1}) : X_i \in B_p\},$$

endowed with the norm

$$\|(X_0, \dots, X_{n-1})\|_{p,a;s} = (\|X_0\|_{p,a}^s + \dots + \|X_{n-1}\|_{p,a}^s)^{1/s},$$

and $\mathbb{C}^n$ endowed with the norm

$$|(a_0, \dots, a_{n-1})|_s = (|a_0|^s + \dots + |a_{n-1}|^s)^{1/s}.$$

We consider the action of $Gl(H)$ on $B_p^{(n)}$, defined by

$$l : Gl(H) \times B_p^{(n)} \longrightarrow B_p^{(n)},\ l_g((X_0, \dots, X_{n-1})) = (gX_0g^*, \dots, gX_{n-1}g^*). \tag{6.1}$$

From now on, we denote with $B_{p,a;s}^{(n)}$ the space $B_p^{(n)}$ endowed with the norm $\|(., \dots, .)\|_{p,a;s}$.

Proposition 6.3.1. *The norm in $B^{(n)}_{p,a;s}$ is invariant for the action of the group of invertible elements. By this we mean that for each $(X_0, \dots, X_{n-1}) \in B^{(n)}_p$, $a \in Gl(H)^+$ and $g \in Gl(H, B_p)$, we have*

$$\|(X_0, \dots, X_{n-1})\|_{p,a;s} = \left\| l_g((X_0, \dots, X_{n-1}))\right\|_{p,gag^*;s}.$$

Proof. See proposition 3.1.5.

□

Theorem 6.3.2. *Let $a, b \in Gl(H)^+, 1 \le p, s < \infty$, $n \in \mathbb{N}$ and $t \in (0, 1)$. Then*

$$(B^{(n)}_{p,a;s}, B^{(n)}_{p,b;s})_{[t]} = B^{(n)}_{p,\gamma_{a,b}(t);s}.$$

Proof. Recall Hadamard's classical three line theorem ([55], page 33):
Let $f(z)$ be a Banach space-valued function, bounded and continuous on the strip S, analytic in the interior, satisfying

$$\|f(z)\|_X \le M_0 \;\; if \;\; Re(z) = 0$$

and

$$\|f(z)\|_X \le M_1 \;\; if \;\; Re(z) = 1,$$

where $\|.\|_X$ denotes the norm of the Banach space X. Then

$$\|f(z)\|_X \le M_0^{1-Re(z)} M_1^{Re(z)}.$$

for all $z \in S$.
In order to simplify, we will only consider the case $n = 2$. The proof below works for n-tuples ($n \ge 3$) with obvious modifications.
By the previous proposition, we have that $\|(X_1, X_2)\|_{[t]}$ is equal to the norm of $a^{-1/2}(X_1, X_2)a^{-1/2}$ interpolated between the norms $\|(.,.)\|_{p,1;s}$ and $\|(.,.)\|_{p,c;s}$. Consequently it is sufficient to prove our statement for these two norms.
The proof consists in showing that $\|(X_1, X_2)\|_{[t]}$ and $\|(X_1, X_2)\|_{p,c^t;s}$ coincide in $B^{(2)}_p$, for all $t \in (0, 1)$,.
Let $t \in (0, 1)$ and $(X_1, X_2) \in B^{(2)}_p$ such that $\|(X_1, X_2)\|_{p,c^t;s} = 1$, and define

$$f(z) = c^{\frac{z}{2}} c^{-\frac{t}{2}} (X_1, X_2) c^{-\frac{t}{2}} c^{\frac{z}{2}} = (f_1(z), f_2(z))$$

Then for each $z \in S$, $f(z) \in B^{(2)}_p$

$$\|f(iy)\|_{p,1;s} = \left\| c^{\frac{iy}{2}} c^{-\frac{t}{2}} (X_1, X_2) c^{-\frac{t}{2}} c^{\frac{iy}{2}} \right\|_{p,1;s} = (\sum_{i=1}^{2} \left\| c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_i c^{-\frac{t}{2}} c^{\frac{iy}{2}} \right\|_p^s)^{1/s} \le 1,$$

and

$$\|f(1+iy)\|_{p,c;s} = (\sum_{i=1}^{2} \left\| c^{\frac{1}{2}} c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_i c^{-\frac{t}{2}} c^{\frac{iy}{2}} c^{\frac{1}{2}} \right\|_{p,c}^{s})^{1/s} \leq 1.$$

Since $f(t) = (X_1, X_2)$ and $f = (f_1, f_2) \in \mathcal{F}(B_p^{(2)})$ we have $\|(X_1, X_2)\|_{[t]} \leq 1$. Thus we have shown that

$$\|(X_1, X_2)\|_{[t]} \leq \|(X_1, X_2)\|_{p,c^t;s}\,.$$

To prove the converse inequality, let $f = (f_1, f_2) \in \mathcal{F}(B_p^{(2)})$, with $f(t) = (X_1, X_2)$ and $(Y_1, Y_2) \in B_q^{(2)}$ with $\|Y_k\|_q \leq 1$, where q is the conjugate exponent for $1 < p < \infty$ (or a compact operator and $q = \infty$ if $p = 1$). For $k = 1, 2$, let

$$g_k(z) = c^{-\frac{z}{2}} Y_k c^{-\frac{z}{2}}.$$

Consider the function $h : S \to (\mathbb{C}^2, |(.,.)|_s)$,

$$h(z) = (tr(f_1(z)g_1(z)), tr(f_2(z)g_2(z))).$$

Since $f(z) \in \mathcal{F}(B_p^{(2)})$, then h is analytic in $\overset{\circ}{S}$ and bounded in S, and

$$h(t) = (tr(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1), tr(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2)).$$

By Hadamard's three line theorem, applied to h and the Banach space $\mathbb{C}^2$ endowed with the norm $|(.,.)|_s$, we have

$$|h(t)|_s \leq max\{\sup_{y\in\mathbb{R}} |h(iy)|_s, \sup_{y\in\mathbb{R}} |h(1+iy)|_s\}.$$

For $j = 0, 1$,

$$\begin{aligned}
\sup_{y\in\mathbb{R}} |h(j+iy)|_s &= \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} |tr(f_k(j+iy)g_k(j+iy))|^s)^{1/s} \\
&= \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} |tr(c^{-j/2} f_k(j+iy) c^{-j/2} g_k(iy))|^s)^{1/s} \\
&\leq \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} \|f_k(j+iy)\|_{p,c^j}^s)^{1/s} \leq \|f\|_{\mathcal{F}(B_p^{(2)})},
\end{aligned}$$

then

$$\begin{aligned}
\|X_1\|_{p,c^t}^s + \|X_2\|_{p,c^t}^s &= \sup_{\substack{\|Y_1\|_q \leq 1 \\ \|Y_2\|_q \leq 1}} \{|tr(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1)|^s + |tr(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2)|^s\} \\
&\leq \sup_{\substack{\|Y_1\|_q \leq 1 \\ \|Y_2\|_q \leq 1}} |h(t)|_s^s \leq \|f\|_{\mathcal{F}(B_p^{(2)})}^s.
\end{aligned}$$

Since the previous inequality is valid for each $f \in \mathcal{F}(B_p^{(2)})$ with $f(t) = (X_1, X_2)$, we have

$$\|(X_1, X_2)\|_{p,c^t;s} \leq \|(X_1, X_2)\|_{[t]}.$$

□

In the special case $n = s = 1$ we obtain

Corolary 6.3.3. *Given $a, b \in Gl(H)^+$ and $1 \leq p < \infty$. Then*

$$(B_{p,a}, B_{p,b})_{[t]} = B_{p,\gamma_{a,b}(t)}.$$

Remark 6.3.4. Note that when a and b commute the curve is given by $\gamma_{a,b}(t) = a^{1-t}b^t$. The previous corollary tells us that the interpolating space, $B_{p,\gamma_{a,b}(t)}$ can be regarded as a weighted p-Schatten space with weight $a^{1-t}b^t$ (see [12], Th. 5.5.3).

By **Theorem B**, we obtain the following result of interpolation:

Corolary 6.3.5. *Let $a, b, c, d \in Gl(H)^+$, $p, s \geq 1$, $n \in \mathbb{N}$ and T a linear operator such that:*
The norm of T is at most M_0 (between the spaces $B^{(n)}_{p,a;s}$ and $B^{(n)}_{p,b;s}$),
The norm of T is at most M_1 (between the spaces $B^{(n)}_{p,c;s}$ and $B^{(n)}_{p,d;s}$).
Then, for all $t \in [0, 1]$ we have

$$\|T(x)\|_{p,\gamma_{b,d}(t);s} \leq M_0^{1-t} M_1^t \|x\|_{p,\gamma_{a,c}(t);s}.$$

The complex interpolation method has been used by authors in the context of operator algebras. For instance:

1. In 1977, Uhlmann [61] discussed the quadratic interpolation and introduced the relative entropy for states of an operator algebra. His quadratic interpolation is reduced to a path generated by the geometric mean and the relative entropy is the derivative of this path. Corach et al. [24] pointed out that this path can be regarded as a geodesic in a manifold of positive invertible elements with a Finsler norm.

2. The theory of L^p spaces associated with general (not necessarily semifinite) von Neumann algebras has been developed by U. Haagerup [37]. Kosaki [43] obtained these spaces via complex interpolation in a special case, when there exists a normal faithful positive functional ϕ on the von Neumann algebra M.

3. Andruchow et al. proved in [3] that if $A \subset B(H)$ is a C^* algebra, a, b two invertible positive elements in A, and $\| \ \|_a$ and $\| \ \|_b$ the corresponding quadratic norms on H induced by them, i.e. $\|x\|_a = \langle ax, x \rangle$, then the complex interpolation method, is also determined by $\gamma_{a,b}$. This curve is the unique geodesic of the manifold of positive invertible elements of A, which joins a and b.

6.4 Clarkson's type inequalities

Consider the linear operator $T_n : B^{(n)}_{p,a;s} \longrightarrow B^{(n)}_{p,b;s}$ given by

$$T_n(X_0, ..., X_{n-1}) = (\sum_{j=0}^{n-1} X_j, \sum_{j=0}^{n-1} \theta_j^1 X_j, ..., \sum_{j=0}^{n-1} \theta_j^{n-1} X_j),$$

where $\theta_0, ..., \theta_{n-1}$ are the n roots of unity, i.e. $\theta_j = e^{\frac{2\pi i j}{n}}$.
We remark that the inequalities (2.1) and (2.3) can be viewed as statements about the norm of T_2. This approach was used by Klaus ([60], page 22).
We use the same idea and the interpolation method to obtain the following inequalities.

Theorem 6.4.1. *For $a, b \in Gl(H)^+, X_0, ..., X_{n-1} \in B_p, 1 \leq p < \infty$ and $t \in [0,1]$, we have*

$$\tilde{k} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p \leq \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \theta_j^k X_j \|_{p,\gamma_{a,b}(t)}^p \leq \tilde{K} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p \tag{6.2}$$

where

$$\tilde{k} = \tilde{k}(p,a,b,t) = \begin{cases} n^{p-1} \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-pt} & \text{if } 1 \leq p \leq 2, \\ n \left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-pt} & \text{if } 2 \leq p < \infty, \end{cases}$$

and

$$\tilde{K} = \tilde{K}(p,a,b,t) = \begin{cases} n \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{pt} & \text{if } 1 \leq p \leq 2, \\ n^{p-1} \left\|a^{1/2}b^{-1}a^{1/2}\right\|^{pt} & \text{if } 2 \leq p < \infty. \end{cases}$$

Proof. We only prove the case $n = 2$ and $1 \leq p \leq 2$, the other cases are similar. We will denote by $\gamma(t) = \gamma_{a,b}(t)$, when no confusion can arise.
Consider the space $B_p^{(2)}$ with the norm:

$$\|(X,Y)\|_{p,a;p} = (\|X\|_{p,a}^p + \|Y\|_{p,a}^p)^{1/p},$$

where $a \in Gl(H)^+$.

By (2.1) (or [14], Th. 2 with $n = 2$) the norm of T_2 is at most $2^{1/p}$ when

$$T : (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) \longrightarrow (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}),$$

and the norm of T_2 is at most $2^{1/p}\|a^{1/2}b^{-1}a^{1/2}\|$ when

$$T : (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) \longrightarrow (B_p^{(2)}, \|(\,.,.\,)\|_{p,b;p}).$$

Therefore, using the complex interpolation, we obtain the following diagram of interpolation for $t \in [0,1]$

$$\begin{array}{ccc} & & (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) \\ & \nearrow^{T} & \\ (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) & \xrightarrow{T_t} & (B_p^{(2)}, \|(\,.,.\,)\|_{p,\gamma(t);p}) \\ & \searrow^{T} & \\ & & (B_p^{(2)}, \|(\,.,.\,)\|_{p,b;p}). \end{array}$$

By **Theorem B**, T_t satisfies

$$\begin{aligned} \|T_t(X,Y)\|_{p,\gamma(t);p} &\leq (2^{1/p}\|a^{1/2}b^{-1}a^{1/2}\|)^t (2^{1/p})^{1-t}\,\|(X,Y)\|_{p,a;p} \\ &= 2^{1/p}\|a^{1/2}b^{-1}a^{1/2}\|^t\,\|(X,Y)\|_{p,a;p}\,. \end{aligned} \tag{6.3}$$

Now applying the Complex method to

$$\begin{array}{ccc} (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) & & \\ & \searrow^{T} & \\ (B_p^{(2)}, \|(\,.,.\,)\|_{p,\gamma(t);p}) & \xrightarrow{T_t} & (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) \\ & \nearrow^{T} & \\ (B_p^{(2)}, \|(\,.,.\,)\|_{p,b;p}) & & \end{array}$$

one obtains

$$\begin{aligned} \|T(X,Y)\|_{p,a;p} &\leq (2^{1/p}\|b^{1/2}a^{-1}b^{1/2}\|)^t (2^{1/p})^{1-t}\,\|(X,Y)\|_{p,\gamma(t);2} \\ &= 2^{1/p}\|b^{1/2}a^{-1}b^{1/2}\|^t\,\|(X,Y)\|_{p,\gamma(t);2}\,. \end{aligned} \tag{6.4}$$

Replacing in (6.4) $X = \frac{Z+W}{2}$ and $Y = \frac{Z-W}{2}$ we obtain

$$\|Z\|_{p,a}^p + \|W\|_{p,a}^p \leq 2^{1-p}\|b^{1/2}a^{-1}b^{1/2}\|^{pt}(\|Z-W\|_{p,\gamma(t)}^p + \|Z+W\|_{p,\gamma(t)}^p), \quad (6.5)$$

or equivanletly

$$2^{p-1}\|b^{1/2}a^{-1}b^{1/2}\|^{-pt}(\|Z\|_{p,a}^p + \|W\|_{p,a}^p) \leq \|Z-W\|_{p,\gamma(t)}^p + \|Z+W\|_{p,\gamma(t)}^p. \quad (6.6)$$

Finally, the inequalities (6.16) and (6.6) complete the proof. □

Theorem 6.4.2. *For $a, b \in Gl(H)^+, X_0, ..., X_{n-1} \in B_p, 1 \leq p < \infty$ and $t \in [0,1]$, we have*

$$k\sum_{j=0}^{n-1}\|X_j\|_{p,a}^2 \leq \sum_{k=0}^{n-1}\|\sum_{j=0}^{n-1}\theta_j^k X_j\|_{p,\gamma_{a,b}(t)}^2 \leq K\sum_{j=0}^{n-1}\|X_j\|_{p,a}^2, \quad (6.7)$$

where

$$k = k(p,a,b,t) = \begin{cases} n^{2-2/p}\left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{if } 1 \leq p \leq 2, \\ n^{2/p}\left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{if } 2 \leq p < \infty, \end{cases}$$

and

$$K = K(p,a,b,t) = \begin{cases} n^{2/p}\left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{if } 1 \leq p \leq 2, \\ n^{2-2/p}\left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{if } 2 \leq p < \infty. \end{cases}$$

Proof. A slight change in the previous proof proves our statement.
We need to consider the space $B_p^{(n)}$ endowed with the norm

$$\|(X_0, ..., X_{n-1})\|_{p,a;2} = (\|X_0\|_{p,a}^2 + .. + \|X_{n-1}\|_{p,a}^2)^{1/2},$$

where $a \in Gl(H)^+$ and the following inequality ([14], Th. 1.):
For $2 \leq p \leq \infty$, we have

$$n^{\frac{2}{p}}\sum_{j=0}^{n-1}\|X_j\|_p^2 \leq \sum_{k=0}^{n-1}\|\sum_{j=0}^{n-1}\theta_j^k X_j\|_p^2 \leq n^{2-\frac{2}{p}}\sum_{j=0}^{n-1}\|X_j\|_p^2.$$

For $0 < p \leq 2$ these two inequalities are reversed.

□

6.5 On the Corach-Porta-Recht Inequality

In [21], Corach, Porta and Recht proved that if S is invertible and selfadjoint in $B(H)$, then for all $X \in B(H)$

$$\|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|.$$

In [42], Kittaneh proved a more general version of the CPR inequality:
For any norm ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ of $B(H)$ and for all $X \in \mathcal{I}$ we have

$$2\|X\|_{\mathcal{I}} \leq \|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}}. \tag{6.8}$$

Remark 6.5.1. $\mathcal{I}$ is a norm ideal of $B(H)$ if $\mathcal{I}$ is an ideal and a Banach space with respect to the norm $\|.\|_{\mathcal{I}}$ satisfying:

1. $\|XTY\|_{\mathcal{I}} \leq \|X\|\|T\|_{\mathcal{I}}\|Y\|$ for $T \in \mathcal{I}$ and $X, Y \in B(H)$,
2. $\|X\|_{\mathcal{I}} = \|X\|$ if T is the rank one.

In particular, condition 1. implies that the norm $\|.\|_{\mathcal{I}}$ is unitarily invariant,

$$\|UXV^*\|_{\mathcal{I}} = \|X\|_{\mathcal{I}}$$

for $X \in \mathcal{I}$ and any $U, V \in U(H)$.

In [57], Seddik obtained the following inequality for any norm ideal $\mathcal{I}$ of $B(H)$

Theorem 6.5.2. *For all* $X \in \mathcal{I}$

$$\|SXS^{-1} - S^{-1}XS\|_{\mathcal{I}} \leq (\|S\|\|S^{-1}\| - 1)\|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}}. \tag{6.9}$$

In [47], Larotonda obtained the following inequality for any norm ideal

Theorem 6.5.3. *([47], Corollary 28) For all* $X \in \mathcal{I}$

$$\|SXS^{-1} - S^{-1}XS\|_{\mathcal{I}} \leq \|L_T - R_T\|_{B(\mathcal{I})}\|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}}, \tag{6.10}$$

where $e^T = |S|$ *and* L_T, R_T *are the left and right multiplication representations of* T *in* $B(\mathcal{I})$, $L_T(U) = TU$ *and* $R_T(U) = UT$.

Here $\|P\|_{B(\mathcal{I})}$ denotes the norm of the linear operator $P : \mathcal{I} \to \mathcal{I}$, that is

$$\|P\|_{B(\mathcal{I})} = \sup\{\|P(x)\|_{\mathcal{I}} : \|x\|_{\mathcal{I}} = 1\},$$

The bound in (6.10) is related to the theory of generalized derivations. If $A, B \in B(H)$ let

$$\delta_{A,B} : X \to \delta_{A,B}(X) := AX - XB = L_A(X) - R_B(X).$$

The theory of generalized derivations has been extensively studied in the literature, see for example [28]. In [59], Stampfli proved the following equality

$$\|\delta_{A,B}\| = \inf\{\|A-\lambda\| + \|B-\lambda\| : \lambda \in \mathbb{C}\}, \tag{6.11}$$

If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a norm ideal in $B(H)$ and $X \in \mathcal{I}$, then for all $\lambda \in \mathbb{C}$

$$\|\delta_{A,B}(X)\|_{\mathcal{I}} = \|(A-\lambda)X + X(B-\lambda)\|_{\mathcal{I}} \leq (\|A-\lambda\| + \|B-\lambda\|)\|X\|_{\mathcal{I}}. \tag{6.12}$$

It follows from (6.11) that

$$\|\delta_{A,B}\|_{B(\mathcal{I})} \leq \|\delta_{A,B}\|.$$

From these facts we get,

$$\begin{aligned} \|SXS^{-1} - S^{-1}XS\|_{\mathcal{I}} &\leq \|L_T - R_T\|_{B(\mathcal{I})}\|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}} \\ &= \|\delta_{T,T}\|_{B(\mathcal{I})}\|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}} \\ &\leq \|\delta_{T,T}\|\|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}}. \end{aligned} \tag{6.13}$$

From (6.13) and (6.9) we obtain that

$$\|SXS^{-1} - S^{-1}XS\|_{\mathcal{I}} \leq \min\{\|\delta_{T,T}\|, \|S\|\|S^{-1}\| - 1\}\|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}}.$$

Note that the bound in Theorem 6.5.3 is a refinement of (6.9). We start by recalling the next

Corolary 6.5.4. *([59], Corollary 1) Let T be a normal operator. Then*

$$\|\delta_{T,T}\| = sup\{\|TX - XT\| : T \in B(H) \text{ and } \|T\| = 1\} = 2r(T),$$

where $r(T) = \lambda_{\max}(T) - \lambda_{\min}(T)$ is the radius of the spectrum of T.

First, we shall assume that S is positive. Then

$$\|\delta_{T,T}\| = \lambda_{\max}(T) - \lambda_{\min}(T) = \log(\lambda_{\max}(S)) - \log(\lambda_{\min}(S)) = \log(\frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}),$$

therefore

$$\|S\| = \lambda_{\max}(S) \qquad \text{and} \qquad \|S^{-1}\| = \frac{1}{\lambda_{\min}(S)}.$$

So
$$\|\delta_{T,T}\| = \log(\frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}) < \frac{\lambda_{\max}(S)}{\lambda_{\min}(S)} - 1 = \|S\|\|S^{-1}\| - 1.$$
Here we use the fact that $\log(t) < t-1$, for all $t > 1$.
In the general case (i.e. S invertible and selfadjoint) we have

$$\|\delta_{T,T}\| = \log(\frac{\lambda_{\max}(|S|)}{\lambda_{\min}(|S|)}) < \||S|\|\||S|^{-1}\| - 1 = \|S\|\|S^{-1}\| - 1,$$

Now, we are ready to state the next

Theorem 6.5.5. *Let $\mathcal{I}$ be a norm ideal, then for all $X \in \mathcal{I}$*

$$\|SXS^{-1} - S^{-1}XS\|_{\mathcal{I}} \leq \|\delta_{T,T}\|\|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}}, \tag{6.14}$$

with $e^T = |S|$.

Note that the inequality holds for any norm ideal $\mathcal{I}$ where the explicit bound $\|\delta_{T,T}\|$ depends only on the operator S and the norm in $B(H)$ and not on the given unitarily invariant norm.
Now, we shall apply the Corollary 6.3.5 to the inequality obtained above for the special case that $\mathcal{I} = B_p$ with $1 \leq p < \infty$.
For $p \geq 1$ fixed, we consider

$$R_{p,S} : B_p \to B_p \qquad R_{p,S}(X) = SXS^{-1} + S^{-1}XS.$$

Corolary 6.5.6. *For $a, b \in Gl(H)^+, X \in B_p$ and $t \in [0,1]$, we have*

$$\|SXS^{-1} + S^{-1}XS\|_p \leq 2\mu\|a\|^{1-t}\|b\|^t\|X\|_{p,\gamma_{a,b}(t)}, \tag{6.15}$$

where $\mu = \|S\|\|S^{-1}\|$.

Proof. We will denote by $\gamma(t) = \gamma_{a,b}(t)$.
The norm of $R_{p,S}$ is at most $2\mu\|a\|$ when

$$R_{p,S} : (B_p, \|.\|_{p,a}) \to (B_p, \|.\|_p),$$

and the norm of $R_{p,S}$ is at most $2\mu\|b\|$ when

$$R_{p,S} : (B_p, \|.\|_{p,b}) \to (B_p, \|.\|_p),$$

Therefore, using the complex interpolation, we obtain the following diagram of interpolation for $t \in [0,1]$

$$\begin{array}{ccc}
(B_p, \|.\|_{p,a}) & & \\
& \searrow^{R} & \\
(B_p, \|.\|_{p,\gamma(t)}) & \xrightarrow{R_t} & (B_p, \|.\|_p) \\
& \nearrow_{R} & \\
(B_p, \|.\|_{p,b}) & &
\end{array}$$

By Corollary 6.3.5,

$$\begin{aligned}
\|R_{p,S}(X)\|_p &\leq (2\mu\|b\|)^t (2\mu\|a\|)^{1-t} \|X\|_{p,\gamma(t)} \\
&= 2\mu\|a\|^{1-t}\|b\|^t\|X\|_{p,\gamma(t)}.
\end{aligned}$$

□

From Theorem 6.5.5, we get that

$$\|SXS^{-1} - S^{-1}XS\|_p \leq \|\delta_{T,T}\| \|SXS^{-1} + S^{-1}XS\|_p \leq \|\delta_{T,T}\| 2\mu\|a\|^{1-t}\|b\|^t\|X\|_{p,\gamma_{a,b}(t)},$$

for all $X \in B_p$ and any $a, b \in Gl(H)^+$.
Finally, we conclude this section with the following statement.

Corolary 6.5.7. *For $X \in B_p$ we have*

$$\|SXS^{-1} - S^{-1}XS\|_p \leq 2\mu\|\delta_{T,T}\|C(X),$$

and

$$2\|X\|_p \leq \|SXS^{-1} + S^{-1}XS\|_p \leq 2\mu C(X),$$

where $\mu = \|S\|\|S^{-1}\|$ and $C(X) = \inf\{\|a\|^{1-t}\|b\|^t\|X\|_{p,\gamma_{a,b}(t)} : t \in [0,1], a, b \in Gl(H)^+\}$.

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