UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Transición de fase para modelos diluidos y grandes desvíos para magnetizaciones no homogéneas

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Nahuel Soprano Loto

Director de tesis: Pablo A. Ferrari
Director Asistente: Inés Armendáriz
Consejero de estudios: Pablo A. Ferrari

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Resumen

Esta tesis contiene dos partes con un tema en común: en cada una de ellas, estudiamos diferentes modelos de mecánica estadística.

En la primera parte, estudiamos modelos diluidos de vecinos próximos con espacio de espines finito, donde el grafo subyacente es un subgrafo aleatorio del reticulado d-dimensional. Más precisamente, proporcionamos condiciones suficientes y necesarias para que ocurra co-existencia de fases mediante técnicas de aglomerado aleatorio.

En la segunda parte, estudiamos un modelo del tipo Ising con interacciones de vecinos próximos ferromagnéticas y potencial cuadrático del tipo Kac asociado a un campo externo no-homogéneo. En este caso, probamos que la energía libre y la presión existen y establecemos resultados de grandes desvíos y equivalencia de arreglos.

Phase transition for dilute models and large deviations for inhomogeneous magnetizations

Abstract

This thesis contains two parts with one topic in common: in each one, we study different statistical-mechanical models.

In the first part, we study dilute nearest-neighbour models with finite spin state, being the underlying graph a random subgraph of the d-dimensional lattice. More precisely, we give necessary and sufficient conditions for phase co-existence to occur via random-cluster techniques.

In the second part, we study an Ising-type model with ferromagnetic nearest-neighbour interactions and quadratic Kac-type potential associated to an inhomogeneous external field. In this case, we prove that the free energy and the pressure exist and establish large deviation and equivalence of ensembles results.

A mis papás y hermanos

Contents

Chapter 1

Phase transition for dilute models

1.1 Introducción

Este capítulo está basado en [AFSL15] y [FSL15].

Estudiamos modelos diluidos discretos de vecinos próximos con espacio de espines finito. Por discreto, nos referimos a que los espines se organizan sobre los vértices del reticulado d-dimensional $\mathbb{Z}^d$; por diluido, nos referimos a que las interacciones entre espines son gobernadas por una subconjunto aleatorio de aristas o desorden

$$F \subset \Big\{\{x, y\} \subset \mathbb{Z}^d : \|x - y\| = 1\Big\}, \tag{1.1}$$

en el sentido de que el Hamiltoniano formal es una función aleatoria definida como

$$H_F(\sigma) := \sum_{\{x,y\} \in F} I\left(\sigma_x, \sigma_y\right), \tag{1.2}$$

donde I es una interacción que depende del modelo, dando lugar a un conjunto aleatorio de medidas de Gibbs de acuerdo con el formalismo de DLR.

Establecemos una serie de condiciones suficientes para la unicidad y la no-unicidad de medidas de Gibbs, que valen para casi toda realización del desorden. La herramienta principal que utilizamos es la representación de aglomerado aleatorio de Edwards-Sokal, introducida en [ES88], junto a la estrategia propuesta por Aizenman, Chayes, Chayes y Newman en su trabajo seminal [ACCN87], en donde vinculan unicidad de medidas de Gibbs con ausencia de percolación en la probabilidad de aglomerado aleatorio asociada.

El criterio de unicidad es general en los modelos: por supuesto, hipótesis sobre la aleatoriedad del desorden son necesarias, pero no se piden hipótesis sobre la estructura del conjunto de espines. En este sentido, este trabajo es una generalización del criterio de unicidad introducido por Alexander y Chayes en [AC97].

Para el criterio de no-unicidad, necesitamos que el modelo de mecánica estadística satisfaga ciertas propiedades estructurales. Por un lado, el conjunto de espines debe ser un grupo Abeliano y, por el otro, la interacción debe ser invariante bajo rotaciones del grupo; este tipo de modelos son llamados modelos de espines Abelianos e incluyen los importantes casos del modelo de Potts, el modelo del reloj generalizado y el modelo de Ashkin-Teller generalizado. Por el otro lado, necesitamos ser capaces de definir una noción de reflexión generalizada sobre el conjunto de espines, condición que necesita ser verificada en cada modelo. La existencia de esta reflexión generalizada permite demostrar un lema combinatorio crucial, al que consideramos como uno de los principales puntos del capítulo.

Un comentario final: las principales dificultades aparecen en volumen finito; una vez que estas dificultades son sobrellevadas, se aplican al caso de volumen infinito de una manera sencilla.

1.2 Introduction

This chapter is based on [AFSL15] and [FSL15].

We study dilute nearest-neighbour lattice statistical-mechanical models with finite set of spins. By lattice, we mean that the spins are assigned to the vertices of the d-dimensional square lattice $\mathbb{Z}^d$; by dilute, we mean that the interactions between the spins are governed by a random subset of edges or disorder

$$F \subset \Big\{\{x,y\} \subset \mathbb{Z}^d : \|x-y\| = 1\Big\}, \tag{1.3}$$

in the sense that the associated formal Hamiltonian is the random function defined by

$$H_F(\sigma) := \sum_{\{x,y\}\in F} I\left(\sigma_x, \sigma_y\right), \tag{1.4}$$

being I and interaction that depends on the model, giving rise to a random set of Gibbs measures according to the DLR formalism.

We establish a set of sufficient conditions for uniqueness and non-uniqueness of Gibbs measure, that hold for almost every realization of the disorder. The main tools we use are the Edwards-Sokal random-cluster representation, introduced in [ES88], and the strategy proposed by Aizenman, Chayes, Chayes and Newman in their seminal work [ACCN87], where they relate uniqueness of Gibbs measure to absence of percolation in the associated random-cluster probability.

The uniqueness criteria is general on the models: of course, hypotheses over the randomness of the disorder are required, but we do not ask for any hypothesis over the structure of the set of spins. In this sense, the present work is a generalization of the uniqueness criteria introduced by Alexander and Chayes in [AC97].

For the non-uniqueness criteria, we require the statistical-mechanical model to satisfy certain structural properties. On the one hand, we need the set of spins to be an Abelian group and the interaction to be invariant under group rotations; these kinds of models are called Abelian spin models and include the important cases of the Potts model, the generalized clock model and the generalized Ashkin-Teller model. On the other hand, we have to be able to define a notion of generalized reflection over the set of spins, a condition that has to be verified for each particular model. The existence of this generalized reflection allows us to prove a crucial combinatorial lemma, that we consider one of the main points of the chapter.

A final comment: the main difficulties appear in finite volume; after these are overcome, they apply to the infinite volume case in an easy way.

1.3 Finite volume

1.3.1 Statistical-mechanical model

In this subsection, we introduce the statistical-mechanical model for an arbitrary finite graph. Let (V, E) be such a graph, where V and E respectively denote the set of vertices and edges. We assume that the edges are non-oriented and that there are no loops nor multiple edges. In addition, we need to introduce the set of spins and the interaction. The set of spins is a finite set S with cardinality $q := |S| \in \mathbb{N} \setminus \{1\}$. The interaction is a function $W : S \times S \to (0, 1]$ such that $W(a, b) = W(b, a)$ for every $a, b \in S$; we will refer to this property as the symmetry of W. For simplicity, we suppose W satisfies the following non-singularity condition: $W(a, b) = 1$ if and only if $a = b$. The interaction $W(a, b)$ plays the role of $e^{-\beta I(a,b)}$, where I is the function appearing in the Hamiltonian (1.4) and $\beta > 0$ is the inverse temperature. The product space S^V is called set of vertex-configurations; its elements are denoted by the letter σ. Fix a (possibly empty) subset $U \subset V$ and suppose there are no edges linking vertices in U: $\{\langle xy\rangle \in E : \{x, y\} \subset U\} = \emptyset$. If $U = \emptyset$, we define $\hat{\mu}_V^{\emptyset} = \hat{\mu}\,(V, \emptyset)$ as the uniform probability over the set of configurations S^V. If $U \neq \emptyset$, a boundary condition is an element $\eta \in S^U$, and we define $\hat{\mu}_V^{\eta} = \hat{\mu}\,(V, U, \eta)$ as the uniform probability over the vertex-configurations that coincide with η in U:

$$\hat{\mu}_V^{\eta}\,(\sigma) = \frac{1}{q^{|V \setminus U|}} \mathbf{1}\,\{\sigma_U = \eta\}\,. \tag{1.5}$$

The Gibbsian specification associated to $U = \emptyset$ is the probability $\mu_V^{\emptyset} = \mu\,(V, E, W, \emptyset)$ on S^V defined by

$$\mu_V^{\emptyset}(d\sigma) := \hat{\mu}_V^{\emptyset}(d\sigma) \frac{1}{Z_{V,E}^{\emptyset}} \prod_{\langle xy\rangle \in E} W(\sigma_x, \sigma_y), \tag{1.6}$$

where

$$Z_{V,E}^{\emptyset} = Z\,(V, E, W, \emptyset) := \int \hat{\mu}_V^{\emptyset}(d\sigma) \prod_{\langle xy\rangle \in E} W(\sigma_x, \sigma_y) \tag{1.7}$$

is the normalizing constant. Analogously, for $U \neq \emptyset$ and $\eta \in S^U$, we define the Gibbsian specification as the probability $\mu_V^{\eta} = \mu\,(V, E, W, U, \eta)$ on S^V given by

$$\mu_V^{\eta}(d\sigma) := \hat{\mu}_V^{\eta}(d\sigma) \frac{1}{Z_{V,E}^{\eta}} \prod_{\langle xy\rangle \in E} W(\sigma_x, \sigma_y), \tag{1.8}$$

with

$$Z_{V,E}^{\eta} = Z(V, E, W, U, \eta) := \int \hat{\mu}_V^{\eta}(d\sigma) \prod_{\langle xy \rangle \in E} W(\sigma_x, \sigma_y). \tag{1.9}$$

In case $\eta_x = a$ for every $x \in U$, we simply write μ_V^a and $Z_{V,E}^a$.

Some examples

The Potts model. The interaction is constructed from the Hamiltonian or energy function and a parameter $\beta > 0$ representing the inverse temperature. In the Potts model, the Hamiltonian H is defined by

$$H(\sigma) = \sum_{\langle xy \rangle \in E} \mathbf{1}\{\sigma_x \neq \sigma_y\}; \tag{1.10}$$

it enumerates the number of discordances between neighbour vertices. The Gibbsian specification $\hat{\mu}_V^{\eta}$ is proportional to $e^{-\beta H(\sigma)}$:

$$\mu_V^{\eta}(\sigma) = \frac{1}{Z_{V,E}^{\eta}} e^{-\beta H(\sigma)} = \frac{1}{Z_{V,E}^{\eta}} \prod_{\langle xy \rangle \in E} e^{-\beta \mathbf{1}\{\sigma_x \neq \sigma_y\}}. \tag{1.11}$$

In this case, the corresponding weight is given by

$$W(a, b) = e^{-\beta \mathbf{1}\{a \neq b\}}. \tag{1.12}$$

The Potts model with $q = 2$ is the well studied Ising model.

The generalized clock model. In this case, S is the set of q equidistant angles defined by

$$S := \left\{ \frac{2\pi i}{q} : i = 0, \ldots, q-1 \right\}. \tag{1.13}$$

The weight function is of the form $W(a, b) = f(\cos(a - b))$ with $f : [-1, 1] \to (0, 1]$ any function satisfying $f(t) = 1$ if and only if $t = 1$. If we take $f(t) = e^{-\beta \mathbf{1}\{t \neq 1\}}$, we recover the Potts model. The classical clock model is obtained when taking $f(t) = e^{-\beta(1-t)}$.

The generalized Ashkin-Teller model. Here, the set of spins is the set $\{-1, 1\}^2$. Let $\tilde{W} : \{-1, 1\}^2 \to (0, 1]$ be such that $\tilde{W}(a) = 1$ if and only if $a = (1, 1)$, and define $W(a, b) = \tilde{W}(ab)$, where the product ab is defined coordinate-wise: $ab = (a_1, a_2)(b_1, b_2) := (a_1 b_1, a_2 b_2)$. In the classical Ashkin-Teller model, as presented in [Gri06] for example, the set of spins is the set containing four elements A, B, C and D, and the interaction is defined by

$$W(A, A) = W(B, B) = W(C, C) = W(D, D) = 1, \tag{1.14}$$

$$W(A, B) = W(C, D) = e^{-\beta J_1} \text{ and} \tag{1.15}$$

$$W(A, C) = W(A, D) = W(B, C) = W(B, D) = e^{-\beta J_2}, \tag{1.16}$$

with $\beta > 0$ and $0 < J_1 < J_2$. This version can be recovered from the general one by identifying $A = (1, 1)$, $B = (1, -1)$, $C = (-1, 1)$ and $D = (-1, -1)$, and defining $\tilde{W}$ by $\tilde{W}(1, 1) = 1$, $\tilde{W}(1, -1) = e^{-\beta J_1}$ and $\tilde{W}(-1, 1) = \tilde{W}(-1, -1) = e^{-\beta J_2}$.

Abelian spin models. The previous three examples are part of a wider family of statistical-mechanical models called Abelian spin models. In each of these models, S is an Abelian group with identity $e \in S$, and the interaction W is defined by

$$W(a, b) = \hat{W}(ab^{-1}), \tag{1.17}$$

where $\hat{W} : S \to [0, 1]$ is an even function (that is $\hat{W}(c) = W(c^{-1})$ for every $c \in S$) such that $\tilde{W}(a) = 1$ if and only if $a = e$. This is referred to as the non-singularity condition over $\tilde{W}$.

Note that the generalized clock model is an Abelian spin model. Indeed, the set $\left\{ \frac{2\pi i}{q} : i = 0, \ldots, q-1 \right\}$ can be identified with the set of q-th roots of unity

$$\mathbb{U}_q := \{a \in \mathbb{C} : a^q = 1\}. \tag{1.18}$$

The condition $\tilde{W}(a) = \tilde{W}(a^{-1})$ implies that $\tilde{W}$ depends only on the real part of a, that is, on the cosine of the argument.

If we take S to be the product group $\mathbb{U}_2 \times \mathbb{U}_2$, we recover the generalized Ashkin-Teller model. Indeed, observe that, in this particular group, we have $a = a^{-1}$ for every $a \in S$, so the product ab^{-1} coincides with the product ab.

Notation 1.3.1. *For $A \subset V$, $\sigma \in S^A$ and $a \in S$, the notation $\sigma \equiv a$ means $\sigma_x = a$ for every $x \in A$.*

Observation 1.3.2 (Rotational invariance of the specifications)**.** *Consider an Abelian spin model with spin set S. For every $a \in S$, let $\varphi_a : S \to S$ be the bijection defined by $\varphi_a b := ab$. We use the same symbol to refer to the action (also a bijection) of φ_a over S^V: $(\varphi_a \sigma)_x := \varphi_a \sigma_x$ for every x. Observe that φ_a preserves W in the sense that $W(b, c) = W(\varphi_a b, \varphi_a c)$ for every $b, c \in S$ and, as a consequence, $\mu_V^\emptyset(\sigma) = \mu_V^\emptyset(\varphi_a \sigma)$ for every $\sigma \in S^V$. Also, for every $a, b \in S$ and every $A \subset V$, the sets*

$$\left\{ \sigma \in S^V : \sigma_U \equiv a, \sigma_A \equiv a \right\} \text{ and } \left\{ \sigma \in S^V : \sigma_U \equiv b, \sigma_A \equiv b \right\} \tag{1.19}$$

are in one-to-one correspondence under the action of $\varphi_{a^{-1}b}$, and hence

$$\mu_V^a \left(\sigma_A \equiv a\right) = \mu_V^b \left(\sigma_A \equiv b\right). \tag{1.20}$$

1.3.2 Edwards-Sokal random-cluster representation

In this subsection, we adapt the material from [ES88] to our purposes.

Let $T := \mathrm{Im}(W) \subset (0, 1]$ and enumerate its elements as $0 < t_0 < \ldots < t_k = 1$. We consider the product set T^E. Elements of T^E are called edge-configurations and denoted by the letter ω.

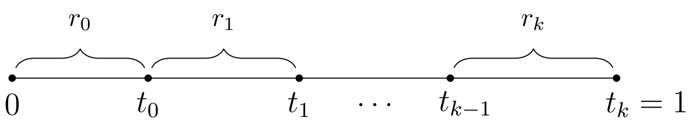

Figure 1.1

Definition 1.3.3. *Given a subset of edges $E' \subset E$, and a pair of configurations $\omega \in T^{E'}$ and $\sigma \in S^V$, we say that ω and σ are compatible if and only if*

$$\omega_{\langle xy \rangle} \leq W(\sigma_x, \sigma_y) \text{ for every } \langle xy \rangle \in E'. \tag{1.21}$$

In this case, we write $\omega \preccurlyeq \sigma$.

Observation 1.3.4. *In the previous definition, if t_0 is achieved at an edge, then this does not impose any restriction over the compatibility condition. More precisely, let $\langle xy \rangle \in E'$ and $\omega \in T^{E'}$ such that $\omega_{\langle xy \rangle} = t_0$, and let $\omega' \in T^{E' \setminus \langle xy \rangle}$ be the projection of ω over $E' \setminus \langle xy \rangle$. Then $\sigma \succcurlyeq \omega$ if and only if $\sigma \succcurlyeq \omega'$. On the other hand, the value t_k forces the corresponding values of the spins to coincide. More precisely, if $\sigma \in S^V$ and $\omega \in T^{E'}$ are such that $\omega_{\langle xy \rangle} = t_k$ for some $\langle xy \rangle \in E'$, then $\sigma_x = \sigma_y$.*

Observation 1.3.5. *Taking $E' = E$ in the previous definition, we can extend the notion of compatibility to the union set $T^E \cup S^V$ in the following way:*

$$\omega \preccurlyeq \omega' \text{ if and only if } \omega_{\langle xy \rangle} \leq \omega'_{\langle xy \rangle} \text{ for every } \langle xy \rangle \in E, \tag{1.22}$$

$$\sigma \preccurlyeq \sigma' \text{ if and only if } W(\sigma_x, \sigma_y) \leq W\left(\sigma'_x, \sigma'_y\right) \text{ for every } \langle xy \rangle \in E, \tag{1.23}$$

$$\sigma \preccurlyeq \omega \text{ if and only if } W(\sigma_x, \sigma_y) \leq \omega_{\langle xy \rangle} \text{ for every } \langle xy \rangle \in E \text{ and} \tag{1.24}$$

$$\omega \preccurlyeq \sigma \text{ if and only if } \omega_{\langle xy \rangle} \leq W(\sigma_x, \sigma_y) \text{ for every } \langle xy \rangle \in E. \tag{1.25}$$

It is easy to see that $\preccurlyeq$ is a pre-order on $T^E \cup S^V$ (it satisfies reflexivity and transitivity). Its restriction to S^V is again a pre-order and its restriction to T^E is a partial order (it satisfies reflexivity, transitivity and antisymmetry). The later one is the partial order used in section 1.6 to define the notion of stochastic domination.

We define the base probability $\hat{\phi}_E = \hat{\phi}(E, W)$ on T^E by

$$\hat{\phi}_E(\omega) := \prod_{\langle xy \rangle \in E} \left[\sum_{i=0}^{k} r_i \mathbf{1}\left\{\omega_{\langle xy \rangle} = t_i\right\} \right], \tag{1.26}$$

where $r_0 := t_0$ and $r_i := t_i - t_{i-1}$ for $i > 0$ (see figure 1.1). In words, the law of $\hat{\phi}_E$ is obtained by independently sampling for each edge an element $t_i \in T$ with probability r_i. For $U = \emptyset$, let $Q^{\emptyset}_{V,E} = Q(V, E, W, \emptyset)$ be the probability on the product space $T^E \times S^V$ defined by

$$Q^{\emptyset}_{V,E}(d(\omega, \sigma)) := \hat{\phi}_E \times \hat{\mu}^{\emptyset}_V(d(\omega, \sigma)) \frac{1}{Z^{\emptyset}_{V,E}} \mathbf{1}\{\omega \preccurlyeq \sigma\}; \tag{1.27}$$

$Q^{\emptyset}_{V,E}$ is the product probability $\hat{\phi}_E \times \hat{\mu}^{\emptyset}_V$ conditioned to the event $(\omega \preccurlyeq \sigma)$. Here $Z^{\emptyset}_{V,E}$ is the same normalizing constant appearing in expression (1.6). For $U \neq \emptyset$, the Edwards-Sokal probability associated to $\eta \in S^U$ is defined by

$$Q^{\eta}_{V,E}\left(d(\omega,\sigma)\right) := \hat{\phi}_E \times \hat{\mu}^{\eta}_V(d(\omega,\sigma))\frac{1}{Z^{\eta}_{V,E}}\mathbf{1}\left\{\omega \preccurlyeq \sigma\right\}. \tag{1.28}$$

Theorem 1.3.6 (Edwards-Sokal coupling)**.** *For $U \neq \emptyset$ and $\eta \in S^U$, μ^{η}_V is the second marginal of $Q^{\eta}_{V,E}$. Analogously, $\mu^{\emptyset}_V$ is the second marginal of $Q^{\emptyset}_{V,E}$.*

Proof. We only prove the first case; the proof with the empty-boundary condition is similar. For every $\langle xy\rangle \in E$, let $\theta_{\langle xy\rangle}$ be the probability on T defined by $\theta_{\langle xy\rangle}(t_i) = r_i$ for every $0 \leq i \leq k$. Under this definition, $\hat{\phi}_E$ is the product probability $\prod_{\langle xy\rangle\in E} \theta_{\langle xy\rangle}$ and

$$t_i = \sum_{j=0}^{i} r_j = \int \theta_{\langle xy\rangle}(d\omega_{\langle xy\rangle})\mathbf{1}\left\{\omega_{\langle xy\rangle} \leq t_i\right\}. \tag{1.29}$$

Then

$$\mu^{\eta}_V(d\sigma) = \hat{\mu}^{\eta}_V\,(d\sigma)\,\frac{1}{Z^{\eta}_{V,E}}\prod_{\langle xy\rangle\in E} W(\sigma_x,\sigma_y) = \tag{1.30}$$

$$\hat{\mu}^{\eta}_V\,(d\sigma)\,\frac{1}{Z^{\eta}_{V,E}}\prod_{\langle xy\rangle\in E}\int \theta_{\langle xy\rangle}\left(d\omega_{\langle xy\rangle}\right)\mathbf{1}\left\{\omega_{\langle xy\rangle} \leq W(\sigma_x,\sigma_y)\right\} = \tag{1.31}$$

$$\hat{\mu}^{\eta}_V\,(d\sigma)\,\frac{1}{Z^{\eta}_{V,E}}\int \hat{\phi}_E(d\omega)\prod_{\langle xy\rangle\in E}\mathbf{1}\left\{\omega_{\langle xy\rangle} \leq W(\sigma_x,\sigma_y)\right\} = \tag{1.32}$$

$$\hat{\mu}^{\eta}_V\,(d\sigma)\,\frac{1}{Z^{\eta}_{V,E}}\int \hat{\phi}_E(d\omega)\mathbf{1}\left\{\omega_{\langle xy\rangle} \leq W(\sigma_x,\sigma_y) \text{ for every } \langle xy\rangle \in E\right\} = \tag{1.33}$$

$$\hat{\mu}^{\eta}_V\,(d\sigma)\,\frac{1}{Z^{\eta}_{V,E}}\int \hat{\phi}_E(d\omega)\mathbf{1}\left\{\omega \succcurlyeq \sigma\right\} = \int \hat{\phi}_E \times \hat{\mu}^{\eta}_V\,(d(\omega,\sigma))\,\frac{1}{Z^{\eta}_{V,E}}\mathbf{1}\left\{\omega \succcurlyeq \sigma\right\} = \tag{1.34}$$

$$\int Q^{\eta}_{V,E}\left(d(\omega,\sigma)\right), \tag{1.35}$$

completing the proof. □

For $U = \emptyset$, the associated random-cluster probability $\phi^{\emptyset}_E = \phi\left(V,E,W,\emptyset\right)$ is defined as the first marginal of $Q^{\emptyset}_{V,E}$. It has the form

$$\phi^{\emptyset}_E(d\omega) = \hat{\phi}_E(d\omega)\frac{1}{Z^{\emptyset}_{V,E}}\hat{\mu}^{\emptyset}_V\left\{\sigma \in S^V : \sigma \succcurlyeq \omega\right\} = \hat{\phi}_E(d\omega)\frac{1}{\tilde{Z}^{\emptyset}_{V,E}}\left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega\right\}\right|, \tag{1.36}$$

with $\tilde{Z}^{\emptyset}_{V,E} := q^{|V|}Z^{\emptyset}_{V,E}$. It is convenient to define the weight

$$\langle\omega\rangle^{\emptyset}_E := \left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega\right\}\right|. \tag{1.37}$$

Analogously, for $U \neq \emptyset$ and $\eta \in S^U$, we define $\phi_E^\eta = \phi(V, E, W, U, \eta)$ to be the first marginal of $Q_{V,E}^\eta$. It has the form

$$\phi_E^\eta(d\omega) = \hat{\phi}_E(d\omega) \frac{1}{\tilde{Z}_{V,E}^\eta} \langle\omega\rangle_E^\eta, \tag{1.38}$$

where $\tilde{Z}_{V,E}^\emptyset := q^{|V \setminus U|} Z_{V,E}^\eta$ and $\langle\omega\rangle_E^\eta := \left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega, \sigma_U = \eta\right\}\right|$. As before, we simply write $\langle\omega\rangle_E^a$ and ϕ_E^a if $\eta_x = a$ for every $x \in U$.

Observation 1.3.7 (monotonicity of the weights)**.** *From the definition of the partial order $\preccurlyeq$ on T^E introduced in observation 1.3.5, it can be easily checked that the functions $\langle \cdot \rangle_E^\emptyset : T^E \to \mathbb{R}$ and $\langle \cdot \rangle_E^\eta : T^E \to \mathbb{R}$ are decreasing:*

$$\langle\omega\rangle_E^\emptyset \geq \langle\omega'\rangle_E^\emptyset \text{ and } \langle\omega\rangle_E^\eta \geq \langle\omega'\rangle_E^\eta \text{ whenever } \omega \preccurlyeq \omega'. \tag{1.39}$$

Observation 1.3.8 (Rotational invariance of the weights)**.** *In the Abelian spin model, for every $a, b \in S$ and $\omega \in T^E$, the sets*

$$\left\{\sigma \in S^V : \sigma_U \equiv a, \sigma \succcurlyeq \omega\right\} \text{ and } \left\{\sigma \in S^V : \sigma_U \equiv b, \sigma \succcurlyeq \omega\right\} \tag{1.40}$$

are in one-to-one correspondence under the action of $\varphi_{ba^{-1}}$ (recall the definition of $\varphi_\cdot$ introduced in Observation 1.3.2). We conclude that the weights $\langle\omega\rangle_E^a$ and $\langle\omega\rangle_E^b$ coincide and, as a consequence, the probabilities ϕ_E^a and ϕ_E^b also do.

Notation 1.3.9. *In general, we use the letter Π to denote projections. For example, $\Pi_{T_E} : T^E \times S^V \to T^E$ is the projection over the first marginal. We also use it to project on sets of vertices or edges. For example, if $E' \subset E$ is an edge subset, $\Pi_{E'} : T^E \to T^{E'}$ denotes the projection on E'.*

In the following observation, the shorthand $Q_{V,E}^\emptyset(\sigma|\omega)$ refers to $Q_{V,E}^\emptyset(\Pi_{S^V} = \sigma | \Pi_{T^E} = \omega)$; the analogous notation is used for $Q_{V,E}^\eta(\sigma|\omega)$. As a general convention, we stablish that a probability conditioned to an event of probability zero is the null measure.

Observation 1.3.10. *As a consequence of the Edwards-Sokal coupling, we have*

$$\mu_V^\emptyset(\sigma) = \sum_{\omega \in T^E} Q_{V,E}^\emptyset(\sigma|\omega)\, \phi_E^\emptyset(\omega) \tag{1.41}$$

for every $\sigma \in S^V$. Also, for every $\sigma \in S^V$ and $\omega \in T^E$, we have

$$Q_{V,E}^\emptyset(\sigma|\omega) = \frac{\mathbf{1}\{\sigma \succcurlyeq \omega\}}{\langle\omega\rangle_E^\emptyset}. \tag{1.42}$$

Then

$$\mu_V^\emptyset(\sigma) = \sum_{\omega \in T^E} \frac{\mathbf{1}\{\sigma \succcurlyeq \omega\}}{\langle\omega\rangle_E^\emptyset} \phi_E^\emptyset(\omega). \tag{1.43}$$

In words, a random vertex-configuration $\sigma \in S^V$ with law $\mu_V^\emptyset$ can be sampled by first sampling a random edge-configuration $\omega \in T^E$ with law $\phi_E^\emptyset$ and then sampling a vertex-configuration over the ones that are compatible with ω. The analogous property holds for $Q_{V,E}^\eta$.

1.3.3 Markov property and positive correlations

Notation 1.3.11. *For $A, B \subset V$ vertex subsets, $E(A, B)$ denotes the set of edges with one vertex in A and the other one in B:*

$$E(A,B) := \{\langle xy\rangle \in E : \{x,y\} \cap A \neq \emptyset, \{x,y\} \cap B \neq \emptyset\}\,. \tag{1.44}$$

To shorten notation, we write $E(A)$ for $E(A,A)$ and $E \cap A$ for $E(A,V)$.

For the rest of this subsection, we suppose $U \neq \emptyset$ and take $\eta \in S^U$. We also fix a non-empty subset $A \subset V \setminus U$. Observe that E can be written as the disjoint union $E(A) \cup E(A^c) \cup E(A, A^c)$.

Observation 1.3.12 (Factorization of the weights)**.** *If $\omega \in T^E$ is such that $\omega_{E(A,A^c)} \equiv t_0$, identity*

$$\langle\omega\rangle_E^\eta = \left\langle \omega_{E(A)}\right\rangle_{E(A)}^{\emptyset} \left\langle \omega_{E(A^c)}\right\rangle_{E(A^c)}^{\eta} \tag{1.45}$$

holds. Also, for $\zeta \in S^A$, we have

$$\left|\left\{\sigma \in S^V : \sigma_U{=}\eta, \sigma_A{=}\zeta, \sigma \succcurlyeq \omega\right\}\right| = \left\langle \omega_{E(A^c)}\right\rangle_{E(A^c)}^{\eta} \mathbf{1}\left\{\zeta \succcurlyeq \omega_{E(A)}\right\}. \tag{1.46}$$

Notation 1.3.13 (Concatenated configurations)**.** *For E' and E'' disjoint subsets of E, and $\omega' \in T^{E'}$ and $\omega'' \in T^{E''}$ edge-configurations, the concatenated edge-configuration $\omega'\omega'' \in T^{E' \cup E''}$ is defined as the one satisfying $\Pi_{E'}(\omega'\omega'') = \omega'$ and $\Pi_{E''}(\omega'\omega'') = \omega''$. The same notation is used for vertex-configurations.*

As a consequence of the previous observation, we obtain the following proposition.

Proposition 1.3.14 (Markov property for the random-cluster probability)**.** *Let $\omega' \in T^{E(A^c)}$ be such that $\phi_E^\eta\left(\omega_{E(A^c)} = \omega'\right) > 0$. Then*

$$\phi_E^\eta\left(\omega_{E(A)} = \omega'' \,\middle|\, \omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} \equiv t_0\right) = \phi_{E(A)}^{\emptyset}\left(\omega''\right) \tag{1.47}$$

for every $\omega'' \in T^{E(A)}$. An immediate consequence is that, if $\mathcal{X} \subset T^E$ is an event depending on $E(A)$, that is if it is of the form $\Pi_{E(A)}^{-1}(\mathcal{Y})$ for some event $\mathcal{Y} \subset T^{E(A)}$, we have

$$\phi_E^\eta\left(\mathcal{X} \,\middle|\, \omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} \equiv t_0\right) = \phi_{E(A)}^{\emptyset}\left(\mathcal{Y}\right). \tag{1.48}$$

Proof. Let $\tilde\omega \in T^{E(A,A^c)}$ be the configuration defined by $\tilde\omega \equiv t_0$. The left hand side of (1.47) is

$$\frac{\phi_E^\eta\left(\omega'\tilde\omega\omega''\right)}{\sum\limits_{\zeta \in T^{E(A)}} \phi_E^\eta\left(\omega'\tilde\omega\zeta\right)} = \frac{\hat\phi_E\left(\omega'\tilde\omega\omega''\right)\langle\omega'\tilde\omega\omega''\rangle_E^\eta}{\sum\limits_{\zeta \in T^{E(A)}} \hat\phi_E\left(\omega'\tilde\omega\zeta\right)\langle\omega'\tilde\omega\zeta\rangle_E^\eta}. \tag{1.49}$$

The terms of the form $\hat\phi_E\left(\omega'\tilde\omega\zeta\right)$ factorize as $\hat\phi_{E(A^c)}\left(\omega'\right)\hat\phi_{E(A,A^c)}\left(\tilde\omega\right)\hat\phi_{E(A)}\left(\zeta\right)$ and, from observation 1.3.12, $\langle\omega'\tilde\omega\zeta\rangle_E^\eta = \langle\omega'\rangle_{E(A^c)}^\eta \langle\zeta\rangle_{E(A)}^\emptyset$ factorizes as well. After cancellation, we get

$$\frac{\hat\phi_{E(A)}\left(\omega''\right)\langle\omega''\rangle_{E(A)}^\emptyset}{\sum\limits_{\zeta \in T^{E(A)}} \hat\phi_{E(A)}\left(\zeta\right)\langle\zeta\rangle_{E(A)}^\emptyset} = \phi_{E(A)}^\emptyset\left(\omega''\right), \tag{1.50}$$

as desired. □

Proposition 1.3.15 (Markov property for the Edwards-Sokal coupling)**.** *Let $\omega' \in T^{E(A^c)}$ be such that $\phi_E^\eta\left(\omega_{E(A^c)} = \omega'\right) > 0$. Then*

$$Q_{V,E}^\eta\left(\sigma_A{=}\sigma' \,\middle|\, \omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} \equiv t_0\right) = \mu_A^\emptyset\left(\sigma'\right) \tag{1.51}$$

for every $\sigma' \in S^A$. As in Proposition 1.3.14, if $\mathcal{X} \subset S^V$ is an event depending on A, that is $\mathcal{X} = \Pi_A^{-1}(\mathcal{Y})$ for some $\mathcal{Y} \subset S^A$, we have

$$Q_{V,E}^\eta\left(\sigma \in \mathcal{X} \,\middle|\, \omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} \equiv t_0\right) = \mu_A^\emptyset\left(\mathcal{Y}\right). \tag{1.52}$$

Proof. Let again $\tilde{\omega} \in T^{E(A,A^c)}$ be the configuration defined by $\tilde{\omega} \equiv t_0$. The left hand side of expression (1.51) can be expanded as

$$\sum_{\omega'' \in T^{E(A)}} Q_{V,E}^\eta\left(\sigma_A = \sigma' \,|\omega'\tilde{\omega}\omega''\right) \phi_E^\eta\left(\omega_{E(A)} = \omega'' \,\middle|\, \omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} = \tilde{\omega}\right). \tag{1.53}$$

From the Markov property for the random-cluster probability (Proposition 1.3.14), we have

$$\phi_E^\eta\left(\omega_{E(A)} = \omega'' \,\middle|\, \omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} = \tilde{\omega}\right) = \phi_{E(A)}^\emptyset(\omega''). \tag{1.54}$$

From observation 1.3.10 and observation 1.3.12, we get

$$Q_{V,E}^\eta\left(\sigma_A = \sigma' \,|\omega'\tilde{\omega}\omega''\right) = \frac{\left|\left\{\sigma \in S^V : \sigma_U = \eta, \sigma_A = \sigma', \sigma \succcurlyeq \omega'\tilde{\omega}\omega''\right\}\right|}{\langle\omega'\tilde{\omega}\omega''\rangle_E^\eta} \tag{1.55}$$

$$= \frac{\mathbf{1}\left\{\sigma' \succcurlyeq \omega''\right\} \langle\omega'\rangle_{E(A^c)}^\eta}{\langle\omega''\rangle_{E(A)}^\emptyset \langle\omega'\rangle_{E(A^c)}^\eta} = \frac{\mathbf{1}\left\{\sigma' \succcurlyeq \omega''\right\}}{\langle\omega''\rangle_{E(A)}^\emptyset}. \tag{1.56}$$

Replace in expression (1.53) and use observation 1.3.10 to conclude. □

The rest of this subsection is dedicated to Abelian spin models (with its associated weight function $\tilde{W}$). We give a positive correlation result that applies if the model satisfies a crucial combinatorial lemma (section 1.3.6 is dedicated to it). To state it, we need to introduce some concepts.

Definition 1.3.16. *For a function $R : S \to S$, let*

$$\mathrm{Fix}(R) := \{a \in S : Ra = a\} \tag{1.57}$$

be the set of fixed points of R. For an element $a \in S$, the hemisphere of a is defined by

$$\mathrm{Hem}(a) = \{b \in S : W(a,b) > W(a,Rb)\}. \tag{1.58}$$

Definition 1.3.17. *A function $R : S \to S$ is a generalized reflection if it is an involution (that is $R^2 = \mathrm{Id}$), it preserves W (that is $W(a,b) = W(Ra,Rb)$ for every $a,b \in S$) and $\mathrm{Hem}(b) \subset \mathrm{Hem}(a)$ for every $a,b \in S$ such that $b \in \mathrm{Hem}(a)$.*

The third condition can be replaced by the following one: the relation $\sim$ defined on $S \setminus \mathrm{Fix}(R)$ by $a \sim b$ if and only if $a \in \mathrm{Hem}(b)$ is an equivalence relation. Indeed, suppose R is a generalized reflection. Reflexivity reads $a \in \mathrm{Hem}(a)$ for every $a \in S \setminus \mathrm{Fix}(R)$, that follows from the non-singularity of W. Symmetry is obvious from the definition of Hem and transitivity follows from the third condition of the definition of generalized reflection. Reciprocally, suppose R is an involution that preserves W and that $\sim$ is an equivalence relation. By definition of $\sim$, the equivalence class of an element $a \in S \setminus \mathrm{Fix}(R)$ is $\mathrm{Hem}(a)$. Also, every element $a \in \mathrm{Fix}(R)$ satisfies $\mathrm{Hem}(a) = \emptyset$ and $a \notin \mathrm{Hem}(b)$ for every $b \in S$. From these properties, if $a, b \in S$ are such that $a \in \mathrm{Hem}(b)$, we conclude that $a, b \notin \mathrm{Fix}(R)$ and that they both are in the same equivalence relation $\mathrm{Hem}(a) = \mathrm{Hem}(b)$. Hence $\mathrm{Hem}(a) \subset \mathrm{Hem}(b)$, as we wanted.

The canonical generalized reflection is given by the function $R_a : S \to S$ defined by $R_a b = ab^{-1}$ for $a \in S$. It automatically satisfies the involution property and preserves W. The third property needs to be verified for each model.

Lemma 1.3.18. *Suppose R is a generalized reflection. Then*

$$\left|\left\{\sigma \in S^V : \sigma_x = a, \sigma_y = a, \sigma \not\succcurlyeq \omega\right\}\right| \geq \left|\left\{\sigma \in S^V : \sigma_x = Ra, \sigma_y = a, \sigma \not\succcurlyeq \omega\right\}\right| \tag{1.59}$$

for every $x, y \in V$, $a \in S$ and $\omega \in T^E$.

After identifying all the vertices in U with the vertex y, we immediately conclude

$$\hat{\mu}_V^a\left(\sigma_x = a, \sigma \not\succcurlyeq \omega\right) \geq \hat{\mu}_V^a\left(\sigma_x = Ra, \sigma \not\succcurlyeq \omega\right) \tag{1.60}$$

for every $x \in V \setminus U$.

Definition 1.3.19. *For two vertices $x, y \in V$ and an edge configuration $\omega \in T^E$, we write $x \overset{\omega}{\longleftrightarrow} y$ (resp. $x \overset{\omega,*}{\longleftrightarrow} y$), to denote there exists a path of edges $\langle x_0x_1\rangle, \langle x_1x_2\rangle, \ldots, \langle x_{l-1}x_l\rangle \in E$ such that $x_0 = x$, $x_l = y$ and $\omega_{\langle x_{i-1}x_i\rangle} = t_k$ (resp. $\omega_{\langle x_{i-1}x_i\rangle} > t_0$) for every $0 < i \leq l$. For two vertex subsets $A, B \subset V$, we write $A \overset{\omega}{\longleftrightarrow} B$ (resp. $A \overset{\omega,*}{\longleftrightarrow} B$) if there exist $x \in A$ and $y \in B$ such that $x \overset{\omega}{\longleftrightarrow} y$ (resp. $x \overset{\omega,*}{\longleftrightarrow} y$).*

Proposition 1.3.20 (Positive correlations)**.** *Suppose R is a generalized reflection. Then*

$$\mu_V^a\left(\sigma_x = a\right) \geq \mu_V^a\left(\sigma_x = Ra\right) + \phi_E^a\left(x \overset{\omega}{\longleftrightarrow} U\right) \tag{1.61}$$

for every $x \in V \setminus U$ and every $a \notin \mathrm{Fix}(R)$.

Lemma 1.3.18 is proven at the end of the chapter in a more general framework. Before going to the proof of Proposition 1.3.20, we give some examples.

> *The Potts model and the generalized clock model.* We recall that, in this case, we have $S = \mathbb{U}_q = \{a \in \mathbb{C} : a^q = 1\}$ and $\tilde{W}(a) = f\left(\mathrm{Real}(a)\right)$ for a function $f : [-1, 1] \to (0, 1]$ satisfying $f(t) = 1$ if and only if $t = 1$. For $a \in S$, the function R_a is the reflection with respect to the line connecting the origin $0 \in \mathbb{C}$ with $e^{i\frac{\arg(a)}{2}}$ (see figure 1.2), and $\mathrm{Fix}(R_a)$ is

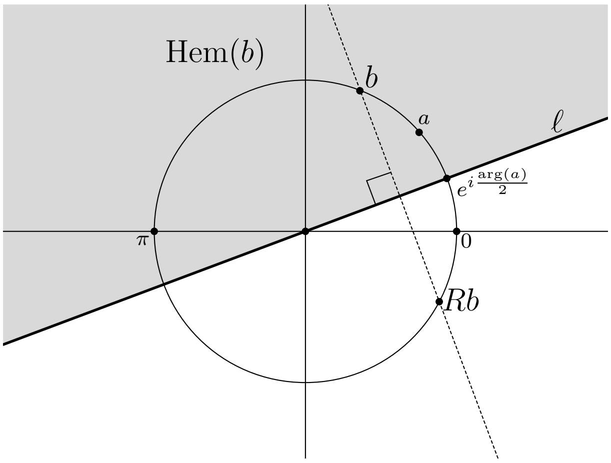

Figure 1.2

the intersection set $S \cap \left\{ \pm e^{i\frac{\arg(a)}{2}} \right\}$. In the Potts model, where $f(t) = \mathbf{1}\{t = 1\} + e^{-\beta}\mathbf{1}\{t \neq 1\}$, we have $\mathrm{Hem}(b) = \{b\}$ for every $b \in S \setminus \mathrm{Fix}(R_a)$, so R_a is a generalized reflection for every $a \in S$. If f is strictly increasing, we have

$$\mathrm{Hem}(b) = \left\{ c \in S : \mathrm{Im}(ce^{-i\frac{\arg(a)}{2}}) > 0 \right\} \tag{1.62}$$

(the intersection between S and the shaded area in figure 1.2) if $\mathrm{Im}\left(be^{-i\frac{\arg(a)}{2}}\right) > 0$ and

$$\mathrm{Hem}(b) = \left\{ c \in S : \mathrm{Im}(ce^{-i\frac{\arg(a)}{2}}) < 0 \right\} \tag{1.63}$$

if $\mathrm{Im}\left(be^{-i\frac{\arg(a)}{2}}\right) < 0$. Indeed, call $t = e^{i\frac{\arg(a)}{2}}$ and suppose we are in the first case, that is $\mathrm{Im}\left(be^{-i\frac{\arg(a)}{2}}\right) = \mathrm{Im}\left(b\bar{t}\right) > 0$ or, equivalently, $-i\left(b\bar{t} - \bar{b}t\right) > 0$. On the one hand,

$$c \in \mathrm{Hem}(b) \iff W(c,b) > W(c, R_a b) \iff \tilde{W}\left(c\bar{b}\right) > \tilde{W}\left(\bar{c}a\bar{b}\right) \iff \tag{1.64}$$

$$f\left(\mathrm{Re}\left(c\bar{b}\right)\right) > f\left(\mathrm{Re}\left(\bar{c}a\bar{b}\right)\right) \iff c\bar{b} + \bar{c}b > \bar{c}a\bar{b} + c\bar{a}b. \tag{1.65}$$

On the other hand, after multiplying by $-i\left(b\bar{t} - \bar{b}t\right)$, it is easy to check that the right-hand side of (1.65) is equivalent to $\mathrm{Im}(ce^{-i\frac{\arg(a)}{2}}) = -i\left(c\bar{t} - \bar{c}t\right) > 0$. Then

$$c \in \mathrm{Hem}(b) \iff ce^{-i\frac{\arg(a)}{2}} > 0 \tag{1.66}$$

and characterization (1.62) follows. One can analogously prove characterization (1.63). As a consequence, R_a is a generalized reflection.

The generalized Ashkin-Teller model. We call a, b, c and $e = (1,1)$ the four elements of $S = \mathbb{U}_2 \times \mathbb{U}_2$. Here a, b and c are indistinguishable in the sense that they satisfy identities $a^2 = b^2 = c^2 = e$, $ab = c$, $ac = b$ and $bc = a$. As every $\tilde{a} \in S$ satisfies $\tilde{a}^{-1} = \tilde{a}$, we have $R_{\tilde{a}} = \varphi_{\tilde{a}}$ in this case, that is $R_{\tilde{a}}(b) = \tilde{a}b$. We claim that R_a is a generalized reflection.

Observe that the non-singularity condition over $\tilde{W}$ implies $e \notin \mathrm{Hem}(a)$ and $a \in \mathrm{Hem}(a)$. If $\mathrm{Hem}(a) = \{a\}$, we are done. If $b \in \mathrm{Hem}(a)$ (the case $c \in \mathrm{Hem}(a)$ is analogue), we have $W(a,b) > W(a, R_a b)$ or, equivalently, $\tilde{W}(c) > \tilde{W}(b)$. As a consequence, $c \notin \mathrm{Hem}(a)$ ($c \in \mathrm{Hem}(a)$ if and only if $\tilde{W}(b) > \tilde{W}(c)$), so $\mathrm{Hem}(a) = \{a, b\}$. It remains to prove that $\mathrm{Hem}(b) \subset \mathrm{Hem}(a)$ or, in other words, that $c, e \notin \mathrm{Hem}(b)$. Again from the non-singularity of $\tilde{W}$, $e \notin \mathrm{Hem}(b)$. Finally, $c \in \mathrm{Hem}(b)$ is equivalent to $\tilde{W}(a) > \tilde{W}(e)$, that contradicts the non-singularity of W. As a is arbitrary, also R_b and R_c are generalized reflections.

Proof of Proposition 1.3.20. First observe that, for every $b \in S$,

$$\mu_V^a(\sigma_x = b) = Q_{V,E}^a\left(\sigma_x = b, x \overset{\omega}{\longleftrightarrow} U\right) + Q_{V,E}^a\left(\sigma_x = b, x \overset{\omega}{\not\longleftrightarrow} U\right) = \tag{1.67}$$

$$Q_{V,E}^a\left(\sigma_x = b \middle| x \overset{\omega}{\longleftrightarrow} U\right) \phi_E^a\left(x \overset{\omega}{\longleftrightarrow} U\right) + Q_{V,E}^a\left(\sigma_x = b \middle| x \overset{\omega}{\not\longleftrightarrow} U\right) \phi_E^a\left(x \overset{\omega}{\not\longleftrightarrow} U\right) = \tag{1.68}$$

$$\mathbf{1}\{a = b\}\, \phi_E^a\left(x \overset{\omega}{\longleftrightarrow} U\right) + Q_{V,E}^a\left(\sigma_x = b \middle| x \overset{\omega}{\not\longleftrightarrow} U\right) \phi_E^a\left(x \overset{\omega}{\not\longleftrightarrow} U\right); \tag{1.69}$$

the last identity follows from observation 1.3.4. Taking $b = a$ and $b = Ra$, and comparing the obtained expressions, the result is reduced to proving that

$$Q_{V,E}^a\left(\sigma_x = Ra \middle| x \overset{\omega}{\not\longleftrightarrow} U\right) \leq Q_{V,E}^a\left(\sigma_x = a \middle| x \overset{\omega}{\not\longleftrightarrow} U\right). \tag{1.70}$$

Of course, it follows if we prove

$$Q_{V,E}^a(\sigma_x = Ra \,|\omega) \leq Q_{V,E}^a(\sigma_x = a \,|\omega) \tag{1.71}$$

for every $\omega \in T^E$. But, from observation 1.3.10 and inequality (1.60),

$$Q_{V,E}^a(\sigma_x = Ra \,|\omega) = \frac{\hat{\mu}_V^a(\sigma_x = Ra, \sigma \not\succcurlyeq \omega)}{\hat{\mu}_V^a(\sigma \not\succcurlyeq \omega)} \leq \frac{\hat{\mu}_V^a(\sigma_x = a, \sigma \not\succcurlyeq \omega)}{\hat{\mu}_V^a(\sigma \not\succcurlyeq \omega)} = Q_{V,E}^a(\sigma_x = a \,|\omega). \tag{1.72}$$

□

Observation 1.3.21. *In the examples described above, R_c is a generalized reflection for every $c \in S$. Note that $R_{ab} b = a$, so we can take $R = R_{ab}$ in Lemma 1.3.18 and in Proposition 1.3.20 to respectively obtain*

$$\left|\left\{\sigma \in S^V : \sigma_x = a, \sigma_y = a, \sigma \not\succcurlyeq \omega\right\}\right| \geq \left|\left\{\sigma \in S^V : \sigma_x = b, \sigma_y = a, \sigma \not\succcurlyeq \omega\right\}\right| \tag{1.73}$$

for every $a, b \in S$ and

$$\mu_V^a(\sigma_x = a) \geq \mu_V^a(\sigma_x = b) + \phi_E^a\left(x \overset{\omega}{\longleftrightarrow} U\right) \tag{1.74}$$

for every $b \neq a$.

1.3.4 Stochastic domination from above

We dominate the random-cluster probability by Bernoulli probabilities. The domination from above works in full generality; additional hypothesis are required while dominating from below in the next subsection.

Definition 1.3.22 (Bernoulli probability)**.** *For $\rho \in [0,1]$, let $B_E^\rho = B(E,\rho)$ be the Bernoulli probability on T^E defined by*

$$B_E^\rho(\omega) := \prod_{\langle xy\rangle \in E} \left[\rho \mathbf{1}\left\{\omega_{\langle xy\rangle} = t_k\right\} + (1-\rho)\mathbf{1}\left\{\omega_{\langle xy\rangle} = t_0\right\}\right]. \tag{1.75}$$

Despite B_E^ρ is supported over the subset $\{t_0, t_k\}^E$, we define it on the hole space T^E for technical convenience. The following result is an immediate application of Holley's theorem; for the statement of this theorems and the definition of stochastic domination, go to subsection 1.6.

Proposition 1.3.23 (Stochastic domination from above)**.** *For every boundary condition $\eta \in S^U$, the stochastic domination*

$$\phi_E^\eta \leq_{st} B_E^{1-r_0} \tag{1.76}$$

holds. The same result holds for the random-cluster probability with empty boundary condition: $\phi_E^\emptyset \leq_{st} B_E^{1-r_0}$.

Proof. If we take $P = \phi_E^\eta$ and $P' = B_E^{1-r_0}$ in Theorem 1.6.1, it is easy to see that condition **(a)** is satisfied. Then the problem is reduced to proving that, for any $\langle xy\rangle \in E$, any $t \in \mathbb{R}$ and any pair of configurations $\omega', \omega'' \in T^{E\setminus\langle xy\rangle}$ satisfying

$$\phi_E^\eta\left(\omega_{E\setminus\langle xy\rangle} = \omega'\right), B_E^{1-r_0}\left(\omega_{E\setminus\langle xy\rangle} = \omega''\right) > 0 \tag{1.77}$$

and $\omega' \preccurlyeq \omega''$, we have

$$\phi_E^\eta\left(\omega_{\langle xy\rangle} \geq t \middle| \omega_{E\setminus\langle xy\rangle} = \omega'\right) \leq B_E^{1-r_0}\left(\omega_{\langle xy\rangle} \geq t \middle| \omega_{E\setminus\langle xy\rangle} = \omega''\right). \tag{1.78}$$

For $t \leq t_0$ and $t > t_k$, this inequality is trivialy satisfied. For $t_0 < t \leq t_k$, the right hand side is simply $1 - r_0$ and we can then restrict to the case $t_0 < t \leq t_1$. In this case, inequality (1.78) can be written as

$$\phi_E^\eta\left(\omega_{\langle xy\rangle} \neq t_0 | \omega_{E\setminus\langle xy\rangle} = \omega'\right) \leq 1 - r_0, \tag{1.79}$$

or as

$$r_0 \leq \phi_E^\eta\left(\omega_{\langle xy\rangle} = t_0 | \omega_{E\setminus\langle xy\rangle} = \omega'\right). \tag{1.80}$$

For every i, let $\omega'_{t_i} \in T^E$ be the concatenated configuration defined by $\Pi_{E\setminus\langle xy\rangle}\left(\omega'_{t_i}\right) = \omega'$ and $\Pi_{\langle xy\rangle}\left(\omega'_{t_i}\right) = t_i$. Last inequality follows from the simple computation

$$\phi_E^\eta\left(\omega_{\langle xy\rangle} = t_0 \middle| \Pi_{E\setminus\langle xy\rangle} = \omega'\right) = \frac{\hat{\phi}_E\left(\omega'_{t_0}\right)\left\langle\omega'_{t_0}\right\rangle_E^\eta}{\sum_{i=0}^k \hat{\phi}_E\left(\omega'_{t_i}\right)\left\langle\omega'_{t_i}\right\rangle_E^\eta} \geq \frac{r_0\left\langle\omega'_{t_0}\right\rangle_E^\eta}{\sum_{i=0}^k r_i\left\langle\omega'_{t_0}\right\rangle_E^\eta} = r_0 \tag{1.81}$$

where, in the inequality, we used the monotonicity of the weights (observation 1.3.7). □

Observation 1.3.24. *In the previous proof, we showed that the conditional probabilities outside a given edge (the ones of the form* $\phi_E^\eta\left(\omega_{\langle xy\rangle} \neq t_0 | \omega_{E\setminus\langle xy\rangle} = \omega'\right)$*) are bounded from above by* $1 - r_0$*. Of course, it is also true if we condition outside a smaller region. More precisely, for an edge subset* $E' \subset E$*, an edge* $\langle xy\rangle \in E \setminus E'$ *and a configuration* $\omega' \in T^{E'}$*, we have*

$$\phi_E^\eta\left(\omega_{\langle xy\rangle} \neq t_0\, | \omega_{E'} = \omega'\right) \leq 1 - r_0. \tag{1.82}$$

As in the previous subsection, we now take $A \subset V \setminus U$. The following proposition shows how to estimate the behaviour of random-cluster probabilities with different boundary conditions in the region $E(A)$.

Proposition 1.3.25. *Let* $\eta, \eta' \in S^U$ *be two boundary conditions and* $\mathcal{X} \subset T^E$ *an event depending on* $E(A)$ *(as defined in Proposition 1.3.14). Then*

$$\left|\phi_E^\eta(\mathcal{X}) - \phi_E^{\eta'}(\mathcal{X})\right| \leq B_E^{1-r_0}\left(A \overset{\omega,*}{\longleftrightarrow} U\right). \tag{1.83}$$

Proof. We follow the proof given in [AC97]. Let ν be the graph distance on (V, E): $\nu(x, x) = 0$ and

$$\nu(x, y) := \min\left\{n \in \mathbb{N} : \exists\, \langle x_0x_1\rangle, \langle x_1x_2\rangle, \dots, \langle x_{n-1}x_n\rangle \in E \text{ such that } x_0 = x, x_n = y\right\}, \tag{1.84}$$

with the convention $\min \emptyset = \infty$. For a vertex subset $B \subset V$ and a vertex $x \in V$, we define

$$\nu(x, B) := \min\left\{\nu(x, y) : y \in B\right\}. \tag{1.85}$$

If, for $n \in \mathbb{N} \cup \{0, \infty\}$, we define $V_n := \{x \in V : d(x, U) = n\}$, then V can be partitioned as $\cup_{n\in\mathbb{N}\cup\{0,\infty\}} V_n$. We give the order $e_1, e_2, \dots, e_{|E\setminus E(A)|}$ to the edges of $E \setminus E(A)$ starting with the ones in $E(U, V_1)$, following with the ones in $E(V_1)$, continuing with the ones in $E(V_1, V_2)$, then with the ones in $E(V_2)$, and so on, finishing with the ones in $E(V_\infty)$. For $N \leq |E \setminus E(A)|$, we define the following growing algorithm. We start by randomly assigning to e_1 either the value (or flag) t_0 or t_k; they are assigned with respectively probabilities r_0 and $1 - r_0$. We continue by choosing the minimum $i \in \{2, 3, \dots, N\}$ such that $e_i \in E(U, V_1)$ or such that one of the extreme vertices of e_i is connected to U by a path of edges that have been already flagged with the value t_k. For every step $1 \leq i \leq N$ of the algorithm, let $\Gamma_i \subset V$ be the support of the edges that have not been flagged yet (the support of an edge subset $E' \subset E$ is defined by

$$\operatorname{supp}(E') := \{x \in V : x \in \{y, z\} \text{ for some } \langle yz\rangle \in E'\}\,.) \tag{1.86}$$

The algorithm stops when all the edges in $E(\Gamma_i, \Gamma_i^c)$ has been flagged with t_0 or when it reaches the N-th step; in the later case, we say that the stop is forced. An outcome of this algorithm is a configuration $\theta_F \in \{t_0, t_k\}^F$, where F is the set of edges that have been flagged when the algorithm stops. Let $\Gamma := \operatorname{supp}(E \setminus F)$ be the support of the "not flagged" edges. Observe that $\Pi_{E(\Gamma,\Gamma^c)}(\theta_F) \equiv t_0$ for every non-forced outcome θ_F. Observe also that, in the case $N = |E \setminus E(A)|$, we have $\Gamma = A$ for every forced outcome. Let X and Y be random elements taking values on T^E with laws ϕ_E^η and $B_E^{1-r_0}$ respectively, and let P be the underlying probability: $P(X=\omega)=\phi_E^\eta(\omega)$ and $P(Y=\omega)=B_E^{1-r_0}(\omega)$ for every $\omega\in T^E$. Under this definition, the probability of obtaining the outcome θ_F is $P(Y_F=\theta_F)$, and

$$\sum_{\theta_F \text{ outcome}} P(Y_F=\theta_F)=1. \tag{1.87}$$

The result follows from the following lemma.

Lemma 1.3.26. *Fix* $1\le N\le |E\setminus E(A)|$ *and consider the growing algorithm defined above. Then*

$$P(X=\cdot)=\sum_{\theta_F \text{ outcome}} P(Y_F=\theta_F)\,\alpha_{\theta_F}(\cdot)\,; \tag{1.88}$$

for every outcome θ_F, α_{θ_F} *is a measure on* T^E *defined by*

$$\alpha_{\theta_F}(\cdot):=\sum_{\substack{\zeta_F\in T^F\\ \zeta_F\le\theta_F}}\lambda_{\theta_F}(\zeta_F)P\left(X=\cdot\,\middle|X_F=\zeta_F\right), \tag{1.89}$$

with $(\lambda_{\theta_F}(\zeta_F))_{\zeta_F\in T^F:\zeta_F\le\theta_F}$ *convex coefficients (that depend on the boundary condition* η*).*

Before giving the proof of this lemma, we see how we conclude from it. Let $N=|E\setminus E(A)|$. We have

$$\phi_E^\eta(\mathcal{X})=P(X\in\mathcal{X})=\sum_{\theta_F \text{ outcome}} P(Y_F=\theta_F)\,\alpha_{\theta_F}(\mathcal{X})\,. \tag{1.90}$$

We say that the outcome θ_F is good if it has not been forced (that is if $\Pi_{E(\Gamma,\Gamma^c)}(\theta_F)\equiv t_0$, as mentioned before). We claim that, for θ_F a good outcome, $\alpha_{\theta_F}(\mathcal{X})$ does not depend on η. More precisely, as $A\subset\Gamma$, $\mathcal{X}$ depends on $E(\Gamma)$ so there exists $\mathcal{Y}_{E(\Gamma)}\subset T^{E(\Gamma)}$ such that $\mathcal{X}=\Pi^{-1}_{E(\Gamma)}\left(\mathcal{Y}_{E(\Gamma)}\right)$. By Proposition 1.3.14,

$$P(X\in\mathcal{X}\,|X_F=\zeta_F)=\phi_E^\eta(\omega\in\mathcal{X}\,|\omega_F=\zeta_F)=\phi^{\emptyset}_{E(\Gamma)}\left(\mathcal{Y}_{E(\Gamma)}\right); \tag{1.91}$$

indeed, if ζ_F is such that $\zeta_F\le\theta_F$, we have $\Pi_{E(\Gamma,\Gamma^c)}(\zeta_F)\equiv t_0$; from the fact that $(\lambda_{\theta_F}(\zeta_F))_{\zeta_F\in T^F:\zeta_F\le\theta_F}$ are convex coefficients, we conclude

$$\alpha_{\theta_F}(\mathcal{X})=\phi^{\emptyset}_{E(\Gamma)}\left(\mathcal{Y}_{E(\Gamma)}\right). \tag{1.92}$$

Then the right hand side of expression (1.90) can be written as

$$\sum_{\theta_F \text{ good}} P(Y_F=\theta_F)\,\phi^{\emptyset}_{E(\Gamma)}\left(\mathcal{Y}_{E(\Gamma)}\right)+\sum_{\theta_F \text{ forced}} P(Y_F=\theta_F)\,\alpha_{\theta_F}(\mathcal{X})\,. \tag{1.93}$$

The first sum does not depend on η; the second one can be controlled in the following way:

$$\sum_{\theta_F \text{ forced}} P(Y_F=\theta_F)\,\alpha_{\theta_F}(\mathcal{X})\le\sum_{\theta_F \text{ forced}} P(Y_F=\theta_F)=B_E^{1-r_0}\left(A\overset{\omega}{\longleftrightarrow}U\right). \tag{1.94}$$

As the analogous expression to (1.93) holds for $\phi_E^{\eta'}(\mathcal{X})$, we have

$$\left|\phi_E^{\eta}(\mathcal{X})-\phi_E^{\eta'}(\mathcal{X})\right|\le 2B_E^{1-r_0}\left(A\overset{\omega}{\longleftrightarrow}U\right), \tag{1.95}$$

as desired.

Proof of Lemma 1.3.26. The proof is by induction.

For $N = 1$, we have two possible outcomes. In this case, we have to prove that

$$P(X = \cdot) = r_0\alpha_{t_0}(\cdot) + (1 - r_0)\alpha_{t_k}(\cdot), \tag{1.96}$$

where $\alpha_{t_0}(\cdot) := P\left(X = \cdot \middle| X_{e_1} = t_0\right)$ and α_{t_k} is a convex combination of the measures $\left(P\left(X = \cdot \middle| X_{e_1} = t_i\right)\right)_{i=0}^{k}$. (We remind that if the conditioning has probability zero, the conditioned probability is not a probability but the null measure.) For $i > 0$, define

$$\lambda_{t_i} := \frac{P(X_{e_1} = t_i)}{1 - r_0} \tag{1.97}$$

and

$$\lambda_{t_0} := 1 - \sum_{i>0} \lambda_{t_i}. \tag{1.98}$$

Observe that $\sum_{i>0} \lambda_{t_i} \leq 1$ thanks to observation 1.3.24. It is easy to check that the convex combination

$$\alpha_{t_k}(\cdot) := \sum_{i=0}^{k} \lambda_{t_i} P\left(X = \cdot \middle| X_{e_1} = t_i\right) \tag{1.99}$$

works.

Suppose the conclusion holds for $N < |E \setminus F|$. Then we can write

$$P(X = \cdot) = \sum_{\theta_F \text{ outcome}} P(Y_F = \theta_F)\, \alpha_{\theta_F}(\cdot), \tag{1.100}$$

with

$$\alpha_{\theta_F}(\cdot) = \sum_{\zeta_F \leq \theta_F} \lambda_{\theta_F}(\zeta_F) P\left(X = \cdot \middle| X_F = \zeta_F\right). \tag{1.101}$$

We can write expression (1.100) as

$$\sum_{\theta_F \text{ outcome}} P(Y_F = \theta_F)\, \alpha_{\theta_F}(\cdot) = \tag{1.102}$$

$$\sum_{\theta_F \text{ good}} P(Y_F = \theta_F)\, \alpha_{\theta_F}(\cdot) + \sum_{\theta_F \text{ forced}} P(Y_F = \theta_F)\, \alpha_{\theta_F}(\cdot). \tag{1.103}$$

For every forced outcome θ_F, let $e = e(\theta_F)$ be the $(N+1)$-th edge to be sampled in the algorithm of $N + 1$ steps (if θ_F is a good outcome for the algorithm of N steps, it is also a good outcome for the one of $N + 1$ steps; in this case, there is no edge to be flagged in the next step). Every outcome of the algorithm of $N + 1$ steps is a good outcome of the algorithm of N steps or is of the form $\theta_F^{t_0}$ or of the form $\theta_F^{t_k}$, with θ_F a forced outcome of the algorithm with N steps and $\theta_F^{t_i}$ the concatenated configuration defined by $\Pi_F\left(\theta_F^{t_i}\right) = \theta_F$ and $\Pi_e\left(\theta_F^{t_i}\right) = t_i$. Then we are done if we are able to write

$$\sum_{\theta_F \text{ forced}} P(Y_F = \theta_F)\, \alpha_{\theta_F}(\cdot) \tag{1.104}$$

as

$$\sum_{\theta_F \text{ forced}} \left(P\left(Y_{F\cup e} = \theta_F^{t_0}\right) \alpha_{\theta_F^{t_0}}(\cdot) + P\left(Y_F = \theta_F^{t_k}\right) \alpha_{\theta_F^{t_k}}(\cdot) \right) \tag{1.105}$$

with

$$\alpha_{\theta_F^{t_i}}(\cdot) = \sum_{\zeta_{F\cup e} \le \theta_F^{t_i}} \lambda_{\theta_F^{t_i}}(\zeta_{F\cup e}) P\left(X = \cdot \middle| X_{F\cup e} = \zeta_{F\cup e}\right), \tag{1.106}$$

$i \in \{0, k\}$. Observe that this expression can be written as

$$\sum_{\zeta_F \le \theta_F} \lambda_{\theta_F^{t_0}}(\zeta_F^{t_0}) P\left(X = \cdot \middle| X_F = \zeta_F, X_e = t_0\right) \tag{1.107}$$

for $i = 0$, and as

$$\sum_{\zeta_F \le \theta_F} \sum_{i=0}^{k} \lambda_{\theta_F^{t_k}}(\zeta_F^{t_i}) P\left(X = \cdot \middle| X_F = \zeta_F, X_e = t_i\right), \tag{1.108}$$

for $i = k$ ($\zeta_F^{t_i}$ defined analogously to $\theta_F^{t_i}$). We can conclude if we prove that identity

$$P\left(Y_F = \theta_F\right) \alpha_{\theta_F}(\cdot) = P\left(Y_{F\cup e} = \theta_F^{t_0}\right) \alpha_{\theta_F^{t_0}}(\cdot) + P\left(Y_F = \theta_F^{t_k}\right) \alpha_{\theta_F^{t_k}}(\cdot) \tag{1.109}$$

holds for every forced θ_F. Define

$$\lambda_{\theta_F^{t_0}}(\zeta_{F\cup e}) := \lambda_{\theta_F}(\zeta_F). \tag{1.110}$$

Define also

$$\lambda_{\theta_F^{t_k}}(\zeta_F^{t_i}) := \lambda_{\theta_F}(\zeta_F) \frac{P\left(X_e = t_i \middle| X_F = \zeta_F\right)}{1 - r_0} \tag{1.111}$$

for $i > 0$ and

$$\lambda_{\theta_F^{t_k}}(\zeta_F^{t_0}) := \lambda_{\theta_F}(\zeta_F) \left(1 - \frac{P\left(X_e > t_0 \middle| X_F = \zeta_F\right)}{1 - r_0}\right). \tag{1.112}$$

As before,

$$\sum_{i>0} \frac{P\left(X_e = t_i \middle| X_F = \zeta_F\right)}{1 - r_0} \le 1 \tag{1.113}$$

follows from observation 1.3.24. After replacing on the right hand side of (1.109) and making some standard computations, we obtain

$$P\left(Y_F = \theta_F\right) \sum_{\zeta_F \le \theta_F} \lambda_{\theta_F}(\zeta_F) \sum_{i \ge 0} P\left(X = \cdot \middle| X_F = \zeta_F, X_e = t_i\right) P\left(X_e = t_i \middle| X_F = \zeta_F\right); \tag{1.114}$$

of course, this is

$$P\left(Y_F = \theta_F\right) \alpha_{\theta_F}(\cdot), \tag{1.115}$$

as we wanted. □

□

1.3.5 Stochastic domination from below

During this subsection, we suppose our model is an Abelian spin model (with its associated weight function $\tilde{W}$). Let

$$\gamma := \frac{r_k}{\sum_{i=0}^{k} t_i \left|\tilde{W}^{-1}(t_i)\right|}. \tag{1.116}$$

Observe that γ is well defined and that $0 < \gamma < 1$. The definition of γ becomes clear in the proof of the following proposition.

Proposition 1.3.27. *Let S be an Abelian spin model and suppose that R_a is a generalized reflection for every $a \in S$. Then the stochastic domination*

$$B_E^{\gamma} \leq_{st} \phi_E^a \tag{1.117}$$

holds for every $a \in S$. The same domination holds for the empty boundary condition case.

Observation 1.3.28. *The hypothesis of constant boundary condition is necessary in the later statement. Indeed, suppose $x, y \in U$ and $\eta \in S^U$ are such that $\eta_x \neq \eta_y$, and suppose there exists a path of edges in E connecting them. In such a case, $\phi_E^{\eta}(\omega_E \equiv t_k) = 0$. On the other hand, $B_E^{\gamma}(\omega_E \equiv t_k) > 0$. As $(\omega_E \equiv t_k) = 0$ is an increasing event, we have $B_E^{\gamma} \not\leq_{st} \phi_E^{\eta}$.*

Proof. We take $P = B_E^{\gamma}$ and $P' = \phi_E^e$ in Theorem 1.6.1. In this case, condition **(b)** is satisfied. Then, in a similar way that in the proof of Proposition 1.3.23, the problem is reduced to proving

$$\gamma \leq \phi_E^a \left(\omega_{\langle xy \rangle} = t_k \left| \omega_{E \setminus \langle xy \rangle} = \omega' \right.\right) \tag{1.118}$$

for every $\langle xy \rangle \in E$ and every $\omega' \in T^{E \setminus \langle xy \rangle}$ such that the conditioning has positive probability.

For every $0 \leq i \leq k$, let $\omega'_{t_i} \in T^E$ be the concatenated configuration defined by $\Pi_{E \setminus \langle xy \rangle}(\omega'_{t_i}) = \omega'$ and $\Pi_{\langle xy \rangle}(\omega'_{t_i}) = t_i$. Under these definitions, the right hand side of (1.118) can be written as

$$\frac{\phi_E^a\left(\omega'_{t_k}\right)}{\sum_{i=0}^{k} \phi_E^a\left(\omega'_{t_i}\right)} = \frac{\hat{\phi}_E\left(\omega'_{t_k}\right)\left\langle \omega'_{t_k} \right\rangle_E^a}{\sum_{i=0}^{k} \hat{\phi}_E\left(\omega'_{t_i}\right)\left\langle \omega'_{t_i} \right\rangle_E^a} = \frac{r_k \left\langle \omega'_{t_k} \right\rangle_E^a}{\sum_{i=0}^{k} r_i \left\langle \omega'_{t_i} \right\rangle_E^a}. \tag{1.119}$$

Inverting this expression and replacing on (1.118), we reduce the problem to proving that

$$\sum_{i=0}^{k} \frac{r_i}{r_k} \frac{\left\langle \omega'_{t_i} \right\rangle_E^a}{\left\langle \omega'_{t_k} \right\rangle_E^a} \leq \gamma^{-1} = \sum_{i=0}^{k} \frac{t_i}{r_k} \left|\tilde{W}^{-1}(t_i)\right|. \tag{1.120}$$

Let (V, E^*) be the auxiliary graph defined from (V, E) by adding all the edges connecting the vertices in U:

$$E^* := E \cup \{\langle xy \rangle : \{x, y\} \subset U\}. \tag{1.121}$$

For every $0 \leq i \leq k$, let $\omega^*_{t_i} \in T^{E^*}$ be the configuration defined by $\Pi_{E^* \setminus E}(\omega^*_{t_i}) \equiv t_k$ and $\Pi_E(\omega^*_{t_i}) = \omega'_{t_i}$. Fix $0 \leq i \leq k$ for a moment. After observation 1.3.4, we get that every $\sigma \in S^V$ such that $\sigma \succcurlyeq \omega^*_{t_i}$ satisfies $\sigma_x = \sigma_y$ for every $x, y \in U$. We can then write

$$\left\langle \omega^*_{t_i} \right\rangle^{\emptyset}_{E^*} = \sum_{b \in S} \left\langle \omega'_{t_i} \right\rangle^{b}_{E}. \tag{1.122}$$

But, from observation 1.3.8, we have $\left\langle \omega'_{t_i} \right\rangle^{b}_{E} = \left\langle \omega'_{t_i} \right\rangle^{c}_{E}$ for every $b, c \in S$, implying $\left\langle \omega^*_{t_i} \right\rangle^{\emptyset}_{E^*} = q \left\langle \omega'_{t_i} \right\rangle^{a}_{E}$. Taking into account this consideration, the left hand side of (1.120) coincides with

$$\sum_{i=0}^{k} \frac{r_i}{r_k} \frac{\left\langle \omega^*_{t_i} \right\rangle^{\emptyset}_{E^*}}{\left\langle \omega^*_{t_k} \right\rangle^{\emptyset}_{E^*}}. \tag{1.123}$$

Observe that, if we define $\omega^* \in T^{E^* \setminus \langle xy \rangle}$ by $\Pi_{E^* \setminus E}(\omega^*) \equiv t_k$ and $\Pi_{E \setminus \langle xy \rangle}(\omega^*) = \omega'$, then

$$\left\langle \omega^*_{t_i} \right\rangle^{\emptyset}_{E^*} = \sum_{j \geq i} \left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, W(\sigma_x, \sigma_y) = t_j \right\} \right| \tag{1.124}$$

for every i. For fixed j, we have

$$\left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, W(\sigma_x, \sigma_y) = t_j \right\} \right| = \tag{1.125}$$

$$\sum_{a \in S} \sum_{\substack{b \in S \\ W(a,b) = t_j}} \left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = a, \sigma_y = b \right\} \right|. \tag{1.126}$$

By rotational invariance, we have

$$\sum_{\substack{b \in S \\ W(a,b) = t_j}} \left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = a, \sigma_y = b \right\} \right| = \tag{1.127}$$

$$\sum_{\substack{b \in S \\ \tilde{W}(a^{-1}b) = t_j}} \left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = a^{-1}b \right\} \right| = \tag{1.128}$$

$$\sum_{\substack{c \in S \\ \tilde{W}(c) = t_j}} \left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = c \right\} \right| \tag{1.129}$$

for every $a \in S$. (Remember $e \in S$ is the unit element.) Replacing on (1.126), we obtain

$$q \sum_{\substack{c \in S \\ \tilde{W}(c) = t_j}} \left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = c \right\} \right|. \tag{1.130}$$

Replacing on the right hand side of (1.124), we get

$$\left\langle \omega^*_{t_i} \right\rangle^{\emptyset}_{E^*} = q \sum_{j \geq i} \sum_{\substack{c \in S \\ \tilde{W}(c) = t_j}} \left| \left\{ \sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = c \right\} \right|. \tag{1.131}$$

From the non-singularity of W, in the particular case $i = k$, the later expression is

$$q\left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = e\right\}\right|. \tag{1.132}$$

Replacing in expression (1.123), we obtain

$$\sum_{i=0}^{k}\sum_{j\geq i}\frac{r_i}{r_k}\sum_{\substack{c\in S\\ \tilde{W}(c)=t_j}}\frac{\left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = c\right\}\right|}{\left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = e\right\}\right|}. \tag{1.133}$$

From observation 1.3.21, the quotient

$$\frac{\left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = c\right\}\right|}{\left|\left\{\sigma \in S^V : \sigma \succcurlyeq \omega^*, \sigma_x = e, \sigma_y = e\right\}\right|} \tag{1.134}$$

is bounded from above by 1; we conclude (1.133) is bounded from above by

$$\sum_{i=0}^{k}\sum_{j\geq i}\frac{r_i}{r_k}\left|\tilde{W}^{-1}(t_j)\right|. \tag{1.135}$$

Finally, inequality (1.120) follows by observing the later expression can be written as $\sum_{i=0}^{k}\frac{t_i}{r_k}\left|\tilde{W}^{-1}(t_i)\right|$ after a change of variables (and using that $t_i = \sum_{j=0}^{i} r_j$).

The proof for the empty boundary condition is similar. □

1.3.6 Combinatorial lemma

In this subsection, we give a more general version to Lemma 1.3.18. We intentionally allow some notation overlapping. Let (V, E) be a finite graph as in section 1.3, and let $x, y \in V$ be two distinguished vertices. Let S be any set (not necessarily finite). Fix a configuration $\omega \in \mathbb{R}^E$ and a symmetric function $W : S \times S \to \mathbb{R}$. As before, by symmetric we mean $W(a, b) = W(b, a)$ for all $a, b \in S$. Fix a function $R : S \to S$ such that $R^2 = \mathrm{Id}_S$ (involution property) and $W(a, b) = W(Ra, Rb)$ for all $a, b \in S$ (R preserves W). R plays the role of a generalized reflection. For each $a \in S$, we define the hyperplane associated to a by

$$\mathrm{Hem}(a) = \{b \in S : W(a, b) > W(a, Rb)\}. \tag{1.136}$$

As R satisfies

$$W(a, Rb) = W(Ra, R^2b) = W(Ra, b) \tag{1.137}$$

for every $a, b \in S$, the hemisphere $\mathrm{Hem}(a)$ can be written as

$$\mathrm{Hem}(a) = \{b \in S : W(a, b) > W(Ra, b)\}. \tag{1.138}$$

Finally, for $a, b \in S$, we define the set

$$L(a, b) = \left\{\sigma \in S^V : \sigma_x = a, \sigma_y = b, \sigma \succcurlyeq \omega\right\} \tag{1.139}$$

where, as usual, we have $\sigma \succcurlyeq \omega$ if and only if $\omega_{\langle uv\rangle} \leq W(\sigma_u, \sigma_v)$ for every $\langle uv\rangle \in E$.

Lemma 1.3.29. *Let $a, b \in S$ such that $W(a,b) \leq W(Ra,b)$. Suppose R satisfies $\mathrm{Hem}(c) \subset \mathrm{Hem}(a)$ for every $c \in \mathrm{Hem}(a)$. Under these hypothesis, there exists an injection $\Phi : L(a,b) \hookrightarrow L(Ra,b)$.*

Taking $b = a$, Lemma 1.3.18 follows immediately.

Before going to the proof of Lemma 1.3.29, we show an application that is important from the point of view of the intuition. Let $S = \mathbb{Z}$ and, for simplicity, let $b = 0$. An integer Lipschitz function is a configuration $\sigma \in \mathbb{Z}^V$ such that $|\sigma_u - \sigma_v| \leq 1$ for every $\langle uv \rangle \in E$. This definition corresponds to the standard definition of Lipschitz function with constant 1, if we consider on V the graph distance ν as defined in the proof of Proposition 1.3.25. Let W be defined as $W(c,c') = -|c - c'|$ and ω as $\omega_{\langle uv \rangle} = -1$ for every $\langle uv \rangle \in E$. According to these definitions, $L(a,b)$ is the set of Lipschitz functions taking the value a in x and 0 in y. Fix $a' \in \mathbb{Z}$ and suppose $0 \leq a' \leq a$. Define the function R by

$$Rc = a + a' - c. \tag{1.140}$$

As R is the reflection with respect to $\frac{a+a'}{2}$, we have $Ra = a'$. Obviously, R is an involution and preserves W. The previous lemma tells us there exists an injection $\Phi : L(a,0) \hookrightarrow L(Ra,0) = L(a',0)$ provided that $|0 - a| \geq |0 - a'|$, inequality that is satisfied by hypothesis, and that $\mathrm{Hem}(c) \subset \mathrm{Hem}(a)$ for every $c \in \mathrm{Hem}(a)$. Last condition follows from the fact that

$$\mathrm{Hem}(c) = \begin{cases} (\frac{a+a'}{2}, \infty) \cap \mathbb{Z} & \text{in case } c > \frac{a+a'}{2} \\ (-\infty, \frac{a+a'}{2}) \cap \mathbb{Z} & \text{in case } c < \frac{a+a'}{2} \\ \emptyset & \text{in case } c = \frac{a+a'}{2} \end{cases}. \tag{1.141}$$

Then the lemma proves inequality

$$|L(a,0)| \leq |L(a',0)|\,. \tag{1.142}$$

Using the fact that

$$|L(b,0)| = |L(-b,0)| \tag{1.143}$$

for every $b \in S$, that holds by symmetry, we deduce that property (1.142) is actually true for every pair of integers a and a' satisfying $|a| \geq |a'|$ (not necessarily non-negative). Last fact can be rephrased in the following way: if we uniformly sample a function over the set $\left\{\sigma \in S^V : \sigma_y = 0, \sigma \succcurlyeq \omega\right\}$, the probability of the event $(\sigma_x = a)$ is decreasing on $|a|$.

Proof. Let $\sigma \in L(a,b)$. We define a dependent-on-σ sequence $A_0 \subset A_1 \subset A_2 \subset \ldots \subset V$ of nested subsets of V. Let $A_0 = \{x\}$ and, for $n \geq 0$, define, in a recursive way,

$$A_{n+1} = A_n \cup \left\{u \in \partial A_n : W(\sigma_u, R\sigma_v) < \omega_{\langle uv \rangle} \text{ for some } v \in A_n\right\}.$$

The set $A_{n+1} \setminus A_n$ has to be read as the set of vertices where incompatibilities appear after applying the transformation R to all the vertices in A_n. We define

$$A = \bigcup_{n \geq 0} A_n$$

and the injection Φ by

$$(\Phi\sigma)_A = (R\sigma)_A \text{ and } (\Phi\sigma)_{V\setminus A} = \sigma_{V\setminus A},$$

where $R\sigma \in S^V$ is the configuration defined by $(R\sigma)_u = R\sigma_u$ for every $u \in V$. We have to see that Φ is well defined and that it is injective.

Good definition. We have to prove that $\Phi\sigma \succcurlyeq \omega$ and that $(\Phi\sigma)_y = b$.

$\Phi\sigma \succcurlyeq \omega$. For $\langle uv \rangle \in E$, we have to see that $W((\Phi\sigma)_u, (\Phi\sigma)_v) \geq \omega_{\langle uv \rangle}$. The case $\{u, v\} \subset A$ follows because R preserves W and the case $\{u, v\} \subset A^c$ because, in this case, the values of the configuration are not modified. If $u \notin A$ and $v \in A$, the concerning inequality must be true because, on the contrary, it would be $u \in A$.

$(\Phi\sigma)_y = b$. We are done if we prove that $y \notin A$. The hypothesis $W(a, b) \leq W(Ra, b)$ can be rephrased as $b \notin \mathrm{Hem}(a)$; it is then enough to prove that $\sigma_u \in \mathrm{Hem}(a)$ for every $u \in A \setminus \{x\}$. We do it by induction. If $A_1 \setminus \{x\} = \emptyset$ we are done; suppose $A_1 \setminus \{x\} \neq \emptyset$ and take $u \in A_1 \setminus \{x\}$. By definition of A_1, we have $W(\sigma_u, Ra) < \omega_{\langle u,y \rangle}$. Also, as $\sigma \succcurlyeq \omega$, we have $\omega_{\langle u,y \rangle} \leq W(\sigma_u, a)$. From this two inequalities, we obtain $W(\sigma_u, Ra) < W(\sigma_u, a)$ or, equivalently, $\sigma_u \in \mathrm{Hem}(a)$. This completes the first step of the induction. Suppose now $\sigma_u \in \mathrm{Hem}(a)$ for every $u \in A_n \setminus \{x\}$. From the hypothesis over R, we have $\mathrm{Hem}(\sigma_u) \subset \mathrm{Hem}(a)$ for every $u \in A_n \setminus \{x\}$. We can suppose $A_{n+1} \setminus A_n \neq \emptyset$ and take $v \in A_{n+1} \setminus A_n$. By definition of A_{n+1}, there exists $w \in A_n$ such that $W(\sigma_v, R\sigma_w) < \omega_{\langle v,w \rangle}$. As in the first step of the induction, from last inequality and the fact that $\sigma \succcurlyeq \omega$, we obtain $\sigma_v \in \mathrm{Hem}(\sigma_w) \subset \mathrm{Hem}(a)$.

Injectivity. For two different configurations $\sigma, \sigma' \in L(a, b)$, we have to prove that $\Phi\sigma \neq \Phi\sigma'$. Call $A, A_0, A_1, A_2, \ldots$ and $A', A'_0, A'_1, A'_2, \ldots$ the associated sets, espectively. If $A = A'$ we are done because R is injective. Suppose $A \neq A'$ and let $n = \min \{k : A_k \neq A'_k\}$; of course $n \geq 1$ and $A_{n-1} = A'_{n-1}$. If there exists $u \in A_{n-1}$ such that $\sigma_u \neq \sigma'_u$, we are done. Suppose $\sigma \overset{A_{n-1}}{=} \sigma'$. Without loss of generality, we assume $A_n \setminus A'_n \neq \emptyset$ and take $u \in A_n \setminus A'_n$. We claim that $(\Phi\sigma)_u \neq (\Phi\sigma')_u$. It must be $\sigma_u \neq \sigma'_u$; on the contrary, because of the definitions of A_n and A'_n, u would be in both A_n and A'_n or in none of them. Then, we are done if we prove that $u \in A'$. Suppose $u \notin A'$. Let $v \in A_{n-1}$ be such that $W(\sigma_u, R\sigma_v) < \omega_{\langle uv \rangle}$. From (1.137) and identity $(\Phi\sigma)_u = R\sigma_u$, we have $W(\sigma_u, R\sigma_v) = W((\Phi\sigma)_u, \sigma_v)$, and then

$$W((\Phi\sigma)_u, \sigma_v) < \omega_{\langle uv \rangle}. \tag{1.144}$$

On the other hand, as $\sigma' \succcurlyeq \omega$, we have $W(\sigma'_u, \sigma'_v) \geq \omega_{\langle uv \rangle}$. But

$$W(\sigma'_u, \sigma'_v) = W((\Phi\sigma')_u, \sigma_v) \tag{1.145}$$

because $u \notin A'$ and $\sigma \overset{A_{n-1}}{=} \sigma'$, and then

$$W((\Phi\sigma')_u, \sigma_v) \geq \omega_{\langle uv \rangle}. \tag{1.146}$$

From inequalities (1.144) and (1.146) we get $(\Phi\sigma)_u \neq (\Phi\sigma')_u$. It completes the proof. □

1.4 Dilute models

In this section, we use the previous results in finite volume to give sufficient conditions for uniqueness and non-uniqueness of Gibbs measure on dilute models. We start by defining some basic notions. We now work in infinite volume: our lattice is constituted by the set of vertices $\mathbb{Z}^d$ with its associated set of edges

$$\mathcal{E} := \{\langle xy \rangle : \|x - y\|_2 = 1\}. \tag{1.147}$$

A local function is a function $f : S^{\mathbb{Z}^d} \to \mathbb{R}$ depending on a finite number of coordinates: there exists a finite subset $A = A(f) \subset \mathbb{Z}^d$ such that $f(\sigma) = f(\sigma')$ for every pair of configurations $\sigma, \sigma' \in S^{\mathbb{Z}^d}$ satisfying $\sigma_A = \sigma'_A$. Our probability space is the product set $S^{\mathbb{Z}^d}$ endowed with the product σ-algebra $\mathcal{F}$, that is the smallest one for which every local function is measurable.

1.4.1 The model

The dilute (or disordered) model is a statistical-mechanical model defined in a random graph. More precisely, the set of vertices is deterministically given by $\mathbb{Z}^d$ and the set of edges is a random subset of $\mathcal{E}$. To properly define it, we introduce the concept of disorder.

Definition 1.4.1 (disorder)**.** *An element $J = \left(J_{\langle xy \rangle} : \langle xy \rangle \in \mathcal{E}\right) \in \{0,1\}^{\mathcal{E}}$ is called a disorder. We identify a disorder J with its associated set of open edges*

$$\mathcal{E}(J) := \left\{\langle xy \rangle \in \mathcal{E} : J_{\langle xy \rangle} = 1\right\}. \tag{1.148}$$

The rest of the edges, that is the ones in which J takes the value 0, are called closed edges.

For fixed $J \in \{0,1\}^{\mathcal{E}}$, we define the notions of specification and Gibbs measure on the graph $\left(\mathbb{Z}^d, \mathcal{E}(J)\right)$. J has to be understood as a random element with law P_p, where P_p is the Bernoulli bond percolation probability with parameter $p \in [0,1]$ (defined on $\{0,1\}^{\mathcal{E}}$). Despite we are not working in a finite graph anymore, we use the same notation concerning subsets of edges: for example, for $A \subset \mathbb{Z}^d$, $\mathcal{E}(J) \cap A$ is the set of edges with at least one extreme vertex in A.

Definition 1.4.2 (specification)**.** *The specification associated to a finite region $\Lambda \subset \mathbb{Z}^d$, a boundary condition $\eta \in S^{\mathbb{Z}^d}$ and a disorder J, is the probability $\mu^\eta_{\Lambda,J}$ defined on $\left(S^{\mathbb{Z}^d}, \mathcal{F}\right)$ by*

$$\mu^\eta_{\Lambda,J}(\sigma) := \frac{1}{Z^\eta_{\Lambda,J}} \mathbf{1}\{\sigma_{\Lambda^c} = \eta_{\Lambda^c}\} \prod_{\langle xy \rangle \in \mathcal{E}(J) \cap \Lambda} W(\sigma_x, \sigma_y). \tag{1.149}$$

Observe that $\mu^\eta_{\Lambda,J}(\sigma)$ is supported on a finite set.

As in the finite case, we write $\mu^a_{\Lambda,J}$ in case $\eta \equiv a$.

Notation 1.4.3. *For a vertex subset $A \subset \mathbb{Z}^d$, we define the boundary of A by*

$$\partial A := \left\{ x \in \mathbb{Z}^d \setminus A : \langle xy \rangle \in \mathcal{E} \text{ for some } y \in A \right\}. \tag{1.150}$$

Observation 1.4.4. *If, in the previous definition, we take $U = \partial\Lambda$, $V = \Lambda \cup \partial\Lambda$ and $E = \mathcal{E}(J) \cap \Lambda$, the specification $\mu^\eta_{\Lambda,J}$ can be identified with the probability $\mu^{\eta_U}_V$ defined in the finite graph (V, E) (as in subsection 1.3) in the sense that, for $\sigma' \in S^V$ such that $\sigma'_U = \eta_U$, we have*

$$\mu^{\eta_U}_V(\sigma') = \mu^\eta_{\Lambda,J}(\sigma_V = \sigma'). \tag{1.151}$$

Definition 1.4.5 (Gibbs measure and phase co-existence)**.** *A Gibbs measure associated to a disorder J is a probability μ_J defined on $\left(S^{\mathbb{Z}^d}, \mathcal{F}\right)$ that satisfies the so called DLR condition:*

$$\int_{S^{\mathbb{Z}^d}} \mu_J(d\sigma) f(\sigma) = \int_{S^{\mathbb{Z}^d}} \mu_J(d\eta) \left(\int_{S^{\mathbb{Z}^d}} \mu^\eta_{\Lambda,J}(d\sigma) f(\sigma) \right) \tag{1.152}$$

for every local function f and every finite region Λ. The set of Gibbs measures associated to J is denoted by $\mathcal{G}_J$. We say that phase co-existence occurs if $|\mathcal{G}_J| > 1$.

As our set of spins is finite, the set of Gibbs measures is non-empty (see [FV]), so we worry only about the two cases $|\mathcal{G}_J| = 1$ and $|\mathcal{G}_J| > 1$.

1.4.2 Uniqueness criteria

In this subsection, we give a sufficient condition for absence of phase co-existence, that is $|\mathcal{G}_J| = 1$. Our criteria is of the quenched type: it holds for P_p-almost every disorder J under certain hypothesis over p. It is in the spirit of the uniqueness criteria given in [AC97]; we show here how the uniqueness criteria appearing in the later article can be generalized (we do not ask for any structure on the set of spins) and how the proof can be simplified by the use of a different random-cluster representation. Alternative methods to prove uniqueness are the ones introduced by Dobrushin [Dob68] and van den Berg and Maes [vdBM94].

Let p_c be the critical bond Bernoulli percolation probability. (The reader who is not familiar with the basic notions in percolation theory, such as critical probability and infinite cluster, can for example consult [Gri99].)

Theorem 1.4.6. *If $p(1 - r_0) < p_c$, then $|\mathcal{G}_J| = 1$ for P_p-almost every disorder J.*

Proof. The product JJ' of two disorders J and J' is defined coordinate-wise, that is the open edges for JJ' are the ones that are open for both J and J'. Take P_{1-r_0} independent of P_p. We claim the subset $\mathcal{X} \subset \{0,1\}^{\mathcal{E}}$ defined by

$$\mathcal{X} := \{J : P_{1-r_0}(J' : JJ' \text{ has an infinite cluster}) = 0\} \tag{1.153}$$

has P_p-probability 1. Indeed, observe that if we sample a disorder J with law P_p and a disorder J' with law P_{1-r_0}, the product disorder JJ' has law $P_{p(1-r_0)}$. By the fact that $P_{p(1-r_0)}$ is sub-critical, the later observation and Fubini's theorem (in that order), we have

$$1 = \int P_{p(1-r_0)}(dJ)\mathbf{1}\left\{J \text{ has an infinite cluster}\right\} = \tag{1.154}$$

$$\int P_p \times P_{1-r_0}(d(J,J'))\mathbf{1}\left\{JJ' \text{ has an infinite cluster}\right\} = \tag{1.155}$$

$$\int P_p(dJ) \int P_{1-r_0}(dJ')\mathbf{1}\left\{JJ' \text{ has an infinite cluster}\right\}. \tag{1.156}$$

Then $\int P_{1-r_0}(dJ')\mathbf{1}\left\{JJ' \text{ has an infinite cluster}\right\} = 1$ for P_p-almost every J or, equivalently, $P_p\left(\mathcal{X}\right) = 1$.

Take a finite subset $\Delta \subset \mathbb{Z}^d$ and a configuration $\zeta \in S^\Delta$. The result follows if we prove that, for every $J \in \mathcal{X}$ and every pair of Gibbs measures $\mu_J, \mu'_J \in \mathcal{G}_J$, identity $\mu_J\left(\sigma_\Delta = \zeta\right) = \mu'_J\left(\sigma_\Delta = \zeta\right)$ holds. From the definition of Gibbs measure, for any finite subset $\Lambda \subset \mathbb{Z}^d$, we have

$$\left|\mu_J\left(\sigma_\Delta = \zeta\right) - \mu'_J\left(\sigma_\Delta = \zeta\right)\right| = \left|\int \mu_J(d\sigma)\mathbf{1}\left\{\sigma_\Delta = \zeta\right\} - \int \mu'_J(d\sigma)\mathbf{1}\left\{\sigma_\Delta = \zeta\right\}\right| = \tag{1.157}$$

$$\left|\int \mu_J(d\eta)\int \mu^\eta_{\Lambda,J}(d\sigma)\mathbf{1}\left\{\sigma_\Delta = \zeta\right\} - \int \mu'_J(d\eta)\int \mu^\eta_{\Lambda,J}(d\sigma)\mathbf{1}\left\{\sigma_\Delta = \zeta\right\}\right| = \tag{1.158}$$

$$\left|\int \mu_J(d\eta)\mu^\eta_{\Lambda,J}\left(\sigma_\Delta = \zeta\right) - \int \mu'_J(d\eta)\mu^\eta_{\Lambda,J}\left(\sigma_\Delta = \zeta\right)\right| \leq \tag{1.159}$$

$$\max_{\eta,\eta' \in S^{\mathbb{Z}^d}} \left\{\mu^\eta_{\Lambda,J}\left(\sigma_\Delta = \zeta\right) - \mu^{\eta'}_{\Lambda,J}\left(\sigma_\Delta = \zeta\right)\right\}. \tag{1.160}$$

The strategy will be to take $\varepsilon > 0$ arbitrary and $\Lambda = \Lambda(\varepsilon)$ large enough such that

$$\left|\mu^\eta_{\Lambda,J}\left(\sigma_\Delta = \zeta\right) - \mu^{\eta'}_{\Lambda,J}\left(\sigma_\Delta = \zeta\right)\right| < \varepsilon. \tag{1.161}$$

For every $n \in \mathbb{N}$, let $\Lambda_n := [-n,n]^d \cap \mathbb{Z}^d$. From (1.153), we can take $n = n(J) \in \mathbb{N}$ such that $\Delta \subset \Lambda_n$ and $P_{1-r_0}\left(\Delta \overset{JJ'}{\longleftrightarrow} \Lambda^c_n\right) \leq \frac{\varepsilon}{4}$. (We use here the analogous to definition 1.3.19: $\Delta \overset{JJ'}{\longleftrightarrow} \Lambda^c_n$ if there is a path of edges connecting Δ with Λ^c_n taking the value 1 in JJ'.) Again from (1.153), we can take $N \in \mathbb{N}$ such that $\Lambda_{n+1} \subset \Lambda_N$ and $P_{1-r_0}\left(\Lambda_{n+1} \overset{JJ'}{\longleftrightarrow} \Lambda^c_N\right) \leq \frac{\varepsilon}{4}\left|\left\{A \subset \Lambda_n\right\}\right|^{-1} = \frac{\varepsilon}{4}2^{-|\Lambda_n|}$ (the choice of this bound appears naturally at the end of the proof). Λ_N plays the role of the subset Λ mentioned before.

Take $U = \partial\Lambda_N$, $V = \Lambda_N \cup \partial\Lambda_N$ and $E = \mathcal{E}(J) \cap \Lambda_N$ (as in subsection 1.3). For a configuration $\omega \in T^E$, the set family

$$\left\{A \subset \Lambda_n : E(A, A^c) \equiv t_0\right\} \tag{1.162}$$

is closed under unions. Let

$$C_\omega := \bigcup \{A \subset \Lambda_n : E(A, A^c) \equiv t_0\} \tag{1.163}$$

be its maximal element, with the convention $C_\omega = \emptyset$ if $\{A \subset \Lambda_n : E(A, A^c) \equiv t_0\} = \emptyset$. As in observation 1.4.4,

$$\mu^{\eta}_{\Lambda,J}\left(\sigma_\Delta = \zeta\right) = \mu^{\eta_U}_{V}\left(\sigma_\Delta = \zeta\right) = \tag{1.164}$$

$$\sum_{\substack{A\subset\Lambda_n\\ \Delta\subset A}} Q^{\eta_U}_{V,E}\left(\sigma_\Delta = \zeta\,|C_\omega = A\right)\phi^{\eta_U}_{E}\left(C_\omega = A\right) + \sum_{\substack{A\subset\Lambda_n\\ \Delta\not\subset A}} Q^{\eta_U}_{V,E}\left(\sigma_\Delta = \zeta, C_\omega = A\right). \tag{1.165}$$

For the second sum, observe that

$$\sum_{\substack{A\subset\Lambda_n\\ \Delta\not\subset A}} Q^{\eta_U}_{V,E}\left(\sigma_\Delta = \zeta, C_\omega = A\right) \le \sum_{\substack{A\subset\Lambda_n\\ \Delta\not\subset A}} Q^{\eta_U}_{V,E}\left(C_\omega = A\right) = \sum_{\substack{A\subset\Lambda_n\\ \Delta\not\subset A}} \phi^{\eta_U}_{E}\left(C_\omega = A\right) = \tag{1.166}$$

$$\phi^{\eta_U}_{E}\left(\Delta \overset{\omega,*}{\longleftrightarrow} \Lambda^c_n\right) \le B^{1-r_0}_{E}\left(\Delta \overset{\omega,*}{\longleftrightarrow} \Lambda^c_n\right) = P_{1-r_0}\left(\Delta \overset{J}{\longleftrightarrow} \Lambda^c_n\right) \le \frac{\varepsilon}{4}; \tag{1.167}$$

in the second inequality, we used Proposition 1.3.23 and the fact that $\left(\Delta \overset{\omega,*}{\longleftrightarrow} \Lambda^c_n\right)$ is an increasing event. For $A \subset \Lambda_n$ such that $\Delta \subset A$, we claim that

$$Q^{\eta_U}_{V,E}\left(\sigma_\Delta = \zeta\,|C_\omega = A\right) = \mu^{\emptyset}_{A}\left(\sigma_\Delta = \zeta\right). \tag{1.168}$$

Indeed, $(C_\omega = A)$ only depends on $E \cap A^c$; let $\mathcal{Y} \subset T^{E\cap A^c}$ such that $(C_\omega = A) = \Pi^{-1}_{E\cap A^c}(\mathcal{Y})$. Observe also that $\Pi_{E(A,A^c)}(\omega') \equiv t_0$ for every $\omega' \in \mathcal{Y}$. As a consequence, $(C_\omega = A)$ is the disjoint union

$$\bigcup_{\omega'\in\Pi_{E(A^c)}(\mathcal{Y})} \left(\omega_{E\cap A^c} = \omega', \omega_{E(A,A^c)} \equiv t_0\right). \tag{1.169}$$

Then

$$Q^{\eta_U}_{V,E}\left(\sigma_\Delta = \zeta\,|C_\omega = A\right) = \tag{1.170}$$

$$\sum_{\omega'\in\Pi_{E(A^c)}(\mathcal{Y})} Q^{\eta_U}_{V,E}\left(\sigma_\Delta = \zeta\,\middle|\omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} \equiv t_0\right)\phi^{\eta_U}_{E}\left(\omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} \equiv t_0\middle| C_\omega = A\right) = \tag{1.171}$$

$$\mu^{\emptyset}_{A}\left(\sigma_\Delta = \zeta\right)\sum_{\omega'\in\Pi_{E(A^c)}(\mathcal{Y})} \phi^{\eta_U}_{E}\left(\omega_{E(A^c)} = \omega', \omega_{E(A,A^c)}\middle| C_\omega = A\right) = \mu^{\emptyset}_{A}\left(\sigma_\Delta = \zeta\right); \tag{1.172}$$

in the second identity, we used the Markov property for the Edwards-Sokal coupling (Proposition 1.3.15) to establish identity

$$\mu^{\emptyset}_{A}\left(\sigma_\Delta = \zeta\right) = Q^{\eta_U}_{V,E}\left(\sigma_\Delta = \zeta\,\middle|\omega_{E(A^c)} = \omega', \omega_{E(A,A^c)} \equiv t_0\right). \tag{1.173}$$

Replace in (1.165) to obtain

$$\mu_V^{\eta_U}(\sigma_\Delta=\zeta)=\sum_{\substack{A\subset\Lambda_n\\ \Delta\subset A}}\mu_A^{\emptyset}(\sigma_\Delta=\zeta)\,\phi_E^{\eta_U}(C_\omega=A)+\text{sth}, \tag{1.174}$$

with $0\le \text{sth}\le \frac{\varepsilon}{4}$. As the analogous formula holds for $\mu_V^{\eta_U'}(\sigma_\Delta=\zeta)$, we have

$$\left|\mu_V^{\eta_U}(\sigma_\Delta=\zeta)-\mu_V^{\eta_U'}(\sigma_\Delta=\zeta)\right|\le \tag{1.175}$$

$$\sum_{\substack{A\subset\Lambda_n\\ \Delta\subset A}}\mu_A^{\emptyset}(\sigma_\Delta=\zeta)\left|\phi_E^{\eta_U}(C_\omega=A)-\phi_E^{\eta_U'}(C_\omega=A)\right|+\frac{\varepsilon}{2}. \tag{1.176}$$

Finally, by Proposition 1.3.25,

$$\left|\phi_E^{\eta_U}(C_\omega=A)-\phi_E^{\eta_U'}(C_\omega=A)\right|\le 2B_E^{1-r_0}\left(\Lambda_n\overset{\omega,*}{\longleftrightarrow}U\right)=2P_{1-r_0}\left(\Lambda_n\overset{J}{\longleftrightarrow}U\right)\le \tag{1.177}$$

$$\frac{\varepsilon}{2}\left|\{A\subset\Lambda_n\}\right|^{-1}. \tag{1.178}$$

We conclude by replacing it in expression (1.176). □

1.4.3 Non-uniqueness criteria

In this subsection, we give sufficient conditions for non-uniqueness of Gibbs measure in the Abelian spin case. Remember the definition of γ given in expression (1.116).

Theorem 1.4.7. *Suppose we are in the Abelian spin case. Suppose also R_a is a generalized reflection for every $a\in S$. If $p\gamma>p_c$, then $|\mathcal{G}_J|\ge q$ for P_p-almost every disorder J.*

In the following proof, we use standard concepts concerning Gibbs measures; they are studied in detail in [FV] for example.

Proof. Analogously to the proof of Theorem 1.4.6, the event $\mathcal{Y}\subset\{0,1\}^{\mathcal{E}}$ defined by

$$\mathcal{Y}:=\{J:P_\gamma(J':JJ'\text{ has an infinite cluster})=1\} \tag{1.179}$$

has P_p-probability 1. It is then enough to prove that $|\mathcal{G}_J|\ge q$ for every $J\in\mathcal{Y}$; fix such a J. Let $x\in\mathbb{Z}^d$ be a vertex belonging to a J-infinite cluster and let
$\delta:=P_\gamma(J':x\text{ belongs to a }JJ'\text{-infinite cluster})>0$ (this choice of δ is possible because of (1.179)). The strategy will be to show that there exists a family $(\mu_J^a)_{a\in S}$ of Gibbs measures satisfying

$$\mu_J^a(\sigma_x=a)\ge\mu_J^a(\sigma_x=b)+\delta \tag{1.180}$$

for $a\ne b$; it immediately follows $\mu_J^a\ne\mu_J^b$ and $|\mathcal{G}_J|\ge q$.

As in the proof of Theorem 1.4.6, let $\Lambda_n := [-n,n]^d \cap \mathbb{Z}^d$ and take $U = \partial\Lambda_n$, $V = \Lambda_n \cup \partial\Lambda_n$ and $E = \mathcal{E}(J) \cap \Lambda_n$. We have

$$\mu^a_{\Lambda_n,J}(\sigma_x = a) = \mu^a_V(\sigma_x = a) \geq \mu^a_V(\sigma_x = b) + \phi^a_E\left(x \overset{\omega}{\longleftrightarrow} U\right) \geq \tag{1.181}$$

$$\mu^a_V(\sigma_x = b) + B^\gamma_E\left(x \overset{\omega}{\longleftrightarrow} U\right) = \mu^a_{\Lambda_n,J}(\sigma_x = b) + P_\gamma\left(J' : x \overset{JJ'}{\longleftrightarrow} U\right) \geq \tag{1.182}$$

$$\mu^a_{\Lambda_n,J}(\sigma_x = b) + \delta; \tag{1.183}$$

we used observation 1.3.21 in the first inequality and Proposition 1.3.27 in the second one. Then

$$\mu^a_{\Lambda_n,J}(\sigma_x = a) \geq \mu^a_{\Lambda_n,J}(\sigma_x = b) + \delta \tag{1.184}$$

for every $n \in \mathbb{N}$. As the set of spins is finite, the space of probabilities defined on $\left(S^{\mathbb{Z}^d}, \mathcal{F}\right)$ is sequentially compact: we can extract a subsequence $\left(\mu^a_{\Lambda_{n'},J}\right)_{n'}$ of $\left(\mu^a_{\Lambda_n,J}\right)_n$ weakly converging to a probability on $\left(S^{\mathbb{Z}^d}, \mathcal{F}\right)$ that we call μ^a_J. The later probability is a Gibbs measure. As the weak convergence is characterized by the finite-volume events, we can take limit as $n' \to \infty$ in expression 1.184 with n replaced by n' to obtain

$$\mu^a_J(\sigma_x = a) \geq \mu^a_J(\sigma_x = b) + \delta. \tag{1.185}$$

It completes the proof. □

1.5 Comparison with the homogeneous case

The homogeneous version of the model is obtained by taking $p = 1$. In this case, non-uniqueness methods such as the Pirogov-Sinai theory [PS75] or reflection positivity (as in Frölich, Israel, Lieb and Simon [FILS78] or Biskup [Bis09]) prove that, for β sufficiently large, there exist at least q different Gibbs measures. Both Pirogov-Sinai theory and reflection positivity strongly depend on the symmetry of the graph, an assumption that breaks down for the properly dilute model $p < 1$.

In the homogeneous Potts model, both uniqueness and non-uniqueness criteria coincide with the ones given in [GHM01]: if $e^{-\beta} < p_c$, uniqueness is guaranteed; if $\frac{1-e^{-\beta}}{1+e^{-\beta}(q-1)} > p_c$, there are at least q Gibbs measures.

It is instructive to analyse the homogeneous classical clock model (remember it is obtained from the generalized clock model by taking $f(t) = e^{-\beta(1-t)}$). In this case, $\gamma = \gamma(\beta)$ is a function of the inverse temperature β and our criteria guarantees non-uniqueness for β such that $p\gamma(\beta) > p_c$. First of all, observe that, as functions of $\beta \in (0,\infty)$, r_k is strictly increasing and t_i is strictly decreasing for every $i < k$, implying $\gamma(\beta)$ is strictly increasing. See figure 1.3 for the graph of γ when $q = 4$. Condition $p\gamma(\beta) > p_c$ is then equivalent to $\beta > \gamma^{-1}\left(\frac{p_c}{p}\right) =: \beta_0$. Using that $\left|\tilde{W}^{-1}(t_i)\right| \leq 2$ for $0 \leq i < k$ and that $2k \leq q$, we get

$$\gamma = \frac{r_k}{1 + \sum_{i=0}^{k-1} t_i \left|\tilde{W}^{-1}(t_i)\right|} \geq \frac{r_k}{1 + 2\sum_{i=0}^{k-1} t_i} \geq \frac{r_k}{1 + 2kt_{k-1}} \geq \frac{r_k}{1 + qt_{k-1}}. \tag{1.186}$$

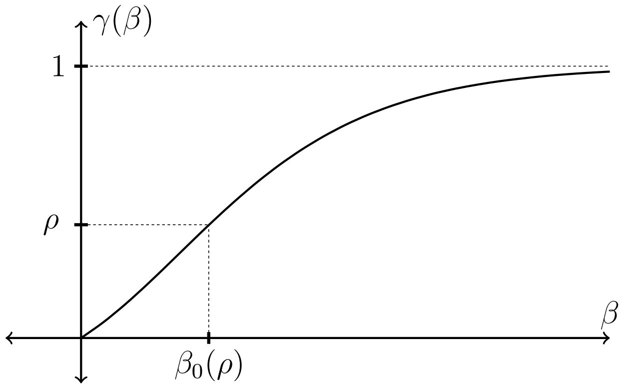

Figure 1.3

Then β_0 is bounded from above by the solution to the equation

$$\frac{p_c}{p} = \frac{r_k}{1 + qt_{k-1}}. \tag{1.187}$$

Using that $r_k = 1 - t_{k-1}$ and that $t_{k-1} = e^{-\beta\left(1-\cos\left(\frac{2\pi}{q}\right)\right)}$, this solution can be explicitly computed as

$$\frac{\log\left(\frac{p+qp_c}{p-p_c}\right)}{1-\cos\left(\frac{2\pi}{q}\right)}. \tag{1.188}$$

If we fix p and d, this expression is of order $q^2\log(q)$ as $q \to \infty$, the same order given by Pirogov-Sinai theory and reflection positivity in the 2-dimensional homogeneous case. If we fix p and q, it is of order

$$\frac{\log\left(1+\frac{1}{d}\right)}{1-\cos\left(\frac{2\pi}{q}\right)} \tag{1.189}$$

as $d \to \infty$, taking into account that $p_c \sim \frac{1}{2d}$. In particular, $\beta_0 \to 0$ as $d \to \infty$. Finally, $\lim_{q\to\infty} \beta_0(q, d, p) = \infty$, implying that our approach is not suitable to study the XY model, that is the model with set of spins $S = S^1 = \{z \in \mathbb{C} : |z| = 1\}$; see van Enter, Külske and Opoku [vEKO11] for results concerning the approximation of the XY model via the clock model. With respect to this assymptotics, for dimension $d \geq 3$ and $p = 1$, reflection positivity computes a threshold β_0 independent of q; see Maes and Shlosman [MS11] for a discussion.

1.6 Appendix: Holley's theorem and stochastic domination

The version of Holley's theorem we invoke is the one appearing in [GHM01], with a slight modification in the hypothesis that results in a slight modification in the proof; for completeness, we include the statement here. We intentionally allow notation overlapping. Let

T be any finite subset of $\mathbb{R}$ and E be any finite set. The set T inherits the order $\leq$ from $\mathbb{R}$, order that induces a partial order $\preccurlyeq$ in the set T^E defined by $\omega \preccurlyeq \omega'$ if and only if $\omega_e \leq \omega'_e$ for every $e \in E$. We say that a function $f : T^E \to \mathbb{R}$ is increasing if $f(\omega) \leq f(\omega')$ whenever $\omega \preccurlyeq \omega'$ and, for two probabilities P and P' defined on T^E, we say that P is stochastically dominated by P' if and only if $P(f) \leq P'(f)$ for every increasing function $f : T^E \to \mathbb{R}$; in that case, we write $P \leq_{st} P'$. We say a probability P on T^E is irreducible if any two configurations $\omega, \omega' \in T^E$ of P-positive probability can be connected via successive coordinate changes only passing through configurations of P-positive probability.

Theorem 1.6.1. *Let P and P' be two probabilities on T^E. Suppose one of the following conditions is satisfied.*

(a) *P' is irreducible and assigns positive probability to the maximal element of T^E.*

(b) *P is irreducible and assigns positive probability to the minimal element of T^E.*

Suppose also that

$$P\left(\omega_e \geq u \middle| \omega_{E\setminus e} = \omega'\right) \leq P'\left(\omega_e \geq u \middle| \omega_{E\setminus e} = \omega''\right) \tag{1.190}$$

for every $u \in \mathbb{R}$, every $e \in E$ and every pair of configurations $\omega', \omega'' \in T^{E\setminus e}$ satisfying $\omega' \preccurlyeq \omega''$ and such that the conditioning has positive probability. Then $P \leq_{st} P'$.

1.7 Descripción del capítulo

Comenzamos por definir el modelo de mecánica estadística en un grafo finito. Para ello, es necesario introducir el conjunto de espines, el grafo y el potencial de vecinos próximos. El conjunto de espines S es simplemente un conjunto finito. El grafo (V, E) también es finito, y tiene un subconjunto $U \subset V$, posiblemente vacío, distinguido de vértices al que denominamos frontera. El potencial de vecinos próximos es una función simétrica $W : S \times S \to (0, 1]$. Desde el punto de vista físico, $W(a, b)$ es de la forma $e^{-\beta I(a,b)}$, donde $\beta > 0$ es la temperatura inversa e $I(a, b)$ es el potencial de vecinos próximos, que depende del modelo que consideramos. Para una configuración $\eta \in S^U$, la probabilidad de Gibbs μ_V^η con condición de frontera es proporcional al coeficiente

$$\mathbf{1}\{\sigma_U = \eta\} \prod_{\langle xy \rangle \in E} W(\sigma_x, \sigma_y). \tag{1.191}$$

El producto que aparece en esta expresión es el llamado factor de Boltzmann.

La probabilidad de aglomerado aleatorio de Edwards y Sokal es una probabilidad sobre el conjunto T^E, donde T es la imagen de la función W. Para definir esta probabilidad, hace falta definir una noción de compatibilidad entre configuraciones de espines $\sigma \in S^V$ y configuraciones de aristas $\omega \in T^E$: σ y ω son compatibles, lo que anotamos como $\sigma \sim \omega$, si toda arista $\langle xy \rangle \in E$

satisface $\omega_{\langle xy\rangle} \leq W(\sigma_x, \sigma_y)$. En particular, cuando $\omega_{\langle xy\rangle} = 1$, debe ocurrir la coincidencia de espines $\sigma_x = \sigma_y$. La probabilidad de aglomerado aleatorio ϕ_E^η es proporcional a

$$\left[\prod_{\langle xy\rangle \in E} \theta_{\langle xy\rangle}(\omega)\right] \left|\left\{\sigma \in S^V : \sigma \sim \omega, \sigma_U = \eta\right\}\right|, \tag{1.192}$$

donde $\prod_{\langle xy\rangle \in E} \theta_{\langle xy\rangle}(\omega)$ es una medida producto que depende de la interacción W y, al igual que antes, $\eta \in S^U$ es una condición de frontera. Las probabilidades de Gibbs y de aglomerado aleatorio estás acopladas en el sentido de que las configuraciones sorteadas de manera conjunta son compatibles.

En la sección 1.3.3, demostramos que la probabilidad de aglomerado aleatorio satisface una propiedad de Markov y que la probabilidad de Gibbs satisface una propiedad de correlación positiva. La propiedad de Markov será utilizada subsección 1.3.4. La propiedad de correlación positiva será utilizada para el criterio de no-unicidad en la subsección 1.4.3. Cabe destacar que, para demostrar esta úlitma propiedad, se utiliza un lema combinatorio que resulta de vital importancia en este capítulo (se vuelve a utilizar en la subsección 1.3.5) y que las hipótesis adicionales de invariancia por rotaciones y existencia de una reflexión generalizada son requeridas.

En la subsección 1.3.4 se muestra cómo la probabilidad de aglomerado aleatorio puede ser dominada estocásticamente por una probabilidad de Bernoulli con parámetro adecuado utilizando el teorema de Holley. Como corolario, se obtiene la proposición 1.3.25, en la que se controlan perturbaciones de la probabilidad de aglomerado aleatorio bajo el cambio de condición de frontera; la herramienta fundamental es un lema introducido por Alexander y Chayes.

En la subsección 1.3.5, bajo las hipótesis de invariancia por rotación y existencia de reflexión generalizada, demostramos la dominación estocástica inversa: la probabilidad de aglomerado aleatorio domina estocásticamente a una probabilidad Bernoulli con parámetro adecuado. Las herramientas utilizadas son el teorema de Holley y el lema combinatorio antes mecionado.

En la sección 1.4 aplicamos los resultados de la sección anterior para dar condiciones suficientes para unicidad y no-unicidad de medidas de Gibbs es modelos diluidos. Un modelo diluido es un modelo de mecánica estadísica en el que el grafo subyacente es el aglomerado infinito que se obtiene al adelgazar las aristas del reticulado d-dimensional $\mathbb{Z}^d$ mediante percolación independiente. Estos criterios se deducen fácilmente de la maquinaria que se desarrolló en la sección anterior: el criterio de unicidad (teorema 1.4.6) de la subsección anterior es consecuencia de la proposición 1.3.25; el criterio de no-unicidad (teorema 1.4.7) se deduce de la proposición 1.3.20.

Finalemente, en la sección 1.5, se compara nuestro método con métodos ya existentes, como la teoría de Pirogov-Sinai y reflexión positiva, en el caso particular en el que el grafo es homogéneo.

Chapter 2

Large deviations for inhomogeneous magnetizations

2.1 Introducción

Este capítulo está basado en [MSLT15].

Estudiamos un modelo con configuraciones del tipo Ising en el toro discreto d-dimensional con interacciones ferromagnéticas de vecinos próximos y potencial cuadrático de Kac asociado a un campo externo no-homogéneo. Más precisamente, el conjunto de configuraciones está dado por $\{-1,1\}^{\Lambda}$, donde el reticulado $\Lambda \subset \mathbb{Z}^d$ es considerado con periodicidad. La distribución de las configuraciones está gobernada por el factor de Boltzmann asociado al Hamiltoniano

$$H_{\Lambda,\gamma,\alpha}(\sigma) := -\sum_{\substack{x,y\in\Lambda \\ \text{vecinos próximos}}} \sigma_x\sigma_y + \sum_{x\in\Lambda}\left(\mathrm{Av}_{\gamma}(\sigma,x)-\alpha(x)\right)^2, \tag{2.1}$$

donde $\alpha : \Lambda \to \mathbb{R}$ es una función no-constante y $\mathrm{Av}_{\gamma}(\sigma,x)$ es un promedio de rango γ^{-1} de la configuración σ alrededor del vértice x. La segunda suma del lado derecho de (2.1) es un potencial cuadrático de Kac que fija la configuración al valor del campo externo α.

En la sección 2.5, demostramos la existencia de la energía libre y de la presión asociadas a nuestro modelo. Con respecto a la primera, un parámetro intermedio l es necesario para el correcto planteo del problema. Estos resultados establecen una conexión entre los fenómenos microscópicos y macroscópicos.

En la sección 2.6, demostramos un resultado de grandes desvíos que da información acerca de las magnetizaciones típicas asociadas al campo externo α. En particular, demostramos que la interacción de Kac da lugar a una biyección entre campos externos y perfiles magnéticos.

Finalmente, en la sección 2.7, establecemos un resultado de equivalencia de arreglos que relaciona las ya definidas energía libre y presión.

2.2 Introduction

This chapter is based on [MSLT15].

In this chapter, we study a model with Ising-type configurations in the d-dimensional discrete torus with ferromagnetic nearest-neighbour interactions and quadratic Kac potential associated to a non-homogeneous external field. More precisely, the set of configurations is given by $\{-1,1\}^{\Lambda}$, where the lattice $\Lambda \subset \mathbb{Z}^d$ is considered with periodicity. The distribution of the configurations is governed by the Boltzmann factor associated to the Hamiltonian

$$H_{\Lambda,\gamma,\alpha}(\sigma) := -\sum_{\substack{x,y\in\Lambda \\ \text{nearest-neighbours}}} \sigma_x\sigma_y + \sum_{x\in\Lambda} \left(\mathrm{Av}_\gamma(\sigma,x) - \alpha(x)\right)^2, \tag{2.2}$$

where $\alpha : \Lambda \to \mathbb{R}$ is a non-constant function and $\mathrm{Av}_\gamma(\sigma,x)$ is an average of range γ^{-1} of the configuration σ around the vertex x. The second sum of the right-hand side of (2.2) is a quadratic Kac potential that fixes the configuration to the value of the external field α.

In section 2.5, we prove the existence of the free energy and the pressure associated to our model. In the former one, an intermediate parameter l is required for the proper definition of the problem. These results stablish a bridge between the microscopic and the macroscopic phenomena.

In section 2.6, we prove a large deviation result that gives information about the typical magnetizations associated to the external field α. In particular, we show that the Kac interaction gives rise to a bijective correspondence between external fields and magnetization profiles.

Finally, in section 2.7, we stablish an equivalence of ensembles result that relates the already defined free energy and pressure.

2.3 Classical results

Let $\mathbb{T} := \left[-\frac{1}{2},\frac{1}{2}\right)^d$ be the d-dimensional torus, $d \geq 2$. For $r \in \mathbb{R}^d$, let $\overline{r}$ be its representative in $\mathbb{T}$, that is the only element of the set

$$\left\{r + (a_1,\ldots,a_d) : (a_1,\ldots,a_d) \in \mathbb{Z}^d\right\} \cap \mathbb{T}. \tag{2.3}$$

We endow $\mathbb{T}$ with the torus distance $d_{\mathbb{T}}$ defined by

$$d_{\mathbb{T}}(r,r') = \left\|\overline{r-r'}\right\|_1; \tag{2.4}$$

this is the distance that determines the notion of continuity in $\mathbb{T}$. For $q \in \mathbb{Z}^+$, we consider a discretisation parameter ε of the form 2^{-q}. From now on, $\lim\limits_{\varepsilon\to 0}$ stands for $\lim\limits_{q\to\infty}$. The microscopic version of $\mathbb{T}$ is the lattice $\Lambda_\varepsilon := (\varepsilon^{-1}\mathbb{T}) \cap \mathbb{Z}^d$. For a non-empty subset $A \subset \mathbb{Z}^d$, let $\Omega_A := \{-1,1\}^A$ be the set of configurations in A. The nearest-neighbour Hamiltonian is the function $H^{nn}_{\Lambda_\varepsilon} : \Omega_{\Lambda_\varepsilon} \to \mathbb{R}$ defined by

$$H^{nn}_{\Lambda_\varepsilon}(\sigma) := -\sum_{\substack{x,y\in\Lambda_\varepsilon \\ x\sim y}} \sigma_x\sigma_y, \tag{2.5}$$

where $x \sim y$ means x and y are nearest-neighbours in the torus, that is $\varepsilon^{-1} d_{\mathbb{T}}(\varepsilon x, \varepsilon y) = 1$. We define also the Hamiltonian with external field $h \in \mathbb{R}$ by

$$H^{nn}_{\Lambda_\varepsilon,h}(\sigma) := H^{nn}_{\Lambda_\varepsilon}(\sigma) - h \sum_{x \in \Lambda_\varepsilon} \sigma_x. \tag{2.6}$$

Its associated Gibbs probability is the probability on $\Omega_{\Lambda_\varepsilon}$ defined by

$$\mu^{nn}_{\Lambda_\varepsilon,h}(\sigma) := \frac{1}{Z^{nn}_{\Lambda_\varepsilon,h}} e^{-\beta H^{nn}_{\Lambda_\varepsilon,h}(\sigma)}, \tag{2.7}$$

where $\beta > 0$ is the inverse temperature and $Z^{nn}_{\Lambda_\varepsilon,h} := \sum_{\sigma \in \Omega_{\Lambda_\varepsilon}} e^{-\beta H^{nn}_{\Lambda_\varepsilon,h}(\sigma)}$ is the grand canonical partition function or normalizing constant. The parameter β is fixed and we omit it in the notation in most of the occasions. For $A \subset \mathbb{Z}^d$ non-empty and finite, $B \subset \mathbb{Z}^d$ containing A and $\sigma \in \Omega_B$, we define the average magnetization of σ in A by $m_A(\sigma) = \frac{1}{|A|} \sum_{x \in A} \sigma_x$. To shorten notation, we simply write $m(\sigma)$ if $A = B$. For $i \in \mathbb{N}$, let

$$V_i := \left\{ \frac{2j - i}{i} : j \in \mathbb{Z} \cap [0, i] \right\}. \tag{2.8}$$

Observe that $V_{|A|}$ is the image of m_A. For $u \in V_{|\Lambda_\varepsilon|}$, we define the finite volume free energy by

$$f_{\Lambda_\varepsilon,\beta}(u) := -\frac{1}{\beta |\Lambda_\varepsilon|} \log \sum_{\substack{\sigma \in \Omega_{\Lambda_\varepsilon} \\ m(\sigma) = u}} e^{-\beta H^{nn}_{\Lambda_\varepsilon}(\sigma)} \tag{2.9}$$

and extend the domain of $f_{\Lambda_\varepsilon,\beta}$ to $[-1, 1]$ by linear interpolation. For $h \in \mathbb{R}$, we define the finite volume pressure by

$$p_{\Lambda_\varepsilon,\beta}(h) := \frac{1}{\beta |\Lambda_\varepsilon|} \log \sum_{\sigma \in \Omega_{\Lambda_\varepsilon}} e^{-\beta H^{nn}_{\Lambda_\varepsilon,h}(\sigma)}. \tag{2.10}$$

We are ready to state a classical result: the existence of the (infinite volume) free energy and pressure.

Theorem 2.3.1 (Free energy and pressure)**.**

The sequence of functions $(f_{\Lambda_\varepsilon,\beta})_\varepsilon$ *point-wise converges to a function* $f_\beta : [-1, 1] \to \mathbb{R}$ *called free energy.* f_β *is convex and continuous, differentiable in the interior of its domain, and its derivative* f'_β *is continuous, satisfies* $\lim_{u \downarrow -1} f'_\beta(u) = -\infty$ *and* $\lim_{u \uparrow 1} f'_\beta(u) = \infty$ *and is bounded on compact subsets of* $(-1, 1)$*. Moreover, the convergence* $\lim_{\varepsilon \to 0} f_{\Lambda_\varepsilon,\beta} = f_\beta$ *is indeed uniform.*

The sequence of functions $(p_{\Lambda_\varepsilon,\beta})_\varepsilon$ *point-wise converges to a function* $p_\beta : \mathbb{R} \to \mathbb{R}$ *called pressure.*

The proof of this result can be found in [FV], with the exception of the uniform convergence of the free energy that is proved in the appendix.

Observe that the first part of the later result could have been stated as

$$\lim_{\varepsilon\to 0} f_{\Lambda_\varepsilon,h,\beta}\left(u\right) = f_\beta\left(u\right) - hu, \tag{2.11}$$

where $f_{\Lambda_\varepsilon,h,\beta}$ is defined as $f_{\Lambda_\varepsilon,\beta}$ with $H^{nn}_{\Lambda_\varepsilon}$ replaced by $H^{nn}_{\Lambda_\varepsilon,h}$. As a first consequence, we obtain a large deviations result. To state it, we define the rate function $I_h : [-1,1] \to \mathbb{R}$ by

$$I_h\left(u\right) := \beta\left[\left(f_\beta\left(u\right) - hu\right) - \min_{u'\in[-1,1]}\left(f_\beta\left(u'\right) - hu'\right)\right]. \tag{2.12}$$

Theorem 2.3.2 (Large deviations)**.** *For every interval* $[a,b] \subset [-1,1]$,

$$\lim_{\varepsilon\to 0}\frac{1}{|\Lambda_\varepsilon|}\log\left(\mu^{nn}_{\Lambda_\varepsilon,h}\left(m_{\Lambda_\varepsilon}\in[a,b]\right)\right) = -\min_{u\in[a,b]} I_h(u). \tag{2.13}$$

The later result has the following interpretation: the typical magnetization averages are the ones that minimizes the function $f_\beta(u) - hu$. Indeed, let $[a,b] \subset [-1,1]$ such that no minimizer of the previous function belongs to $[a,b]$: $\left(\arg\min_{u\in[-1,1]}\{f_\beta(u) - hu\}\right) \cap [a,b] = \emptyset$. Then there exists a positive constant $C > 0$ such that

$$\frac{1}{|\Lambda_\varepsilon|}\log\left(\mu^{nn}_{\Lambda_\varepsilon,h}\left(m\left(\sigma\right)\in[a,b]\right)\right) \le -C \tag{2.14}$$

for ε small enough. Equivalently,

$$\mu^{nn}_{\Lambda_\varepsilon,h}\left(m\left(\sigma\right)\in[a,b]\right) \le e^{-C|\Lambda_\varepsilon|}. \tag{2.15}$$

In words, the probability of the average magnetization to lie in $[a,b]$ is exponentially small in the volume of the box.

Another corollary to Theorem 2.3.1 is the following equivalence of ensembles result that essentially asserts that the free energy and the pressure provide the same information.

Theorem 2.3.3 (Equivalence of ensembles at the level of potentials)**.** *The free energy and the pressure are each other's Legendre transform:*

$$f_\beta(u) = \sup_{h\in\mathbb{R}}\{hu - p_\beta(h)\} \tag{2.16}$$

for every $u \in [-1,1]$ *and*

$$p_\beta(h) = \sup_{u\in[-1,1]}\{hu - f_\beta(u)\} \tag{2.17}$$

for every $h \in \mathbb{R}$.

We finish this section with a brief analysis of the free energy f_β. There exists a critical value $\beta_c > 0$ that separates the behaviour of the model into the following two scenarios: if $\beta \leq \beta_c$, f_β is strictly convex; if $\beta > \beta_c$, there exists $m_\beta \in (0,1)$ such that f_β is strictly convex outside $(-m_\beta, m_\beta)$ and is constant in $[-m_\beta, m_\beta]$. We can consider both cases together by defining m_β as zero if $\beta \leq \beta_c$. If $\beta \leq \beta_c$, the function $f_\beta(u) - hu$ is strictly convex for every $h \in \mathbb{R}$, and the supremum in (2.17) is attained in only one point. By differentiating, this point is the solution of equation $f'_\beta(u) = h$. The later observation gives a one-to-one relation between external fields $h \in \mathbb{R}$ and magnetizations $u \in (-1,1)$. If $\beta > \beta_c$, the one-to-one relation can only be established between external fields in $\mathbb{R} \setminus \{0\}$ and magnetizations in $[-1,1] \setminus [-m_\beta, m_\beta]$, and the external field $h = 0$ is identified with the magnetization $u = 0$ in the sense of the observation made after Theorem 2.3.2.

Let $\varphi : \mathbb{R} \to (-1,1)$ be the function defined by

$$\varphi(h) := \lim_{\varepsilon \to 0} \int \mu^{nn}_{\Lambda_\varepsilon, h}(d\sigma)\sigma_0, \tag{2.18}$$

where $0 = (0, \ldots, 0)$ is the origin in $\mathbb{Z}^d$. φ is well defined (the limit exists and is in $(-1,1)$), is odd ($\varphi(h) = -\varphi(-h)$ for every $h \in \mathbb{R}$) and strictly increasing.

As in the previous chapter, an infinite volume Gibbs measures is a probability defined on the space $\Omega_{\mathbb{Z}^d}$ endowed with the σ-algebra $\mathcal{F}$ generated by the local functions. For every $u \in [-1,1] \setminus [-m_\beta, m_\beta]$, there exists a unique infinite volume Gibbs probability μ^{nn}_h associated to the corresponding external field $h := f'_\beta(u)$, and the following convergence in probability holds:

$$\lim_{\varepsilon \to 0} \mu^{nn}_h \left(|m_{\Lambda_\varepsilon} - u| > \delta \right) = 0 \tag{2.19}$$

for every $\delta > 0$. (Indeed, a stronger result is known; we state it as we will use it.) The probability μ^{nn}_h can be obtained as a thermodynamic limit: for every event E that depends on a finite number of coordinates, the limit $\mu^{nn}_h(E) = \lim_{\varepsilon \to 0} \mu^{nn}_{\Lambda_\varepsilon, h}(E)$ holds. (2.19) is another way of expressing that the external field h fixes the magnetization u.

All these results can be found in [FV]; they will have their adapted-to-our-model correlates in the following sections.

2.4 The model

In the model we study, in addition to the nearest-neighbour interactions, we consider interactions at an intermediate or mesoscopic scale. To define them, we introduce a function $\phi : \mathbb{R}^d \to [0, \infty)$ that we assume to be even ($\phi(r) = \phi(-r)$ for every $r \in \mathbb{R}^d$), to have compact support, to be Riemann integrable and to integrate 1. The hypothesis of Riemann integrability guarantees it is bounded. The function ϕ, together with a small parameter $0 < \gamma < 1$, is used to determine the strength of the intermediate interactions. With this in mind, we define the function $J_{\Lambda_\varepsilon, \gamma} : \Lambda_\varepsilon \times \Lambda_\varepsilon \to \mathbb{R}$ by

$$J_{\Lambda_\varepsilon, \gamma}(x, y) := \gamma^d \phi\left(\gamma \varepsilon^{-1} \overline{\varepsilon(x-y)} \right). \tag{2.20}$$

As we are treating inhomogeneous external fields, we consider, instead of a number $h \in \mathbb{R}$, a continuous function $\alpha \in \mathscr{C}(\mathbb{T}, \mathbb{R})$ (we recall the notion of continuity is the one associated to the distance $d_{\mathbb{T}}$). Similarly, instead of homogeneous magnetizations $u \in [-1, 1]$, we treat inhomogeneous magnetizations $u \in \mathscr{C}(\mathbb{T}, (-1, 1))$. In many occasions, α and u will be constant functions; in those cases, we use the same letter to denote the constant value they take. The associated Hamiltonian to ε, γ and α is the function $H_{\Lambda_\varepsilon,\gamma,\alpha} : \Omega_{\Lambda_\varepsilon} \to \mathbb{R}$ defined by

$$H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) := H^{nn}_{\Lambda_\varepsilon}(\sigma) + \sum_{x \in \Lambda_\varepsilon} \left(\sum_{y \in \Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x, y)\, \sigma_y - \alpha(\varepsilon x) \right)^2 . \tag{2.21}$$

The sum $\sum_{y \in \Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x, y)\, \sigma_y$ has to be understood as an average around x. The second term is the quadratic Kac potential that induces the profile of the configurations to follow α. The associated Gibbs probability is defined by

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) := \frac{1}{Z_{\Lambda_\varepsilon,\gamma,\alpha}} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)}, \tag{2.22}$$

where $Z_{\Lambda_\varepsilon,\gamma,\alpha}$ is the grand canonical partition function.

2.5 Free energy and pressure

This section is devoted to properly defining the free energy and the pressure. In contrast to the nearest-neighbour case described in section 2.3 where they are functions with domains $[-1, 1]$ and $\mathbb{R}$, they are functionals with domains $\mathscr{C}(\mathbb{T}, (-1, 1))$ and $\mathscr{C}(\mathbb{T}, \mathbb{R})$ in our case.

Let l be an intermediate scale of the form 2^{-p}, $p \in \mathbb{Z}^+$. From now on, $\lim_{l \to 0}$ stands for $\lim_{p \to \infty}$. Let $\mathscr{P}_l$ be the natural partition of $\mathbb{T}$ into cubes of side length l, that is the partition into the cubes inside $\mathbb{T}$ of the form $\prod_{i=1}^{d} [la_i, l(a_i + 1))$, $a_i \in \mathbb{Z}^d$ for every i. $\mathscr{P}_l$ has $n := l^{-d}$ elements that we denote by $A_1, \ldots, A_n$. For $\sigma \in \Omega_{\Lambda_\varepsilon}$ and $i \in \{1, \ldots, n\}$, let σ_i be its projection over $(\varepsilon^{-1} A_i) \cap \mathbb{Z}^d$, and observe that $\left|(\varepsilon^{-1} A_i) \cap \mathbb{Z}^d\right| = (\varepsilon^{-1} l)^d$. For $g \in \mathscr{C}(\mathbb{T}, \mathbb{R})$, the notation g_l will represent the coarse-graining approximation at scale l of g, that is the function defined by

$$g_l(r) = g_i := \fint_{A_i} u = \frac{1}{|A_i|} \int_{A_i} u \tag{2.23}$$

for $r \in A_i$. For $u \in [-1, 1]$, let $\lceil u \rceil_i$ be the magnetization in V_i that best fits u from above:

$$\lceil u \rceil_i := \min \{ t \in V_i : t \geq u \} . \tag{2.24}$$

We define the set of configurations whose magnetizations are close to $u \in \mathscr{C}(\mathbb{T}, (-1, 1))$ at coarse-graining scale l by

$$\Omega_{\Lambda_\varepsilon,l}(u) := \left\{ \sigma \in \Omega_{\Lambda_\varepsilon} : m(\sigma_i) = \lceil u_i \rceil_{(\varepsilon^{-1} l)^d} \text{ for every } i \in \{1, \ldots, n\} \right\} . \tag{2.25}$$

We are ready to state the main result of the chapter.

Theorem 2.5.1 (Free energy)**.** *For $u \in \mathscr{C}(\mathbb{T}, (-1,1))$ and $\alpha \in \mathscr{C}(\mathbb{T}, \mathbb{R})$,*

$$\lim_{l\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0} -\frac{1}{\beta|\Lambda_\varepsilon|}\log\sum_{\sigma\in\Omega_{\Lambda_\varepsilon,l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} = \int_{\mathbb{T}}\left[f_\beta(u)+(u-\alpha)^2\right]. \tag{2.26}$$

This limit is the free energy associated to our model and is denoted by $F_\alpha(u)$.

We start by proving an homogeneous version of this theorem in which the intermediate coarse-graining parameter l is not taken into consideration; instead of it, we consider a microscopic parameter L. For $u \in V_{|\Lambda_\varepsilon|}$, we introduce the finite volume free energy associated to the Hamiltonian (2.21) by

$$F_{\Lambda_\varepsilon,\gamma,\alpha}(u) := -\frac{1}{\beta|\Lambda_\varepsilon|}\log\sum_{\substack{\sigma\in\Omega_{\Lambda_\varepsilon}\\ m(\sigma)=u}} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)}. \tag{2.27}$$

Lemma 2.5.2. *For $\alpha \in \mathscr{C}(\mathbb{T}, \mathbb{R})$ constant and $u \in (-1,1)$,*

$$\lim_{\gamma\to 0}\lim_{\varepsilon\to 0} F_{\Lambda_\varepsilon,\gamma,\alpha}\left(\lceil u\rceil_{|\Lambda_\varepsilon|}\right) = f_\beta(u)+(u-\alpha)^2. \tag{2.28}$$

Proof of Lemma 2.5.2. We first show that the limit $\lim_{\varepsilon\to 0} F_{\Lambda_\varepsilon,\gamma,\alpha}\left(\lceil u\rceil_{|\Lambda_\varepsilon|}\right)$ exists for fixed γ, we then exhibit an approximation result, and we finally establish upper and lower bounds.

- *Step 1: existence of the limit* $\lim_{\varepsilon\to 0} F_{\Lambda_\varepsilon,\gamma,\alpha}\left(\lceil u\rceil_{|\Lambda_\varepsilon|}\right)$ *for fixed $\gamma > 0$.* We need the following continuity lemma that controls the difference between the values $F_{\Lambda_\varepsilon,\gamma,\alpha}$ takes in neighbour points; its proof can be found in section 2.8.

Lemma 2.5.3 (Continuity lemma)**.**

$$\max_{u\in V_{|\Lambda_\varepsilon|}\setminus\{-1\}}\left|F_{\Lambda_\varepsilon,\gamma,\alpha}(u)-F_{\Lambda_\varepsilon,\gamma,\alpha}\left(u-\frac{2}{|\Lambda_\varepsilon|}\right)\right| = O\left(\gamma^{-1}\frac{\log|\Lambda_\varepsilon|}{|\Lambda_\varepsilon|}\right). \tag{2.29}$$

During the rest of this step, we use the notations $\Lambda_q := \Lambda_\varepsilon$ (recalling ε is of the form 2^{-q}), $u_q := \lceil u\rceil_{|\Lambda_q|}$ and $F_q := F_{\Lambda_q,\gamma,\alpha}$. We are done if we prove that $(F_q(u_q))_q$ is bounded from below and that, for every q, inequality

$$F_{q+1}(m_{q+1}) \le F_q(m_q) + a_q \tag{2.30}$$

holds for a sequence $(a_q)_q$ of positive real numbers satisfying $\sum_q a_q < \infty$.

Observe that

$$\sum_{\substack{\sigma\in\Omega_{\Lambda_q}\\ m(\sigma)=u_q}} e^{-\beta H_{\Lambda_q,\gamma,\alpha}(\sigma)} \le \sum_{\substack{\sigma\in\Omega_{\Lambda_q}\\ m(\sigma)=u_q}} e^{-\beta H^{nn}_{\Lambda_q}(\sigma)} \le \sum_{\sigma\in\Omega_{\Lambda_q}} e^{-\beta H^{nn}_{\Lambda_q}(\sigma)}. \tag{2.31}$$

Take logarithm and divide by $-\beta|\Lambda_q|$ to obtain

$$F_q(u_q) \ge -\frac{1}{\beta|\Lambda_q|}\log\sum_{\sigma\in\Omega_{\Lambda_q}} e^{-\beta H^{nn}_{\Lambda_q}(\sigma)}. \tag{2.32}$$

The right-hand side of this expression converges to minus the pressure with zero external field $-p_\beta(0) \in \mathbb{R}$ and the fact that $(F_q(u_q))_q$ is bounded from below follows.

Inequality (2.30) follows after estimating

$$F_{q+1}(m_{q+1}) - F_q(m_q) = [F_{q+1}(m_{q+1}) - F_{q+1}(m_q)] + [F_{q+1}(m_q) - F_q(m_q)]. \tag{2.33}$$

To find an upper bound for $F_{q+1}(m_q) - F_q(m_q)$, we can proceed by using the same sub-additive argument appearing in sub-section 2.8. Indeed, by that argument and the fact that the difference

$$H_{\Lambda_{q+1},\gamma,\alpha}(\sigma) - \sum_{i=1}^{2^d} H_{\Lambda_{q,i},\gamma,\alpha}(\sigma_i) \tag{2.34}$$

is $O(\gamma^{-1}2^q)$, it follows that $F_{q+1}(m_q) - F_q(m_q) \le O(\gamma^{-1}2^{-q})$; for every i, the Hamiltonian $H_{\Lambda_{q,i},\gamma,\alpha}$ is considered with periodic boundary conditions (this is possible because α is constant). To control $F_{q+1}(m_{q+1}) - F_{q+1}(m_q)$, we need Lemma 2.5.3. The upper bound

$$F_{q+1}(m_{q+1}) - F_{q+1}(m_q) \le O\left(\gamma^{-1}\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right) \tag{2.35}$$

follows after using this lemma repeatedly as m_{q+1} can be reached from m_q by moving trough consecutive elements of $V_{|\Lambda_{q+1}|}$ in a finite number of steps. To conclude, we define

$$a_q := O\left(\gamma^{-1}2^{-q}\right) + O\left(\gamma^{-1}\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right) \tag{2.36}$$

and observe that $\sum_q a_q < \infty$.

- *Step 2: coarse-graining approximation.* We fix a microscopic parameter L of the form 2^m, $m \in \mathbb{Z}^+$. The set $\{\varepsilon^{-1}A : A \in \mathscr{P}_{\varepsilon L}\}$ has $N := \left(|\Lambda_\varepsilon|\, L^{-d}\right)$ elements that we call $B_1, \ldots, B_N$. For $i \in \{1, \ldots, N\}$, let $\Delta_i := B_i \cap \mathbb{Z}^d$ be the discrete version of B_i. We extend the domain of $J_{\Lambda_\varepsilon,\gamma}$ to $(\varepsilon^{-1}\mathbb{T}) \times (\varepsilon^{-1}\mathbb{T})$ in the natural way:

$$J_{\Lambda_\varepsilon,\gamma}(r,r') := \gamma^d\phi\left(\gamma\varepsilon^{-1}\overline{\varepsilon(r-r')}\right). \tag{2.37}$$

For $i, j \in \{1, \ldots, N\}$, let

$$\hat{J}_{\Lambda_\varepsilon,\gamma}(i,j) := \fint_{B_i\times B_j} d(r,r')\, J_{\Lambda_\varepsilon,\gamma}(r,r') = \frac{1}{L^{2d}}\int_{B_i}\int_{B_j} dr\, dr'\, J_{\Lambda_\varepsilon,\gamma}(r,r'). \tag{2.38}$$

For $x, y \in \Lambda_\varepsilon$ such that $x \in \Delta_i$ and $y \in \Delta_j$, $\hat{J}_{\Lambda_\varepsilon,\gamma}(x,y)$ stands for $\hat{J}_{\Lambda_\varepsilon,\gamma}(i,j)$.

Let us introduce the following notation for the quadratic part of the Hamiltonian (2.21):

$$W_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) := \sum_{x\in\Lambda_\varepsilon}\left(\sum_{y\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y)\,\sigma_y - \alpha(\varepsilon x)\right)^2. \tag{2.39}$$

The idea is to approximate the nearest-neighbour part of the Hamiltonian $H_{\Lambda_\varepsilon}^{nn}(\sigma)$ by $\sum_{i=1}^{N} H_{\Delta_i}^{nn}(\sigma_i)$, where σ_i is the projection of σ over Δ_i and $H_{\Delta_i}^{nn}$ is considered with periodic boundary conditions, and the quadratic term $W_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)$ by

$$\hat{W}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) := \sum_{x\in\Lambda_\varepsilon}\left(\sum_{y\in\Lambda_\varepsilon}\hat{J}_{\Lambda_\varepsilon,\gamma}(x,y)\sigma_y - \alpha\right)^2. \tag{2.40}$$

The Hamiltonian $H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)$ will then be approximated by

$$\xi := -\frac{1}{\beta\,|\Lambda_\varepsilon|}\log \sum_{\substack{\sigma\in\Omega_{\Lambda_\varepsilon}\\ m(\sigma)=\lceil u\rceil_{|\Lambda_\varepsilon|}}} \exp\left\{-\beta\sum_{i=1}^{N} H_{\Delta_i}^{nn}(\sigma_i) - \beta\hat{W}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)\right\}. \tag{2.41}$$

To approximate the nearest-neighbour part, observe that, as there are $dL^{d-1}N$ interactions of the nearest-neighbour type between the boxes $\Delta_1,\ldots,\Delta_N$, we have

$$H_{\Lambda_\varepsilon}^{nn}(\sigma) = \sum_{i=1}^{N} H_{\Delta_i}^{nn}(\sigma_i) + O\left(L^{-1}\,|\Lambda_\varepsilon|\right). \tag{2.42}$$

To approximate the quadratic part, we need to control the difference

$$\sum_{x\in\Lambda_\varepsilon}\left|\left(\sum_{y\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y)\sigma_y - \alpha\right)^2 - \left(\sum_{y\in\Lambda_\varepsilon}\hat{J}_{\Lambda_\varepsilon,\gamma}(x,y)\sigma_y - \alpha\right)^2\right|. \tag{2.43}$$

For each $x\in\Lambda_\varepsilon$, both arguments $\sum_{y\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y)\,\sigma_y - h$ and $\sum_{y\in\Lambda_\varepsilon}\hat{J}_{\Lambda_\varepsilon,\gamma}(x,y)\,\sigma_y - h$ are uniformly bounded in ε, γ and σ. As the function $t\mapsto t^2$ restricted to bounded sets is Lipschitz, it is enough to control the difference between these arguments. Applying this argument to (2.43), we get that it is enough to dominate

$$\sum_{x\in\Lambda_\varepsilon}\sum_{y\in\Lambda_\varepsilon}\left|J_{\Lambda_\varepsilon,\gamma}(x,y) - \hat{J}_{\Lambda_\varepsilon,\gamma}(x,y)\right|. \tag{2.44}$$

For $i,j\in\{1,\ldots,N\}$, let

$$\overline{J}_{\Lambda_\varepsilon,\gamma}(i,j) := \sup_{r\in B_i,r'\in B_j} J_{\Lambda_\varepsilon,\gamma}(r,r') \tag{2.45}$$

and

$$\underline{J}_{\Lambda_\varepsilon,\gamma}(i,j) := \inf_{r\in B_i,r'\in B_j} J_{\Lambda_\varepsilon,\gamma}(r,r'). \tag{2.46}$$

As before, let $\overline{J}_{\Lambda_\varepsilon,\gamma}(x,y) := \overline{J}_{\Lambda_\varepsilon,\gamma}(i,j)$ and $\underline{J}_{\Lambda_\varepsilon,\gamma}(x,y) := \underline{J}_{\Lambda_\varepsilon,\gamma}(i,j)$ for $x\in\Delta_i$ and $y\in\Delta_j$. Under these definitions, (2.44) is controlled by

$$L^{2d}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\overline{J}_{\Lambda_\varepsilon,\gamma}(i,j) - \underline{J}_{\Lambda_\varepsilon,\gamma}(i,j)\right). \tag{2.47}$$

Suppose, without loss of generality, that B_1 is the square containing the origin $0=(0,\dots,0)$. By rotational invariance, the later expression coincides with

$$L^{2d}N\sum_{i=1}^{N}\left(\overline{J}_{\Lambda_\varepsilon,\gamma}(1,i)-\underline{J}_{\Lambda_\varepsilon,\gamma}(1,i)\right). \tag{2.48}$$

For $i\in\{1,\dots,n\}$, the subset $B_i-B_1:=\left\{\varepsilon^{-1}\overline{\varepsilon(r-r')}:r\in B_i,r'\in B_1\right\}$ is the union of the 2^d cubes whose $d_{\varepsilon^{-1}\mathbb{T}}$-topological closures intersect B_1, where $d_{\varepsilon^{-1}\mathbb{T}}$ is the distance defined on $\varepsilon^{-1}\mathbb{T}$ by $d_{\varepsilon^{-1}\mathbb{T}}(r,r'):=d_{\mathbb{T}}(\varepsilon r,\varepsilon r')$. (The difference B_i-B_1 should not be confused with the standard set difference, the one that would have been denoted by $B_i\setminus B_1$.) Under this definition, identity

$$\overline{J}_{\Lambda_\varepsilon,\gamma}(1,i)-\underline{J}_{\Lambda_\varepsilon,\gamma}(1,i)=\gamma^d\left(\sup_{r\in\gamma(B_i-B_1)}\phi(r)-\inf_{r\in\gamma(B_i-B_1)}\phi(r)\right) \tag{2.49}$$

holds. Then (2.48) can be written as

$$\frac{|\Lambda_\varepsilon|}{2^d}\sum_{i=1}^{N}(2\gamma L)^d\left(\sup_{r\in\gamma(B_i-B_1)}\phi(r)-\inf_{r\in\gamma(B_i-B_1)}\phi(r)\right). \tag{2.50}$$

Let

$$s_1(\varepsilon,\gamma,L):=\sum_{i=1}^{N}(2\gamma L)^d\left(\sup_{r\in\gamma(B_i-B_1)}\phi(r)-\inf_{r\in\gamma(B_i-B_1)}\phi(r)\right). \tag{2.51}$$

As $|\gamma(B_i-B_1)|=(2L\gamma)^d$ and as ϕ has compact support, $s_1(\varepsilon,\gamma,L)$ is (dominated by) the sum of 2^d differences between upper and lower Darboux sums associated to ϕ (when ε is small enough), so $\lim_{\gamma\to0}\lim_{\varepsilon\to0}s_1(\varepsilon,\gamma,L)=0$. Rounding out,

$$\sum_{x\in\Lambda_\varepsilon}\left(\sum_{y\in\Lambda_\varepsilon}J_{\Lambda_\varepsilon,\gamma}(x,y)\sigma_y-h\right)^2=\sum_{x\in\Lambda_\varepsilon}\left(\sum_{y\in\Lambda_\varepsilon}\hat{J}_{\Lambda_\varepsilon,\gamma}(x,y)\sigma_y-h\right)^2+s_1(\varepsilon,\gamma,L)\,O(|\Lambda_\varepsilon|). \tag{2.52}$$

From (2.42) and (2.52), we get

$$F_{\Lambda_\varepsilon,\gamma,\alpha}\left(\lceil u\rceil_{|\Lambda_\varepsilon|}\right)=\xi+O\left(L^{-1}\right)+s_1(\varepsilon,\gamma,L)\,O(1). \tag{2.53}$$

As L can be chosen as large as desired, we reduced the problem to proving that

$$\lim_{\gamma\to0}\lim_{\varepsilon\to0}\xi=f_\beta(u)+(m-h)^2. \tag{2.54}$$

We finally observe that, after defining

$$\tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j):=L^d\hat{J}_{\Lambda_\varepsilon,\gamma}(i,j), \tag{2.55}$$

ξ can be written as

$$-\frac{1}{\beta|\Lambda_\varepsilon|}\log\sum_{\substack{u_1,\dots,u_N\in V_{L^d}\\ \frac{1}{N}\sum_{i=1}^N u_i=\lceil u\rceil_{|\Lambda_\varepsilon|}}}\left(\prod_{i=1}^{N}\sum_{\substack{\sigma_i\in\Omega_{\Delta_i}\\ m(\sigma_i)=u_i}}e^{-\beta H^{nn}_{\Delta_i}(\sigma_i)}\right)\exp\left\{-\beta L^d\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j)u_j-\alpha\right)^2\right\}. \tag{2.56}$$

As the convergence to the free energy f_β is uniform (Theorem 2.3.1), the sum $\sum_{\substack{\sigma_i \in \Omega_{\Delta_i} \\ m(\sigma_i)=u_i}} e^{-\beta H^{nn}_{\Delta_i}(\sigma_i)}$ can be approximated by $e^{-\beta L^d f_\beta(u_i)}$ with error $e^{\beta L^d s_2(L)}$, $s_2(L)$ vanishing as L goes to infinity, so the error while approximating the product $\prod_{i=1}^{N} \sum_{\substack{\sigma_i \in \Omega_{\Delta_i} \\ m(\sigma_i)=u_i}} e^{-\beta H^{nn}_{\Delta_i}(\sigma_i)}$ is $\left(e^{\beta L^d s_2(L)}\right)^N = e^{\beta |\Lambda_\varepsilon| s_2(L)}$; this error vanishes after taking log, dividing by $|\Lambda_\varepsilon|$ and choosing L large. Finally, we reduced the problem to establishing the convergence of

$$-\frac{1}{\beta |\Lambda_\varepsilon|} \log \sum_{\substack{u_1,\ldots,u_N \in V_{L^d} \\ \frac{1}{N}\sum_{i=1}^N u_i = \lceil u \rceil_{|\Lambda_\varepsilon|}}} \left(\prod_{i=1}^{N} e^{-\beta L^d f_\beta(u_i)} \right) \exp\left\{ -\beta L^d \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j) u_j - \alpha \right)^2 \right\}. \tag{2.57}$$

- *Step 3: upper and lower bounds.* Let $G : [-1,1]^N \to \mathbb{R}$ be the function defined by

$$G(u_1,\ldots,u_N) = \frac{1}{N}\sum_{i=1}^{N} f_\beta(u_i) + \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j) u_j - \alpha\right)^2. \tag{2.58}$$

Lemma 2.5.4. *For every $(u_1,\ldots,u_N) \in [-1,1]^N$,*

$$G(u_1,\ldots,u_N) \geq f_\beta\left(\frac{1}{N}\sum_{i=1}^{N} u_i\right) + \left(\frac{1}{N}\sum_{i=1}^{N} u_i - \alpha\right)^2. \tag{2.59}$$

Proof of Lemma 2.5.4. From the convexity of f_β, we have

$$\frac{1}{N}\sum_{i=1}^{N} f_\beta(u_i) \geq f_\beta\left(\frac{1}{N}\sum_{i=1}^{N} u_i\right). \tag{2.60}$$

To deal with the other term, the notation $C_i := \sum_{j=1}^{N} \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j) u_j$ will be helpful. As $\sum_{i=1}^{N} \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j) = 1$ for every j (and ε small enough),

$$\sum_{i=1}^{N} C_i = \sum_{j=1}^{N} u_j \sum_{i=1}^{N} \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j) = \sum_{j=1}^{N} u_j. \tag{2.61}$$

From the convexity of the function $t \mapsto (t-\alpha)^2$,

$$\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j) u_j - \alpha\right)^2 = \frac{1}{N}\sum_{i=1}^{N}(C_i-\alpha)^2 \geq \left(\frac{1}{N}\sum_{i=1}^{N} C_i - \alpha\right)^2 = \left(\frac{1}{N}\sum_{i=1}^{N} u_i - \alpha\right)^2. \tag{2.62}$$

The proof finishes after putting (2.60) and (2.62) together. □

We start by showing that

$$\liminf_{\gamma\to 0}\lim_{\varepsilon\to 0} F_{\Lambda_\varepsilon,\gamma,\alpha}\left(\lceil u\rceil_{|\Lambda_\varepsilon|}\right) \geq f_\beta(u) + (u-\alpha)^2 . \tag{2.63}$$

From step 2, it is enough to prove that expression (2.57) satisfies this inequality.

As the number of elements of the sum appearing in expression (2.57) is $O(|\Lambda_\varepsilon|)$, that vanishes after taking logarithm and dividing by $\beta|\Lambda_\varepsilon|$, the problem is reduced to controlling

$$\min_{\substack{u_1,\dots,u_N\in V_{L^d}\\ \frac{1}{N}\sum_{i=1}^N u_i=\lceil u\rceil_{|\Lambda_\varepsilon|}}} \left\{\frac{1}{N}\sum_{i=1}^N f_\beta(u_i) + \frac{1}{N}\sum_{i=1}^N\left(\sum_{j=1}^N \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j)u_j - \alpha\right)^2\right\}. \tag{2.64}$$

From Lemma 2.5.4, expression (2.64) is bounded from below by

$$f_\beta\left(\lceil u\rceil_{|\Lambda_\varepsilon|}\right) + (\lceil u\rceil_{|\Lambda_\varepsilon|}-\alpha)^2, \tag{2.65}$$

that converges to $f_\beta(u)+(u-\alpha)^2$, and the lower bound (2.63) follows.

To show that

$$\limsup_{\gamma\to 0}\lim_{\varepsilon\to 0} F_{\Lambda_\varepsilon,\gamma,\alpha}\left(\lceil u\rceil_{|\Lambda_\varepsilon|}\right) \leq f_\beta(u) + (u-\alpha)^2 , \tag{2.66}$$

we again control (2.57) but, instead of substituting the sum over $u_1,\dots,u_N$ with the supremum times the cardinality of the sum, we choose particular values $\tilde{u}_1,\dots,\tilde{u}_N$. The idea is that these values should be as close to $\lceil u\rceil_{|\Lambda_\varepsilon|}$ as possible and satisfy

$$\frac{1}{N}\sum_{i=1}^N \tilde{u}_i = \lceil u\rceil_{|\Lambda_\varepsilon|}. \tag{2.67}$$

Let u^- and u^+ be the best possible approximations of $\lceil u\rceil_{|\Lambda_\varepsilon|}$ in V_{L^d} respectively from below and from above:

$$u^- := \max\{t\in V_{L^d} : t\leq u\} \qquad u^+ := \min\{t\in V_{L^d} : t\geq u\}. \tag{2.68}$$

Notice that $u^+ - u^- \leq \frac{2}{L^d}$. We define

$$\tilde{u}_i := \begin{cases} u^+ \text{ if } i=1 \\ u^- \text{ if } i\in\{2,\dots,N-1\} \text{ and } \frac{1}{i-1}\sum_{j=1}^{i-1}\tilde{u}_j > \lceil u\rceil_{|\Lambda_\varepsilon|} \\ u^+ \text{ if } i\in\{2,\dots,N-1\} \text{ and } \frac{1}{i-1}\sum_{j=1}^{i-1}\tilde{u}_j \leq \lceil u\rceil_{|\Lambda_\varepsilon|} \\ N\lceil u\rceil_{|\Lambda_\varepsilon|} - \sum_{j=1}^{N-1}\tilde{u}_j \text{ if } i=N \end{cases}. \tag{2.69}$$

Notice that identity (2.67) is satisfied by construction. Notice also that

$$\left|\tilde{u}_i - \lceil u\rceil_{|\Lambda_\varepsilon|}\right| \leq 2L^{-d} \quad \forall i\in\{1,\dots,N\}. \tag{2.70}$$

As $u \in (-1,1)$, we can chose $[a,b] \subset (-1,1)$ such that $\lceil u \rceil_{|\Lambda_\varepsilon|} \in [a,b]$ and $\tilde{u}_i \in [a,b]$ for every i, every ε small enough and every L large enough. As f'_β is bounded in $[a,b]$ (Theorem 2.3.1) and the function $t \mapsto t^2$ is Lipschitz over bounded sets, it follows that

$$G(\tilde{u}_1, \ldots, \tilde{u}_N) = G\left(\lceil u \rceil_{|\Lambda_\varepsilon|}, \ldots, \lceil u \rceil_{|\Lambda_\varepsilon|}\right) + O\left(L^{-d}\right). \tag{2.71}$$

The error term $O(L^{-d})$ is not a problem because L can be chosen arbitrarily large. By using again the fact that $\sum_i \tilde{J}_{\Lambda_\varepsilon,\gamma}(j,i) = 1$ for every j, it follows that

$$\lim_{\varepsilon \to 0} G\left(\lceil u \rceil_{|\Lambda_\varepsilon|}, ..., \lceil u \rceil_{|\Lambda_\varepsilon|}\right) = f_\beta(u) + (u-\alpha)^2. \tag{2.72}$$

This concludes the proof of step 3 and, with it, the proof of the lemma. □

Proof of Theorem 2.3.1. We recall the definitions of $l = 2^{-p}$, $\mathscr{P}_l$ and $\{A_1, \ldots, A_n\}$. We add the following ones: for every $i \in \{1, \ldots, n\}$, let $B_i := \varepsilon^{-1} A_i$ and $\Delta_i := B_i \cap \mathbb{Z}^d$ be the magnified and discrete versions of A_i, respectively. (This notation overlaps the one used while making the microscopic analysis in the proof of the previous lemma.) As before, for $\sigma \in \Omega_{\Lambda_\varepsilon}$, σ_i denotes its projection over Δ_i.

The first step is to establish the approximation

$$\sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \approx \sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha_l}(\sigma)}, \tag{2.73}$$

where α_l is the coarse-grained image of α defined in subsection 2.5. As α is a continuous function defined on a compact set, it is uniformly continuous and then

$$\lim_{l \to 0} \|\alpha - \alpha_l\|_\infty = 0. \tag{2.74}$$

Then, as the function $t \mapsto t^2$ is Lipschitz on bounded sets,

$$H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) = H_{\Lambda_\varepsilon,\gamma,\alpha_l}(\sigma) + O\left(|\Lambda_\varepsilon|\right) s\left(l\right) \tag{2.75}$$

with $s(l)$ vanishing as l goes to zero. It follows that

$$\sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} = e^{O(|\Lambda_\varepsilon|)\delta(l)} \sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha_l}(\sigma)}. \tag{2.76}$$

The error vanishes after taking log, dividing by $|\Lambda_\varepsilon|$ and taking $\lim_{l \to 0}$.

The next step is to stablish the approximation

$$\sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha_l}(\sigma)} \approx \sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} \prod_{i=1}^{n} e^{-\beta H_{\Delta_i,\gamma,\alpha_i}(\sigma_i)}, \tag{2.77}$$

where $\alpha_i := ⨍_{A_i} \alpha$ (as defined in subsection 2.5) and $H_{\Delta_i,\gamma,\alpha_i}$ is defined with periodic boundary conditions. As there are $O\left(\varepsilon^{-d+1} l^{-1}\right)$ interactions of the nearest-neighbour type between the boxes $\Delta_1, \ldots, \Delta_n$, we have

$$H^{nn}_{\Lambda_\varepsilon}(\sigma) = \sum_{i=1}^{n} H^{nn}_{\Delta_i}(\sigma_i) + O\left(\varepsilon^{-d+1} l^{-1}\right). \tag{2.78}$$

(As before, $H^{nn}_{\Delta_i}(\sigma_i)$ takes into consideration the periodic boundary conditions.) As there are $O\left(\varepsilon^{-d+1}\gamma^{-1}l^{-1}\right)$ interactions between the boxes $\Delta_1, \ldots, \Delta_n$ in the quadratic part of the Hamiltonian, we have

$$\sum_{x\in\Lambda_\varepsilon}\left(\sum_{y\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y)\sigma_y - \alpha_l(\varepsilon x)\right)^2 = \tag{2.79}$$

$$\sum_{i=1}^{n}\sum_{x\in\Delta_i}\left(\sum_{y\in\Delta_i} J_{\Delta_i,\gamma}(x,y)\sigma_y - \alpha_l(\varepsilon x)\right)^2 + O\left(\varepsilon^{-d+1}\gamma^{-1}l^{-1}\right). \tag{2.80}$$

In the later expression, $J_{\Delta_i,\gamma}(x,y)$ is considered with periodic boundary conditions. From the previous observations, we have

$$\sum_{\sigma\in\Omega_{\Lambda_\varepsilon,l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} = e^{O(\varepsilon^{-d+1}l^{-1})+O(\varepsilon^{-d+1}\gamma^{-1}l^{-1})} \sum_{\sigma\in\Omega_{\Lambda_\varepsilon,l}(u)}\prod_{i=1}^{n} e^{-\beta H_{\Delta_i,\gamma,\alpha_i}(\sigma_i)} \tag{2.81}$$

for every $\sigma\in\Omega_{\Lambda_\varepsilon}$. The error vanishes when we take logarithm, divide by the volume and send ε to zero.

Approximations (2.73) and (2.77) imply approximation

$$\sum_{\sigma\in\Omega_{\Lambda_\varepsilon,l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \approx \sum_{\sigma\in\Omega_{\Lambda_\varepsilon,l}(u)}\prod_{i=1}^{N} e^{-\beta H_{\Delta_i,\gamma,\alpha_l}(\sigma_i)} = \prod_{i=1}^{N}\sum_{\substack{\sigma\in\Omega_{\Delta_i}\\ m(\sigma)=u_i}} e^{-\beta H_{\Delta_i,\gamma,\alpha_i}(\sigma_i)}. \tag{2.82}$$

For every $i\in\{1,\ldots,n\}$, the Hamiltonian $H_{\Delta_i,\gamma,\alpha_i}$ is of the type studied in the previous lemma. Take log, divide by $-\beta|\Lambda_\varepsilon|$ and take $\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}$ to obtain

$$\sum_{i=1}^{N}|A_i|\left[f_\beta(u_i)+(u_i-\alpha_i)^2\right]. \tag{2.83}$$

Finally, take $\lim_{l\to 0}$ to obtain $\int_{\mathbb{T}}\left[f_\beta(u)+(u-\alpha)^2\right]$. It completes the proof. □

Once the free energy is well defined, we are ready to define the pressure.

Theorem 2.5.5 (Pressure)**.**

$$\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\frac{1}{\beta|\Lambda_\varepsilon|}\log Z_{\Lambda_\varepsilon,\gamma,\alpha} = -\min_{u\in\mathscr{C}(\mathbb{T},(-1,1))} F_\alpha(u). \tag{2.84}$$

This limit is the pressure associated to our model and is denoted by $P(\alpha)$.

Proof. We first treat the constant case and follow with the general one.

- *Case α constant.* We have to prove that

$$\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\frac{1}{\beta|\Lambda_\varepsilon|}\log\sum_{\sigma\in\Omega_{\Lambda_\varepsilon}} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} = -\min_{u\in(-1,1)}\left\{f_\beta(u)+(u-\alpha)^2\right\}. \tag{2.85}$$

Observation 2.5.6. *We give a similar discussion to the one given at the end of Theorem 2.3.3. As f_β is convex and $(\alpha - u)^2$ is strictly convex, $f_\beta(u) + (\alpha - u)^2$ is strictly convex. Differentiating $f_\beta(u) + (\alpha - u)^2$ and equating to zero, we obtain*

$$\frac{1}{2} f'_\beta(u) + u = \alpha. \tag{2.86}$$

As f_β is convex and its derivative f'_β is increasing and continuous in $(-1,1)$, $\frac{1}{2} f'_\beta(u) + u$ is strictly increasing; this fact joint with the continuity of f'_β and identities $\lim_{u\downarrow -1} f'_\beta(u) = -\infty$ and $\lim_{u\uparrow 1} f'_\beta(u) = \infty$ imply there is only one solution to equation (2.86). We conclude that the minimum appearing in (2.85) is achieved in only one point $\tilde{u}(\alpha)$. Reciprocally, to every $u \in (-1,1)$ we associate the solution to equation (2.86) $\widetilde{\alpha}(u) := \frac{1}{2} f'_\beta(u) + u$. We have established a one-to-one correspondence between $(-1,1)$ and $\mathbb{R}$ that, in contrast to the nearest-neighbour case, holds for every value of the inverse temperature β.

We use the same definitions than in the proof of Lemma 2.5.2: $L = 2^m$, $N := (\varepsilon L)^{-d}$, $\{B_1, \dots, B_N\}$ and $\{\Delta_1, \dots, \Delta_N\}$.

The existence of the limit $\lim_{\varepsilon\to 0}$ with fixed γ can be proved with the same sub-additivity argument appearing at the beginning of sub-section 2.8 (this case is even easier because a continuity argument is not necessary).

As in step 2 (of the proof of Lemma 2.5.2), we can approximate the left-hand side of (2.85) by

$$\xi := \frac{1}{\beta |\Lambda_\varepsilon|} \log \sum_{u_1,\dots,u_N \in V_{L^d}} \left(\prod_{i=1}^N e^{-\beta L^d f_\beta(u_i)} \right) \exp\left\{ -\beta L^d \sum_{i=1}^N \left(\sum_{j=1}^N \tilde{J}_{\Lambda_\varepsilon,\gamma}(i,j) u_j - \alpha \right)^2 \right\}. \tag{2.87}$$

We first prove that

$$\limsup_{\gamma\to 0} \lim_{\varepsilon\to 0} \xi \le - \min_{u\in\{-1,1\}} \left\{ f_\beta(u) + (u-\alpha)^2 \right\} (= -f_\beta(\tilde{u}(\alpha)) - (\tilde{u}(\alpha) - \alpha)^2). \tag{2.88}$$

Controlling the sum by its cardinality times the supremum as in step 3 (of the proof of Lemma 2.5.2), we simplify the problem to controlling

$$- \min_{u_1,\dots,u_N \in V_{L^d}} G(u_1, \dots, u_N). \tag{2.89}$$

By Lemma 2.5.4, (2.89) is bounded from above by

$$- \min_{u_1,\dots,u_N \in V_{L^d}} \left\{ f_\beta\left(\frac{1}{N} \sum_{i=1}^N u_i \right) + \left(\frac{1}{N} \sum_{i=1}^N u_i - \alpha \right)^2 \right\}. \tag{2.90}$$

The later quantity is controlled by

$$- \min_{u\in[-1,1]} \left\{ f_\beta(u) + (u-\alpha)^2 \right\}, \tag{2.91}$$

and the upper bound follows.

By controlling the sum by only one of its summands, we have

$$\xi \le G(m,\dots,m) = f_\beta(m) + (m-\alpha)^2 \tag{2.92}$$

for every $m \in V_{L^d}$. Since m is arbitrary and $\bigcup_L V_{L^d}$ is dense in $[-1,1]$, it follows that

$$\liminf_{\gamma\to 0} \lim_{\varepsilon\to 0} \frac{1}{\beta|\Lambda_\varepsilon|} \log \sum_{\sigma\in\Omega_{\Lambda_\varepsilon}} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \ge -\min_{m\in[-1,1]} \left\{ f_\beta(u) + (m-\alpha)^2 \right\}, \tag{2.93}$$

completing the proof of the homogeneous case.

- *General case.* We use the same notation than in the proof of Theorem 2.5.1: $\{B_1,\dots,B_n\}$ and $\{\Delta_1,\dots,\Delta_n\}$. Proceeding as in the mentioned proof, the problem is reduced to computing the limit

$$\lim_{l\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0} \frac{1}{\beta|\Lambda_\varepsilon|} \log \prod_{i=1}^n \sum_{\sigma_i\in\Omega_{\Delta_i}} e^{-\beta H_{\Delta_i,\gamma,\alpha_i}(\sigma_i)} = \tag{2.94}$$

$$\lim_{l\to 0} \sum_{i=1}^n \frac{1}{n} \lim_{\gamma\to 0}\lim_{\varepsilon\to 0} \frac{1}{\beta|\Delta_i|} \log \sum_{\sigma_i\in\Omega_{\Delta_i}} e^{-\beta H_{\Delta_i,\gamma,\alpha_i}(\sigma_i)} = \tag{2.95}$$

$$-\lim_{l\to 0} \sum_{i=1}^n |A_i| \min_{u\in(-1,1)} \left\{ f_\beta(u) - (u-\alpha_i)^2 \right\} = -\lim_{l\to 0} \int_{\mathbb{T}} \left[f_\beta(\tilde u(\alpha_l)) + (\tilde u(\alpha_l) - \alpha_l)^2 \right], \tag{2.96}$$

where $\tilde u : \mathbb{R} \to (-1,1)$ is the bijective function defined in observation 2.5.6 and $\tilde u(\alpha_l)$ denotes a composition of functions. In the second identity, we used the previous case. By dominated convergence and the continuity of $\tilde u$, the later expression coincides with

$$-\int_{\mathbb{T}} \left[f_\beta(\tilde u(\alpha)) + (\tilde u(\alpha)-\alpha)^2 \right]. \tag{2.97}$$

Finally, as

$$f_\beta(\tilde u(\alpha)) + (\tilde u(\alpha)-\alpha)^2 = -\min_{u\in\mathscr{C}(\mathbb{T},(-1,1))} \left\{ f_\beta(u) + (u-\alpha)^2 \right\}, \tag{2.98}$$

(2.97) coincides with

$$-\min_{u\in\mathscr{C}(\mathbb{T},(-1,1))} \int_{\mathbb{T}} dr \left[f_\beta(u) + (u-\alpha)^2 \right], \tag{2.99}$$

and the result follows. □

2.6 Large deviation principle

For $\alpha \in \mathscr{C}(\mathbb{T},\mathbb{R})$, let $I_\alpha : \mathscr{C}(\mathbb{T},(-1,1)) \to \mathbb{R}$ be the rate functional defined by

$$I_\alpha(u) := \beta \left[F_\alpha(u) - \min_{v\in\mathscr{C}(\mathbb{T},(-1,1))} F_\alpha(v) \right]. \tag{2.100}$$

The following large deviation result is an immediate corollary of theorems 2.5.1 and 2.5.5.

Theorem 2.6.1 (Large deviations)**.**

$$\lim_{l\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Lambda_\varepsilon|}\log\mu_{\Lambda_\varepsilon,\gamma,\alpha}(\Omega_{\Lambda_\varepsilon,l}(u)) = -I_\alpha(u). \tag{2.101}$$

As a byproduct of the large deviations, we can estimate the observed magnetization while considering an external field $\alpha\in\mathscr{C}(\mathbb{T},\mathbb{R})$: at a region larger than γ^{-1} around any macroscopic point $r\in\mathbb{T}$, we observe $\widetilde{u}(\alpha(r))$. This is the content of the next result. To state it precisely, we need to introduce some background.

We consider a continuous function $\omega:\mathbb{T}\to\mathbb{R}$ that plays the role of a test function. Let $R\gg\gamma^{-1}$ be a mesoscopic parameter. More formally, we consider $R:(0,1)\to(0,\infty)$ as a function of γ satisfying $\lim_{\gamma\to 0}\gamma R(\gamma)=\infty$. For every $x\in\Lambda_\varepsilon$, we consider the ball B_x centred at x with radius R defined by

$$B_x := \left\{y\in\Lambda_\varepsilon : \varepsilon^{-1}d_{\mathbb{T}}(\varepsilon x,\varepsilon y)\le R\right\}. \tag{2.102}$$

Let X be the random variable defined by

$$X(\sigma) := \frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}\omega(\varepsilon x)\, m_{B_x}(\sigma). \tag{2.103}$$

X plays the role of the ergodic average of the function m_{B_x} weighted with ω. The following result is a "sort of" convergence in probability of X to $\int_{\mathbb{T}}\omega u$. We say "sort of" because the probability space varies as ε goes to zero.

Theorem 2.6.2. *Let $\alpha\in\mathscr{C}(\mathbb{T},\mathbb{R})$ and let $u:=\tilde{u}(\alpha)\in\mathscr{C}(\mathbb{T},(-1,1))$ its associated magnetization. Then, for every $\delta>0$,*

$$\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\left|X-\int_{\mathbb{T}}\omega u\right|>\delta\right)=0. \tag{2.104}$$

Proof. As usual, we proceed by cases.

- *Case α and ω constant.* As α is constant, so it is u. In this case, we have to prove that

$$\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\left|\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}m_{B_x}(\sigma)-u\right|>\delta\right)=0. \tag{2.105}$$

Actually, we will prove the stronger exponential bound

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\left|\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}m_{B_x}(\sigma)-u\right|>\delta\right)\le e^{-O(1)|\Lambda_\varepsilon|\delta^3}, \tag{2.106}$$

that holds for ε and γ small enough.

To shorten notation, we write $U_{x,\gamma}(\sigma)$ for $\sum_{y\in\Lambda_\varepsilon}J_{\Lambda_\varepsilon,\gamma}(x,y)\,\sigma_y$. The next step is to stablish the approximation

$$\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}|m_{B_x}(\sigma)-u| = \frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}|U_{x,\gamma}(\sigma)-u| + s(\gamma)\,O(1)+O\left(\gamma^{-1}R^{-1}\right), \tag{2.107}$$

for $s(\gamma)$ vanishing as γ goes to zero. We first observe that, for every $x \in \Lambda_\varepsilon$ (and ε small enough), we have

$$\sum_{y\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y) = \sum_{y\in\Lambda_\varepsilon} \gamma^d \phi(\gamma y) = \int_{\mathbb{R}^d} \phi + s(\gamma) = 1 + s(\gamma), \tag{2.108}$$

and then

$$m_{B_x}(\sigma) = \frac{1}{|B_x|}\sum_{y\in B_x}\sigma_y = \frac{1}{|B_x|}\sum_{y\in B_x}\left[\sum_{z\in\Lambda_\varepsilon} J_\gamma(z,y) - s(\gamma)\right]\sigma_y = \tag{2.109}$$

$$\frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y + s(\gamma)\,O(1). \tag{2.110}$$

For the first adding of the later expression, we have

$$\frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y = \frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in B_x} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y + O\left(\gamma^{-1}R^{-1}\right). \tag{2.111}$$

Indeed,

$$\frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y - \frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in B_x} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y = \tag{2.112}$$

$$\frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in\partial(B_x,\gamma^{-1})} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y = \frac{1}{|B_x|}\sum_{z\in\partial(B_x,\gamma^{-1})}\sum_{y\in B_x} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y, \tag{2.113}$$

where $\partial(B_x,\gamma^{-1})$ is the set containing the vertices of $\Lambda_\varepsilon \setminus B_x$ whose distance to B_x is less than γ^{-1}. As $\sum_{y\in B_x} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y$ is $O(1)$, the right-hand side of (2.113) is $O(\gamma^{-1}R^{-1})$, implying (2.111). Similarly,

$$\frac{1}{|B_x|}\sum_{y\in B_x} U_{y,\gamma}(\sigma) = \frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in B_x} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y + O\left(\gamma^{-1}R^{-1}\right). \tag{2.114}$$

Putting (2.111) and (2.114) together, we obtain

$$\frac{1}{|B_x|}\sum_{y\in B_x}\sum_{z\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(z,y)\,\sigma_y = \frac{1}{|B_x|}\sum_{y\in B_x} U_{y,\gamma}(\sigma) + O\left(\gamma^{-1}R^{-1}\right). \tag{2.115}$$

Replacing (2.115) in (2.110), we obtain

$$m_{B_x}(\sigma) = \frac{1}{|B_x|}\sum_{y\in B_x} U_{y,\gamma}(\sigma) + s(\gamma)\,O(1) + O\left(\gamma^{-1}R^{-1}\right). \tag{2.116}$$

(2.107) follows immediately from the later expression.

From (2.107), taking γ small enough, we get

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}|m_{B_x}(\sigma) - u| > \delta\right) \le \mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}|U_{x,\gamma}(\sigma) - u| > \frac{\delta}{2}\right). \tag{2.117}$$

As δ is arbitrary, the problem is reduced to controlling

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}|U_{x,\gamma}(\sigma)-u|>\delta\right). \tag{2.118}$$

From inequality,

$$\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}|U_{x,\gamma}(\sigma)-u|\le O(1)\frac{|\{x:|U_{x,\gamma}(\sigma)-u|>\delta/2\}|}{|\Lambda_\varepsilon|}+\frac{\delta}{2}, \tag{2.119}$$

it follows that

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}|U_{x,\gamma}(\sigma)-u|>\delta\right)\le \tag{2.120}$$

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{|\Lambda_\varepsilon|}|\{x\in\Lambda_\varepsilon:|U_{x,\gamma}(\sigma)-u|>\delta/2\}|>O(1)\,\delta\right). \tag{2.121}$$

We reduced the problem to the following lemma, whose proof can be found in the appendix.

Lemma 2.6.3. *For every $c,\delta>0$,*

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(|\{x\in\Lambda_\varepsilon:|U_{x,\gamma}-u|>\delta\}|>c|\Lambda_\varepsilon|\right)\le e^{-|\Lambda_\varepsilon|\beta c\delta^2/2} \tag{2.122}$$

for γ and ε small enough.

- *General case.* Let n, $\{B_1,\dots,B_n\}$ and $\{\Delta_1,\dots,\Delta_n\}$ as in the general cases of the proof of theorems 2.3.1 and 2.5.5.

  As u and ω are uniformly continuous,

$$\left|\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}m_{B_x}\omega(\varepsilon x)-\int_{\mathbb{T}}u\omega\right|\le \tag{2.123}$$

$$\left|\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}m_{B_x}\omega_l(\varepsilon x)-\int_{\mathbb{T}}u_l\omega_l\right|+s(l) \tag{2.124}$$

with $\lim_{l\to0}s(l)=0$. For $x\in\Delta_i$, let $\tilde m_{B_x}$ be the magnetization considering periodic boundary conditions in Δ_i. As $\tilde m_{B_x}$ coincides with m_{B_x} if the distance between x and $\Lambda_\varepsilon\setminus\Delta_i$ is larger than R, (2.125) is controlled by

$$\left|\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}\tilde m_{B_x}\omega_l(\varepsilon x)-\int_{\mathbb{T}}u_l\omega_l\right|+O\left(\varepsilon l^{d-1}R\right)+s(l). \tag{2.125}$$

Also

$$(2.125)=\left|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{|\Delta_i|}\sum_{x\in\Delta_i}\tilde m_{B_x}\omega_i-\frac{1}{n}\sum_{i=1}^{n}u_i\omega_i\right|+O\left(\varepsilon l^{d-1}R\right)+s(l)\le \tag{2.126}$$

$$\|\omega\|_\infty\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{|\Delta_i|}\sum_{x\in\Delta_i}\tilde m_{B_x}-u_i\right|+O\left(\varepsilon l^{-1}R\right)+s(l). \tag{2.127}$$

As δ is arbitrary, we reduced the problem to proving that

$$\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{|\Delta_i|}\sum_{x\in C_\varepsilon}\tilde{m}_{B_x}-u_i\right|>\delta\right). \tag{2.128}$$

To make the notations more compact, let

$$Y_i:=\left|\frac{1}{|\Delta_i|}\sum_{x\in\Delta_i}\tilde{m}_{B_x}-u_i\right|. \tag{2.129}$$

We then need to control

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{n}\sum_{i=1}^{n}Y_i>\delta\right). \tag{2.130}$$

We notice that

$$\frac{1}{n}\sum_{i=1}^{n}Y_i\leq 2\left\|g\right\|_\infty\frac{|\{i:Y_i>\delta/2\}|}{n}+\delta/2, \tag{2.131}$$

so (2.130) is bounded by

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{n}|\{i:Y_i>\delta/2\}|>\frac{\delta}{4\left\|g\right\|_\infty}\right). \tag{2.132}$$

As δ is arbitrary, it is enough to control

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{n}|\{i:Y_i>\delta\}|>c\right) \tag{2.133}$$

with $c:=\dfrac{\delta}{8\left\|g\right\|_\infty}$.

In order to apply the result given in the first step (the case α and ω constant), we substitute α by α_l, neglect the interactions between the boxes $\Delta_1,\ldots,\Delta_n$ and put periodic boundary conditions in each of them; as a consequence, we obtain a new probability $\tilde{\mu}_{\Lambda_\varepsilon,\gamma,\alpha_l}$ that satisfies

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}(B)\leq e^{|\Lambda_\varepsilon|O(l)+O(\varepsilon^{-d+1}l^{-1})+O(\varepsilon^{-d+1}l^{-1}\gamma^{-1})}\tilde{\mu}_{\Lambda_\varepsilon,\gamma,\alpha_l}(B) \tag{2.134}$$

for every event B. Let $\lceil cN\rceil$ be the roof of cN: $\lceil cN\rceil:=\min\{i\in\mathbb{N}:i\geq cN\}$. Under these conventions, for γ small enough,

$$\tilde{\mu}_{\Lambda_\varepsilon,\gamma,\alpha_l}\left(|\{i:Y_i>\delta\}|\geq cN\right)\leq\binom{N}{\lceil cN\rceil}\prod_{i=1}^{\lceil cN\rceil}\tilde{\mu}_{\Delta_i,\gamma,\alpha_i}(Y_i>\delta)\leq e^{N\delta O(1)}e^{-|\Lambda_\varepsilon|\delta^3O(1)}; \tag{2.135}$$

in the last inequality, we used (2.106) (here is where we need γ to be small enough) and Stirling's formula. From (2.134) and (2.135), we obtain

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{n}|\{i:Y_i>\delta\}|>c\right)\leq \tag{2.136}$$

$$\exp\left\{|\Lambda_\varepsilon|O(l)+O(\varepsilon^{-d+1}l^{-1})+O(\varepsilon^{-d+1}l^{-1}\gamma^{-1})+N\delta O\left(1\right)-\left|\Lambda_\varepsilon\right|\delta^3O\left(1\right)\right\}. \tag{2.137}$$

For l and γ small enough, the coefficient of ε^{-d} inside the exponential is negative and thus

$$\lim_{\varepsilon\to 0}\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(\frac{1}{N}\sum_{i=1}^{N}Y_i>\delta/2\right)=0, \tag{2.138}$$

completing the proof. □

2.7 Equivalence of ensembles

The following result is the analogue to Theorem 2.3.3.

Theorem 2.7.1 (Equivalence of ensembles)**.** *For $\alpha \in \mathscr{C}(\mathbb{T}, \mathbb{R})$, identity*

$$P(\alpha) = \max_{u \in \mathcal{C}(\mathbb{T},(-1,1))} -F_{\alpha}(u) \tag{2.139}$$

holds. Conversely, for $u \in \mathscr{C}(\mathbb{T}, (-1,1))$,

$$\int_{\mathbb{T}} f_{\beta}(u) = \max_{\alpha \in \mathscr{C}(\mathbb{T},\mathbb{R})} \left\{ -P(\alpha) - \int_{\mathbb{T}} (u-\alpha)^2 \right\}. \tag{2.140}$$

Proof. We have already proved (2.139) in Theorem 2.5.5, so we just need to prove (2.140). As in the proof of Theorem 2.5.5, (2.139) can be read as

$$P(\alpha) = -F_{\alpha}(\tilde{u}(\alpha)). \tag{2.141}$$

Given $u \in \mathscr{C}(\mathbb{T}, (-1,1))$ and taking $\alpha = \tilde{\alpha}(u)$, we get

$$P(\tilde{\alpha}(u)) = -F_{\tilde{\alpha}(u)}(u). \tag{2.142}$$

From the definition of $F_{\tilde{\alpha}(u)}$, it follows that

$$\int_{\mathbb{T}} f_{\beta}(u) = -P\left(\tilde{h}(u)\right) - \int_{\mathbb{T}} [u - \tilde{\alpha}(u)]. \tag{2.143}$$

It remains to show that

$$\int_{\mathbb{T}} f_{\beta}(u) \geq -\int_{\mathbb{T}} (u-\alpha)^2 - P(\alpha) \tag{2.144}$$

for every $\alpha \in \mathscr{C}(\mathbb{T}, (-1,1))$. Observe that

$$\sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} e^{-\beta H^{nn}_{\Lambda_\varepsilon}(\sigma)} = Z_{\Lambda_\varepsilon,\gamma,\alpha} \sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} \frac{e^{-\beta\left[H^{nn}_{\Lambda_\varepsilon}(\sigma) + W_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)\right]}}{Z_{\Lambda_\varepsilon,\gamma,\alpha}} e^{\beta W_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \leq \tag{2.145}$$

$$Z_{\Lambda_\varepsilon,\gamma,\alpha} \sum_{\sigma \in \Omega_{\Lambda_\varepsilon, l}(u)} e^{\beta W_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)}, \tag{2.146}$$

with $W_{\Lambda_\varepsilon,\gamma,\alpha}$ defined as in (2.39). With computations identical to the ones appearing in the proof of Theorem 2.5.1, it follows that

$$\lim_{l \to 0} \lim_{\varepsilon \to 0} -\frac{1}{\beta |\Lambda_\varepsilon|} \log \sum_{\sigma \in \Omega_{l,\varepsilon}(u)} e^{-\beta H^{nn}_{\Lambda_\varepsilon}(\sigma)} = \int_{\mathbb{T}} f_{\beta}(u) \tag{2.147}$$

$$\lim_{l \to 0} \lim_{\gamma \to 0} \lim_{\varepsilon \to 0} \frac{1}{\beta |\Lambda_\varepsilon|} \log \sum_{\sigma \in \Omega_{l,\varepsilon}(u)} e^{\beta W_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} = \int_{\mathbb{T}} (u-\alpha)^2. \tag{2.148}$$

Dividing expression 2.145 by $-\beta |\Lambda_\varepsilon|$, taking limit and using the later identities, we obtain the desired inequality (2.144), and the result follows. □

2.8 Appendix

Proof of the uniform convergence in Theorem 2.3.1

During this sub-section, we use the notation $\Lambda_q = \Lambda_\varepsilon$ (recalling ε is of the form 2^{-q}). Let $\left\{\Lambda_{q,1}, \ldots, \Lambda_{q,2^d}\right\}$ be the natural partition of Λ_{q+1} into translations of Λ_q. For $\sigma \in \Omega_{\Lambda_{q+1}}$, let σ_i be its projection over $\Lambda_{q,i}$.

For $u \in V_{|\Lambda_q|}$, let

$$a_q := \sum_{\substack{\sigma\in\Omega_{\Lambda_q}\\ m(\sigma)=u}} e^{-\beta H^{nn}_{\Lambda_q}(\sigma)}. \tag{2.149}$$

As, for $\sigma \in \Omega_{\Lambda_{q+1}}$, we have $m(\sigma) = u$ if $m(\sigma_i) = u$ for every i, and as there are $O\left(\varepsilon^{-d+1}\right)$ nearest-neighbour interactions between the boxes $\Lambda_{q,1}, \ldots, \Lambda_{q,2^d}$,

$$a_{q+1} \geq \sum_{\substack{\sigma_1\in\Omega_{\Lambda_{q,1}}\\ m(\sigma_1)=u}} \cdots \sum_{\substack{\sigma_{2^d}\in\Omega_{\Lambda_{q,2^d}}\\ m\left(\sigma_{2^d}\right)=u}} e^{O\left(\varepsilon^{-d+1}\right)} \prod_{i=1}^{2^d} e^{-\beta H^{nn}_{\Lambda_{q,i}}(\sigma_i)} = e^{O\left(\varepsilon^{-d+1}\right)} a_q^{2^d}. \tag{2.150}$$

Taking logarithm and dividing by $-\beta\left|\Lambda_{q+1}\right|$, we get

$$f_{\Lambda_{q+1},\beta}(u) \leq f_{\Lambda_q,\beta}(u) + O(\varepsilon). \tag{2.151}$$

For a configuration $\sigma \in \Omega_{\Lambda_{q+1}}$, let $N^+(\sigma) := |\{x \in \Lambda_{q+1} : \sigma_x = 1\}|$ be its associated number of pluses. There is a one-to-one correspondence between $V_{|\Lambda_{q+1}|}$ and the set $[0, |\Lambda_{q+1}|] \cap \mathbb{Z}$ containing all the possible number of pluses. Let u and u' be consecutive elements of $V_{|\Lambda_{q+1}|}$ such that $u < u'$, and let i and $i+1$ be respectively their associated number of pluses. Then

$$\sum_{\substack{\sigma\in\Omega_{\Lambda_{q+1}}\\ m(\sigma)=u'}} e^{-\beta H^{nn}_{\Lambda_{q+1}}(\sigma)} = \sum_{\substack{\sigma\in\Omega_{\Lambda_{q+1}}\\ N^+(\sigma)=i+1}} e^{-\beta H^{nn}_{\Lambda_{q+1}}(\sigma)} = \frac{1}{i+1} \sum_{\substack{\sigma\in\Omega_{\Lambda_{q+1}}\\ N^+(\sigma)=i}} \sum_{\substack{\sigma'\in\Omega_{\Lambda_{q+1}}\\ \sigma'\geq\sigma\\ N^+(\sigma')=i+1}} e^{-\beta H^{nn}_{\Lambda_{q+1}}(\sigma')}, \tag{2.152}$$

where $\sigma' \geq \sigma$ means $\sigma'_x \geq \sigma_x$ for every $x \in \Lambda_{q+1}$. In the later sum, the configurations σ' are perturbations in one site of the configurations σ. As every site has 2^d neighbours,

$$H^{nn}_{\Lambda_{q+1}}(\sigma') \geq H^{nn}_{\Lambda_{q+1}}(\sigma) - 2^{d+1}. \tag{2.153}$$

Replacing in (2.152), using the fact that $|\{\sigma' : \sigma' \geq \sigma, N^+(\sigma') = i+1\}| = |\Lambda_{q+1}| - i$ and the bound $\frac{|\Lambda_{q+1}|-i}{i+1} \leq |\Lambda_{q+1}|$, we get

$$\sum_{\substack{\sigma\in\Omega_{\Lambda_{q+1}}\\ m(\sigma)=u'}} e^{-\beta H^{nn}_{\Lambda_{q+1}}(\sigma)} \leq \left|\Lambda_{q+1}\right| e^{\beta 8} \sum_{\substack{\sigma\in\Omega_{\Lambda_{q+1}}\\ m(\sigma)=u}} e^{-\beta H^{nn}_{\Lambda_{q+1}}(\sigma)}. \tag{2.154}$$

Taking logarithm and dividing by $-\beta\left|\Lambda_{q+1}\right|$, we get

$$f_{\Lambda_{q+1},\beta}(u) - f_{\Lambda_{q+1},\beta}(u') \leq O\left(\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right). \tag{2.155}$$

The same bound for $f_{\Lambda_{q+1},\beta}(u') - f_{\Lambda_{q+1},\beta}(u)$ can be obtained by replacing the number of pluses N^+ by the number of minuses N^-. Then

$$\left|f_{\Lambda_{q+1},\beta}(u) - f_{\Lambda_{q+1},\beta}(u')\right| = O\left(\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right). \tag{2.156}$$

For $u \in V_{|\Lambda_{q+1}|}$, let u_- and u_+ be the elements of $V_{|\Lambda_q|}$ that best approximates u respectively from below and from above:

$$u_- := \max\left\{u' \in V_{|\Lambda_q|} : u' \le u\right\} \qquad u_+ := \min\left\{u' \in V_{|\Lambda_q|} : u' \ge u\right\}. \tag{2.157}$$

For $u \in V_{|\Lambda_{q+1}|} \setminus V_{|\Lambda_q|}$, using (2.156) repeatedly, we get

$$f_{\Lambda_{q+1},\beta}(u) \le f_{\Lambda_{q+1},\beta}(u_-) \wedge f_{\Lambda_{q+1},\beta}(u_+) + O\left(\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right), \tag{2.158}$$

where $a \wedge b$ stands for $\min\{a, b\}$. From (2.151), the later expression is bounded from above by

$$f_{\Lambda_q,\beta}(u_-) \wedge f_{\Lambda_q,\beta}(u_+) + O(\varepsilon) + O\left(\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right) \le f_{\Lambda_q,\beta}(u) + O(\varepsilon) + O\left(\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right). \tag{2.159}$$

Let $a_q := O(\varepsilon) + O\left(\frac{\log|\Lambda_{q+1}|}{|\Lambda_{q+1}|}\right)$ and observe that $a := \sum_i a_i$ is finite. From the estimates of the previous paragraph and the fact that $f_{\Lambda_q,\beta}$ is defined by linear interpolation, we get

$$f_{\Lambda_{q+1},\beta}(u) \le f_{\Lambda_q,\beta}(u) + a_q \tag{2.160}$$

for every $u \in [-1, 1]$ and every q. Let $g_q := f_{\Lambda_q,\beta} - \sum_{i=0}^{q-1} a_i$. Inequality (2.160) implies

$$g_{q+1}(u) \le g_q(u) \tag{2.161}$$

for every $u \in [-1, 1]$. The point-wise convergence of the free energy given in Theorem 2.3.1 guarantees the point-wise convergence of $(g_q)_q$ to $f_\beta - a$. Then $(g_q)_q$ is a sequence of continuous functions defined on a compact set that converges point-wise and in a monotonic way to $f_\beta - a$. Under these hypothesis, Dini's theorem asserts that the convergence is uniform; we deduce the uniform convergence of $f_{\Lambda_q,\beta}$ to f_β.

Proof of Lemma 2.5.3

We consider identity (2.152) with $H^{nn}_{\Lambda_q}$ replaced by $H_{\Lambda_q,\gamma,\alpha}$. While comparing $H_{\Lambda_q,\gamma,\alpha}(\sigma)$ with $H_{\Lambda_q,\gamma,\alpha}(\sigma')$, the nearest-neighbour part of the Hamiltonian can be treated as in sub-section 2.8. To treat the quadratic part, observe that, as every vertex interacts with $O(\gamma^{-1})$ vertices, we have

$$H_{\Lambda_q,\gamma,\alpha}(\sigma') \ge H_{\Lambda_q,\gamma,\alpha}(\sigma) - O\left(\gamma^{-1}\right). \tag{2.162}$$

We can now repeat the arguments of sub-section 2.8 to conclude.

Proof of Lemma 2.6.3

It is convenient to introduce a notation for the random variable "density of sites with the wrong magnetization": for $\delta > 0$, let

$$D\left(\Lambda_\varepsilon, \delta\right) := \frac{1}{|\Lambda_\varepsilon|} \left|\left\{x \in \Lambda_\varepsilon : |U_{x,\gamma} - u| > \delta\right\}\right|. \tag{2.163}$$

Under this definition, the statement of the lemma is the following: for $u \in (-1, 1)$ and $\alpha := \tilde{\alpha}(u)$,

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(D\left(\Lambda_\varepsilon, \delta\right) > c\right) \leq e^{-|\Lambda_\varepsilon|\beta c\delta^2/2}. \tag{2.164}$$

We proceed by cases.

- *Case 1:* $u \in [-m_\beta, m_\beta]$. Inequality

$$H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) \geq H^{nn}_{\Lambda_\varepsilon}(\sigma) + c\,|\Lambda_\varepsilon|\,\delta^2 \tag{2.165}$$

holds for every $\sigma \in \left(D\left(\Lambda_\varepsilon, \delta\right) > c\right)$; then

$$\sum_{\sigma \in (D(\Lambda_\varepsilon,\delta)>c)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \leq e^{-\beta c|\Lambda_\varepsilon|\delta^2} \sum_{\sigma\in\Omega_{\Lambda_\varepsilon}} e^{-\beta H^{nn}_{\Lambda_\varepsilon}(\sigma)}. \tag{2.166}$$

As $u \in [-m_\beta, m_\beta]$, Theorem 2.3.3 tells us $p_\beta(0) = f_\beta(u)$; then identity

$$\sum_{\sigma\in\Omega_{\Lambda_\varepsilon}} e^{-\beta H^{nn}_{\Lambda_\varepsilon}(\sigma)} = e^{-\beta|\Lambda_\varepsilon|\left[f_\beta(u)+s_1(\varepsilon)\right]} \tag{2.167}$$

holds for $s_1(\varepsilon)$ vanishing as ε goes to zero. Putting (2.166) and (2.167) together, we obtain

$$\sum_{\sigma \in (D(\Lambda_\varepsilon,\delta)>c)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \leq e^{-\beta|\Lambda_\varepsilon|\left[c\delta^2+f_\beta(u)+s_1(\varepsilon)\right]}. \tag{2.168}$$

As $f'_\beta(u) = 0$ in this case, we have $\alpha = u$ and $P(\alpha) = -f_\beta(u)$ (see observation 2.5.6); then

$$Z_{\Lambda_\varepsilon,\gamma,\alpha} = e^{-\beta|\Lambda_\varepsilon|\left[f_\beta(u)+s_2(\varepsilon,\gamma)\right]} \tag{2.169}$$

for $s_2(\varepsilon, \delta)$ vanishing as ε and γ go to zero. Dividing (2.168) by (2.169), we obtain

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(D\left(\Lambda_\varepsilon, \delta\right) > c\right) \leq e^{-\beta|\Lambda_\varepsilon|\left[c\delta^2+s_1(\varepsilon)-s_2(\varepsilon,\gamma)\right]}. \tag{2.170}$$

The proof of case 1 finishes after observing that the later expression is bounded by $e^{-\beta|\Lambda_\varepsilon|c\delta^2/2}$ if ε and γ are small enough.

- *Case 2:* $u \notin [-m_\beta, m_\beta]$. Subtracting and adding u and expanding, we get

$$H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) = \tag{2.171}$$

$$H^{nn}_{\Lambda_\varepsilon}(\sigma) + \sum_{x\in\Lambda_\varepsilon} \left[U_{x,\gamma}(\sigma) - u\right]^2 + 2(u-\alpha)\sum_{x\in\Lambda_\varepsilon} U_{x,\gamma}(\sigma) - 2u(u-\alpha)|\Lambda_\varepsilon| + (u-\alpha)^2|\Lambda_\varepsilon|. \tag{2.172}$$

As the constants are irrelevant in the computation of probabilities, the probability associated to the Hamiltonian

$$\tilde{H}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) := H^{nn}_{\Lambda_\varepsilon}(\sigma) + \sum_{x\in\Lambda_\varepsilon}\left[U_{x,\gamma}(\sigma) - u\right]^2 + 2(u-\alpha)\sum_{x\in\Lambda_\varepsilon} U_{x,\gamma}(\sigma) \tag{2.173}$$

is also $\mu_{\Lambda_\varepsilon,\gamma,\alpha}$:

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}(D(\Lambda_\varepsilon,\delta) > c) = \frac{\sum_{\sigma\in(D(\Lambda_\varepsilon,\delta)>c)} e^{-\beta\tilde{H}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)}}{\tilde{Z}_{\Lambda_\varepsilon,\gamma,\alpha}}. \tag{2.174}$$

($\tilde{Z}_{\Lambda_\varepsilon,\gamma,\alpha}$ is the grand canonical partition function associated to the Hamiltonian $\tilde{H}_{\Lambda_\varepsilon,\gamma,\alpha}$.)

As u and α satisfy

$$f'_\beta(u) = -2(u-\alpha), \tag{2.175}$$

the value $h := -2(u-\alpha)$ present in $\tilde{H}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)$ is the value of the external field associated to u in the nearest-neighbour case. As in (2.108),

$$\sum_{y\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y) = \sum_{y\in\Lambda_\varepsilon}\gamma^d\phi(\gamma y) = \int_{\mathbb{R}^d}\phi + s(\gamma) = 1 + s(\gamma) \tag{2.176}$$

for every $x\in\Lambda_\varepsilon$ and ε small enough, where $s(\gamma)$ vanishes as γ goes to zero, implying

$$\sum_{y\in\Lambda_\varepsilon} U_{x,\gamma}(\sigma) = \sum_{y\in\Lambda_\varepsilon}\sum_{x\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y)\,\sigma_x = \sum_{x\in\Lambda_\varepsilon}\sigma_x\sum_{y\in\Lambda_\varepsilon} J_{\Lambda_\varepsilon,\gamma}(x,y) = \sum_{x\in\Lambda_\varepsilon}\sigma_x + s(\gamma)\,O(1)\,|\Lambda_\varepsilon|. \tag{2.177}$$

We can replace in (2.173) to obtain

$$\tilde{H}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) = H^{nn}_{\Lambda_\varepsilon,h}(\sigma) + s(\gamma)\,O(1)\,|\Lambda_\varepsilon| + \sum_{x\in\Lambda_\varepsilon}\left(U_{x,\gamma}(\sigma) - u\right)^2. \tag{2.178}$$

To control the numerator of the right-hand side of (2.174), we observe that, for $\sigma\in(D(\Lambda_\varepsilon,\delta) > c)$, we have

$$\sum_{x\in\Lambda_\varepsilon}\left(U_{x,\gamma}(\sigma) - u\right)^2 \geq c\delta^2\,|\Lambda_\varepsilon|, \tag{2.179}$$

implying

$$\sum_{\sigma\in(D(\Lambda_\varepsilon,\delta)>c)} e^{-\beta\tilde{H}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \leq e^{-\beta|\Lambda_\varepsilon|\left(c\delta^2 + O(1)s(\gamma)\right)}\sum_{\sigma\in\Omega_{\Lambda_\varepsilon}} e^{-\beta H^{nn}_{\Lambda_\varepsilon,h}(\sigma)}. \tag{2.180}$$

To control the denominator, observe that, for $c',\zeta > 0$ and $\sigma\in(D(\Lambda_\varepsilon,\zeta)\leq c')$,

$$\sum_{x\in\Lambda_\varepsilon}\left(U_{x,\gamma}(\sigma) - u\right)^2 \leq O(1)\,c'\,|\Lambda_\varepsilon| + \zeta^2\,|\Lambda_\varepsilon|, \tag{2.181}$$

implying

$$\tilde{Z}_{\Lambda_\varepsilon,\gamma,\alpha} \geq \sum_{\sigma\in(D(\Lambda_\varepsilon,\zeta)\leq c')} e^{-\beta\tilde{H}_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \geq e^{-\beta|\Lambda_\varepsilon|[O(1)(s(\gamma)+c')+\zeta]} \sum_{\sigma\in(D(\Lambda_\varepsilon,\zeta)\leq c')} e^{-\beta H^{nn}_{\Lambda_\varepsilon,h}(\sigma)}. \tag{2.182}$$

Replacing in (2.174), we get

$$\mu_{\Lambda_\varepsilon,\gamma,\alpha}\left(D\left(\Lambda_\varepsilon,\delta\right)>c\right) \leq \frac{e^{-\beta|\Lambda_\varepsilon|\left(c\delta^2+O(1)s(\gamma)-O(1)c'-\zeta^2\right)}}{\mu^{nn}_{\Lambda_\varepsilon,h}\left(D\left(\Lambda_\varepsilon,\zeta\right)\leq c'\right)}. \tag{2.183}$$

$O\left(1\right)s\left(\gamma\right)+O\left(1\right)c'+\zeta^2$ can be chosen to be as small as desired as c' and ζ are arbitrary. We are done if we prove that the denominator can be chosen to be as close to one as desired; that is the content of the following lemma.

Lemma 2.8.1. *Let $u\in[-1,1]\setminus[-m_\beta,m_\beta]$ and $h:=f'_\beta\left(u\right)$ the associated value of the external field for the nearest-neighbour case (as in the analysis after Theorem 2.3.3). Then*

$$\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\mu^{nn}_{\Lambda_\varepsilon,h}\left(D\left(\Lambda_\varepsilon,\zeta\right)>c'\right)=0 \tag{2.184}$$

for every $\zeta, c'>0$.

Proof of Lemma 2.8.1. For $\sigma\in\left(D\left(\Lambda_\varepsilon,\zeta\right)>c'\right)$, we have

$$\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}\left|U_{x,\gamma}\left(\sigma\right)-u\right|\geq c'\zeta; \tag{2.185}$$

then

$$\mathbb{E}_{\mu^{nn}_{\Lambda_\varepsilon,h}}\left(\frac{1}{|\Lambda_\varepsilon|}\sum_{x\in\Lambda_\varepsilon}\left|U_{x,\gamma}-u\right|\right)\geq c\zeta\mu^{nn}_{\Lambda_\varepsilon,h}\left(D\left(\Lambda_\varepsilon,\zeta\right)>c'\right). \tag{2.186}$$

By rotational invariance, the left-hand side of (2.186) coincides with $\mathbb{E}_{\mu^{nn}_{\Lambda_\varepsilon,h}}\left(\left|U_{0,\gamma}-u\right|\right)$. As the random variable $\left|U_{0,\gamma}-u\right|$ depends on a finite number of coordinates, the later expectation converges to $\mathbb{E}_{\mu^{nn}_h}\left(\left|U_{0,\gamma}-u\right|\right)$ as ε goes to zero (μ^{nn}_h as in section 2.3). Then we are done if we prove that

$$\lim_{\gamma\to 0}\mathbb{E}_{\mu^{nn}_h}\left(\left|U_{0,\gamma}-u\right|\right)=0. \tag{2.187}$$

In this particular case in which $\sup_{\gamma\in(0,1)}\left\|U_{0,\gamma}-u\right\|_\infty<\infty$, the concerned convergence in mean is implied by the following convergence in probability:

$$\lim_{\gamma\to 0}\mu^{nn}_h\left(\left|U_{0,\gamma}-u\right|>\delta\right)=0 \tag{2.188}$$

for every $\delta>0$.

Let L be a microscopic parameter of the form 2^m, $J_\gamma:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be the function defined by $J_\gamma\left(r,r'\right):=\gamma^d\phi\left(\gamma\left(r-r'\right)\right)$ (J_γ is the infinite-volume version of $J_{\Lambda_\varepsilon,\gamma}$), S_γ be the smallest square of the form $[-aL,aL)^d$, $a\in\mathbb{N}$, containing the support of $J_\gamma\left(0,\cdot\right)$, $\{B_1,\ldots,B_N\}$ be the natural partition of S_γ into boxes of side L and $\{\Delta_1, \ldots, \Delta_N\}$ be their discrete versions, assuming Δ_1 is the one that contains the origin. For $i, j \in \{1, \ldots, N\}$, let

$$\tilde{J}_\gamma (i, j) := L^d \fint_{B_i \times B_j} d(r, r') J_\gamma (r, r'). \tag{2.189}$$

As in (2.56), the approximation $U_{0,\gamma} = \sum_{i=1}^{N} \tilde{J}_\gamma (1, i) m_{\Delta_i} + s(\gamma, L)$ holds, with $s(\gamma, L)$ vanishing as γ goes to zero for every fixed L . As δ is arbitrary, the problem is reduced to proving that

$$\lim_{\gamma \to 0} \mu_h^{nn} \left(\sum_{i=1}^{N} \tilde{J}_\gamma (1, i) |m_{\Delta_i} - u| > \delta \right) = 0. \tag{2.190}$$

For $\zeta, c > 0$ and $\sigma \in \Omega_{\mathbb{Z}^d}$ such that $\frac{1}{N} |\{i \in \{1, \ldots, N\} : |m_{\Delta_i}(\sigma) - u| > \zeta\}| \leq c$, we have

$$\sum_{i=1}^{N} \tilde{J}_{\Lambda_\varepsilon, \gamma} (1, i) |m_{\Delta_i}(\sigma) - u| \leq cO(1) + \zeta. \tag{2.191}$$

Take $\zeta = \delta/2$ and $c = \dfrac{\delta}{2O(1)}$ to obtain the inclusion

$$\left(\sum_{i=1}^{N} \tilde{J}_{\Lambda_\varepsilon, \gamma} (1, i) |m_{\Delta_i} - u| > \delta \right) \subset \left(\frac{1}{N} |\{i \in \{1, \ldots, N\} : |m_{\Delta_i} - u| > \zeta\}| > c \right). \tag{2.192}$$

The problem is then reduced to proving that the convergence

$$\lim_{\gamma \to 0} \mu_h^{nn} \left(\frac{1}{N} \sum_{i=1}^{N} \mathbf{1} \{|m_{\Delta_i} - u| > \zeta\} > c \right) = 0 \tag{2.193}$$

holds for every $\zeta, c > 0$.

Take L sufficiently large such that

$$\mu_h^{nn} (|m_{\Delta_1} - u| > \zeta) \leq c \tag{2.194}$$

(the discussion at the end of section 2.3 guarantees this choice of L is possible). From the multidimensional ergodic theorem (theorem 14.A8 of [Geo11]) applied to the function $\mathbf{1}\{|m_{\Delta_1} - u| > \delta\}$, we obtain the following μ_h^{nn}-almost-surely convergence:

$$\lim_{\gamma \to 0} \frac{1}{N} \sum_{i=1}^{N} \mathbf{1} \{|m_{\Delta_i} - u| > \zeta\} = \mu_h^{nn} (|m_{\Delta_1} - u| > \zeta). \tag{2.195}$$

Finally, (2.194) and (2.195) imply (2.193), completing the proof. □

2.9 Descripción del capítulo

Para describir el fenómeno macroscópico, se introduce el toro d-dimensional $\mathbb{T} := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$. Las configuraciones microscópicas viven en una discretización de dicho toro. Para definirla, se introduce un parámetro pequeño ε de la forma 2^{-q}, con $q \in \mathbb{N}$. De esta manera, las configuraciones son elementos del espacio producto $\sigma \in \{-1,1\}^{\Lambda_\varepsilon}$, con $\Lambda_\varepsilon := (\varepsilon^{-1}\mathbb{T}) \cap \mathbb{Z}^d$.

El Hamiltoniano asociado a nuestro sistema está conformado por dos partes. La primera es la correspondiente a las interacciones ferromagnéticas de vecinos próximos con periodicidad:

$$H^{nn}_{\Lambda_\varepsilon}(\sigma) := - \sum_{\substack{x,y \in \Lambda_\varepsilon \\ x,y \text{ vecinos próximos}}} \sigma_x \sigma_y. \tag{2.196}$$

La segunda es del tipo mesoscópica (interacción de alcance intermedio), y será la que tendrá en cuenta el campo externo. Para definirla, se introduce un parámetro $\gamma > 0$ y un campo externo no-homogéneo $\alpha : \mathbb{T} \to \mathbb{R}$. El parámetro γ da lugar a un promedio de rango γ^{-1}: $\mathrm{Av}_\gamma(\sigma, x)$ representa el promedio de la configuración $\sigma \in \{-1,1\}^{\Lambda_\varepsilon}$ alrededor de una bola de radio γ^{-1} centrada en el vértice $x \in \Lambda_\varepsilon$. Finalmente, el Hamiltoniano queda definido como

$$H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma) := H^{nn}_{\Lambda_\varepsilon}(\sigma) + \sum_{x \in \Lambda_\varepsilon} \left[\mathrm{Av}_\gamma(\sigma, x) - \alpha(\varepsilon x)\right]^2, \tag{2.197}$$

y su factor de Boltzmann asociado como

$$e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)}, \tag{2.198}$$

donde $\beta > 0$ es la temperatura inversa.

La primera parte de la sección 2.5 consiste en definir la manera adecuada de fijar un perfil magnético no homogéneo de manera canónica. Para ello, es necesario introducir un parámetro intermedio l de la forma 2^{-p}, con $p \in \mathbb{N}$. Este parámetro da lugar a una partición de $\mathbb{T}$ en N cubos, a los que denominamos $\{A_1, \ldots, A_N\}$. Sean $\{\Delta_1, \ldots, \Delta_N\}$ sus correspondientes versiones microscópicas. Dado un perfil magnético no-homogéneo $u : \mathbb{T} \to (-1,1)$, definimos las configuraciones fijas a u de manera canónica a escala l como

$$\Omega_{\Lambda_\varepsilon,l}(u) := \left\{ \sigma \in \{-1,1\}^{\Lambda_\varepsilon} : m_{\Delta_i}(\sigma) = \left[⨍_{A_i} dr\, u(r) \right] \text{ para todo } i \right\}, \tag{2.199}$$

donde $\left[⨍_{A_i} dr\, u(r) \right]$ es un elemento aproximante de $⨍_{A_i} dr\, u(r)$ en la imagen de m_{Δ_i}. Una vez definido cómo fijar un perfil magnético de manera canónica, se procede a demostrar el primer resultado: la existencia de la energía libre. Más precisamente, se demuestra que el límite

$$\lim_{l \to 0} \lim_{\gamma \to 0} \lim_{\varepsilon \to 0} -\frac{1}{\beta |\Lambda_\varepsilon|} \log \sum_{\sigma \in \Omega_{\Lambda_\varepsilon,l}(u)} e^{-\beta H_{\Lambda_\varepsilon,\gamma,\alpha}(\sigma)} \tag{2.200}$$

existe y que converge a la integral

$$F_\alpha(u) = \int_{\mathbb{T}} dr \left[f_\beta(u(r)) + (u(r) - \alpha(r))^2 \right], \tag{2.201}$$

donde f_β es la energía libre que se obtiene si solo se consideran interacciones de vecinos próximos. El segundo resultado de esta sección es la existencia de la presión. Más precisamente, demostramos que el límite

$$\lim_{\gamma \to 0} \lim_{\varepsilon \to 0} \frac{1}{\beta |\Lambda_\varepsilon|} \log Z_{\Lambda_\varepsilon,\gamma,\alpha} \tag{2.202}$$

existe y coincide con

$$P(\alpha) = - \inf_{u \text{ perfil magnético}} F_\alpha(u) . \tag{2.203}$$

Más aún, demostramos que este ínfimo se alcanza en una función $\tilde{u}$, mostramos cómo encontrar $\tilde{u}$ a partir de α, y demostramos que existe una relación biunívoca entre campos externos y perfiles magnéticos.

En la sección 2.6, como consecuencia de la existencia de la energía libre y la presión, obtenemos el resultado de grandes desvíos

$$\lim_{l\to 0} \lim_{\gamma\to 0} \lim_{\varepsilon\to 0} \frac{1}{|\Lambda_\varepsilon|} \log \mu_{\Lambda_\varepsilon,\gamma,\alpha}(\Omega_{\Lambda_\varepsilon,l}(u)) = -I_\alpha(u), \tag{2.204}$$

donde I_α es el funcional de tasa definido como

$$I_\alpha(u) := \beta \left[F_\alpha(u) - \min_{v \text{ perfil magnético}} F_\alpha(v) \right]. \tag{2.205}$$

Como consecuencia de este resultado, podemos estimar la magnetización observada en presencia del campo externo α. Para ello, consideramos un parámetro $R \gg \gamma^{-1}$ y definimos el promedio ergódico pesado

$$X(\sigma) := \frac{1}{|\Lambda_\varepsilon|} \sum_{x\in\Lambda_\varepsilon} \omega(\varepsilon x)\, m_{B_x}(\sigma) , \tag{2.206}$$

donde $\omega : \mathbb{T} \to \mathbb{R}$ es una función test y B_x es la bola de radio R centrada en $x \in \Lambda_\varepsilon$. El segundo resultado de esta sección es el siguiente límite en probabilidad: para todo $\delta > 0$,

$$\lim_{\gamma\to 0} \lim_{\varepsilon\to 0} \mu_{\Lambda_\varepsilon,\gamma,\alpha} \left(\left| X - \int_{\mathbb{T}} dr\, \omega(r) u(r) \right| > \delta \right) = 0. \tag{2.207}$$

Para finalizar, en la sección 2.7, establecemos el siguiente resultado de equivalencia de arreglos, que establece una relación entre la energía libre y la presión:

$$P(\alpha) = \max_{u \text{ perfil magnético}} -F_\alpha(u) \tag{2.208}$$

$$\int_{\mathbb{T}} dr\, f_\beta(u(r)) = \max_{\alpha \text{ campo externo}} \left\{ -P(\alpha) - \int_{\mathbb{T}} dr\, (u(r) - \alpha(r))^2 \right\}. \tag{2.209}$$

Chapter 3

Conclusiones

A modo de conclusión, se presentan en este apartado las principales virtudes de nuestros métodos, se las comparan con métodos existentes y se comentan aspectos de índole cualitativas de los modelos considerados.

Con respecto al primer capítulo, la principal ventaja de nuestros métodos es la aplicación en grafos irregulares y, en particular, aleatorios. Además de los casos mencionados en la tesis, se pueden considerar grafos que presentan no solo aleatoriedad en las aristas sino también aleatoriedad en la distribución espacial de sus vértices. Un ejemplo posible es en el que los vértices se distribuyen de acuerdo a un proceso puntual de Poisson homogéneo y éstos se vinculan entre sí cuando están a distancia menor que cierto radio R. Si R es suficientemente grande, el grafo en cuestión tiene un único aglomerado infinito. Si adelgazamos las aristas mediante percolación Bernoulli con parámetro suficientemente bajo, todos los aglomerados aleatorios resultantes son finitos; es entonces aplicable el criterio de unicidad. Si, en cambio, las aristas son adelgazadas tomando un parámetro cercano a 1, el grafo resultante tiene una única componente infinita; el criterio de no-unicidad es el que nos da información en este caso. La propiedad abstracta que debe satisfacer el grafo es que tenga sentido adelgazar las aristas de forma independiente. Nuestro método abarca entonces una gran familia de grafos en donde el problema no se sabía resolver. Cabe hacer una comparación con el método de Pirogov-Sinai, el cual da un criterio de unicidad en grafos regulares pero no requiere la simetría en el conjuntos de espines que nosotros necesitamos; la generalidad que ganamos en la estructura del grafo, la perdemos en la estructura del conjunto de espines.

En relación al segundo capítulo, es remarcable la aparición del término cuadrático en la energía libre que surge a partir de las interacciones mesoscópicas o coeficiente cuadrático de Kac. Desde el punto de vista cualitativo, este hecho implica que la energía libre es estrictamente convexa, es decir, que no se produce el fenómeno de co-existencia de fases presente en el modelo de Ising ferromagnético de vecinos próximos a baja temperatura, lo cual es interesante desde el punto de vista del modelado, ya que nos informa acerca de cómo fijar un perfil magnético mediante un campo externo bajo la presencia de las mencionadas interacciones intermedias o mesoscópicas, y más aún, nos da una receta explícita para hacerlo.

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