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Desigualdades polinomiales en espacios de Banach

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Resumen

En esta tesis estudiamos desigualdades para el producto de polinomios en espacios de Banach. Nos enfocamos principalmente en los llamados *factor problem* y *plank problem.*

El *factor problem* consiste en buscar cotas inferiores para la norma del producto de polinomios de grados previamente fijados. Estudiamos este problema en diferentes contextos. Consideramos el producto de funciones lineales (i.e. polinomios homogéneos de grado uno), polinomios homogéneos y no homogéneos de grados arbitrarios. También investigamos el problema en diferentes espacios: finito e infinito dimensionales, espacios L_p, las clases de Schatten $\mathcal{S}_p$ y ultraproductos de espacios de Banach, entre otros. En algunos casos, como en los espacios L_p y las clases de Schatten $\mathcal{S}_p$, obtenemos cotas inferiores óptimas, mientras que en otros sólo estimamos la cota inferior óptima.

En un espacio de Banach X, el *plank problem* para polinomios consiste en encontrar condiciones sobre escalares no negativos $a_1, \ldots, a_n$ que aseguren que para cualquier conjunto de polinomios de norma uno $P_1, \ldots, P_n : X \to \mathbb{K}$ exista un vector z de norma uno tal que

$$|P_i(z)| \geq a_i^{\deg(P_i)} \text{ para } i = 1, \ldots, n.$$

Aplicamos las cotas inferiores obtenidas para el producto de polinomios al estudio de este problema y obtenemos condiciones suficientes para espacios de Banach complejos. También obtenemos condiciones menos restrictivas para ciertos espacios de Banach, como los espacios L_p o las clases Schatten $\mathcal{S}_p$.

Palabras clave: polinomios, espacios de Banach, ultraproductos, normas, desigualdades polinomiales, constantes de polarización.

Polynomial inequalities on Banach spaces

Abstract

In this thesis we study inequalities for the product of polynomials on Banach spaces. We focus mainly on the so called *factor problem* and *plank problem*.

The factor problem is the problem of finding lower bounds for the norm of the product of polynomials of some prescribed degrees. We study this problem in different contexts. We consider the product of linear functions (i.e. homogeneous polynomials of degree one), homogeneous and non homogeneous polynomials of arbitrary degrees. We also study this problem on different spaces: finite and infinite dimensional spaces, L_p spaces, Schatten classes $\mathcal{S}_p$ and ultraproducts of Banach spaces, among others. In some case, like in the L_p spaces and the Schatten classes $\mathcal{S}_p$, we obtain optimal lower bounds, while for other spaces we only give some estimates of the optimal lower bounds.

On a Banach space X, the plank problem for polynomials consists in finding conditions on nonnegative scalars $a_1, \ldots, a_n$ ensuring that for any set of norm one polynomials $P_1, \ldots, P_n : X \to \mathbb{K}$ there is an element in the ball of X such that

$$|P_i(z)| \geq a_i^{\deg(P_i)} \text{ for } i = 1, \ldots, n.$$

We apply our lower bounds for products of polynomials to study the plank problem, and obtain sufficient conditions for complex Banach spaces. We also obtain some less restrictive conditions for some particular Banach spaces, like the L_p spaces or Schatten classes $\mathcal{S}_p$.

Key words: polynomials, Banach spaces, ultraproducts, norms, polynomial inequalities, polarization constants.

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Contents

Introduction

Polynomials are a fundamental notion in mathematics, having a major importance in several fields. One of the most important asset polynomials have is the large amount of functions they can approximate and that they are much easier to handle than these functions.

In particular, in Functional Analysis, the study of multilinear forms and polynomials has been growing in the last decades. These concepts are closely related to other aspects of the modern theory of Banach spaces, such as local theory, operators ideals and the geometry of Banach spaces.

In this work we focus on studying inequalities for the product of polynomials on Banach spaces. We study what is sometimes called the *factor problem* and its applications to a geometrical problem called the *plank problem.* Although these problems are related, they are of an independent interest and will be treated accordingly.

Below, we provide a brief introduction to the factor problem and the plank problem, as well as some of the results already known regarding these problems.

Polarization constants

As a first step, let us introduce the polarization constants, which can be regarded as a particular case of the factor problem. It is immediate that if $\psi_1, \ldots, \psi_n$ are continuous linear functions over a Banach spaces then

$$\|\psi_1 \cdots \psi_n\| \leq \|\psi_1\| \cdots \|\psi_n\|,$$

where $\psi_1 \cdots \psi_n$ is the polynomial given by the pointwise product of the linear functions and the norm consider is the uniform norm over the ball of the space. In [RT] R. Ryan and B. Turett proved a sort of reverse inequality: for each n there is a constant K_n such that the reverse inequality

$$\|\psi_1\| \cdots \|\psi_n\| \leq K_n \|\psi_1 \cdots \psi_n\|,$$

holds for every Banach space and any set of n continuous linear function. Later, as a corollary of Theorem 3 from [BST], it was obtained that the optimal K_n with such property, for complex Banach spaces, is n^n.

The nth polarization constant $\mathbf{c}_n(X)$ of a Banach space X is defined as the best constant such that

$$\|\psi_1\| \cdots \|\psi_n\| \leq \mathbf{c}_n(X) \|\psi_1 \cdots \psi_n\|,$$

for any set of linear functions $\psi_1, \ldots, \psi_n \in X^*$. From the mentioned results in [RS] and [BST] we obtain that $\mathbf{c}_n(X) \leq n^n$ for every complex Banach space X.

Regarding Hilbert spaces, Arias-de-Reyna in [A] proved that $\mathbf{c}_n(H) = n^{\frac{n}{2}}$ for any complex Hilbert space, provided that $\dim(H) \geq n$. It follows, by a complexification argument, that for a real Hilbert space the nth polarization constant is at most $2(2n)^{\frac{n}{2}}$ (see [RS]).

In every other case, the exact value of the polarization constants $\{\mathbf{c}_n(X)\}_{n\in\mathbb{N}}$ are unknown, even for finite dimensional Banach and Hilbert spaces. It is conjectured, for example, that the result of Arias-de-Reyna holds for real Hilbert spaces.

Related to the concept of the nth polarization constant, the polarization constant $\mathbf{c}(X)$ of the Banach space X is defined as

$$\mathbf{c}(X) = \lim_{n\to\infty} \mathbf{c}_n(X)^{\frac{1}{n}}.$$

That is, this constant is determined by the behaviour of the nth polarization constants when n is large. The polarization constant was studied for the d-dimensional real Hilbert spaces $\mathbb{R}^d$ by García-Vázquez and Villa in [GV]. The authors found the exact value of $\mathbf{c}(\mathbb{R}^d)$ and proved that its order is $\sqrt{d}$. This result was later extended to complex Hilbert spaces by A. Pappas and S. G. Révész in [PR].

We devote Chapter 2 to analyze these constants. We study the polarization constant of finite dimensional spaces, in particular the spaces $\ell_p^d(\mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We prove that the order of $\mathbf{c}(\ell_p^d(\mathbb{K}))$, for $2 < p < \infty$ is $\sqrt{d}$ while for $p < 2$ the order is $\sqrt[p]{d}$. In the case $p = \infty$ we show that the order of $\mathbf{c}(\ell_\infty^d(\mathbb{K}))$ is greater than or equal to $\sqrt{d}$, but less than $d^{\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$.

We also give some estimates on the norm of the product of linear functions on $\ell_\infty^d(\mathbb{C})$, thus obtaining bounds for the nth polarization constant $\mathbf{c}_n(\ell_\infty^d(\mathbb{C}))$.

The Factor Problem

As previously mentioned, the problem of finding the nth polarization constants can be regarded as a particular case of the factor problem.

The factor problem consists in finding optimal lower bounds for the norm of the product of polynomials, of some prescribed degrees, using the norm of the polynomials. This problem has been studied in several spaces, considering a wide variety of norms. For example, in [B], W. D. Boyd proved that for any set of polynomials $P_1, \ldots, P_n$, of degrees $k_1 \ldots, k_n$, over the field of complex numbers, there is a constant C, depending only on n, such that:

$$\|P_1\| \cdots \|P_n\| \leq C^{\sum_{i=1}^n k_i} \|P_1 \cdots P_n\|.$$

Here the norm is the supremum over the unit disk and

$$C = \exp\left(\frac{n}{\pi}\int_0^{\frac{\pi}{2}} \ln(2\cos(t/2))\ \ dt\right).$$

In [BBEL], B. Beauzamy, E. Bombieri, P. Enflo and H. L. Montgomery studied the factor problem for homogeneous polynomials on $\mathbb{C}^d$, considering the norm

$$\left[\sum_{|\alpha|=k} a_\alpha z^\alpha\right]_2 = \sqrt{\sum_{|\alpha|=k} \frac{\alpha!}{k}|a_\alpha|^2},$$

which later was named Bombieri's norm. The authors proved that given two polynomials P and Q, of degrees k and l respectively, then

$$[P]_2[Q]_2 \leq \sqrt{\frac{(k+l)!}{k!l!}}[PQ]_2.$$

In Chapter 3 we study the factor problem on Banach spaces, considering the standard uniform norm for polynomials. That is, given a Banach space X, over a field $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), and natural numbers $k_1, \ldots, k_n$, our objective is to find the optimal constant M, depending only on these natural numbers and the space X, such that, given any set of continuous scalar polynomials $P_1, \ldots, P_n : X \to \mathbb{K}$, of degrees $k_1, \ldots, k_n$, the inequality

$$\|P_1\| \cdots \|P_n\| \leq M\|P_1 \cdots P_n\|$$

holds, where $\|P\| = \sup_{\|x\|_X \leq 1} |P(x)|$.

In this direction C. Benítez, Y. Sarantopoulos and A. Tonge [BST] proved that the inequality from above holds with constant

$$M = \frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}$$

for any complex Banach space. As pointed out before, a corollary from this result is that $\mathbf{c}_n(X) \leq n^n$ for any complex Banach space. The authors also showed that this is the best universal constant, since there are polynomials on ℓ_1 for which equality prevails. However, as we will see later, in some cases it is possible to improve this bound.

In Chapter 3 we study the factor problem on several spaces. For L_p spaces, with $1 < p < 2$, we obtain the exact value of the optimal constant and prove that in this case the optimal constant is

$$M = \sqrt[p]{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}. \tag{1}$$

For L_p spaces, with $2 < p$ we give some estimates for the optimal constant. Exploiting the similarities between the L_p spaces and the Schatten classes $\mathcal{S}_p$, we are able to transport some results, obtained for the L_p spaces, to the Schatten classes. In particular we prove that for the Schatten classes the optimal constant is (1).

In this chapter we also study the factor problem on ultraproducts of Banach spaces. Given an ultraproduct of Banach spaces $(X_i)_{\mathfrak{U}}$ we prove that under certain conditions the best constant for this ultraproduct is the limit of the best constants for the spaces X_i. Using similar ideas, we show that given a Banach space X such that X^{**} has the metric approximation property, then the best constant for X and X^{**} is the same.

The Plank Problem

Let us begin by recalling the original plank problem posed in the early 1930's by Alfred Tarski [Tar1, Tar2].

> Given a convex body $K \subset \mathbb{R}^d$ of minimal width 1, if K is covered by n planks with widths $a_1, \ldots, a_n$, is it true that $\sum a_i \geq 1$?

Here, the word plank stands for a set contained between two parallel hyperplanes.

The solution to this problem was given by T. Bang in [Ban]. The author also presented a similar question to Tarski's Plank Problem.

> When the convex body is covered with planks, is it true that the sum of the relative widths is greater than or equal to 1?

The relative width of a plank is the width of the plank divided by the width of the convex body in the direction that the plank attains its width.

This question remains unanswered in the general case, but for centrally symmetric convex bodies the solution was given by K. Ball in [Ba1], where he proved (slightly more than) the following:

> If $(\phi_j)_{j\in\mathbb{N}}$ is a sequence of norm one linear functionals on a real Banach space X and $(a_j)_{j\in\mathbb{N}}$ is a sequence of non-negative numbers whose sum is less than 1, then there is a unit vector $x \in X$ for which $|\phi_j(x)| \geq a_j$ for every $j \in \mathbb{N}$.

To realize that this is a sharp result, it is enough to consider $X = \ell_1(\mathbb{R})$ and the vectors of the standard basis of its dual, $\ell_\infty(\mathbb{R})$. However, it is reasonable to try to improve this constraint when we restrict ourselves to some special Banach spaces. For example, given $\{\phi_1, \ldots, \phi_n\}$ a set of orthonormal linear functionals defined on a Hilbert space H, it is clear that for any set of real numbers $\{a_1, \ldots, a_n\}$ such that $\sum_{j=1}^n a_j^2 \leq 1$, it is possible to find a unit vector $x \in H$ satisfying $|\phi_j(x)| \geq a_j$ for $j = 1, \dots, n$. Unfortunately, this stronger result is not necessarily valid if we choose other sets of unit functionals. Nonetheless, the situation is quite different if we consider complex Hilbert spaces, as K. Ball showed in [Ba2]:

> If $(\phi_j)_{j\in\mathbb{N}}$ is a sequence of norm one linear functionals on a complex Hilbert space H and $(a_j)_{j\in\mathbb{N}}$ is a sequence of non-negative numbers satisfying $\sum_{j\geq 1} a_j^2 = 1$, then there is a unit vector $z \in H$ for which $|\phi_j(z)| > a_j$ for every $j \in \mathbb{N}$.

In particular, this result implies the result of Arias-de-Reyna about the polarization constants mentioned above.

Using results from [BST, P] related to the factor problem, A. Kavadjiklis and S. G. Kim [KK] studied a plank type problem for polynomials on Banach spaces. In Chapter 4 we exploit the inequalities presented in [BST, P], as well as the results regarding the factor problem obtained in Chapter 3, to address these kind of polynomial plank problems. We aim to give sufficient conditions such that if $a_1, \dots, a_n$, are positive numbers fulfilling these conditions then, for any set of polynomials $P_1, \dots, P_n$ on a Banach space X, of degrees $k_1, \dots, k_n$, there is a norm one vector z such that $|P_j(z)| \geq a_j^{k_j}$ for $j = 1, \dots, n$.

We prove that for any complex Banach space, a suficient condition is

$$\sum_{i=1}^n a_i < \frac{1}{n^{n-1}}.$$

We also prove that when we restrict ourselves to L_p or the Schatten classes $\mathcal{S}_p$, with $1 \leq p \leq 2$, a sufficient condition for homogeneous polynomials is

$$\sum_{i=1}^n a_i^p < \frac{1}{n^{n-1}}.$$

We also address the problem on finite dimensional spaces.

Chapter 1

Preliminaries

We devote this chapter to fix some notation that will be used throughout this work, as well as to establish some theoretical content which will serve as a basis for the development of the following chapters.

1.1 Polinomials on Banach spaces

We start with some notation. Throughout this thesis, we will use E, F, X and Y to denote Banach spaces and H for Hilbert spaces. Given a Banach space X, we denote its topological dual by X^*, the norm on the space by $\| \cdot \|_X$ or, when it is clear from context, $\| \cdot \|$. All the Banach spaces considered will be either over the complex field $\mathbb{C}$ or the real field $\mathbb{R}$, we write $\mathbb{K}$ when we mean either.

The elements of the Banach spaces will usually be represented by the letters x, y and z, while the elements of its dual by ϕ, φ and ψ or z^*. We use both notations $\psi(x)$ and $\langle x, \psi \rangle$ for the value of a linear function ψ on an element x. The later will be used mostly when we have both situations: x fixed and ψ variable and the inverse, x variable and ψ fixed. For multilinear operators between two spaces, we will use T and reserve the letters P and Q for polynomials.

As usual, B_X will stand for the closed unit ball $\{x \in X : \|x\| \leq 1\}$ and S_X for the unit sphere $\{x \in X : \|x\| = 1\}$. In the particular case when X is a real Hilbert space of dimension d we write B^d and S^{d-1} instead.

For a couple of vectors $z = (z_1, \ldots, z_d) \in \mathbb{K}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we use the standard notation

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d} \text{ and } |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

In order to introduce the notion of polynomials on Banach spaces we start by recalling the definition of multilinear operators. If $X_1, \ldots, X_k$ and Y are Banach spaces, then a map

$$T : X_1 \times \ldots \times X_k \to Y,$$

is a $k-$linear operator if it is linear in each variable. The space of $k-$linear continuous operators from $X_1 \times \ldots \times X_k$ to Y is denoted by $\mathcal{L}(X_1, \ldots, X_k; Y)$ and it is a Banach space endowed with the uniform norm

$$\|T\| = \sup\{\|T(x_1, \ldots, x_k)\|_Y : x_i \in B_{X_i},\ i = 1, \ldots, k\}.$$

When $X_1 = \ldots = X_k = X$ we denote the space of $k-$linear continuous operators by $\mathcal{L}({}^kX; Y)$ and by $\mathcal{L}_s({}^kX; Y)$ the spaces of $k-$linear continuous and symmetric operators. Moreover, if $Y = \mathbb{K}$ we omit it from the notation and write $\mathcal{L}({}^kX)$ and by $\mathcal{L}_s({}^kX)$. Recall that an operator T is symmetric if for every permutation $\sigma \in S_k$ we have

$$T(x_1, \ldots, x_k) = T(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \quad \forall x_1, \ldots, x_k \in X.$$

Definition 1.1.1. Given $k \in \mathbb{N}$, a mapping $P : X \to Y$ is a continuous k-homogeneous polynomial if there is a continuous $k-$linear operator $T \in \mathcal{L}({}^kX; Y)$ for which

$$P(x) = T(x, \ldots, x).$$

A function $Q : X \to Y$ is a continuous polynomial of degree k if it can be written as

$$Q = \sum_{l=0}^{k} Q_l,$$

with Q_l $(0 \leq l \leq k)$ an $l-$homogeneous polynomial and $Q_k \neq 0$, where a 0-homogeneous polynomial is a constant.

Note that for a finite dimensional space $(\mathbb{K}^d, \|\cdot\|)$, this definition agrees with the standard definition of a polynomial on several variables, where a mapping $P : \mathbb{K}^d \to \mathbb{K}$ is a polynomial of degree k if it can be written as

$$P(z) = \sum_{|\alpha| \leq k} c_\alpha z^\alpha,$$

where the coefficients c_α belongs to $\mathbb{K}$ and there is α_0 such that $c_{\alpha_0} \neq 0$ and $|\alpha_0| = k$. The polynomial is k-homogeneous if $c_\alpha = 0$ for every α such that $|\alpha| < k$.

Example 1.1.2. If $\psi_1, \ldots, \psi_n : X \to \mathbb{K}$ are linear functions, the pointwise product

$$\prod_{i=1}^{n} \psi_i : X \to \mathbb{K}$$

is an homogeneous polynomial of degree n.

The space of continuous $k-$homogeneous polynomials on a Banach space X will be denoted by $\mathcal{P}(^kX;Y)$. It is a Banach space under the uniform norm

$$\|P\|_{\mathcal{P}(^kX;Y)} = \sup_{\|z\|_X=1} \|P(z)\|_Y.$$

When $Y = \mathbb{K}$ we omit it from the notation and write $\mathcal{P}(^kX)$.

Remark 1.1.3. It is easy to see that if P and Q are (homogeneous) continuous scalar polynomial of degrees k and l over X then, the pointwise product PQ, is a (homogeneous) continuous polynomial of degree $k+l$, and $\|PQ\| \leq \|P\|\|Q\|$.

Although for a continuous homogeneous polynomial P there are several k-linear continuous operators T such that $P(x) = T(x,\ldots,x)$, the polarization formula, stated below, assures us that there is only one k-linear continuous symmetric function with this property. We will denote this function by $\check{P}$.

Theorem 1.1.4 (Polarization Formula)**.** *Let $P \in \mathcal{P}(^kX;Y)$ and $T \in \mathcal{L}_s(^kX;Y)$ be such that*

$$P(x) = T(x,\ldots,x).$$

Then, for any $x_0 \in X$, we have the following formula for the operator T:

$$T(x_1,\ldots,x_k) = \frac{1}{k!2^k} \sum_{\varepsilon_1,\ldots,\varepsilon_k \in \{-1,1\}} \varepsilon_1 \cdots \varepsilon_k P\left(x_0 + \sum_{j=1}^{k} \varepsilon_j x_j\right).$$

For details on this classical result, as well as a deeper introduction to polynomials on Banach spaces, we refer the reader to the survey by Richard Aron [Arn] on this topic.

The Aron-Berner extension

Given a Banach space X, a natural way of extending a linear function $f : X \to \mathbb{K}$ to a w^*-continuous linear functional on its bidual X^{**} is to define

$$f(z) = \lim_\alpha f(x_\alpha),$$

where $\{x_\alpha\}$ is any net in X that w^*-converges to z. Arens [Ar] generalized this to $k-$linear operators and R. Aron and P. Berner [AB] to homogeneous polynomials, with the method described below.

Given a k-linear function $T : X \times \ldots \times X \to \mathbb{K}$ its Aron-Berner extension $AB(T)$ is a k-linear operator over X^{**}. First, define a function $\bar{z} : \mathcal{L}_s^k(X) \to \mathcal{L}_s^{k-1}(X)$ for each $z \in X^{**}$ as

$$\bar{z}(T)(x_1, \ldots, x_{k-1}) = z(T(x_1, \ldots, x_{k-1}, -)).$$

Similarly, we can define $\bar{z} : \mathcal{L}_s^l(X) \to \mathcal{L}_s^{l-1}(X)$ for any $l \in \mathbb{N}$. Thus, the Aron-Berner extension of T is defined as

$$AB(T)(z_1, \ldots, z_k) = \bar{z}_1 \circ \cdots \circ \bar{z}_k(T).$$

This extension is not symmetric in general. Actually, it does depend on the order we do the composition $\bar{z}_1 \circ \cdots \circ \bar{z}_k$. If we take a permutation $\sigma \in S_k$, the extension $AB_\sigma(T)$ defined as

$$AB_\sigma(T)(z_1, \ldots, z_k) = \bar{z}_{\sigma(1)} \circ \cdots \circ \bar{z}_{\sigma(k)}(T),$$

may be diferent from $AB(T)$. But the restriction of $AB(T)$ to the diagonal does not depend on the order of the composition. That is, the value $AB_\sigma(T)(z, \ldots, z)$ is the same for every permutation $\sigma \in S_k$.

Then, given a polynomial $P \in \mathcal{P}^k(X)$, its Aron-Berner extension $AB(P)$ is defined using the Aron-Berner extension of the symmetric k-linear function $\check{P}$ associated to P

$$AB(P)(z) = AB(\check{P})(z, \ldots, z).$$

Davie and Gamelin [DG] proved that the Aron-Berner extension preserves the norm. That is $\|AB(P)\| = \|P\|$. Moreover, they extended Goldstine's theorem proving that for each $z \in B_{X^{**}}$ there is a net $(x_\alpha) \subset B_X$ such that

$$P(x_\alpha) \to AB(P)(z) \ \forall P \in \mathcal{P}^k(X).$$

For more details on the Aron-Berner extension, as well as extensions of polynomials in general, we refer the reader to the survey by I. Zalduendo [Z].

The Mahler measure of a polynomial

We end this introductory section about polynomials mentioning other quantities associated to a polynomial that can be related to its norm: the Mahler measure and the length of a polynomial.

The Mahler measure of a polynomial $P : \ell_\infty^d(\mathbb{C}) \to \mathbb{C}$ is a powerful tool introduced by K. Mahler in [Ma]. In that article Mahler gave a simple proof of the Gelfand-Mahler inequality using this measure. The Mahler measure $M(P)$ of a polynomial $P : \mathbb{C}^d \to \mathbb{C}$, P not identically zero, is the geometric mean of $|P|$ over the d-dimensional torus $\mathbb{T}^d$ with respect to the Lebesgue measure:

$$M(P) = \exp\left\{ \int_{\mathbb{T}^d} \ln |P(t)| d\lambda(t) \right\}.$$

Remark 1.1.5. It is easy to see that the Mahler measure is multiplicative. That is, if $P, Q : \ell_\infty^d(\mathbb{C}) \to \mathbb{C}$ are polynomials, then

$$M(PQ) = M(P)M(Q).$$

Other quantities related to polynomials can be compared with the Mahler measure, such as the norm of a polynomial and its length (defined below). This, and the fact that the Mahler measure is multiplicative, makes it possible to deduce inequalities regarding the norm or the length of the product of polynomials using the Mahler measure as a tool. A prime example of this is the article [DM], where the authors give several relations between a variety of norms and the Mahler measure.

Definition 1.1.6. Let $P : \ell_\infty^d(\mathbb{C}) \to \mathbb{C}$ be a polynomial of degree k given by $P(z) = \sum_{|\alpha|\leq k} a_\alpha z^\alpha$. Its length $L(P)$ is defined as

$$L(P) = \sum_{|\alpha|\leq k} |a_\alpha|.$$

The following lemma establishes a relation between the norm of a polynomial, its length and its Mahler measure.

Lemma 1.1.7. *Let $P : \ell_\infty^d(\mathbb{C}) \to \mathbb{C}$ be a polynomial of degree k, then*

a) $M(P) \leq \|P\| \leq L(P)$.

b) $L(P) \leq 2^{dk} M(P)$.

The proof of the first part of this lemma is rather immediate. The second part can be deduced from Proposition 5 of [BL].

More details on the Mahler measure can be found in the work of of M. Bertin and M. Lalín [BL] and the references therein.

1.2 Remez type inequalities

In this section, we introduce Remez type inequalities for polynomials. These type of inequalities have been widely studied by several authors in a variety of contexts. In this section we restrict our attention to Remez type inequalities for multivariate polynomials in order to, later on, obtain results on the factor problem as an application of these Remez type inequalities.

The objective of Remez type inequalities is to give bounds for classes of functions over some fixed set, given that the modulus of the functions is bounded on some subset of prescribed measure. For example, the original inequality of Remez states the following.

> Take $a > 0$ and a polynomial $P : [-1, 1+a] \to \mathbb{R}$ of degree k such that
> $$\sup_{t \in V} |P(t)| \leq 1$$
> for some measurable subset $V \subseteq [-1, 1+a]$, with $|V| \geq 2$, where $|V|$ stands for the Lebesgue measure of V. Then
> $$\sup_{t \in [-1,1+a]} |P(t)| \leq \sup_{t \in [-1,1+a]} |T_k(t)|,$$
> where T_k stands for the Chebyshev polynomial of degree k.

This inequality, combined with some properties of the Chebyshev polynomials, produces the following corollary, which most applications of Remez inequality use.

Corollary 1.2.1. *Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree k, $I \subset \mathbb{R}$ be an interval and $V \subseteq I$ an arbitrary measurable set, then*

$$\sup_{t \in I} |P(t)| \leq \left(\frac{4|I|}{|V|} \right)^k \sup_{t \in V} |P(t)|. \tag{1.1}$$

Its proof is analogue to the proof of Proposition 1.2.3 which can be found in [BG].

We are interested in inequalities similar to (1.1), but for polynomials on several variables, which will be an important tool to study inequalities for the product of multivariate polynomials.

Y. Brudnyi and M. Ganzburg studied Remez type inequalities for polynomials on several variables in [BG]. As the original result of Remez, they stated their main result in terms of the Chebyshev polynomials.

Theorem 1.2.2. *Let X be a d-dimensional real space, λ a positive number and $P : X \to \mathbb{R}$ a polynomial of degree k such that*

$$\sup_{t \in V} |P(t)| \leq 1$$

for some measurable subset $V \subseteq B_X$, with $\mu(V) \geq \lambda$, where μ is the normalized Lebesgue measure over B_X. Then

$$\|P\| \leq T_k \left(\frac{1 + (1-\lambda)^{\frac{1}{d}}}{1 - (1-\lambda)^{\frac{1}{d}}} \right).$$

Just as in the applications of the original Remez inequality, rather than using Theorem 1.2.2, we will use next proposition (see inequality (8) from [BG]), which is a corollary from the main result of [BG].

Proposition 1.2.3. *Let X be a d-dimensional real space and $P : X \to \mathbb{R}$ be a polynomial of degree k. Given any Lebesgue measurable subset $V \subseteq B_X$, we have*

$$\sup_{z \in B_X} |P(z)| \leq \frac{1}{2} \left(\frac{4d}{\mu(V)} \right)^k \sup_{z \in V} |P(z)|,$$

where μ is the normalized Lebesgue measure over B_X.

As an immediate consequence of this result, we have the following inequality (see inequality (14) from [BG]). If P is a norm one polynomial of degree k over a finite d-dimensional Banach space X, then

$$\mu(\{z \in B_X : |P(z)| \leq t\}) \leq 4d \left(\frac{t}{2} \right)^{\frac{1}{k}}, \tag{1.2}$$

for any $0 < t < 1$.

To end this section we give an alternative bound for $\mu(\{z \in B_X : |P(z)| \leq t\})$ to the one given in (1.2), which holds when the space X is a finite dimensional real Hilbert space and $P : X \to \mathbb{R}$ is an homogeneous polynomial.

Proposition 1.2.4. *Let $P : \ell_2^d(\mathbb{R}) \to \mathbb{R}$ be a norm one homogeneous polynomial of degree k. Given any positive number t such that $4t^{\frac{1}{k}} \leq 1$, we have*

$$\mu(\{z \in B^d : |P(z)| \leq t\}) \leq 1 - (1 - 4t^{\frac{1}{k}})^d. \tag{1.3}$$

To prove this proposition we need the following auxiliary calculation regarding the gamma function.

Lemma 1.2.5. *For any natural number d,*

$$\int_0^{\pi/2} 2^d \cos^d(t) \sin^{d-2}(t) dt = \sqrt{\pi} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

Proof. First, let us consider

$$\Gamma(s)\Gamma(z) = \int_0^\infty \int_0^\infty t^{s-1} u^{z-1} e^{-(t+u)} dt du.$$

Making the change of variables $t = x^2$, $s = y^2$, and then using polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, we obtain

$$
\begin{aligned}
\int_0^\infty \int_0^\infty t^{s-1} u^{z-1} e^{-(t+u)} dt du &= \int_0^\infty \int_0^\infty x^{2s-2} u^{2z-2} e^{-(x^2+y^2)} 2x dx 2y dy \\
&= 4 \int_0^\infty \int_0^\infty x^{2s-1} y^{2z-1} e^{-(x^2+y^2)} dx dy \\
&= 4 \left(\int_0^{\pi/2} \cos^{2s-1}(\theta) \sin^{2z-1}(\theta) d\theta \right) \\
&\quad \left(\int_0^\infty r^{2s+2z-1} e^{-r^2} dr \right) \\
&= 4 \left(\int_0^{\pi/2} \cos^{2s-1}(\theta) \sin^{2z-1}(\theta) d\theta \right) \frac{\Gamma(s+z)}{2}.
\end{aligned}
$$

Therefore

$$\frac{\Gamma(s)\Gamma(z)}{2\Gamma(s+z)} = \left(\int_0^{\pi/2} \cos^{2s-1}(\theta) \sin^{2z-1}(\theta) d\theta \right).$$

Taking $s = \frac{d+1}{2}$ and $z = \frac{d-1}{2}$ we deduce the formula

$$\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{2\Gamma(d)} = \left(\int_0^{\pi/2} \cos^{d}(\theta) \sin^{d-2}(\theta) d\theta \right).$$

Now, we want to see that

$$\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{2\Gamma(d)} = \frac{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}{2^d\Gamma\left(\frac{d}{2}\right)}.$$

To do this we use Legendre's Duplication Formula

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z).$$

Applying this formula to $z = \frac{d}{2}$ we obtain

$$\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(d)} = \frac{2^{1-d}\sqrt{\pi}}{\Gamma\left(\frac{d}{2}\right)},$$

therefore

$$
\begin{aligned}
\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(d)} \frac{\Gamma\left(\frac{d-1}{2}\right)}{2} &= \frac{2^{1-d}\sqrt{\pi}}{\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma\left(\frac{d-1}{2}\right)}{2} \\
&= \frac{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}{2^d\Gamma\left(\frac{d}{2}\right)},
\end{aligned}
$$

as desired.

$\square$

In the proof of Proposition 1.2.4 we will use spherical coordinates in $\mathbb{R}^d$, but centred on $-e_1$. That is, if $(x_1, \ldots, x_d) \in \mathbb{R}^d$, then

$$x_1 + 1 = r\cos(\alpha_1)$$
$$x_2 = r\sin(\alpha_1)\cos(\alpha_2)$$
$$x_3 = r\sin(\alpha_1)\sin(\alpha_2)\cos(\alpha_3)$$
$$\vdots$$
$$x_{d-1} = r\sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)\cdots\sin(\alpha_{d-2})\cos(\alpha_{d-1})$$
$$x_d = r\sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)\cdots\sin(\alpha_{d-2})\sin(\alpha_{d-1}).$$

We will also use the fact that the equation

$$\sum_{i\geq 1} x_i^2 = 1$$

in terms on these coordinates can be written as

$$\begin{aligned} 1 &= ((x_1+1)-1)^2 + \sum_{i>1} x_i^2 \\ &= 1^2 - 2(x_1+1) + \sum_{i\geq 1} x_i^2 \\ &= 1 - 2r\cos(\alpha_1) + r^2. \end{aligned}$$

Or equivalently

$$r = 2\cos(\alpha_1). \tag{1.4}$$

Proof of Proposition 1.2.4. Without loss of generality, we may assume $|P(-e_1)| = 1$, where e_1 stand for the first vector of the canonical basis, $(1, 0, \ldots, 0)$.

Let $E = \{x \in B^d : |P(x)| \leq t\}$. First we are going to estimate the Lebesgue measure $|E|$ of the set E by integrating the characteristic function $1_E(x)$ of the set E over B^d.

Before proceeding with the proof, let us set some notation to lighten up the writing. We use α for the vector $(\alpha_1, \ldots, \alpha_{d-1}, \alpha_d)$, $\tilde{\alpha}$ for $(\alpha_2, \ldots, \alpha_{d-1})$ and $d\tilde{\alpha}$ will stand for $d\alpha_2 \cdots d\alpha_{d-1}$. Define the set

$$A = \{\tilde{\alpha} : 0 \leq \alpha_i < \pi \text{ if } 2 \leq i \leq d-2 \text{ and } 0 \leq \alpha_{d-1} < 2\pi\}$$

and the function $f : A \to \mathbb{R}$ as

$$f(\tilde{\alpha}) = \sin^{d-3}(\alpha_2)\sin^{d-4}(\alpha_3)\cdots\sin^2(\alpha_{d-3})\sin(\alpha_{d-2}).$$

Note that the area differential dS of S^{d-2} in terms of f is

$$dS = f(\tilde{\alpha})d\tilde{\alpha}.$$

The idea will be to integrate first on the radius r with the angle α fixed, in order to apply the one dimensional Remez result (1.1) to obtain an upper bound. Then we integrate over α:

$$\begin{aligned} |E| &= \int_{B^d} 1_E(x)dx \\ &= \int_A \int_0^{\pi/2} \int_0^{2\cos(\alpha_1)} 1_E(x) r^{d-1} \sin^{d-2}(\alpha_1) f(\tilde{\alpha}) dr d\alpha_1 d\tilde{\alpha} \qquad (1.5) \\ &= \int_A f(\tilde{\alpha}) \int_0^{\pi/2} \int_0^{2\cos(\alpha_1)} 1_E(x) r^{d-1} dr \sin^{d-2}(\alpha_1) d\alpha_1 d\tilde{\alpha}. \end{aligned}$$

In (1.5) we use spherical coordinates centred in $-e_1$ and the fact that, by (1.4), the set B^d can be written as

$$\{(r, \alpha_1, \tilde{\alpha}) : 0 \le r \le 2\cos(\alpha_1),\ 0 \le \alpha_1 \le \frac{\pi}{2} \text{ and } \tilde{\alpha} \in A\}.$$

We need an upper bound for

$$\int_0^{2\cos(\alpha_1)} 1_E(x) r^{d-1} dr.$$

For α fixed, consider the set $R = \{(r, \alpha) : 0 \le r \le 2\cos(\alpha_1)\}$. By inequality (1.1) we have

$$\sup\{|P(x)| : x \in R\} \le \left(\frac{4|R|}{|R \cap E|}\right)^k \sup\{|P(x)| : x \in R \cap E\}.$$

Since $-e_1$ belongs to R, we have $\sup\{|P(x)| : x \in R\} = 1$. Given that $|R| = 2\cos(\alpha_1)$ and

$$\sup\{|P(x)| : x \in R \cap E\} \le \sup\{|P(x)| : x \in E\} = t$$

we conclude that

$$|R \cap E| \leq 8\cos(\alpha_1)t^{\frac{1}{k}}.$$

Then, since the map $r \mapsto r^{d-1}$ is increasing, we have

$$\begin{aligned}
\int_0^{2\cos(\alpha_1)} 1_E(x) r^{d-1} dr &= \int_{R\cap E} r^{d-1} dr \\
&\leq \int_{2\cos(\alpha_1)-8\cos(\alpha_1)t^{\frac{1}{k}}}^{2\cos(\alpha_1)} r^{d-1} dr \\
&= \frac{1}{d}\left[(2\cos(\alpha_1))^d - (2\cos(\alpha_1)(1-4t^{\frac{1}{k}}))^d\right] \\
&= \frac{1-(1-4t^{\frac{1}{k}})^d}{d}(2\cos(\alpha_1))^d.
\end{aligned}$$

Therefore, combining this with Lemma 1.2.5, we obtain

$$\begin{aligned}
|E| &\leq \int_A f(\tilde{\alpha}) \int_0^{\pi/2} \frac{1-(1-4t^{\frac{1}{k}})^d}{d} 2^d \cos^d(\alpha_1)\sin^{d-2}(\alpha_1) d\alpha_1 d\tilde{\alpha} \\
&= |S^{d-2}| \frac{1-(1-4t^{\frac{1}{k}})^d}{d} \sqrt{\pi} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.
\end{aligned}$$

Recall that

$$|B^d| = \frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{\frac{d}{2}}} \text{ and } S^{d-2} = \frac{(d-1)\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}+1\right)}.$$

As a consequence,

$$\begin{aligned}
\mu(E) &= \frac{|E|}{|B^d|} = \frac{1-(1-4t^{\frac{1}{k}})^d}{d} \frac{|S^{d-2}|}{|B^d|} \sqrt{\pi} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \\
&= \frac{1-(1-4t^{\frac{1}{k}})^d}{d} \frac{(d-1)\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d-1}{2}+1\right)\pi^{\frac{d}{2}}} \sqrt{\pi} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \\
&= 1-(1-4t^{\frac{1}{k}})^d, \qquad (1.6)
\end{aligned}$$

where in (1.6) we use that

$$\frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d}{2}\right)} = \frac{d}{2} \quad \text{and} \quad \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}+1\right)} = \frac{2}{d-1}.$$

□

1.3 Resumen en castellano del Capítulo I

En este capítulo fijamos gran parte de la notación que va a ser utilizada a lo largo de la tesis. También introducimos el concepto de polinomios en espacios de Banach y parte del contenido teórico sobre este tema, necesario para el desarrollo de los capítulos posteriores. Este contenido teórico consiste principalmente en la extensión de Aron-Berner de un polinomio, la medida de Mahler $M(P)$ de un polinomio y desigualdades tipo Remez para polinomios en varias variables.

Extension de Aron-Berner Dado un espacio de Banach E y un polinomio continuo $P : E \to \mathbb{K}$, la extensión de Aron-Berner $AB(P) : E^{**} \to \mathbb{K}$ es un polinomio continuo, con norma igual a la de P, que coincide con P sobre E (pensado como subespacio de E^{**}).

Medida de Mahler La medida de Mahler de un polinomio escalar no nulo P en $\mathbb{C}^d$ es la media geométrica de P sobre el toro d-dimensional $\mathbb{T}^d$, considerando la medida de Lebesgue, es decir

$$M(P) = \exp\left\{\int_{\mathbb{T}^d} \ln |P(t)| d\lambda(t)\right\}.$$

La medida de Mahler es una herramienta de mucha utilidad a la hora de probar desigualdades para la norma del producto de polinomios ya que es multiplicativa, y que esta cantidad puede relacionarse con la norma de un polinomio.

Desigualdades tipo Remez Las desigualdades tipo Remez consisten en dar cotas para una familia de funciones (por ejemplo, polinomios, polinomios trigonométricos, etc.) definidos sobre cierto conjunto V, dado que para cualquiera de estas funciones se tiene una cota para algún subconjunto $U \subset V$, de una medida previamente fijada. En nuestro caso la familia de funciones van a ser los polinomios de grado k en un espacio de Banach finito dimensional y V va a ser la bola unidad del espacio. En este contexto utilizamos los resultados presentados en [BG] por Y. Brudnyi y M. Ganzburg.

Chapter 2

Polarization constants

Given a Banach space X, its nth polarization constant is defined as the smallest constant $\mathbf{c}_n(X)$ such that for any set of n linear functions $\{\psi_j\}_{j=1}^n \subseteq X^*$, we have

$$\|\psi_1\| \cdots \|\psi_n\| \leq \mathbf{c}_n(X) \|\psi_1 \cdots \psi_n\|. \tag{2.1}$$

Where $\psi_1 \cdots \psi_n$ is the n-homogeneous polynomial given by the pointwise product of $\psi_1, \ldots, \psi_n$. Recall that, as pointed out in Remark 1.1.3, the reverse inequality always holds with constant one for any set of linear functions.

Related to this concept the polarization constant $\mathbf{c}(X)$ of X is defined as

$$\mathbf{c}(X) = \lim_{n\to\infty} (\mathbf{c}_n(X))^{\frac{1}{n}}.$$

These constants have been studied by several authors. Among the works on this topic, in [RT] the authors proved that for each n there is a constant K_n such that $\mathbf{c}_n(X) \leq K_n$ for every Banach space X. As a corollary from a result of [BST] it is easy to see that the best possible constant K_n, for complex Banach spaces, is n^n.

In [A] Arias-de-Reyna proved that if X is a complex Hilbert space, of dimension greater or equal than n, then

$$\mathbf{c}_n(X) = n^{\frac{n}{2}}.$$

This result holds for real Hilbert spaces and $n \leq 5$ (see [PR], Theorem 2), but it is not known if it is true for every natural number n.

As for the polarization constant, it was studied for real Hilbert spaces in the article [GV] by García-Vázquez and Villa. The authors found its exact value and proved that, when the dimension d is large, the order of this constant is $\sqrt{d}$. This result was later extended to complex Hilbert spaces by A. Pappas and S. G. Révész in [PR].

In this chapter we study the nth polarization constants, as well as the polarization constant, of finite dimensional Banach spaces. A Banach space X is finite dimensional if and only if $\mathbf{c}(X) < \infty$ (see [RS], Theorem 12), therefore the finite dimensional is the only setting worth studying the polarization constants.

We devote the first section to develop a method to estimate the polarization constant of a finite dimensional space. In a subsequent section, we apply this method to the finite dimensional spaces $\ell_p^d(\mathbb{K})$, obtaining asymptotically optimal results on d. In the final section we study these problems for the finite dimensional spaces $\ell_\infty^d(\mathbb{C})$, and give some estimates for its nth polarization constants.

It is important to remark that this terminology it is not standard an in some works the polarization constant stands for a different constant, see for example [Di] and [LR].

2.1 Polarization constants of finite dimensional spaces

In this section we present a general method for estimating polarization constants. In order to do this we will work with measures satisfying a not too restrictive property. We call such measures *admissible*.

Definition 2.1.1. Let X be a Banach space and λ a measure over a subet $K \subseteq B_X$. We say that λ is admissible if

$$\int_K \ln |\langle x, \psi \rangle| d\lambda(x)$$

is finite for every $\psi \in X^*$ and the functions $g_m : S_{X^*} \to \mathbb{R}$ defined as

$$g_m(\psi) = \int_K \max\{\ln |\langle x, \psi \rangle|, -m\} d\lambda(x),$$

converge uniformly to the function $g : S_{X^*} \to \mathbb{R}$, defined as

$$g(\psi) = \int_K \ln |\langle x, \psi \rangle| d\lambda(x).$$

For example, for H a Hilbert space, the Lebesgue measure over S_H is admissible, since the functions g_m are constant functions that converge to the constant function g.

The main result of this section is the following theorem.

Theorem 2.1.2. *Given a finite dimensional Banach space X, let $K \subseteq B_X$, μ and η be admissible probability measures over K and S_{X^*} respectively. Then there are*

$x_0 \in S_X$ and $\psi_0 \in S_{X^*}$, depending on μ and η, such that

$$\exp\left\{-\int_{S_{X^*}} \ln|\langle x_0, \psi\rangle| \; d\eta(\psi)\right\} \leq \mathbf{c}(X) \leq \exp\left\{-\int_K \ln|\langle x, \psi_0\rangle| \; d\mu(x)\right\}.$$

We will treat separately the lower and the upper bound. Let us sketch some of the ideas behind this method. First we focus in the lower bound. By a compactness argument, for a finite dimensional Banach X and for each natural number n, there are linear functions $\psi_1^n, \ldots, \psi_n^n \in S_{X^*}$ such that

$$\|\psi_1^n \cdots \psi_n^n\| = \mathbf{c}_n(X)^{-1}. \tag{2.2}$$

Suppose that $x_n \in B_X$ is a point where $\psi_1^n \cdots \psi_n^n$ attains its norm. That is, x_n is a point such that $\|\psi_1^n \cdots \psi_n^n\| = |\psi_1^n \cdots \psi_n^n(x)|$. Then

$$\|\psi_1^n \cdots \psi_n^n\|^{\frac{1}{n}} = \exp\left\{\frac{1}{n}\sum_{i=1}^n \ln|\psi_i^n(x_n)|\right\}.$$

If we consider the functions $f_n : S_{X^*} \to \mathbb{K}$ defined as $f_n(\varphi) = \ln|\varphi(x_n)|$ and η_n the probability measure over S_{X^*} defined as

$$\eta_n = \frac{1}{n}\sum_{i=1}^n \delta_{\psi_i^n},$$

then we have:

$$\frac{1}{n}\sum_{i=1}^n \ln\left(|\psi_i^n(x_n)|\right) = \int_{S_{X^*}} f_n(\psi) \; d\eta_n.$$

The idea now is to take a subsequence $\{n_k\}$ such that η_{n_k} w^*-converges to some probability measure η and such that $\{x_{n_k}\}$ converges to some $x_0 \in S_X$. All this will give us an estimate of $\mathbf{c}(X)$ in terms of η and the function $f_0 : S_{X^*} \to \mathbb{K}$, defined as $f_0(\varphi) = \ln|\varphi(x_0)|$.

On the road we will find some problems. For example, the functions $\{f_n\}_{n\in\mathbb{N}}$ are not continuous. Because of these problems, we will obtain a lower bound of $\mathbf{c}(X)$ rather than its exact value. Another issue with this method is that finding a set of functions such that we have equality in (2.2) is not a trivial problem. As a consequence, it is not easy to determine x_0 nor η. So, an alternative procedure is the following: we start by fixing our measure η beforehand and choose the sets of linear functions $\psi_1^n, \ldots, \psi_n^n$ to obtain this particular η as a w^*-limit of the measures η_n. These sets of linear functions may not be the ones that give equality in (2.2), but they will clearly satisfy

$$\|\psi_1^n \cdots \psi_n^n\| \geq \mathbf{c}_n(X)^{-1},$$

which is precisely what we need to obtain the desired lower bounds.

The sharpness of the bounds obtained with this method will depend on the adequate selection of a probability measure η.

Lower bounds

In the sequel, for a measure space (K,ν) and an integrable function $f : K \to \mathbb{R}$ we will use the notation
$$\mu(f) = \int_K f(\omega)\ d\nu(\omega).$$

We need the following auxiliary lemma due to A. Pappas and S. G. Révész (see [PR], Lemma 4).

Lemma 2.1.3. *Let η be any probability measure over S_{X^*}. There is a sequence of sets of norm one linear functions $\{\psi_1^n,\dots,\psi_n^n\}_{n\in\mathbb{N}}$ over X such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(\psi_j^n) = \int_{S_{X^*}} f(\psi)\ d\eta(\psi)$$

for any continuous function $f : S_{X^*} \to \mathbb{R}$. In other words, if we consider the measures $\eta_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{\psi_j^n}$, the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ w^*-converges to η.

We remark that, although the result in [PR] is stated for X a Hilbert space and η the normalized Lebesgue measure, the proof works in the more general setting of our statement.

Now we are ready to state our method to obtain lower estimates of $\mathbf{c}(X)$.

Proposition 2.1.4. *Given a finite dimensional Banach space X and an admissible probability measure η over S_{X^*}, there is a point $x_0 \in S_X$, depending on η, such that*
$$\mathbf{c}(X) \geq \exp\left\{-\int_{S_{X^*}} \ln|\langle x_0,\psi\rangle|\ d\eta(\psi)\right\}.$$

Proof. Take a sequence of sets of norm one linear functions $\{\psi_1^n,\dots\psi_n^n\}_{n\in\mathbb{N}}$ as in Lemma 2.1.3, and consider the measures $\eta_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{\psi_j^n}$. Let $x_n \in S_X$ be a point where $\prod_{j=1}^{n}\psi_j^n$ attains its norm. We may assume $\left\|\prod_{j=1}^{n}\psi_j^n\right\|$ converges, otherwise we work with a subsequence. With the same argument we may assume that there is $x_0 \in S_X$ such that $x_n \to x_0$.

Since
$$c_n(X)\Big\|\prod_{j=1}^{n}\psi_j^n\Big\| \geq 1,$$
we need an upper bound of $\lim\limits_{n\to\infty}\Big\|\prod\limits_{j=1}^{n}\psi_j^n\Big\|^{\frac{1}{n}}$.

For every $n, m \in \mathbb{N}_0$ consider the functions $f_n : S_{X^*} \to \mathbb{R}$ and $f_{n,m} : S_{X^*} \to \mathbb{R}$ defined by

$$f_n(\psi) = \ln(|\langle x_n, \psi \rangle|)$$

$$f_{n,m}(\psi) = \max\{f_n(\psi), -m\}.$$

Using that $f_{n,m} \geq f_n$ we obtain

$$\begin{aligned}
\left\| \prod_{j=1}^{n} \psi_j^n \right\|^{\frac{1}{n}} &\geq \prod_{j=1}^{n} |\langle x_n, \psi_j^n \rangle|^{\frac{1}{n}} \\
&= \exp\left\{ \frac{1}{n} \sum_{j=1}^{n} \ln\left(|\langle x_n, \psi_j^n \rangle| \right) \right\} \\
&= \exp\left\{ \frac{1}{n} \sum_{j=1}^{n} f_n(\psi_j^n) \right\} \\
&= \exp\left\{ \eta_n(f_n) \right\} \\
&\leq \exp\left\{ \eta_n(f_{n,m}) \right\}.
\end{aligned}$$

Fixed m, since $x_n \to x_0$, it is easy to check that the functions $\{f_{n,m}\}$ converge uniformly to $f_{0,m}$ as $n \to \infty$. Also, we know that $\{\eta_n\}$ w^*-converges to η. This altogether gives that $\{\eta_n(f_{n,m})\}$ converges to $\eta(f_{0,m})$ and then

$$\lim_{n\to\infty} \left\| \prod_{j=1}^{n} \psi_j^n \right\|^{\frac{1}{n}} \leq \exp\left\{ \eta(f_{0,m}) \right\}.$$

Since this holds for arbitrary m and η is admissible, taking limit on m we obtain

$$\lim_{n\to\infty} \left\| \prod_{j=1}^{n} \psi_j^n \right\|^{\frac{1}{n}} \leq \exp\{\eta(f_0)\} = \exp\left\{ \int_{S_{X^*}} \ln|\langle x_0, \psi \rangle| \; d\eta(\psi) \right\},$$

as desired.

$\square$

Remark 2.1.5. In the previous proof we only use from the Definition 2.1.1 that

$$\int_{S_{X^*}} \max\{\ln|\langle x_0, \varphi \rangle|, -m\} \; d\eta(\varphi) \to \int_{X^*} \ln|\langle x_0, \varphi \rangle| \; d\eta(\varphi),$$

that is, we only needed pointwise convergence for the point $x_0 \in S_{(X^*)^*}$, rather than uniform convergence on $S_{(X^*)^*}$. So, for the lower bounds, it is enough to ask for pointwise convergence. To obtain this it is enough to have

$$\int_{S_{X^*}} \ln |\langle x_0, \psi\rangle| \; d\eta(\psi) < \infty$$

and apply the Dominated Convergence Theorem.

Upper bounds

For the upper bounds we will obtain a slightly better result, since we will get upper bounds for $\mathbf{c}_n(X)$ rather than for $\mathbf{c}(X)$.

Proposition 2.1.6. *Given a finite dimensional Banach space X, $K \subseteq B_X$, and an admissible probability measure μ over K, there is a point $\psi_0 \in S_{X^*}$, depending on μ, such that*

$$\mathbf{c}_n(X) \leq \exp\left\{-n \int_K \ln |\langle x, \psi_0\rangle| \; d\mu(x)\right\}.$$

Proof. Consider the function $g : S_{X^*} \to \mathbb{R}$ defined as

$$g(\psi) = \int_{S_X} \ln |\langle x, \psi\rangle| \; d\mu(x).$$

We start by showing that g is continuous. For every natural number m define $g_m : S_{X^*} \to \mathbb{R}$ by

$$g_m(\psi) = \int_{S_X} \max\{-m, \ln |\langle x, \psi\rangle|\} \; d\mu(x).$$

Given that μ is admissible, $\{g_m\}_{m\in\mathbb{N}}$ converges uniformly to g and therefore, since each g_m is continuous, g is continuous. Since g is continuous and S_{X^*} is compact, there is $\psi_0 \in S_{X^*}$ a global minimum of g.

Recall that $\mathbf{c}_n(X)$ is the smallest constant such that

$$1 = \prod_{j=1}^{n} \|\psi_j\| \leq \mathbf{c}_n(X) \left\| \prod_{j=1}^{n} \psi_j \right\|$$

for any set of linear functions $\psi_1, \ldots \psi_n \in S_{X^*}$. So we need to prove that

$$\exp\left\{n \int_K \ln |\langle x, \psi_0\rangle| \; d\mu(x)\right\} \leq \left\| \prod_{j=1}^{n} \psi_j \right\|.$$

Using that μ is a probability measure and that ψ_0 minimizes g we obtain

$$\begin{aligned}
\left\|\prod_{j=1}^{n} < -, \psi_j >\right\| &= \exp\left\{\ln\left(\sup_{x\in K}\prod_{j=1}^{n}|\langle x,\psi_j\rangle|\right)\right\} \\
&= \exp\left\{\sup_{x\in K}\sum_{j=1}^{n}\ln|\langle x,\psi_j\rangle|\right\} \\
&\geq \exp\left\{\int_K\sum_{j=1}^{n}\ln|\langle x,\psi_j\rangle|\ d\mu(x)\right\} \\
&= \exp\left\{\sum_j\int_K\ln|\langle x,\psi_j\rangle|\ d\mu(x)\right\} \\
&\geq \exp\left\{n\int_K\ln|\langle x,\psi_0\rangle|\ d\mu(x)\right\}
\end{aligned}$$

as desired. $\square$

Remark 2.1.7. In the previous proof we used that μ is admissible only to prove that g has a global minimum.

2.2 Polarization constants of $\ell_p^d(\mathbb{K})$ spaces

In this section we apply the method developed in the previous section, stated in Theorem 2.1.2, to estimate the asymptotic behaviour of the polarization constants $\mathbf{c}(\ell_p^d(\mathbb{K}))$, for d large. To describe the asymptomatic behaviour of two sequences of positive numbers $\{a_d\}_{d\in\mathbb{N}}$ and $\{b_d\}_{d\in\mathbb{N}}$ we use the notation $a_d \prec b_d$ to indicate that there is a constant $L > 0$ such that $a_d \leq Lb_d$. The notation $a_d \asymp b_d$ means that $a_d \prec b_d$ and $a_d \succ b_d$.

When we consider a d-dimensional (real or complex) Hilbert space H, taking in Theorem 2.1.2 $\mu = \eta$ the normalized Lebesgue measure over $S_H = S_{H^*}$, we recover the main result from [PR]:

$$\mathbf{c}(H) = \exp\left\{-\int_{S_H}|\langle x,\psi_0\rangle|dS(x)\right\}, \tag{2.3}$$

where dS stands for the surface differential of S_H considering the normalized Lebesgue measure. If we call

$$L(d,\mathbb{K}) = \int_{S_H}|\langle x,\psi_0\rangle|dS(x),$$

a standard computation (see [PR]) gives:

$$-L(d,\mathbb{R}) = \begin{cases} \sum_{j=1}^{(d-2)/2} \frac{1}{2j} + \ln(2) & \text{if} \quad d \equiv 0(2) \\ \sum_{j=1}^{(d-3)/2} \frac{1}{2j+1} & \text{if} \quad d \equiv 1(2) \end{cases} \quad \text{and} \quad -L(d,\mathbb{C}) = \frac{1}{2}\sum_{j=1}^{d-1} \frac{1}{j}.$$

In particular $\mathbf{c}(H) \asymp \sqrt{d}$.

The main difficulty on applying our method to a d-dimensional Banach space X, is the need of candidates of measures η and μ. The idea behind the method would suggest that the measure η on S_{X^*} needed to obtain a sharp lower bound is the one induced by the sequence of sets of norm one linear functions $\{\psi_1^n, \dots, \psi_n^n\}_{n\in\mathbb{N}}$ such that

$$\|\psi_1^n \cdots \psi_n^n\| = c_n(X)^{-1}.$$

Although, as mentioned before, finding these linear functions is not an easy task, it seems reasonable to assume that these linear functions have to be spread out in S_{X^*}. In the particular case when X is a Hilbert space, due to the symmetry of the sphere, we may even expect that they are uniformly distributed across the sphere. Note that when we consider sets of linear functions uniformly distributed across the sphere, they induce the normalized Lebesgue measure which, as observed before, is an optimal choice of η for our method.

But this argument is no longer valid for the spaces ℓ_p^d with $p \neq 2$. If $\frac{1}{p} + \frac{1}{q} = 1$ the sphere of the dual of ℓ_p^d is $S_{\ell_q^d}$ and, unlike in the Hilbert case, we do not have symmetry that suggests that the linear functions will be uniformly distributed across the sphere. For example, when $p < 2$, by the geometry of the sphere, it is likely that the linear functions are more concentrated around the points $e_1, \dots, e_n$ than around points of the form $\sum \lambda_i e_i$, with $|\lambda_i| = \frac{1}{d^{1/q}}$. At least this is the case for $n \leq d$, as we will see later in Chapter 3, or for $n = dk$, as we will see later in this section.

On the other hand, for $p > 2$, one may expect that the linear functions will be concentrated around the points of the form $\sum \lambda_i e_i$ rather than around the canonical basis.

Then, for the spaces ℓ_p^d we will choose a measure η reflecting the previous reasoning and try to obtain the best possible lower bound, taking into consideration that we will not have control over the vector x_0 mentioned on Theorem 2.1.2.

For the upper bound we will choose a measure μ similar to η, and try to get an upper bound as close as we can to our lower bound.

The following is the main result of this section and gives the asymptotic behaviour of the polarization constants $\mathbf{c}(\ell_p^d(\mathbb{K}))$ as d goes to infinity. We devote the rest of this section to its proof.

Theorem 2.2.1. *Let $1 \leq p < \infty$, then*

$$\mathbf{c}(\ell_p^d(\mathbb{K})) \asymp \begin{cases} \sqrt{d} & \textit{if} \quad p \geq 2 \\ \\ \sqrt[p]{d} & \textit{if} \quad p \leq 2. \end{cases}$$

For $p = \infty$ we have the following estimate

$$\sqrt{d} \prec \mathbf{c}(\ell_\infty^d(\mathbb{K})) \prec d^{\frac{1}{2}+\varepsilon} \quad \textit{for every } \varepsilon > 0.$$

In order to prove Theorem 2.2.1 we need some auxiliary calculations. Next lemma is a particular case of Lemma 2.8 of [CGP].

Lemma 2.2.2. *Given $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $1 \leq p < \infty$, we have*

$$\int_{S_{\ell_2^d(\mathbb{K})}} \|t\|_p^p dS_d(t) \asymp d^{1-\frac{p}{2}}.$$

Proof. We give the proof for the case $\mathbb{K} = \mathbb{C}$. The real case is similar and its proof can be found in Lemma 2.8 of [CGP].

Consider the Gaussian measure γ over $\mathbb{C}^d$ defined as

$$\gamma(A) = \int_A e^{\frac{-\|z\|^2}{2}} \; dz.$$

Then we have

$$\begin{aligned} \int_{\mathbb{C}^d} \|z\|_p^p \; d\gamma(z) &= \int_{\mathbb{C}^d} \sum_{j=1}^n |z_j|^p \; d\gamma(z) \\ &= d \int_{\mathbb{C}^d} |z_1|^p \; d\gamma(z) \\ &\asymp d. \end{aligned}$$

On the other hand, using polar coordinates, we obtain

$$\begin{aligned} \int_{\mathbb{C}^d} \|z\|_p^p \; dz &= \frac{|S^{2d-1}|}{(2\pi)^d} \int_0^{+\infty} \int_{S_{\ell_2^d(\mathbb{C})}} \|z\|_p^p dS(z) r^{2d-1+p} e^{\frac{-r^2}{2}} dr \\ &= \int_{S_{\ell_2^d(\mathbb{C})}} \|z\|_p^p dS(z) \frac{|S^{2d-1}|}{(2\pi)^d} \int_0^{+\infty} r^{2d-1+p} e^{\frac{-r^2}{2}} dr. \end{aligned}$$

Therefore we have to show that

$$\frac{|S^{2d-1}|}{(2\pi)^d}\int_0^{+\infty} r^{2d-1+p}e^{\frac{-r^2}{2}}\,dr \asymp d^{\frac{p}{2}}.$$

But this follows from an easy computation using Stirling's formula:

$$\Gamma(t+1)\asymp\sqrt{2\pi t}\left(\frac{t}{e}\right)^t.$$

Indeed, making the substitution $r^2=u$ and recalling that $|S^{2d-1}|=\frac{2d\pi^d}{\Gamma(d+1)}$, we obtain

$$\begin{aligned}
\frac{|S^{2d-1}|}{(2\pi)^d}\int_0^{+\infty} r^{2d-1+p}e^{\frac{-r^2}{2}}\,dr &= \frac{2d\pi^d}{\Gamma(d+1)}\frac{1}{(2\pi)^d}\int_0^{+\infty}(2u)^{\frac{2d+p-2}{2}}e^{-u}du\\
&= \frac{2}{\Gamma(d+1)}2^{\frac{p}{2}}\Gamma\left(d+\frac{p}{2}\right)\\
&\asymp \frac{d}{\sqrt{2\pi d}(d/e)^d}\sqrt{2\pi\left(d+\frac{p}{2}-1\right)}\left(\frac{d+\frac{p}{2}-1}{e}\right)^{d+\frac{p}{2}-1}\\
&= d\sqrt{\frac{d+\frac{p}{2}-1}{d}}\left(\frac{d+\frac{p}{2}+1}{d}\right)^d\left(\frac{d+\frac{p}{2}+1}{e}\right)^{\frac{p}{2}-1}\\
&\asymp d^{\frac{p}{2}}
\end{aligned}$$

as desired. □

Using this lemma we are able to prove the following.

Lemma 2.2.3. *Let $1\le p<\infty$. Then*

$$\exp\left\{\int_{S_{\ell_2^d(\mathbb{R})}}\ln\left(\frac{1}{\|z\|_p}\right)dS(z)\right\}\asymp d^{\frac{1}{2}-\frac{1}{p}}.$$

Proof. For the upper bound, using Jensen's inequality and equality (0.6) from [Pi], we have

$$\begin{aligned}
\int_{S_{\ell_2^d(\mathbb{R})}} \ln\left(\frac{1}{\|z\|_p}\right) dS(z) &= \frac{1}{d}\int_{S_{\ell_2^d(\mathbb{R})}} \ln\left(\frac{1}{\|z\|_p^d}\right) dS(z) \\
&\leq \frac{1}{d}\ln\left(\int_{S_{\ell_2^d(\mathbb{R})}} \left(\frac{1}{\|z\|_p^d}\right) dS(z)\right) \\
&= \frac{1}{d}\ln\left(\frac{|B_{\ell_p^d}|}{|B_{\ell_2^d}|}\right) \\
&= \ln\left(\left(\frac{|B_{\ell_p^d}|}{|B_{\ell_2^d}|}\right)^{\frac{1}{d}}\right).
\end{aligned}$$

Then, by (1.18) of [Pi],

$$\ln\left(\left(\frac{|B_{\ell_p^d}|}{|B_{\ell_2^d}|}\right)^{\frac{1}{d}}\right) \asymp \ln\left(\frac{d^{-\frac{1}{p}}}{d^{-\frac{1}{2}}}\right) = \ln\left(d^{\frac{1}{2}-\frac{1}{p}}\right).$$

Therefore,

$$\exp\left\{\int_{S_{\ell_2^d(\mathbb{R})}} \ln\left(\frac{1}{\|z\|_p}\right) dS(z)\right\} \prec d^{\frac{1}{2}-\frac{1}{p}}.$$

For the lower bound, we will use again Jensen's inequality, and Lemma 2.2.2.

$$\begin{aligned}
\int_{S_{\ell_2^d(\mathbb{R})}} \ln\left(\frac{1}{\|z\|_p}\right) dS(z) &= \int_{S_{\ell_2^d(\mathbb{R})}} -\frac{1}{p}\ln\left(\|z\|_p^p\right) dS(z) \\
&\geq -\frac{1}{p}\ln\left(\int_{S_{\ell_2^d(\mathbb{R})}} \|z\|_p^p dS(z)\right) \\
&\asymp -\frac{1}{p}\ln\left(d^{1-\frac{p}{2}}\right) \\
&= \ln\left(d^{\frac{1}{2}-\frac{1}{p}}\right).
\end{aligned}$$

Therefore

$$\exp\left\{\int_{S_{\ell_2^d(\mathbb{R})}} \ln\left(\frac{1}{\|z\|_p}\right) dS(z)\right\} \succ d^{\frac{1}{2}-\frac{1}{p}}. \qquad \square$$

Remark 2.2.4. It follows from the proof of the previous lemma, since Lemma 2.2.2 also holds for complex spaces, that

$$\exp\left\{\int_{S_{\ell_2^d(\mathbb{C})}} \ln\left(\frac{1}{\|z\|_p}\right) dS(z)\right\} \succ d^{\frac{1}{2}-\frac{1}{p}},$$

for $1 \leq p < \infty$.

Now we are ready to prove our main result of this section.

Proof of Theorem 2.2.1. For this proof we will have to consider the cases $p > 2$ and $p < 2$ ($p = 2$ is already known), and also the subcases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. In order to have a better organization we divide the proof in different parts. Throughout this proof we consider $\frac{1}{p} + \frac{1}{q} = 1$.

Step I: $\mathbf{c}(\ell_p^d(\mathbb{R})) \succ \sqrt{d}$ for $2 < p \leq \infty$. As mentioned before, we want to consider a measure related to the geometry of the sphere $S_{\ell_p^d(\mathbb{R})}$. That being said, we also want a measure that can be easily related to the Lebesgue measure of S^{d-1}, given that for Hilbert spaces the polarization constant is known.

Consider then the measure η on $S_{\ell_p^d(\mathbb{R})}$ defined as

$$\eta(A) = \int_{H(A)} \frac{1}{|DH^{-1}(\varphi)|} dS(\varphi),$$

where $H : S_{\ell_q^d} \to S^{d-1}$ is defined as $H(\psi) = \frac{\psi}{\|\psi\|_2}$. That is, we choose η such that for any integrable function $f : S_{\ell_q^d} \to \mathbb{R}$, we have

$$\int_{S_{\ell_q^d}} f(\psi) \; d\eta(\psi) = \int_{S^{d-1}} f\left(\frac{\varphi}{\|\varphi\|_q}\right) dS(\varphi). \tag{2.4}$$

Using that the normalized Lebesgue measure is admissible, and its close relation with η, it is easy to see that η is admissible. Then, by Theorem 2.1.2, there is $x_0 \in S_{\ell_p^d}$ such that

$$\mathbf{c}(\ell_p^d(\mathbb{R})) \geq \exp\left\{-\int_{S_{\ell_q^d(\mathbb{R})}} \ln(|\langle x_0, \psi\rangle|) \; d\eta(\psi)\right\}.$$

We need then an upper bound for

$$\int_{S_{\ell_q^d(\mathbb{R})}} \ln(|\langle x_0, \psi\rangle|) \; d\eta(\psi).$$

By (2.4), we have

$$\begin{aligned}
\int_{S_{\ell_q^d}} \ln(|\langle x_0, \psi\rangle|) \; d\eta(\psi) &= \int_{S^{d-1}} \ln\left(\left|\langle x_0, \frac{\varphi}{\|\varphi\|_q}\rangle\right|\right) dS(\varphi) \\
&= \int_{S^{d-1}} \ln\left(\left|\langle x_0 \frac{\|x_0\|_2}{\|x_0\|_2}, \frac{\varphi}{\|\varphi\|_q}\rangle\right|\right) dS(\varphi) \\
&= \int_{S^{d-1}} \ln\left(\left|\langle \frac{x_0}{\|x_0\|_2}, \varphi\rangle\right|\right) dS(\varphi) \\
&\quad + \int_{S^{d-1}} \ln\left(\frac{1}{\|\varphi\|_q}\right) dS(\varphi) + \ln(\|x_0\|_2).
\end{aligned}$$

Then, using (2.3), Lemma 2.2.3 and that $x_0 \in S_{\ell_p^d}$, with $p > 2$, we obtain

$$\begin{aligned}
\mathbf{c}(\ell_p^d(\mathbb{R})) &\geq \mathbf{c}(\ell_2^d(\mathbb{R})) \exp\left\{-\int_{S^{d-1}} \ln\left(\frac{1}{\|\varphi\|_q}\right) dS(z)\right\} \frac{1}{\|x_0\|_2} \\
&\succ \mathbf{c}(\ell_2^d(\mathbb{R})) d^{\frac{1}{q}-\frac{1}{2}} d^{\frac{1}{p}-\frac{1}{2}} \\
&= \mathbf{c}(\ell_2^d(\mathbb{R})) \\
&\asymp \sqrt{d}.
\end{aligned}$$

Step II: $\mathbf{c}(\ell_p^d(\mathbb{R})) \prec \sqrt{d}$ for $2 < p < \infty$. As before, define the measure μ on $S_{\ell_p^d}$ by

$$\mu(A) = \int_{G(A)} \frac{1}{|DG^{-1}(z)|} dS(z),$$

where $G : S_{\ell_p^d} \to S^{d-1}$ is defined as $G(z) = \frac{z}{\|z\|_2}$.

Proceeding as in the previous case, we obtain

$$\mathbf{c}(\ell_p^d(\mathbb{R})) \leq \mathbf{c}(\ell_2^d(\mathbb{R})) \exp\left\{-\int_{S^{d-1}} \ln\left(\frac{1}{\|z\|_p}\right) dS(z)\right\} \frac{1}{\|\psi_0\|_2},$$

where ψ_0 is some point in $S_{\ell_q^d}$. Using Lemma 2.2.3 and the fact that $q < 2$ we conclude

$$\begin{aligned}
\mathbf{c}(\ell_p^d(\mathbb{R})) &\prec \mathbf{c}(\ell_2^d(\mathbb{R}))d^{\frac{1}{p}-\frac{1}{2}}d^{\frac{1}{q}-\frac{1}{2}} \\
&= \mathbf{c}(\ell_2^d(\mathbb{R})) \\
&\asymp \sqrt{d}.
\end{aligned}$$

Step III: $\mathbf{c}(\ell_\infty^d(\mathbb{R})) \prec d^{\frac{1}{2}+\varepsilon}$. As before, it is easy to prove that

$$\mathbf{c}(\ell_\infty^d(\mathbb{R})) \leq \mathbf{c}(\ell_2^d(\mathbb{R})) \exp\left\{-\int_{S^{d-1}} \ln\left(\frac{1}{\|z\|_\infty}\right) dS(z)\right\} \frac{1}{\|\psi_0\|_2},$$

for some $\psi_0 \in S_{\ell_1^d}$.

Combining this with the fact that, for any $r \geq 1$, we have

$$\begin{aligned}
\exp\left\{-\int_{S^{d-1}} \ln\left(\frac{1}{\|z\|_\infty}\right) dS(z)\right\} &\leq \exp\left\{-\int_{S^{d-1}} \ln\left(\frac{1}{\|z\|_r}\right) dS(z)\right\} \\
&\prec d^{\frac{1}{r}-\frac{1}{2}},
\end{aligned}$$

we obtain the desired result.

Step IV: $\mathbf{c}(\ell_p^d(\mathbb{R})) \prec \sqrt[p]{d}$ for $p < 2$. Following the proof of the case $p > 2$, we obtain

$$\mathbf{c}(\ell_p^d(\mathbb{R})) \leq \mathbf{c}(\ell_2^d(\mathbb{R})) \exp\left\{-\int_{S^{d-1}} \ln\left(\frac{1}{\|z\|_p}\right) dS(z)\right\} \frac{1}{\|\psi_0\|_2}.$$

But in this case ψ_0 is some point in $S_{\ell_q^d}$, with $q > 2$. Therefore, applying Lemma 2.2.3 we obtain

$$\begin{aligned}
\mathbf{c}(\ell_p^d(\mathbb{R})) &\prec \mathbf{c}(\ell_2^d(\mathbb{R}))d^{\frac{1}{p}-\frac{1}{2}}1 \\
&\asymp \sqrt[p]{d}.
\end{aligned}$$

Step V: $\mathbf{c}(\ell_p^d(\mathbb{R})) \succ \sqrt[p]{d}$ for $p < 2$. Note that in this case, the previous procedure would lead us to

$$\mathbf{c}(\ell_p^d(\mathbb{R}) \succ \sqrt[q]{d}.$$

Therefore, we need an alternative proof for this case. It is enough to find a subsequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\mathbf{c}_{n_k}(\ell_p^d(\mathbb{R})) \succ d^{\frac{n_k}{p}}.$$

Let us consider the subsequence $n_k = dk$. For each k consider the set of norm one linear functions

$$\{\underbrace{e_1 \cdots e_1}_{k \text{ times}} \cdots \underbrace{e_d \cdots e_d}_{k \text{ times}}\} \subseteq S_{\ell_q^d},$$

that is, we consider k copies of each vector of the canonical basis. Then we have

$$\begin{aligned} \mathbf{c}_{n_k}(\ell_p^d(\mathbb{R})) &\geq \|(e_1)^k \cdots (e_d)^k\|^{-1} \\ &= \sqrt[p]{\frac{(k+\cdots+k)^{k+\cdots+k}}{k^k\cdots k^k}} \qquad (2.5) \\ &= \sqrt[p]{\frac{(dk)^{dk}}{k^{dk}}} \\ &= \sqrt[p]{d^{dk}} \\ &= \sqrt[p]{d^{n_k}}, \end{aligned}$$

where (2.5) follows using Lagrange multipliers, as we will show in more detail in Chapter 3.

Note that in this case we proved that $\mathbf{c}(\ell_p^d(\mathbb{R}) \geq \sqrt[q]{d}$, rather than $\mathbf{c}(\ell_p^d(\mathbb{R}) \succ \sqrt[q]{d}$.

The complex case: The upper bound follows as in the real case using Remark 2.2.4 instead of Lemma 2.2.3

For the lower bound, by a complexification argument, is easy to see that

$$\mathbf{c}_n(\ell_p^d(\mathbb{R})) \leq 2^{n-1}\mathbf{c}_n(\ell_p^d(\mathbb{C})).$$

Indeed, given linear functions $\psi_1, \cdots, \psi_n \in \ell_p^d(\mathbb{R})$, consider the natural complexification of the space $\ell_p^d(\mathbb{R})$ that gives $\ell_p^d(\mathbb{C})$ and the complexifications $\tilde{\psi}_1, \cdots, \tilde{\psi}_n$ of the linear functions $\psi_1, \cdots, \psi_n$. Then we have

$$\begin{aligned}\prod_{i=1}^{n} \|\psi_i\| &= \prod_{i=1}^{n} \|\tilde{\psi}_i\| \\ &\leq \mathbf{c}_n(\ell_p^d(\mathbb{C})) \left\| \prod_{i=1}^{n} \tilde{\psi}_i \right\| \\ &\leq \mathbf{c}_n(\ell_p^d(\mathbb{C})) 2^{n-1} \left\| \prod_{i=1}^{n} \psi_i \right\|. \qquad (2.6)\end{aligned}$$

In (2.6) we use inequality (6) from [MST]. Therefore, by definition of $\mathbf{c}_n(\ell_p^d(\mathbb{R}))$, we conclude that $\mathbf{c}_n(\ell_p^d(\mathbb{R})) \leq 2^{n-1}\mathbf{c}_n(\ell_p^d(\mathbb{C}))$.

Then $\mathbf{c}(\ell_p^d(\mathbb{R})) \leq 2\mathbf{c}(\ell_p^d(\mathbb{C})) \asymp \mathbf{c}(\ell_p^d(\mathbb{C}))$. □

2.3 On the nth polarization constant of $\ell_\infty^d(\mathbb{C})$

In this section we study the nth polarization constant of the complex finite dimensional spaces $\ell_\infty^d(\mathbb{C})$.

In the first part, we use a probabilistic approach to prove the existence of linear functionals whose product has small norm. This provides a lower bound for $\mathbf{c}_n(\ell_\infty^d(\mathbb{C}))$.

In the second part, using the Mahler measure of a polynomial, we obtain a lower bound for the product of linear functions, which depends on the coefficients of the linear functions. Despite this dependence, this lower bound will allow us to deduce a general lower bound, and thus, an upper bound for $\mathbf{c}_n(\ell_\infty^d(\mathbb{C}))$.

Finally, we will end this section with a remark on the polarization constants of the 2-dimensional space $\ell_\infty^2(\mathbb{C})$.

Lower bound for $\ell_\infty^d(\mathbb{C})$

Our objective is to prove the existence of linear functionals $\varphi_1, \ldots, \varphi_n : \ell_\infty^d \to \mathbb{C}$ such that the norm of the product is small in comparison with the product of the norms. As a corollary, we obtain a lower bound for $\mathbf{c}_n(\ell_\infty^d(\mathbb{C}))$.

The probabilistic techniques we use in this section are an adaptation, to our problem, of techniques used by H. Boas in [Bo].

Let us start by setting some notation. We fix (Ω, Σ, P) a probability space where there is $\{\varepsilon_k^j\}_{j,k}$, with $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, d\}$, a family of independent Bernoulli random variables over Ω. That is, $\{\varepsilon_k^j\}_{j,k}$ are independent random variables such that $P(\varepsilon_k^j = 1) = P(\varepsilon_k^j = -1) = \frac{1}{2}$ for $j = 1, \dots, n$ and $k = 1, \dots, d$. For any $t \in \Omega$ and $j \in \{1, \dots, n\}$ we define the linear function $\varphi_j(\cdot, t) : \ell_\infty^d \to \mathbb{C}$ as $\varphi_j(z, t) = \sum_{k=1}^d \varepsilon_k^j(t) z_k$ and $F : \ell_\infty^d \times \Omega \to \mathbb{C}$ by

$$F(z, t) = \prod_{j=1}^n \varphi_j(z, t) = \sum_{k_1, \dots, k_n = 1}^n \varepsilon_{k_1}^1 \cdots \varepsilon_{k_n}^n z_{k_1} \cdots z_{k_n}$$

We aim to prove the existence of some $t_0 \in \Omega$ such that the norm $\left\| \prod_{j=1}^n \varphi_j(\cdot, t_0) \right\| = \|F(\cdot, t_0)\|$ is small. To see this we are going to prove that the probability of $\|F(\cdot, t)\|$ not being small is less than one. To do this we need some auxiliary lemmas, related to the function F and the geometry of the space $\ell_\infty^d(\mathbb{C})$.

Lemma 2.3.1. *For any natural number N, the d dimensional torus $\mathbb{T}^d$ can be covered up with N^d balls of $\ell_\infty^d(\mathbb{C})$, with center on $\mathbb{T}^d$ and radius $\frac{\pi}{N}$.*

To prove this lemma it is enough to consider the balls centred at $(e^{2\pi i \frac{j_1}{N}}, \dots, e^{2\pi i \frac{j_d}{N}})$, with $j_1, \dots, j_d \in \{1, \dots, N\}$.

Lemma 2.3.2. *Given a norm one vector $z_0 \in \ell_\infty^d(\mathbb{C})$ and positive numbers λ and R, we have*

$$P(|F(z_0, t)| > R) \leq 4 e^{-\lambda R + \frac{\lambda^2}{2} d^n}.$$

Proof. Since

$$\begin{aligned} P(|F(z_0, t)| > R) &\leq P(|\operatorname{Re} F(z_0, t)| > R) + P(|\operatorname{Im} F(z_0, t)| > R) \\ &\leq P(\operatorname{Re} F(z_0, t) > R) + P(\operatorname{Re} F(z_0, t) < R) \\ &\quad + P(\operatorname{Im} F(z_0, t) > R) + P(\operatorname{Im} F(z_0, t) < R), \end{aligned}$$

by the symmetry of the problem it is enough to see that

$$P(\operatorname{Re} F(z_0, t) > R) \leq e^{-\lambda R + \frac{\lambda^2}{2} d^n}. \tag{2.7}$$

To prove (2.7) we are going to use the exponential Chebyshev's inequality. That is, if f is a real function over a probability space Ω, R and λ are positive numbers, then

$$P(f(t) > R) \leq e^{-\lambda R} \mathbb{E}(e^{\lambda f(t)}), \tag{2.8}$$

where $\mathbb{E}(g)$ is the expectation value of the function g.

Since we are going to apply this formula to $f = \mathrm{Re}(F)$, we need an upper bound of $\mathbb{E}(e^{\lambda \mathrm{Re}(F)})$. If we write $z = (z_1, \ldots, z_d)$, then we have

$$
\begin{aligned}
\mathbb{E}(e^{\lambda \mathrm{Re}(F(z,t))}) &= \mathbb{E}\left(e^{\mathrm{Re}(\lambda \sum_{k_1,\ldots,k_n=1}^{n} \varepsilon_{k_1}^1 \cdots \varepsilon_{k_n}^n z_{k_1} \cdots z_{k_n})}\right) \\
&= \prod_{k_1,\ldots,k_n=1}^{n} \mathbb{E}\left(e^{\mathrm{Re}(\lambda \varepsilon_{k_1}^1 \cdots \varepsilon_{k_n}^n z_{k_1} \cdots z_{k_n})}\right) \qquad (2.9) \\
&= \prod_{k_1,\ldots,k_n=1}^{n} \int_\Omega \left(e^{\mathrm{Re}(\lambda \varepsilon_{k_1}^1 \cdots \varepsilon_{k_n}^n z_{k_1} \cdots z_{k_n})}\right) dt \\
&= \prod_{k_1,\ldots,k_n=1}^{n} \left(\frac{1}{2} e^{\mathrm{Re}(\lambda z_{k_1} \cdots z_{k_n})} + \frac{1}{2} e^{\mathrm{Re}(-\lambda z_{k_1} \cdots z_{k_n})}\right) \\
&= \prod_{k_1,\ldots,k_n=1}^{n} \cosh(\mathrm{Re}(\lambda z_{k_1} \cdots z_{k_n})) \\
&\le \prod_{k_1,\ldots,k_n=1}^{n} e^{\frac{\lambda^2}{2} \mathrm{Re}((\lambda z_{k_1} \cdots z_{k_n}))^2} \\
&\le \prod_{k_1,\ldots,k_n=1}^{n} e^{\frac{\lambda^2}{2}} \\
&= e^{\frac{\lambda^2}{2} d^n},
\end{aligned}
$$

where in (2.9) we use the independence of the Rademacher functions.

Combining this with (2.8) we obtain the desired result. $\square$

Lemma 2.3.3. *For any pair of norm one vectors $z, w \in \ell_\infty^d(\mathbb{C})$ and any $t \in \Omega$, we have*

$$|F(w,t) - F(z,t)| \le n d^{\frac{n}{2}} \|F(\cdot,t)\| \|w - z\|.$$

Proof. Fixed $t \in \Omega$, take $\check{F}(\cdot,t)$ the n-linear continuous symmetric function associated to the polynomial $F(\cdot,t)$. When m of the parameters of $\check{F}(\cdot,t)$ are all the same

$x \in \ell_\infty^d(\mathbb{C})$ we will note x^m instead of $\underbrace{x, \cdots, x}_{m \text{ times}}$. Then we have

$$\begin{aligned}
|F(w,t) - F(z,t)| &\leq |\check{F}(w^n,t) - \check{F}(z^n,t)| \\
&= |\sum_{i=1}^{n} \check{F}(w^{n-i+1}, z^{i-1}, t) - \check{F}(w^{n-i}, z^i, t)| \\
&= |\sum_{i=1}^{n} \check{F}(w-z, w^{n-i}, z^{i-1}, t)| \\
&\leq \sum_{i=1}^{n} |\check{F}(w-z, w^{n-i}, z^{i-1}, t)| \\
&\leq n\|\check{F}(\cdot,t)\|_{\mathcal{L}(^n\ell_\infty^d(\mathbb{C}))}\|w-z\| \\
&\leq nd^{\frac{n}{2}}\|\check{F}(\cdot,t)\|_{\mathcal{L}(^n\ell_2^d(\mathbb{C}))}\|w-z\| \\
&= nd^{\frac{n}{2}}\|F(\cdot,t)\|_{\mathcal{P}(^n\ell_2^d(\mathbb{C}))}\|w-z\| \qquad (2.10) \\
&\leq nd^{\frac{n}{2}}\|F(\cdot,t)\|_{\mathcal{P}(^n\ell_\infty^d(\mathbb{C}))}\|w-z\|,
\end{aligned}$$

where in (2.10) we use that if P is a homogeneous polynomial over a Hilbert space then $\|P\| = \|\check{P}\|$ (see [Di], Proposition 1.44). □

Lemma 2.3.4. *For any positive number R*

$$P(\|F(\cdot,t)\| > 2\sqrt{2}R) < [8nd^{\frac{n}{2}}]^d 4e^{-\lambda R + \frac{\lambda^2}{2}d^n},$$

where $[\cdot]$ stands for integer part.

Proof. By Lemma 2.3.1, there is a family of points $\{w_1, \dots, w_{[8nd^{\frac{n}{2}}]^d}\} \subseteq \mathbb{T}^d$ such that for any $z \in \mathbb{T}^d$, for some $i = 1, \dots, [8nd^{\frac{n}{2}}]^d$, we have $z \in B\left(w_i, \frac{\pi}{[8nd^{\frac{n}{2}}]}\right)$.

For any fixed $t \in \Omega$, by the maximum modulus principle, there is $z_0 \in \mathbb{T}^d$ such that

$$\|F(\cdot,t)\| = |F(z_0,t)|.$$

Let i be such that $\|w_i - z_0\| \leq \frac{\pi}{[8nd^{\frac{n}{2}}]} \leq \frac{1}{2nd^{\frac{n}{2}}}$. By Lemma 2.3.3

$$|F(w_i,t) - F(z_0,t)| \leq \|F(\cdot,t)\|nd^{\frac{n}{2}}\|w_i - z_0\| < \|F(\cdot,t)\|\frac{1}{2}.$$

Therefore

$$\frac{\|F(\cdot,t)\|}{2} < |F(w_i,t)|.$$

We conclude that

$$\|F(\cdot,t)\| < \max\{2|F(w_i,t)| : i = 1,\dots,[8nd^{\frac{n}{2}}]^d\}.$$

Since $t \in \Omega$ was arbitrary, we have

$$\begin{aligned} P(\|F(\cdot,t)\| > 2\sqrt{2}R) &< P(\max\{|F(w_i,t)| : i = 1,\dots,[8nd^{\frac{n}{2}}]^d\} > \sqrt{2}R) \\ &\leq \sum_{i=1}^{[8nd^{\frac{n}{2}}]^d} P(|F(w_i,t)| > \sqrt{2}R) \\ &\leq \sum_{i=1}^{[8nd^{\frac{n}{2}}]^d} 4e^{-\lambda R + \frac{\lambda^2}{2}d^n} \qquad (2.11) \\ &= [8nd^{\frac{n}{2}}]^d 4e^{-\lambda R + \frac{\lambda^2}{2}d^n}, \end{aligned}$$

where in (2.11) we used Lemma 2.3.2. □

Proposition 2.3.5. *For the space $\ell_\infty^d(\mathbb{C})$ we have the following lower bound for its nth polarization constant*

$$\mathbf{c}_n(\ell_\infty^d(\mathbb{C})) \geq \frac{\sqrt{d^n}}{4\sqrt{\ln([8nd^{\frac{n}{2}}]^d 4)}}.$$

Proof. Take in Lemma 2.3.4

$$\lambda = \frac{\sqrt{2\ln([8nd^{\frac{dn}{2}}]^d 4)}}{\sqrt{d^n}} \text{ and } R = \sqrt{2d^n \ln([8nd^{\frac{n}{2}}]^d 4)}.$$

Then

$$\begin{aligned} P(\|F(\cdot,t)\| > 2\sqrt{2}R) &< [8nd^{\frac{n}{2}}]^d 4e^{-\lambda R + \frac{\lambda^2}{2}d^n} \\ &= [8nd^{\frac{n}{2}}]^d 4e^{-\ln([8nd^{\frac{n}{2}}]^d 4)} \\ &= 1. \end{aligned}$$

Therefore, there is $t_0 \in \Omega$, such that

$$\begin{aligned}
\|\prod_{j=1}^n \varphi_j(t_0)\| &= \|F(\cdot, t_0)\| \\
&\leq 2\sqrt{2}R \\
&= 4\sqrt{\ln([8nd^{\frac{n}{2}}]^d 4)}\sqrt{d^n} \\
&= \frac{4\sqrt{\ln([8nd^{\frac{n}{2}}]^d 4)}}{\sqrt{d^n}} d^n \\
&= \frac{4\sqrt{\ln([8nd^{\frac{n}{2}}]^d 4)}}{\sqrt{d^n}} \prod_{j=1}^n \|\varphi_j(t_0)\|,
\end{aligned}$$

which ends the proof. □

Remark 2.3.6. Note that in particular, the previous result assures us that

$$\begin{aligned}
\mathbf{c}(\ell_\infty^d(\mathbb{C})) &\geq \lim_{n\to\infty} \frac{\sqrt{d}}{\left(4\sqrt{\ln([8nd^{\frac{n}{2}}]^d 4)}\right)^{\frac{1}{n}}} \\
&\geq \lim_{n\to\infty} \frac{\sqrt{d}}{\left(4\sqrt{\ln((9nd)^{\frac{dn}{2}} 4)}\right)^{\frac{1}{n}}} \\
&= \lim_{n\to\infty} \frac{\sqrt{d}}{\left(4\sqrt{\frac{dn}{2}\ln((9nd)4)}\right)^{\frac{1}{n}}} \\
&= \sqrt{d}.
\end{aligned}$$

This improves the bound from Theorem 2.2.1, where we only had the asymptotic behaviour $\mathbf{c}(\ell_\infty^d(\mathbb{C})) \succ \sqrt{d}$.

An application of Mahler measure.

In this subsection we use the Mahler measure to obtain a lower bound for the product of linear functions over the complex Banach spaces $\ell_\infty^d(\mathbb{C})$. The difference with the previous results of this section is that this lower bound will depend on the coefficients of the linear functions. To obtain this inequality we need the following lemma, regarding the Mahler measure of a linear function.

Lemma 2.3.7. *Let $\psi : \mathbb{C}^d \to \mathbb{C}$ be a non zero linear function given by $\psi(z) = \sum_{j=1}^d a_j z_j$. Then we have the following lower bound for its Mahler measure*

$$M(\psi) \geq \max\{|a_j| : j = 1 \ldots, d\}.$$

To prove this lemma we are going to use next equality, consequence of Jensen's formula (see [BL])

$$\int_0^1 \ln\left(\left|e^{2\pi i t} - a\right|\right) dt = \ln^+(|a|), \tag{2.12}$$

where $\ln^+(x) = \max\{\ln(x), 0\}$.

Proof. We prove the result by induction on d. For $d = 1$ it is immediate that $M(\psi) = |a_1|$.

Now, let us assume the result holds for $d-1$. Without loss of generality we may assume $a_1, \ldots, a_d \in \mathbb{R}_{\geq 0}$ and $a_1 \geq a_2 \geq \ldots \geq a_d$. If $a_2 = 0$ the result follows as in the case $d = 1$, so let us assume that $a_2 \neq 0$. Using the inductive hypothesis and the definition of the Mahler measure we obtain

$$\begin{aligned}
\ln(M(\psi)) &= \int_0^1 \cdots \int_0^1 \int_0^1 \ln\left(\left|\sum_{j=1}^d a_j e^{2\pi i t_j}\right|\right) dt_1 dt_2 \cdots dt_d \\
&= \int_0^1 \cdots \int_0^1 \int_0^1 \ln\left(a_1 \left|e^{2\pi i t_1} + \sum_{j=2}^d \frac{a_j}{a_1} e^{2\pi i t_j}\right|\right) dt_1 dt_2 \cdots dt_d \\
&= \int_0^1 \cdots \int_0^1 \ln(a_1) + \ln^+\left(\left|\sum_{j=2}^d \frac{a_j}{a_1} e^{2\pi i t_j}\right|\right) dt_2 \cdots dt_d \qquad (2.13) \\
&\geq \ln(a_1) + \int_0^1 \cdots \int_0^1 \ln\left(\left|\sum_{j=2}^d \frac{a_j}{a_1} e^{2\pi i t_j}\right|\right) dt_2 \cdots dt_d \\
&\geq \ln(a_1) + \ln\left(\frac{a_2}{a_1}\right), \qquad (2.14)
\end{aligned}$$

where in (2.13) we integrate the variable t_1 using (2.12), and in (2.14) we use the inductive hypothesis.

Then

$$M(\psi) \geq a_1 \frac{a_2}{a_1} = a_1,$$

as desired. □

As an immediate consequence of this lemma we obtain the following.

Proposition 2.3.8. *Take non zero linear functions $\psi_1, \ldots, \psi_n : \ell_\infty^d(\mathbb{C}) \to \mathbb{C}$, with $\psi_k = \sum_{j=1}^d a_{j,k} z_j$. Then*

$$\|\psi_1 \cdots \psi_n\| \geq \max\{|a_{j,1}| : j = 1, \ldots, d\} \cdots \max\{|a_{j,d}| : j = 1, \ldots, d\}.$$

Proof. Using that the Mahler measure is less than or equal to the norm and that the Mahler measure is multiplicative, we have

$$\begin{aligned}\|\psi_1 \cdots \psi_n\| &\geq M(\psi_1 \cdots \psi_n)\\ &= M(\psi_1) \cdots M(\psi_n)\\ &\geq \max\{|a_{j,1}| : j = 1, \ldots, d\} \cdots \max\{|a_{j,d}| : j = 1, \ldots, d\}. \qquad \square\end{aligned}$$

Remark 2.3.9. Note that, if $\psi_k = \sum_{j=1}^d a_{j,k} z_j$, then

$$\max\{|a_{j,d}| : j = 1 \ldots, d\} \geq \frac{\|\psi_k\|}{d}.$$

Therefore, from the previous proposition, we deduce

$$\mathbf{c}_n(\ell_\infty^d(\mathbb{C})) \leq d^n \text{ and } \mathbf{c}(\ell_\infty^d(\mathbb{C})) \leq d.$$

A remark on two dimensional spaces

We end this section with some observations about this problem for 2-dimensional spaces. In [AnRe] A. Anagnostopoulos and S. G. Révész found a relation between the nth polarization constant of a two dimensional Hilbert space and the nth Chebyshev constant of S^1 or S^2, depending if the space is real or complex. Let us recall the definition of the nth Chebyshev constant $M_n(K)$ for a compact set K on a Banach space:

$$M_n(K) = \inf_{y_1, \ldots, y_n \in K} \sup_{y \in K} \|y_1 - y\| \cdots \|y_n - y\|.$$

Anagnostopoulos and S. G. Révész showed that

$$\mathbf{c}_n(\ell_2^2(\mathbb{K})) = \begin{cases} \frac{2^n}{M_n(S^1)} & \text{if} \quad \mathbb{K} = \mathbb{R} \\ \\ \frac{2^n}{M_n(S^2)} & \text{if} \quad \mathbb{K} = \mathbb{C}. \end{cases}$$

In a minor contribution to the study of these constants, in the following we give an elementary proof that

$$\mathbf{c}_2(\ell_2^2(\mathbb{C})) = 2,$$

and characterizes the linear functions needed to have an equality.

Proposition 2.3.10. *Given two linear functions $\psi, \varphi : \ell_\infty^2(\mathbb{C}) \to \mathbb{C}$, we have*

$$\|\psi\varphi\| \geq \frac{1}{2}\|\psi\| \, \|\varphi\|.$$

We remark that this result is already known (see Proposition 18 of [RS]). But our proof characterizes the linear functions needed to have an equality.

Proof. Let us write $\psi(x,y) = xa + yb$. Multiplying the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ for complex numbers of modulus one, we may assume $a, b \geq 0$. Replacing φ by $\lambda\varphi$, with $|\lambda| = 1$, we may also assume $\varphi(x,y) = xc + yde^{i\theta}$ with $c, d \geq 0$ and $\theta \in [-\pi, \pi)$.

We claim that $|\psi\varphi(e^{i\theta/2}, 1)| \geq \frac{1}{2}(ac + bd + ad + bc) = \frac{1}{2}\|\psi\|\|\varphi\|$. Let us prove this.

$$\begin{aligned} |\psi\varphi(e^{i\theta/2}, 1)| &= |(ae^{i\theta/2} + b)(ce^{i\theta/2} + de^{i\theta})| \\ &= |(ac + bd)e^{i\theta} + (ade^{i3\theta/2} + bce^{i\theta/2})| \\ &= |(ac + bd) + (ade^{i\theta/2} + bce^{-i\theta/2})|. \end{aligned}$$

Since $\frac{\theta}{2}$ and $-\frac{\theta}{2}$ belong to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the real part of the complex number $ade^{i\theta/2} + bce^{-i\theta/2}$ is non negative. Therefore,

$$\begin{aligned} |(ac + bd) + (ade^{i\theta/2} + bce^{-i\theta/2})| &\geq \left(|(ac + bd)|^2 + |ade^{i\theta/2} + bce^{-i\theta/2}|^2\right)^{\frac{1}{2}} \\ &\geq \left((ac + bd)^2 + (ad - bc)^2\right)^{\frac{1}{2}} \qquad (2.15) \\ &= \left((ac - bd)^2 + (ad + bc)^2\right)^{\frac{1}{2}}. \qquad (2.16) \end{aligned}$$

From (2.15) we conclude that $|(ac + bd)e^{i\theta} + ade^{i3\theta/2} + bce^{i\theta/2}|$ is greater than $ac + bd$, and from (2.16) that it is greater than $ad + bc$. Thus we have

$$\begin{aligned} |(ac + bd)e^{i\theta} + ade^{i3\theta/2} + bce^{i\theta/2}| &\geq \max\{(ac + bd), (ad + bc)\} \\ &\geq \frac{1}{2}(ac + bd + ad + bc), \end{aligned}$$

as desired. ☐

From the proof of last result, it follows that equality holds if and only if all the inequalities in the proof are equalities. Therefore,

$$\max\{(ac + bd), (ad + bc)\} = \frac{1}{2}(ac + bd + ad + bc), \qquad (2.17)$$

and then

$$\begin{aligned} \left((ac + bd)^2 + (ad - bc)^2\right)^{1/2} &= \left((ac - bd)^2 + (ad + bc)^2\right)^{1/2} \\ &= \max\{(ac + bd), (ad + bc)\}. \qquad (2.18) \end{aligned}$$

Equation (2.17) implies $ac+bd = ad+bc$, which, combined with (2.18) gives

$$ad = bc$$

$$ac = bd.$$

Therefore $a = b$ and $c = d$.

The last condition to have an equality is that

$$\begin{aligned} |(ac+bd)+(ade^{i\theta/2}+bce^{-i\theta/2})| &= \left(|(ac+bd)|^2+|ade^{i\theta/2}+bce^{-i\theta/2}|^2\right)^{\frac{1}{2}} \\ &= \left((ac+bd)^2+(ad-bc)^2\right)^{\frac{1}{2}}, \end{aligned}$$

which only happens if $\theta = -\pi$.

Therefore, to have $\|\psi\varphi\| = \frac{1}{2}\|\psi\|\,\|\varphi\|$, a necessary and sufficient condition is that the linear functions ψ, φ are given by orthogonal vectors $(a,b),(c,d)$ with $|a| = |b|$ and $|c| = |d|$.

2.4 Resumen en castellano del Capítulo II

En este capítulo introducimos la noción de la n-ésima constante de polarización $\mathbf{c}_n(X)$ de un espacio de Banach X. La misma es la menor constante para la cual se tiene la siguiente desigualdad

$$\|\psi_1\|\cdots\|\psi_n\| \leq \mathbf{c}_n(X)\,\|\psi_1\cdots\psi_n\|,$$

para cualquier conjunto de n elementos $\psi_1\cdots\psi_n$ de X^*.

Relacionado con este concepto, también definimos la constante de polarización de X como

$$\mathbf{c}(X) = \lim_{n\to\infty}(\mathbf{c}_n(X))^{\frac{1}{n}}.$$

Con respecto a los resultados previos sobre estas constantes, en el artículo [A] Arias-de-Reyna prueba que si X es un espacio de Hilbert complejo de dimensión mayor o igual que n, entonces

$$\mathbf{c}_n(X) = n^{\frac{n}{2}}.$$

Este resultado es valido en el caso real para $n \leq 5$ (ver [PR]), pero no se sabe si en general vale (aunque se conjetura que si).

En cuanto a la constante de polarización, García-Vázquez y Villa en [GV] hallan su valor exacto para espacios de Hilbert reales de dimensión d. Estas constantes tiene un orden de $\sqrt{d}$ para d suficientemente grande. Posteriormente A. Pappas y S. G. Révész extienden este resultado al caso complejo en [PR].

En este capítulo estudiamos estas constantes. Nuestro resultado principal respecto a la constante de polarización es que, para los espacio $\ell_p^d(\mathbb{K})$, con $\mathbb{K} = \mathbb{R}$ o $\mathbb{C}$, esta constante es del orden de $\sqrt{d}$ si $2 < p < \infty$ y del orden de $\sqrt[p]{d}$ si $1 \leq p < 2$. Para $\ell_\infty^d(\mathbb{K})$ probamos que el orden de la constante de polarización es al menos $\sqrt{d}$ y que, para cualquier $\varepsilon > 0$, es menor que $d^{\frac{1}{2}+\varepsilon}$.

Sobre a la n-ésima constante de polarización de $\ell_\infty^d(\mathbb{C})$ probamos, utilizando métodos probabilísticos, la existencia de funciones lineales $\psi_1 \cdots \psi_n : \ell_\infty^d(\mathbb{C}) \to \mathbb{C}$ tales que

$$\|\psi_1\| \cdots \|\psi_n\| \geq \frac{\sqrt{d^n}}{4\sqrt{\ln([8nd^{\frac{n}{2}}]^d 4)}} \|\psi_1 \cdots \psi_n\|.$$

En particular, esto nos dice que

$$\frac{\sqrt{d^n}}{4\sqrt{\ln([8nd^{\frac{n}{2}}]^d 4)}} \leq \mathbf{c}_n(\ell_\infty^d(\mathbb{C})).$$

Tomando raíz n-ésima y límite en n, concluimos además

$$\sqrt{d} \leq \mathbf{c}(\ell_\infty^d(\mathbb{C})).$$

En cuanto a cotas superiores para $\mathbf{c}_n(\ell_\infty^d(\mathbb{C}))$, utilizando la medida de Mahler, obtenemos una demostración elemental de la siguiente cota

$$\mathbf{c}_n(\ell_\infty^d(\mathbb{C})) \leq d^n.$$

Chapter 3

The factor problem

In this chapter we study a generalization to polynomials of the problem of finding the nth polarization constant, studied in Chapter 2. The problem object of our study is some times called the factor problem. On a Banach space X, this problem consists in finding the best constant M such that, for any set of continuous scalar polynomials $P_1, \ldots, P_n$ over X, of some prescribed degrees, the following inequality holds

$$\|P_1\| \cdots \|P_n\| \leq M \|P_1 \cdots P_n\|. \tag{3.1}$$

The constant will necessarily depend on X and on the degrees of the polynomials.

The factor problem has been studied by several authors. In [BST], C. Benítez, Y. Sarantopoulos and A. Tonge proved that, for continuous polynomials of degrees $k_1, \ldots, k_n$, inequality (3.1) holds with constant

$$M = \frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}},$$

for any complex Banach space. The authors also showed that this is the best universal constant, since there are polynomials on ℓ_1 for which equality prevails. For complex Hilbert spaces and homogeneous polynomials, D. Pinasco proved in [P] that the optimal constant is

$$M = \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}. \tag{3.2}$$

We will address the factor problem in both the homogeneous and the non homogeneous context. Most of the content of this chapter belongs to the articles [CPR, RO].

3.1 Notation and basic results

In this brief first section, we introduce some notation and give some basic results.

Definition 3.1.1. For a Banach space X, let $D(X, k_1, \ldots, k_n)$ denote the smallest constant that satisfies (3.1) for polynomials of degree $k_1, \ldots, k_n$. We also define $C(X, k_1, \ldots, k_n)$ as the smallest constant that satisfies (3.1) for homogeneous polynomials of degree $k_1, \ldots, k_n$.

Several of our results will have two parts. The first one involving the constant $C(X, k_1, \ldots, k_n)$, for homogeneous polynomials, and the second one involving the constant $D(X, k_1, \ldots, k_n)$, for arbitrary polynomials. Whenever the proof of both parts are similar, we limit to prove only one of them.

When working with the constants $C(X, k_1, \ldots, k_n)$ and $D(X, k_1, \ldots, k_n)$, the following characterization may result handy.

Lemma 3.1.2. *a) The constant $C(X, k_1, \ldots, k_n)$ is the biggest constant M such that, given any $\varepsilon > 0$, there exist a set of homogeneous continuous polynomials $\{P_j\}_{j=1}^n$ with $deg(P_j) \leq k_j$ such that*

$$M \left\| \prod_{j=1}^n P_j \right\| \leq (1+\varepsilon) \prod_{j=1}^n \|P_j\|. \tag{3.3}$$

b) The constant $D(X, k_1, \ldots, k_n)$ is the biggest constant satisfying (3.3) *for arbitrary polynomials.*

Proof. We only prove item *b*). First, let us see that if M is strictly bigger than $D(X, k_1, \ldots, k_n)$, then M does not satisfy (3.3). Take $\varepsilon > 0$ such that

$$\frac{1+\varepsilon}{(1-\varepsilon)^n} D(X, k_1, \ldots, k_n) < M$$

and suppose that there is a set of polynomials $\{P_j\}_{j=1}^n$ on X, with $deg(P_j) = l_j \leq k_j$, satisfying (3.3) for this ε. For each j, take $x_j \in B_X$ such that $|P_j(x_j)| \geq \|P_j\|(1-\varepsilon)$. Now we define the polynomials $\tilde{P}_j : X \to \mathbb{K}$ by $\tilde{P}_j(x) = P_j(x)(\psi_j(x))^{k_j - l_j}$, where $\psi_j(x) \in S_{X^*}$ is such that $|\psi_j(x_j)| = 1$. The polynomial $\tilde{P}_j$ has degree k_j and

$$\|\tilde{P}_j\| \geq |\tilde{P}_j(x_j)| \geq \|P_j\|(1-\varepsilon).$$

Therefore

$$\begin{aligned}
\frac{1+\varepsilon}{(1-\varepsilon)^n} D(X,k_1,\dots,k_n) \left\| \prod_{j=1}^n \tilde{P}_j \right\| &< M \left\| \prod_{j=1}^n \tilde{P}_j \right\| \\
&\leq M \left\| \prod_{j=1}^n P_j \right\| \\
&\leq (1+\varepsilon) \prod_{j=1}^n \|P_j\| \\
&\leq \frac{1+\varepsilon}{(1-\varepsilon)^n} \prod_{j=1}^n \|\tilde{P}_j\|,
\end{aligned}$$

which is a contradiction, since

$$D(X,k_1,\dots,k_n) \left\| \prod_{j=1}^n \tilde{P}_j \right\| \geq \prod_{j=1}^n \|\tilde{P}_j\|$$

by definition.

Now we prove that $D(X,k_1,\dots,k_n)$ satisfies the condition (3.3) of the lemma. Given any $\varepsilon > 0$, we have that $\frac{D(X,k_1,\dots,k_n)}{1+\varepsilon} < D(X,k_1,\dots,k_n)$. Then, there is a set of polynomials $\{P_j\}_{j=1}^n$ with $deg(P_j) = k_j$, such that

$$\frac{D(X,k_1,\dots,k_n)}{1+\varepsilon} \left\| \prod_{j=1}^n P_j \right\| < \prod_{j=1}^n \|P_j\|.$$

Otherwise $\frac{D(X,k_1,\dots,k_n)}{1+\varepsilon}$ would be a smaller constant than $D(X,k_1,\dots,k_n)$ satisfying (3.1). □

Remark 3.1.3. Following the proof of the previous Lemma, it is clear that we can take the polynomials $\{P_j\}_{j=1}^n$ with $deg(P_j) = k_j$, instead of $deg(P_j) \leq k_j$. Later on we will use both versions of the Lemma.

Remark 3.1.4. The constants $C(X,k_1,\dots,k_n)$ and $D(X,k_1,\dots,k_n)$ are increasing in k_i for $i = 1,\dots,n$. This fact follows using the same procedure to replace a polynomial of degree l_i by a polynomial of degree k_i, with $k_i \geq l_i$, used in the proof of last lemma.

3.2 The factor problem on L_p spaces

In this section we study the factor problem on complex L_p spaces. In what follows, we focus our attention on infinite dimensional spaces or spaces whose dimension is greater than the number of polynomials considered. In a subsequent section we will treat the factor problem in the context where the number of polynomials is greater than the dimension of the space.

As mentioned above, the problem for the spaces L_1 and L_2 was solved in [BST] and [P] respectively. We will focus then on the case $1 < p < 2$, and we will show that

$$C(L_p, k_1 \dots, k_n) = \sqrt[p]{\frac{(k_1 + \dots + k_n)^{(k_1+\dots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}. \tag{3.4}$$

For $p > 2$ we will only give some estimates on the constant $C(L_p, k_1 \dots, k_n)$. Finally, exploiting the similarities between L_p spaces and the Schatten classes S_p, we will transport some of our result to these spaces.

Banach-Mazur distance

We now introduce an important concept that will be used in this section: the Banach-Mazur distance.

If X and Y are isomorphic Banach spaces, their *Banach-Mazur distance* (see, for example, [Pi, Chapter 1] or [T]) is defined as

$$\begin{aligned} d(X,Y) &= \inf\{\|u\| \, \|u^{-1}\| \mid u : X \to Y \text{ isomorphism}\} \\ &= \inf\{\|u^{-1}\| \mid u : X \to Y \text{ norm one isomorphism}\}. \end{aligned}$$

Note that this infimum is in fact a minimum when the dimension of the spaces X and Y is finite.

Related with this concept, we define

$$d_n(X) := \sup\{d(E, \ell_2^n) : E \text{ subspace of } X \text{ with } \dim E = n\}. \tag{3.5}$$

From Corollary 5 in [L], we obtain

$$d_n(L_p(\Omega, \mu)) \leq n^{|1/p-1/2|}, \tag{3.6}$$

whenever $L_p(\Omega, \mu)$ has dimension at least n.

The finite dimensional setting

Using the result of Pinasco (3.2) on Hilbert spaces and the Banach-Mazur distance between L_p spaces and Hilbert spaces, we will deduce (3.4). This will work for polynomials of the same degree (see Remark 3.2.2). For polynomials of arbitrary degree more work will be required.

The proof of the following lemma is inspired by Proposition 1 of [RS].

Lemma 3.2.1. *Let X be a Banach space and let $d_n(X)$ be defined as in* (3.5)*. Then,*

$$C(X, k_1, \ldots, k_n) \leq \sqrt{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}} d_n(X)^{\sum_{i=1}^n k_i}}.$$

Proof. We need to see that given any set $P_1, \ldots, P_n : X \to \mathbb{C}$ of homogeneous polynomials of degree $k_1, \ldots, k_n$, then

$$\sqrt{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}} d_n(X)^{\sum_{i=1}^n k_i}} \|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}X)} \geq \|P_1\|_{\mathcal{P}(^{k_1}X)} \cdots \|P_n\|_{\mathcal{P}(^{k_n}X)}.$$

To simplify the notation let us define $\mathbf{k} := \sum_{i=1}^n k_i$. Given $\varepsilon > 0$, we can take a set of norm one vectors $\{x_1, \ldots, x_n\} \subset X$ such that $|P_j(x_j)| > (1 - \varepsilon) \|P_j\|_{\mathcal{P}(^{k_j}X)}$, for $1 \leq j \leq n$. Let $E \subset X$ be any $n-$dimensional subspace containing $\{x_1, \ldots, x_n\}$ and let $T : \ell_2^n \to E$ be a norm one isomorphism with $\|T^{-1}\| = d_n(X)$. We have

$$\begin{aligned}
\|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}X)} &\geq \|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}E)} \geq \|(P_1 \circ T) \cdots (P_n \circ T)\|_{\mathcal{P}(^{\mathbf{k}}\ell_2^n)} \\
&\geq \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{\mathbf{k}^{\mathbf{k}}}} \|(P_1 \circ T)\|_{\mathcal{P}(^{k_1}\ell_2^n)} \cdots \|(P_n \circ T)\|_{\mathcal{P}(^{k_n}\ell_2^n)} \qquad (3.7) \\
&\geq \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{\mathbf{k}^{\mathbf{k}}}} \frac{1}{\|T^{-1}\|^{\mathbf{k}}} \|P_1\|_{\mathcal{P}(^{k_1}E)} \cdots \|P_n\|_{\mathcal{P}(^{k_n}E)} \\
&> \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{\mathbf{k}^{\mathbf{k}}}} d_n(X)^{-\mathbf{k}} (1 - \varepsilon)^n \|P_1\|_{\mathcal{P}(^{k_1}X)} \cdots \|P_n\|_{\mathcal{P}(^{k_n}X)},
\end{aligned}$$

where (3.7) follows from (3.2). Since ε was arbitrary we conclude the desire inequality. □

Remark 3.2.2. If we restrict ourselves to the spaces $L_p(\Omega,\mu)$ and polynomials *with the same degree*, we can combine Lemma 3.2.1 with Lewis' result (3.6) to obtain

$$n^{nk/p}\|P_1\cdots P_n\|_{\mathcal{P}(^{kn}L_p(\Omega,\mu))} \geq \|P_1\|_{\mathcal{P}(^{k}L_p(\Omega,\mu))}\cdots\|P_n\|_{\mathcal{P}(^{k}L_p(\Omega,\mu))} \tag{3.8}$$

for $1\leq p\leq 2$. For $2\leq p\leq\infty$ we have

$$n^{nk/q}\|P_1\cdots P_n\|_{\mathcal{P}(^{kn}L_p(\Omega,\mu))} \geq \|P_1\|_{\mathcal{P}(^{k}L_p(\Omega,\mu))}\cdots\|P_n\|_{\mathcal{P}(^{k}L_p(\Omega,\mu))},$$

where q is the conjugate exponent of p.

Note that (3.8) is precisely (3.1) with the constant given in (3.4). In order to extend this result to a general case, where the polynomials have arbitrary degrees, it is convenient to consider another particular case. In the sequel we will say that $P,Q:\ell_p^d\to\mathbb{C}$ *depend on different variables* if it is possible to find disjoint subsets $I,J\subset\{1,2,\ldots,d\}$, such that $P\left(\sum_{i=1}^d a_ie_i\right)=P\left(\sum_{i\in I}a_ie_i\right)$ and $Q\left(\sum_{i=1}^d a_ie_i\right)=Q\left(\sum_{i\in J}a_ie_i\right)$, for all $\{a_i\}_{i=1}^d\subset\mathbb{C}$.

Lemma 3.2.3. *Let $\{P_i\}_{i=1}^n$ be homogeneous polynomials of degrees $\{k_i\}_{i=1}^n$ on ℓ_p^d, depending on different variables. Then*

$$\sqrt[p]{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}}\|P_1\cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}\ell_p^d)} = \|P_1\|_{\mathcal{P}(^{k_1}\ell_p^d)}\cdots\|P_n\|_{\mathcal{P}(^{k_n}\ell_p^d)}.$$

Proof. First, we prove this lemma for two polynomials P and Q of degrees k and l. We may assume that P depends on the first r variables and Q on the last $d-r$ ones. Given $z\in\ell_p^d$, we can write $z=x+y$, where x and y are the projections of z on the first r and the last $d-r$ coordinates respectively. We then have

$$|P(z)Q(z)|=|P(x)Q(y)|\leq\|P\|_{\mathcal{P}(^k\ell_p^d)}\,\|Q\|_{\mathcal{P}(^l\ell_p^d)}\,\|x\|_p^k\,\|y\|_p^l.$$

Since $\|z\|_p^p=\|x\|_p^p+\|y\|_p^p$, we can estimate the norm of PQ as follows:

$$\begin{aligned}\|PQ\|_{\mathcal{P}(^{k+l}\ell_p^d)} &= \sup_{\|z\|_p=1}|P(z)Q(z)|\\ &\leq \sup_{|a|^p+|b|^p=1}|a|^k\,|b|^l\,\|P\|_{\mathcal{P}(^k\ell_p^d)}\,\|Q\|_{\mathcal{P}(^l\ell_p^d)}\\ &= \sqrt[p]{\frac{k^k\,l^l}{(k+l)^{(k+l)}}}\,\|P\|_{\mathcal{P}(^k\ell_p^d)}\,\|Q\|_{\mathcal{P}(^l\ell_p^d)},\end{aligned}$$

the last equality being a simple application of Lagrange multipliers. In order to see that this inequality is actually an equality, take x_0 and y_0 norm-one vectors where P and Q respectively attain their norms, each with nonzero entries only in the coordinates in which the corresponding polynomial depends. If we define

$$z_0 = \sqrt[p]{\frac{k}{k+l}}\, x_0 + \sqrt[p]{\frac{l}{k+l}}\, y_0,$$

then z_0 is a norm one vector which satisfies

$$|P(z_0)Q(z_0)| = \sqrt[p]{\frac{k^k\, l^l}{(k+l)^{(k+l)}}}\, \|P\|_{\mathcal{P}(^k\ell_p^d)}\, \|Q\|_{\mathcal{P}(^l\ell_p^d)}.$$

We prove the general statement by induction on n. We assume the result is valid for $n-1$ polynomials and we know that it is also valid for two. We omit the subscripts in the norms of the polynomials to simplify the notation. We then have

$$\begin{aligned}
\left\|\prod_{i=1}^{n} P_i\right\| &= \sqrt[p]{\frac{k_n^{k_n}\,\left(\sum_{i=1}^{n-1} k_i\right)^{\sum_{i=1}^{n-1} k_i}}{\left(\sum_{i=1}^{n} k_i\right)^{\sum_{i=1}^{n} k_i}}}\, \left\|\prod_{i=1}^{n-1} P_i\right\|\, \|P_n\| \\
&= \sqrt[p]{\frac{k_n^{k_n}\,\left(\sum_{i=1}^{n-1} k_i\right)^{\sum_{i=1}^{n-1} k_i}}{\left(\sum_{i=1}^{n} k_i\right)^{\sum_{i=1}^{n} k_i}}}\, \sqrt[p]{\frac{\prod_{i=1}^{n-1} k_i^{k_i}}{\left(\sum_{i=1}^{n-1} k_i\right)^{\sum_{i=1}^{n-1} k_i}}}\, \left(\prod_{i=1}^{n-1} \|P_i\|\right) \|P_n\| \\
&= \sqrt[p]{\frac{\prod_{i=1}^{n} k_i^{k_i}}{\left(\sum_{i=1}^{n} k_i\right)^{\sum_{i=1}^{n} k_i}}}\, \prod_{i=1}^{n} \|P_i\|.
\end{aligned}$$

□

Now we are ready to prove the main result of this section.

Theorem 3.2.4. *Given* $1 < p < 2$, *then*

$$C(\ell_p^d(\mathbb{C}), k_1, \dots, k_n) \leq \sqrt[p]{\frac{\left(\sum_{i=1}^{n} k_i\right)^{\sum_{i=1}^{n} k_i}}{\prod_{i=1}^{n} k_i^{k_i}}}.$$

Moreover, if $d \geq n$ *the equality holds.*

Proof. We need to prove that if $\{P_i\}_{i=1}^n$ are homogeneous polynomials of degrees $\{k_i\}_{i=1}^n$ on ℓ_p^d, then

$$\sqrt[p]{\frac{\left(\sum_{i=1}^{n} k_i\right)^{\sum_{i=1}^{n} k_i}}{\prod_{i=1}^{n} k_i^{k_i}}}\|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}\ell_p^d)} \geq \|P_1\|_{\mathcal{P}(^{k_1}\ell_p^d)} \cdots \|P_n\|_{\mathcal{P}(^{k_n}\ell_p^d)}. \tag{3.9}$$

We first prove the inequality for two polynomials: we take homogeneous polynomials P and Q of degrees k and l. If $k = l$, the result follows from Remark 3.2.2. Let us suppose $k > l$. Moving to ℓ_p^{d+1} if necessary, we take a norm one polynomial S, of degree $j = k - l$, depending on different variables than the polynomials P and Q. An example of such a polynomial is $(e'_{d+1})^j$. In the following, we identify ℓ_p^d with a subspace of ℓ_p^{d+1} in the natural way. We use Lemma 3.2.3 for equalities (3.10) and (3.12), and inequality (3.8) for inequality (3.11) to obtain:

$$\begin{aligned}
\|PQ\|_{\mathcal{P}(^{k+l}\ell_p^d)} &= \|PQ\|_{\mathcal{P}(^{k+l}\ell_p^{d+1})}\,\|S\|_{\mathcal{P}(^{d}\ell_p^{d+1})} \\
&= \sqrt[p]{\frac{((k+l)+j)^{(k+l)+j}}{(k+l)^{(k+l)}j^j}}\,\|PQS\|_{\mathcal{P}(^{2k}\ell_p^{d+1})} \qquad (3.10)\\
&\geq \sqrt[p]{\frac{((k+l)+j)^{(k+l)+j}}{(k+l)^{(k+l)}j^j}}\,\frac{1}{4^{k/p}}\,\|P\|_{\mathcal{P}(^{k}\ell_p^{d+1})}\,\|QS\|_{\mathcal{P}(^{k}\ell_p^{d+1})} \qquad (3.11)\\
&= \sqrt[p]{\frac{(2k)^{2k}l^l j^j}{(k+l)^{(k+l)}\,d^d\,4^k k^k}}\|P\|_{\mathcal{P}(^{k}\ell_p^{d+1})}\|Q\|_{\mathcal{P}(^{l}\ell_p^{d+1})}\|S\|_{\mathcal{P}(^{d}\ell_p^{d+1})} \qquad (3.12)\\
&= \sqrt[p]{\frac{k^k l^l}{(k+l)^{k+l}}}\,\|P\|_{\mathcal{P}(^{k}\ell_p^{d})}\,\|Q\|_{\mathcal{P}(^{l}\ell_p^{d})}.
\end{aligned}$$

The proof of the general case continues by induction on n as in the previous lemma.

To see that the equality holds for $d \geq n$, consider for each $i = 1, \ldots, n$ the polynomial $P_i = (e'_i)^{k_i}$. From Lemma 3.2.3 we obtain an equality in (3.9). □

Remark 3.2.5. That the polynomials depend on different variables is a sufficient condition to have an equality on (3.9), but it is not a necessary one. For example, on Hilbert spaces, we can see this on ℓ_2^3 taking the polynomials

$$P_1(x, y, z) = x^2 + \frac{z^2}{2} \text{ and } P_2(x, y, z) = y^2 + \frac{z^2}{2}.$$

In this example, each P_i can be written as the sum of two polynomials $S_i + R_i$ depending on different variables, such that S_1 and S_2 also depend on different variables and $\|P_i\| = \|S_i\|$. We do not know if all the cases in which we have an equality in (3.9) are of this type (for some adequate orthogonal coordinate system on the Hilbert space).

Theorem 3.2.4 holds also for ℓ_p. This is a consequence of the following: if we have a polynomial $P \in \mathcal{P}({}^k\ell_p)$ then

$$\|P\|_{\mathcal{P}({}^k\ell_p)} = \lim_{N\to\infty} \|P \circ i_d\|_{\mathcal{P}({}^k\ell_p^d)},$$

where i_d is the canonical inclusion of ℓ_p^d in ℓ_p. The proof of this fact is rather standard. Anyway, in the next subsection we will show that Theorem 3.2.4 holds for spaces $L_p(\mu)$, which comprises ℓ_p as a particular case.

Spaces L_p and Schatten classes

Now we show that the results obtained for $\ell_p(\mathbb{C})$ can be extended to complex spaces $L_p(\Omega, \mu)$ and to the Schatten classes $\mathcal{S}_p$ for $1 \le p \le 2$. We will sometimes omit parts of the proofs which are very similar to those in the previous results.

Let (Ω, μ) be a measure space. From now on, the notation $\Omega = \bigsqcup_{i=1}^n A_i$ will mean that it is possible to decompose the set Ω as the union of measurable subsets $\{A_i\}_{1\le i \le n}$, such that $\mu(A_i), \mu(A_j) > 0$ and $\mu(A_i \cap A_j) = 0$ for all $1 \le i < j \le n$. Next lemma is the analogue to Lemma 3.2.3 for L_p spaces.

Lemma 3.2.6. *Let $P, Q : L_p(\Omega, \mu) \to \mathbb{C}$ be homogeneous polynomials of degree k and l respectively. Suppose that $\Omega = A_1 \sqcup A_2$, and that for all $f \in L_p(\Omega, \mu)$ $P(f) = P(f\mathcal{X}_{A_1})$ and $Q(f) = Q(f\mathcal{X}_{A_2})$. Then we have*

$$\sqrt[p]{\frac{(k+l)^{(k+l)}}{k^k\, l^l}} \|PQ\|_{\mathcal{P}({}^{k+l}L_p(\Omega,\mu))} = \|P\|_{\mathcal{P}({}^kL_p(\Omega,\mu))}\, \|Q\|_{\mathcal{P}({}^lL_p(\Omega,\mu))}.$$

Proof. Given $f \in L_p(\Omega, \mu)$ we write it as $f = f\mathcal{X}_{A_1} + f\mathcal{X}_{A_2}$ and then

$$\begin{aligned} |P(f)Q(f)| =& |P(f\mathcal{X}_{A_1})Q(f\mathcal{X}_{A_2})| \\ \le& \|P\|_{\mathcal{P}({}^kL_p(\Omega,\mu))}\, \|Q\|_{\mathcal{P}({}^lL_p(\Omega,\mu))} \|f\mathcal{X}_{A_1}\|_p^k\, \|f\mathcal{X}_{A_2}\|_p^l. \end{aligned}$$

Given $\varepsilon > 0$, we can take norm one functions $f_0, g_0 \in L_p(\Omega, \mu)$ such that

$$|P(f_0)| > \|P\|_{\mathcal{P}({}^kL_p(\Omega,\mu))} - \varepsilon \quad \text{and} \quad |Q(g_0)| > \|Q\|_{\mathcal{P}({}^kL_p(\Omega,\mu))} - \varepsilon.$$

By the hypotheses on P and Q we may assume that $f_0 = f_0\mathcal{X}_{A_1}$ and $g_0 = g_0\mathcal{X}_{A_2}$. We clearly have

$$\left\| \sqrt[p]{\frac{k}{k+l}}\, f_0 + \sqrt[p]{\frac{l}{k+l}}\, g_0 \right\|_p = 1.$$

Now we can, modulo ε, proceed as in the proof of Lemma 3.2.3, and then let ε go to zero to obtain the desired result. $\square$

Combining this lemma with the fact that $d_n(L_p(\mu)) = n^{|1/p-1/2|}$ we obtain the next result.

Theorem 3.2.7. *Given $1 < p < 2$ and a measure μ over an space Ω, then*

$$C(L_p(\Omega,\mu), k_1, \ldots, k_n) \leq \sqrt[p]{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}}.$$

Moreover, if Ω admits a decomposition as $\Omega = A_1 \sqcup \ldots \sqcup A_n$, then the equality holds.

Proof. We need to prove that if $\{P_i\}_{i=1}^n$ are homogeneous polynomials of degrees $\{k_i\}_{i=1}^n$, then

$$\sqrt[p]{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}} \|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}L_p(\Omega,\mu))} \geq \|P_1\|_{\mathcal{P}(^{k_1}L_p(\Omega,\mu))} \cdots \|P_n\|_{\mathcal{P}(^{k_n}L_p(\Omega,\mu))}.$$

We prove this for two polynomials. Let P and Q be homogeneous polynomials of degree k and l. If $k = l$, the result follows from Remark 3.2.2. Then, we can assume $k > l$. Let us define an auxiliary measure space (Ω', μ') by adding a point $\{c\}$ to Ω. The measure μ' in Ω' is given by $\mu'(U) = \mu(U)$ if $U \subseteq \Omega$, and $\mu'(U) = \mu(U \cap \Omega) + 1$ whenever $c \in U$. It is clear that we have $\Omega' = \Omega \sqcup \{c\}$. Let us consider the polynomials P', Q' and S of degree k, l and $d = k - l$ respectively, defined on $L_p(\Omega', \mu')$ by $P'(f) = P(f|_\Omega)$, $Q'(f) = Q(f|_\Omega)$ and $S(f) = (f(c))^d$. Observe that $\|S\|_{\mathcal{P}(^dL_p(\Omega',\mu'))} = 1$. The polynomials $P'Q'$ and S are in the conditions of Lemma 3.2.6. Proceeding as in the proof of Theorem 3.2.4, we have

$$\begin{aligned}
\|PQ\|_{\mathcal{P}({}^{k+l}L_p(\Omega,\mu))} =& \|P'Q'\|_{\mathcal{P}({}^{k+l}L_p(\Omega',\mu'))}\,\|S\|_{\mathcal{P}({}^{d}L_p(\Omega',\mu'))} \\
=& \sqrt[p]{\frac{((k+l)+d)^{(k+l)+d}}{(k+l)^{(k+l)}d^d}}\;\|P'Q'S\|_{\mathcal{P}({}^{2k}L_p(\Omega',\mu'))} \\
\geq& \sqrt[p]{\frac{((k+l)+d)^{(k+l)+d}}{(k+l)^{(k+l)}d^d}}\;\frac{1}{4^{k/p}}\,\|P'\|_{\mathcal{P}({}^{k}L_p(\Omega',\mu'))}\,\|Q'S\|_{\mathcal{P}({}^{k}L_p(\Omega',\mu'))} \\
=& \sqrt[p]{\frac{k^k\,l^l}{(k+l)^{(k+l)}}}\;\|P'\|_{\mathcal{P}({}^{k}L_p(\Omega',\mu'))}\,\|Q'\|_{\mathcal{P}({}^{l}L_p(\Omega',\mu'))}\,\|S\|_{\mathcal{P}({}^{d}L_p(\Omega',\mu'))} \\
=& \sqrt[p]{\frac{k^k\,l^l}{(k+l)^{k+l}}}\;\|P\|_{\mathcal{P}({}^{k}L_p(\Omega,\mu))}\,\|Q\|_{\mathcal{P}({}^{l}L_p(\Omega,\mu))}.
\end{aligned}$$

The general case follows by induction exactly as in the proof of Lemma 3.2.3, and the equality when Ω admits a decomposition as $\Omega = A_1 \sqcup \ldots \sqcup A_n$ is analogous to that of Theorem 3.2.4 when $d \geq n$. $\square$

Now we show how the previous proofs can be adapted to obtain the corresponding results for the Schatten classes. Let $\{P_i\}_{i=1}^n$ be $k-$homogeneous polynomials on $\mathcal{S}_p = \mathcal{S}_p(H)$, the p-Schatten class of operators on the Hilbert space H. In the article [T] Tomczak-Jaegermann proved that $d_n(\mathcal{S}_p) \leq n^{|1/p-1/2|}$ (see Corollary 2.10). Then, by Lemma 3.2.1, we have

$$n^{nk/p}\|P_1\cdots P_n\|_{\mathcal{P}({}^{nk}\mathcal{S}_p(H))} \geq \|P_1\|_{\mathcal{P}({}^{k}\mathcal{S}_p(H))}\cdots\|P_n\|_{\mathcal{P}({}^{k}\mathcal{S}_p(H))}.$$

Suppose that $H = H_1 \oplus H_2$ (an orthogonal sum) and let $\pi_1, \pi_2 : H \to H$ be the orthogonal projections onto H_1 and H_2 respectively. If the homogeneous polynomials $P, Q : \mathcal{S}_p(H) \to \mathbb{C}$ satisfy

$$P(s) = P(\pi_1 \circ s \circ \pi_1) \text{ and } Q(s) = Q(\pi_2 \circ s \circ \pi_2) \text{ for all } s \in \mathcal{S}_p,$$

we can think of P and Q as depending on different variables. Moreover, for each $s \in \mathcal{S}_p(H)$, it is rather standard to see that

$$\|\pi_1 \circ s \circ \pi_1\|_{\mathcal{S}_p}^p + \|\pi_2 \circ s \circ \pi_2\|_{\mathcal{S}_p}^p = \|\pi_1 \circ s \circ \pi_1 + \pi_2 \circ s \circ \pi_2\|_{\mathcal{S}_p}^p. \tag{3.13}$$

Also, we have

$$\pi_1 \circ s \circ \pi_1 + \pi_2 \circ s \circ \pi_2 = \frac{1}{2}\Big(s + (\pi_1 - \pi_2) \circ s \circ (\pi_1 - \pi_2)\Big).$$

By the ideal property of Schatten norms, the last operator has norm (in $\mathcal{S}_p$) not greater than $\|s\|_{\mathcal{S}_p}$. We then have

$$\|\pi_1 \circ s \circ \pi_1\|_{\mathcal{S}_p}^p + \|\pi_2 \circ s \circ \pi_2\|_{\mathcal{S}_p}^p \leq \|s\|_{\mathcal{S}_p}^p. \tag{3.14}$$

Now, with (3.13) and (3.14) at hand, we can follow the proof of Lemma 3.2.3 to obtain the analogous result for Schatten classes.

Finally, the trick of adding a variable in Theorem 3.2.4 or a singleton in Theorem 3.2.7 can be performed for Schatten classes just taking the orthogonal sum of H with a (one dimensional) Hilbert space. As a consequence, mimicking the proof of Theorem 3.2.4 we obtain the following.

Theorem 3.2.8. *Given $1 \leq p \leq 2$ and H a Hilbert space, then*

$$C(S_p(H), k_1, \ldots, k_n) \leq \sqrt[p]{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}}.$$

Moreover, when dim$(H) \geq n$ the equality holds.

The non homogeneous case

In order to study the constant $D(X, k_1, \ldots, k_n)$, using the previous results, we need a couple of auxiliary results. We could not find these basic results in the literature.

Lemma 3.2.9. *Let P be a polynomial on a complex Banach space X with deg$(P) = k$. Given any point in $x \in X$, we have*

$$|P(x)| \leq \max\{\|x\|, 1\}^k \|P\|.$$

Proof. If P is homogeneous the result is rather obvious since we have the inequality

$$\|P(x)\| \leq \|x\|^k \|P\|.$$

Suppose that $P = \sum_{l=0}^k P_l$ with P_l an $l-$homogeneous polynomial. Consider the space $X \oplus_\infty \mathbb{C}$ and the polynomial $\tilde{P} : X \oplus_\infty \mathbb{C} \to \mathbb{C}$ defined as

$$\tilde{P}(x, \lambda) = \sum_{l=0}^k P_l(x) \lambda^{k-l}.$$

The polynomial $\tilde{P}$ is homogeneous of degree k. It is clear that $\|P\| \leq \|\tilde{P}\|$, since if $x \in B_X$ then $(x, 1) \in B_{X \oplus_\infty \mathbb{C}}$ and $|P(x)| = |\tilde{P}(x, 1)|$.

The reverse inequality, $\|P\| \geq \|\tilde{P}\|$, also holds. Indeed, for any norm one vector (x, δ), by the maximum modulus principle, there is $\lambda \in \mathbb{C}$, with $|\lambda| = 1$ such that

$$|\tilde{P}(x, \delta)| \leq |\tilde{P}(x, \lambda)|.$$

Using that λ has modulus one we obtain

$$\begin{aligned}
|\tilde{P}(x,\lambda)| &= \left|\sum_{l=0}^{k} \lambda^{k-l} P_l(x)\right| \\
&= \left|\lambda^k \sum_{l=0}^{k} \lambda^{-l} P_l(x)\right| \\
&= \left|\sum_{l=0}^{k} P_l(\lambda^{-1} x)\right| \\
&= |P(\lambda^{-1}x)| \\
&\leq \|P\|.
\end{aligned}$$

Then we conclude that $\|\tilde{P}\| \leq \|P\|$.

Since $\tilde{P}$ is homogeneous, we have

$$|P(x)| = |\tilde{P}(x,1)| \leq \|(x,1)\|^k \|\tilde{P}\| = \max\{\|x\|,1\}^k \|P\|. \qquad \square$$

As an immediate corollary of the previous lemma, we have.

Corollary 3.2.10. *Let P be as in the previous lemma and $T : Y \to X$ a continuous operator with $\|T\| \geq 1$. Then*

$$\|P \circ T\| \leq \|P\| \|T\|^k.$$

Proof. Given $y \in B_Y$ we need to see that

$$|P(T(y))| \leq \|P\| \|T\|^k. \tag{3.15}$$

But $\|T(y)\| \leq \|T\|$, then equation (3.15) follows from Lemma 3.2.9, taking $x = T(y)$ and the fact that $\max\{\|x\|, 1\} \leq \|T\|$. $\square$

Now we use the results on the constant $C(X, k_1, \ldots, k_n)$ to deduce some results for $D(X, k_1, \ldots, k_n)$, whenever $X = L_p(\mu)$ or $\mathcal{S}_p(H)$, with $1 \leq p \leq 2$.

Proposition 3.2.11. *Let $X = L_p(\mu)$ or $S_p(H)$, with $1 \leq p \leq 2$, then*

$$D(X, k_1, \ldots, k_n) \leq \sqrt[p]{2^{\sum_{i=1}^n k_i} \frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}}.$$

Proof. We need to prove that if $P_1, \dots, P_n : X \to \mathbb{C}$ are continuous polynomials of degrees $k_1, \dots, k_n$, then

$$\sqrt[p]{2^{\sum_{i=1}^n k_i} \frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}} \|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}X)} \geq \|P_1\|_{\mathcal{P}(^{k_1}X)} \cdots \|P_n\|_{\mathcal{P}(^{k_n}X)}.$$

For each i write $P_i = \sum_{l=0}^{k_i} P_{i,l}(x)$ with $P_{i,l}(x)$ an l homogeneous polynomial. Consider the space $Y = X \oplus_p \mathbb{C}$ and the polynomials $\tilde{P}_i : Y \to \mathbb{C}$ defined by

$$\tilde{P}_i(x, \lambda) = \sum_{l=0}^{k} P_{i,l}(x)\lambda^{k-l}.$$

These polynomials are homogeneous polynomials. If we consider the space $Z = X \oplus_\infty \mathbb{C}$ then $B_Y \subseteq B_Z \subseteq \sqrt[p]{2} B_Y$. Therefore

$$\begin{aligned}
\|P_1\|_{\mathcal{P}(^{k_1}X)} \cdots \|P_n\|_{\mathcal{P}(^{k_n}X)} &= \|\tilde{P}_1\|_{\mathcal{P}(^{k_1}Z)} \cdots \|\tilde{P}_n\|_{\mathcal{P}(^{k_n}Z)} &(3.16)\\
&\leq (\sqrt[p]{2})^{k_1} \|\tilde{P}_1\|_{\mathcal{P}(^{k_1}Y)} \cdots (\sqrt[p]{2})^{k_n} \|\tilde{P}_n\|_{\mathcal{P}(^{k_n}Y)} \\
&\leq \sqrt[p]{2^{\sum_{i=1}^n k_i}} \sqrt[p]{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}} \|\tilde{P}_1 \cdots \tilde{P}_n\|_{\mathcal{P}(^{\mathbf{k}}Y)} \\
&\leq \sqrt[p]{2^{\sum_{i=1}^n k_i} \frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}} \|\tilde{P}_1 \cdots \tilde{P}_n\|_{\mathcal{P}(^{\mathbf{k}}Z)} \\
&= \sqrt[p]{2^{\sum_{i=1}^n k_i} \frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}} \|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}X)}, &(3.17)
\end{aligned}$$

where in (3.16) and (3.17) we used that we have an equality due to the maximum modulus principle, as seen in the proof of Lemma 3.2.9. □

Also, it is worth mentioning that an analogue of Lemma 3.2.1 holds for non homogeneous polynomials, that is:

Lemma 3.2.12. *Let X be a Banach space and let $d_n(X)$ be as defined in* (3.5)*, the following holds*

$$D(X, k_1, \dots, k_n) \leq D(\ell_2, k_1, \dots, k_n) d_n(X)^{\sum_{i=1}^n k_i}.$$

Proof. We need to see that given any set $P_1, \dots, P_n : X \to \mathbb{C}$ of continuous polynomials of degree $k_1, \dots, k_n$ the following holds:

$$D(\ell_2, k_1, \dots, k_n) d_n(X)^{\sum_{i=1}^n k_i} \|P_1 \cdots P_n\|_{\mathcal{P}(^{\mathbf{k}}X)} \geq \|P_1\|_{\mathcal{P}(^{k_1}X)} \cdots \|P_n\|_{\mathcal{P}(^{k_n}X)}.$$

As in the proof of Lemma 3.2.1, given $\varepsilon > 0$, we can take a set of norm one vectors $\{x_1, \ldots, x_n\} \subset X$ such that $|P_j(x_j)| > (1-\varepsilon)\,\|P_j\|_{\mathcal{P}(^{k_j}X)}$, for $1 \leq j \leq n$, E a $n-$dimensional subspace containing $\{x_1, \ldots, x_n\}$ and $T : \ell_2^n \to E$ a norm one isomorphism with $\|T^{-1}\| = d_n(X)$. Then we have

$$\begin{aligned}
\|P_1 \cdots P_n\|_{\mathcal{P}(X)} &\geq \|P_1 \cdots P_n\|_{\mathcal{P}(E)} \geq \|(P_1 \circ T) \cdots (P_n \circ T)\|_{\mathcal{P}(\ell_2^n)} \\
&\geq D(\ell_2, k_1, \ldots, k_n)\, \|(P_1 \circ T)\|_{\mathcal{P}(\ell_2^n)} \cdots \|(P_n \circ T)\|_{\mathcal{P}(\ell_2^n)} \\
&\geq D(\ell_2, k_1, \ldots, k_n)\, \frac{1}{\|T^{-1}\|^{\mathbf{k}}}\, \|P_1\|_{\mathcal{P}(E)} \cdots \|P_n\|_{\mathcal{P}(E)} \qquad (3.18) \\
&> D(\ell_2, k_1, \ldots, k_n)\;\; d_n(X)^{-\mathbf{k}} (1-\varepsilon)^n \|P_1\|_{\mathcal{P}(X)} \cdots \|P_n\|_{\mathcal{P}(X)},
\end{aligned}$$

where in (3.18) we use Corollary 3.2.10, since $\|T\| = 1$ implies $\|T^{-1}\| \geq 1$. Taking $\varepsilon \to 0$ we end the proof. $\square$

Remarks on the case $p > 2$

We end this section with some observations on the constant $C(\ell_p^d, k_1, \ldots, k_n)$ for $p > 2$. From [BST], [P] and Theorem 3.2.4 we know that

$$C(\ell_p^d, k_1, \ldots, k_n) = \sqrt[p]{\frac{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i^{k_i}}},$$

provided that $1 \leq p \leq 2$ and $d \geq n$. In [RS, Proposition 8], the authors show that the best constant for products of linear functionals on an infinite dimensional Banach space is worse than the corresponding one for Hilbert spaces. In our notation, that is

$$C(\ell_2, 1, \ldots, 1) \leq C(X, 1, \ldots, 1),$$

for every infinite dimensional Banach space X. Next theorem, together with Theorems 3.2.4 and 3.2.7, shows that the same holds for products of homogeneous polynomials in ℓ_p^d and L_p spaces, provided that the dimension is greater than or equal to the number of factors. That is, the constant for Hilbert spaces is better than the constant of any other L_p space for homogeneous polynomials of any degree, even in the finite dimensional setting.

Theorem 3.2.13. *For $d \geq n$ and $2 \leq p \leq \infty$, we have*

$$C(\ell_2^d, k_1, \ldots, k_n) \leq C(\ell_p^d, k_1, \ldots, k_n) \leq \left(n^{k_1 + \cdots + k_n}\right)^{\frac{1}{2} - \frac{1}{p}}\; C(\ell_2^d, k_1, \ldots, k_n).$$

The same holds for $L_p(\Omega, \mu)$ whenever Ω admits a decomposition as in Theorem 3.2.7.

Proof. The second inequality is a direct consequence of Lemma 3.2.1, so let us show the first one. Consider the linear forms on ℓ_p^d defined by the vectors

$$g_j = \left(1, e^{\frac{2\pi i j}{d}}, e^{\frac{2\pi i 2j}{d}}, e^{\frac{2\pi i 3j}{d}}, \ldots, e^{\frac{2\pi i (d-1)j}{d}}\right) \text{ for } j = 1, \ldots, n.$$

These are orthogonal vectors in ℓ_2^d. We can choose an orthogonal coordinate system such that the g_i's depend on different variables (we are in ℓ_2^d). So by Lemma 3.2.3, inequality (3.1) holds as an equality with the constant for Hilbert spaces given in (3.2):

$$\|g_1^{k_1}\|_{\mathcal{P}(^{k_1}\ell_2^d)} \cdots \|g_n^{k_n}\|_{\mathcal{P}(^{k_n}\ell_2^d)} = C(\ell_2^d, k_1, \ldots, k_n) \, \|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(^{\mathbf{k}}\ell_2^d)}.$$

For products of orthogonal linear forms this equality was observed in [A] and for the general case (with arbitrary powers) in Remark 4.2 of [P].

On the other hand, we have the equalities

$$\|g_j^{k_j}\|_{\mathcal{P}(^{k_j}\ell_2^d)} = (d^{1/2})^{k_j} \text{ and } \|g_l^{k_j}\|_{\mathcal{P}(^{k_j}\ell_p^d)} = (d^{1-\frac{1}{p}})^{k_j},$$

which gives $\|g_j^{k_j}\|_{\mathcal{P}(^{k_j}\ell_2^d)} = (d^{\frac{1}{p}-\frac{1}{2}})^{k_j}\|g_l^{k_j}\|_{\mathcal{P}(^{k_j}\ell_p^d)}$. Combining all this, if we define $\mathbf{k} = \sum_{i=1}^n k_i$, we obtain the following:

$$\begin{aligned}
C(\ell_2^d, k_1, \ldots, k_n) &= \frac{\|g_1^{k_1}\|_{\mathcal{P}(^{k_1}\ell_2^d)} \cdots \|g_n^{k_n}\|_{\mathcal{P}(^{k_n}\ell_2^d)}}{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(^{\mathbf{k}}\ell_2^d)}} \\
&= \frac{(d^{\frac{1}{p}-\frac{1}{2}})^{k_1}\|g_1^{k_1}\|_{\mathcal{P}(^{k_1}\ell_p^d)} \cdots (d^{\frac{1}{p}-\frac{1}{2}})^{k_n}\|g_n^{k_n}\|_{\mathcal{P}(^{k_n}\ell_p^d)}}{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(^{\mathbf{k}}\ell_2^d)}} \\
&\leq \frac{(d^{\frac{1}{p}-\frac{1}{2}})^{k_1}\|g_1^{k_1}\|_{\mathcal{P}(^{k_1}\ell_p^d)} \cdots (d^{\frac{1}{p}-\frac{1}{2}})^{k_n}\|g_n^{k_n}\|_{\mathcal{P}(^{k_n}\ell_p^d)}}{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(^{\mathbf{k}}\ell_p^d)} d^{\left(\frac{1}{p}-\frac{1}{2}\right)\mathbf{k}}} \\
&= \frac{\|g_1^{k_1}\|_{\mathcal{P}(^{k_1}\ell_p^d)} \cdots \|g_n^{k_n}\|_{\mathcal{P}(^{k_n}\ell_p^d)}}{\|g_1^{k_1} \cdots g_n^{k_n}\|_{\mathcal{P}(^{\mathbf{k}}\ell_p^d)}} \\
&\leq C(\ell_p^d, k_1, \ldots, k_n).
\end{aligned}$$

This shows the statement for ℓ_p^d. Since the space $L_p(\Omega, \mu)$, with our assumptions on Ω, contains a 1-complemented copy of ℓ_p^d, the statement for $L_p(\Omega, \mu)$ readily follows. □

3.3 The factor problem on finite dimensional spaces

The main results in Section 3.2 on the factor problem are optimal whenever the dimension d of the underlying spaces is at least n (the number of polynomials). Then, it is reasonable to look for better constants for spaces with finite dimension and a large number of polynomials. In this section we seek to improve our results on the factor problem in the case when n is much larger than d.

General case

We start studying the factor problem for any d dimensional Banach space. Although the following estimate can be improved in some particular cases, its value relies on its generality.

Proposition 3.3.1. *Let X be a $d-$dimensional Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then*

$$D(X, k_1, \ldots, k_n) \leq \frac{(C_{\mathbb{K}} 4ed)^{\sum_{i=1}^n k_i}}{2^{\frac{n}{C_{\mathbb{K}}}}}, \tag{3.19}$$

where $C_{\mathbb{R}} = 1$ and $C_{\mathbb{C}} = 2$.

Remark 3.3.2. If $X = \ell_1^d$, $k_1 = k_2 = \ldots = k_n$ and $n = dm$, last proposition states that

$$D(\ell_1^d, k, \ldots, k) \leq \frac{(C_{\mathbb{K}} 4ed)^{\sum_{i=1}^n k_i}}{2^{\frac{n}{C_{\mathbb{K}}}}}.$$

Consider the following set of polynomials on ℓ_1^d

$$\{\underbrace{e_1^k \cdots e_1^k}_{m \text{ times}} \cdots \underbrace{e_d^k \cdots e_d^k}_{m \text{ times}}\},$$

that is, we consider m copies of each vector of the canonical basis of ℓ_∞^d to the k. With this particular set of polynomials it is easy to see that

$$D(\ell_1^d, k, \ldots, k) \geq d^{dmk} = d^{\sum_{i=1}^n k_i}.$$

Therefore, if we want the best constant $M(d)$ such that

$$D(X, k, \ldots, k) \leq M(d)^{\sum_{i=1}^n k_i},$$

last proposition, as a general result for any Banach space, is tangentially sharp on d, since, as we just saw, for the space ℓ_1^d the order of the best constant $M(d)$ is greater than or equal to d.

To prove this proposition, we will need the following lemma.

Lemma 3.3.3. *Let $P : X \to \mathbb{R}$ be a norm one polynomial of degree k. Then*

$$\int_0^{+\infty} \mu(\{\boldsymbol{z} \in B_X : |P(\boldsymbol{z})| \leq e^{-t}\}) \leq -\ln\left(\frac{2}{(4d)^k}\right) + k,$$

where μ is the normalized Lebesgue measure over B_X.

Proof. To simplify notation let us write

$$V_t = \{\mathbf{z} \in B_X : |P(\mathbf{z})| \leq e^{-t}\}.$$

Then, using the Remez type inequality (1.2), we have

$$\begin{aligned}
\int_0^{+\infty} \mu(V_t)dt &= \int_0^{-\ln\left(\frac{2}{(4d)^k}\right)} \mu(V_t)dt + \int_{-\ln\left(\frac{2}{(4d)^k}\right)}^{+\infty} \mu(V_t)dt \\
&\leq \int_0^{-\ln\left(\frac{2}{(4d)^k}\right)} 1dt + \int_{-\ln\left(\frac{2}{(4d)^k}\right)}^{+\infty} 4d\frac{e^{\frac{-t}{k}}}{2^{\frac{1}{k}}}dt \\
&= -\ln\left(\frac{2}{(4d)^k}\right) + \frac{4d}{2^{\frac{1}{k}}}(-k)e^{\frac{-t}{k}}\Bigg|_{-\ln\left(\frac{2}{(4d)^k}\right)}^{+\infty} \\
&= -\ln\left(\frac{2}{(4d)^k}\right) - \frac{(-k)4d}{2^{\frac{1}{k}}}\exp\left\{\frac{\ln\left(\frac{2}{(4d)^k}\right)}{k}\right\} \\
&= -\ln\left(\frac{2}{(4d)^k}\right) - \frac{(-k)4d}{2^{\frac{1}{k}}}\left(\frac{2}{(4d)^k}\right)^{\frac{1}{k}} \\
&= -\ln\left(\frac{2}{(4d)^k}\right) + k. \quad \square
\end{aligned}$$

Proof of Proposition 3.3.1. Given $P_1, \ldots, P_n : X \to \mathbb{K}$ polynomials of degree $k_1, \ldots, k_n$, we have to prove that

$$\|P_1 \cdots P_n\| \geq \frac{2^{\frac{n}{C_{\mathbb{K}}}}}{(C_{\mathbb{K}}4ed)^{\sum_{i=1}^n k_i}} \|P_1\| \cdots \|P_n\|.$$

We start with the real case. We may assume all the polynomials have norm one. Using Lemma 3.3.3 we have:

$$\begin{aligned}
\ln(\|P_1\cdots P_n\|) &= \sup_{\mathbf{z}\in B_X} \ln\left(\prod_{i=1}^n |P_i(\mathbf{z})|\right) \\
&\geq \int_{B_X} \ln\left(\prod_{i=1}^n |P_i(\mathbf{z})|\right)\ d\mu(\mathbf{z}) \\
&= \sum_{i=1}^n \int_{B_X} \ln|P_i(\mathbf{z})|\ d\mu(\mathbf{z}) \\
&= -\sum_{i=1}^n \int_{B_X} -\ln|P_i(\mathbf{z})|\ d\mu(\mathbf{z}) \\
&= -\sum_{i=1}^n \int_0^{+\infty} \mu(\{\mathbf{z}\in B_X : |P_i(\mathbf{z})| \leq e^{-t}\})dt \\
&\geq \sum_{i=1}^n \ln\left(\frac{2}{(4d)^{k_i}}\right) - k_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|P_1\cdots P_n\| &\geq \exp\left\{\sum_{i=1}^n \ln\left(\frac{2}{(4d)^{k_i}}\right) - k_i\right\} \\
&= \prod_{i=1}^n \frac{2}{(4d)^{k_i}}\frac{1}{e^{k_i}} \\
&= \frac{2^n}{(4de)^{\sum_{i=1}^n k_i}},
\end{aligned}$$

as desired.

To prove the complex case we will use the real case. Let X be a $d-$dimensional complex Banach space and $P_1,\ldots,P_n : X\to\mathbb{C}$ polynomials of degree $k_1,\ldots,k_n$. Take Y the $2d-$dimensional real Banach space obtained from thinking X as a real space, and consider the polynomials $Q_1,\ldots,Q_n : Y\to\mathbb{R}$, of degrees $2k_1,\ldots,2k_n$, defined as

$$Q_i(\mathbf{z}) = |P_i(\mathbf{z})|^2, \quad i=1,\ldots,n.$$

Applying inequality (3.19) for polynomials on a real Banach space to these polynomials we obtain

$$\begin{aligned}\|P_1\|^2 \cdots \|P_n\|^2 \frac{2^n}{(8de)^{\sum_{i=1}^n 2k_i}} &= \|Q_1\| \cdots \|Q_n\| \frac{2^n}{(8de)^{\sum_{i=1}^n 2k_i}} \\ &\leq \|Q_1 \cdots Q_n\| = \|P_1 \cdots P_n\|^2,\end{aligned}$$

which ends the proof. □

Hilbert spaces

A cornerstone on the proof of last proposition was the use of the Remez type inequality (1.2) to obtain Lemma 3.3.3. But when we restrict ourselves to homogeneous polynomials over Hilbert spaces we can prove the following sharper lemma.

Lemma 3.3.4. *Let $P : \ell_2^d(\mathbb{R}) \to \mathbb{R}$ be a norm one homogeneous polynomial of degree k. Then*

$$\int_0^{+\infty} \mu(\{\boldsymbol{z} \in B : |P(\boldsymbol{z})| \leq e^{-t}\}) \leq k(\ln(4) + H_d),$$

where H_d stands for the dth harmonic number $\sum_{k=1}^d \frac{1}{k}$.

Proof. As before, we use the notation $V_t = \{\mathbf{z} \in B : |P(\mathbf{z})| \leq e^{-t}\}$. Using (1.3), instead (1.2), we obtain

$$
\begin{aligned}
\int_0^{+\infty} \mu(V_t)dt &= \int_0^{-kln\left(\frac{1}{4}\right)} \mu(V_t)dt + \int_{-kln\left(\frac{1}{4}\right)}^{+\infty} \mu(V_t)dt \\
&\leq \int_0^{-kln\left(\frac{1}{4}\right)} 1dt + \int_{-kln\left(\frac{1}{4}\right)}^{+\infty} 1-(1-4e^{\frac{-t}{k}})^d dt \\
&= \int_0^{-kln\left(\frac{1}{4}\right)} 1dt + \int_{-kln\left(\frac{1}{4}\right)}^{+\infty} -\sum_{j=1}^{d}\binom{d}{j}(-4)^j e^{\frac{-tj}{k}} dt \\
&= kln\left(\frac{1}{4}\right) - \sum_{j=1}^{d}\binom{d}{j}(-4)^j \left[\frac{-ke^{\frac{-tj}{k}}}{j}\right]_{-kln\left(\frac{1}{4}\right)}^{+\infty} \\
&= -kln\,(4) - \sum_{j=1}^{d}\binom{d}{j}(-4)^j \frac{ke^{ln\left(\frac{1}{4}\right)j}}{j} \\
&= -kln\,(4) - \sum_{j=1}^{d}\binom{d}{j}(-4)^j \frac{k}{j4^j} \\
&= -kln\,(4) + k\sum_{j=1}^{d}\binom{d}{j}(-1)^{j-1}\frac{1}{j} \\
&= -kln\,(4) + kH_d.
\end{aligned}
$$

□

Then, using Lemma 3.3.4, instead of Lemma 3.3.3, we obtain the following, sharper bound for homogeneous polynomials on Hilbert spaces.

Proposition 3.3.5. *Let H be a d dimensional (real or complex) Hilbert space, then*

$$
C(H, k_1, \ldots, k_n) \leq \left(\frac{e^{H_d C_{\mathbb{K}}}}{4}\right)^{\sum_{i=1}^n k_i}, \tag{3.20}
$$

where $C_{\mathbb{R}} = 1$, $C_{\mathbb{C}} = 2$ and H_d stands for the dth harmonic number.

Two dimensional spaces

In this subsection, we study the factor problem on a 2-dimensional Hilbert space using the polarization constants from Chapter 2 as a tool. In order to do this we need the following lemma, which states that every homogeneous polynomial on $\mathbb{C}^2$ is a product of linear functions.

Lemma 3.3.6. *Let $P : \mathbb{C}^2 \to \mathbb{C}$ be a k-homogeneous polynomial. Then, there are k linear functions $\varphi_1, \dots, \varphi_k : \mathbb{C}^2 \to \mathbb{C}$ such that*

$$P = \prod_{i=1}^{k} \varphi_i.$$

Proof. We proceed by induction on k. Since the case $k = 1$ is trivial, let us prove the result for $k > 1$, assuming the result holds for $k - 1$. Take $B = \{b_1, b_n\}$ a basis of $\mathbb{C}^2$ such that $P(b_1) = 0$ and write P in terms of this basis

$$P(z) = \sum_{|\alpha|=k} a_\alpha [z]_B^\alpha,$$

where $[z]_B = (t_1, t_2)$ stands for the coordinates of z in the basis B. Using that P is zero on b_1 it is easy to deduce that the coefficient $a_{(k,0)}$ is zero:

$$0 = P(b_1) = \sum_{|\alpha|=k} a_\alpha (1,0)^\alpha = a_{(k,0)}.$$

We conclude then, that

$$P(z) = \sum_{|\alpha|=k, \alpha \neq (k,0)} a_\alpha [z]_B^\alpha = t_2 \sum_{|\alpha|=k, \alpha \neq (k,0)} a_\alpha [z]_B^{\alpha-(0,1)}.$$

Applying the inductive hypothesis to the $(k-1)$-homogeneous polynomial

$$Q(z) = \sum_{|\alpha|=k, \alpha \neq (k,0)} a_\alpha [z]_B^{\alpha-(0,1)}$$

we obtain the desired result. $\square$

Proposition 3.3.7. *Let H be the complex 2-dimensional Hilbert space, then*

$$C(H, k_1, \dots, k_n) \leq (e^{\frac{1}{2}})^{\sum_{i=1}^n k_i}. \tag{3.21}$$

Proof. Given $P_1, \dots, P_n : H \to \mathbb{C}$ norm one homogeneous polynomials of degrees $k_1, \dots, k_n$, we have to prove that

$$\|P_1 \cdots P_n\| (e^{\frac{1}{2}})^{\sum_{i=1}^n k_i} \geq 1.$$

By Lemma 3.3.6 we know that

$$P_i = L_i \varphi_{i,1} \cdots \varphi_{i,k_i} \text{ for } i = 1, \dots, n,$$

where $\varphi_{i,j}$ are norm one linear functions and

$$L_i = \frac{1}{\|\varphi_{i,1} \cdots \varphi_{i,k_i}\|} \geq 1.$$

Then, using Proposition 2.1.6 with μ the normalized Lebesgue measure, we obtain

$$\begin{aligned}
\|P_1 \cdots P_n\| &= \left\| \prod_{i=1}^{n} \left(L_i \prod_{j=1}^{k_i} \varphi_{i,j} \right) \right\| \\
&= \left(\prod_{i=1}^{n} L_i \right) \left\| \prod_{i=1}^{n} \prod_{j=1}^{k_i} \varphi_{i,j} \right\| \\
&\geq \left\| \prod_{i=1}^{n} \prod_{j=1}^{k_i} \varphi_{i,j} \right\| \\
&\geq e^{-\sum_{i=1}^{n} k_i \; L(2,\mathbb{C})} \\
&= \frac{1}{(e^{\frac{1}{2}})^{\sum_{i=1}^{n} k_i}},
\end{aligned}$$

which ends the proof. □

The space $\ell_\infty^d(\mathbb{C})$

As a final remark regarding finite dimensional spaces, we study the space $\ell_\infty^d(\mathbb{C})$, to obtain an alternative result on the factor problem for this space.

Proposition 3.3.8. *For the complex Banach space $\ell_\infty^d(\mathbb{C})$ we have*

$$D(\ell_\infty^d(\mathbb{C}), k_1, \dots, k_n) \leq 2^{d \sum_{i=1}^{n} k_i}. \tag{3.22}$$

Proof. To prove this result we use the Mahler measure and the length of a polynomial (see Section 1.1). By the properties listed in Lemma 1.1.7, given any set of polynomials $P_1, \ldots, P_n : \ell_\infty^d(\mathbb{C}) \to \mathbb{C}$ of degree $k_1, \ldots, k_n$, we have

$$\begin{aligned}\|P_1\| \cdots \|P_n\| &\leq L(P_1) \cdots L(P_n)\\ &\leq 2^{d\sum_{i=1}^n k_i} M(P_1) \cdots M(P_n)\\ &= 2^{d\sum_{i=1}^n k_i} M(P_1 \cdots P_n)\\ &\leq 2^{d\sum_{i=1}^n k_i} \|P_1 \cdots P_n\|,\end{aligned}$$

which ends the proof. □

Remark 3.3.9. This alternative result is far from optimal for d large. If we compare it with the constant $\frac{(C_{\mathbb{K}} 4ed)^{\sum_{i=1}^n k_i}}{2^{\frac{n}{C_{\mathbb{K}}}}}$ obtained in Proposition 3.3.1, applied to the complex space $\ell_\infty^d(\mathbb{C})$, the constant $\frac{(8ed)^{\sum_{i=1}^n k_i}}{2^{\frac{n}{2}}}$ in (3.19) is better than the one in (3.22) if $8de < 2^d$, that is, for $d > 7$. On the other hand, if $d \leq 6$ then

$$\frac{(8ed)^{\sum_{i=1}^n k_i}}{2^{\frac{n}{2}}} \geq \left(\frac{8ed}{\sqrt{2}}\right)^{\sum_{i=1}^n k_i} \geq 2^{d\sum_{i=1}^n k_i}.$$

That is, the constant (3.22) is better for $d \leq 6$. For $d = 7$ which constant is better depends on the number and the degree of the polynomials.

3.4 The factor problem on ultraproducts of Banach spaces

In this section we study the factor problem in the context of ultraproducts of Banach spaces.

The results of this section are partially motivated by the work of M. Lindström and R. A. Ryan in [LR]. In that article they studied, among other things, the problem of finding the optimal constant $K_n(X)$ such that for any n-homogeneous continuous polynomial $P : X \to \mathbb{K}$ then

$$\|\check{P}\| \leq K_n(X)\|P\|,$$

where $\check{P}$ is the symmetric linear function associated to P. They found a relation between the constant $K_n((X_i)_{\mathfrak{U}})$ for the the ultraproduct $(X_i)_{\mathfrak{U}}$ and the constant $K_n(X_i)$ for each space X_i. The main objective of this section is to do an analogous analysis for our problem (3.1). That is, to find a relation between the factor problem for the space $(X_i)_{\mathfrak{U}}$ and the factor problem for the spaces X_i.

Ultraproducts of Banach spaces

The ultraproduct construction is a fundamental method of model theory that has had an impact in several other branches of mathematics, like algebra, set theory and analysis among others. In particular in Banach spaces, the use of ultraproducts has led to the solution of some open problems in local theory of Banach spaces and in the theory of operators ideals on Banach spaces. Also, the study of ultraproducts on Banach spaces has given a new perspective on several known results on local theory and made clear the relation between these results and some infinite dimensional results. Hence the importance of ultraproducts of Banach spaces on Functional Analysis and our interest on studying the factor problem in this context.

We start with some basic notations and results needed to define ultraproducts of Banach spaces.

Definition 3.4.1. A filter $\mathfrak{U}$ on a family I is a collection of non empty subsets of I, closed by finite intersections and inclusions. An ultrafilter is maximal filter.

Proposition 3.4.2. *If $\mathfrak{U}$ is an ultrafilter on I, given any set $A \subset I$, either $A \in \mathfrak{U}$ or $A^c \in \mathfrak{U}$ (but not both).*

Proof. First let us prove that both A and A^c can not be elements of $\mathfrak{U}$. If this is the case, then $\phi = A \cap A^c \in \mathfrak{U}$, which is a contradiction since $\mathfrak{U}$ is a filter.

Now let us see that at least one of them is in $\mathfrak{U}$. If every element of $\mathfrak{U}$ intersects A^c then

$$\tilde{\mathfrak{U}} = \{W : A^c \cap V \subseteq W, A^c \subseteq W \text{ or } V \subseteq W, \text{ for some } V \in \mathfrak{U}\},$$

is a filter and $A^c \in \tilde{\mathfrak{U}}$. Therefore, by maximality of $\mathfrak{U}$, we conclude that $\mathfrak{U} = \tilde{\mathfrak{U}}$ and $A^c \in \mathfrak{U}$.

Now let us assume that there is $U \in \mathfrak{U}$ such that $U \cap A^c = \phi$, that is $U \subseteq A$, therefore, since $\mathfrak{U}$ is closed by inclusions, $A \in \mathfrak{U}$. □

Definition 3.4.3. Let $\mathfrak{U}$ be an ultrafilter on I and X a topological space. We say that the limit of $(x_i)_{i\in I} \subseteq X$ with respect to $\mathfrak{U}$ is x if for every open neighbourhood U of x the set $\{i \in I : x_i \in U\}$ is an element of $\mathfrak{U}$. We denote

$$\lim_{i,\mathfrak{U}} x_i = x.$$

The following is Proposition 1.5 from [H]. For the sake of completeness we prove it, since in [H] its proof is omitted.

Proposition 3.4.4. *Let $\mathfrak{U}$ be an ultrafilter on I, X a compact Hausdorff space and $(x_i)_{i\in I} \subseteq X$. Then, the limit of $(x_i)_{i\in I}$ with respect to $\mathfrak{U}$ exists and is unique.*

Proof. Let us start proving the existence. Assume that there is no limit of $(x_i)_{i\in I}$. Then, for every $x \in X$, there is an open neighbourhood U_x of x such that

$$\{i \in I : x_i \in U\} \notin \mathfrak{U}.$$

Then, by Proposition 3.4.2, $\{i \in I : x_i \notin U_x\} \in \mathfrak{U}$.

By compactness, there are $x_1, \ldots, x_n$ such that

$$X = \bigcup_{j=1}^{n} U_{x_j}.$$

Therefore

$$\phi = \bigcap_{j=1}^{n} U_{x_j}^c.$$

This implies that

$$\phi = \bigcap_{j=1}^{n} \{i \in I : x_i \notin U_{x_j}\}.$$

But $\bigcap_{j=1}^{n} \{i \in I : x_i \notin U_{x_j}\} \in \mathfrak{U}$, given that it is a finite intersection of elements of $\mathfrak{U}$, which is impossible, since $\phi \notin \mathfrak{U}$.

The uniqueness follows from the fact that if $x \neq y$ and U and V are disjoint open neighbourhoods of x and y respectively, then $\{i \in I : x_i \in U\}$ and $\{i \in I : x_i \in V\}$ are disjoint sets, therefore, at most one can belong to $\mathfrak{U}$. $\square$

Later on, we are going to need the next basic Lemma about limits of ultraproducts.

Lemma 3.4.5. *Let $\mathfrak{U}$ be an ultrafilter on I and $\{x_i\}_{i\in I}$ a family of real numbers. Assume that the limit of $(x_i)_{i\in I} \subseteq \mathbb{R}$ with respect to $\mathfrak{U}$ exists and let r be a real number such that there is a subset U of $\{i : r < x_i\}$, with $U \in \mathfrak{U}$. Then*

$$r \leq \lim_{i,\mathfrak{U}} x_i.$$

Proof. Let us assume that $r > \lim_{i,\mathfrak{U}} x_i$. Then, the interval $(-\infty, r)$ is a neighbourhood of $\lim_{i,\mathfrak{U}} x_i$, therefore, by definition, the set $W = \{i : x_i < r\}$ is an element of $\mathfrak{U}$. Then $\phi = W \cap U \in \mathfrak{U}$, which is a contradiction. $\square$

We are now able to define the ultraproduct of Banach spaces. Given an ultrafilter $\mathfrak{U}$ on I and a family of Banach spaces $(X_i)_{i\in I}$, take the Banach space $\ell_\infty(I, X_i)$ of norm bounded families $(x_i)_{i\in I}$ with $x_i \in X_i$ and norm

$$\|(x_i)_{i\in I}\| = \sup_{i\in I} \|x_i\|.$$

The ultraproduct $(X_i)_{\mathfrak{U}}$ is defined as the quotient space $\ell_\infty(I, X_i)/\sim$ where

$$(x_i)_{i\in I} \sim (y_i)_{i\in I} \Leftrightarrow \lim_{i,\mathfrak{U}} \|x_i - y_i\| = 0.$$

Observe that Proposition 3.4.4 assures us that this limit exists for every pair $(x_i)_{i\in I}, (y_i)_{i\in I} \in \ell_\infty(I, X_i)$. We denote the class of $(x_i)_{i\in I}$ in $(X_i)_{\mathfrak{U}}$ by $(x_i)_{\mathfrak{U}}$.

The following result is the polynomial version of Definition 2.2 from [H] (see also Proposition 2.3 from [LR]). The reasoning behind is almost the same.

Proposition 3.4.6. *Given two ultraproducts $(X_i)_{\mathfrak{U}}$, $(Y_i)_{\mathfrak{U}}$ and a family of continuous homogeneous polynomials $\{P_i : X_i \to Y_i\}_{i\in I}$ of degree k with*

$$\sup_{i\in I} \|P_i\| < \infty,$$

the map $P : (X_i)_{\mathfrak{U}} \longrightarrow (Y_i)_{\mathfrak{U}}$ defined by $P((x_i)_{\mathfrak{U}}) = (P_i(x_i))_{\mathfrak{U}}$ is a continuous homogeneous polynomial of degree k. Moreover $\|P\| = \lim_{i,\mathfrak{U}} \|P_i\|$.

If $\mathbb{K} = \mathbb{C}$, the hypothesis of homogeneity can be omitted, but in this case the degree of P can be lower than k.

Proof. Let us start with the homogeneous case. Write $P_i(x) = T_i(x, \dots, x)$, with T_i a $k-$linear continuous symmetric function. Define $T : (X_i)_{\mathfrak{U}}^k \longrightarrow (Y_i)_{\mathfrak{U}}$ by

$$T((x_i^1)_{\mathfrak{U}}, \dots, (x_i^k)_{\mathfrak{U}}) = (T_i(x_i^1, \cdots, x_i^k))_{\mathfrak{U}}.$$

T is well defined since, by the polarization formula, $\sup_{i\in I} \|T_i\| \leq \sup_{i\in I} \frac{k^k}{k!} \|P_i\| < \infty$.

Using that for each coordinate the maps T_i are linear, we see that the map T is linear in each coordinate, and thus it is a $k-$linear function. Given that

$$P((x_i)_{\mathfrak{U}}) = (P_i(x_i))_{\mathfrak{U}} = (T_i(x_i, \dots, x_i))_{\mathfrak{U}} = T((x_i)_{\mathfrak{U}}, \dots, (x_i)_{\mathfrak{U}}),$$

we conclude that P is a $k-$homogeneous polynomial.

To see the equality of the norms for every i choose a norm one element $x_i \in X_i$ where P_i almost attains its norm, and from there it is easy to deduce that $\|P\| \geq \lim_{i,\mathfrak{U}} \|P_i\|$. For the other inequality we use that

$$|P((x_i)_{\mathfrak{U}})| = \lim_{i,\mathfrak{U}} |P_i(x_i)| \leq \lim_{i,\mathfrak{U}} \|P_i\| \|x_i\|^k = \left(\lim_{i,\mathfrak{U}} \|P_i\|\right) \|(x_i)_{\mathfrak{U}}\|^k.$$

Now we deal with the non homogeneous case. For each $i \in I$ write $P_i = \sum_{l=0}^{k} P_{i,l}$, with $P_{i,l}$ $(0 \leq l \leq k)$ an $l-$homogeneous polynomial. Take the direct sum $X_i \oplus_\infty \mathbb{C}$ of X_i and $\mathbb{C}$, endowed with the norm $\|(x, \lambda)\| = \max\{\|x\|, |\lambda|\}$.

Consider the polynomial $\tilde{P}_i : X_i \oplus_\infty \mathbb{C} \to Y_i$ defined as

$$(x, \lambda) \to \sum_{l=0}^{k} P_{i,l}(x)\lambda^{k-l}.$$

The polynomial $\tilde{P}_i$ is an homogeneous polynomial of degree k and, using the maximum modulus principle as in the proof of Lemma 3.2.9, it is easy to see that $\|P_i\| = \|\tilde{P}_i\|$.

By the homogeneous case, we have that the polynomial $\tilde{P} : (X_i \oplus_\infty \mathbb{C})_{\mathfrak{U}} \to (Y_i)_{\mathfrak{U}}$ defined as $\tilde{P}((x_i, \lambda_i)_{\mathfrak{U}}) = (\tilde{P}_i(x_i, \lambda_i))_{\mathfrak{U}}$, is a continuous homogeneous polynomial of degree k and $\|\tilde{P}\| = \lim_{i,\mathfrak{U}} \|\tilde{P}_i\| = \lim_{i,\mathfrak{U}} \|P_i\|$.

Via the identification

$$(X_i \oplus_\infty \mathbb{C})_{\mathfrak{U}} = (X_i)_{\mathfrak{U}} \oplus_\infty \mathbb{C}$$

$$(x_i, \lambda_i)_{\mathfrak{U}} \leftrightarrow ((x_i)_{\mathfrak{U}}, \lim_{i,\mathfrak{U}} \lambda_i)$$

we have that the polynomial $Q : (X_i)_{\mathfrak{U}} \oplus_\infty \mathbb{C} \to \mathbb{C}$ defined as $Q((x_i)_{\mathfrak{U}}, \lambda) = \tilde{P}((x_i, \lambda)_{\mathfrak{U}})$, is a continuous homogeneous polynomial of degree k and $\|Q\| = \|\tilde{P}\|$.

Then, the polynomial $P((x_i)_{\mathfrak{U}}) = Q((x_i)_{\mathfrak{U}}, 1)$ is a continuous polynomial of degree at most k and $\|P\| = \|Q\| = \lim_{i,\mathfrak{U}} \|P_i\|$. Finally, observe that if $\lim_{i,\mathfrak{U}} \|P_{i,k}\| = 0$ then the degree of P is lower than k. $\square$

Note that, in the last proof, we can take the same approach used for non homogeneous polynomials in the real case, but we would not have the same control over the norms.

We refer the reader to Heinrich's article [H] and the references therein for a more exhaustive exposition on ultraproducts and their importance on the theory of Banach spaces.

Main result

The main result of this section, stated below, involves spaces that have the 1+ uniform approximation property. Recall that a space X has the 1+ uniform approximation property if for every $n \in \mathbb{N}$, there exists $m = m(n)$ such that for every subspace $M \subset X$ with $dim(M) = n$ and every $\varepsilon > 0$ there is an operator $T \in \mathcal{L}(X, X)$ with $T|_M = id$, $\operatorname{rank}(T) \leq m$ and $\|T\| \leq 1 + \varepsilon$ (i.e. for every $\varepsilon > 0$ X has the $1 + \varepsilon$ uniform approximation property).

Theorem 3.4.7. *If $\mathfrak{U}$ is an ultrafilter on a family I and $(X_i)_{\mathfrak{U}}$ is an ultraproduct of complex Banach spaces with the 1+ uniform approximation property, then*

(a) $C((X_i)_{\mathfrak{U}}, k_1, \dots, k_n) = \lim_{i,\mathfrak{U}} (C(X_i, k_1, \dots, k_n))$.

(b) $D((X_i)_{\mathfrak{U}}, k_1, \dots, k_n) = \lim_{i,\mathfrak{U}} (D(X_i, k_1, \dots, k_n))$.

In order to prove this Theorem we need the next Lemma, due to Heinrich [H].

Lemma 3.4.8. *Given an ultraproduct of Banach spaces $(X_i)_{\mathfrak{U}}$, if each X_i has the 1+ uniform approximation property then $(X_i)_{\mathfrak{U}}$ has the metric approximation property.*

Proof of Theorem 3.4.7. Throughout this proof we regard the space $(\mathbb{C})_{\mathfrak{U}}$ as $\mathbb{C}$ via the identification $(\lambda_i)_{\mathfrak{U}} = \lim_{i,\mathfrak{U}} \lambda_i$.

First, we are going to see that $D((X_i)_{\mathfrak{U}}, k_1, \dots, k_n) \geq \lim_{i,\mathfrak{U}} (D(X_i, k_1, \dots, k_n))$. To do this we only need to prove that $\lim_{i,\mathfrak{U}} (D(X_i, k_1, \dots, k_n))$ satisfies (3.3). Given $\varepsilon > 0$ we need to find a set of polynomials $\{P_j\}_{j=1}^n$ on $(X_i)_{\mathfrak{U}}$ with $deg(P_j) \leq k_j$ such that

$$\lim_{i,\mathfrak{U}} (D(X_i, k_1, \dots, k_n)) \left\| \prod_{j=1}^n P_j \right\| \leq (1 + \varepsilon) \prod_{j=1}^n \|P_j\| .$$

By Remark 3.1.3 we know that for each $i \in I$ there is a set of polynomials $\{P_{i,j}\}_{j=1}^{n}$ on X_i, with $deg(P_{i,j}) = k_j$, such that

$$D(X_i, k_1, \ldots, k_n) \left\| \prod_{j=1}^{n} P_{i,j} \right\| \leq (1+\varepsilon) \prod_{j=1}^{n} \|P_{i,j}\| .$$

Replacing $P_{i,j}$ by $P_{i,j}/\|P_{i,j}\|$ we may assume that $\|P_{i,j}\| = 1$. Define the polynomials $\{P_j\}_{j=1}^{n}$ on $(X_i)_{\mathfrak{U}}$ by $P_j((x_i)_{\mathfrak{U}}) = (P_{i,j}(x_i))_{\mathfrak{U}}$. Then, by Proposition 3.4.6, $deg(P_j) \leq k_j$ and

$$\begin{aligned} \lim_{i,\mathfrak{U}} (D(X_i, k_1, \ldots, k_n)) \left\| \prod_{j=1}^{n} P_j \right\| &= \lim_{i,\mathfrak{U}} \left(D(X_i, k_1, \ldots, k_n) \left\| \prod_{j=1}^{n} P_{i,j} \right\| \right) \\ &\leq \lim_{i,\mathfrak{U}} \left((1+\varepsilon) \prod_{j=1}^{n} \|P_{i,j}\| \right) \\ &= (1+\varepsilon) \prod_{j=1}^{n} \|P_j\|, \end{aligned}$$

as desired.

Proving that $D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \leq \lim_{i,\mathfrak{U}} (D(X_i, k_1, \ldots, k_n))$ if each X_i has the 1+ uniform approximation property is not as straightforward. Given $\varepsilon > 0$, let $\{P_j\}_{j=1}^{n}$ be a set of polynomials on $(X_i)_{\mathfrak{U}}$ with $deg(P_j) = k_j$ such that

$$D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \left\| \prod_{j=1}^{n} P_j \right\| \leq (1+\varepsilon) \prod_{j=1}^{n} \|P_j\|.$$

Take $K \subseteq B_{(X_i)_{\mathfrak{U}}}$ the finite set $K = \{x_1, \ldots, x_n\}$, where x_j is such that

$$|P_j(x_j)| > \|P_j\|(1-\varepsilon) \text{ for } j = 1, \ldots, n.$$

Since each X_i has the 1+ uniform approximation property, by Lemma 3.4.8, $(X_i)_{\mathfrak{U}}$ has the metric approximation property. Therefore, there exists a finite rank operator $S : (X_i)_{\mathfrak{U}} \to (X_i)_{\mathfrak{U}}$ such that $\|S\| \leq 1$ and

$$\|P_j - P_j \circ S\|_K < |P_j(x_j)|\varepsilon \text{ for } j = 1, \ldots, n.$$

Now, define the polynomials $Q_1, \ldots, Q_n$ on $(X_i)_{\mathfrak{U}}$ as $Q_j = P_j \circ S$. Then,

$$\left\| \prod_{j=1}^{n} Q_j \right\| \leq \left\| \prod_{j=1}^{n} P_j \right\|$$

and also

$$\|Q_j\|_K > |P_j(x_j)| - \varepsilon |P_j(x_j)| = |P_j(x_j)|(1-\varepsilon) \geq \|P_j\|(1-\varepsilon)^2.$$

The construction of these polynomials is a slight variation of Lemma 3.1 from [LR]. Therefore, we have

$$\begin{aligned} D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \left\| \prod_{j=1}^n Q_j \right\| &\leq D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \left\| \prod_{j=1}^n P_j \right\| \\ &\leq (1+\varepsilon) \prod_{j=1}^n \|P_j\| . \end{aligned} \tag{3.23}$$

Since S is a finite rank operator, the polynomials $\{Q_j\}_{j=1}^n$ have the advantage that are finite type polynomials. This will allow us to construct polynomials on $(X_i)_{\mathfrak{U}}$ which are the limit of polynomials on the spaces X_i. For each j write $Q_j = \sum_{t=1}^{m_j} (\psi_{j,t})^{r_{j,t}}$ with $\psi_{j,t} \in (X_i)_{\mathfrak{U}}^*$, and consider the spaces

$$N = \mathrm{span}\{\mathrm{x_1}, \ldots, \mathrm{x_n}\} \subset (\mathrm{X_i})_{\mathfrak{U}} \text{ and } \mathrm{M} = \mathrm{span}\{\psi_{\mathrm{j,t}}\} \subset (\mathrm{X_i})_{\mathfrak{U}}^*.$$

By the local duality of ultraproducts (see Theorem 7.3 from [H]) there exists $T : M \to (X_i^*)_{\mathfrak{U}}$ an $(1+\varepsilon)-$isomorphism such that

$$JT(\psi)(x) = \psi(x) \ \forall x \in N, \ \forall \psi \in M,$$

where $J : (X_i^*)_{\mathfrak{U}} \to (X_i)_{\mathfrak{U}}^*$ is the canonical embedding. Let $\phi_{j,t} = JT(\psi_{j,t})$ and consider the polynomials $\bar{Q}_1, \ldots, \bar{Q}_n$ on $(X_i)_{\mathfrak{U}}$ given by $\bar{Q}_j = \sum_{t=1}^{m_j} (\phi_{j,t})^{r_{j,t}}$. Clearly $\bar{Q}_j$ coincides with Q_j on N. Since $K \subseteq N$, we have the following lower bound for the norm of each polynomial

$$\|\bar{Q}_j\| \geq \sup_{x \in K} |\bar{Q}_j(x)| = \sup_{x \in K} |Q_j(x)| > \|P_j\|(1-\varepsilon)^2. \tag{3.24}$$

Now, let us find an upper bound for the norm of the product $\| \prod_{j=1}^n \bar{Q}_j \|$. Let $x = (x_i)_{\mathfrak{U}}$ be any point in $B_{(X_i)_{\mathfrak{U}}}$. Then, we have

$$\begin{aligned} \left| \prod_{j=1}^n \bar{Q}_j(x) \right| &= \left| \prod_{j=1}^n \sum_{t=1}^{m_j} (\phi_{j,t}(x))^{r_{j,t}} \right| = \left| \prod_{j=1}^n \sum_{t=1}^{m_j} (JT\psi_{j,t}(x))^{r_{j,t}} \right| \\ &= \left| \prod_{j=1}^n \sum_{t=1}^{m_j} ((JT)^* \hat{x}(\psi_{j,t}))^{r_{j,t}} \right| . \end{aligned}$$

Since $(JT)^*\hat{x}$ belongs to M^*

$$\|(JT)^*\hat{x}\| = \|JT\|\|x\| \leq \|J\|\|T\|\|x\| < 1+\varepsilon$$

and $M^* = (X_i)^{**}_{\mathfrak{U}}/M^{\perp}$, we can choose $z^{**} \in (X_i)^{**}_{\mathfrak{U}}$ with

$$\|z^{**}\| < \|(JT)^*\hat{x}\| + \varepsilon < 1+2\varepsilon,$$

such that $\prod_{j=1}^n \sum_{t=1}^{m_j} ((JT)^*\hat{x}(\psi_{j,t}))^{r_{j,t}} = \prod_{j=1}^n \sum_{t=1}^{m_j} (z^{**}(\psi_{j,t}))^{r_{j,t}}$. By Goldstine's Theorem there exists a net $\{z_\alpha\} \subseteq (X_i)_{\mathfrak{U}}$ w^*-convergent to z in $(X_i)^{**}_{\mathfrak{U}}$ with $\|z_\alpha\| = \|z^{**}\|$. In particular, $\psi_{j,t}(z_\alpha)$ converges to $z^{**}(\psi_{j,t})$. If we call $\mathbf{k} = \sum_{j=1}^n k_j$, by Lemma 3.2.9 and the fact that $\|z_\alpha\| < (1+2\varepsilon)$, we have

$$\left\|\prod_{j=1}^n Q_j\right\| (1+2\varepsilon)^{\mathbf{k}} \geq \left|\prod_{j=1}^n Q_j(z_\alpha)\right| = \left|\prod_{j=1}^n \sum_{t=1}^{m_j} ((\psi_{j,t})(z_\alpha))^{r_{j,t}}\right|. \tag{3.25}$$

Combining this with the fact that

$$\begin{aligned} \left|\prod_{j=1}^n \sum_{t=1}^{m_j} ((\psi_{j,t})(z_\alpha))^{r_{j,t}}\right| \quad &\longrightarrow \quad \left|\prod_{j=1}^n \sum_{t=1}^{m_j} (z^{**}(\psi_{j,t}))^{r_{j,t}}\right| \\ &= \quad \left|\prod_{j=1}^n \sum_{t=1}^{m_j} ((JT)^*\hat{x}(\psi_{j,t}))^{r_{j,t}}\right| \\ &= \quad \left|\prod_{j=1}^n \bar{Q}_j(x)\right|, \end{aligned}$$

we conclude that $\left\|\prod_{j=1}^n Q_j\right\| (1+2\varepsilon)^{\mathbf{k}} \geq |\prod_{j=1}^n \bar{Q}_j(x)|$.

Since the choice of x was arbitrary we obtain

$$\begin{aligned} D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \left\|\prod_{j=1}^n \bar{Q}_j\right\| \quad &\leq \quad (1+2\varepsilon)^{\mathbf{k}} D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \left\|\prod_{j=1}^n Q_j\right\| \\ &\leq \quad (1+2\varepsilon)^{\mathbf{k}}(1+\varepsilon) \prod_{j=1}^n \|P_j\| \qquad (3.26) \\ &< \quad (1+2\varepsilon)^{\mathbf{k}}(1+\varepsilon) \frac{\prod_{j=1}^n \|\bar{Q}_j\|}{(1-\varepsilon)^{2n}}. \qquad (3.27) \end{aligned}$$

In (3.26) and (3.27) we use (3.23) and (3.24) respectively. The polynomials $\bar{Q}_j$ are not only of finite type, these polynomials are also generated by elements of $(X_i^*)_{\mathfrak{U}}$.

This will allow us to write them as limits of polynomials on X_i. For any i, consider the polynomials $\bar{Q}_{i,1},\ldots,\bar{Q}_{i,n}$ on X_i defined by $\bar{Q}_{i,j}=\sum_{t=1}^{m_j}(\phi_{i,j,t})^{r_{j,t}}$, where the functionals $\phi_{i,j,t}\in X_i^*$ are such that $(\phi_{i,j,t})_{\mathfrak{U}}=\phi_{j,t}$. Then $\bar{Q}_j(x)=\lim_{i,\mathfrak{U}}\bar{Q}_{i,j}(x)\ \forall x\in(X_i)_{\mathfrak{U}}$ and, as in the proof of Proposition 3.4.6, is easy to see that $\|\bar{Q}_j\|=\lim_{i,\mathfrak{U}}\|\bar{Q}_{i,j}\|$. Therefore

$$\begin{aligned} D((X_i)_{\mathfrak{U}},k_1,\ldots,k_n)\lim_{i,\mathfrak{U}}\left\|\prod_{j=1}^n\bar{Q}_{i,j}\right\| &= D((X_i)_{\mathfrak{U}},k_1,\ldots,k_n)\left\|\prod_{j=1}^n\bar{Q}_j\right\| \\ &< \frac{(1+\varepsilon)(1+2\varepsilon)^{\mathbf{k}}}{(1-\varepsilon)^{2n}}\prod_{j=1}^n\|\bar{Q}_j\| \\ &= \frac{(1+\varepsilon)(1+2\varepsilon)^{\mathbf{k}}}{(1-\varepsilon)^{2n}}\prod_{j=1}^n\lim_{i,\mathfrak{U}}\|\bar{Q}_{i,j}\|. \end{aligned}$$

To simplify the notation let us call $\lambda=\frac{(1+\varepsilon)(1+2\varepsilon)^{\mathbf{k}}}{(1-\varepsilon)^{2n}}$. Take $L>0$ such that

$$D((X_i)_{\mathfrak{U}},k_1,\ldots,k_n)\lim_{i,\mathfrak{U}}\left\|\prod_{j=1}^n\bar{Q}_{i,j}\right\|<L<\lambda\prod_{j=1}^n\lim_{i,\mathfrak{U}}\|\bar{Q}_{i,j}\|.$$

Since $\left(-\infty,\frac{L}{D((X_i)_{\mathfrak{U}},k_1,\ldots,k_n)}\right)$ and $\left(\frac{L}{\lambda},+\infty\right)$ are neighbourhoods of $\lim_{i,\mathfrak{U}}\left\|\prod_{j=1}^n\bar{Q}_{i,j}\right\|$ and $\prod_{j=1}^n\lim_{i,\mathfrak{U}}\|\bar{Q}_{i,j}\|$ respectively, and $\prod_{j=1}^n\lim_{i,\mathfrak{U}}\|\bar{Q}_{i,j}\|=\lim_{i,\mathfrak{U}}\prod_{j=1}^n\|\bar{Q}_{i,j}\|$, by definition of $\lim_{i,\mathfrak{U}}$, the sets

$$A=\left\{i_0: D((X_i)_{\mathfrak{U}},k_1,\ldots,k_n)\left\|\prod_{j=1}^n\bar{Q}_{i_0,j}\right\|<L\right\}\text{ and }B=\left\{i_0:\lambda\prod_{j=1}^n\|\bar{Q}_{i_0,j}\|>L\right\}$$

are elements of $\mathfrak{U}$. Since $\mathfrak{U}$ is closed by finite intersections, $A\cap B\in\mathfrak{U}$. If we take any element $i_0\in A\cap B$ then, for any $\delta>0$, we have

$$\begin{aligned} \frac{1}{\lambda}D((X_i)_{\mathfrak{U}},k_1,\ldots,k_n)\left\|\prod_{j=1}^n\bar{Q}_{i_0,j}\right\| &\le \frac{L}{\lambda} \\ &\le \prod_{j=1}^n\|\bar{Q}_{i_0,j}\| \\ &< (1+\delta)\prod_{j=1}^n\|\bar{Q}_{i_0,j}\| \end{aligned}$$

Then, since δ is arbitrary, the constant $\frac{1}{\lambda}D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n)$ satisfies (3.3) for the space X_{i_0} and therefore, by Lemma 3.1.2,

$$\frac{1}{\lambda}D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \leq D(X_{i_0}, k_1, \ldots, k_n).$$

This holds for any i_0 in $A \cap B$. Since $A \cap B \in \mathfrak{U}$, by Lemma 3.4.5,

$$\frac{1}{\lambda}D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \leq \lim_{i,\mathfrak{U}} D(X_i, k_1, \ldots, k_n).$$

Using that $\lambda \to 1$ when $\varepsilon \to 0$ we conclude

$$D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \leq \lim_{i,\mathfrak{U}} D(X_i, k_1, \ldots, k_n). \quad \square$$

Remark 3.4.9. From the proof of Theorem 3.4.7, it follows that if we remove the 1+ uniform approximation property hypothesis, then we have

(a) $C((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \geq \lim_{i,\mathfrak{U}}(C(X_i, k_1, \ldots, k_n))$.

(b) $D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \geq \lim_{i,\mathfrak{U}}(D(X_i, k_1, \ldots, k_n))$.

Note that in the proof of Theorem 3.4.7 the only steps where we need the Banach spaces to be complex are at the beginning, where we use Proposition 3.4.6, and in the inequality (3.25), where we use Lemma 3.2.9. But both results hold for homogeneous polynomials on a real Banach space. Then, copying the proof of Theorem 3.4.7 we obtain the following result for real spaces.

Theorem 3.4.10. *If $\mathfrak{U}$ is an ultrafilter on a family I and $(X_i)_{\mathfrak{U}}$ is an ultraproduct of real Banach spaces with the 1+ uniform approximation property, then*

$$C((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) = \lim_{i,\mathfrak{U}}(C(X_i, k_1, \ldots, k_n)).$$

Same as before, if the remove the 1+ uniform approximation property hypothesis, we have

$$C((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \geq \lim_{i,\mathfrak{U}}(C(X_i, k_1, \ldots, k_n)).$$

As a final comment of this subsection, we mention two types of spaces for which the results from above can be applied.

Example 3.4.11. Corollary 9.2 from [H] states that any Orlicz space $L_\Phi(\mu)$, with μ a finite measure and Φ an Orlicz function with regular variation at ∞, has the 1+ uniform projection property, which is stronger than the 1+ uniform approximation property.

Example 3.4.12. In [PeR] Section two, A. Pełczyński and H. Rosenthal proved that any $\mathcal{L}_{p,\lambda}-$space $(1 \leq \lambda < \infty)$ has the $1+\varepsilon-$uniform projection property for every $\varepsilon > 0$ (which is stronger than the $1+\varepsilon-$uniform approximation property), therefore, any $\mathcal{L}_{p,\lambda}-$space has the 1+ uniform approximation property.

The factor problem on biduals

Similar to Corollary 3.3 from [LR], a straightforward corollary of Theorem 3.4.7 is that for any complex Banach space X with 1+ uniform approximation property $C(X, k_1, \ldots, k_n) = C(X^{**}, k_1, \ldots, k_n)$ and $D(X, k_1, \ldots, k_n) = D(X^{**}, k_1, \ldots, k_n)$. Using that X^{**} is $1-$complemented in some suitable ultrafilter $(X)_\mathfrak{U}$ the result is rather obvious. For a construction of the suitable ultrafilter see [LR].

But following the proof of Theorem 3.4.7, and using the principle of local reflexivity applied to X^* instead of the local duality of ultraproducts, we can prove a stronger result. Let us first let us recall the principle of local reflexivity.

Theorem 3.4.13 (Local Reflexvity)**.** *For any pair of finite dimensional subspaces $E \subseteq X^{**}$ and $F \subseteq X^*$, and any positive number ε, there is a continuous linear operator*

$$T : E \to X$$

with the following properties

1. $1 - \varepsilon \leq \|T\| \leq 1 + \varepsilon$.

2. $T(J(x)) = x$ *for any* $x \in X$ *such that* $J(x) \in E$*, where* $J : X \to X^{**}$ *is the canonical embedding.*

3. $f(T(e)) = e(f)$ *for each* $f \in F$.

For more details on this result see [DF] Section 6.6. Using this theorem we can prove the following.

Theorem 3.4.14. *Let X be a complex Banach space such that X^{**} has the metric approximation property, then*

(a) $C(X^{**}, k_1, \dots, k_n) = C(X, k_1, \dots, k_n)$.

(b) $D(X^{**}, k_1, \dots, k_n = D(X, k_1, \dots, k_n))$.

Proof. The inequality $D(X^{**}, k_1, \dots, k_n) \geq D(X, k_1, \dots, k_n)$ is a corollary of Theorem 3.4.7 (using the suitable ultrafilter mentioned above).

Let us prove that if X^{**} has the metric approximation property, then

$$D((X^{**}, k_1, \dots, k_n) \leq D(X, k_1, \dots, k_n).$$

Given $\varepsilon > 0$, let $\{P_j\}_{j=1}^n$ be a set of polynomials on X^{**} with $deg(P_j) = k_j$ such that

$$D(X^{**}, k_1, \dots, k_n) \left\| \prod_{j=1}^n P_j \right\| \leq (1+\varepsilon) \prod_{j=1}^n \|P_j\|.$$

As in the proof of Theorem 3.4.7, since X^{**} has the metric approximation, we can construct finite type polynomials $Q_1, \dots, Q_n$ on X^{**} with $deg(Q_j) = k_j$, such that $\|Q_j\|_K \geq \|P_j\|(1-\varepsilon)^2$ for some finite set $K \subseteq B_{X^{**}}$ and

$$D(X^{**}, k_1, \dots, k_n) \left\| \prod_{j=1}^n Q_j \right\| < (1+\varepsilon) \prod_{j=1}^n \|P_j\|.$$

Suppose that $Q_j = \sum_{t=1}^{m_j} (\psi_{j,t})^{r_{j,t}}$ and consider the spaces $N = \mathrm{span}\{\mathrm{K}\}$ and $M = \mathrm{span}\{\psi_{\mathrm{j,t}}\}$. By the principle of local reflexivity, applied to X^* (considering N as a subspace of $(X^*)^*$ and M as a subspace of $(X^*)^{**}$), there is an $(1+\varepsilon)-$isomorphism $T : M \to X^*$ such that

$$JT(\psi)(x) = \psi(x) \ \forall x \in N, \ \forall \psi \in M \cap X^* = M,$$

where $J : X^* \to X^{***}$ is the canonical embedding.

Let $\phi_{j,t} = JT(\psi_{j,t})$ and consider the polynomials $\bar{Q}_1, \dots, \bar{Q}_n$ on X^{**} defined by $\bar{Q}_j = \sum_{t=1}^{m_j} (\phi_{j,t})^{r_{j,t}}$. Following the proof of Theorem 3.4.7, one obtains

$$D(X^{**}, k_1, \dots, k_n) \left\| \prod_{j=1}^n \bar{Q}_j \right\| < (1+\delta) \frac{(1+\varepsilon)(1+2\varepsilon)^{\mathbf{k}}}{(1-\varepsilon)^{2n}} \prod_{j=1}^n \|\bar{Q}_j\|$$

for every $\delta > 0$. Since each $\bar{Q}_j$ is generated by elements of $J(X^*)$, by Goldstine's Theorem, the restriction of $\bar{Q}_j$ to X has the same norm and the same is true for $\prod_{j=1}^n \bar{Q}_j$. Then

$$D(X^{**}, k_1, \ldots, k_n) \left\| \prod_{j=1}^n \bar{Q}_j\big|_X \right\| < (1+\delta) \frac{(1+\varepsilon)(1+2\varepsilon)^{\mathbf{k}}}{(1-\varepsilon)^{2n}} \prod_{j=1}^n \| \bar{Q}_j\big|_X \|.$$

By Lemma 3.1.2 we conclude that

$$\frac{(1-\varepsilon)^{2n}}{(1+\varepsilon)(1+2\varepsilon)^{\mathbf{k}}} D(X^{**}, k_1, \ldots, k_n) \leq D(X, k_1, \ldots, k_n).$$

Given that the choice of ε is arbitrary and that

$$\frac{(1-\varepsilon)^{2n}}{(1+\varepsilon)(1+2\varepsilon)^{\mathbf{k}}} \to 1$$

when ε tends to 0 we conclude that

$$D(X^{**}, k_1, \ldots, k_n) \leq D(X, k_1, \ldots, k_n). \quad \square$$

Remark 3.4.15. If we remove the metric approximation property hypothesis In last theorem we have

(a) $C(X^{**}, k_1, \ldots, k_n) \geq C(X, k_1, \ldots, k_n)$.

(b) $D(X^{**}, k_1, \ldots, k_n \geq D(X, k_1, \ldots, k_n))$.

We can also get a similar result for the bidual of a real space.

Theorem 3.4.16. *Let X be a real Banach space. Then*

(a) $C(X^{**}, k_1, \ldots, k_n) \geq C(X, k_1, \ldots, k_n)$.

(b) $D(X^{**}, k_1, \ldots, k_n) \geq D(X, k_1, \ldots, k_n)$.

If X^{**} has the metric approximation property, equality holds in (a).

Proof. The proof of item (a) is the same that in the complex case. To prove (b) we will show that given an arbitrary $\varepsilon > 0$, there is a set of polynomials $\{P_j\}_{j=1}^n$ on X^{**} with $deg(P_j) \leq k_j$ such that

$$D(X, k_1, \dots, k_n) \left\| \prod_{j=1}^n P_j \right\| \leq (1+\varepsilon) \prod_{j=1}^n \|P_j\| \,.$$

Take $\{Q_j\}_{j=1}^n$ a set of polynomials on X with $deg(Q_j) = k_j$ such that

$$D(X, k_1, \dots, k_n) \left\| \prod_{j=1}^n Q_j \right\| \leq (1+\varepsilon) \prod_{j=1}^n \|Q_j\| \,.$$

Consider now the polynomials $P_j = AB(Q_j)$, where $AB(Q_j)$ is the Aron-Berner extension of Q_j. Since $AB\left(\prod_{j=1}^n P_j\right) = \prod_{j=1}^n AB(P_j)$, using that the Aron-Berner extension is norm preserving (see [DG]) we have

$$\begin{aligned} D(X, k_1, \dots, k_n) \left\| \prod_{j=1}^n P_j \right\| &= D(X, k_1, \dots, k_n) \left\| \prod_{j=1}^n Q_j \right\| \\ &\leq (1+\varepsilon) \prod_{j=1}^n \|Q_j\| \\ &= (1+\varepsilon) \prod_{j=1}^n \|P_j\| \end{aligned}$$

as desired.

□

3.5 Resumen en castellano del Capítulo III

En este capítulo estudiamos el *factor problem* en espacios de Banach. En un espacio de Banach X, este problema consiste en encontrar la constante optima M tal que, para cualquier conjunto de polinomios escalares continuos en X, de grados previamente fijados, valga la siguiente desigualdad

$$\|P_1\| \cdots \|P_n\| \leq M \|P_1 \cdots P_n\|. \tag{3.28}$$

La constante M dependerá del espacio X y de los grados de los polinomios. Este problema se puede considerar una generalización a polinomios del problema de hallar las constantes de polarización n-ésimas estudiado en el Capítulo 2.

Si llamamos $D(X, k_1, \ldots, k_n)$ a la mejor constante para la cual vale (3.28) para polinomios de grados $k_1, \ldots, k_n$ y $C(k_1, \ldots, k_n)$ la mejor constante cuando nos restringimos a polinomios homogéneos, en [BST], C. Benítez, Y. Sarantopoulos y A. Tonge probaron que

$$D(X, k_1, \ldots, k_n) \leq \frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}.$$

Más aún, cuando $X = \ell_1$ vale la igualdad. Mientras que D. Pinasco probó en [P] que

$$C(H, k_1, \ldots, k_n) = \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}},$$

si H es un espacio de Hilbert complejo de dimensión al menos n.

Para espacios L_p, con $1 < p < 2$, valiéndonos del resultado de D. Pinasco y la distancia de Banach-Mazur entre espacios L_p y espacios de Hilbert, en este capítulo probamos que

$$C(L_p(\mu), k_1, \ldots, k_n) \leq \sqrt[p]{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}.$$

Si la dimensión del espacio es al menos n, entonces vale la igualdad. Usando las similitudes entre los espacios L_p y las clases Schatten $\mathcal{S}_p$ probamos que este resultado también es valido para $\mathcal{S}_p$.

Para ultraproductos de espacios de Banach probamos que si $(X_i)_{i\in I}$ es una familia de espacios de Banach complejos y $\mathfrak{U}$ es un ultrafiltro en I, entonces, si $(X_i)_{\mathfrak{U}}$ es el ultraproducto de espacios de Banach respecto del ultrafiltro $\mathfrak{U}$, tenemos

(a) $C((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \geq \lim_{i,\mathfrak{U}}(C(X_i, k_1, \ldots, k_n))$.

(b) $D((X_i)_{\mathfrak{U}}, k_1, \ldots, k_n) \geq \lim_{i,\mathfrak{U}}(D(X_i, k_1, \ldots, k_n))$.

Si los espacios tienen la 1+ propiedad de aproximación uniforme, vale la igualdad. El item (a) también es válido para espacios reales.

Relacionado con biduales, utilizando el principio de reflexividad local, probamos que si X es un espacio de Banach complejo, entonces

(a) $C(X^{**}, k_1, \ldots, k_n) \geq C(X, k_1, \ldots, k_n)$.

(b) $D(X^{**}, k_1, \ldots, k_n \geq D(X, k_1, \ldots, k_n))$.

Si X^{**} tiene la propiedad de aproximación métrica, vale la igualdad. Para espacios reales siguen valiendo las desigualdades, pero si X^{**} tiene la propiedad de aproximación métrica sólo probamos la igualdad en (a) (para espacios reales).

Chapter 4

The plank problem

In this chapter we study a polynomial version of Tarski's plank problem. The original plank problem consisted in proving that given n norm one linear functionals $\psi_1, \ldots, \psi_n$ on a Banach space X and positive numbers $a_1, \ldots, a_n$, with $\sum_{i=1}^n a_i < 1$, there is a norm one vector $z_0 \in X$ such that $|\psi_i(z_0)| > a_i$ for $i = 1, \ldots, n$. This problem was solved by K. Ball in [Ba1]. Moreover, in [Ba2], Ball proved that for complex Hilbert spaces the condition on the positive numbers can be replaced by $\sum_{i=1}^n a_i^2 < 1$.

By a polynomial plank problem we mean to give sufficient conditions such that for any set of positive real numbers $a_1, \ldots, a_n$, fulfilling these conditions, and any set of norm one scalar polynomials $P_1, \ldots, P_n$ over a Banach space X, of degrees $k_1, \ldots, k_n$, there is an element $z \in B_X$ for which $|P_j(z)| \geq a_j^{k_j}$ for $j = 1, \ldots, n$.

Using results from [BST, P], A. Kavadjiklis and S. G. Kim [KK] studied a plank type problem for polynomials on Banach spaces in the particular case when all the polynomials are homogeneous polynomials with the same degree and all the positive numbers $a_1, \ldots, a_n$ are equal. Our objective is to obtain more general results in this direction. In order to achieve this, we exploit the inequalities for the norm of the product of polynomials studied in Chapter 3.

4.1 Main results

Our first main plank type result, and our most general one, is the following.

Theorem 4.1.1. *Let X be a complex Banach space and $P_1, \ldots, P_n : X \to \mathbb{C}$ be norm one polynomials of degrees $k_1, \ldots, k_n$. Given $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ satisfying*

$\sum_{i=1}^n a_i < \frac{1}{n^{n-1}}$, there is $z_0 \in B_X$ such that

$$|P_i(z_0)| \geq a_i^{k_i} \text{ for } i = 1, \ldots, n.$$

Moreover, if X is finite dimensional, this also holds for $\sum_{i=1}^n a_i = \frac{1}{n^{n-1}}$.

The proof of this theorem will make use of the inequality for the product of polynomials by C. Benítez, Y. Sarantopoulos and A. Tonge (see [BST]) cited on Chapter 3. For $P_1, \ldots, P_n$ as in the theorem, we have the following lower bound for the norm of their product:

$$\|P_1 \cdots P_n\| \geq \frac{\prod_{i=1}^n k_i^{k_i}}{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}} \|P_1\| \cdots \|P_n\|. \tag{4.1}$$

Although the constant of (4.1) is optimal for general complex Banach spaces, as seen in Chapter 3, in some cases, like L_p spaces or Schatten classes $\mathcal{S}_p$, better constants can be obtained. Namely, Theorem 3.2.4 states that if X is the complex space $L_p(\mu)$ or $\mathcal{S}_p$, with $1 \leq p \leq 2$, the following version of (4.1) for homogeneous polynomials holds:

$$\|P_1 \cdots P_n\| \geq \sqrt[p]{\frac{\prod_{i=1}^n k_i^{k_i}}{(\sum_{i=1}^n k_i)^{\sum_{i=1}^n k_i}}} \|P_1\| \cdots \|P_n\|. \tag{4.2}$$

Using (4.2) instead of (4.1) we obtain the following plank result.

Theorem 4.1.2. *Let X be the complex Banach space $L_p(\mu)$ or $\mathcal{S}_p$, with $1 \leq p \leq 2$, and $P_1, \ldots, P_n : X \to \mathbb{C}$ be norm one homogeneous polynomials of degrees $k_1, \ldots, k_n$. Given $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ satisfying $\sum_{i=1}^n a_i^p < \frac{1}{n^{n-1}}$, there is $z_0 \in B_X$ such that*

$$|P_i(z_0)| \geq a_i^{k_i} \text{ for } i = 1, \ldots, n.$$

Moreover, if X is finite dimensional, this also holds for $\sum_{i=1}^n a_i^p = \frac{1}{n^{n-1}}$.

For the proof of Theorems 4.1.1 and 4.1.2 we need the following two technical lemmas, whose proofs will be given at the end of this section.

Lemma 4.1.3. *Given n positive integers $k_1, \ldots, k_n$, the set*

$$\left\{ \frac{1}{\sum_{i=1}^n k_i r_i} (k_1 r_1, \ldots, k_n r_n) : r_1, \ldots, r_n \in \mathbb{N} \right\} \tag{4.3}$$

is dense in $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$.

Lemma 4.1.4. *Given $b_1, \ldots, b_n \in \mathbb{R}_{\geq 0}$, with $\sum_{i=1}^n b_i = \frac{1}{n^{n-1}}$, there is an element $(t_1, \ldots, t_n) \in \mathbb{R}_{>0}^n$ such that*

$$\sum_{i=1}^n t_i = 1 \quad \textit{and} \quad t_1^{t_1} \cdots t_n^{t_n} \geq b_i^{t_i} \textit{ for } i = 1, \ldots, n.$$

Note that the inequalities in Theorem 4.1.1 and (4.1) are exactly the ones in Theorem 4.1.2 and (4.2) when we take $p = 1$. With this in mind, it is easy to see that, if we put $p = 1$ in the proof of Theorem 4.1.2 below, we obtain the proof of Theorem 4.1.1.

Proof of Theorem 4.1.2. Choose $b_i > a_i^p$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n b_i = \frac{1}{n^{n-1}}$. By Lemma 4.1.4, we can take an element $(t_1, \ldots, t_n) \in \mathbb{R}_{>0}^n$ with $\sum_{i=1}^n t_i = 1$ and

$$t_1^{t_1} \cdots t_n^{t_n} \geq b_i^{t_i} \text{ for } i = 1, \ldots, n.$$

We claim that there is $\delta > 0$ such that for any positive integer N, we can choose $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$ so that, if we call $s_i = \frac{k_i r_i}{\sum_{j=1}^n k_j r_j}$, we have

$$s_i \geq \delta \quad \text{and} \quad s_1^{s_1} \cdots s_n^{s_n} \geq b_i^{s_i} \left(1 - \frac{1}{N}\right) \qquad \text{for } i = 1, \ldots, n.$$

In the notation we omit the dependence on N for simplicity. Indeed, using the convention that the function $t \mapsto t^t$ is one at $t = 0$, we have the finite family of continuous functions $x \mapsto x_1^{x_1} \cdots x_n^{x_n}$ and $x \mapsto b_i^{x_i}$, $i = 1, \ldots, n$, defined on the compact set $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$. Then, the density of the set (4.3) in this compact set is just what we need for finding r such that $s_1^{s_1} \cdots s_n^{s_n} \geq b_i^{s_i} \left(1 - \frac{1}{N}\right)$. Since we have to take s_i close enough to t_i we also may assume $s_i \geq \frac{t_i}{2}$. Then, we define $\delta := \min\left\{\frac{t_i}{2} : i = 1, \ldots, n\right\}$.

On the other hand, by (4.2), we have

$$\left\| \prod_{i=1}^n P_i^{r_i} \right\| \geq \left(\frac{\prod_{i=1}^n (k_i r_i)^{k_i r_i}}{(\sum_{i=1}^n k_i r_i)^{\sum_{i=1}^n k_i r_i}} \right)^{\frac{1}{p}}.$$

So, for all $N \in \mathbb{N}$, we can take $\mathbf{z}_N \in B_X$ such that

$$|\left(P_1(\mathbf{z}_N)\right)^{r_1} \cdots \left(P_n(\mathbf{z}_N)\right)^{r_n}| \geq \left(\frac{\prod_{i=1}^n (k_i r_i)^{k_i r_i}}{(\sum_{i=1}^n k_i r_i)^{\sum_{i=1}^n k_i r_i}} \right)^{\frac{1}{p}} \left(1 - \frac{1}{N}\right).$$

Since each polynomial has norm one, this gives for each $i = 1, \dots, n$:

$$|\big(P_i(\mathbf{z}_N)\big)^{r_i}| \geq \left(\frac{\prod_{j=1}^n (k_j r_j)^{k_j r_j}}{(\sum_{j=1}^n k_j r_j)^{\sum_{j=1}^n k_j r_j}} \right)^{\frac{1}{p}} \left(1 - \frac{1}{N}\right).$$

Therefore,

$$\begin{aligned} |P_i(\mathbf{z}_N)| &\geq \left(\frac{\prod_{j=1}^n (k_j r_j)^{k_j r_j}}{(\sum_{j=1}^n k_j r_j)^{\sum_{j=1}^n k_j r_j}} \right)^{\frac{k_i}{k_i r_i p}} \left(1 - \frac{1}{N}\right)^{\frac{1}{r_i}} \\ &= (s_1^{s_1} \cdots s_n^{s_n})^{\frac{k_i}{s_i p}} \left(1 - \frac{1}{N}\right)^{\frac{1}{r_i}} \\ &\geq \left(b_i^{s_i} \left(1 - \frac{1}{N}\right)\right)^{\frac{k_i}{s_i p}} \left(1 - \frac{1}{N}\right) \\ &= b_i^{\frac{k_i}{p}} \left(1 - \frac{1}{N}\right)^{\frac{k_i}{s_i p} + 1} \\ &\geq b_i^{\frac{k_i}{p}} \left(1 - \frac{1}{N}\right)^{\frac{k_i}{\delta p} + 1}. \end{aligned} \tag{4.4}$$

But recall that $b_i > a_i^p$, $i = 1, \dots, n$. Since δ does not depend on N, we can take N large enough such that

$$b_i^{\frac{k_i}{p}} \left(1 - \frac{1}{N}\right)^{\frac{k_i}{\delta p} + 1} \geq a_i^{k_i},$$

which ends the proof of the general case.

For the finite dimensional case, we need to deal with the case $\sum_{i=1}^n a_i^p = \frac{1}{n^{n-1}}$. For this, we take $b_i = a_i^p$ and proceed as in the proof of the general case up to (4.4). We can take, by the finite dimension of our space, a limit point $\mathbf{z}_0 \in B_X$ of the sequence $\{\mathbf{z}_N\}_{N \in \mathbb{N}}$. Then, by continuity, we have

$$|P_i(\mathbf{z}_0)| \geq b_i^{\frac{k_i}{p}} = a_i^{k_i},$$

as desired. $\square$

Proof of Lemmas 4.1.3 and 4.1.4

In this subsection we prove Lemmas 4.1.3 and 4.1.4. In fact, Lemma 4.1.3 is rather simple, so most of our efforts will be focused on proving Lemma 4.1.4.

Proof of Lemma 4.1.3. Take a set of rational numbers $t_1, \ldots, t_n \in \mathbb{Q}$, with $t_i > 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n t_i = 1$. Write $t_i = \frac{q_i}{p}$ with $q_1, \ldots, q_n \in \mathbb{N}$ and $p = \sum_{i=1}^n q_i$. Let $M = \prod_{i=1}^n k_i$ and take r_i such that $k_i r_i = q_i M$. Then

$$\frac{1}{\sum_{i=1}^n k_i r_i}(k_1 r_1, \ldots, k_n r_n) = \frac{1}{\sum_{i=1}^n q_i M}(q_1 M, \ldots, q_n M) = \left(\frac{q_1}{p}, \ldots, \frac{q_n}{p}\right).$$

Now, the density of rational numbers gives the desired result. $\square$

For the proof of Lemma 4.1.4 we need two more results. In the following, we use the conventions that the function $t \mapsto t^t$ is one at $t = 0$ and that the function $t \mapsto \frac{1}{\ln(t)}$ is zero at $t = 0$, to extend these maps continuously at zero.

Lemma 4.1.5. *The function $g(x_1, \ldots, x_n) = x_1^{x_1} \cdots x_n^{x_n}$ attains its minimum on the set*

$$A = \{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i = 1,\ x_i \geq 0\}$$

at $(\frac{1}{n}, \ldots, \frac{1}{n})$.

Proof. Since that $x \mapsto x\ln(x)$ is convex, using Jensen's inequality we obtain

$$\begin{aligned} \ln(g(x_1, \ldots, x_n)) &= \sum_{i=1}^n x_i \ln(x_i) \\ &\geq n\frac{\sum_{i=1}^n x_i}{n} \ln\left(\frac{\sum_{i=1}^n x_i}{n}\right) \\ &= \ln\left(\frac{1}{n}\right), \end{aligned}$$

As a consequence

$$g(x_1, \ldots, x_n) \geq \frac{1}{n} = g\left(\frac{1}{n}, \ldots, \frac{1}{n}\right),$$

which ends the proof. $\square$

Lemma 4.1.6. *Given $C \in \left(0, \frac{1}{e^2}\right]$ and $n \in \mathbb{N}_{\geq 2}$, the function*

$$f(x_1, \ldots, x_n) = -\sum_{i=1}^n \frac{1}{\ln(x_i)}$$

attains its unique maximim on the set

$$A = \{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i = C,\ x_i \geq 0\}$$

at $\left(\frac{C}{n}, \ldots \frac{C}{n}\right)$. Moreover, when $n = 2$ we can take $C \in (0, \frac{1}{2}]$.

Proof. We start with the case $n = 2$ and $C \leq \frac{1}{e^2}$. Using that $x_2 = C - x_1$, this problem can be translated into finding the maximum of $h(s) = -\frac{1}{\ln(s)} - \frac{1}{\ln(C-s)}$ on the interval $[0, C]$. Our candidates to maximum are $h(0) = -\frac{1}{\ln(C)}$, $h(C) = -\frac{1}{\ln(C)}$ and the critical points in $(0, C)$. Since $h'(s) = \frac{1}{s\ln^2(s)} - \frac{1}{(C-s)\ln^2(C-s)}$, the critical points are the solutions of

$$\frac{1}{s\ln^2(s)} - \frac{1}{(C-s)\ln^2(C-s)} = 0. \tag{4.5}$$

Now, given that $C \leq \frac{1}{e^2}$, it is easy to see that $\frac{1}{s\ln^2(s)}$ is decreasing and $\frac{1}{(C-s)\ln^2(C-s)}$ is increasing, and therefore we have at most one solution, which happens to be $s = \frac{C}{2}$. It can only be a local maximum and therefore is the global maximum. For $\frac{1}{e^2} < C \leq \frac{1}{2}$ there can be up to two critical points in the interior, but we can reason similarly.

Now assume that $n > 2$. The condition $\sum_{i=1}^n x_i = C$ implies that each x_i is at most $1/e^2$. Since the function $x \mapsto -\frac{1}{\ln(x)}$ is concave on $\left[0, \frac{1}{e^2}\right]$, using Jensen's inequality we obtain

$$\begin{aligned}\sum_{i=1}^n \frac{-1}{\ln(x_i)} &\leq n\frac{-1}{\ln\left(\frac{\sum_{i=1}^n x_i}{n}\right)} \\ &= \frac{-n}{\ln\left(\frac{C}{n}\right)} \\ &= f\left(\frac{C}{n}, \ldots, \frac{C}{n}\right),\end{aligned}$$

and equality holds if and only if $x_1 = x_2 = \ldots = x_n$. □

Proof of Lemma 4.1.4. Since the case $n = 1$ is trivial, let us assume $n \geq 2$. We first assume that $b_1, \ldots, b_n$ are strictly positive. Consider the function

$$f(x_1, \ldots, x_n) = -\frac{1}{\ln(x_1)} - \ldots - \frac{1}{\ln(x_n)}.$$

By Lemma 4.1.6, we have

$$f(b_1, \ldots, b_n) \leq -\frac{n}{\ln\left(\frac{1/n^{n-1}}{n}\right)} = \frac{1}{\ln(n)}.$$

Since the function f is increasing in each variable $x_i \in [0,1)$, we have

$$f\left(\max\left\{b_1, \frac{1}{n^n}\right\}, \ldots, \max\left\{b_n, \frac{1}{n^n}\right\}\right) \geq f\left(\frac{1}{n^n}, \ldots, \frac{1}{n^n}\right) = \frac{1}{\ln(n)}.$$

Then, we can take $b_i \leq \tilde{b_i} \leq \max\left\{b_i, \frac{1}{n^n}\right\}$ such that

$$f(\tilde{b_1}, \ldots, \tilde{b_n}) = -\frac{1}{\ln(\tilde{b_1})} - \ldots - \frac{1}{\ln(\tilde{b_n})} = \frac{1}{\ln(n)}.$$

Now we define $t_i := -\frac{\ln(n)}{\ln(\tilde{b_i})}$. By the choice of $\tilde{b_i}$ $(i = 1, \ldots, n)$, we have $\sum_{i=1}^n t_i = 1$. Note also that, since $\ln(n) > 0$ and $\ln(\tilde{b_i}) \leq \ln\left(\max\left\{b_i, \frac{1}{n^n}\right\}\right) < 0$, each t_i is positive.

Now we use Lemma 4.1.5 to get

$$t_1^{t_1} \cdots t_n^{t_n} \geq \frac{1}{n} = \tilde{b_i}^{-\frac{\ln(n)}{\ln(\tilde{b_i})}} = \tilde{b_i}^{t_i} \geq b_i^{t_i},$$

as desired.

If $b_{i_0} = 0$ for some i_0, we do not have $b_1 = b_2 = \ldots = b_n$, therefore

$$f(b_1, \ldots, b_n) < \frac{1}{\ln(n)}.$$

Given that f is continuous, we can take positive numbers $b'_1, \ldots, b'_n$ such that $b'_i \geq b_i$ and that

$$f(b'_1, \ldots, b'_n) \leq \frac{1}{\ln(n)}.$$

Then, we proceed as in the previous case using these numbers. $\square$

4.2 The plank problem on finite dimensional spaces

We remark that the constants in inequalities (4.1) and (4.2) are optimal when the dimension of the underlying spaces are at least n (the number of polynomials). In this section we use sharper results on the factor problem, for spaces with finite dimension and a large number of polynomials, to obtain better plank type results in this setting. In this direction, we have (3.19), (3.20), (3.21) and (3.22). To make use of these lower bounds for the product of polynomials to obtain plank type results, we need the following proposition.

Proposition 4.2.1. *Let X be a finite dimensional Banach space, n a natural number and suppose we have a constant $K \in \left(0, \frac{1}{\sqrt[n]{ne^2}}\right]$ such that for any set $P_1, \ldots, P_n$ of norm one (arbitrary or homogeneous) polynomials on X we have*

$$\|P_1 \cdots P_n\| \geq K^{\sum_{i=1}^n k_i} \|P_1\| \cdots \|P_n\|,$$

where $k_1, \ldots, k_n$ are the degrees of the polynomials. Then, given $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$, with $\sum_{i=1}^n a_i \leq nK^n$, there is $z_0 \in B_X$ such that

$$|P_i(z_0)| \geq a_i^{k_i} \text{ for } i = 1, \ldots, n.$$

We omit the proof of this proposition since it is analogous to the proof of Theorems 4.1.1 and 4.1.2, replacing Lemma 4.1.4 with the following lemma.

Lemma 4.2.2. *Let n be a natural number and $K \in \left(0, \frac{1}{\sqrt[n]{ne^2}}\right]$. Given non negative numbers $b_1, \ldots, b_n$, with $\sum_{i=1}^n b_i = nK^n$, there is an element $(t_1, \ldots, t_n) \in \mathbb{R}^n_{>0}$ such that*

$$\sum_{i=1}^n t_i = 1 \quad \text{and} \quad K^{\frac{1}{t_i}} \geq b_i \text{ for } i = 1, \ldots, n.$$

Proof. Let us first assume $b_1, \ldots, b_n$ are strictly positive and define $s_i := \frac{\ln(K)}{\ln(b_i)}$, or equivalently $b_i = K^{\frac{1}{s_i}}$. If we show that $\sum_{j=1}^n s_j \leq 1$, then we can take $t_i \geq s_i$ such that $\sum_{j=1}^n t_j = 1$ and, since $x \mapsto K^{\frac{1}{s}}$ is increasing, we have

$$K^{\frac{1}{t_i}} \geq K^{\frac{1}{s_i}} = b_i.$$

Let us see then, that $\sum_{j=1}^n s_j \leq 1$. The condition $\sum_{j=1}^n b_j = nK^n$ implies $b_i \leq \frac{1}{e^2}$ for each $i = 1, \ldots, n$. Since the function $x \mapsto \frac{\ln(K)}{\ln(x)}$ is concave on $\left[0, \frac{1}{e^2}\right]$, using Jensen's inequality we have

$$\sum_{j=1}^n s_i = \sum_{j=1}^n \frac{\ln(K)}{\ln(b_j)} \leq n \frac{\ln(K)}{\ln\left(\frac{\sum_{j=1}^n b_j}{n}\right)} = 1.$$

If $b_{i_0} = 0$ for some i_0, we define $s_i = 0$ whenever $b_i = 0$ and $s_i := \frac{\ln(K)}{\ln(b_i)}$ otherwise. Since in this case we do not have $b_1 = b_2 = \ldots = b_n$, proceeding as in the previous case we obtain

$$\sum_{j=1}^n s_i < 1.$$

This allow us to take each t_i strictly greater than s_i (and, in particular, strictly positive as desired), satisfying $\sum_{j=1}^{n} t_j = 1$. We go on as above to obtain the result. $\square$

Combining Proposition 4.2.1 with inequalities (3.19), (3.20), (3.21) and (3.22), we obtain the following plank result for finite dimensional spaces.

Proposition 4.2.3. *Let X be a d-dimensional Banach space over $\mathbb{K}$, $P_1, \dots, P_n : X \to \mathbb{K}$ a set of norm one polynomials of degrees $k_1, \dots, k_n$ and let $K > 0$ be defined as follows:*

$$K = \frac{1}{C_{\mathbb{K}} 4ed} \text{ for } X \text{ any } d\text{-dimensional Banach space;} \tag{4.6}$$

$$K = \min\left\{\frac{1}{\sqrt[n]{ne^2}}, \frac{4}{e^{H_d C_{\mathbb{K}}}}\right\} \text{ for homogeneous polynomials and } X \text{ a Hilbert space;} \tag{4.7}$$

$$K = \min\left\{\frac{1}{\sqrt[n]{ne^2}}, \frac{1}{e^{\frac{1}{2}}}\right\} \text{ for homogeneous polynomials and } X = \ell_2^2(\mathbb{C}); \tag{4.8}$$

$$K = \min\left\{\frac{1}{\sqrt[n]{ne^2}}, \frac{1}{2^d}\right\} \text{ for } X = \ell_\infty^d(\mathbb{C});, \tag{4.9}$$

where $C_{\mathbb{R}} = 1$, $C_{\mathbb{C}} = 2$ and H_d is the dth harmonic number. Then, given $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ satisfying $\sum_{i=1}^{n} a_i \leq nK^n$, there is $z_0 \in B_X$ such that

$$|P_i(z_0)| \geq a_i^{k_i} \text{ for } i = 1, \dots, n.$$

In some cases, like in Hilbert spaces, last proposition give us two alternative constants. Also, for finite dimensional spaces the results of Section 4.1.1 can be applied. So, let us compare them.

Note that the constant nK^n from Proposition 4.2.1, is bigger than the constant $\frac{1}{n^{n-1}}$ from Theorem 4.1.1 whenever $n > K^{-1}$. As a consequence, for complex $d-$dimensional spaces, the constant (4.6) is better than the one of Theorem 4.1.1 when $n > 8de$. With a similar analysis, it is easy to see that for $\ell_p^d(\mathbb{C})$, $1 \leq p \leq 2$, for large values of n, the constant in (4.6) is better than the constant in Theorem 4.1.2.

For $\ell_\infty^d(\mathbb{C})$, the constant(4.9) is better than (4.6) whenever $2^d < 8de$, that is, for $d \leq 7$.

Finally, the constant (4.7) has the following asymptotic behaviour

$$\frac{4}{e^{H_d C_{\mathbb{K}}}} \asymp \frac{1}{d}.$$

Therefore, the constant (4.6), applied to homogeneous polynomials on Hilbert spaces, for d large enough, is similar to the constant (4.7).

Remark 4.2.4. It is natural to compare the results described in this chapter to previous work. First, it is easy to see that for polynomials of degree one, we are far from recovering the optimal results of K. Ball on the plank problem. On the other hand, the value of our results relies on the generality in which they can be stated. They can be applied for polynomials of arbitrary (and different) degrees, and a large range of positive numbers $a_1, \ldots, a_n$. Moreover, most of them also work for non homogeneous polynomials. In this way, we extend, and sometimes improve, previous work in the subject. For example, Theorem 5 of [KK] can be recovered from Theorem 4.1.1 and Theorem 4.1.2 as a particular case, taking polynomials of the same degrees, all the scalars with the same value, etc.

4.3 Further remarks

We end this chapter with some remarks on the polynomial plank problem in two particular cases. The first remark is when in the set of polynomials $P_1, \ldots, P_n : X \to \mathbb{R}$ all the polynomials are of the form $P_i = \varphi_i^{k_i}$, for some linear functions $\varphi_1, \ldots, \varphi_n \in X^*$. In this case, an immediate corollary from Ball's result on the plank problem (see [Ba1]) is that for any set of positive numbers $a_1, \ldots, a_n$, with $\sum_{i=1}^n a_i < 1$, there is $\mathbf{z}_0 \in B_X$ such that

$$|P_i(\mathbf{z}_0)| \geq a_i^{k_i} \text{ for } i = 1, \ldots, n.$$

Similarly, for polynomials on a complex Hilbert space that are the powers of a linear functions it is enough to check that $\sum_{i=1}^n a_i^2 \leq 1$.

Finally, let us state the next plank type result for two polynomials.

Proposition 4.3.1. *Let H be a finite dimensional complex Hilbert space and $P_1, P_2 : H \to \mathbb{C}$ norm one homogeneous polynomials of degree k_1 and k_2 such that one of them is the power of a linear function. If a_1 and a_2 are positive numbers such that $a_1^2 + a_2^2 \leq 1$, then there is $\boldsymbol{z}_0 \in B_H$ such that*

$$|P_i(\boldsymbol{z}_0)| \geq a_i^{k_i} \textit{ for } i = 1, 2.$$

Proof. Let us assume $a_i = \sqrt{\frac{k_i}{k_1+k_2}}$. If that is not the case, we can consider the polynomials $P_i^{r_i}$ for some suitable natural numbers as done in the proof of Theorems 4.1.1 and 4.1.2.

Write $P_2 = \phi^{k_2}$ and take $\mathbf{w}_1 \in S_H$ such that $|P_1(w_1)| = 1$. Assume that $\phi(\cdot) = \langle \cdot\,, \mathbf{w}\rangle$ and take $\mathbf{w}_2 \in S_H$ orthogonal to $\mathbf{w}_1$ such that $\mathbf{w} \in \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. We may assume $\mathbf{w} = \alpha\mathbf{w}_1 + \beta\mathbf{w}_2$ with α and β positive numbers, otherwise we multiply $\mathbf{w}_1$ and $\mathbf{w}_2$ by complex numbers of modulus one. Then $1 = \|\phi\|^2 = \alpha^2 + \beta^2$.

Now let us consider the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined as

$$f(\lambda) = P\left(\left(\frac{k_1}{k_1+k_2}\right)^{1/2}\mathbf{w}_1 + \lambda\left(\frac{k_2}{k_1+k_2}\right)^{1/2}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2)\right).$$

Since $f(0) = \left(\frac{k_1}{k_1+k_2}\right)^{\frac{k_1}{2}}$, by the maximum modulus principle applied to f on the set

$$\left\{\lambda : \left\|\left(\frac{k_1}{k_1+k_2}\right)^{1/2}\mathbf{w}_1 + \lambda\left(\frac{k_2}{k_1+k_2}\right)^{1/2}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2)\right\|_2^2 \leq 1\right\},$$

there is λ_0 such that

- $\left\|\left(\frac{k_1}{k_1+k_2}\right)^{1/2}\mathbf{w}_1 + \lambda_0\left(\frac{k_2}{k_1+k_2}\right)^{1/2}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2)\right\|_2^2 = 1$
- $\left|P\left(\left(\frac{k_1}{k_1+k_2}\right)^{1/2}\mathbf{w}_1 + \lambda_0\left(\frac{k_2}{k_1+k_2}\right)^{1/2}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2)\right)\right| = |f(\lambda_0)| \geq \left(\frac{k_1}{k_1+k_2}\right)^{\frac{k_1}{2}}$.

Then, our candidate for $\mathbf{z}_0$ is $\left(\frac{k_1}{k_1+k_2}\right)^{1/2}\mathbf{w}_1 + \lambda_0\left(\frac{k_2}{k_1+k_2}\right)^{1/2}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2)$. We already know that $|P_1(\mathbf{z}_0)| \geq a_1^{k_1}$, so it remains to see that $|P_2(\mathbf{z}_0)| \geq a_2^{k_2}$. Note that

$$\begin{aligned}|P_2(\mathbf{z}_0)| &= \left|\phi^{k_2}\left(\left(\frac{k_1}{k_1+k_2}\right)^{1/2}\mathbf{w}_1 + \lambda_0\left(\frac{k_2}{k_1+k_2}\right)^{1/2}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2)\right)\right| \\ &= \left|\alpha\left(\frac{k_1}{k_1+k_2}\right)^{1/2} + \lambda_0\left(\frac{k_2}{k_1+k_2}\right)^{1/2}\right|^{k_2}.\end{aligned}$$

Then, we only need to see that

$$\left|\alpha\left(\frac{k_1}{k_1+k_2}\right)^{1/2} + \lambda_0\left(\frac{k_2}{k_1+k_2}\right)^{1/2}\right|^2 \geq \frac{k_2}{k_1+k_2}.$$

But since

$$\begin{aligned} 1 &= \left\| \left(\frac{k_1}{k_1+k_2}\right)^{1/2} \mathbf{w}_1 + \lambda_0 \left(\frac{k_2}{k_1+k_2}\right)^{1/2} (\alpha \mathbf{w}_1 + \beta \mathbf{w}_2) \right\|_2^2 \\ &= \left\| \mathbf{w}_1 \left(\left(\frac{k_1}{k_1+k_2}\right)^{1/2} + \lambda_0 \alpha \left(\frac{k_2}{k_1+k_2}\right)^{1/2} \right) + \mathbf{w}_2 \lambda_0 \left(\frac{k_2}{k_1+k_2}\right)^{1/2} \beta \right\|_2^2, \end{aligned}$$

we have

$$\begin{aligned} 1 &= \left| \left(\frac{k_1}{k_1+k_2}\right)^{1/2} + \lambda_0 \alpha \left(\frac{k_2}{k_1+k_2}\right)^{1/2} \right|^2 + \beta^2 |\lambda_0|^2 \left(\frac{k_2}{k_1+k_2}\right) \\ &= \frac{k_1 + |\lambda_0|^2 k_2 (\alpha^2+\beta^2)}{k_1+k_2} + \frac{\alpha (k_2 k_1)^{1/2} (\lambda_0 + \overline{\lambda_0})}{k_1+k_2} \\ &= \frac{k_1 + |\lambda_0|^2 k_2 + \alpha (k_2 k_1)^{1/2} (\lambda_0 + \overline{\lambda_0})}{k_1+k_2}. \end{aligned}$$

We deduce then, that

$$k_2 = |\lambda_0|^2 k_2 + \alpha (k_2 k_1)^{1/2} (\lambda_0 + \overline{\lambda_0}).$$

Using this equality, we obtain the desired result:

$$\begin{aligned} \left| \alpha \left(\frac{k_1}{k_1+k_2}\right)^{1/2} + \lambda_0 \left(\frac{k_2}{k_1+k_2}\right)^{1/2} \right|^2 &= \frac{\alpha^2 k_1 + |\lambda_0|^2 k_2 + \alpha (k_2 k_1)^{1/2} (\lambda_0 + \overline{\lambda_0})}{k_1+k_2} \\ &= \frac{\alpha^2 k_1 + k_2}{k_1+k_2} \geq \frac{k_2}{k_1+k_2}. \qquad \square \end{aligned}$$

4.4 Resumen en castellano del Capítulo IV

En este Capítulo tratamos una versión polinomial del *plank problem* de Tarski. El problema original consistía en probar que, dadas n funcionales lineales $\psi_1, \ldots, \psi_n$ de norma uno en un espacio de Banach X y n números no negativos $a_1 \ldots, a_n$, cuya suma es menor a uno, existe un vector de norma uno $z \in X$ tal que $|\psi_i(z)| > a_i$. Este problema fue resuelto por K. Ball en [Ba1]. Más aún, en [Ba2], Ball probó que para espacios de Hilbert complejos la condición $\sum_{i=1}^n a_i < 1$ se puede reemplazar por $\sum_{i=1}^n a_i^2 < 1$.

Nuestro objetivo consiste en dar condiciones suficientes tales que para cualquier conjunto de números no negativos $a_1, \ldots, a_n$, cumpliendo estas condiciones, y cualquier conjunto de polinomios escalares de norma uno $P_1, \ldots, P_n$ en un espacio de

Banach X, de grados $k_1, \ldots, k_n$, exista un vector $z \in B_X$ para el cual $|P_j(z)| \geq a_j^{k_j}$ para $j = 1, \ldots, n$.

Usando los resultados sobre el *factor problem* del Capítulo 3, obtenemos varios resultados en esta dirección. Por ejemplo, probamos que para cualquier espacio de Banach una condición suficiente es que

$$\sum_{i=1}^{n} a_i < \frac{1}{n^{n-1}}. \tag{4.10}$$

Restringiéndonos a casos particulares, a costa de perdida de generalidad, obtenemos mejores condiciones. Por ejemplo, cuando nos restringimos a polinomios homogéneos en espacios L_p o las clases Schatten $\mathcal{S}_p$, con $1 \leq p \leq 2$, probamos que una condición suficiente es

$$\sum_{i=1}^{n} a_i^p < \frac{1}{n^{n-1}}. \tag{4.11}$$

En el contexto de espacios finito dimensionales, probamos que si X es de dimensión d, entonces se puede tomar la condición

$$\sum_{i=1}^{n} a_i \leq nK^n,$$

donde, si H_d es el número armónico d, $C_{\mathbb{R}} = 1$ y $C_{\mathbb{C}} = 2$, la constante K puede ser cualquiera de las siguientes:

$$\begin{aligned}
K &= \frac{1}{C_{\mathbb{K}} 4ed} \text{ para cualquier espacio } d\text{-dimensional} \\
K &= \min\left\{ \frac{1}{\sqrt[n]{ne^2}}, \frac{4}{e^{H_d C_{\mathbb{K}}}} \right\} \text{ si } X \text{ es un espacio de Hilbert} \\
K &= \min\left\{ \frac{1}{\sqrt[n]{ne^2}}, \frac{1}{e^{\frac{1}{2}}} \right\} \text{ para polinomios homogeneos y } X = \ell_2^2(\mathbb{C}) \\
K &= \min\left\{ \frac{1}{\sqrt[n]{ne^2}}, \frac{1}{2^d} \right\} \text{ si } X = \ell_\infty^d(\mathbb{C}).
\end{aligned}$$

Aunque estos resultado están lejos de recuperar los resultados óptimos de K. Ball para el *plank problem* original (cuando los polinomios considerados son de hecho funciones lineales), su valor radica en su generalidad. Estos resultados pueden ser aplicados a polinomios de grados arbitrarios (y diferentes), y son válidos para una amplia gamma de números positivos $a_1, \ldots, a_n$. Mas aún, la mayoría son resultados para polinomios no necesariamente homogéneos. En este sentido, extendemos, y a veces mejoramos, resultados previos en este tema. Por ejemplo, el Teorema 5 de [KK] puede ser recuperado a partir de (4.10) y (4.11) como un caso particular, tomando todos los polinomios del mismo grado, todos los números positivos iguales, etc.

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