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Acerca del rango y propiedades topológicas de algunos operadores no lineales

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Acerca del rango y propiedades topológicas de algunos operadores no lineales

Estudiamos el siguiente tipo de problemas de segundo orden:

$$u'' = g(x,u) + p(x) \qquad x \in (a,b) \subset \mathbb{R},$$

donde $g \in C([a,b] \times \mathbb{R}^N, \mathbb{R}^N)$.

El objetivo principal de esta tesis es estudiar, bajo distintas condiciones de contorno, qué funciones $p \in L^2((a,b),\mathbb{R}^N)$ garantizan la existencia de solución. Donde la definición de solución será dada en cada caso.

En otras palabras, analizamos la imagen del operador semilineal $S(u) = u'' - g(x,u)$, considerado como un operador continuo de $H \subset H^2((a,b),\mathbb{R}^N)$ a $L^2((a,b),\mathbb{R}^N)$, donde H es un subespacio cerrado que depende de las condiciones de contorno.

En primer lugar, estudiamos problemas resonantes bajo condiciones periódicas, que generalizan, por un lado, la ecuación del péndulo forzado y, por otro, las condiciones de Landesman-Lazer. Consideramos el caso variacional $S(u) = u'' - \nabla G(u)$, para el cual logramos caracterizar $Im(S)$ y dar algunas de sus propiedades topológicas.

En segundo lugar, estudiamos problemas con condiciones de contorno de radiación, es decir,

$$u'(0) = a_0 u(0), \ \ u'(1) = a_1 u(1),$$

con $a_0, a_1 > 0$. Encontramos una condición de Hartman generalizada que garantiza existencia de solución. En particular, si g es superlineal, probamos que el operador S es suryectivo. Para este caso, estudiamos también condiciones necesarias y suficientes para la unicidad o multiplicidad de soluciones. Logramos obtener resultados más precisos para el caso $N = 1$ empleando métodos topológicos y variacionales y Teorema de la Función Implícita.

Palabras clave: Problemas de contorno no lineales; Soluciones periódicas; Métodos variacionales; Métodos topológicos; Ecuaciones elípticas semilineales.

2010 MSC: 34B15, 34C25, 35A15, 35A16, 35J61.

On the range and topological properties of some nonlinear operators

We study the following type of second order problems:

$$u'' = g(x, u) + p(x) \qquad x \in (a, b) \subset \mathbb{R},$$

where $g \in C([a, b] \times \mathbb{R}^N, \mathbb{R}^N)$.

The thesis is devoted to the following problem: which functions $p \in L^2((a, b), \mathbb{R}^N)$ guarantee the existence of solution under different boundary conditions? Where, in each case, the definition of solution will be given.

In other words, we try to characterize and prove different properties of the range of the semilinear operator $S(u) := u'' - g(x, u)$, regarded as a continuous function from $H \subset H^2((a, b), \mathbb{R}^N)$ to $L^2((a, b), \mathbb{R}^N)$, where H is a closed subspace depending on the boundary conditions.

Firstly, we study resonant periodic problems that generalize, on the one hand, the forced pendulum equation and, on the other hand, the Landesman-Lazer conditions. For the variational case $S(u) = u'' - \nabla G(u)$ we give a characterization of the set $Im(S)$ and prove some of its topological properties.

Secondly, we consider the so-called radiation boundary conditions, namely

$$u'(0) = a_0 u(0), \;\; u'(1) = a_1 u(1),$$

with $a_0, a_1 > 0$. We obtain a generalized Hartman condition that ensures the existence of solution. In particular, if g is a superlinear function, we prove that S is onto. For this case, we study sufficient and necessary conditions for uniqueness or multiplicity of solutions. More accurate results are obtained for the scalar case $N = 1$, using variational and topological methods and Implicit Function Theorem.

Keywords: Nonlinear boundary value problems; Periodic solutions; Variational methods; Topological and monotonicity methods; Semilinear elliptic equations.

2010 MSC: 34B15, 34C25, 35A15, 35A16, 35J61.

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Contents

Introducción

El Análisis No lineal es un área en la Matemática que tiene un gran número de aplicaciones. En este trabajo se estudian sistemas no lineales de ecuaciones diferenciales de segundo orden. En particular, problemas de contorno de la forma:

$$Lu = Nu \qquad \text{en } (a,b) \subset \mathbb{R},$$

en donde L un operador diferencial lineal y N un operador no lineal. Trabajamos con el operador de segundo orden $L = u''$ y con no linealidades de la forma $Nu = g(x,u) + p(x)$. Dependiendo del contexto, fueron estudiadas diferentes condiciones de borde. Consideramos $p \in L^2$ y $g \in C([a,b] \times \mathbb{R}^N, \mathbb{R}^N)$.

El objetivo principal de este trabajo es, en primer lugar, generalizar y extender resultados previos a un sistema de ecuaciones, descripto en el Capítulo 3. Primero trabajamos con un sistema diferencial no lineal de segundo orden para el caso en el que g sea un gradiente y no dependa de x:

$$u'' + \nabla G(u) = p(x), \qquad x \in (0,T),$$

con $p \in L^2((0,T), \mathbb{R}^N)$, y condiciones de borde periódicas:

$$\left\{ \begin{array}{l} u(0) = u(T) \\ u'(0) = u'(T) \end{array} \right. .$$

Observemos que esta ecuación es una generalización de la ecuación del péndulo. Nuestro objetivo es estudiar la imagen del operador semilineal $S(u) = u'' - \nabla G(u)$, considerado como un operador continuo de $H^2((0,T), \mathbb{R}^N)$ a $L^2((0,T), \mathbb{R}^N)$. Descomponemos el espacio L^2 como la suma directa de $\mathbb{R}^N$ y las funciones de promedio cero, es decir,

$$\begin{array}{rcccc} L^2((0,T), \mathbb{R}^N) & = & \mathbb{R}^N & \oplus & \tilde{L}^2, \\ p & = & \overline{p} & + & \tilde{p}, \end{array}$$

donde $\tilde{L}^2 := \{v \in L^2((0,T),\mathbb{R}^N) : \overline{v} = 0\}$, $\overline{p} := \frac{1}{T}\int_0^T p(t)dt$ y $\tilde{p} = p - \overline{p}$. En este sentido, estudiamos el conjunto

$$\mathcal{I}(\tilde{p}) := \{\overline{p} \in \mathbb{R}^N : \overline{p} + \tilde{p} \in Im(S)\},$$

para $\tilde{p} \in \tilde{L}^2$ dada. Logramos probar varios resultados interesantes, como por ejemplo: hallamos condiciones para que $\overline{p} \in \mathcal{I}(\tilde{p})$ y para que un punto sea interior de $\mathcal{I}(\tilde{p})$, además vimos que bajo ciertas hipótesis se puede probar que $\mathcal{I}(\tilde{p}) = Im(\nabla G)$. Uno de los resultados más importantes que obtuvimos fue una generalización de un trabajo de Castro [20] para la ecuación del péndulo; más precisamente, probamos que si ∇G es periódica y consideramos $\mathcal{I}$ como una función de $\tilde{L}^2$ al conjunto de subconjuntos compactos de $\mathbb{R}^N$ (considerado con la topología Hausdorff), entonces es continua.

Estas ideas fueron plasmadas en [5] y serán discutidas en profundidad en el Capítulo 3.

Nuestro próximo paso es trabajar con el siguiente problema escalar:

$$u'' = g(x,u) + p(x), \qquad x \in (0,1),$$

donde $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ es continua y superlineal y $p \in L^2((0,1),\mathbb{R})$. Por simplicidad, suponemos que $g \in C^1$ respecto de u. Sin pérdida de generalidad, podemos suponer que $g(x,0) = 0$ para todo $x \in [0,1]$. Bajo condiciones de contorno de tipo radiación:

$$u'(0) = a_0 u(0), \quad u'(1) = a_1 u(1),$$

con $a_0, a_1 > 0$.

Esta ecuación es una generalización de un modelo de Painlevé II de electrodifusión de dos iones, donde $g(x,u) = \frac{1}{2}u^3 + (a+bx)u$ para a, b, p, a_0 y a_1 unas constantes específicas del problema. En un trabajo reciente [10] Amster, Kwong y Rogers probaron existencia y multiplicidad de soluciones utilizando métodos variacionales y el problema fue modelado numéricamente.

Logramos generalizar varios resultados de [10] en el Capítulo 4. A diferencia de ese capítulo, en el Capítulo 5, las herramientas principales que utilizamos para probar estos resultados fueron los métodos topológicos, en particular, trabajamos con super y subsoluciones, Teorema de la Función Implícita y método de shooting. Estos resultados forman parte de [7].

Se puede encontrar una explicación más profunda sobre estas condiciones de borde en el Capítulo 2

Finalmente, en el Capítulo 6, logramos generalizar los resultados anteriores en el contexto de un sistema de ecuaciones, es decir:

$$\begin{cases} u'' = g(x, u) \\ u'(0) = a_0 u(0), \ u'(1) = a_1 u(1), \end{cases}$$

donde $g : [0, 1] \times \mathbb{R}^N \to \mathbb{R}^N$ es una función continua y superlineal, i.e.,

$$\lim_{|u| \to +\infty} \frac{g(x, u).u}{|u|^2} = +\infty,$$

uniformemente en $x \in [0, 1]$, bajo las mismas condiciones de contorno.

Probamos existencia de solución a través de métodos variacionales, similares a los utilizados en el trabajo [10] mencionado anteriormente. Además, encontramos una condición generalizada de Hartman para probar existencia en un contexto no variacional.

Esta tesis está organizada de la siguiente manera:

En el próximo Capítulo, se presenta la matemática necesaria para entender por completo los resultados aquí presentados. Está dividido en una sección de preliminares analíticos y otra de preliminares topológicos. En la primera se enuncian resultados de inmersión de espacios de Sobolev junto con algunos otros resultados relevantes del análisis funcional. En la segunda, se repasan los teoremas de punto fijo, el método de super y subsoluciones y el de shooting, entre otros resultados importantes.

El Capítulo 2 es una breve historia de los dos principales problemas tratados en este trabajo: el estudio del rango de un operador semilineal y los problemas con condiciones de radiación. Aquí, las principales referencias son explicadas con más detalle y se presentan las dificultades principales de cada problema.

En el Capítulo 3 damos resultados para el problema periódico. En el Capítulo 4, aplicamos métodos variacionales para extender algunos resultados de [10], mientras que en el Capítulo 5 aplicamos métodos topológicos. El Capítulo 6 contiene una generalización a sistemas de los resultados obtenidos en el Capítulo 4.

Introduction

Nonlinear Analysis is an area of Mathematics that has a great number of applications. In particular, our objects of study are Boundary Value Problems (BVP) of the following type:

$$Lu = Nu,$$

where L is a Linear Differential Operator and N a nonlinear one. We will work only with Second Order Operators and our results will be for the case $L = u''$, with nonlinearities of the form $Nu = g(x, u) + p(x)$. Different boundary conditions are studied depending on the context. We will work with $p \in L^2((0,T), \mathbb{R}^N)$ and $g \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$.

The main goal of this work is, at first, to generalize and extend some previous results, described in Chapter 3. We work with a second order nonlinear ordinary differential system, where the nonlinearity g is a gradient depending only on u:

$$u'' + \nabla G(u) = p(x), \qquad x \in (0,T),$$

with $p \in L^2((0,T), \mathbb{R}^N)$ and Periodic Boundary Conditions:

$$\begin{cases} u(0) = u(T) \\ u'(0) = u'(T) \end{cases}.$$

We study the range of the semilinear operator $S(u) = u'' - \nabla G(u)$, regarded as a continuous function from $H^2((0,T), \mathbb{R}^N)$ to $L^2((0,T), \mathbb{R}^N)$. We decompose $L^2((0,T), \mathbb{R}^N)$ as the orthogonal sum of $\mathbb{R}^N$ and the set of zero-average functions, namely,

$$\begin{array}{rcccc} L^2((0,T), \mathbb{R}^N) & = & \mathbb{R}^N & \oplus & \tilde{L}^2, \\ p & = & \overline{p} & + & \tilde{p}, \end{array}$$

with $\tilde{L}^2 := \{v \in L^2((0,T), \mathbb{R}^N) : \overline{v} = 0\}$, $\overline{p} := \frac{1}{T}\int_0^T p(x)dx$ and $\tilde{p} = p - \overline{p}$. In this sense, we study the set

$$\mathcal{I}(\tilde{p}) := \{\overline{p} \in \mathbb{R}^N : \overline{p} + \tilde{p} \in \mathrm{Im}(S)\},$$

for a given $\tilde{p} \in \tilde{L}^2$. We are able to obtain several results, such as: a basic criterion which ensures that $\overline{p} \in \mathbb{R}^N$ belongs to $\mathcal{I}(\tilde{p})$ for some given $\tilde{p}$, sufficient conditions for a point $\overline{p}_0 \in \mathcal{I}(\tilde{p})$ to be interior and we prove that, if G is strictly convex and satisfies some accurate growth assumptions, then $\mathcal{I}(\tilde{p}) = Im(\nabla G)$ for all $\tilde{p} \in \tilde{L}^2$. One of the most important results is an extension of a well known result by Castro [20] for the pendulum equation; more precisely, we prove that if ∇G is periodic then $\mathcal{I}$ regarded as a function from $\tilde{L}^2$ to the set of compacts subsets of $\mathbb{R}^N$ (equipped with the Hausdorff metric) is continuous.

These results were proved in [5] and are thoroughly discussed in Chapter 3.

Our next step is to work with the following scalar problem:

$$u'' = g(x, u) + p(x), \qquad x \in (0, 1),$$

where $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and superlinear, i.e.,

$$\lim_{|u| \to +\infty} \frac{g(x, u)}{u} = +\infty,$$

uniformly in $x \in [0, 1]$, $g \in C^1$ with respect to u and $p \in L^2((0, 1), \mathbb{R})$. Without loss of generality we can assume that $g(x, 0) = 0$ for all $x \in [0, 1]$. Under the following Radiation Boundary Conditions:

$$u'(0) = a_0 u(0), \;\; u'(1) = a_1 u(1), \tag{1}$$

with $a_0, a_1 > 0$.

This is a generalization of a particular case of interest, where $g(x, u) = \frac{1}{2}u^3 + (a + bx)u$ for a, b, p, a_0 and a_1 some specific constants, it is a Painlevé II model in two-ion electrodiffusion. In a recent work [10], Amster, Kwong and Rogers applied variational methods in order to treat this class of two-ion BVPs, and the problem was modeled numerically. It is worth recording that the boundary conditions differ from the standard Robin-type conditions in the crucial fact that both a_0 and a_1 are positive.

We were able to generalize several results of [10] in Chapter 4. Unlike what was done in that chapter, in Chapter 5 our main tool to prove the results were Topological Methods, in particular, we worked with upper and lower solutions, Implicit Function Theorem and shooting method. These results are part of a submitted article [7].

A further explanation about the radiation boundary conditions can be found in Chapter 2.

Finally, in Chapter 6, we generalize the previous results in the context of a system of equations, i.e.,

$$\begin{cases} u'' = g(x, u) + p(x) \\ u'(0) = a_0 u(0), \; u'(1) = a_1 u(1), \end{cases}$$

where $g : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous superlinear function, namely

$$\lim_{|u|\to+\infty} \frac{g(x,u).u}{|u|^2} = +\infty,$$

uniformly in $x \in [0,1]$ and $p \in L^2$. Under radiation boundary conditions, as before.

We obtain an existence result via variational methods, similar to those used in [10], mentioned above. Moreover, we find a generalized Hartman condition to prove existence of solution in a nonvariational setting.

This thesis is organized as follows:

In the next Chapter, we give the mathematics needed to fully understand the results mentioned in this introduction. It is divided in a topological section, in which fixed point theorems, upper and lower solutions and shooting method, among other important results, are described; and an analytical section, where Sobolev spaces are revised and some functional analysis results are enumerated.

Chapter 2 is a brief history of the two main type of problems we worked with: Range of semilinear operators and Radiation Boundary Conditions. Here, the main references are described with more detail and the difficulties of the problems are presented.

The Chapter 3 is devoted to the periodic problem and the study of $\mathcal{I}(\tilde{p})$ mentioned before. In Chapter 4 we extend some of the results in [10] in a variational setting, while in Chapter 5 topological methods are applied. Chapter 6 is devoted to generalize the existence results obtained in Capter 4 to a system of equations.

Chapter 1

Preliminaries

This section is meant to present the mathematical background needed to appreciate and understand the concepts that will be used throughout the work.

We divide the preliminaries in two parts: an analytical one with classical results in Sobolev spaces and some Nonlinear Functional Analysis, and a topological one, where we give definitions and ideas from Topological Methods.

1.1 Analytical Preliminaries

1.1.1 Poincaré Inequalities

We have the classical Sobolev inequalities that give an answer to the embedding problems.

A particular case of this is the well-known Poincaré inequality:

Theorem 1.1.1 (Poincaré). *Assume $1 \leq p \leq \infty$, and $u \in W_0^{1,p}(a,b)$. Then there exists a constant $C = C(p)$ such that we have the estimate*

$$\|u\|_{L^p(a,b)} \leq C\|u'\|_{L^p(a,b)}.$$

Recall that the Sobolev space $W^{1,p}(a,b)$ is defined as

$$W^{1,p}(a,b) = \{u \in L^p(\Omega) : u, u' \in L^p(a,b)\},$$

and

$$W_0^{1,p}(a,b) := \left\{u \in W^{1,p}(a,b) : \exists \{u_m\}_{m=1}^{\infty} \subset C_c^{\infty}(a,b), \text{ such that } u_m \to u \text{ in } W^{1,p}(a,b)\right\}.$$

Let us recall the definition of the average:

Definition 1.1.2. We define the average of a function as:

$$\overline{u} := \frac{1}{b-a}\int_a^b u(x)dx.$$

Note that if the function is periodic, i.e. $u(x+T) = u(x)$ for all $x \in \mathbb{R}$, then the average is also defined as before.

Remark 1.1.3. The average is the orthogonal projection to the Kernel of the operator $Lu = -u''$ in $L^2(0,T)$ under Periodic Boundary Conditions.

Theorem 1.1.4. *Let us recall the Wirtinger Inequality:*

$$\|u - \overline{u}\|_{L^p(a,b)} \leq C\|u'\|_{L^p(a,b)}.$$

For more on this, see Brézis [18].

1.1.2 Functional analysis results

In this section we enumerate a series of results that we use freely in the rest of this work. We begin with the Mean-Value Theorem for Vector-Valued integrals:

Theorem 1.1.5. *If $\gamma \in C([0,T],\Omega)$, with $\Omega \subset \mathbb{R}^N$, then*

$$\overline{\gamma} = \frac{1}{T}\int_0^T \gamma(x)dx \in \overline{co(\Omega)},$$

where $co(\Omega)$ is the convex hull of Ω.

Definition 1.1.6. Given $A \in \mathbb{R}^N$, we define the *Convex Hull* of A as the smallest convex set that contains A. Formally, the convex hull may be defined as the intersection of all convex sets containing A or as the set of all convex combinations of points in A.

Here we also recall Fredholm's Alternative Theorem.

Theorem 1.1.7. *Let E be a Banach space and $T : E \to E$ a linear compact operator. Then for any $\lambda \neq 0$, we have*

1) *The equation $(T - \lambda I)v = 0$ has a nonzero solution.*

or

2) *The equation $(T-\lambda I)v = f$ has a unique solution v for any element f.*

In the second case, the solution v depends continuously on f.

The Fredholm alternative can be restated as follows: any $\lambda \neq 0$ which is not an eigenvalue of a compact operator is in the resolvent, i.e., $T - \lambda I$, has a continuous inverse. The basic special case is when E is finite-dimensional, in which case any non degenerate matrix is invertible.

In particular, a variant of Theorem 1.1.7 is the following: if $T - \lambda I : E \to E$ is continuous, linear and injective, then it is an isomorphism.

Let us adapt the latter observation to Radiation Boundary Conditions.

Theorem 1.1.8. *Let*

$$X = \left\{u \in H^2(0,1)/u'(0) = a_0 u(0), u'(1) = a_1 u(1)\right\},$$

for given a_0, $a_1 > 0$. Let $f \in C([0,1])$ and assume that the linear operator $L : X \to L^2(0,1)$ given by $L(u) = u'' - f(\cdot)u$ is a monomorphism. Then L is an isomorphism.

Proof:
Let u_0 u_1 be solutions of

$$u''(x) - f(x)u(x) = 0 \quad x \in (0,1), \tag{1.1}$$

such that

$$\begin{cases} u_0'(0) = a_0 u_0(0), \\ u_0'(1) \neq a_1 u_0(1) \end{cases} \text{and} \begin{cases} u_1'(0) \neq a_0 u_1(0), \\ u_1'(1) = a_1 u_1(1). \end{cases}$$

Since L is a monomorphism, $\{u_0, u_1\}$ is linearly independent.

Let $\varphi \in L^2(0,1)$ and

$$u(x) = \frac{1}{W}\left(u_0(x)\int_x^1 u_1(t)\varphi(t)dt + u_1(x)\int_0^x u_0(t)\varphi(t)dt\right),$$

where

$$W = \begin{vmatrix} u_0 & u_0' \\ u_1 & u_1' \end{vmatrix} = u_0 u_1' - u_1 u_0'.$$

Since u_0 and u_1 satisfy (1.1), it is easy to see that W is constant.
Claim: $u \in X$ and $u'' - f(x)u = \varphi$.

Indeed, $u' = \dfrac{1}{W}\left(u_0'\displaystyle\int_x^1 u_1\varphi dt + u_1'\int_0^x u_0\varphi dt\right)$. So

$$u'(0) = \frac{1}{W}u_0'(0)\int_0^1 u_1\varphi dt = a_0\frac{1}{W}u_0(0)\int_0^1 u_1\varphi dt = a_0 u(0).$$

In a similar way, we can prove $u'(1) = a_1 u(1)$.

On the other hand, since u_0 and u_1 satisfy (1.1), we have

$$\begin{aligned} u'' &= \frac{1}{W}\left(u_0''\int_x^1 u_1\varphi dt - u_0'u_1\varphi + u_1''\int_0^x u_0\varphi dt + u_1'u_0\varphi\right) \\ &= f\frac{1}{W}\left(u_0\int_x^1 u_1\varphi dt + u_1\int_0^x u_0\varphi dt\right) + \varphi\frac{1}{W}(u_1'u_0 - u_0'u_1) \\ &= fu + \varphi. \end{aligned}$$

Due to the Open Mapping Theorem, the result follows. □

The following uniqueness result has been proven by Lazer in [33] using a lemma on bilinear forms. Firstly, let us recall the definition:

Definition 1.1.9. A symmetric matrix $M \in \mathbb{R}^{N\times N}$ is said to be positive positive-semidefinite if $x^t \cdot M \cdot x \geq 0$ for every $x \in \mathbb{R}^N$. We write $M \geq 0$.

For $M, N \in \mathbb{R}^{N\times N}$ we write $M \geq N$ if $M - N \geq 0$.

Theorem 1.1.10. *Let Q be a real $N \times N$ symmetric matrix valued function with elements defined, continuous and 2π-periodic on the real line. Suppose there exist real constant symmetric $A, B \in \mathbb{R}^{N\times N}$ such that*

$$A \leq Q(x) \leq B, \quad x \in [0, 2\pi], \tag{1.2}$$

and such that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ denote the eigenvalues of A and B respectively and there exist integers $N_k \geq 0$, $k = 1, \ldots, N$, such that

$$N_k^2 < \lambda_k \leq \mu_k < (N_k + 1)^2. \tag{1.3}$$

Then, there exists no non-trivial $2\pi-$periodic solution of the vector differential equation

$$w'' + Q(x)w = 0. \tag{1.4}$$

One form of the Hahn–Banach theorem is known as the Geometric Hahn-Banach Theorem.

Theorem 1.1.11. *Let K be a convex set having a nonempty interior in a real normed vector space X. Suppose V is a linear variety in X containing no interior points of K. Then there is a closed hyperplane in X containing V but containing no interior points of K; i.e., there is an element $x^* \in X^*$ and a constant c such that $\langle v, x^*\rangle = c$ for all $v \in V$ and $\langle k, x^*\rangle < c$ for all $k \in int(K)$.*

1.1.3 Implicit Function Theorem

In this section we state a very important result, the Implicit Function Theorem. First we review the main facts of differentiation in Banach spaces.

Let X and Y be Banach spaces, and $U \subset X$ be an open subset of X. A function $f : U \to Y$ is called Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $A : X \to Y$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} = 0.$$

The limit here is meant in the usual sense of a limit of a function defined on a metric space, using X and Y as the two metric spaces, and the above expression as the function of argument h in X. If there exists such an operator A, it is unique, so we write $Df(x) = A$ and call it the (Fréchet) derivative of f at x. A function f that is Fréchet differentiable for any point of U is said to be C^1 if the function

$$Df : U \to \mathcal{L}(X, Y); \quad x \mapsto Df(x)$$

is continuous, where we denote with $\mathcal{L}(X, Y)$ the set of continuous linear functions from X to Y.

Theorem 1.1.12. *Let X, Y and Z be Banach spaces, and let U and V be open subsets of X and Y respectively. Let $F \in C^r(U \times V, Z)$, $r \geq 1$. Fix $(x_0, y_0) \in U \times V$ and assume $D_x F(x_0, y_0) \in \mathcal{L}(X, Z)$ is an isomorphism. Then there exists an open neighbourhood $U_1 \times V_1 \subset U \times V$ of (x_0, y_0) such that for each $y \in V_1$ there exists an unique $(\xi(y), y) \in U_1 \times V_1$ satisfying $F(\xi(y), y) = F(x_0, y_0)$ and $\xi(y_0) = x_0$. Furthermore, $\xi \in C^r(V_1, U_1)$.*

1.1.4 Resonant Problems

Here we give a short introduction to resonant problems. Let us consider the general nonlinear problem:

$$Lu = Nu,$$

where L is a differential operator and N is a nonlinear operator. Boundary conditions are also present, and they define the space where the operator is defined. For example the scalar problem

$$u'' = f(x, u, u') \qquad x \in (0, T),$$

with $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ a bounded function. If L is invertible in a suitable space then the problem is called *non-resonant*. A simple example of a non resonant problem is the previous equation under Dirichlet

Boundary Conditions, $u(0) = u(T) = 0$. In this case, $Lu = u''$ and $ker(L) = 0$. The problem reduces to a fixed point problem:

$$u = L^{-1}Nu$$

and fixed point theory can be applied directly.

If, on the other hand, L is not invertible, then the problem is called *resonant*. This is the case if in the previous example we consider Neumann, or periodic conditions, where $ker(L)$ is non trivial. If $Lu = -u''$, under periodic boundary conditions, the Kernel of L is the subspace of constant functions. This is a case of resonance in the first eigenvalue (in this case 0). This denomination comes from the following:

If we consider the eigenvalue problem

$$-u'' = \lambda u$$

with periodic conditions, then it is not hard to see that the eigenvalues are:

$$\lambda_k = \left(\frac{2k\pi}{T}\right)^2, \qquad k = 0, 1, \cdots .$$

The first eigenvalue is 0, and the associated eigenspace is the space of constant functions. For more of this see Amster [3], Chapter 2.

The pioneer work on resonant problems in the direction of our studies is from Landesman and Lazer [32]. They studied the following scalar problem: Let $\Omega \subset \mathbb{R}^N$ a bounded domain, we find a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} Lu + \alpha u + g(u) = h(x) & \text{in } \Omega \\ u = 0 \quad \partial\Omega, \end{cases} \tag{1.5}$$

where $L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a^{ij} \left(\frac{\partial}{\partial x_j}\right)$ is a second order, self adjoint, uniformly elliptic operator.

By a weak solution of (1.5) the authors mean an $H_0^1(\Omega)$ solution of

$$u = \alpha Tu + T[g(u) - h], \tag{1.6}$$

where $T : L^2(\Omega) \to L^2(\Omega)$ and Tf is the unique solution of the linear problem:

$$\begin{cases} Lu = -f & \text{in } \Omega \\ u = 0 \quad \partial\Omega. \end{cases} \tag{1.7}$$

The following result was proven:

Theorem 1.1.13. *Let $w \in H_0^1(\Omega)$, a non trivial solution ($w \neq 0$) of $u = \alpha T u$, that is, a weak solution of*

$$\begin{cases} Lu + \alpha u = 0 & \text{in } \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{1.8}$$

Assume that the space of solutions of $u = \alpha T u$ has dimension 1, i.e. every solution is of the form cw; that the limits

$$\lim_{s \to +\infty} g(s) = g_+, \qquad \lim_{s \to -\infty} g(s) = g_-$$

exist and are finite and that

$$g_- \leq g(s) \leq g_+ \qquad \forall s. \tag{1.9}$$

Define $\Omega^+ = \{x \in \Omega : w(x) > 0\}$, $\Omega^- = \{x \in \Omega : w(x) < 0\}$. The inequalities

$$g_- \int_{\Omega^+} |w| dx - g_+ \int_{\Omega^-} |w| dx \leq \langle h, w \rangle \leq g_+ \int_{\Omega^+} |w| dx - g_- \int_{\Omega^-} |w| dx, \tag{1.10}$$

are necessary and the strict inequalities are sufficient for the existence of a weak solution of the boundary value problem (1.5).

Moreover, if (1.9) is replaced by the slightly stronger condition:

$$g_- < g(s) < g_+ \qquad \forall s, \tag{1.11}$$

then the strict inequalities are both necessary and sufficient for the existence of at least one solution of the boundary value problem (1.5).

Thus, the following result, adapted from a theorem given by Nirenberg in [41] for elliptic systems, may be regarded as a natural extension of the Landesman-Lazer Theorem:

Theorem 1.1.14. *Let $g : \mathbb{R}^N \to \mathbb{R}^N$ continuous and bounded, assume the radial limits $g_v := \lim_{r \to +\infty} g(rv)$ exist uniformly respect to $v \in S^{N-1}$. Then the problem*

$$\begin{cases} u'' + g(u) = p(x) \\ u(0) = u(T), u'(0) = u'(T), \end{cases}$$

has at least one T-periodic solution if the following conditions hold:

1. *$g_v \neq \overline{p}$, for any $v \in S^{N-1}$.*
2. *$\deg_B(g - \overline{p}, B_R(0), 0) \neq 0$ for $R \gg 0$.*

Here, $\deg_B$ refers to the Brouwer degree, it will be introduced later in this chapter, in Section 1.2.6.

In Chapters 4, 5 and 6 we work with radiation boundary conditions

$$u'(0) = a_0 u(0), u'(1) = a_1 u(1), \quad \text{with } a_0, a_1 > 0.$$

Let us see the condition on a_0 and a_1 for resonance at 0. Consider the problem

$$\begin{cases} u'' = 0 \\ u'(0) = a_0 u(0), u'(1) = a_1 u(1). \end{cases} \tag{1.12}$$

The solutions are: $u(x) = mx + b$, with $m, b \in \mathbb{R}$ such that

$$\begin{cases} m = a_0 b \\ m = a_1(m + b). \end{cases}$$

Hence, problem (1.12) has nontrivial solution if and only if

$$\frac{a_0}{1 + a_0} \neq a_1. \tag{1.13}$$

Many of the results shall assume $a_1 \geq a_0$ which, in particular, is a non-resonance condition.

The difference between our conditions and the standard Robin conditions, namely

$$au(0) - bu'(0) = 0, \quad au(1) + bu'(1) = 0,$$

where a and b are non-zero constants, may be briefly shown as follows. From standard results (e.g. Leray-Schauder theorem, see Theorem 1.2.5 below), the existence of nontrivial solutions of

$$\begin{cases} u'' = \lambda(g(x, u) + p(x)) \\ u'(0) = a_0 u(0), u'(1) = a_1 u(1). \end{cases} \tag{1.14}$$

would be easily deduced if there exists a number M such that all the solutions of (1.14) with $\lambda \in [0, 1]$ satisfy $\|u\|_\infty < M$. When $a_1 < 0$, suppose for example that

$$u(x_0) = \max_{x \in [0,1]} u(x) = M > 0.$$

If $x_0 \in (0, 1)$, then $u''(x_0) = \lambda(g(x_0, M) + p(x_0)) > 0$, a contradiction. If $x_0 = 0$, the we deduce that $u'(0) > 0$, a contradiction. If $x_0 = 1$, $u'(1) = a_1 u(1) < 0$, again, a contradiction.

However, when $a_1 > 0$, even on the non-resonant case, there is no contradiction for $x_0 = 1$.

1.1.5 Variational methods

In [39], Mawhin and Willem give the following introduction to variational methods.

A real function φ of a real variable which is bounded below on the real line needs to have an infimum. If we call *minimizing sequence* for φ any sequence $(a_k)_k$ such that

$$\varphi(a_k) \to_{k\to\infty} \inf \varphi,$$

a necessary condition for a to be such that $\varphi(a) = \inf \varphi$ is that φ has a minimizing sequence which converges to a. Without suitable assumptions on φ this condition can not be sufficient.

Definition 1.1.15. Let X be a normed space. A *minimizing sequence* for a function $\varphi : X \to (-\infty, +\infty]$ is a sequence (u_k) such that

$$\varphi(u_k) \to_{k\to\infty} \inf \varphi.$$

A function $\varphi : X \to (-\infty, +\infty]$ is *lower semi-continuous*, l.s.c, (resp. *weakly lower semi-continuous*, w.l.s.c.) if

$$u_k \to u \quad \Rightarrow \quad \liminf \varphi(u_k) \geq \varphi(u)$$

$$(\text{resp. } u_k \rightharpoonup u \quad \Rightarrow \quad \liminf \varphi(u_k) \geq \varphi(u)).$$

Theorem 1.1.16. *If φ is w.l.s.c. on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

The existence of a bounded minimizing sequence will be in particular ensured when φ is *coercive*, i.e., such that

$$\varphi(u) \to +\infty \quad \text{if } \|u\| \to \infty.$$

Definition 1.1.17. A function $\varphi : X \to (-\infty, +\infty]$ is *convex* (resp. *strictly convex*) if

$$\varphi(sx + (1-s)y) \leq s\varphi(x) + (1-s)\varphi(y) \quad \text{for all } s \in (0,1),\ x, y \in X.$$

$$(\text{resp. } \varphi(sx+(1-s)y) < s\varphi(x)+(1-s)\varphi(y) \quad \text{for all } s \in (0,1),\ x, y \in X).$$

In view of Theorem 1.1.16, it is important to obtain sufficient conditions for weak lower semi-continuity. We shall obtain such a condition from the following result.

Theorem 1.1.18. *If X is a normed space and $\varphi : X \to (-\infty, +\infty]$ is l.s.c and convex, then φ is w.l.s.c.*

Remark 1.1.19. The latter result is useful to our purpose, since we work with functionals of the form

$$J(u) = \int_0^T \left(\frac{|u'(x)|^2}{2} + F(x, u(x)) \right) dx,$$

where te function F depends on the context. J is continuously differentiable and w.l.s.c. on $H^1(0, T)$ as the sum of a convex continuous function and of a weakly continuous one.

In Chapter 4 we shall make use of a linking theorem by Rabinowitz. Firstly, let us recall the following definitions.

Definition 1.1.20. Let X be a Banach space and $J \in C^1(X, \mathbb{R})$, then it is said that

- $(u_n) \subset X$ is a *Palais-Smale sequence* if there exists a constant $c > 0$ such that $|J(u_n)| \leq c$ for all $n \in \mathbb{N}$ and $DJ(u_n) \longrightarrow_{n \to +\infty} 0$,
- J satisfies the *Palais-Smale condition (PS)* if any Palais-Smale sequence has a convergent subsequence in X.

Theorem 1.1.21 (Rabinowitz, [44])**.** *Let X be a Banach space and $J \in C^1(X, \mathbb{R})$ satisfy (PS). Assume $X = X_1 \oplus X_2$, with dim$(X_1) < \infty$ and*

$$\max_{u \in X_1 : \|u\|_X = r} J(u) < \inf_{u \in X_2} J(u) =: \rho \tag{1.15}$$

for some $r > 0$. Then J has at least one critical point $u_1 \in X$ such that $J(u_1) \geq \rho$.

1.1.6 Legendre and Fenchel transforms

In [39], Mawhin and Willem give the following ideas about the Legendre and Fenchel transforms.

The *Legendre transform* F^* of a smooth function $F \in C^1(\mathbb{R}^N, \mathbb{R})$ is defined by the implicit formula

$$F^*(v) = v \cdot u - F(u),$$

$v = \nabla F(u)$ when $\nabla F(u)$ is invertible. It has the following property

$$\sum_{i=1}^N D_i F^*(v) dvi = dF^*(v) = \sum_{i=1}^N (v_i du_i + u_i dv_i - D_i F(u) du_i) = \sum_{i=1}^N u_i dv_i,$$

or,

$$u = \nabla F^*(v),$$

so that F^* is such that

$$(\nabla F)^{-1} = \nabla F^*.$$

Its geometrical meaning is the following: the tangent hyperplane to the graph of F with normal $[v, -1]$ is given by

$$\left\{[w, s] \in \mathbb{R}^{N+1} : s = w \cdot v - F^*(v)\right\}.$$

Thus, the graph of F can be described in a dual way, either as a set of points or as an envelope of tangent hyperplanes.

The *Fenchel trasform* extends the Legendre transform to not necessarily smooth convex functions by using affine minorants instead of tangent hyperplanes. To motivate the analytical definition of the Fenchel transform of F we can notice that, when F is convex, the function $\tilde{F}_v : u \mapsto v \cdot u - F(u)$ is concave and the definition of the Legendre transform just expresses that u is a critical point of $\tilde{F}_v$, and hence, the global maximum of $\tilde{F}_v$ is achieved at u. Consequently,

$$F^*(v) = \sup_{w \in \mathbb{R}^N} \{w \cdot v - F(w)\}$$

and the right-hand side member of this equality, which is defined as an element of $(-\infty, +\infty]$, without the smoothness and invertibility conditions required by the Legendre transform is, by definition, the Fenchel transform of the convex l.s.c. function F. The reciprocity property between ∇F and ∇F^*, which loses its meaning for a non-smooth convex F or a non-smoth F^*, can be recovered in terms of the subdifferential of a convex function G, i.e., a subset of $\mathbb{R}^N$ associated to G at u and which reduces to $\{\nabla G\}$ when G is differentiable at u.

1.2 Topological Preliminaries

1.2.1 Fixed Point Theorems

In this section, we give a brief enumeration of the most important fixed point theorems, which are the cornerstones of the Topological Methods for solving nonlinear problems.

The classical proof of existence and uniqueness of solution for an ordinary differential equation with initial conditions relies in the Picard method of successive approximation, in his PhD thesis (1917) Banach proved that Picard method was in fact a particular case of a much more general result. First we recall the following definition.

Definition 1.2.1. Let X, Y be two metric spaces, we say that $T : X \to Y$ is a contraction if there exists $\alpha < 1$ such that:

$$\forall\, x, y \in X, \quad d_Y(Tx, Ty) \leq \alpha d_X(x, y).$$

We state here the famous Banach Fixed Point Theorem:

Theorem 1.2.2 (Banach). *Let X be a complete metric space and let $T : X \to X$ a contraction. Then, T has a unique fixed point $\hat{x}$. Moreover, $\hat{x}$ can be calculated in an iterative way from the sequence $x_{n+1} = T(x_n)$, starting from any $x_0 \in X$.*

Another important Fixed Point Theorem is due to Brouwer:

Theorem 1.2.3 (Brouwer). *Let $B = B_1(0) \subset \mathbb{R}^N$ and $f \in C(\overline{B}, \overline{B})$. Then there exists $x \in \overline{B}$ such that $f(x) = x$.*

Although Theorem 1.2.3 is valid for any set homeomorphic to the unit ball $\overline{B} \subset \mathbb{R}^N$, Kakutani (1943) showed that it is not true for infinite dimensional spaces. Some additional hypothesis is needed for the operator T.

J. Schauder, around 1930, proved another Fixed Point Theorem, this time for infinite dimensional spaces:

Theorem 1.2.4 (Schauder). *Let $(E, \|\cdot\|)$ be a normed space and let C be a closed convex and bounded subset of E. If $T : C \to C$ is a continuous function such that $T(C)$ is relatively compact ($\overline{T(C)}$ is compact), then T has at least a fixed point.*

The last fixed point theorem in this enumeration is an extension of the previous one, and has important applications in nonlinear problems. It was stated and proved by Leray and Schauder in 1934. We give here a particular case, due to Schauder:

Theorem 1.2.5 (Leray-Schauder). *Let T be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \lambda T x \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then T has at least a fixed point in X.

1.2.2 Hausdorff metric

In Chapter 3 we we shall need, among other things, to measure the distance between compact subsets of $\mathbb{R}^N$. Therefore, we introduce the Hausdorff metric, which is defined on the space of nonempty closed bounded subsets of a metric space.

Let (X, d) be a metric space. Let $\mathcal{H}$ be the collection of all nonempty closed, bounded subsets of X. If A, $B \in \mathcal{H}$ define the *Hausdorff distance* between A and B by

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}.$$

This definition of the Hausdorff metric, while sometimes useful for symbolic manipulation, has a reformulation which is more visually appealing. Given $A \subset \mathcal{H}$, let the ε-expansion of A be the union of all ε-open balls around points in A. We denote it by A_ε; that is,

$$A_\varepsilon = \bigcup_{x\in A} B(x, \varepsilon).$$

Then $d_H(A, B)$ is defined as the smallest ε that allows the expansion of A to cover B and vice versa:

$$d_H(A, B) = \inf\{\varepsilon > 0/B \subset A_\varepsilon \text{ and } A \subset B_\varepsilon\}.$$

Proposition 1.2.6. *The above two definitions of the Hausdorff metric are equivalent.*

Proposition 1.2.7. *The function d is a metric on $\mathcal{H}$.*

1.2.3 Upper and lower solutions for radiation boundary conditions

In this section we extend the well known method of upper and lower solutions to radiation boundary conditions. Let us consider the problem

$$\left\{\begin{array}{l} u''(x) = g(x, u) + p(x), \\ u'(0) = a_0 u(0), \quad u'(1) = a_1 u(1), \end{array}\right. \tag{1.16}$$

where $g \in C([0, 1] \times \mathbb{R})$, $p \in L^2(0, 1)$ and $a_0, a_1 > 0$.

Let us adapt the definition of upper and lower solutions to our purpose:

A function $\alpha \in H^2([0, 1])$ is a *lower solution* of (1.16) if it satisfies

$$\left\{\begin{array}{l} \alpha'' \geq g(x, \alpha) + p \quad \text{a. e. } x \in (0, 1) \\ \alpha'(0) \geq a_0\alpha(0), \quad \alpha'(1) \leq a_1\alpha(1). \end{array}\right.$$

In the same way, a function $\beta \in H^2([0, 1])$ is an *upper solution* of (1.16) if it satisfies

$$\left\{\begin{array}{l} \beta'' \leq g(x, \beta) + p \quad \text{a. e. } x \in (0, 1) \\ \beta'(0) \leq a_0\beta(0), \quad \beta'(1) \geq a_1\beta(1). \end{array}\right.$$

Now we can prove the following original result. It is an adaptation of a classical theorem to radiation boundary conditions.

Theorem 1.2.8. *Assume there exist lower and upper solutions of (1.16) α and β, such that $\alpha \leq \beta$ on $[0,1]$. Let*

$$\mathcal{C} = \left\{ u \in H^2([0,1]) / \alpha \leq u \leq \beta, \text{ on } [0,1] \right\}.$$

Then, problem (1.16) has at least one classical solution in $\mathcal{C}$.

Proof:
Let us consider the following modified problem for $\lambda, \mu > 0$

$$\begin{cases} u'' - \lambda u = g(x, P(x,u)) - \lambda P(x,u) + p(x) \\ u'(0) - \mu u(0) = (a_0 - \mu) P(0, u(0)), \\ u'(1) + \mu u(1) = (a_1 + \mu) P(1, u(1)), \end{cases} \tag{1.17}$$

where P is the truncation function defined by

$$P(x,u) = \begin{cases} \alpha(x) & \text{if } \alpha(x) > u \\ u & \text{if } \alpha(x) \leq u \leq \beta(x) \\ \beta(x) & \text{if } \beta(x) < u. \end{cases}$$

Firstly, let us prove that problem (1.17) has at least one solution. Note that now we have classical Robin boundary conditions, and it is well-known that the following problem for $\mu > 0$

$$\begin{cases} u'' - \lambda u = \varphi(x) \\ u'(0) - \mu u(0) = C_0 \\ u'(1) + \mu u(1) = C_1 \end{cases}$$

has a unique solution $u \in H^2(0,1)$ for each $\varphi \in L^2$, C_0 and $C_1 \in \mathbb{R}$; moreover, the operator $L^2(0,1) \times \mathbb{R}^2 \to H^2$ given by $(\varphi, C_0, C_1) \mapsto u$ is continuous. Hence, the fixed point operator associated to equation (1.17) is compact.

Note that, since P is a bounded function, the right-hand side of (1.17) is bounded, so, by Schauder's Fixed Point Theorem (see [3] or Theorem 1.2.4 in this chapter) we deduce the existence of at least one solution u satisfying the Robin boundary conditions.

Now, let us prove that the solution u of (1.17) is also a solution of (1.16). Note that it is enough to prove that $\alpha(x) \leq u(x) \leq \beta(x)$ for all $x \in [0,1]$. Suppose by contradiction that there exists $x \in [0,1]$ such that $u(x) > \beta(x)$. Let $x_0 \in [0,1]$ be such that

$$\max_{x \in [0,1]} (u - \beta) = u(x_0) - \beta(x_0) > 0.$$

Then $P(x_0, u(x_0)) = \beta(x_0)$. If $x_0 \in (0,1)$, then there exists $\delta > 0$ such that $u > \beta$ in $(x_0, x_0 + \delta)$. By definition of upper solution, if $x \in (x_0, x_0 + \delta)$ then we have

$$u''(x) - \lambda u(x) = g(x, \beta(x)) + A(x) - \lambda \beta(x) \geq \beta''(x) - \lambda \beta(x).$$

Since $(u-\beta)'(x_0)=0$, integrating the previous inequality we get

$$(u-\beta)'(x) \geq \lambda \int_{x_0}^{x} (u-\beta)(t)dt > 0,$$

for $x \in (x_0, x_0+\delta)$. It is a contradiction since x_0 is a maximum.

If $x_0 = 0$, then $P(0,u(0)) = \beta(0)$. Since 0 is a maximum, $(u-\beta)'(0) \leq 0$. Then, due to the boundary condition satisfied by u and β in 0, we have

$$\begin{aligned} 0 &\leq \beta'(0) - u'(0) = \beta'(0) - a_0\beta(0) + \mu\beta(0) - \mu u(0) = \\ &= \beta'(0) - a_0\beta(0) + \mu(\beta(0) - u(0)) < \beta'(0) - a_0\beta(0) \leq 0, \end{aligned}$$

a contradiction.

Similarly, if $x_0 = 1$, then we find a contradiction using the boundary condition satisfied by u and β in 1.

This proves that $u(x) \leq \beta(x)$ for all $x \in [0,1]$. In a similar way, it is easy to see that $\alpha(x) \leq u(x)$. □

1.2.4 Shooting method

In this section, we shall describe a tool for the study of boundary value problems, usually known as the shooting method. In very general terms, the method can be summarized in two steps:

1. Solve an initial value problem with a free parameter λ.

2. Find an appropriate value of λ such that the obtained solution satisfies the desired boundary condition.

The task looks really simple, but it requires some qualitative analysis on the behavior of the solutions of the initial value problem, according to the variations in the parameter λ.

As an example, let us consider the following second-order equation with homogeneous Dirichlet conditions

$$\begin{cases} u''(x) = f(x, u(x)) \\ u(0) = u(1) = 0. \end{cases}$$

Let us suppose $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and locally Lipschitz with respect to u, which guarantees that for any $\lambda \in \mathbb{R}$ the initial value problem

$$\begin{cases} u''(x) = f(x, u(x)) \\ u(0) = 0, \quad u'(0) = \lambda \end{cases} \tag{1.18}$$

has a unique solution u_λ, defined in a maximal nontrivial interval $I_\lambda = [0, M)$, with $M = M(\lambda) \in (0, +\infty]$. In general, we cannot know if $M(\lambda) > 1$, although on the set $\{\lambda : M(\lambda) > 1\}$, the function $\lambda \mapsto u_\lambda(1)$ is continuous. This is due to the continuous dependence with respect to the initial values.

It is clear that we are looking for a value λ such that $u_\lambda(1)$ is well defined and $u_\lambda(1) = 0$. In other words, we are looking for a zero of the function T defined by

$$T(\lambda) := u_\lambda(1).$$

Due to the continuity mentioned before, it is enough to find an interval $\Lambda = [\lambda_*, \lambda^*]$ such that $T(\lambda)$ is well defined for all $\lambda \in \Lambda$ and, moreover, $T(\lambda_*) \leq 0 \leq T(\lambda^*)$ or vice versa.

For instance, if f is bounded, the solutions of (1.18) are defined in $[0, 1]$. Moreover, direct integration of the equation yields

$$u_\lambda'(x) = \lambda + \int_0^x f(s, u_\lambda(s))ds.$$

This says that, if $\lambda \geq \|f\|_\infty$, then

$$u_\lambda'(x) \geq \lambda - x\|f\|_\infty \geq 0$$

for $x \leq 1$. Thus u_λ is nondecreasing, and hence $T(\lambda) > 0$. In a similar way, if $\lambda < -\|f\|_\infty$, then $T(\lambda) < 0$. Which allows to deduce that $T(\lambda) = 0$ for some $\lambda \in [-\|f\|_\infty, \|f\|_\infty]$.

For more sophisticated examples, see e.g. [3], Chapter 1.

1.2.5 Hartman condition

In 1960, Hartman [29] showed that the second order system in $\mathbb{R}^N$ for a vector function $u : [0, 1] \to \mathbb{R}^N$

$$\begin{cases} u'' = f(x, u, u'), \\ u(0) = u_0, \quad u(1) = u_1, \end{cases}$$

with $f : [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ continuous, has at least one solution when f satisfies a Nagumo-like condition and

$$f(x, v, w) \cdot v + |w|^2 > 0 \quad \text{for all } (x, v, w) \in [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N, \tag{1.19}$$

with $|v| = R$, $v \cdot w = 0$, for some $R \geq |u_0|, |u_1|$.

In our case, since g does not depend on u', condition (1.19) reads

$$g(x, u).u > 0, \quad \text{for } |u| = R.$$

It is easy to see, for the case $g = g(x, u)$, that the result can be extended for any convex, bounded, open subset of $\mathbb{R}^N$, replacing u by an outer-pointing normal unit vector.

In this work, we extend Hartman condition in a different direction, which allows the convex set (in our case, a ball) to depend on the value of x.

1.2.6 The Topological Degree

Roughly speaking, the topological degree is an algebraic count of the zeros of a continuous function $f : \overline{U} \to E$ where U is an open and bounded subset of a Banach space E, and f does not vanish on ∂U. Let $E = \mathbb{R}^N$. A function $f \in C^1$ has 0 as a regular value, if the differential $Df(x) : \mathbb{R}^N \to \mathbb{R}^N$ is surjective for every $x \in f^{-1}(0)$. We define Brouwer degree of f as

$$\deg_B(f, U, 0) := \sum_{x \in f^{-1}(0) \cap U} \operatorname{sgn}(J_f(x)),$$

where J_f denotes the Jacobian of f, namely $J_f(x) = \det Df(x)$. This definition can be extended in an appropriate way for $f \in C$ with $f \neq 0$ on ∂U.

Further generalization for infinite dimensional spaces is given by the Leray–Schauder degree, which is defined for Fredholm operators $f : U \to E$ of the type $f = I - K$ with K compact. In particular, when the range of K is contained in a finite dimensional subspace $V \subset E$, the Leray–Schauder degree is defined by

$$\deg_{LS}(f, U, 0) := \deg_B(f|_V, U \cap V, 0).$$

More remarkable properties of the degree can be found for example in [36], however, in Chapter 6, we will only mention two of them:

1. If $\deg_{LS}(f, U, 0) \neq 0$, then f vanishes in U.
2. *Homotopy invariance:* if $F : \overline{U} \times [0, 1] \to E$ is continuous such that $I - F(\cdot, \lambda)$ is compact for all λ and $F(u, \lambda) \neq 0$ for $u \in \partial U$ and $\lambda \in [0, 1]$, then $\deg_{LS}(F(\cdot, \lambda), U, 0)$ does not depend on λ.

For a proof of the results stated in this section, refer to the books of Amster [3] or Teschl [46], where they give a more detailed analysis of this subject. The first appearance of this notion was in 1911 in a work from Brouwer [19].

Resumen del Capítulo 1

En este capítulo se presentan algunos resultados necesarios para comprender la tesis. La mayoría de los resultados son conocidos. Está dividido en dos secciones, una de preliminares analíticos y otra de preliminares topológicos.

En la Subsección 1.1.1 enunciamos las desigualdades de Poincaré y de Wirtinger, además, difinimos el promedio de una función.

La Subsección 1.1.2 está dedicada a recordar algunos resultados conocidos del Análisis Funcional. Entre ellos se encuentran: el teorema del valor medio, la Alternativa de Fredholm, un resultado de unicidad de Lazer para matrices periódicas y la versión geométrica del teorema de Hahn-Banach. Además, probamos que el operador lineal continuo e inyectivo $L : X \to L^2(0,1)$ dado por $L(u) = u'' - f(\cdot)u$, donde

$$X = \left\{ u \in H^2(0,1)/u'(0) = a_0 u(0), u'(1) = a_1 u(1) \right\},$$

es un isomorfismo.

En la Subsección 1.1.3, revisamos el concepto de diferenciación en espacios de Banach y enunciamos el Teorema de la Función Implícita en ese contexto.

La Subsección 1.1.4 contiene una introducción a los problemas resonantes. Se introducen las conocidas condiciones de Landesman-Lazer y una generalización de Nirenberg. Además, encontramos la condición de resonancia para el operador $L(u) = u''$ bajo condiciones de contorno de radiación.

La Subsección 1.1.5 está dedicada a los métodos variacionales. Contiene definiciones y teoremas clásicos, entre los que se encuentra un teorema linking de Rabinowitz.

En la Subsección 1.1.6 describimos, de manera informal, las transformadas de Fenchel y de Legendre.

En la Subsección 1.2.1, enunciamos los teoremas de punto fijo de Banach, Brouwer, Schauder y Leray-Schauder.

En la Subsección 1.2.2 introducimos dos definiciones equivalentes de la distancia de Hausdorff entre subconjuntos no vacíos, cerrados y acotados de un espacio métrico.

En la Subsección 1.2.3 probamos un resultado de existencia de solución para un problema de segundo orden con condiciones de contorno de radiación utilizando el método de super y subsoluciones.

En la Subsección 1.2.4 presentamos una breve descripción del método de shooting y un ejemplo.

La Subsección 1.2.5 contiene una descripción de la condición de Hartman estricta.

Finalmente, en la Subsección 1.2.6 presentamos una breve descripción y algunas propiedades del grado topológico de Brouwer y de Leray-Schauder.

Chapter 2

A brief survey of the problems

2.1 On the range of semilinear operators

In Chapter 3 we shall consider the following problem. For a vector function $u : \mathbb{R} \to \mathbb{R}^N$ we consider the system

$$\begin{aligned} u''(x) + \nabla G(u(x)) &= p(x) \\ u(x) &= u(x+T), \end{aligned}$$

where $G : \mathbb{R}^N \to \mathbb{R}$ is a C^1 function. We are interested in finding all possible T-periodic forcing terms $p(x)$ for which there is at least one solution. In other words, we shall examine the range of the semilinear operator $S : H^2_{\text{per}} \to L^2([0,T],\mathbb{R}^N)$ given by $Su = u'' + \nabla G(u)$, where

$$H^2_{\text{per}} = \{u \in H^2([0,T],\mathbb{R}^N); u(0) - u(T) = u'(0) - u'(T) = 0\}.$$

Writing $p(x) = \overline{p} + \widetilde{p}(x)$, where $\overline{p} := \frac{1}{T}\int_0^T p(x)\,dx$, we shall present several results concerning the topological structure of the set

$$\mathcal{I}(\widetilde{p}) = \{\overline{p} \in \mathbb{R}^N; \overline{p} + \widetilde{p} \in \text{Im}(S)\}.$$

2.1.1 Bounded ∇G

Let $G \in C^1(\mathbb{R}^N,\mathbb{R})$. A well known result establishes that if ∇G is bounded, then the Dirichlet problem

$$u'' + \nabla G(u) = p(x) \tag{2.1}$$

$$u(0) = u(T) = 0 \tag{2.2}$$

has at least one solution for any $p \in L^2([0,T],\mathbb{R}^N)$; that is to say, the operator $S : H^2 \cap H_0^1([0,T],\mathbb{R}^N) \to L^2([0,T],\mathbb{R}^N)$ given by

$$S(u) := u'' + \nabla G(u),$$

is surjective. The boundedness condition ensures that the nonlinearity does not interact with the spectrum of L, hence the associated linear operator $Lu := -u''$ is invertible; thus, a simple proof follows as a straightforward application of Schauder's fixed point theorem.

The situation is different at resonance, as mentioned in Section 1.1.4 in the previous chapter, when the associated linear operator is non-invertible. In particular, if we consider the periodic problem for (2.1), then integrating we have

$$\frac{1}{T}\int_0^T \nabla G(u(t))\,dt = \overline{p}.$$

Thus, the geometric version of the Hahn-Banach Theorem (Theorem 1.1.11 in the previous chapter), implies that a necessary condition for the existence of solutions is that $\overline{p} \in \mathrm{co}(\mathrm{Im}(\nabla G))$, where 'co' stands for the convex hull. In particular, if we decompose $L^2([0,T],\mathbb{R}^N)$ as the orthogonal sum of $\mathbb{R}^N$ and the set $\widetilde{L}^2$ of zero-average functions; i.e.,

$$\begin{array}{rcccc} L^2((0,T),\mathbb{R}^N) & = & \mathbb{R}^N & \oplus & \tilde{L}^2, \\ p & = & \overline{p} & + & \tilde{p}, \end{array}$$

with

$$\widetilde{L}^2 := \{v \in L^2([0,T],\mathbb{R}^N);\ \overline{v} = 0\},$$

then the range of S, now defined on H^2_{per}, is contained in $\mathrm{co}(\mathrm{Im}(\nabla G)) \oplus \widetilde{L}^2$. Thus, it is useful to study, for a given $\widetilde{p} \in \widetilde{L}^2$, the set

$$\mathcal{I}(\widetilde{p}) := \{\overline{p} \in \mathbb{R}^N : \overline{p} + \widetilde{p} \in \mathrm{Im}(S)\} \subset \mathrm{co}(\mathrm{Im}(\nabla G)).$$

When ∇G is bounded it can be proven, generalizing the arguments given in [28] for a scalar equation, that $\mathcal{I}(\widetilde{p})$ is non-empty and connected; if ∇G is also periodic, then $\mathcal{I}(\widetilde{p})$ is compact (see e.g. [30]). For example, a quite precise description of this set can be given when the radial limits

$$\lim_{s\to+\infty} \nabla G(sv) := \Gamma(v)$$

exist uniformly for $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. In this case, a well-known result by Nirenberg [41] (see Theorem 1.1.14 in the previous chapter) implies that all the interior points of the field $\Gamma : S^{N-1} \to \mathbb{R}^N$ (i. e. those points $\overline{p}$ such that the winding number of Γ with respect to $\overline{p}$ is nonzero) is contained in $\mathcal{I}(\widetilde{p})$. If also $\mathrm{co}(\mathrm{Im}(\nabla G)) \subset \mathrm{Int}(\Gamma)$, then the condition $Index(\Gamma, \overline{p}) \neq 0$ is both necessary and sufficient, indeed:

$$\mathrm{Im}(S) = \mathrm{Int}(\Gamma) \oplus \widetilde{L}^2.$$

2.2 Radiation boundary conditions

The following equation

$$u''(x) = Ku(x)^3 + L(x)u(x) + A, \tag{2.3}$$

where K and A are some given positive constants and $L(x) := a_0^2 + (a_1^2 - a_0^2)x$, of Painlevé II type was derived independently by Grafov and Chernenko [27] and Bass [15] in the context of two-ion electrodiffusion. Two-point Dirichlet and periodic boundary value problems (BVPs) for this Painlevé II equation and a non-integrable generalization were successively investigated in [11] and [12]. A Neumann BVP for a Painlevé II equation depending on the Dirichlet boundary values of the solution was recently studied in [8] and [9] by a two-dimensional shooting method.

In [17] novel flux quantization aspects associated with the iterative action of the Bäcklund transformations were investigated. Exact analytic expressions were obtained for the electric field and ionic concentrations in well-stirred reservoirs exterior to the junction boundaries. Radiation boundary conditions, namely

$$u'(0) = a_0u(0), \quad u'(1) = a_1u(1), \tag{2.4}$$

applied to two-point BVPs for the Painlevé II equation were derived in this connection. Unlike the standard Robin condition, the coefficients a_0 and a_1 in the radiation boundary condition (2.4) are assumed to be positive.

In a recent work, [10], the following results were proven via variational methods.

Theorem 2.2.1. *Problem (2.3)-(2.4) has exactly one negative solution. Moreover, there are at most two positive solutions, and the set of all solutions is bounded in the C^2-norm.*

Theorem 2.2.2.

1. *If $a_1 \geq a_0$, then (2.3)-(2.4) has a unique solution.*
2. *If $a_1 < a_0$, then there exist positive constants $A_* < A^*$ such that the following hold.*
   (a) *If $A < A_*$, then (2.3)-(2.4) admits at least three classical solutions.*
   (b) *If $A > A^*$, then (2.3)-(2.4) has a unique solution.*

Since a_0 and a_1 are positive, the associated functional J is coercive over a subspace $H \subset H^1(0,1)$ of codimension 1 and $-J$ is coercive over a linear complement of H. This geometry explains the nature of the results in [10], where it was proven that the functional is in fact coercive over the whole space and, in consequence, it achieves a minimum. The global minimum corresponds to the negative solution mentioned in Theorem 2.2.1. The multiplicity part (Theorem 2.2.2, case $2a$) was proven using a linking theorem by Rabinowitz (see Theorem 1.1.21).

In [10], some examples were presented using the shooting method; here, we shall demonstrate that this tool can be useful for getting theoretical results.

Resumen del Capítulo 2

Este capítulo contiene una breve historia de los dos principales problemas tratados en la tesis: el estudio del rango de un operador semilineal y los problemas con condiciones de contorno de radiación y un término superlineal. Las principales referencias son explicadas con más detalle y se presentan las dificultades principales de cada problema.

En la Sección 2.1 se plantea el problema, estudiar el rango del operador semilineal $S : H^2_{\text{per}} \to L^2([0,T],\mathbb{R}^N)$ dado por $Su = u'' + \nabla G(u)$, donde

$$H^2_{\text{per}} = \{u \in H^2([0,T],\mathbb{R}^N); u(0) - u(T) = u'(0) - u'(T) = 0\}.$$

En la Subsección 2.1.1 presentamos antecedentes del problema para ∇G acotada.

La Sección 2.2 está dedicada a describir brevemente el origen de las condiciones de contorno de radiación y los antecedentes de un modelo de Painlevé II de electrodifusión de dos iones.

Chapter 3

Range of semilinear operators for systems at resonance

3.1 Introduction

In this chapter we study the following problem: for a vector function $u : \mathbb{R} \to \mathbb{R}^N$ we consider the system

$$\begin{cases} u'' + \nabla G(u) = p(x), & x \in (0, T) \\ \quad u(0) = u(T), & u'(0) = u'(T), \end{cases} \tag{3.1}$$

where $G : \mathbb{R}^N \to \mathbb{R}$ is a C^1 function.

We study the range of the semilinear operator

$$S : H^2_{\text{per}} \to L^2([0, T], \mathbb{R}^N),$$

given by $Su = u'' + \nabla G(u)$, where

$$H^2_{\text{per}} = \{u \in H^2([0, T], \mathbb{R}^N); u(0) - u(T) = u'(0) - u'(T) = 0\}.$$

Writing $p(x) = \overline{p} + \widetilde{p}(x)$, where $\overline{p} := \frac{1}{T}\int_0^T p(x)\, dx$, we present several results concerning the topological structure of the set

$$\mathcal{I}(\widetilde{p}) = \{\overline{p} \in \mathbb{R}^N; \overline{p} + \widetilde{p} \in \text{Im}(S)\}.$$

This chapter is organized as follows. In the next section, we prove a basic criterion which ensures that $\overline{p} \in \mathbb{R}^N$ belongs to $\mathcal{I}(\widetilde{p})$ for some given $\widetilde{p}$. In Section 3.3, we give sufficient conditions for a point $\overline{p}_0 \in \mathcal{I}(\widetilde{p})$ to be interior. In Section 3.4, we extend a well known result by Castro [20] for the pendulum equation; more precisely, we prove that if ∇G is periodic, namely for every $j = 1, \ldots, N$ there exists $T_j > 0$ such that $\nabla G(u + T_j e_j) = \nabla G(u)$, then $\mathcal{I}$ regarded as a function from $\widetilde{L}^2$ to the set of compacts subsets of $\mathbb{R}^N$ (equipped with the Hausdorff metric) is continuous. Finally, in Section 3.5, we prove that if G is strictly convex and satisfies some accurate growth assumptions, then $\mathcal{I}(\widetilde{p}) = \mathrm{Im}(\nabla G)$ for all $\widetilde{p}$.

The results from this chapter were published in [5].

3.2 A basic criterion for general G

Proposition 3.2.1. *Given $\tilde{p} \in \tilde{L}^2$. Let $\overline{p} \in \mathbb{R}^N$ and define $\psi_{\overline{p}} : \mathbb{R}^N \to \mathbb{R}$ by $\psi_{\overline{p}}(u) := \overline{p} \cdot u - G(u)$. Assume that:*

1. *$\psi_{\overline{p}}$ is bounded from below,*
2. *$\liminf_{|u|\to+\infty} \psi_{\overline{p}}(u) > \inf_{u\in\mathbb{R}^N} \psi_{\overline{p}}(u) + \frac{T}{8\pi^2}\|\widetilde{p}\|^2_{L^2(0,T)}$.*

Then $\overline{p} \in \mathcal{I}(\widetilde{p})$.

Proof:
Consider the functional

$$J : H^1_{\mathrm{per}} := \{u \in H^1([0,T], \mathbb{R}^N) : u(0) = u(T)\} \to \mathbb{R}$$

given by

$$J(u) := \int_0^T \frac{|u'(x)|^2}{2} + \psi_{\overline{p}}(u(x)) + \widetilde{p}(x) \cdot u(x)\, dx.$$

It is readily seen that J is continuously Fréchet differentiable, and

$$DJ(u)(v) = \int_0^T u'(x) \cdot v'(x) - \nabla G(u(x)) \cdot v(x) + p(x) \cdot v(x)\, dx, \quad (3.2)$$

where $p(x) = \overline{p} + \tilde{p}(x)$. Thus, if u is a minimum of J, u is a weak solution of (3.1), and by standard arguments we deduce that it is a strong solution. Also, it is known, from Remark 1.1.19, that J is weakly lower semi-continuous; thus, due to Theorem 1.1.16 of Chapter 1, it suffices to prove that J has a bounded minimizing sequence. Without loss of generality, we may suppose that $G(0) = 0$.

Claim 1: $-\infty < \inf J \leq T \inf \psi_{\overline{p}} \leq 0$.

Indeed, let us recall from Chapter 1, 1.1.4, the well known Wirtinger inequality:

$$\|u - \overline{u}\|^2_{L^2} \leq \left(\frac{T}{2\pi}\right)^2 \|u'\|^2_{L^2}. \quad (3.3)$$

From Cauchy-Schwarz inequality we deduce:

$$J(u) \geq \frac{1}{2}\|u'\|_{L^2}^2 - \|\widetilde{p}\|_{L^2}\|u - \overline{u}\|_{L^2} + \int_0^T \psi_{\overline{p}}(u(x))\,dx.$$

Thus, from (3.3),

$$J(u) \geq \frac{1}{2}\left(\|u'\|_{L^2} - \frac{T}{2\pi}\|\widetilde{p}\|_{L^2}\right)^2 - \frac{T^2}{8\pi^2}\|\widetilde{p}\|_{L^2}^2 + T \inf_{u\in\mathbb{R}^N} \psi_{\overline{p}} \tag{3.4}$$

and the first inequality is proven. For the second inequality, it is sufficient to observe that

$$\inf_{u\in H^1_{\mathrm{per}}} J(u) \leq \inf_{u\in\mathbb{R}^N} J(u) = T \inf_{u\in\mathbb{R}^N} \psi_{\overline{p}}(u).$$

The third inequality is obvious since $\psi_{\overline{p}}(0) = 0$.

Next, consider a sequence $(u_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} J(u_n) = \inf J$.

<u>Claim 2:</u> The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1_{per}.

From the previous claim, for any given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$J(u_n) < T \inf \psi_{\overline{p}} + \varepsilon, \quad \text{for all } n \geq n_0. \tag{3.5}$$

Then (3.4) yields

$$\left(\|u_n'\|_{L^2} - \frac{T}{2\pi}\|\widetilde{p}\|_{L^2}\right)^2 < \frac{T^2}{4\pi^2}\|\widetilde{p}\|_{L^2}^2 + 2\varepsilon$$

so

$$\|u_n'\|_{L^2}^2 < \frac{T}{\pi}\|\widetilde{p}\|_{L^2}\|u_n'\|_{L^2} + 2\varepsilon.$$

Hence, there exists $\tau > 0$, independent of n, such that $\|u_n'\|_{L^2} \leq \frac{T}{\pi}\|\widetilde{p}\|_{L^2} + \tau$.

As before,

$$J(u_n) \geq \frac{1}{2}\left(\|u_n'\|_{L^2} - \frac{T}{2\pi}\|\widetilde{p}\|_{L^2}\right)^2 - \frac{T^2}{8\pi^2}\|\widetilde{p}\|_{L^2}^2 + \int_0^T \psi_{\overline{p}}(u_n(x))\,dx,$$

and from (3.5) we deduce that

$$\int_0^T \psi_{\overline{p}}(u_n(x))\,dx \leq \frac{T^2}{8\pi^2}\|\widetilde{p}\|_{L^2}^2 + T \inf \psi_{\overline{p}} + \varepsilon. \tag{3.6}$$

Suppose that $\|u_n\|_{H^1} \to \infty$, then from the bound for $\|u_n'\|_{L^2}$ and the standard inequality

$$\|u_n - \overline{u}_n\|_\infty \leq \frac{\sqrt{T}}{2}\|u_n'\|_{L^2}$$

we deduce that $|\overline{u}_n| \to \infty$ and $|u_n(x)| \to \infty$ uniformly in x. Thus, for a given $\delta > 0$ there exists $n_1 \geq n_0$ such that $\psi_{\overline{p}}(u_n(x)) \geq \liminf_{|u|\to\infty} \psi_{\overline{p}}(u) - \frac{\delta}{T}$ for all $n \geq n_1$. Hence

$$\int_0^T \psi_{\overline{p}}(u_n(x))\, dx \geq T \liminf_{|u|\to\infty} \psi_{\overline{p}}(u) - \delta \quad \text{for all } n \geq n_1.$$

Then, by (3.6)

$$T \liminf_{|u|\to\infty} \psi_{\overline{p}}(u) \leq T \inf \psi_{\overline{p}} + \frac{T^2}{8\pi^2} \|\widetilde{p}\|_{L^2}^2 + \varepsilon + \delta, \tag{3.7}$$

which contradicts hypothesis 2 when $\varepsilon + \delta$ is small enough. So $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1_{per}. $\square$

Remark 3.2.2. In particular, if

$$\liminf_{|u|\to+\infty} \psi_{\overline{p}}(u) - \inf \psi_{\overline{p}} \geq r > 0,$$

then $\overline{p} \oplus \tilde{B}_r(0) \subset \operatorname{Im}(S)$, where $\tilde{B}_r(0) \subset \widetilde{L}^2$ denotes the open ball of radius r centered at 0.

Example 3.2.3. Suppose that

$$\limsup_{|u|\to\infty} \frac{G(u)}{|u|} \leq -R < 0.$$

Then $B_R(0) \subseteq \mathcal{I}(\widetilde{p})$ for any $\widetilde{p}$.

Indeed, if $|\overline{p}| < R$ let $\varepsilon = \frac{R-|\overline{p}|}{2}$ and fix r_0 such that $\frac{G(u)}{|u|} < -R + \varepsilon$ for $|u| \geq r_0$. Hence

$$\psi_{\overline{p}}(u) = |u| \left(\frac{u}{|u|} \cdot \overline{p} - \frac{G(u)}{|u|} \right) > |u|(R - \varepsilon - |\overline{p}|) = \varepsilon |u| \to +\infty$$

as $|u| \to \infty$ and the result follows. Proposition 3.2.1 is still applicable for example if

$$\limsup_{|u|\to\infty} \frac{G(u)}{|u|} \leq 0 \quad \text{and} \quad \limsup_{|u|\to\infty, u\in\mathcal{C}} \frac{G(u)}{|u|} = -R < 0$$

with

$$\mathcal{C} := \{u \in \mathbb{R}^N : u \cdot w > -c|u|\}$$

for some $w \in S^{N-1}$ and $c \in (0,1)$. In this case, $\mathcal{I}(\widetilde{p})$ contains all the vectors $\overline{p} \in B_R(0)$ such that the angle between $\overline{p}$ and $-w$ is smaller than $\frac{\pi}{2} - \arccos(c)$.

3.3 Interior points

In this section we give sufficient conditions for a point $\overline{p}_0 \in \mathcal{I}(\widetilde{p})$ to be interior. Roughly speaking, we shall prove that if the Hessian matrix of G does not interact with the spectrum of the operator $Lu := -u''$, then $\mathcal{I}(\widetilde{p})$ is a neighborhood of $\overline{p}_0$. More precisely:

Theorem 3.3.1. *Let us assume that $G \in C^2(\mathbb{R}^N, \mathbb{R})$ and let $\overline{p}_0 \in \mathcal{I}(\widetilde{p})$ for some $\widetilde{p} \in \widetilde{L}^2$. Further, let u_0 be a solution of (3.1) for $\overline{p} = \overline{p}_0$ and assume there exist symmetric matrices A, $B \in \mathbb{R}^{N\times N}$ such that*

$$A \leq d^2G(u_0(x)) \leq B \quad x \in [0, T]$$

and

$$\left(\frac{2\pi N_k}{T}\right)^2 < \lambda_k \leq \mu_k < \left(\frac{2\pi(N_k+1)}{T}\right)^2$$

for some integers $N_k \geq 0$, $k = 1, \ldots, N$, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ are the respective eigenvalues of A and B. Then there exists an open set $U \subset \mathbb{R}^N$ such that $\overline{p}_0 \in U \subseteq \mathcal{I}(\widetilde{p})$.

Proof:
Let us consider the operator

$$F : H^2_{\text{per}} \times \mathbb{R}^N \to L^2,$$
$$(u, \overline{p}) \mapsto u'' + \nabla G(u) - \widetilde{p} - \overline{p},$$

then clearly $F(u_0, \overline{p}_0) = 0$.

On the other hand, F is Fréchet differentiable, and its differential with respect to u at $(u_0, \overline{p}_0)$ is computed by

$$\begin{aligned} D_uF(u_0,\overline{p}_0)(\varphi) &= \lim_{t\to 0} \frac{F(u_0 + t\varphi, \overline{p}_0) - F(u_0, \overline{p}_0)}{t} \\ &= \lim_{t\to 0} \frac{t\varphi'' + \nabla G(u_0 + t\varphi) - \nabla G(u_0)}{t} \\ &= \varphi'' + \lim_{t\to 0} \frac{\nabla G(u_0 + t\varphi) - \nabla G(u_0)}{t} \\ &= \varphi'' + d^2G(u_0)\varphi \end{aligned}$$

Taking $Q(x) = d^2G(u_0(x))$ in Theorem 1.1.10, we deduce that $D_uF(u_0, \overline{p}_0) : H^2_{\text{per}} \to L^2$ is a monomorphism; furthermore, it is easy to see that it is a Frendholm's operator of index 0, thus from the Fredholm's Alternative (see e. g. [22] or Theorem 1.1.7 on Chapter 1) we conclude that $D_uF(u_0, \overline{p}_0)$ is an isomorphism.

By the Implicit Function Theorem (see Theorem 1.1.12), there exists an open neighborhood U of $\overline{p}_0$ and a unique function $u : U \to H^2_{\rm per}$ such that

$$F(u(\overline{p}), \overline{p}) = 0, \quad \text{for all } \overline{p} \in U \text{ and } u(\overline{p}_0) = u_0.$$

Thus $U \subset \mathcal{I}(\widetilde{p})$ and the proof is complete. $\square$

A simple computation shows that a similar result is obtained when $d^2G(u_0(x))$ lies at the left of the first eigenvalue.

Theorem 3.3.2. *Let us assume that $G \in C^2(\mathbb{R}^N, \mathbb{R})$ and let $\overline{p}_0 \in \mathcal{I}(\widetilde{p})$ for some $\widetilde{p} \in \widetilde{L}^2$. Further, let u_0 be a solution of (3.1) for $\overline{p} = \overline{p}_0$ and assume*

1. *$d^2G(u_0(x)) \leq 0$ for all x.*
2. *There exists $A \subset [0, T]$ with $meas(A) > 0$ such that $d^2G(u_0(x)) < 0$ for $x \in A$.*

Then there exists an open set $U \subset \mathbb{R}^N$ such that $\overline{p}_0 \in U \subseteq \mathcal{I}(\widetilde{p})$.

Proof:
As in the proof of the previous theorem, it suffices to prove that $L\varphi := \varphi'' + d^2G(u_0)\varphi$ is a monomorphism.

Suppose that $L\varphi = 0$, then

$$0 = -\int_0^T L\varphi(x)\cdot\varphi(x)\, dx = \int_0^T |\varphi'(x)|^2\, dx - \int_0^T d^2G(u_0(x))\varphi(x)\cdot\varphi(x)\, dx,$$

Then

$$\begin{aligned}\int_0^T |\varphi'(x)|^2\, dx &= \int_0^T d^2G(u_0(x))\varphi(x)\cdot\varphi(x)\, dx \\ &\leq \int_A d^2G(u_0(x))\varphi(x)\cdot\varphi(x)\, dx,\end{aligned}$$

and we conclude that $\varphi \equiv 0$. $\square$

The following corollary is immediate.

Corollary 3.3.3. *Let $\widetilde{p} \in \widetilde{L}^2$ and assume that*

$$d^2G(u) < 0 \ \textit{for all} \ u \in \mathbb{R}^N$$

or that

$$A \leq d^2G(u) \leq B \ \textit{for all} \ u \in \mathbb{R}^N$$

with A and B as in Theorem 3.3.1. Then $\mathcal{I}(\widetilde{p})$ is open.

3.4 Continuity of the function $\mathcal{I}$

In this section we assume that ∇G is periodic, as in Chapter 2, and give a characterization of the set $\mathcal{I}(\widetilde{p})$ which, in particular, will allow us to prove the continuity of the function $\mathcal{I} : \widetilde{L} \to \mathcal{K}(\mathbb{R}^N)$, where $\mathcal{K}(\mathbb{R}^N)$ denotes the set of compact subsets of $\mathbb{R}^N$ equipped with the Hausdorff metric. In fact, we prove a little more.

Theorem 3.4.1. *Assume that $G \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies:*

1. *∇G is periodic, that is: for every $j = 1, \ldots, N$ there exists $T_j > 0$ such that $\nabla G(u + T_j e_j) = \nabla G(u)$.*
2. *There exists a discrete set $S \subset \mathbb{R}^N$ such that*

$$(\nabla G(u) - \nabla G(v)) \cdot (u - v) < \left(\frac{2\pi}{T}\right)^2 \|u - v\|_{L^2}^2 \quad \text{for } u, v \in \mathbb{R}^N \backslash S. \tag{3.8}$$

Then for every $\widetilde{p} \in \widetilde{L}^2$ there exists a periodic function $F_{\widetilde{p}} \in C(\mathbb{R}^N, \mathbb{R}^N)$ such that $\mathcal{I}(\widetilde{p}) = \mathrm{Im}(F_{\widetilde{p}})$. Furthermore, if $\widetilde{p}_n \to \widetilde{p}$ weakly in $\widetilde{L}^2$, then $\mathcal{I}(\widetilde{p}_n) \to \mathcal{I}(\widetilde{p})$ for the Hausdorff metric.

Remark 3.4.2. In particular, since it is the range of a continuous periodic function, it follows that $\mathcal{I}(\widetilde{p})$ is compact and arcwise connected. As mentioned in Chapter 2, integrating (3.1) we have

$$\frac{1}{T} \int_0^T \nabla G(u(x))\, dx = \overline{p}.$$

Thus, the geometric version of the Hahn-Banach Theorem (Theorem 1.1.11 in Chapter 1), implies that a necessary condition for the existence of solutions is that $\overline{p} \in \mathrm{co}(\mathrm{Im}(\nabla G))$. Hence $\mathcal{I}(\widetilde{p}) \subset \mathrm{co}(\mathrm{Im}(\nabla G))$.

For convenience, let us consider the decomposition $H^1_{\mathrm{per}} = \mathbb{R}^N \oplus \tilde{H}^1_{\mathrm{per}}$, where $\tilde{H}^1_{\mathrm{per}} = H^1_{\mathrm{per}} \cap \widetilde{L}^2$, and denote the functional defined in Section 3.2 by $J_p : H^1_{\mathrm{per}} \to \mathbb{R}$,

$$J_p(u) = \int_0^T \frac{|u'(x)|^2}{2} - G(u(x)) + p(x) \cdot u(x) dx.$$

The proof of Theorem 3.4.1 shall be based on a series of lemmas. From now on, we shall assume that $G \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies hypotheses 1. and 2..

Lemma 3.4.3. *For each $z \in \mathbb{R}^N$ and $p \in L^2([0,T],\mathbb{R}^N)$, there exists a unique $\phi(z,p) \in \tilde{H}^1_{\text{per}}$ such that*

$$DJ_p(z+\phi(z,p))(v) = 0 \quad \textit{for all } v \in \tilde{H}^1_{\text{per}}. \tag{3.9}$$

Moreover, if $z_n \to z$ in $\mathbb{R}^N$, then $\phi(z_n,p) \to \phi(z,p)$ and if $\tilde{p}_n \to \tilde{p}$ weakly in $\tilde{L}^2$, then $\phi(z,\tilde{p}_n) \to \phi(z,\tilde{p})$.

Proof:
Let us first prove the uniqueness of $\phi(z,p)$. Suppose $u_1,\ u_2 \in \tilde{H}^1_{\text{per}}$ are such that

$$DJ_p(z+u_1)(v) = 0 = DJ_p(z+u_2)(v) \quad \text{for all } v \in \tilde{H}^1_{\text{per}}.$$

Taking $v = u_1 - u_2$, using (3.2) it follows that

$$\int_0^T |(u_1-u_2)'|^2\,dx = \int_0^T (\nabla G(z+u_1)-\nabla G(z+u_2))\cdot(u_1-u_2)\,dx. \tag{3.10}$$

This fact, (3.3) and (3.8) imply that $u_1 = u_2$.

Next we prove the existence of $\phi(z,p)$. Let $I_z : \tilde{H}^1_{\text{per}} \to \mathbb{R}$ be the functional defined by $I_z(v) = J_p(z+v)$, then I_z is weakly lower semicontinuous and

$$\begin{aligned} I_z(v) &= \frac{1}{2}\|v'\|^2_{L^2} + \int_0^T p(x)\cdot(z+v(x)) - G(z+v(x))\,dx \\ &\geq \frac{1}{2}\|v'\|^2_{L^2} - \|\widetilde{p}\|_{L^2}\|v\|_{L^2} + T(\overline{p}\cdot z - \|G\|_\infty). \end{aligned} \tag{3.11}$$

It follows that I_z is coercive , hence, by Theorem 1.1.16, it achieves an absolute minimum, which satisfies (3.9).

Finally, let $z_n \to z$ and suppose that $\phi(z_n,p) \not\to \phi(z,p)$. From (3.11), the sequence $(\phi(z_n,p))_n$ is bounded in $\tilde{H}^1_{\text{per}}$. Taking a subsequence, if necessary, we may assume that it converges weakly to some $w \in H^1_{\text{per}}$, uniformly and $\|\phi(z_n,p)-\phi(z,p)\|_{H^1} \geq \varepsilon > 0$ for all n. Passing to the limit in the equalities

$$DJ_p(z_n+\phi(z_n,p))(v) = 0 \quad \text{for all } v \in \tilde{H}^1_{\text{per}}$$

we deduce that $DJ_p(z+w)(v) = 0$ for all $v \in \tilde{H}^1_{\text{per}}$ and hence $w = \phi(z,p)$. Moreover, as

$$J_p(z_n+\phi(z_n,p)) \leq J_p(z_n+\phi(z,p)) \quad \text{and} \quad J_p(z+\phi(z,p)) \leq J_p(z+\phi(z_n,p))$$

for all n, we deduce that

$$\limsup_{n\to\infty}\int_0^T |\phi(z_n,p)'|^2\,dx \le \int_0^T |\phi(z,p)'|^2\,dx \le \liminf_{n\to\infty}\int_0^T |\phi(z_n,p)'|^2\,dx$$

and hence $\|\phi(z_n,p)'\|_{L^2} \to \|\phi(z,p)'\|_{L^2}$. Thus,

$$\begin{aligned}&\|\phi(z_n,p)' - \phi(z,p)'\|_{L^2}^2 =\\ &= \|\phi(z_n,p)'\|_{L^2}^2 + \|\phi(z,p)'\|_{L^2}^2 - 2\int_0^T \phi(z_n,p)'\cdot\phi(z,p)'\,dx \to 0\end{aligned}$$

as $n\to\infty$, which contradicts the fact that $\phi(z_n,p)\not\to\phi(z,p)$.

In a similar way, we can see that, if $\tilde{p}_n \to \tilde{p}$ weakly in $\tilde{L}^2$, then $\phi(z,\tilde{p}_n)\to\phi(z,\tilde{p})$ when $n\to\infty$. □

Lemma 3.4.4. *The function $\phi(\cdot,p)$ depends only on $\widetilde{p}$.*

Proof:
Let $c\in\mathbb{R}^N$, then

$$\begin{aligned}&DJ_{p+c}(z+\phi(z,p))(v) =\\ &= \int_0^T \phi(z,p)'\cdot v' - \nabla G(z+\phi(z,p))\cdot v + (p+c)\cdot v\,dx\\ &= \int_0^T \phi(z,p)'\cdot v' - \nabla G(z+\phi(z,p))\cdot v + p\cdot v\,dx = 0\end{aligned}$$

for all $v\in\tilde{H}^1_{\text{per}}$. From uniqueness, we deduce that $\phi(\cdot,p)=\phi(\cdot,p+c)$. □

The following lemma will allow us to reduce the problem of finding a critical point in H^1_{per} to a finite-dimensional problem.

Lemma 3.4.5. *The element $z+v\in\mathbb{R}^N\oplus\tilde{H}^1_{\text{per}}$ is a critical point of J_p if and only if $v=\phi(z,p)$ and $DJ_p(z+\phi(z,p))(y+v)=0$ for all $y\in\mathbb{R}^N$.*

Proof:
By Lemma 3.4.3, if $z+v$ is a critical point of J_p, then $v=\phi(z,p)$. $DJ_p(z+\phi(z,p))(y+v)=0$ for every $y\in\mathbb{R}^N$ and hence z is a critical point of $J_p(\cdot+\phi(\cdot,p))$.

Conversely, suppose $v=\phi(z,p)$ and $DJ_p(z+\phi(z,p))(y+v)=0$ for all $y\in\mathbb{R}^N$. For $u\in H^1_{\text{per}}$, let us write $u=\overline{u}+\tilde{u}$ with $\overline{u}\in\mathbb{R}^N$ and $\tilde{u}\in\tilde{H}^1_{\text{per}}$. Then

$$DJ_p(z+v)(u) = DJ_p(z+\phi(z,p))(\overline{u}+\tilde{u}) = 0,$$

so $z+v$ is a critical point of J_p. □

Lemma 3.4.6. *The function $\phi(\cdot, p)$ is periodic.*

Proof:
Let $z \in \mathbb{R}^N$. From the periodicity of ∇G we deduce that

$$DJ_p(z + T_j e_j + \phi(z,p))(v) = DJ_p(z + \phi(z,p))(v) = 0$$

for all $v \in \tilde{H}^1_{\text{per}}$. By Lemma 3.4.3, $\phi(z + T_j e_j, p) = \phi(z,p)$. $\square$

The following proposition will provide a proof of Theorem 3.4.1.

Proposition 3.4.7. *Let $\widetilde{p} \in \widetilde{L}^2$ and define the function $F_{\widetilde{p}} : \mathbb{R}^N \to \mathbb{R}^N$ by*

$$F_{\widetilde{p}}(z) = \int_0^T \nabla G(z + \phi(z, \widetilde{p})(x))\, dx.$$

Then $F_{\widetilde{p}}$ is continuous and $\mathcal{I}(\widetilde{p}) = \mathrm{Im}(F_{\widetilde{p}})$. Moreover, if $\widetilde{p}^n \in \widetilde{L}^2$ converges weakly to some $\widetilde{p} \in \widetilde{L}^2$, then $\mathcal{I}(\widetilde{p}^n)$ converges to $\mathcal{I}(\widetilde{p})$ for the Hausdorff topology.

Proof:
The continuity of $F_{\widetilde{p}}$ is clear from the continuity of $\phi(\cdot, \widetilde{p})$ and the embedding $\tilde{H}^1_{\text{per}} \hookrightarrow C([0,T], \mathbb{R}^N)$. Let us prove that $\mathcal{I}(\widetilde{p}) = \mathrm{Im}(F_{\widetilde{p}})$. According to Lemma 3.4.4, problem (3.1) has a weak solution if and only if there exists $z \in \mathbb{R}^N$ such that $DJ_p(z + \phi(z, \widetilde{p}))(y) = 0$ for every $y \in \mathbb{R}^N$. From the definition of J_p, this is equivalent to

$$0 = \int_0^T -\nabla G(z + \phi(z, \widetilde{p})) \cdot y + p \cdot y \, dx = y \cdot \int_0^T -\nabla G(z + \phi(z, \widetilde{p})) + \overline{p}\, dx$$

for all $y \in \mathbb{R}^N$. Thus, the problem has a solution if and only if

$$\overline{p} = \frac{1}{T} \int_0^T \nabla G(z + \phi(z, \widetilde{p}))\, dx,$$

for some $z \in \mathbb{R}^N$; that is, $\overline{p} \in \mathrm{Im}(F_{\widetilde{p}})$.

Finally, suppose that $\widetilde{p}^n \to \widetilde{p}$ weakly in $\widetilde{L}^2$ and denote $J_n := J_{\widetilde{p}^n}$, $J := J_{\widetilde{p}}$, $\phi_n(\cdot) := \phi(\cdot, \widetilde{p}^n)$, $\phi(\cdot) := \phi(\cdot, \widetilde{p})$, $F_n := F_{\widetilde{p}^n}$ and $F := F_{\widetilde{p}}$.

We claim that $F_n \to F$ pointwise. Indeed, for fixed $x \in \mathbb{R}^N$, from Lemma 3.4.3 we know that, if $n \to \infty$, then $\phi_n(z) \to \phi(z)$. As ∇G is continuous, we deduce from the Lebesgue's dominated convergence theorem that $F_n(z) \to F(z)$.

To prove that $\mathcal{I}(\widetilde{p}^n) \to \mathcal{I}(\widetilde{p})$ as $n \to \infty$ for the Hausdorff topology, we need to see that:

(i) $\sup_{\overline{q}^n \in \mathcal{I}(\widetilde{p}^n)} \mathrm{dist}(\overline{q}^n, \mathcal{I}(\widetilde{p})) \to 0$,

(ii) $\sup_{\overline{q}\in\mathcal{I}(\widetilde{p})} \operatorname{dist}(\overline{q}, \mathcal{I}(\widetilde{p}^n)) \to 0$.

For (i), denote $S_n = \sup_{\overline{q}^n\in\mathcal{I}(\widetilde{p}^n)} \operatorname{dist}(\overline{q}^n, \mathcal{I}(\widetilde{p}))$ and let $\overline{p}^n \in \mathcal{I}(\widetilde{p}^n)$ be chosen in such a way that $\operatorname{dist}(\overline{p}^n, \mathcal{I}(\widetilde{p})) \geq S_n - \frac{1}{n}$. We shall prove that $\operatorname{dist}(\overline{p}^n, \mathcal{I}(\widetilde{p})) \to 0$. By contradiction, suppose there exists a subsequence, still denoted $\{\overline{p}^n\}$, such that

$$\operatorname{dist}(\overline{p}^n, \mathcal{I}(\widetilde{p})) \geq \varepsilon > 0. \tag{3.12}$$

Moreover, we know that $\mathcal{I}(\widetilde{p}) \subset \operatorname{co}(\operatorname{Im}(\nabla G))$; in particular, taking a convergent subsequence if necessary we may suppose that $\overline{p}^n \to \overline{p}$ for some $\overline{p} \in \mathbb{R}^N$. For each n, let $u_n \in H^1_{\text{per}}$ be a solution of the problem for $\overline{p}^n$. From the periodicity of ∇G, we may assume that the sequence $\{\overline{u}_n\}$ is bounded in $\mathbb{R}^N$. Thus, $\{u_n\}$ is bounded in H^1_{per} and

$$\int_0^T u_n' \cdot v' - \nabla G(u_n) \cdot v + (\widetilde{p}^n + \overline{p}^n) \cdot v \, dx = 0 \tag{3.13}$$

for all $v \in H^1_{\text{per}}$. Taking again a subsequence, we may assume that $u_n \to u_0$ weakly in H^1_{per} and hence

$$\int_0^T u_0' \cdot v' - \nabla G(u_0) \cdot v + (\widetilde{p} + \overline{p}) \cdot v \, dx = 0$$

for all $v \in H^1_{\text{per}}$. Then u_0 is a weak solution of (3.1) with $p = \widetilde{p} + \overline{p}$ and $\overline{p} \in \mathcal{I}(\widetilde{p})$, which contradicts (3.12). Thus $\operatorname{dist}(\overline{p}^n, \mathcal{I}(\widetilde{p})) \to 0$ and consequently $S_n \to 0$.

Next we prove (ii). Denote now $S_n = \sup_{\overline{q}\in\mathcal{I}(\widetilde{p})} \operatorname{dist}(\overline{q}, \mathcal{I}(\widetilde{p}^n))$ and take $\overline{q}^n \in \mathcal{I}(\widetilde{p})$ such that $\operatorname{dist}(\overline{q}^n, \mathcal{I}(\widetilde{p}^n)) \geq S_n - \frac{1}{n}$. As before, suppose there exists a subsequence, still denoted $\{\overline{q}^n\}$, such that

$$\operatorname{dist}(\overline{q}^n, \mathcal{I}(\widetilde{p}^n)) \geq \varepsilon > 0. \tag{3.14}$$

Passing to a subsequence if necessary, there exist $\overline{q} \in \mathcal{I}(\widetilde{p}) = \operatorname{Im}(F)$ and $n_1 \in \mathbb{N}$ such that $\operatorname{dist}(\overline{q}^n, \overline{q}) < \frac{\varepsilon}{2}$ for all $n \geq n_1$. Fix $z_0 \in \mathbb{R}^N$ such that $F(z_0) = \overline{q}$ and let $\overline{p}^n = F_n(z_0) \in \mathcal{I}(\widetilde{p}^n)$. As $F_n(z_0) \to F(z_0)$, there exists $n_2 \in \mathbb{N}$ such that $\operatorname{dist}(\overline{p}^n, \overline{q}) < \frac{\varepsilon}{2}$ for all $n \geq n_2$. Take $n_0 = \max\{n_1, n_2\}$ and hence

$$\operatorname{dist}(\overline{q}^n, \mathcal{I}(\widetilde{p}^n)) \leq \operatorname{dist}(\overline{q}^n, \overline{p}^n) \leq \operatorname{dist}(\overline{q}^n, \overline{q}) + \operatorname{dist}(\overline{q}, \overline{p}^n) < \varepsilon$$

for $n \geq n_0$. This contradicts (3.14), so we conclude that $S_n \to 0$. $\square$

3.5 Characterization of $\mathcal{I}$ for convex G

The main result of this section reads as follows.

Theorem 3.5.1. *Assume that G is strictly convex (recall Definition 1.1.17) and that*

1. *There exist $\alpha < \left(\frac{2\pi}{T}\right)^2$ and $\beta \in \mathbb{R}$ such that*

$$G(u) \leq \frac{\alpha}{2}|u|^2 + \beta \quad \text{for all } u \in \mathbb{R}^N. \tag{3.15}$$

2. *For every $a \in \mathbb{R}^N$ there exists $r_0 > 0$ such that*

$$\frac{\partial G}{\partial w}(rw + x) \geq \frac{\partial G}{\partial w}(a) \tag{3.16}$$

for all $r \geq r_0$, $w \in S^{N-1}$ and $|x| \leq C$, where $C = C(a, \widetilde{p})$ is the constant defined below in (3.22).

Then $\mathcal{I}(\widetilde{p}) = \mathrm{Im}(\nabla G)$.

Remark 3.5.2. Consider the following example: $T = 1$, $G = 2\pi^2|u|^2$ and $\tilde{p} = \sin(2\pi x)(1, ..., 1)$, it is readily seen that $\mathcal{I}(\tilde{p}) = \emptyset$. In view of the example, we can conclude that the hypothesis on α in condition (3.15) is sharp.

Remark 3.5.3. Let us give an intuitive idea of hypothesis *2.*. The condition (3.16) is “almost” a consequence of the convexity hypothesis in the following sense: if we assume $G \in C^2$ and write $w_r = w + \frac{x-a}{r}$, then

$$\frac{\partial G}{\partial w}(rw + x) - \frac{\partial G}{\partial w}(a) = rw_r^t d^2G(\xi_r)w,$$

for some ξ_r between a and $rw + x$. Since x moves within a compact set, w_r tends to w when $r \to +\infty$, then it is reasonable to assume that the value of the difference is positive for large r.

For example, if we write

$$w_r^t d^2G(\xi_r)w = (w_r - w)^t d^2G(\xi_r)w + w^t d^2G(\xi_r)w,$$

then it is easy to see that (3.16) holds if d^2G is bounded and its smallest eigenvalue is always greater than a positive constant.

Proof of Theorem 3.5.1: Firstly, we shall prove the inclusion $\mathrm{Im}(\nabla G) \subseteq \mathcal{I}(\widetilde{p})$. For simplicity, from the rescaling $v(x) = u(\frac{T}{2\pi}x)$ we may assume that $T = 2\pi$. Let $K : \widetilde{L}^2 \to H^2 \cap \widetilde{L}^2$ be the inverse of the operator $Lu := u''$, namely $Kh = u$, where u is the unique solution of the problem

$$\begin{cases} u'' = h \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \\ \overline{u} = 0. \end{cases}$$

<u>Claim 1:</u> $\int_0^{2\pi} Kh(x) \cdot h(x)\,dx + \int_0^{2\pi} |h(x)|^2\,dx \geq 0$.

Indeed, from (3.3) it is seen that

$$\int_0^{2\pi} |(Kh)'(x)|^2\,dx = -\int_0^{2\pi} Kh(x) \cdot h(x)\,dx \leq \|(Kh)'\|_{L^2}\|h\|_{L^2},$$

which implies that $\|(Kh)'\|_{L^2} \leq \|h\|_{L^2}$, and hence

$$-\int_0^{2\pi} Kh(x) \cdot h(x)\,dx = \int_0^{2\pi} |(Kh)'(x)|^2\,dx \leq \|h\|_{L^2}^2.$$

For $\overline{p} \in \mathrm{Im}(\nabla G)$, fix $a \in \mathbb{R}^N$ such that $\nabla G(a) = \overline{p}$, and define the functions

$$F(x,u) := G(u) - p(x) \cdot u;$$

and, for given $\varepsilon > 0$,

$$F_\varepsilon(x,u) := G(u) - p(x) \cdot u + \frac{\varepsilon}{2}|u|^2$$

where $p(x) = \widetilde{p}(x) + \overline{p}$. Next, consider the Fenchel transform F_ε^* of the function F_ε defined as

$$F_\varepsilon^*(x,v) = \max_{w \in \mathbb{R}^N} \left(v \cdot w - F_\varepsilon(x,w) \right). \tag{3.17}$$

Observe that F_ε^* is well defined, since F_ε is convex; hence a unique global maximum w is achieved and satisfies the following properties:

1. $v = \nabla F_\varepsilon(x,w)$,
2. $w = \nabla F_\varepsilon^*(x,v)$,
3. $v \cdot w - F_\varepsilon(x,w) = F_\varepsilon^*(x,v)$.

Properties 1 and 2 are known as *Fenchel duality* (see Section 1.1.6 in Chapter 1 or [39]).

Define the functional $I_\varepsilon : \widetilde{L}^2 \to \mathbb{R}$ given by

$$I_\varepsilon(v) = \int_0^{2\pi} \frac{1}{2} Kv(x) \cdot v(x) + F_\varepsilon^*(x, v(x))\,dx.$$

From (3.17) and (3.15),

$$\begin{aligned} F_\varepsilon^*(x,v) \geq |v|^2 - F_\varepsilon(x,v) &= |v|^2 + p\cdot v - \frac{\varepsilon}{2}|v|^2 - G(v) \\ &\geq |v|^2 + p\cdot v - \frac{\varepsilon+\alpha}{2}|v|^2 - \beta \end{aligned}$$

and using Claim 1, Cauchy-Schwarz Inequality and the fact that $v \in \widetilde{L}^2$ we deduce:

$$I_\varepsilon(v) \geq -\frac{1}{2}\int_0^{2\pi} |v(x)|^2\,dx + \int_0^{2\pi} |v(x)|^2 + \widetilde{p}(x)\cdot v(x) - \frac{\varepsilon+\alpha}{2}|v(x)|^2 - \beta\,dx;$$

that is,

$$I_\varepsilon(v) \geq \frac{1-\alpha-\varepsilon}{2}\|v\|_{L^2}^2 - \|\widetilde{p}\|_{L^2}\|v\|_{L^2} - 2\pi\beta. \tag{3.18}$$

Thus I_ε is coercive for $\varepsilon < 1-\alpha$ and hence it achieves a minimum u_ε. As K is self-adjoint, it is easy to verify that

$$\int_0^{2\pi} [Ku_\varepsilon(x) + \nabla F_\varepsilon^*(x, u_\varepsilon(x))]\cdot \varphi(x)\,dx = 0, \text{ for all } \varphi \in \widetilde{L}^2.$$

Then $K(u_\varepsilon) + \nabla F_\varepsilon^*(s,u_\varepsilon) = A \in \mathbb{R}^N$. Let $v_\varepsilon = \nabla F_\varepsilon^*(s,u_\varepsilon) = A - K(u_\varepsilon)$, then by the Fenchel duality $u_\varepsilon = \nabla F_\varepsilon(s,v_\varepsilon)$. In other words, $u_\varepsilon = \nabla G(v_\varepsilon) - p(x) + \varepsilon v_\varepsilon$.

On the other hand, $v_\varepsilon'' = (-K(u_\varepsilon))'' = -u_\varepsilon$; hence, v_ε satisfies

$$\begin{cases} v_\varepsilon'' + \nabla G(v_\varepsilon) + \varepsilon v_\varepsilon = p(x) \\ v_\varepsilon(0) = v_\varepsilon(2\pi), \quad v_\varepsilon'(0) = v_\varepsilon'(2\pi). \end{cases} \tag{3.19}$$

Moreover, recall the Legendre transform of F (Section 1.1.6 in Chapter 1), defined by

$$F^*(x,v) = \sup_{w\in\mathbb{R}^N} (v\cdot w - F(x,w))$$

then it is obvious that $F_\varepsilon^* \leq F^*$. As u_ε is the minimum, it follows that

$$I_\varepsilon(u_\varepsilon) \leq I_\varepsilon(-\widetilde{p}) = \int_0^{2\pi} \frac{1}{2}K\widetilde{p}(x)\cdot\widetilde{p}(x) + F^*(x, -\widetilde{p}(x))\,dx. \tag{3.20}$$

For fixed x, let $\Psi(y) := -\widetilde{p}\cdot y - F(x,y) = \overline{p}\cdot y - G(y)$, then

$$\nabla\Psi(y) = -\widetilde{p} - \nabla F(x,y) = \overline{p} - \nabla G(y).$$

Thus, a is a critical point of Ψ and, as Ψ is strictly concave, we conclude that a is the absolute maximum. Then

$$-\widetilde{p}\cdot a - F(x,a) = \max_{w\in\mathbb{R}^N} (-\widetilde{p}(x)\cdot w - F(x,w)) = F^*(x, -\widetilde{p}(x)).$$

Hence, from (3.20) and the fact that $\widetilde{p} \in \widetilde{L}^2$ we obtain:

$$I_\varepsilon(u_\varepsilon) \leq \int_0^{2\pi} \frac{1}{2} K\widetilde{p}(x) \cdot \widetilde{p}(x) - F(x,a)\,dx$$
$$= 2\pi\,(a \cdot \nabla G(a) - G(a)) - \frac{1}{2}\|(K\widetilde{p})'\|_{L^2}^2. \tag{3.21}$$

Fixing $c < (1-\alpha)/2$, we conclude from (3.18) that if ε is small enough then

$$c\|u_\varepsilon\|_{L^2}^2 - \|\widetilde{p}\|_{L^2}\|u_\varepsilon\|_{L^2} \leq 2\pi\,(a \cdot \nabla G(a) - G(a) + \beta) - \frac{1}{2}\|(K\widetilde{p})'\|_{L^2}^2.$$

As $v_\varepsilon'' = -u_\varepsilon$, it follows that $\tilde{v}_\varepsilon$ is bounded for the H^2 norm; in particular,

$$\|\tilde{v}_\varepsilon\|_\infty \leq C \tag{3.22}$$

for some constant C, depending only on $\widetilde{p}$ and a.

Let us prove now that $\overline{v}_\varepsilon$ is bounded. By direct integration of (3.19) we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} \nabla G(v_\varepsilon(x))dx + \varepsilon\overline{v}_\varepsilon = \overline{p}. \tag{3.23}$$

Writing $\overline{v}_\varepsilon = rw$, where $r = |\overline{v}_\varepsilon|$ and $|w| = 1$, and multiplying (3.23) by w, we obtain

$$\varepsilon r + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial G}{\partial w}(rw + \tilde{v}_\varepsilon(x))\,dx = \overline{p} \cdot w = \nabla G(a) \cdot w = \frac{\partial G}{\partial w}(a).$$

As $|\tilde{v}_\varepsilon(x)| \leq C$, for $r \geq r_0$ inequality (3.16) yields:

$$0 = \varepsilon r + \frac{1}{2}\int_0^{2\pi} \left(\frac{\partial G}{\partial w}(rw + \tilde{v}_\varepsilon(x)) - \frac{\partial G}{\partial w}(a)\right) dx \geq \varepsilon r,$$

a contradiction. So, $|\overline{v}_\varepsilon| \leq r_0$ and v_ε is bounded for the H^2 norm.

From the compact embedding $H^2([0,2\pi],\mathbb{R}^N) \hookrightarrow C^1([0,2\pi],\mathbb{R}^N)$, there exists a sequence $\{v_{\varepsilon_n}\}_{n\in\mathbb{N}}$ that converges in $C^1([0,2\pi],\mathbb{R}^N)$ to some function v. From (3.19),

$$\int_0^{2\pi} \big(v_{\varepsilon_n}''(x) + \nabla G(v_{\varepsilon_n}(x)) + \varepsilon_n v_{\varepsilon_n}(x)\big) \cdot \varphi(x)\,dx = \int_0^{2\pi} p(x) \cdot \varphi(x)\,dx$$

for all $\varphi \in \widetilde{L}^2$. Integrating by parts and passing to the limit, we obtain:

$$-\int_0^{2\pi} v'(x) \cdot \varphi'(x)\,dx + \int_0^{2\pi} \nabla G(v(x)) \cdot \varphi(x)\,dx = \int_0^{2\pi} p(x) \cdot \varphi(x)\,dx.$$

Then v is a solution of (3.1).

Finally, let us prove that $\mathcal{I}(\widetilde{p}) \subseteq \mathrm{Im}(\nabla G)$. As previously mentioned, we know that $\mathcal{I}(\widetilde{p}) \subseteq \mathrm{co}(\mathrm{Im}(\nabla G))$, so it remains to see that $\mathrm{Im}(\nabla G)$ is convex.

Claim 2: If $F \in C^1(\mathbb{R}^N, \mathbb{R})$ is strictly convex, then

$$0 \in \mathrm{Im}(\nabla F) \quad \iff \lim_{|x|\to+\infty} F(x) = +\infty.$$

The sufficiency is obvious. In order to prove the necessity, assume that $\nabla F(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$ and for each $w \in S^{N-1}$ define $\Phi_w(t) := \frac{\partial F}{\partial w}(x_0 + tw)$. From the convexity of F we deduce that Φ_w is strictly increasing. Furthermore, the function $\Phi : S^{N-1} \times [0, +\infty) \to \mathbb{R}$ given by $\Phi(w, t) := \Phi_w(t)$ is continuous and $\Phi(w, 1) > 0$ for all $w \in S^{N-1}$. Hence, there exists a constant $c > 0$, such that $\Phi_w(1) \geq c > 0$ for all $w \in S^{N-1}$. As Φ_w is strictly increasing, we conclude that $\Phi_w(t) > c$ for all $t > 1$. Thus,

$$F(x_0 + Rw) - F(x_0 + w) = R\nabla F(x_0 + \xi w) \cdot w = R\frac{\partial F}{\partial w}(x_0 + \xi w) \geq cR.$$

We conclude that $F(x_0 + Rw) \geq F(x_0 + w) + cR$ and the claim is proved.

Next, let us consider y_1, $y_2 \in Im(\nabla G)$ and $y = a_1 y_1 + a_2 y_2$, with $a_1 + a_2 = 1$ and a_1, $a_2 \geq 0$. Define

$$F(x) = G(x) - y \cdot x = a_1\,(G(x) - y_1 \cdot x) + a_2(G(x) - y_2 \cdot x).$$

As $G(x) - y_1 \cdot x$ and $G(x) - y_2 \cdot x$ are strictly convex, it follows from Claim 2 that both functions tend to $+\infty$ as $|x| \to \infty$, and hence

$$\lim_{|x|\to+\infty} F(x) = +\infty. \tag{3.24}$$

Using Claim 2 again, (3.24) implies that $0 \in \mathrm{Im}(\nabla F) = \mathrm{Im}(\nabla G - y)$, then $y \in \mathrm{Im}(\nabla G)$ and so completes the proof. $\square$

Resumen del Capítulo 3

En este capítulo estudiamos el siguiente problema: para una función vectorial $u : \mathbb{R} \to \mathbb{R}^N$ consideramos el sistema

$$\begin{cases} u'' + \nabla G(u) = p(x), & x \in (0,T) \\ u(0) = u(T), & u'(0) = u'(T), \end{cases}$$

donde $G : \mathbb{R}^N \to \mathbb{R}$ es una función de clase C^1.

Estudiamos la imagen del operador semilineal

$$S : H^2_{\text{per}} \to L^2([0,T], \mathbb{R}^N),$$

dado por $Su = u'' + \nabla G(u)$, donde

$$H^2_{\text{per}} = \{u \in H^2([0,T], \mathbb{R}^N); u(0) - u(T) = u'(0) - u'(T) = 0\}.$$

Escribimos $p(x) = \overline{p} + \widetilde{p}(x)$, donde $\overline{p} := \frac{1}{T}\int_0^T p(x)\,dx$, presentamos varios resultados respecto a la estructura topológica del conjunto

$$\mathcal{I}(\widetilde{p}) = \{\overline{p} \in \mathbb{R}^N; \overline{p} + \widetilde{p} \in \text{Im}(S)\}.$$

Este capítulo está organizado de la siguiente manera. En la Sección 3.2 probamos un criterio básico que asegura que, bajo ciertas hipótesis, $\overline{p} \in \mathbb{R}^N$ pertenece al conjunto $\mathcal{I}(\widetilde{p})$ para una $\widetilde{p}$ dada.

En la Sección 3.3 damos condiciones suficientes para que un punto $\overline{p}_0 \in \mathcal{I}(\tilde{p})$ sea interior. A grandes rasgos, probamos que si el Hessiano de G no interactúa con el espectro del operador $L(u) = -u''$, entonces $\mathcal{I}(\tilde{p})$ es un entorno de $\overline{p}_0$.

En la Sección 3.4, extendemos un resultado conocido de Castro [20] para la ecuación del péndulo; más precisamente, probamos que si ∇G es periódica, entonces $\mathcal{I}$, considerado como una función del espacio $\widetilde{L}^2$ al conjunto de subconjuntos compactos de $\mathbb{R}^N$ (equipado con la distancia de Hausdorff) es continuo.

Finalmente, en la Sección 3.5, probamos que si G es estrictamente convexa y satisface ciertas condiciones, entonces podemos caracterizar el conjunto $\mathcal{I}(\tilde{p})$. Más precisamente $\mathcal{I}(\widetilde{p}) = \text{Im}(\nabla G)$ para toda $\widetilde{p}$.

Chapter 4

Radiation boundary conditions, a variational approach

4.1 Introduction

In this chapter we want to find appropriate extensions to the general case of the results obtained in [10]. Namely, we consider the following equation for a function $u : [0,1] \to \mathbb{R}$

$$u''(x) = g(x,u) + p(x), \tag{4.1}$$

where $p \in L^2(0,1)$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and superlinear, that is:

$$\lim_{|u| \to +\infty} \frac{g(x,u)}{u} = +\infty \tag{4.2}$$

uniformly in $x \in [0,1]$, without loss of generality we can assume that $g(x,0) = 0$ for all $x \in [0,1]$; under the following radiation boundary conditions

$$u'(0) = a_0 u(0), \quad u'(1) = a_1 u(1), \tag{4.3}$$

with a_0, $a_1 > 0$.

By 'solution' we mean a function $u \in H^2(0,1)$ satisfying (4.1)-(4.3). It is clear that if $p \in C([0,1])$ then u is a classical C^2 solution.

As in [10], we shall study existence, uniqueness and multiplicity of solutions using variational methods and a linking theorem.

This chapter is organized as follows. In the following section we introduce a variational formulation for problem (4.1)-(4.3) and we establish an existence result. In Section 4.3 we prove that under certain hypothesis, a global minimum of the associated functional J corresponds to a negative solution of the problem; moreover, we study the behavior of the solutions when $p(x) \geq p_0 \gg 0$ a.e. $x \in [0,1]$, we conclude that problem (4.1)-(4.3) has a unique solution. Section 4.4 is devoted to prove uniqueness and multiplicity results. For the multiplicity results, we prove that the functional satisfies the hypotheses of a linking-type theorem, which allows us to prove the existence of at least one extra local minimum and a saddle-type critical point.

Some of the results from this chapter were published in [6].

4.2 Variational Setting

Let us define the functional $J : H^1(0,1) \to \mathbb{R}$ by

$$J(u) = \int_0^1 \left(\frac{1}{2}(u')^2 + G(x,u) + p(x)u \right) dx + \frac{a_0}{2} u(0)^2 - \frac{a_1}{2} u(1)^2,$$

where $G(x,u) = \int_0^u g(x,s)ds$. It is readily seen that $J \in C^1(H^1(0,1), \mathbb{R})$, with

$$DJ(u)(v) = \int_0^1 (u'v' + g(x,u)v + pv)\, dx + a_0 u(0)v(0) - a_1 u(1)v(1).$$

Remark 4.2.1. $u \in H^1(0,1)$ is a critical point of J if and only if u is a solution of (4.1)-(4.3).

Indeed, let u be a critical point of J. By considering the equality $DJ(u)(v) = 0$ for all $v \in H_0^1(0,1)$ it is deduced that $u \in H^2(0,1)$ and satisfies (4.1).

Let us see that u satisfies (4.3). Given an arbitrary $v \in H^1(0,1)$, integrating by parts equation $DJ(u)(v) = 0$, we get:

$$u'(1)v(1) - u'(0)v(0) = a_1 u(1)v(1) - a_0 u(0)v(0).$$

Take $v \in H^1(0,1)$ such that $0 = v(0) \neq v(1)$, we conclude that $u'(1) = a_1 u(1)$; similarly, if $0 = v(1) \neq v(0)$, $u'(0) = a_0 u(0)$.

Conversely, assume $u \in H^2$ is a solution of (4.1)-(4.3). Let $v \in H^1(0,1)$ arbitrary, multiply (4.1) by v, integration by parts yields $DJ(u)(v) = 0$. Hence, u is a critical point of J.

Remark 4.2.2. It is clear that if $p \in C([0,1])$ then u is a classical C^2 solution.

4.2.1 Existence result

Theorem 4.2.3. *Assume (4.2) holds, then problem (4.1)-(4.3) has at least one solution.*

Proof:
We claim that J achieves a global minimum in $H^1(0,1)$, which is a critical point of J and, hence, a solution of our problem (4.1)-(4.3). By Theorem 1.1.18 in Chapter 1, J is weakly lower semi-continuous. Therefore, to prove our claim it is enough to see that J is coercive.

Since g is superlinear, given $M > 0$, there exists $K \in \mathbb{R}$ such that

$$G(x,u) > Mu^2 + K, \tag{4.4}$$

for all $u \in \mathbb{R}$.

On the other hand, since $2ab \le a^2 + b^2$ for all $a, b \in \mathbb{R}$, given $R > 0$ we have

$$2uu' = 2(\sqrt{R}u)\left(\frac{1}{\sqrt{R}}u'\right) \le Ru^2 + \frac{1}{R}u'^2.$$

So, let us set $\varphi(x) = x$, by the previous inequality

$$\begin{aligned} u(1)^2 = \varphi u^2\Big|_0^1 = \int_0^1 (\varphi u^2)' dx = \int_0^1 u^2 dx + 2\int_0^1 xuu' dx \le \\ \le (1+R)\|u\|_{L^2}^2 + \frac{1}{R}\|u'\|_{L^2}^2. \end{aligned} \tag{4.5}$$

Thus, by (4.5) and (4.4),

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|u'\|^2 + \int_0^1 (G(x,u) + p(x)u)\, dx + \frac{a_0}{2}u(0)^2 - \frac{a_1}{2}u(1)^2 \\ &\ge \frac{1}{2}\|u'\|^2 + \left(M - \frac{1}{2}\right)\|u\|_{L^2}^2 + K - \frac{1}{2}\|p\|_{L^2}^2 - \frac{a_1(1+R)}{2}\|u\|_{L^2}^2 - \frac{a_1}{2R}\|u'\|_{L^2}^2 \\ &\ge \left(\frac{1}{2} - \frac{a_1}{2R}\right)\|u'\|_{L^2}^2 + \left(M - \frac{1 + a_1(1+R)}{2}\right)\|u\|_{L^2}^2 + K - \frac{1}{2}\|p\|_{L^2}^2. \end{aligned}$$

Let $\tilde{K} = K - \frac{1}{2}\|p\|_{L^2}^2$. Choose $R > a_1$ and then $M > \dfrac{1 + a_1(1+R)}{2}$, thus, there exists $C > 0$ such that $J(u) \ge C\|u\|_{H^1(0,1)}^2 + \tilde{K}$. □

4.3 Behavior of the solutions

Our aim in this section is to prove some results regarding the behavior of the solutions of problem (4.1)-(4.3), such as the sign, to know how many negative solutions there are and how do they behave when p is large.

Proposition 4.3.1. *Assume*

$h_1)$ $\frac{1}{2u}\int_{-u}^{u} g(x,s)ds + p(x) \geq 0$ *for all* $x \in [0,1]$ *and* $u \in \mathbb{R}_{>0}$.

There exists u_0 *such that* $J(u_0) = \inf_{u \in H^1} J(u)$ *and* $u_0 \leq 0$. *Moreover, if there exists* $C > 0$ *such that* $p \geq C > 0$ *a.e. in a neighborhood of some* x_0*, then* $u(x_0) < 0$.

Proof:
It suffices to observe, for all u, that by $h_1)$,

$$\begin{aligned} & J(u) - J(-|u|) = \\ & = \int_{\{x\in[0,1]/u(x)>0\}} (G(x,u(x)) - G(x,-u(x)) + 2p(x)u(x))dx = \\ & = \int_{\{x\in[0,1]/u(x)>0\}} 2u(x)\left(\frac{1}{2u(x)}\int_{-u(x)}^{u(x)} g(x,s)ds + p(x)\right)dx \geq 0. \end{aligned}$$

Moreover, let $u_0 \leq 0$ be a solution and suppose that there exists $x_0 \in [0,1]$ such that $p(x) \geq C > 0$ a.e. x in a neighborhood of x_0 and that $u_0(x_0) = 0$. If $x_0 \in [0,1)$, $u_0(x_0) = 0$ and $u_0'(x_0) = 0$. Since $g(x,0) = 0$ and $p \geq C > 0$ near x_0, take $x > x_0$ small enough such that $\displaystyle\int_{x_0}^{x} u_0''(s)ds > 0$. We have,

$$u_0'(x) - u_0'(x_0) = \int_{x_0}^{x} u_0''(s)ds > 0,$$

hence $u_0'(x) > 0$ for $x > x_0$, a contradiction.

If $x_0 = 1$, similarly, we can take x large enough such that $\int_x^1 u_0''(s)ds > 0$. Hence $u_0'(x) < 0$ for x near 1. This says that u_0 decreases in a neighborhood of 1, a contradiction with $u_0(1) = 0$. □

Proposition 4.3.2. *There exists* $p_0 > 0$ *such that (4.1)-(4.3) has no positive nor sign-changing solutions if* $p(x) \geq p_0$ *for a.e.* $x \in [0,1]$.

Proof:
Due to the superlinearity of g, for each $M \geq 0$ we may define the quantity

$$N_M := \inf_{x\in[0,1],u\geq 0} \{g(x,u) - Mu\} > -\infty.$$

Then

$$g(x,u) \geq Mu + N_M, \tag{4.6}$$

for $u \geq 0$. For $M > 0$ to be determined, fix $p_0 > -N_M$ and let u be a solution of (4.1)-(4.3) for $p \geq p_0$ such that $u(x) \geq 0$ for some $x \in [0,1]$. In view of (4.6), the inequality $u''(x) \geq g(x, u(x)) + p_0$ implies that

$$u''(x) > Mu(x) \tag{4.7}$$

when $u(x) \geq 0$. We deduce that, if $x_0 \in [0,1]$ is such that $u(x_0)$ and $u'(x_0)$ are nonnegative, then $u(x)$ and $u'(x)$ are strictly positive for $x > x_0$. Multiply (4.7) by u' and integrate to obtain, for $x > x_0$:

$$u'(x)^2 > u'(x_0)^2 + M(u(x)^2 - u(x_0)^2). \tag{4.8}$$

If $u(0) > 0$, then $u'(0) > 0$ and

$$u(1)^2 - u(0)^2 = \int_0^1 2u(x)u'(x)dx > 2a_0 u(0)^2.$$

Thus,

$$u(1)^2 - u(0)^2 > \frac{2a_0}{1+2a_0} u(1)^2 \tag{4.9}$$

and fixing $M = a_1^2 \frac{1+2a_0}{2a_0}$ we obtain, from (4.8) and (4.9):

$$a_1^2 u(1)^2 > M \frac{2a_0}{1+2a_0} u(1)^2 = a_1^2 u(1)^2.$$

This contradiction proves that there are no positive solutions when $p_0 > -N_M$.

On the other hand, if $u(0) \leq 0$ then u vanishes at a (unique) $x_0 > 0$, with $u'(x_0) \geq 0$. Fix $M = a_1^2$, then (4.8) yields

$$a_1^2 u(1)^2 = u'(1)^2 > u'(x_0)^2 + a_1^2 u(1)^2 \geq a_1^2 u(1)^2,$$

a contradiction. □

As we shall see, all possible solutions tend uniformly to $-\infty$ as p tends uniformly to $+\infty$. In order to emphasize the dependence on p, for fixed p, any solution shall be denoted u_p, despite the fact that it might not be unique.

Proposition 4.3.3. *$u_p \to -\infty$ uniformly when $p \to +\infty$ uniformly.*

Proof:
From the previous proposition, we know that there exists $p_0 > 0$ such that if $p(x) \geq p_0$ for a.e. $x \in [0,1]$, there are no positive nor sign-changing solutions of problem (4.1)-(4.3). Hence, if p is large enough, we can assume that u_p is strictly negative.

Let $x_p \in [0,1]$ be the point where the absolute maximum of u_p is achieved. Since $u_p'(1) = a_1 u_p(1) < 0$, u_p decreases in a neighborhood of 1. Hence u_p cannot have a maximum at 1, thus $x_p < 1$.

Suppose there exists $M > 0$ such that for all p_0 there exists p such that $p(x) \geq p_0$ for a.e. x and $u_p(x_p) > -M$. Take p_0 large enough, such that,

$$g(x,u) + p_0 > Ma_0, \quad \text{for all } u \geq -(1+a_0)M. \tag{4.10}$$

Since $u_p''(x_p) > 0$, if $x_p \in (0,1)$ it would be a minimum, a contradiction, hence $x_p = 0$.

Let us define the maximum $\delta \leq 1$ such that $u_p''(x) \geq 0$ for all $x \in [0,\delta]$. Thus

$$u_p'(x) - u_p'(0) = \int_0^x u_p''(s)ds \geq 0 \quad \text{for } x \leq \delta,$$

then $u_p'(x) \geq u_p'(0) = a_0 u_p(0) > -Ma_0$, for $x \leq \delta$. Hence, since $\delta \leq 1$,

$$u_p(\delta) - u_p(0) = \int_0^\delta u_p'(s)ds > -\delta M a_0 \geq -Ma_0.$$

Then $u_p(\delta) > -M(1+a_0)$ and by (4.10), $u_p''(\delta) > 0$. Hence $\delta = 1$ and, in particular, $u_p(x) > -M(1+a_0)$ and from (4.10) $u_p''(x) > Ma_0$ holds for all x. Then

$$u_p'(1) - u_p'(0) = \int_0^1 u_p''(s)ds > Ma_0,$$

hence $u_p'(1) > 0$, a contradiction. □

Proposition 4.3.4. *Assume*

h_2) $\dfrac{g(x,u)+p(x)}{u}$ *is strictly non increasing in* $u < 0$ *and a.e.* $x \in [0,1]$.

Then, there exists at most one negative solution of (4.1)-(4.3).

When $g(x,\cdot) \in C^1(\mathbb{R})$, hypothesis h_2) reads

$$\frac{\partial g}{\partial u}(x,u) > \frac{g(x,u)+p(x)}{u}, \quad \text{for } u < 0 \text{ and a.e. } x \in [0,1].$$

This condition is well known in the literature (see e.g. [21]). When assumed for all $u \in \mathbb{R}$, it implies that if $u_0 \neq 0$ is a critical point of the associated functional J, then u_0 is transversal to the Nehari manifold (introduced after the pioneering work [40]), namely

$$\mathcal{N} := \{u \in H^1(0,1) \setminus \{0\} : DJ(u) \cdot u = 0\}.$$

Indeed, setting $I(u) := DJ(u) \cdot u$ it is readily seen that $T_{u_0}\mathcal{N} = ker(DI(u_0))$ and $DI(u_0) \cdot u_0 > 0$.

For the particular case, studied in [10], of problem

$$u''(x) = Ku(x)^3 + L(x)u(x) + A,$$

where K and A are given constants and $L(x) := a_0^2 + (a_1^2 - a_0^2)x$, condition h_2) simply reads $\frac{A}{u^3} < 2K$ and is trivially satisfied when $u < 0$.

Proof:
Let u_1 and u_2 be negative solutions and suppose for example that $u_1(0) < u_2(0)$. Note that, by hypothesis h_2), if $u_1 < u_2$ over $[0, x_0)$, then

$$u_1''(x) = \frac{g(x, u_1(x)) + p(x)}{u_1(x)} u_1(x) < \frac{g(x, u_2(x)) + p(x)}{u_2(x)} u_1(x)$$

and hence

$$u_1''(x)u_2(x) > u_1(x)u_2''(x) \qquad x < x_0.$$

We conclude that

$$u_1'(x_0)u_2(x_0) > u_1(x_0)u_2'(x_0), \tag{4.11}$$

and a contradiction yields if $x_0 = 1$. Otherwise, we may suppose that $u_1(x_0) = u_2(x_0)$ and $u_1'(x_0) \geq u_2'(x_0)$ which, again, contradicts (4.11). □

If we put together Propositions 4.3.2, 4.3.3 and 4.3.4, then we get the following uniqueness result.

Corollary 4.3.5. *Assume h_2) holds. There exists p_0 such that, if $p(x) \geq p_0$ for all x, then problem (4.1)-(4.3) has a unique solution.*

Sharper bounds for p can be obtained when some specific conditions are assumed on a_0 and a_1.

Remark 4.3.6. If $a_1 \leq \frac{a_0}{a_0+1}$ then there are no positive solutions if $p \geq p_0$. Indeed, if u is a positive solution and $z := \ln u$, then

$$z''(x) + z'(x)^2 = \left(g(x, e^{z(x)}) + p(x)\right) e^{-z(x)} > 0$$

$$z'(0) = a_0, \quad z'(1) = a_1.$$

Observe that if $z'(x_0) = 0$ then $z''(x_0) > 0$; thus, z' cannot vanish and

$$1 \leq \frac{1}{a_1} - \frac{1}{a_0} = \frac{1}{z'(1)} - \frac{1}{z'(0)} = -\frac{z''(\xi)}{z'(\xi)^2} < 1.$$

Remember, from Chapter 1, (1.13) is a non resonance condition.

For the particular case of problem

$$u''(x) = Ku(x)^3 + L(x)u(x) + A,$$

where K, A and L are given as before, $a_1 \le a_0$ yields uniqueness, thus, it suffices to consider the case $a_1 > a_0$, for which it is directly seen that there are no positive solutions if $A \ge a_1\sqrt{\frac{8}{K}}$, and no sign changing solutions if $A \ge \frac{2(a_1^2-a_0^2)^{3/2}}{3\sqrt{3K}}$. Also, it is deduced that if u is a positive solution, then $u(0) < u(1) < \frac{4a_1^2}{AK}$ and if u is a sign changing solution then $u(1) < a_1\sqrt{\frac{2}{K}}$. Since there is a unique negative solution (see Theorem 4.3.4) and the graphs of different solutions do not cross each other (see Lemma 5.4.6 below), the latter bounds for $u(1)$ provide an alternative proof of the fact, established in [10], that the the set of solutions is bounded.

4.4 Uniqueness and multiplicity results

For the following results, suppose that $g(x,\cdot) \in C^1(\mathbb{R})$. Let us define Φ as the unique solution of the linear problem

$$\begin{cases} -\Phi'' + \frac{\partial g}{\partial u}(x,0)\Phi = 0 \\ a_0\Phi(0) = \Phi'(0) = a_0. \end{cases} \tag{4.12}$$

Lemma 4.4.1. *Assume*

$h_3)$ $\frac{\partial g}{\partial u}(x,0) \ge 0$ *for all* $x \in [0,1]$.

The following conditions are equivalent.

1. $\Phi'(1) < a_1\Phi(1)$.
2. $M(u) < \frac{a_1}{2}u(1)^2 - \frac{a_0}{2}u(0)^2, \quad$ *for some* $u \in H^1(0,1)$,

where $M(u) := \int_0^1 \left(\frac{1}{2}(u')^2 + \frac{1}{2}\frac{\partial g}{\partial u}(x,0)u^2\right)dx$.

Proof:
If $\Phi'(1) < a_1\Phi(1)$ is true, then, it is enough to take $u = \Phi$. Since Φ satisfies (4.12), integration by parts yields

$$2M(\Phi) = \int_0^1 (\Phi'^2 + \Phi''\Phi)dx = \Phi'\Phi\Big|_0^1 - \int_0^1 \Phi''\Phi dx + \int_0^1 \Phi''\Phi dx =$$
$$= \Phi'(1)\Phi(1) - \Phi'(0)\Phi(0) < a_1\Phi(1)^2 - a_0\Phi(0)^2.$$

Conversely, define $\tilde{J}(u) = M(u) + \frac{a_0}{2}u(0)^2$. It is clear, from h_3), that $\tilde{J}$ is coercive and that $\tilde{J}(u) > 0$ if $u \neq 0$, take the minimum of $\tilde{J}$ restricted to the set $\{u \in H^1(0,1) : u(1) = 1\}$, let us call u_1 to that minimum. Then

$$D\tilde{J}(u_1)(v) = \int_0^1 \left(u_1'v' + \frac{\partial g}{\partial u}(x,0)u_1 v\right) dx + a_0 u_1(0)v(0) = \mu v(1),$$

for all $v \in H^1(0,1)$, where μ is a Lagrange multiplier. Take $v \in C_0^\infty([0,1])$, we deduce that $u_1 \in H^2(0,1)$ and $u_1'' = \frac{\partial g}{\partial u}(x,0)u_1$. Integrating by parts, the previous equation reads

$$v(0)(a_0u_1(0) - u_1'(0)) + v(1)(u'(1) - \mu) + \int_0^1 \left(-u_1'' + \frac{\partial g}{\partial u} g(x,0)u_1\right) v dx = 0,$$

for all $v \in H^1(0,1)$. Take $v \in H^1(0,1)$ such that $0 = v(0) \neq v(1)$, we conclude that $u_1'(1) = \mu$. Take $0 = v(1) \neq v(0)$, then $u_1'(0) = a_0u_1(0)$.

Finally, we conclude that u_1 satisfies

$$\begin{cases} -u_1'' + \frac{\partial g}{\partial u}(x,0)u_1 = 0 \\ u_1'(0) = a_0u_1(0), \quad u_1'(1) = \mu. \end{cases} \tag{4.13}$$

Since $u_1(1) = 1$ and satisfies (4.13), integrating by parts yields

$$\begin{aligned} 2\tilde{J}(u_1) &= \int_0^1 \left(u_1'^2 + \frac{\partial g}{\partial u}(x,0)u_1^2\right) dx + a_0u_1(0)^2 = \\ &= u_1'u_1\Big|_0^1 + \int_0^1 \left(-u_1'' + \frac{\partial g}{\partial u}(x,0)u_1\right) u_1 dx + a_0u_1(0)^2 = \mu. \end{aligned}$$

Note that, without loss of generality, we may assume that $u(1) = 1$ in *2.*, then $\mu = 2\tilde{J}(u_1) < a_1$. Finally, $\Phi'(1) < a_1\Phi(1)$ follows from the fact that $\Phi(x) = \frac{u_1(x)}{u_1(0)}$. □

Theorem 4.4.2 (Uniqueness)**.** *Assume h_3) holds and*

h_4) $\Phi'(1) \geq a_1\Phi(1)$,

h_5) $\frac{\partial g}{\partial u}(x,u) \geq \frac{\partial g}{\partial u}(x,0)$ *for all* $u \neq 0$, $x \in [0,1]$ *and the inequality is strict for* $x \in I$, *with* $I \subset [0,1]$ *of positive measure.*

Then problem (4.1)-(4.3) has a unique solution.

<u>*Proof*</u>:
Suppose we have two solutions, u and $\tilde{u}$, for problem (4.1)-(4.3). Consider $\varphi = u - \tilde{u}$, then φ satisfies

$$\varphi''(x) = g(x,u) - g(x,\tilde{u}).$$

Since g is C^1, due to the Mean Value Theorem, we have

$$\varphi''(x) = \frac{\partial g}{\partial u}(x,\xi)\varphi, \tag{4.14}$$

for some ξ between u and $\tilde{u}$. Moreover, we can choose ξ to be measurable. Multiplying (4.14) by φ, the hypothesis h_5) and integrating by parts, yields

$$\begin{aligned}\int_0^1 \frac{\partial g}{\partial u}(x,0)\varphi^2 dx < &\int_0^1 \frac{\partial g}{\partial u}(x,\xi)\varphi^2 dx \\ &= \int_0^1 \varphi''\varphi dx = \varphi'\varphi\Big|_0^1 - \int_0^1 (\varphi')^2 dx \\ &= a_1\varphi(1)^2 - a_0\varphi(0)^2 - \int_0^1 (\varphi')^2 dx.\end{aligned}$$

Hence, we have

$$2M(\varphi) < a_1\varphi(1)^2 - a_0\varphi(0)^2,$$

a contradiction with Lemma 4.4.1, which concludes the proof. □

On the contrary, when h_4) does not hold, Lemma 4.4.1 combined with a linking theorem gives us the following multiplicity result.

Theorem 4.4.3 (Multiplicity, two different solutions)**.** *Assume h_3) holds and*

h_6) $\Phi'(1) < a_1\Phi(1)$,

h_7) $G(x,u) \geq 0$ *for all* $u \in \mathbb{R}$ *and a.e.* $x \in [0,1]$.

Then, there exists a constant $p_1 > 0$ such that, if $\|p\|_{L^2} < p_1$, problem (4.1)-(4.3) has at least two classical solutions.

Proof:
For the main part of the proof we shall make use of a linking theorem by Rabinowitz (see [44] or Theorem 1.1.21). Let $X = H^1(0,1)$, $X_1 = span\{\Phi\}$, where Φ is the solution of (4.12), and

$$X_2 = \left\{u \in H^1(0,1) : u(1) = 0\right\}.$$

First, let us prove that $X = X_1 \oplus X_2$. Due to h_3), Φ is strictly non decreasing, then $\Phi(1) > 0$ and, thus, $X_1 \cap X_2 = \{0\}$. Moreover, all $u \in H^1(0,1)$ can be written $u = a\Phi + u - a\Phi$, with $a = \frac{u(1)}{\Phi(1)}$.

Let us see that (1.15) holds. First, let us recall that the statement $\Phi'(1) < a_1\Phi(1)$ is equivalent to the condition

$$2M(u) < a_1 u(1)^2 - a_0 u(0)^2, \quad \text{for some } u \in H^1(0,1),$$

where $M(u) = \int_0^1 \left(\frac{1}{2}(u')^2 + \frac{1}{2}\frac{\partial g}{\partial u}(x,0)u^2\right) dx$.

On the one hand, due to Poincaré Inequality, h_7) and rearranging terms, we obtain

$$\inf_{u\in X_2} J(u) = \inf_{u\in X_2}\left(\int_0^1 \left(\frac{1}{2}(u')^2 + G(x,u) + pu\right) dx + \frac{a_0}{2}u(0)^2\right)$$

$$\geq \inf_{u\in X_2}\int_0^1 \left(\frac{1}{2}(u')^2 + p(u-u(0)) + \frac{a_0}{2}u(0)^2 + pu(0)\right) dx$$

$$\geq \inf_{u\in X_2}\left\{\frac{1}{2}\|u'\|_{L^2}^2 - \frac{1}{\pi}\|p\|_{L^2}\|u'\|_{L^2} + \frac{a_0}{2}u(0)^2 - \|p\|_{L^2}u(0)\right\}$$

$$= \inf_{u\in X_2}\left\{\frac{1}{2}\left(\|u'\|_{L^2} - \|p\|_{L^2}\right)^2 - \frac{\|p\|_{L^2}^2}{2\sqrt{\pi}} + \frac{1}{2}\left(\sqrt{a_0}u(0) - \frac{\|p\|_{L^2}}{\sqrt{a_0}}\right)^2 - \frac{\|p\|_{L^2}^2}{2a_0}\right\}$$

$$\geq -\|p\|_{L^2}^2\left(\frac{1}{2\sqrt{\pi}} + \frac{1}{2a_0}\right).$$

On the other hand, for $u = \lambda\Phi$ we obtain

$$J(\lambda\Phi) = \int_0^1 \left(\frac{\lambda^2}{2}(\Phi')^2 + G(x,\lambda\Phi) + \lambda p\Phi\right) dx + \frac{a_0}{2}\lambda^2\Phi(0)^2 - \frac{a_1}{2}\lambda^2\Phi(1)^2 =$$

$$= \lambda^2\left(M(\Phi) + \frac{a_0}{2}\Phi(0)^2 - \frac{a_1}{2}\Phi(1)^2\right) + \int_0^1 (R(x,\lambda\Phi) + \lambda p\Phi)dx.$$

where R is the remainder in the second order Taylor expansion of $G(x,\cdot)$. Fix $\delta > 0$ such that

$$M(\Phi) + \frac{a_0}{2}\Phi(0)^2 - \frac{a_1}{2}\Phi(1)^2 + \delta\|\Phi\|_{L^2}^2 < 0,$$

and (using e.g. dominated convergence) choose $\lambda > 0$ small enough such that $\int_0^1 R(x,\lambda\Phi)\, dx \leq \delta\lambda^2\|\Phi\|_{L^2}^2$.

Then

$$J(\lambda\Phi) < \lambda^2\left(M(\Phi) + \frac{a_0}{2}\Phi(0)^2 - \frac{a_1}{2}\Phi(1)^2 + \delta\|\Phi\|_{L^2}^2\right) + \lambda\|p\|_{L^2}\|\Phi\|_{L^2}.$$

Thus, if $\|p\|_{L^2}$ is small enough, then (1.15) holds.

To conclude the proof, let us verify that J satisfies the (PS) condition. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(0,1)$ such that $|J(u_n)| \leq c$ and $DJ(u_n) \to 0$, we want to see that it has a convergent subsequence.

Since J is coercive, as proven in Theorem 4.2.3, $|J(u_n)| \leq c$ implies that there exists $K > 0$ such that $\|u_n\|_{H^1(0,1)} \leq K$ for all $n \in \mathbb{N}$, then, taking a subsequence if needed, we may assume that there exists $u \in H^1(0,1)$ such that $u_n \to u$ weakly in $H^1(0,1)$ and uniformly. Since DJ is w.l.s.c. and $DJ(u_n)(u) \to 0$, then

$$0 \geq DJ(u)(u) = \int_0^1 \big((u')^2 + g(x,u)u + pu\big)\, dx + a_0 u(0)^2 - a_1 u(1)^2.$$

Due to $DJ(u_n)(u_n) \to 0$, it is seen that $\|u_n'\|_{L^2}^2 \to \|u'\|_{L^2}^2$. Since

$$\|u_n' - u'\|_{L^2}^2 = \|u_n'\|_{L^2}^2 + \|u'\|_{L^2}^2 - 2\int_0^1 u_n' u' dx \to 0,$$

we conclude that $u_n \to u$ for the H^1 norm.

We know from Theorem 4.2.3 that J achieves a global minimum at some $u_0 \in H^1(0,1)$ and, hence, the linking theorem used above provides a second solution u_1 such that $J(u_1) \geq \rho > J(u_0)$, which implies that $u_0 \neq u_1$. □

Combining Theorem 4.4.3 and Proposition 4.3.1 we are able to obtain a third different solution.

Theorem 4.4.4 (Multiplicity, three different solutions)**.** *Assume that h_1), h_3) to h_7) hold and that there exists $C > 0$ such that $p(x) \geq C$ for a.e. x in a neighborhood of* 1*. Then there exists $p_1 > 0$ such that, if $\|p\|_{L^2} < p_1$, then problem (4.1)-(4.3) has at least three classical solutions.*

Proof:
From the properties of the functional, we deduce that the infimum

$$\min_{\{u \in H^1(0,1): u(1) \geq 0\}} J(u)$$

is achieved at some u_2.

Let $u_0 \leq 0$ be a global minimizer of J. Since $p(x) \geq C$ for a.e. x in a neighborhood of 1, we know from Proposition 4.3.1 that $u_0(1) < 0$, hence $u_2 \neq u_0$.

Since $\lambda\Phi(1) > 0$, where λ is chosen in the proof of Theorem 4.4.3, it follows that $J(u_2) < \rho$. Since $J(u_1) \geq \rho$, $u_2 \neq u_1$.

Due to (1.15), $u_2 \notin X_2$, and hence $u_2(1) > 0$. This means that $u_2 \notin \partial\{u \in H^1(0,1) : u(1) \geq 0\}$. We conclude that u_2 is a critical point of J, hence a solution of (4.1)-(4.3), and $u_2 \neq u_0$, u_1. □

Resumen del Capítulo 4

En este capítulo buscamos extensiones al caso general de los resultados obtenidos en [10]. Es decir, consideramos la siguiente ecuación para una función $u : [0,1] \to \mathbb{R}$

$$u''(x) = g(x,u) + p(x),$$

donde $p \in L^2(0,1)$ y $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ es continua y superlineal, es decir:

$$\lim_{|u| \to +\infty} \frac{g(x,u)}{u} = +\infty$$

uniformemente para $x \in [0,1]$, sin pérdida de generalidad, podemos asumir que $g(x,0) = 0$ para toda $x \in [0,1]$; bajo las siguiente condiciones de contorno de tipo radiación

$$u'(0) = a_0 u(0), \quad u'(1) = a_1 u(1),$$

con a_0, $a_1 > 0$.

Como en [10], estudiamos existencia, unicidad y multiplicidad de solución utilizando métodos variacionales y un teorema linking.

Este capítulo está organizado de la siguiente manera. En la Sección 4.2 introducimos la formulación variacional del problema, definimos el funcional $J : H^1(0,1) \to \mathbb{R}$ dado por

$$J(u) = \int_0^1 \left(\frac{1}{2}(u')^2 + G(x,u) + p(x)u \right) dx + \frac{a_0}{2} u(0)^2 - \frac{a_1}{2} u(1)^2,$$

donde $G(x,u) = \int_0^u g(x,s)ds$. Además, probamos que el problema tiene solución, ya que J es coerciva.

En la Sección 4.3 probamos que bajo ciertas hipótesis, que un mínimo global de la funcional J corresponde a una solución negativa del problema; más aún, estudiamos el comportamiento de las soluciones cuando p es grande, concluimos que el problema tiene una única solución.

La Sección 4.4 está dedicada a probar resultados de multiplicidad y unicidad. Para la multiplicidad, probamos que la funcional J satisface las hipótesis de un teorema de tipo linking de Rabinowitz, lo que permite probar la existencia de al menos otro mínimo local y un punto crítico de tipo silla.

Chapter 5

Radiation boundary conditions revisited, a topological approach

5.1 Introduction

In the previous chapter, we studied problem (4.1)-(4.3) via variational methods. The goal of the present chapter is two-fold. On the one hand, we are interested in studying the problem with a different approach, namely, topological methods. On the other hand, we intend to get a better understanding of the results in [10]; in particular, we shall provide alternative proofs of some facts for

$$\begin{cases} u''(x) = Ku(x)^3 + L(x)u(x) + A \\ u'(0) = a_0 u(0), \ u'(1) = a_1 u(1), \end{cases} \tag{5.1}$$

where K and A are some given positive constants and $L(x) := a_0^2 + (a_1^2 - a_0^2)x$, which shows a deep connection between the variational structure of the problem and the topological methods here employed. For simplicity, we shall assume that g is of class C^1 with respect to u.

We consider the following equation for a function $u : [0, 1] \to \mathbb{R}$

$$u''(x) = g(x, u) + p(x), \tag{5.2}$$

where $p \in L^2(0, 1)$ and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and superlinear, namely:

$$\lim_{|u| \to +\infty} \frac{g(x, u)}{u} = +\infty$$

uniformly for $x \in [0, 1]$; under radiation boundary conditions:

$$u'(0) = a_0 u(0), \quad u'(1) = a_1 u(1), \tag{5.3}$$

where a_0, $a_1 > 0$.

By 'solution' we mean a function $u \in H^2(0,1)$ satisfying (5.2)-(5.3). It is clear that if $p \in C([0,1])$ then u is a classical C^2 solution.

This chapter is organized as follows. In the following section, we shall establish a general existence result using the well known method of upper and lower solutions. In Section 5.3, we prove an uniqueness result and introduce a remark regarding Corollary 4.3.5. Section 5.4 is devoted to prove several uniqueness and multiplicity results by combining different techniques such as comparison, the Implicit Function Theorem and the shooting method. Finally, in Section 5.5, we end the chapter with some comments and open questions about the behavior of the shooting operator.

5.2 Upper and lower solutions, existence result

Recall from Section 1.2.3 in Chapter 1, that $\alpha, \beta \in H^2(0,1)$ are respectively a lower and an upper solution of (4.1) if

$$\alpha''(x) \geq g(x, \alpha(x)) + p(x), \qquad \beta''(x) \leq g(x, \beta(x)) + p(x)$$

for a.e. $x \in [0,1]$. If, moreover, $\alpha \leq \beta$ and

$$\alpha'(0) - a_0\alpha(0) \geq 0 \geq \beta'(0) - a_0\beta(0), \ \ \alpha'(1) - a_1\alpha(1) \leq 0 \leq \beta'(1) - a_1\beta(1),$$

then we shall say that (α, β) is an ordered pair of a lower and an upper solution of (5.2)-(5.3).

Theorem 5.2.1 (Existence)**.** *Problem (5.2)-(5.3) has at least one solution.*

Proof:
Let Φ be a second primitive of p and set

$$\alpha(x) := -e^{ax^2+bx+c} + \Phi(x), \quad \beta(x) := e^{mx^2+c} + \Phi(x),$$

with $2a + b > a_1$, $b < a_0$, $m > \frac{a_1}{2}$ and $c \gg 0$. It is readily verified that (α, β) is an ordered pair of a lower and an upper solution of (5.2)-(5.3). Hence by Theorem 1.2.8, there exists $u \in H^2([0,1])$ such that $\alpha(x) \leq u(x) \leq \beta(x)$ for $x \in [0,1]$, and so completes the proof. $\square$

5.3 Uniqueness

By using a comparison argument we shall prove, under appropriate assumptions, that the solution is unique.

Theorem 5.3.1 (Uniqueness). *Assume that*

$H_1)$ $a_1 < a_0$,

$H_2)$ $\frac{\partial g}{\partial u}(x,u) \geq a_1^2$ *for all* $x \in [0,1]$ *and* $u \in \mathbb{R}$.

Then problem (5.2)-(5.3) has a unique solution. Moreover, the result still holds if $a_1 = a_0$ *and* $H_2)$ *is strict for all* $u \in \mathbb{R}$ *and* $x \in I$, *where* $I \subset [0,1]$ *has positive measure.*

It is worth noticing that $H_2)$ may be replaced by the condition (still valid for non-differentiable g) that the function $\xi_x(u) := g(x,u) - a_1^2 u$ is nondecreasing for all x. If, furthermore, ξ_x is strictly increasing for all x belonging to a positive measure set, then uniqueness also holds when $a_0 = a_1$. This is clearly satisfied when $g(x,u) = Ku^3 + L(x)u$ with L as before; hence, the referred uniqueness result obtained in [10] for (5.1) (see Chapter 2), can be regarded as a consequence of Theorem 5.3.1.

Proof:
Let u_1 and u_2 be solutions of (5.2)-(5.3) and define $w := u_1 - u_2$. Then w is a solution of the following linear problem

$$\begin{cases} z''(x) = \frac{\partial g}{\partial u}(x,\xi(x))z(x) \\ z'(0) = a_0 z(0), \quad z'(1) = a_1 z(1), \end{cases} \tag{5.4}$$

for some measurable $\xi(x)$ between $u_1(x)$ and $u_2(x)$.

If $w(0) = 0$, then $w'(0) = 0$ and consequently $w \equiv 0$. Otherwise, we may assume that $w(0) > 0$ and hence $w > 0$ in $[0,1]$. Let $v(x) = e^{a_1 x}$, then

$$v''(x)w(x) = a_1^2 v(x)w(x) \leq \frac{\partial g}{\partial u}(x,\xi(x))w(x)v(x) = w''(x)v(x).$$

Integrating by parts, we deduce that

$$(a_0 - a_1)w(0) \leq 0,$$

which contradicts $H_1)$. If, instead, we assume that $a_0 = a_1$ and $H_2)$ is strict over some $I \subset [0,1]$ of positive measure, then the inequality $v''w \leq vw''$ is strict over I and hence

$$v'(1)w(1) - v'(0)w(0) < v(1)w'(1) - v(0)w'(0),$$

a contradiction. $\square$

5.4 Multiplicity

As a complement of the previous result, multiplicity of solutions can be proven when $a_1 \geq a_0$ and p satisfies some smallness condition. The latter condition may be dropped if $a_1 \gg 0$.

Theorem 5.4.1 (Positive and negative solutions for $\|p\|_\infty$ small)**.** *Assume*

$H_3)$ $a_1 \geq a_0$,

$H_4)$ $\frac{\partial g}{\partial u}(x,0) < a_1^2$ *for all* $x \in [0,1]$.

Then there exists a continuous function $q : [0,1] \to (0,+\infty)$ *such that (5.2)-(5.3) has at least one positive solution* u_P*, provided that* $p(x) \leq q(x)$ *for a.e.* $x \in [0,1]$*, and one negative solution* u_N*, provided that* $p(x) \geq -q(x)$ *for a.e.* $x \in [0,1]$.

It is worthy to notice that, for problem (5.1), condition $H_4)$ does not hold when $x = 1$.

Proof:
From $H_4)$ we can fix $\varepsilon > 0$ such that $\frac{\partial g}{\partial u}(x,u) < a_1^2$ for $0 < u < \varepsilon$. Next, fix $c > a_1 - \ln \varepsilon$ and define

$$\alpha(x) := e^{a_1 x - c}.$$

From the Mean Value Theorem,

$$g(x,\alpha(x)) = \frac{\partial g}{\partial u}(x,\xi(x))\alpha(x)$$

where $0 < \xi(x) < \alpha(x) < \varepsilon$, then

$$\alpha''(x) = a_1^2\alpha(x) > g(x,\alpha(x)).$$

Let $q : [0,1] \to \mathbb{R}$ be given by $q(x) := a_1^2\alpha(x) - g(x,\alpha(x))$, then from $H_4)$ we deduce that α is a lower solution if $p(x) \leq q(x)$. Once p is fixed, it suffices to consider an upper solution $\beta \gg 0$ as in the proof of Theorem 5.2.1. By Theorem 1.2.8, there exists a positive solution u_P for problem (5.2)-(5.3) with $\alpha \leq u_P \leq \beta$.

Next, define the change of variables $v(x) := -u(x)$ so the problem becomes

$$\begin{cases} v''(x) = -g(x,-v(x)) - p(x) \\ v'(0) = a_0 v(0), \quad v'(1) = a_1 v(1). \end{cases} \tag{5.5}$$

The function $f(x,v) := -g(x,-v)$ is superlinear, with $f(x,0) = 0$ and $\frac{\partial f}{\partial v}(x,0) = \frac{\partial g}{\partial u}(x,0)$ for all $x \in [0,1]$. Hence, if $p(x) \geq -q(x)$ then problem (5.5) has a positive solution v_P, and $u_N = -v_P$ is a negative solution of (5.2)-(5.3).

$\square$

Remark 5.4.2. As mentioned before, condition H_4) is not satisfied in the case of equation (5.1). However, the existence of u_P is directly verified taking the lower solution $\alpha(x) = e^{\frac{2L(x)^{3/2}}{3(a_1^2-a_0^2)}-b}$ for A small and $b \gg 0$.

Remark 5.4.3. Note that hypotheses H_3) and H_4) can be compared with h_6) of Theorem 4.4.3 in Chapter 4 in the following sense.

Remember that Φ is the solution of the linear problem

$$\begin{cases} -\Phi'' + \frac{\partial g}{\partial u}(x,0)\Phi = 0 \\ a_0\Phi(0) = \Phi'(0) = a_0, \end{cases}$$

hence, due to H_4), direct comparison with $v(x) = e^{a_1 x}$ yields

$$\Phi'' v < v''\Phi.$$

Integrating by parts and due to H_3) we get

$$(\Phi'(1) - a_1\Phi(1))v(1) < a_0 - a_1 \leq 0,$$

thus $\Phi'(1) < a_1\Phi(1)$, which is hypothesis h_6)!

5.4.1 Implicit Function Theorem, multiplicity for $\|p\|_{L^2}$ small

The next result proves that the smallness condition on p can be improved by the use of Implicit Function Theorem.

Theorem 5.4.4 (Multiplicity of solutions for $\|p\|_{L^2}$ small)**.** *Assume that H_3) and H_4) hold and let u_P, u_N be the solutions of (5.2)-(5.3) given by Theorem 5.4.1 for $p = 0$. Further, assume:*

H_5) $\frac{\partial g}{\partial u}(x,u) \geq 0$

for all the pairs (x,u) with $x \in [0,1]$ and $u = u_P(x)$ or $u = u_N(x)$.

H_6) *Condition $\frac{\partial g}{\partial u} > \frac{g(x,u)}{u}$ holds for all the pairs (x,u) with $x \in [0,1]$ and $u = u_P(x)$ or $u = u_N(x)$.*

H_8) $\frac{\partial g}{\partial u}(x,0) \geq 0$ *for all $x \in [0,1]$.*

Then there exists $p_1 > 0$ such that problem (5.2)-(5.3) has at least three solutions, provided that $\|p\|_{L^2} \leq p_1$.

Proof:
Consider the space

$$X = \{u \in H^2(0,1) : u'(0) = a_0 u(0),\ u'(1) = a_1 u(1)\}$$

equipped with the standard H^2-norm, and $F : X \times L^2(0,1) \to L^2(0,1)$ given by

$$F(u,p) = u'' - g(\cdot, u) - p.$$

In this setting, solving problem (5.2)-(5.3) is equivalent to find solutions to the equation $F(u,p) = 0$. It is readily verified that F is of class C^1, with

$$D_u F(u,0)(\varphi) = \varphi'' - \frac{\partial g}{\partial u}(\cdot, u)\varphi.$$

In order to prove that $D_u F(u_P, 0)$ is a monomorphism, suppose that $\varphi \in X \setminus \{0\}$ satisfies $\varphi'' = \frac{\partial g}{\partial u}(\cdot, u_P)\varphi$. Since $\frac{\partial g}{\partial u}(\cdot, u_P) \geq 0$, we may assume that $\varphi(x) > 0$ for all x and, from H_6), we obtain:

$$u_P''(x)\varphi(x) = g(x, u_P(x))\varphi(x) < \frac{\partial g}{\partial u}(x, u_P(x))\varphi(x)u_P(x) = \varphi''(x)u_P(x).$$

Integration by parts yields

$$u_P'\varphi\Big|_0^1 < u_P\varphi'\Big|_0^1,$$

a contradiction since $u_P, \varphi \in X$. Thus, $D_u F(u_P, 0)$ is a monomorphism and, by Theorem 1.1.8, we conclude that it is an isomorphism. In a similar way, we deduce that $D_u F(u_N, 0)$ is also an isomorphism.

Finally, let us prove that $D_u F(0,0)$ is a monomorphism and consequently and isomorphism. Suppose that $\varphi \in X \setminus \{0\}$ is such that $D_u F(0,0)(\varphi) = 0$. We may assume, without loss of generality, that $\varphi(0) > 0$ and hence, by H_8), $\varphi > 0$. Let $v(x) = e^{a_1 x}$, then from H_4) we deduce:

$$v''(x)\varphi(x) = a_1^2 v(x)\varphi(x) > \frac{\partial g}{\partial u}(x,0)\varphi(x)v(x) = \varphi''(x)v(x).$$

Upon integration, we obtain:

$$a_1 v(1)\varphi(1) - a_1 v(0)\varphi(0) > a_1\varphi(1)v(1) - a_0\varphi(0)v(0).$$

This, in turn, implies $(a_0 - a_1)\varphi(0)v(0) > 0$, which contradicts H_3).

Since $F(0,0) = F(u_P, 0) = F(u_N, 0) = 0$, the Implicit Function Theorem guarantees that, taking smaller neighborhoods of 0, u_P and u_N if needed, equation $F(u,p) = 0$ has at least three different solutions when $\|p\|_{L^2}$ is small. □

Remark 5.4.5. In the previous proof, it is observed that assumptions H_5) and H_6) might be replaced by any other guaranteeing that $D_uF(u_P, 0)$ and $D_uF(u_N, 0)$ are isomorphisms. For example, we may have assumed that H_2) holds (strictly if $a_0 = a_1$) for all the pairs $(x, u_P(x))$ and $(x, u_N(x))$. However, in view of H_4) this would be an artificial condition since it could not be assumed for all (x, u).

5.4.2 Shooting method, multiplicity of solutions for a_1 large

In this section, we shall define a shooting operator that will allow us to obtain different conclusions concerning existence and uniqueness or multiplicity of the solutions of (5.2)-(5.3). With this aim, let us firstly state the following lemma, which ensures that, under a monotonicity assumption, the graphs of two different solutions of equation (4.1), with initial condition $u'(0) = a_0u(0)$, do not intersect. More precisely,

Lemma 5.4.6. *Let u_1 and u_2 be solutions of (4.1) satisfying the first boundary condition of (4.3), defined over an interval $[0, b]$ such that $u_1(0) > u_2(0)$ and $u_1'(0) > u_2'(0)$ and assume that H_5) holds for all $x \in [0, b]$ and all u between $u_1(x)$ and $u_2(x)$. Then $u_1 > u_2$ on $[0, b]$.*

Proof:
Set $u(x) = u_1(x) - u_2(x)$, then $u''(x) = \theta(x)u(x)$ on $[0, b)$, where $\theta(x) := \frac{\partial g}{\partial u}(x, \xi(x)) \geq 0$. Thus, the result follows since $u(0), u'(0) > 0$. $\square$

Next, we define a shooting operator as follows. For each fixed $\lambda \in \mathbb{R}$, let u_λ be the unique solution of problem

$$\begin{cases} u''(x) = g(x, u(x)) + p(x) \\ u(0) = \lambda, \quad u'(0) = a_0\lambda \end{cases} \tag{5.6}$$

and define the function $T : \mathcal{D} \to \mathbb{R}$, by

$$T(\lambda) = \frac{u_\lambda'(1)}{u_\lambda(1)},$$

where $\mathcal{D} \subset \mathbb{R}$ is the set of values of λ such that the corresponding u_λ solution of (5.6) is defined on $[0, 1]$ and $u_\lambda(1) \neq 0$. Thus, solutions of (5.2)-(5.3) that do not vanish on $x = 1$ can be characterized as the functions u_λ, where $\lambda \in \mathcal{D}$ is such that $T(\lambda) = a_1$.

Let us fix a constant $R \gg 0$. By comparision arguments, we can prove that if λ is large enough, the solution u_λ is positive and non decreasing and it reaches the value R at some $x_0 < 1$. Similarly, for $\lambda \ll 0$, the solution u_λ reaches the value $-R$.

Therefore, the function

$$\phi_R(\lambda) = \begin{cases} -R & \text{if there exists } x_0 < 1/u_\lambda(x_0) \le -R \\ u_\lambda(1) & \text{if } -R < u_\lambda < R \text{ in } [0,1) \\ R & \text{if there exists} x_0 < 1/u_\lambda(x_0) \ge R \end{cases}$$

is well defined and continuous. Then, for each $r \in (-R, R)$, there exists λ such that $u_\lambda(1) = r$. By Lemma 5.4.6, this value of λ is unique when H_5) is assumed for all $u \in \mathbb{R}$. In particular, there exists a unique λ_0 such that $u_{\lambda_0} = 0$. Thus, we conclude that

$$\mathcal{D} = (\lambda_*, \lambda_0) \cup (\lambda_0, \lambda^*)$$

for some $\lambda_* \ge -\infty$ and $\lambda^* \le +\infty$.

Throughout the rest of the section we shall also assume a condition that guarantees that $\lambda_0 < 0$, namely:

H_7) $g(x,u) + p(x) \ge C > 0$ for a.e. $x \in [0,1]$ and $u \ge 0$.

Furthermore, H_7) implies that if $\lambda > 0$ then u_λ is positive and if $\lambda_0 \le \lambda \le 0$ then u_λ vanishes exactly once in $[0,1]$. In particular, $u_{\lambda_0} < 0$ on $[0,1)$ and, since $u''_{\lambda_0}(x) > 0$ when x is close to 1, we conclude that $u'_{\lambda_0}(1) > 0$. In consequence:

$$\lim_{\lambda \to \lambda_0^-} T(\lambda) = -\infty, \qquad \lim_{\lambda \to \lambda_0^+} T(\lambda) = +\infty.$$

We claim that also

$$\lim_{\lambda \to (\lambda^*)^-} T(\lambda) = +\infty, \qquad \lim_{\lambda \to (\lambda_*)^+} T(\lambda) = +\infty.$$

Indeed, observe firstly that, because solutions of (5.6) do not cross each other, $\lim_{\lambda \to (\lambda^*)^-} u_\lambda(1) = +\infty$. On the other hand, multiplying (4.1) by u' it is easy to see, given $M > 0$ that

$$|u'_\lambda(1)| \ge \sqrt{M}\mathcal{O}(|u_\lambda(1)|)$$

for $|\lambda|$ sufficiently large. This implies that

$$|T(\lambda)| = \left| \frac{u'_\lambda(1)}{u_\lambda(1)} \right| > \sqrt{M}$$

and the claim follows.

The previous considerations show the existence of $\lambda_{min} \in (\lambda_0, \lambda^*)$ such that $T(\lambda_{min}) \le T(\lambda)$ for all $\lambda \in (\lambda_0, \lambda^*)$. The value $a_{min} := T(\lambda_{min})$ depends on p. If $p \ge p_0$ and $a_{min} \le a_1$, then we would have a positive solution, which contradicts Theorem 4.3.2, then $a_{min} > a_1$.

In this setting, we can easily prove the following result.

Theorem 5.4.7 (Multiplicity of solutions for a_1 large)**.** *Assume that H_5) is satisfied for all $u \in \mathbb{R}$, H_7) holds and that $p \in L^\infty(0,1)$. Then (5.2)-(5.3) has at least three different solutions, provided that a_1 is large enough. Moreover, at least one of the solutions is negative.*

<u>*Proof*</u>:
By continuity, there exists $\lambda < \lambda_0$ such that $T(\lambda) = a_1$; the corresponding u_λ is a negative solution of (5.2)-(5.3). Moreover, if a_1 is large enough, then $a_{min} < a_1$ and the equation $T(\lambda) = a_1$ has at least two solutions in (λ_0, λ^*).

$\square$

Remark 5.4.8. For the particular case of problem (5.1), the computations mentioned in Remark 4.3.6 provide a lower bound for a_{min}. Also, it is readily seen that $a_{min} > \min\{a_0, a_1\}$. This provides an alternative proof of the fact that the problem has no positive nor sign-changing solutions when $a_1 \leq a_0$. Indeed, fix $\lambda \in (\lambda_0, \lambda^*)$ and let $v(x) := e^{rx}$, where $r := \min\{a_0, a_1\}$. Setting $x_0 := \inf\{x \in [0,1] : u_\lambda(y) > 0 \text{ in } [x,1]\}$ we deduce, for $x \in (x_0, 1)$, that

$$v(x)u_\lambda''(x) > v(x)L(x)u_\lambda(x) > v(x)r^2u_\lambda(x) = v''(x)u_\lambda(x).$$

Thus,

$$v(1)[u_\lambda'(1) - ru_\lambda(1)] > v(x_0)[u_\lambda'(x_0) - ru_\lambda(x_0)] \geq 0$$

and we conclude that $T(\lambda) = \frac{u_\lambda'(1)}{u_\lambda(1)} > r$.

A similar result holds for the general case, namely: if there exists $r \leq a_0$ such that $g(x,u) + p(x) > r^2u$ for all $u > 0$ and all x, then $a_{min} > r$.

Remark 5.4.9. Theorem 5.4.4 is sharp, in the following sense. Assume that $p = 0$ and $\dfrac{\partial g}{\partial u}(x,u) > \dfrac{g(x,u)}{u}$ holds for $u \neq 0$, then the problem has at most one negative solution and at most one positive solution. This follows from the fact that T is stricty decreasing in $(\lambda_*, 0)$ and strictly increasing in $(0, \lambda^*)$. Indeed, assume for example that $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let $0 < u_{\lambda_1} < u_{\lambda_2}$ be the corresponding solutions of problem (5.6). Because $p = 0$, condition $\dfrac{\partial g}{\partial u}(x,u) > \dfrac{g(x,u)}{u}$ implies that $\dfrac{g(x,u)}{u}$ is strictly nondecreasing for $u > 0$, so

$$u_{\lambda_2}(x)u_{\lambda_1}''(x) = g(x, u_{\lambda_1}(x))u_{\lambda_2}(x) < g(x, u_{\lambda_2}(x))u_{\lambda_1}(x) = u_{\lambda_2}''(x)u_{\lambda_1}(x).$$

Thus,

$$u_{\lambda_2}(1)u_{\lambda_1}'(1) < u_{\lambda_2}'(1)u_{\lambda_1}(1),$$

and we conclude that $T(\lambda_1) < T(\lambda_2)$. In a similar way, we deduce that T is strictly decreasing in $(\lambda_*, 0)$.

For the particular case of equation (5.1), when $A = 0$ it is clear that T is an even function and can be extended continuously to $\lambda = 0$. As in Remark 5.4.8, it is seen that if $\lambda > 0$ then $T(\lambda) > \min\{a_0, a_1\}$. This implies that $T(0) \geq \min\{a_0, a_1\}$; thus the problem for $A = 0$ has only the trivial solution if $a_0 \geq a_1$.

On the other hand, if $s > \max\{a_0, a_1\}$ and $0 < \lambda \ll 1$, then $u_\lambda'' < s^2 u_\lambda$ and hence $T(\lambda) < s$. Hence, $T(0) \leq \max\{a_0, a_1\}$. In particular, when $a_0 < a_1$ it is directly seen that $T(0) < a_1$ and the problem has exactly 3 solutions.

5.5 Comments and open questions

We shall end this chapter with some comments and open questions about the behavior of T. With this aim, let us compute

$$T'(\lambda) = \frac{\partial}{\partial\lambda}\left(\frac{u_\lambda'(1)}{u_\lambda(1)}\right) = \frac{u_\lambda(1)\frac{\partial u_\lambda'}{\partial\lambda}(1) - u_\lambda'(1)\frac{\partial u_\lambda}{\partial\lambda}(1)}{u_\lambda(1)^2}$$

and set $w := \dfrac{\partial u_\lambda}{\partial\lambda}$, then

$$T'(\lambda) = \frac{u_\lambda(1)w'(1) - u_\lambda'(1)w(1)}{u_\lambda(1)^2}.$$

Moreover, observe that w solves the linear problem

$$\begin{cases} w''(x) = \frac{\partial g}{\partial u}(x, u_\lambda(x))w(x) \\ w(0) = 1, \quad w'(0) = a_0, \end{cases} \tag{5.7}$$

and hence

$$\begin{aligned} u_\lambda(1)w'(1) - u_\lambda'(1)w(1) &= \int_0^1 \left(u_\lambda(x)w''(x) - u_\lambda''(x)w(x)\right)dx = \\ &= \int_0^1 \left(u_\lambda(x)\frac{\partial g}{\partial u}(x, u_\lambda(x)) - g(x, u_\lambda(x)) - p(x)\right)w(x)dx. \end{aligned} \tag{5.8}$$

In particular, if h_2) of Chapter 4 holds, namely $\frac{\partial g}{\partial u}(x, u) > \frac{g(x,u)+p(x)}{u}$ for a.e. $x \in [0, 1]$ and $u \in \mathbb{R}_{<0}$, then T is strictly decreasing for $\lambda < \lambda_0$. Indeed, from H_5) and (5.7), we deduce that $w(x) > 0$ for all $x \in [0, 1]$. Moreover, $u_\lambda(x) < 0$ for all $x \in [0, 1]$; hence, from $\frac{\partial g}{\partial u}(x, u) > \frac{g(x,u)+p(x)}{u}$ and (5.8) we conclude that $T'(\lambda) < 0$. This proves, again, that the problem has exactly one negative solution.

Remark 5.5.1. More precise conclusions can be obtained for the particular case of equation (5.1), in which the function T depends on the value a_1. Indeed, setting w as before, we deduce:

$$u_\lambda(1)w'(1) - u'_\lambda(1)w(1) = \int_0^1 [2Ku_\lambda(x)^3 - A]w(x)\,dx.$$

The proof that T is decreasing for $\lambda < \lambda_0$ is now straightforward. Furthermore, observe that if $2K\lambda^3 \geq A$, then $2Ku_\lambda(x)^3 > A$ for all $x > 0$; thus, if $\sqrt[3]{\frac{A}{2K}} < \lambda^*$ then T increases strictly in $\left(\sqrt[3]{\frac{A}{2K}}, \lambda^*\right)$.

Also, observe that if $a_1 > a_0$ and p is sufficiently small, then the solution u_0 of (5.6) for $\lambda = 0$ satisfies $u_0(x) < 1$ for all $x \leq 1$ which, in turn, implies that $u_0''(x) < (K + a_1^2)u_0(x) + A$. Take $v(x) := \frac{A}{c^2}(\cosh(cx) - 1)$, where $c := \sqrt{K + a_1^2}$, then $u_0(1) < v(1) = kA$, where $k := \frac{\cosh(cx)-1}{c^2}$. Hence, if p is small and $0 \leq \lambda \ll 1$, then $2Ku_\lambda(x)^3 < 2K(kA)^3 < A$ and we conclude that $T'(\lambda) < 0$. Furthermore, for $\lambda_0 < \lambda < 0$ it is seen that $u_\lambda(x) < 0$ up to some x_0 and $0 < u_\lambda(x) < u_0(x)$ for $x > x_0$. We deduce, when A is small, that $2Ku_\lambda^3 < A$ and hence $T' < 0$ also on $(\lambda_0, 0)$, and the problem has at most one sign-changing solution.

Next, consider the function

$$\alpha(x) = e^{\frac{2L(x)^{3/2}}{3(a_1^2 - a_0^2)} - b}.$$

Making A smaller if necessary, a simple computation shows that α is a lower solution of (5.1) for $b \gg 0$, as seen in Remark 5.4.2. Take $\lambda = \alpha(0)$, then it is verified that $\alpha > u_\lambda > 0$ and consequently $\alpha'' u_\lambda > \alpha u_\lambda''$. It is deduced that $\alpha'(1)u_\lambda(1) > \alpha(1)u'_\lambda(1)$, that is: $T(\lambda) < a_1$.

Thus, when $a_1 > a_0$ and A is small, there are two possible situations:

1. $T(0) \geq a_1$ and the problem has at least two nonnegative solutions (and no sign-changing solutions).

2. $T(0) < a_1$ and the problem has at least one positive solution and a (unique) sign-changing solution.

As mentioned, if $A = 0$ and $a_1 > a_0$ then $T(0) < a_1$; thus, the second situation holds for $A > 0$ sufficiently small.

Similar considerations show that T is increasing for $\lambda \geq 0$ when A is sufficiently large. In this case, $\lambda_{min} \in (\lambda_0, 0)$ and $a_{min} > a_1$, which implies that the problem has a unique (negative) solution. It is readily seen, however, that if $A \gg 0$ then $\lambda^* < 0$, so the statement that T increases for $\lambda > 0$ becomes empty.

5.5.1 Open questions

1. Numerical experiments for the particular case (5.1) suggest that $T'' > 0$ for $\lambda > \lambda_0$. If this is true, then an exact multiplicity result yields, depending on whether a_{min} is smaller, equal or larger than a_1. It would be interesting if this fact could be verified for the general case, under appropriate conditions, using the differential equation for $z := \frac{\partial w}{\partial \lambda} = \frac{\partial^2 u_\lambda}{\partial \lambda^2}$, namely

$$z''(x) = \frac{\partial g}{\partial u}(x, u_\lambda(x))z(x) + \frac{\partial^2 g}{\partial u^2}(x, u_\lambda(x))w(x)^2, \quad z(0) = z'(0) = 0.$$

2. How does the graph of T vary with respect to p? Suppose for simplicity that p is a constant and let $y := \frac{\partial u_\lambda}{\partial p}$. Then

$$y'' = \frac{\partial g}{\partial u}(x, u_\lambda(x))y + 1 \qquad y(0) = y'(0) = 0$$

and the sign of $\frac{\partial T}{\partial p}$ coincides with the sign of the integral

$$\int_0^1 \left(u_\lambda(x)\frac{\partial g}{\partial u}(x, u_\lambda(x)) - g(x, u_\lambda(x)) - p \right) y(x) + u_\lambda(x)\, dx.$$

If $\frac{\partial g}{\partial u}(x,u) > \frac{g(x,u)+p(x)}{u}$ holds for $u < 0$, then $\frac{\partial T}{\partial p} < 0$ for $\lambda < \lambda_0$. This is consistent with the fact, proven in [10] and easily extended to the general case, that the (unique) negative solution tends uniformly to $-\infty$ as $p \to +\infty$.

For problem (5.1), $\frac{\partial g}{\partial u}(x,u) > \frac{g(x,u)+p(x)}{u}$ is not satisfied for all $u > 0$, although it holds when $2Ku^3 > A$. In consequence, $\frac{\partial T}{\partial A} > 0$ for $\lambda \geq \sqrt[3]{\frac{A}{2K}}$.

Resumen del Capítulo 5

En el capítulo anterior, estudiamos el siguiente problema

$$\begin{cases} u''(x) = Ku(x)^3 + L(x)u(x) + A \\ u'(0) = a_0 u(0),\ u'(1) = a_1 u(1), \end{cases}$$

con métodos variacionales. El objetivo para este capítulo consta de dos partes. Por un lado, estamos interesados en estudiar el problema desde otro enfoque, utilizando métodos topológicos. Por el otro lado, buscamos entender mejor los resultados de [10]; en particular, damos pruebas alternativas para p constante y $g(x,u) = Ku(x)^3 + L(x)u(x)$, donde K es una constante dada y $L(x) := a_0^2 + (a_1^2 - a_0^2)x$. Lo que muestra una profunda conexión entre la estructura variacional del problema y los métodos topológicos empleados.

Este capítulo está organizado de la siguiente manera. En la Sección 5.2 establecemos un resultado general de existencia utilizando el método de super y subsoluciones.

En la Sección 5.3, probamos un resultado de unicidad e introducimos una observación respecto al Corolario 4.3.5.

La Sección 5.4 está dedicada a probar varios resultados de unicidad y multiplicidad de solución combinando distintas técnicas, tales como comparación, el Teorema de la Función Implícita y el método de shooting.

Finalmente, en la Sección 5.5, terminamos el capítulo con algunos comentarios y preguntas abiertas sobre el comportamiento del operador de shooting.

Chapter 6

Second order system with radiation boundary conditions

6.1 Introduction

Our aim in this chapter is to generalize some of the results of chapters 4 and 5 to a system of equations. Namely, we study the following system for a vector function $u : [0, 1] \subset \mathbb{R} \to \mathbb{R}^N$ satisfying

$$\begin{cases} u'' = g(x, u) + p(x) \\ u'(0) = a_0 u(0), \ u'(1) = a_1 u(1), \end{cases} \tag{6.1}$$

where $g : [0, 1] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous superlinear function, namely

$$\lim_{|u| \to +\infty} \frac{g(x, u) \cdot u}{|u|^2} = +\infty,$$

uniformly in $x \in [0, 1]$ and $p \in L^2$.

As we will give a general existence result, there is no need to separate $p \in L^2$. Hence, we do not assume $g(x, 0) = 0$ for all $x \in [0, 1]$. For simplicity, we assume $g \in C([0, 1] \times \mathbb{R}^N)$, although the result holds for the general case.

Notice that Theorem 4.2.3 can be easily generalized in this context.

Theorem 6.1.1. *Problem (6.1), where $g = \nabla G$, where $G : [0, 1] \times \mathbb{R}^N \to \mathbb{R}$ is a superlinear C^1 function with respect to u, has at least one solution.*

The proof of this result is a direct adaptation of the one of Theorem (4.2.3).

This chapter is organized as follows. In Section 6.2 we introduce a generalized Hartman condition to prove and existence result in a nonvariational setting. We conclude in Section 6.3 with some open questions and future line of investigation.

6.2 Generalized Hartman condition

In this section, we prove an existence result of (6.1) in a general nonvariational setting.

Let us note that, due to the superlinearity of g, the classical strict Hartman condition (see Section 1.2.5 in Chapter 1) is satisfied, i.e., there exists a constant $R > 0$ such that

$$g(x, u).u > 0$$

for all $x \in [0, 1]$ and $|u| = R$. However, this fact is not enough to prove existence of solution.

Let $R : [0, 1] \to \mathbb{R}$, given by $R(x) = e^{ax^2+bx+c}$, where a, b and $c \in \mathbb{R}$ are constants. If $b < a_0$ and $2a + b > a_1$, then

$$R'(0) < a_0 R(0) \quad \text{and} \quad R'(1) > a_1 R(1). \tag{6.2}$$

Moreover, by the superlinearity of g, we can choose $c \gg 0$ such that

$$g(x, v).v > R''(x)R(x) + R'(x)^2, \quad \text{if } |v| = R(x), \tag{6.3}$$

for all $x \in (0, 1)$.

Observe that (6.3) is a generalized Hartman condition, since it depends on the value of x.

Theorem 6.2.1. *Problem (6.1) admits at least one classical solution u such that $|u(x)| < R(x)$, for all $x \in [0, 1]$.*

Proof:
For each $x \in [0, 1]$, let us consider the following truncation function $P_x : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$P_x(u) = \begin{cases} u & \text{if } |u| < R(x) \\ R(x)\frac{u}{|u|} & \text{if } |u| \geq R(x). \end{cases}$$

Let us consider the following modified problem for λ, $\mu > 0$

$$\begin{cases} u''(x) - \lambda u(x) = g(x, P_x(u(x))) - \lambda P_x(u(x)) \\ u'(0) - \mu u(0) = (a_0 - \mu)P_0(u(0)), \\ u'(1) + \mu u(1) = (a_1 + \mu)P_1(u(1)). \end{cases} \tag{6.4}$$

Let us note that the boundary conditions in (6.4) are the classical Robin conditions, as in the proof of Theorem 1.2.8, and since the right hand side of the equation is bounded, so by Schauder's Fixed Point Theorem (see Theorem 1.2.4), problem (6.4) has at least one solution u.

Finally, let us prove that u is a solution of the original problem (6.1). We want to see that $|u(x)| < R(x)$ for all $x \in [0,1]$.

Let $\phi : [0,1] \to \mathbb{R}$,

$$\phi(x) = |u(x)| - R(x).$$

Suppose ϕ reaches its absolute maximum at x_0 such that $\phi(x_0) \geq 0$. We have tree cases:

$\underline{x_0 = 0}$: in this case, $|u(0)| \geq R(0)$, then $P_0(u(0)) = R(0)\frac{u(0)}{|u(0)|}$. ϕ has a maximum at 0 and then decreases, so $\phi'(0) \leq 0$. Then

$$\begin{aligned} 0 \geq \phi'(0) &= \frac{u(0).u'(0)}{|u(0)|} - R'(0) = \\ &= \frac{u(0)}{|u(0)|}.\left(\frac{(a_0-\mu)R(0)u(0) + \mu u(0)\,|u(0)|}{|u(0)|}\right) - R'(0) \geq \\ &\geq \frac{u(0)}{|u(0)|^2}.((a_0-\mu)R(0)u(0) + \mu R(0)u(0)) - R'(0) = \\ &= \frac{u(0)}{|u(0)|^2}.a_0R(0)u(0) - R'(0) = a_0R(0) - R'(0), \end{aligned}$$

thus $R'(0) \geq a_0 R(0)$, a contradiction with (6.2).

$\underline{x_0 = 1}$: it is analogous to the previous case.

$\underline{x_0 \in (0,1)}$: in this case, $P_{x_0}(u(x_0)) = R(x_0)\frac{u(x_0)}{|u(x_0)|}$, $\phi'(x_0) = 0$ and $\phi''(x_0) \leq 0$. Then

$$0 \geq \phi''(x_0) = \frac{|u'(x_0)|^2 + u(x_0).u''(x_0)}{|u(x_0)|} - \frac{(u(x_0).u'(x_0))^2}{|u(x_0)|^3} - R''(x_0). \tag{6.5}$$

On the one hand, using (6.4) and the inequality $|u(x_0)| \geq R(x_0)$ we have

$$\begin{aligned} &\frac{|u'(x_0)|^2 + u(x_0).u''(x_0)}{|u(x_0)|} \geq \\ &\geq \frac{u(x_0).\left(g\left(x_0, R(x_0)\frac{u(x_0)}{|u(x_0)|}\right) - \lambda R(x_0)\frac{u(x_0)}{|u(x_0)|} + \lambda u(x_0)\right)}{|u(x_0)|} = \\ &= \frac{u(x_0)}{|u(x_0)|}.g\left(x_0, R(x_0)\frac{u(x_0)}{|u(x_0)|}\right) - \lambda R(x_0) + \lambda\,|u(x_0)| \geq \\ &\geq \frac{1}{R(x_0)}R(x_0)\frac{u(x_0)}{|u(x_0)|}g\left(x_0, R(x_0)\frac{u(x_0)}{|u(x_0)|}\right). \end{aligned} \tag{6.6}$$

On the other hand, since $\phi'(x_0) = 0$, we have that $\dfrac{(u(x_0).u'(x_0))^2}{|u(x_0)|^2} = R'(x_0)^2$. Hence

$$-\frac{(u(x_0).u'(x_0))^2}{|u(x_0)|^3} = -\frac{R'(x_0)^2}{|u(x_0)|} \geq -\frac{R'(x_0)^2}{R(x_0)}. \tag{6.7}$$

Replacing (6.6) and (6.7) in (6.5) we get

$$0 \geq \frac{1}{R(x_0)} R(x_0) \frac{u(x_0)}{|u(x_0)|} g\left(x_0, R(x_0)\frac{u(x_0)}{|u(x_0)|}\right) - \frac{R'(x_0)^2}{R(x_0)} - R''(x_0).$$

Then

$$R(x_0)\frac{u(x_0)}{|u(x_0)|} g\left(x_0, R(x_0)\frac{u(x_0)}{|u(x_0)|}\right) \leq R'(x_0)^2 + R''(x_0)R(x_0),$$

a contradiction with (6.3).

We have proved that $|u(x)| < R(x)$ for all $x \in [0,1]$, hence we can conclude that u is a classical solution of (6.1).

□

6.3 Future line of investigation, open questions

To conclude, this section presents some guidelines for future research related to this chapter.

Besides existence, it would be interesting to study whether or not the uniqueness/multiplicity results of the preceding chapters can be extended to a system of equations. In this direction, the following remark should be of help.

Remark 6.3.1. Let K be an appropriate fixed point operator for problem (6.1), then it is not difficult to prove that $\deg_{LS}(I-K, \Omega, 0) = 1$ for certain "large" open bounded subset $\Omega \subset C([0,1])$ containing 0. Thus, the condition for multiplicity given in Theorem 5.4.4, namely $\frac{\partial g}{\partial u}(x,0) < a_1^2$ for all $x \in [0,1]$ may be understood in this context by noticing that it yields, when $p = 0$, that $\deg_{LS}(I-K, B_r(0), 0) = -1$ for $r > 0$ small enough. By continuity, this latter degree is still equal to -1 for p near the origin; hence, the excision property of the degree yields the existence of more solutions in $\Omega \setminus B_r(0)$: of at least one, and generically two. This situation may be interpreted in the shooting setting by looking at the right branch of the graph of T: the "typical" multiplicity situation occurs when $a_1 > a_{min}$ and the only case in which there is only one extra solution (apart from the negative one) corresponds to $a_1 = a_{min}$.

The previous consideration might give the key for an extension to a system, for which the methods in the preceding chapters seem difficult to generalize. For a future work, it seems possible to prove that if the degree of certain mapping $Dg(x, 0)$ is different from 1 then the problem has more than one solution when p is close to 0. It is interesting to ask the following question: for the case $N > 1$, is it possible to prove, under appropriate assumptions, the existence of more than three solutions?

Resumen del Capítulo 6

Nuestro objetivo para este capítulo es generalizar los resultados de existencia obtenidos en los capítulos 4 y 5 a un sistema de ecuaciones. Es decir, estudiamos el siguiente sistema para una función vectorial $u : [0,1] \subset \mathbb{R} \to \mathbb{R}^N$,

$$\begin{cases} u'' = g(x,u) + p(x) \\ u'(0) = a_0 u(0),\ u'(1) = a_1 u(1), \end{cases}$$

donde $a_0, a_1 > 0$, $g : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ es continua y superlineal, es decir:

$$\lim_{|u| \to +\infty} \frac{g(x,u) \cdot u}{|u|^2} = +\infty,$$

uniformemente en $x \in [0,1]$ y $p \in L^2$.

Como damos un resultado general de existencia, no es necesario separar el término p, por lo tanto, no podemos asumir que $g(x,0) = 0$ para toda $x \in [0,1]$. Por simplicidad, suponemos que $g \in C([0,1] \times \mathbb{R}^N)$, aunque el resultado es verdadero en el contexto más general.

Notemos que podemos generalizar el teorema de existencia para el caso variacional fácilmente.

Theorem 6.3.2. *El problema, para $g = \nabla G$, donde $G : [0,1] \times \mathbb{R}^N \to \mathbb{R}$ es una función C^1 con respecto a u y superlineal, tiene al menos una solución.*

Este capítulo está organizado de la siguiente manera. En la Sección 6.2 introducimos una condición de Hartman generalizada para pobar existencia de solución en un contexto no variacional.

Concluímos con la Sección 6.3, donde presentamos preguntas abiertas y futuras líneas de investigación.

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