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Una teoría de 2-pro-objetos, una teoría de 2-categorías de 2-modelos y la estructura de 2-modelos para $2\text{-}\mathcal{P}ro(\mathcal{C})$

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Resumen. En los 60, Grothendieck desarrolla la teoría de pro-objetos de una categoría. La propiedad fundamental de $\mathsf{Pro}(\mathsf{C})$ es que se tiene un embedding $\mathsf{C} \xrightarrow{c} \mathsf{Pro}(\mathsf{C})$, $\mathsf{Pro}(\mathsf{C})$ tiene límites cofiltrantes pequeños, y estos son libres en el sentido de que para cualquier otra categoría E con límites cofiltrantes pequeños, la precomposición con c determina una equivalencia de categorías $\mathcal{C}at(\mathsf{Pro}(\mathsf{C}), \mathsf{E})_+ \simeq \mathcal{C}at(\mathsf{C}, \mathsf{E})$, (el "+" indica la subcategoría plena formada por los funtores que preservan límites cofiltrantes).

En este trabajo, desarrollamos la teoría de pro-objetos "2-dimensional". Dada una 2-categoría $\mathcal{C}$, definimos la 2-categoría $2\text{-}\mathcal{P}ro(\mathcal{C})$ cuyos objetos llamamos 2-pro-objetos. Probamos que $2\text{-}\mathcal{P}ro(\mathcal{C})$ tiene todas las propiedades básicas esperadas relativizadas adecuadamente al caso 2-categórico, incluyendo la propiedad universal correspondiente. Damos una definición de "closed 2-model 2-category" adecuada y demostraciones de sus propiedades básicas. Dejamos para un trabajo futuro la construcción de su categoría homotópica. Finalmente, probamos que nuestra 2-categoría $2\text{-}\mathcal{P}ro(\mathcal{C})$ tiene una estructura de "closed 2-model 2-category" si $\mathcal{C}$ la tiene.

Parte de la motivación de este trabajo fue desarrollar un contexto teórico para manipular el nervio de Čech en teoría de homotopía, [3], en particular en teoría de la forma fuerte, [23]. El nervio de Čech está indexado por las categorías de cubrimientos e hipercubrimientos con morfismos dados por los refinamientos, que no son categorías filtrantes pero sí determinan 2-categorías 2-filtrantes en las cuales el nervio de Čech también está definido, manda las 2-celdas en homotopías, y determina un 2-pro-objeto sobre los conjuntos simpliciales. Usualmente, el nervio de Čech debe ser considerado como un 2-pro-objeto en la categoría homotópica, perdiendo la información codificada en las homotopías explícitas.

Palabras claves. 2-pro-objeto, 2-filtrante, pseudo-límite, bi-límite, 2-cofinal, 2-categoría de 2-modelos.

A theory of 2-pro-objects, a theory of 2-model 2-categories and the 2-model structure for $2\text{-}\mathcal{P}ro(\mathcal{C})$

Abstract. In the sixties, Grothendieck developed the theory of pro-objects over a category. The fundamental property of the category $\mathsf{Pro}(\mathsf{C})$ is that there is an embedding $\mathsf{C} \xrightarrow{c} \mathsf{Pro}(\mathsf{C})$, $\mathsf{Pro}(\mathsf{C})$ is closed under small cofiltered limits, and these are free in the sense that for any category E closed under small cofiltered limits, pre-composition with c determines an equivalence of categories $\mathcal{C}at(\mathsf{Pro}(\mathsf{C}), \mathsf{E})_+ \simeq \mathcal{C}at(\mathsf{C}, \mathsf{E})$, (the "+" indicates the full subcategory of the functors that preserve cofiltered limits).

In this work we develop a "2-dimensional" pro-object theory. Given a 2-category $\mathcal{C}$, we define the 2-category $2\text{-}\mathcal{P}ro(\mathcal{C})$ whose objects we call 2-pro-objects. We prove that $2\text{-}\mathcal{P}ro(\mathcal{C})$ has all the expected basic properties adequately relativized to the 2-categorical setting, including the corresponding universal property. We give an adecuate definition of closed 2-model 2-category and demonstrations of its basic properties. We leave for a future work the construction of its homotpy 2-category. Finally, we prove that our 2-category $2\text{-}\mathcal{P}ro(\mathcal{C})$ has a closed 2-model 2-category structure provided that $\mathcal{C}$ has one.

Part of the motivation of this work was to develop a conceptual framework to handle the Čech nerve in homotopy theory, [3], in particular in strong shape theory, [23]. The Čech nerve is indexed by the categories of covers and of hypercovers, with cover refinments as morphisms, which are not filtered categories, but determine 2-filtered 2-categories on which the Čech nerve is also defined, sends 2-cells into homotopies, and determines a 2-pro-object of simplicial sets. Usually, the Čech nerve has to be considered as a pro-object in the homotopy category, loosing the information encoded in the explicit homotopies.

Key words. 2-pro-object, 2-filtered, pseudo-limit, bi-limit, 2-cofinal, 2-model 2-category.

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Introducción

La teoría de pro-objetos comenzó en Francia en los años 60 en el *Seminaire de Geometrie Algebrique du Bois-Marie* llevado a cabo por Alexander Grothendieck y otros matemáticos. Este seminario fue un fenómeno único de investigación y tuvo lugar entre los años 1960 y 1969 en el IHS cerca de París. En [1] se reúnen parte de las notas de estos seminarios. La categoría $\mathsf{Pro}(\mathsf{C})$ de pro-objetos de una categoría C se define allí. Aquí también se demuestran sus propiedades básicas y se da una caracterización de la misma por propiedad universal:

> *El funtor canónico* $\mathsf{C} \xrightarrow{c} \mathsf{Pro}(\mathsf{C})$ *es 2-universal respecto de los funtores de* C *en una categoría con límites cofiltrantes, más explícitamente: Dada* E *una categoría con límites cofiltrantes*
>
> $$\mathsf{Hom}(\mathsf{Pro}(\mathsf{C}), \mathsf{E})_+ \xrightarrow{c^*} \mathsf{Hom}(\mathsf{C}, \mathsf{E})$$
>
> *es una equivalencia de categorías (aquí el "+" indica la subcategoría plena formada por aquellos funtores que preservan límites cofiltrantes).*

En esa misma época, Daniel Quillen desarrollaba la teoría de categorías de modelos [27], vastamente utilizada en teoría de homotopía. Las categorías de modelos de Quillen permiten construir la categoría homotópica $\mathsf{Ho}(\mathsf{C})$ asociada a una categoría C. Esta categoría se obtiene invirtiendo formalmente la clase de morfismos formada por las equivalencias débiles de una estructura de modelos de Quillen de C. Las categorías homotópicas así obtenidas tienen la ventaja de tener muchas buenas propiedades que las hacen muy útiles en la práctica.

El nervio de *Čech* de un cubrimiento es una herramienta de base en ciertos desarrollos de la teoría de homotopía, y en teoría de la forma ([3], [24]). Dado un sitio C (por ejemplo, el reticulado $O(X)$ de los abiertos de un espacio topológico X), los cubrimientos (tomando como morfismos los refinamientos) forman una categoría $\mathsf{COV}(C)$ que no es cofiltrante, por lo cual el nervio de *Čech*, que es un funtor $\mathsf{COV}(C) \xrightarrow{\check{C}} \mathsf{SS}$, no determina un pro-objeto en la categoría de los conjuntos simpliciales y no se pueden utilizar las herramientas de la teoría de pro-objetos. Este problema se resuelve pasando a la categoría homotópica $\mathsf{Ho}(\mathsf{SS})$. Los cubrimientos ordenados bajo refinamiento sí forman una categoría cofiltrante $\mathsf{cov}(C)$, y dados dos refinamientos, los morfismos inducidos entre los nervios son homotópicos, por lo que se tiene un pro-objeto $\mathsf{cov}(C) \xrightarrow{\check{C}} \mathsf{Ho}(\mathsf{SS})$. Este pasaje módulo homotopía pierde la información dada por las homotopías explícitas asociadas a los refinamientos, haciendo que la teoría no sea suficientemente fina en muchas aplicaciones. En la teoría de la forma fuerte, las homotopías explícitas no se descartan pero el contexto conceptual de la teoría de pro-objetos se pierde para el nervio de *Čech*.

La teoría de 2-categorías se remonta a los años 70 pero viene teniendo un gran auge en los últimos tiempos. La mayoría de los resultados y construcciones básicos de la teoría de categorías han sido generalizados al contexto 2-categórico (ver por ejemplo [20] or [21]). Es esencial para nuestro trabajo la definición de 2-categoría 2-filtrante [17], reformulada en [11], así como también es fundamental la noción de pseudo-límites y, en particular, la construcción explícita de pseudo-colímites 2-filtrantes de categorías dada por Dubuc y Street en ese trabajo.

La teoría de pro-objetos 2-categórica ha demostrado ser de gran interés en sí misma y ha planteado muchos problemas interesantes de la teoría de 2-categorías. Nuestra motivación original para estudiar estos temas fue la de dar las herramientas necesarias para poder trabajar en teoría de la forma fuerte con el nervio de Čech y no perder información pasando módulo homotopía como sucede en teoría de la forma, ni tampoco tener que reemplazar el nervio de Čech por el menos conveniente nervio de Vietoris como se hace actualmente en teoría de la forma fuerte. Esta motivación provino de observar que si bien, como ya mencionamos, la categoría $\mathsf{COV}(C)$ no es cofiltrante, sí determina una 2-categoría 2-cofiltrante sobre la cual el nervio de *Čech* está definido y determina un 2-funtor mandando las 2-celdas en homotopías. Esto motivó nuestra definición de 2-pro-objeto que hará que el nervio de Čech, que no era un pro-objeto simplicial, sí resulte un 2-pro-objeto simplicial. La teoría de 2-pro-objetos fue de hecho una teoría muy interesante en sí misma y requirió mucho trabajo en teoría de 2-categorías, posponiendo las aplicaciones previstas para un trabajo futuro.

Estructuración del trabajo En esta tesis desarrollamos una teoría de pro-objetos 2-dimensional. También damos una noción de 2-funtor 2-cofinal que nos permite probar la versión 2-categórica de los teoremas de reindexación de pro-objetos. Por último damos una noción de “closed 2-bmodel 2-category” y demostramos que nuestra 2-categoría 2-$\mathcal{P}ro(C)$ satisface esta definición.

La sección 1 está dedicada a fijar notación y dejar en claro los resultados básicos de la teoría de 2-categorías que usaremos a lo largo de la tesis. La mayoría de estos resultados son conocidos, sin embargo hay algunos (para los cuales damos demostraciones explícitas) que no parecen encontrarse en la literatura. En 1.2 probamos que los pseudo-límites (cónicos) en las 2-categorías de 2-funtores $\mathcal{H}om(C,\mathcal{D})$, $\mathcal{H}om_p(C,\mathcal{D})$ y $p\mathcal{H}om_p(C,\mathcal{D})$ (definición 1.1.19) y los bi-límites en $p\mathcal{H}om_p(C,\mathcal{D})$ se calculan punto a punto. Este resultado, si bien era esperable, necesita indefectiblemente una demostración. En 1.3 definimos la noción de pseudo-funtor 2-cofinal entre 2-categorías y probamos ciertas propiedades que usaremos en la sección 3 para demostrar las propiedades de reindexación de 2-pro-objetos. En 1.4 construimos un 2-funtor asociado via un pseudo-funtor 2-cofinal a un pseudo-funtor dado. Este resultado tiene interés independiente y será usado en la sección 5. Finalmente, en 1.5 consideramos la noción de funtores flexibles dada en [4] y enunciamos una caracterización de los mismos muy útil e independiente del adjunto a izquierda de la inclusión $\mathcal{H}om(C, \mathcal{D}) \to \mathcal{H}om_p(C, \mathcal{D})$ (Proposición 1.5.3). Usando esta caracterización, el pseudo lema de Yoneda dice directamente que los 2-funtores representables son flexibles. Se sigue también que el 2-funtor asociado a cualquier 2-pro-objeto es flexible, lo cual tiene consecuencias importantes en la teoría de 2-pro-objetos.

En la sección 2 se encuentran algunos de los resultados claves de este trabajo. En 2.1, dada una 2-categoría C definimos la 2-categoría 2-$\mathcal{P}ro(C)$ cuyos objetos llamamos 2-pro-objetos. Un 2-pro-objeto de C es un 2-funtor a valores en C (o diagrama en C) indexado por una 2-categoría 2-cofiltrante. Nuestra teoría va más allá de la teoría de categorías enriquecidas porque en la definición de morfismos, en lugar de usar 2-límites estrictos, usamos la noción no estricta de pseudo-límites, que es usualmente la de interés práctico. También en 2.1, establecemos la fórmula básica que describe los morfismos y las 2-celdas entre 2-pro-objetos en términos de pseudo-límites y pseudo-colímites de las categorías de morfismos de C. Inspirados en la definición hallada en [3] de que un morfismo en la categoría original represente a un morfismo de pro-objetos, introducimos en 2.2 la noción de que un morfismo y una 2-celda en C representen un morfismo y una 2-celda en 2-$\mathcal{P}ro(C)$ respectivamente. También demostramos propiedades técnicas de los 2-pro-objetos que permiten hacer cálculos con ellos y, en particular, son necesarias en la demostración del teorema que establece que la 2-categoría 2-$\mathcal{P}ro(C)$ tiene pseudo-límites 2-cofiltrantes. En 2.3, construimos una 2-categoría 2-filtrante que sirve como 2-categoría de índices para el pseudo-límite 2-cofiltrante de 2-pro-objetos (Definición 2.3.1 y proposición 2.3.3). Esto también fue inspirado por una construcción con el mismo propósito hallada en [3] para el caso 1-dimensional, pero que en el caso 2-dimensional resulta ser mucho más compleja. Nos vimos forzados a recurrir a esta complicada construcción debido a que el tratamiento conceptual hecho en [1] no puede ser aplicado al caso 2-dimensional. Esto se debe a que un 2-funtor a valores en la 2-categoría de categorías *Cat* no es el pseudo-colímite (cónico) de 2-funtores 2-representables indexado por su 2-diagrama, como sí pasa en el caso 1-dimensional. Finalmente, en 2.4, enunciamos y demostramos la propiedad universal de 2-$\mathcal{P}ro(C)$ (Teorema 2.4.6), de una manera inédita incluso si se aplica al caso clásico de la teoría de pro-objetos.

También consideramos en esta sección la 2-categoría 2-$\mathcal{P}ro_p(C)$ que es "retract pseudo-equivalent" a 2-$\mathcal{P}ro(C)$, 2.1.5, hecho que se sigue de que los 2-funtores a valores en *Cat* asociados a 2-pro-objetos son flexibles. Esta 2-categoría será esencial en la sección 5 y probará ser interesante en sí misma.

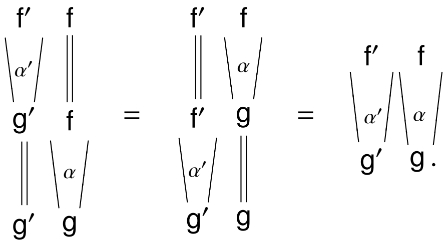
La mayor parte de los resultados de las secciones 1 y 2 fueron publicados en [8].

En la sección 3 probamos los teoremas de reindexación de pro-objetos para el caso 2-categórico. Esta sección está inspirada en los resultados análogos en el caso 1-dimensional dados en [3] pero, como pasaba con los resultados de la sección 2, su versión 2-categórica supone un desafío mayor. El primer resultado es una versión 2-categórica de un resultado debido a Deligne [1, Expose I, 8.1.6] que es clave en el caso 1-dimensional en el desarrollo de la estructura de modelos de la categoría $\mathsf{Pro}(\mathsf{C})$ [12]. El enunciado 1-dimensional establece que todo pro-objeto es isomorfo a uno indexado por un poset cofinito y filtrante. Nuestra versión establece que todo 2-pro-objeto es equivalente a uno indexado por un poset cofinito y filtrante. El segundo resultado establece que todo morfismo de 2-pro-objetos puede ser levantado salvo equivalencia a un morfismo entre 2-pro-objetos indexados por un poset cofinito y filtrante. Esto es un caso particular de un tercer resultado que establece que todo diagrama finito en $2\text{-}\mathcal{P}ro(\mathcal{C})$ puede ser levantado salvo equivalencia a un diagrama finito de 2-pro-objetos indexados por un poset cofinito y filtrante. Es clave para estos resultados la noción de pseudo-funtor 2-cofinal dada en la sección 1. Toda esta sección será usada para probar el teorema central de la sección 5.

En la sección 4 introducimos las nociones inéditas de "closed 2-model 2-category" y "closed 2-bmodel 2-category" y enunciamos y demostramos algunos lemas y proposiciones que usaremos más adelante. Nuestra noción es más fuerte que las "fibration structures" de Pronk ([26]) pues es una versión 2-dimensional de los axiomas de Quillen completos para "closed model categories". También difiere en el hecho importante de que no asumimos la elección de una factorización global privilegiada dada de forma pseudo-funtorial sino que estipulamos, como Quillen, solo la existencia de factorizaciones para cada flecha. La mayoría de los resultados de esta sección son generalizaciones al contexto de 2-categorías de enunciados bien conocidos de la teoría de "closed model categories".

Para terminar, en la sección 5 probamos uno de los teoremas centrales de esta tesis (5.2.5) que establece que si $\mathcal{C}$ es una "closed 2-bmodel 2-category", entonces $2\text{-}\mathcal{P}ro(\mathcal{C})$ también lo es. Para lograrlo, fue necesario demostrar primero los teoremas 5.1.14 and 5.2.4 que establecen respectivamente que la 2-categoría $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ (definición 1.1.19) y la 2-categoría $2\text{-}\mathcal{P}ro_p(\mathcal{C})$ son "closed 2-bmodel 2-categories" si J es un poset cofinito y filtrante y $\mathcal{C}$ es de una "closed 2-bmodel 2-category". Las propiedades de reindexación probadas en la sección 3 son claves para obtener 5.2.4 a partir de 5.1.14.

Notación Además del usual "pegado" de diagramas, usaremos el *Cálculo de ascensores* para expresiones que denotan 2-celdas (comparar con la notación usada en [14, 3.10, 3.17]). Esta es una notación muy gráfica inventada por Eduardo Dubuc en 1969 para escribir ecuaciones con transformaciones naturales entre funtores. En este trabajo usamos los ascensores para escribir ecuaciones con 2-celdas en 2-categorías. Los objetos se omiten, las 2-celdas se escriben con celdas, y las 2-celdas identidades como una doble línea. Es importante remarcar que cuando una 2-celda entre flechas distintas es la identidad, de todas formas se escribe como una 2-celda etiquetada por "=". Por ejemplo, la 2-celda estructural de un 2-funtor visto como caso particular de un pseudo-funtor. Las composiciones se leen de arriba para abajo y de derecha a izquierda. La ecuación 1.1.3 es la igualdad básica para el cálculo de ascensores:



Esto permite mover celdas de arriba hacia abajo y viceversa cuando no hay obstáculos, como si fueran ascensores. Con esto movemos celdas para formar configuraciones que den nuevas ecuaciones a partir de ecuaciones válidas.

Introduction

Pro-object theory started in the sixties in France with the *Seminaire de Geometrie Algebrique du Bois-Marie* conducted by Alexander Grothendieck and other mathematicians. This seminaire was a unique research phenomenon and took place between years 1960 and 1969 in the IHS near Paris. [1] consists on some of the notes of this seminaires. The category $\mathsf{Pro}(\mathsf{C})$ of pro-objects of a category C is defined there. The authors also prove the basic properties of this category and give a characterization by universal property:

> *The canonical functor* $\mathsf{C} \xrightarrow{c} \mathsf{Pro}(\mathsf{C})$ *is 2-universal over the functors from* C *into a category closed under cofiltered limits, more explicitly: Given a category* E *closed under cofiltered limits*
>
> $$\mathsf{Hom}(\mathsf{Pro}(\mathsf{C}), \mathsf{E})_+ \xrightarrow{c^*} \mathsf{Hom}(\mathsf{C}, \mathsf{E})$$
>
> *is an equivalence of categories (here the "+" indicates the full subcategory of those functors that preserve cofiltered limits).*

By the same time, Daniel Quillen developed model category theory [27] which was widely applied in homotopy theory. Quillen's model categories are useful to construct the homotopy category $\mathsf{Ho}(\mathsf{C})$ associated to a category C. This category is obtained by formally turning the class of weak equivalences of the model structure into isomorphisms. Homotopy categories associated to a model category have many good properties that make them very useful in practice.

The *Čech* nerve associated to a covering is a fundamental tool in some developments in homotopy theory and shape theory ([3], [24]). Given a site C (for example, the lattice $O(X)$ formed by the opened sets of a topological space X), coverings (taking refinements as morphisms) form a category $\mathsf{COV}(C)$ that fails to be cofiltered and so the *Čech* nerve, that is a functor $\mathsf{COV}(C) \xrightarrow{\check{C}} \mathsf{SS}$, does not determine a pro-object over simplicial sets, and pro-object theory can't be applied to this setting. This problem is solved by working in the homotopy category $\mathsf{Ho}(\mathsf{SS})$. Coverings under refinement does form a cofiltered category (poset) $\mathsf{cov}(C)$, and given two refinements, the induced morphisms between the nerves are homotopic, so there is a pro-object $\mathsf{cov}(C) \xrightarrow{\check{C}} \mathsf{Ho}(\mathsf{SS})$. Working in the homotopy category has the disadvantage that information given by the explicit homotopies associated to the refinements gets lost, making the theory not enough refined for many applications. In strong shape theory, the explicit homotopies are not discarded, but the conceptual framework of the theory of pro-objects is lost for the *Čech* nerve.

2-category theory goes back to the seventies but it's been having a heyday lately. Most of the results and basic constructions of category theory had been generalized to the 2-categorical context (see for example [20], [21]). It is essential to our work the definition of 2-filtered 2-category [17], reformulated in [11]. It is also key to our work the notion of pseudo-limit and, in particular, the explicit construction of 2-filtered pseudo-colimits of categories given by Dubuc and Street in that paper.

2-categorical pro-object theory had proved to be very interesting itself and had raised many interesting problems in 2-category theory. Our original motivation to begin with this work was to give the needed tools to be able to work with the Čech nerve in strong shape theory so no information is lost by working modulo homotopy as it happens in shape theory. This motivation came from observing that although, as we mentioned before, the category $\mathsf{COV}(C)$ is not cofiltered, it determines a 2-cofiltered 2-category over which the Čech nerve is defined and determines a 2-functor sending 2-cells into homotopies. This encouraged our definition of 2-pro-object that would make the Čech nerve, that was not a simplicial pro-object, a simplicial 2-pro-object. The tl llamado *Mardešić trick* debido a heory of 2-pro-objects was in fact a very interesting theory itself and it required much work in 2-category theory, postponing its intended applications to future work.

Work structure In this thesis, we develop a 2-dimensional pro-object theory. We also give a 2-cofinal pseudo-functor notion that allows as to prove the 2-categorical version of pro-objects reindexing properties. Finally, we give a notion of closed 2-bmodel 2-category and we prove that our 2-category 2-$\mathcal{P}ro(C)$ has a closed 2-bmodel structure.

Section 1 is intended to fix notation and set down some basic results from 2-category theory that we will use all along this thesis. Most of this results are well known, although there are some of them (for which we give explicit proofs) that seem not to be in the literature. In 1.2 we prove that (conical) pseudo-limits in the 2-categories $\mathcal{H}om(C, \mathcal{D})$, $\mathcal{H}om_p(C, \mathcal{D})$ and $p\mathcal{H}om_p(C, \mathcal{D})$ (definition 1.1.19) and bi-limits in $p\mathcal{H}om_p(C, \mathcal{D})$ are computed pointwise. These result, though expected, necessarily requires demonstration. In 1.3 we define the notion of 2-cofinal pseudo-functor between 2-categories and prove some properties that we will use in section 3 to prove 2-pro-objects reindexing properties. In 1.4 we construct a 2-functor associated to a given pseudo-functor via a 2-cofinal pseudo-functor. This result has independent interest and we will use it in section 5. Finally, in 1.5 we consider the notion of flexible functor given in [4] and we state a characterization of them that is very useful and independent of the left adjoint of the inclusion $\mathcal{H}om(C, \mathcal{D}) \to \mathcal{H}om_p(C, \mathcal{D})$ (Proposition 1.5.3). Using this characterization, the pseudo Yoneda lemma says that representable 2-functors are flexible. It also follows that the 2-functor associated to any 2-pro-object is flexible, fact which has important consequences in the theory of 2-pro-objects.

In section 2 are some of the most important results of this thesis. In 2.1, given a 2-category C we define the 2-category 2-$\mathcal{P}ro(C)$ whose objects we call 2-pro-objects.

A 2-pro-object over C is a 2-functor landing on C (or a diagram in C) indexed by a 2-cofiltered 2-category. Our theory goes beyond enriched category theory because in the definition of morphisms, instead of using strict 2-limits, we use the non-strict notion of pseudo-limits, which is usually the one of practical interest. In 2.1, we establish the basic formula describing morphisms and 2-cells between 2-pro-objects in terms of a pseudo-limit of pseudo-colimits of categories. Inspired on the definition found in [3] of a morphism of the original category representing a pro-objects morphism, in 2.2 we introduce the notion of a morphism or a 2-cell in C being a representative of a morphism or a 2-cell in 2-$\mathcal{P}ro(C)$ respectively. We also prove some technical properties of 2-pro-objects that allow us to make calculations with them, and, in particular, are needed in the proof of the theorem that states the 2-category 2-$\mathcal{P}ro(C)$ is closed under 2-cofiltered pseudo-limits. In 2.3, we construct a 2-filtered 2-category that will be the index 2-category of the 2-cofiltered pseudo-limit of 2-pro-objects (Definition 2.3.1 and Theorem 2.3.3). This was also inspired by a construction with the same purpouse in the 1-dimensional case found in [3], but the 2-categorical case turned out to be significantly more complicated. We were forced to make this complicated construction because the conceptual treatment made in [1] can't be applied to the 2-categorical setting. This is due to the fact that a 2-functor landing in the 2-category of categories Cat is not the (conical) pseudo-colimit of representable 2-functors indexed by its 2-diagram, as it is in the 1-dimensional case. Finally, in 2.4, we state and prove the universal property of 2-$\mathcal{P}ro(C)$ (Theorem 2.4.6), in an original way even applied to the classical pro-object theory.

We also consider in this section a 2-category 2-$\mathcal{P}ro_p(C)$ which is retract pseudo-equivalent to 2-$\mathcal{P}ro(C)$, 2.1.5, fact that follows from the flexible nature of the category-valued 2-functor associated to a 2-pro-object. This 2-category will be essential in section 5, and may prove to be interesting in itself.

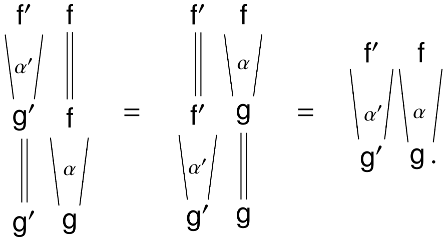
Most of the results of sections 1 and 2 have been published [8].

In section 3 we prove reindexing properties of pro-objects in the 2-categorical case. This section is inspired in the 1-dimensional analogous results given in [3], but, as happened with results of section 2, its 2-categorical version suppose a greater challenge. The first result is a 2-categorical version of a result due to Deligne [1, Expose I, 8.1.6] and that is key to develop the closed 2-bmodel structure for $\mathsf{Pro}(\mathsf{C})$ in the 1-dimensional case treated in [12]. The 1-dimensional statement establishes that every pro-object is isomorphic to a pro-object indexed by a cofinite and filtered poset. Our version establishes that every 2-pro-object is equivalent to a 2-pro-object indexed by a cofinite and filtered poset. The second result establishes that every morphism of 2-pro-objects can be lifted up to equivalence to a morphism between 2-pro-objects indexed by a cofinite and filtered poset. This is a particular case of the third result that establishes that every finite diagram in 2-$\mathcal{P}ro(C)$ can be lifted up to equivalence to a diagram of 2-pro-objects indexed by a cofinite and filtered poset. It is key for these results the notion of 2-cofinal pseudo-functor given in section 1. All this section will be used to prove the central theorems of section 5.

In section 4 we introduce original notions of closed 2-model and closed 2-bmodel 2-category and state some lemmas and propositions that we are going to use later. Our notion is stronger than Pronk's "fibration structures" ([26]) since it is a 2-dimensional version of the full Quillen's axioms for closed model structures. It also differs in the important fact that we do not assume the choice of a privileged global factorization given in a pseudo-functorial way, but stipulates, as Quillen does, only the existence of factorizations for each arrow. Most of the results of this section are generalizations to the context of 2-categories of well known statements about closed model categories.

To conclude, in section 5, we prove one of the central theorems of this thesis (5.2.5) which establishes that if $\mathcal{C}$ is a closed 2-bmodel 2-category, then so is 2-$\mathcal{P}ro(\mathcal{C})$. For this result, it was necessary to prove first theorems 5.1.14 and 5.2.4 which establish that the 2-category $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ (definition 1.1.19) and the 2-category 2-$\mathcal{P}ro_p(\mathcal{C})$ are closed 2-bmodel 2-categories respectively if $\mathcal{C}$ is (J will be a cofinite and filtered poset with a unique initial object). Reindexing properties proved in section 3 were key to obtain 5.2.4 from 5.1.14.

Notation In addition to the usual "pasting" of diagrams, we will use the *Elevators calculus* for expressions denoting 2-cells (compare with the notation used in [14, 3.10, 3.17]). This is a very graphic notation created by Eduardo Dubuc in 1969 to write down equations with natural transformations between functors. In this thesis, we use elevators to write down equations with 2-cells in 2-categories. Objects are omitted, 2-cells are denoted by cells and identity 2-cells as a double line. It is important to remark that when a 2-cell between different arrows is the identity, it is still written as a 2-cell with "=" as label. For example, the structural 2-cell of a 2-functor viewed as a particular case of a pseudo-functor. Compositions must be read from top to bottom and from right to left. Equation 1.1.3 is the basic equality for elevators calculus:



This allows to move cells up and down when there are no obstacles, as if they were elevators. In this way, we move cells to form configurations that fit valid equations in order to prove a new equation out of known ones.

Contents

1 Preliminaries on 2-categories

We distinguish between *small* and *large* sets. For us *legitimate* categories are categories with small hom sets, also called *locally small*. We freely consider without previous warning illegitimate categories with large hom sets, for example the category of all (legitimate) categories, or functor categories with large (legitimate) exponent. They are legitimate as categories in some higher universe, or they can be considered as convenient notational abbreviations for extended collections of data. In fact, questions of size play no overt role in this work, except that we elect for simplicity to consider only small 2-pro-objects. We will explicitly mention whether the categories are legitimate or small when necessary. We reserve the notation *Cat* for the legitimate 2-category of small categories, and we will denote $\mathcal{CAT}$ the illegitimate category (or 2-category) of all legitimate categories.

1.0.1. Notation. 2-Categories will be denoted with the "mathcal" font $\mathcal{C}$, $\mathcal{D}$, $\dots$, pseudo-functors (in particular 2-functors) with the capital "mathff" font, F, G, ... and pseudo-natural transformations (in particular 2-natural transformations) and modifications with the Greek alphabet. For objects in a 2-category, we will use capital "mathff" font C, $\mathsf{D}, \dots$, for arrows in a 2-category, small case letters in "mathff" font f, g, $\dots$, and we will use the Greek alphabet for 2-cells. However, when a 2-category is intended to be used as the index 2-category of a 2-diagram, we will use small case letters i, j, $\dots$ to denote its objects, and small case letters u, v, $\dots$ to denote its arrows. Categories will be denoted with capital "mathff" font C, $\mathsf{D}, \dots$, objects in a category with capital letters C, $D, \dots$ and arrows in a category with small case letters f, g, $\dots$.

We begin with some background material on 2-categories. Most of this is standard, but some results (for which we provide proofs) do not appear to be in the literature. We also set notation and terminology as we will explicitly use in this thesis.

1.1 Basic theory

Let *Cat* be the category of small categories. By a 2-category, we mean a *Cat* enriched category. A 2-functor, a 2-fully-faithful 2-functor, a 2-natural transformation and a 2-equivalence of 2-categories, are a *Cat*-functor, a *Cat*-fully-faithful functor, a *Cat*-natural transformation and a *Cat*-equivalence respectively. For an extended treatment on enriched category theory see [18].

In the sequel we will call *2-category* a structure satisfying the following descriptive definition free of the size restrictions implicit above. As usual, given a 2-category, we denote horizontal composition by juxtaposition, and vertical composition by "$\circ$".

1.1.1. 2-Category. *A* 2-category $\mathcal{C}$ *consists on objects or 0-cells* C, D *... , arrows or 1-cells* f, g *... , and 2-cells* α, β, *... .*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, shift left=2, "\mathsf{f}"] \arrow[r, shift right=2, "\mathsf{g}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D}
\end{tikzcd}$$

The objects and the arrows form a category (called the underlying category of C), with composition (called "horizontal") denoted by juxtaposition. For a fixed C *and* D*, the arrows between them and the 2-cells between these arrows form a category* $C(\mathsf{C}, \mathsf{D})$ *under "vertical" composition, denoted by "$\circ$". There is also an associative horizontal composition between 2-cells denoted by juxtaposition, with units* $id_{id_{\mathsf{C}}}$*. The following is the basic 2-category diagram:*

$$\begin{tikzcd}[row sep=small]
 & {} & {} \\
\mathsf{C} \arrow[r, shift left=6, "\mathsf{f}"] \arrow[r, "\mathsf{g}" description] \arrow[r, shift right=6, "\mathsf{h}"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathsf{D} \arrow[r, shift left=6, "\mathsf{f}'"] \arrow[r, "\mathsf{g}'" description] \arrow[r, shift right=6, "\mathsf{h}'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathsf{E}
\end{tikzcd} \tag{1.1.2}$$

with the equations $(\beta'\beta) \circ (\alpha'\alpha) = (\beta' \circ \alpha')(\beta \circ \alpha)$, $id_{\mathsf{f}'} id_{\mathsf{f}} = id_{\mathsf{f}'\mathsf{f}}$.

In particular it follows that given $\begin{tikzcd} \mathsf{C} \arrow[r, shift left=2, "\mathsf{f}"] \arrow[r, shift right=2, "\mathsf{g}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, shift left=2, "\mathsf{f}'"] \arrow[r, shift right=2, "\mathsf{g}'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & \mathsf{E} \end{tikzcd}$, we have:

$$(\alpha'\, id_{\mathsf{g}}) \circ (id_{\mathsf{f}'}\, \alpha) = (id_{\mathsf{g}'}\, \alpha) \circ (\alpha'\, id_{\mathsf{f}}) = (\alpha'\alpha). \tag{1.1.3}$$

We consider juxtaposition more binding than "$\circ$", thus $\alpha\beta \circ \gamma$ means $(\alpha\beta) \circ \gamma$. We will abuse notation by writing f instead of id_{f} for arrows f when there is no risk of confusion.

1.1.4. Dual 2-Category. *If C is a 2-category, we denote by C^{op} the 2-category with the same objects as C but with $C^{op}(\mathsf{C}, \mathsf{D}) = C(\mathsf{D}, \mathsf{C})$, i.e. we reverse the 1-cells but not the 2-cells.*

1.1.5 Remark. The category of all categories *Cat* has a 2-category structure given by the following:

- Its objects are the categories.
- Its arrows are the functors.
- Its 2-cells are the natural transformations.

With the notation of (1.1.2), the composition between functors and the vertical composition between natural transformations are the usual ones. And the horizontal composition between natural transformations is given by $(\alpha'\alpha)_C = \alpha'_{\mathsf{g}C} \circ \mathsf{f}'(\alpha_C)$ for $C \in \mathsf{C}$.

One can easily check that this gives a 2-category structure. □

1.1.6. Equivalence. *An arrow* $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D}$ *in a 2-category* $\mathcal{C}$ *is said to be an* equivalence *in* $\mathcal{C}$ *if there exist another arrow* $\mathsf{D} \xrightarrow{\mathsf{g}} \mathsf{C} \in \mathcal{C}$ *and invertible 2-cells* $\mathsf{fg} \overset{\alpha}{\Longrightarrow} id_{\mathsf{D}}$, $\mathsf{gf} \overset{\beta}{\Longrightarrow} id_{\mathsf{C}}$.

1.1.7. Notation. We will denote equivalences by $\simeq$ and isomorphisms by $\cong$.

1.1.8 Remark. Equivalences in *Cat* are usual equivalences of categories. □

1.1.9. 2-functor. *A* 2-functor $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *between 2-categories is an enriched functor over Cat. As such, sends objects to objects, arrows to arrows and 2-cells to 2-cells, strictly preserving all the structure.*

1.1.10. Pseudo-functor. *A* pseudo-functor $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *between 2-categories is a correspondence that sends objects to objects, arrows to arrows and 2-cells to 2-cells, preserving all the structure up to invertible 2-cells* $\mathsf{FgFf} \Longrightarrow \mathsf{F(gf)}$ *and* $id_{\mathsf{FC}} \Longrightarrow \mathsf{F}(id_{\mathsf{C}})$ *instead of equalities. More explicitly, it is given by the following data:*

- *For each object* $\mathsf{C} \in \mathcal{C}$*, an object* $\mathsf{F(C)} \in \mathcal{D}$*. We will abuse notation and write* FC *when there is no risk of confusion.*

- *For each hom-category* $\mathcal{C}(\mathsf{C}, \mathsf{D})$*, a functor* $\mathsf{F}_{\mathsf{C},\mathsf{D}} : \mathcal{C}(\mathsf{C}, \mathsf{D}) \longrightarrow \mathcal{D}(\mathsf{FC}, \mathsf{FD})$.

  We will abuse notation and write Ff *instead of* $\mathsf{F}_{\mathsf{C},\mathsf{D}}(\mathsf{f})$ *and* $\mathsf{F}\alpha$ *instead of* $\mathsf{F}_{\mathsf{C},\mathsf{D}}(\alpha)$ *for* $\begin{tikzcd} \mathsf{C} \arrow[r, shift left=2, "\mathsf{f}"] \arrow[r, shift right=2, "\mathsf{g}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{D} \end{tikzcd} \in \mathcal{C}$ *when there is no risk of confusion.*

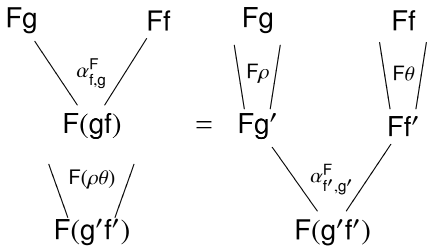
- *For each object* $\mathsf{C} \in \mathcal{C}$*, an invertible 2-cell* $\alpha^{\mathsf{F}}_{\mathsf{C}} : id_{\mathsf{FC}} \Longrightarrow \mathsf{F}(id_{\mathsf{C}}) \in \mathcal{D}$.

- *For each triplet* $\mathsf{C}, \mathsf{D}, \mathsf{E}$ *of objects of* $\mathcal{C}$*, a natural isomorphism:*

$$\begin{tikzcd}
\mathcal{C}(\mathsf{C}, \mathsf{D}) \times \mathcal{C}(\mathsf{D}, \mathsf{E}) \arrow[r, "\mathsf{F} \times \mathsf{F}"] \arrow[d, "c"'] \arrow[dr, phantom, "\cong \Downarrow \alpha^{\mathsf{F}}"] & \mathcal{D}(\mathsf{FC}, \mathsf{FD}) \times \mathcal{D}(\mathsf{FD}, \mathsf{FE}) \arrow[d, "c"] \\
\mathcal{C}(\mathsf{C}, \mathsf{E}) \arrow[r, "\mathsf{F}"'] & \mathcal{D}(\mathsf{FC}, \mathsf{FE})
\end{tikzcd}$$

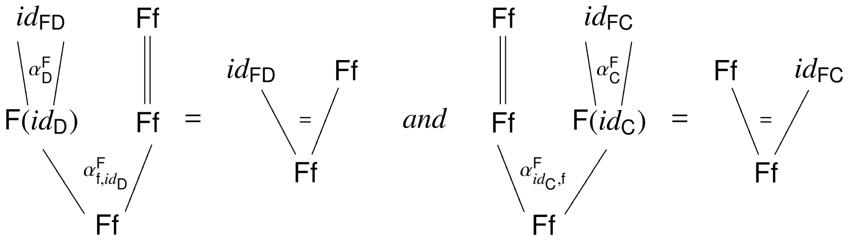
where c denotes the composition functors.

More explicitly, α^{F} *consists on an invertible 2-cell* $\begin{tikzcd} \mathsf{FC} \arrow[r, shift left=2, "\mathsf{FgFf}"] \arrow[r, shift right=2, "\mathsf{F(gf)}"'] \arrow[r, phantom, "\Downarrow \alpha^{\mathsf{F}}_{\mathsf{f},\mathsf{g}}"] & \mathsf{FE} \end{tikzcd}$ *for each configuration* $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D} \xrightarrow{\mathsf{g}} \mathsf{E} \in \mathcal{C}$ *such that* $\forall$ $\begin{tikzcd} \mathsf{C} \arrow[r, shift left=2, "\mathsf{f}"] \arrow[r, shift right=2, "\mathsf{f}'"'] \arrow[r, phantom, "\Downarrow \theta"] & \mathsf{D} \arrow[r, shift left=2, "\mathsf{g}"] \arrow[r, shift right=2, "\mathsf{g}'"'] \arrow[r, phantom, "\Downarrow \rho"] & \mathsf{E} \end{tikzcd}$,
$\mathsf{F}(\rho\theta) \circ \alpha^{\mathsf{F}}_{\mathsf{f},\mathsf{g}} = \alpha^{\mathsf{F}}_{\mathsf{f}',\mathsf{g}'} \circ \mathsf{F}\rho\mathsf{F}\theta$

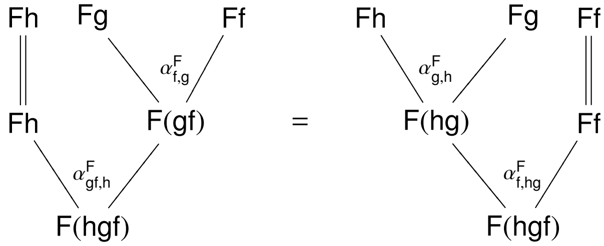


All this data must satisfy the following equalities:

- *For each* $C \xrightarrow{f} D \in \mathcal{C}$*,*



- *For each configuration* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \in \mathcal{C}$*,*



1.1.11 Remark. A 2-functor is a pseudo-functor such that α^{F}_{C} is the equality for each $C \in \mathcal{C}$ and $\alpha^{F}_{f,g}$ is the equality for each $C \xrightarrow{f} D \xrightarrow{g} E \in \mathcal{C}$. □

1.1.12. Pseudo-essentially surjective on objects. *A pseudo functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is said to be* pseudo-essentially surjective on objects *if for each* $D \in \mathcal{D}$*, there exist* $C \in \mathcal{C}$ *and an equivalence* $FC \longrightarrow D \in \mathcal{D}$*.*

1.1.13. 2-fully-faithful. *A 2-functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is said to be* 2-fully-faithful *if for each* $C,\ D \in \mathcal{C}$*,* $F : \mathcal{C}(C, D) \longrightarrow \mathcal{D}(FC, FD)$ *is an isomorphism of categories.*

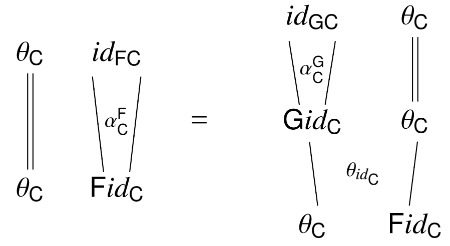
1.1.14. Pseudo-fully-faithful. *A pseudo-functor* $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *is said to be* pseudo-fully-faithful *if for each* $\mathsf{C},\ \mathsf{D} \in \mathcal{C}$, $\mathsf{F} : \mathcal{C}(\mathsf{C}, \mathsf{D}) \longrightarrow \mathcal{D}(\mathsf{FC}, \mathsf{FD})$ *is an equivalence of categories.*

1.1.15. Pseudo-natural transformation. *A* pseudo-natural transformation $\theta : \mathsf{F} \Rightarrow \mathsf{G} : \mathcal{C} \to \mathcal{D}$ *between pseudo-functors consists of a family of arrows* $\left\{\mathsf{FC} \overset{\theta_\mathsf{C}}{\to} \mathsf{GC}\right\}_{\mathsf{C}\in\mathcal{C}}$ *and a family of invertible 2-cells* $\left\{\mathsf{Gf}\theta_\mathsf{C} \overset{\theta_\mathsf{f}}{\Longrightarrow} \theta_\mathsf{D}\mathsf{Ff}\right\}_{\mathsf{C}\overset{\mathsf{f}}{\to}\mathsf{D}\in\mathcal{C}}$

$$\begin{tikzcd}
\mathsf{FC} \arrow[r, "\theta_\mathsf{C}"] \arrow[d, "\mathsf{Ff}"'] \arrow[dr, phantom, "\cong \Downarrow \theta_\mathsf{f}"] & \mathsf{GC} \arrow[d, "\mathsf{Gf}"] \\
\mathsf{FD} \arrow[r, "\theta_\mathsf{D}"'] & \mathsf{GD}
\end{tikzcd}$$

satisfying the following conditions:

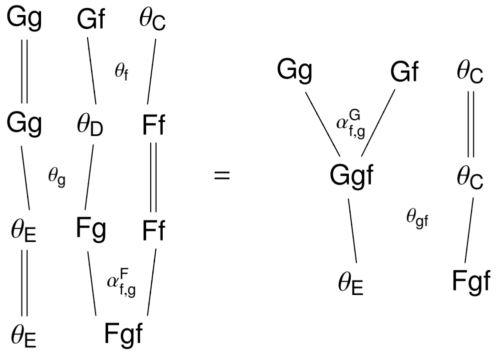
PN0. For each $\mathsf{C} \in \mathcal{C}$, $\theta_\mathsf{C}\alpha^\mathsf{F}_\mathsf{C} = \theta_{id_\mathsf{C}} \circ \alpha^\mathsf{G}_\mathsf{C}\theta_\mathsf{C}$, *i.e.*



i.e.

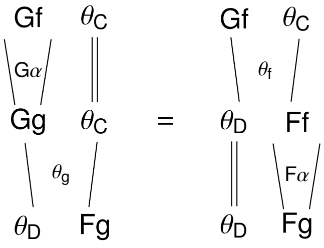
$$\begin{tikzcd}
\mathsf{FC} \arrow[r, bend left, "id_\mathsf{FC}"] \arrow[r, bend right, "\mathsf{F}id_\mathsf{C}"'] \arrow[r, phantom, "\cong \Downarrow \alpha^\mathsf{F}_\mathsf{C}"] & \mathsf{FC} \arrow[r, "\theta_\mathsf{C}"] & \mathsf{GC}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathsf{FC} \arrow[r, "\theta_\mathsf{C}"] \arrow[dr, bend right, "\mathsf{F}id_\mathsf{C}"'] & \mathsf{GC} \arrow[r, bend left, "id_\mathsf{GC}"] \arrow[r, bend right, "\mathsf{G}id_\mathsf{C}"'] \arrow[r, phantom, "\cong \Downarrow \alpha^\mathsf{G}_\mathsf{C}"] \arrow[d, phantom, "\cong \Downarrow \theta_{id_\mathsf{C}}"] & \mathsf{GC} \\
 & \mathsf{FC} \arrow[ur, bend right, "\theta_\mathsf{C}"'] &
\end{tikzcd}$$

PN1. For each $\mathsf{C} \overset{\mathsf{f}}{\to} \mathsf{D} \overset{\mathsf{g}}{\to} \mathsf{E}$, $\theta_\mathsf{E}\alpha^\mathsf{F}_{\mathsf{f},\mathsf{g}} \circ \theta_\mathsf{g}\mathsf{Ff} \circ \mathsf{Gg}\theta_\mathsf{f} = \theta_\mathsf{gf} \circ \alpha^\mathsf{G}_{\mathsf{f},\mathsf{g}}\theta_\mathsf{C}$, *i.e.*



i.e.
$$\begin{tikzcd}
 & FC \arrow[r, "\theta_C"] \arrow[d, "Ff"'] \arrow[dd, bend right=70, "Fgf"'] \arrow[dr, phantom, "\cong \Downarrow \theta_f"] & GC \arrow[d, "Gf"] \\
\overset{\alpha^F_{f,g}}{\underset{\cong}{\Leftarrow}} & FD \arrow[r, "\theta_D"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\cong \Downarrow \theta_g"] & GD \arrow[d, "Gg"] \\
 & FE \arrow[r, "\theta_E"'] & GE
\end{tikzcd}
=
\begin{tikzcd}
FC \arrow[r, "\theta_C"] \arrow[dd, "Fgf"'] \arrow[ddr, phantom, "\cong \Downarrow \theta_{gf}"] & GC \arrow[dd, "Ggf"'] \arrow[dr, bend left, "Gf"] \arrow[dd, phantom, shift left=6ex, "\overset{\alpha^G_{f,g}}{\underset{\cong}{\Leftarrow}}"] & \\
 & & GD \arrow[dl, bend left, "Gg"] \\
FE \arrow[r, "\theta_E"'] & GE &
\end{tikzcd}$$

PN2. For each $C \overset{f}{\underset{g}{\underset{\Downarrow \alpha}{\rightrightarrows}}} D \in \mathcal{C}$, $\theta_g \circ G\alpha\theta_C = \theta_D F\alpha \circ \theta_f$, *i.e.*



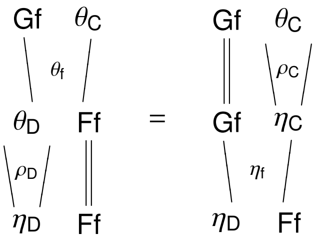
$$\begin{tikzcd}
FC \arrow[r, "\theta_C"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\cong \Downarrow \theta_g"] & GC \arrow[d, bend right, "Gg"'] \arrow[d, bend left, "Gf"] \arrow[d, phantom, "\overset{G\alpha}{\underset{\cong}{\Leftarrow}}"] \\
FD \arrow[r, "\theta_D"'] & GD
\end{tikzcd}
=
\begin{tikzcd}
FC \arrow[r, "\theta_C"] \arrow[d, bend right, "Fg"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "\overset{F\alpha}{\underset{\cong}{\Leftarrow}}"] \arrow[dr, phantom, "\cong \Downarrow \theta_f"] & GC \arrow[d, "Gf"] \\
FD \arrow[r, "\theta_D"'] & GD
\end{tikzcd}$$

As a particular case, we have the notion of pseudo-natural transformation between 2-functors.

1.1.16. 2-Natural transformation. *A 2-natural transformation θ between 2-functors is a pseudo-natural transformation such that θ_f is the equality for each $f \in \mathcal{C}$. Equivalently,* *it is a Cat-enriched natural transformation, that is, a natural transformation between the functors determined by* F *and* G*, such that for each 2-cell* $\mathsf{C} \overset{\mathsf{f}}{\underset{\mathsf{g}}{\overset{\Downarrow \alpha}{\rightrightarrows}}} \mathsf{D}$*, the equation* $\mathsf{G}\alpha\theta_{\mathsf{C}} = \theta_{\mathsf{D}}\mathsf{F}\alpha$ *holds.*

1.1.17. Modification. *Given pseudo-functors* F *and* G *from* C *to* $\mathcal{D}$ *(as a particular case* F *and* G *might be 2-functors), a* modification $\mathsf{F} \overset{\theta}{\underset{\eta}{\overset{\Downarrow \rho}{\rightrightarrows}}} \mathsf{G}$ *between pseudo-natural transformations is a family* $\left\{\theta_{\mathsf{C}} \overset{\rho_{\mathsf{C}}}{\Longrightarrow} \eta_{\mathsf{C}}\right\}_{\mathsf{C}\in C}$ *of 2-cells of* $\mathcal{D}$ *such that:*

PM. For each $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D} \in C$, $\rho_{\mathsf{D}}\mathsf{Ff} \circ \theta_{\mathsf{f}} = \eta_{\mathsf{f}} \circ \mathsf{Gf}\rho_{\mathsf{C}}$, *i.e.*



i.e.

$$\begin{tikzcd}
\mathsf{FC} \arrow[r, "\theta_{\mathsf{C}}"] \arrow[d, "\mathsf{Ff}"'] \arrow[dr, phantom, "\cong \Downarrow \theta_{\mathsf{f}}"] & \mathsf{GC} \arrow[d, "\mathsf{Gf}"] & = & \mathsf{FC} \arrow[r, bend left, "\theta_{\mathsf{C}}"] \arrow[r, bend right, "\eta_{\mathsf{C}}"'] \arrow[r, phantom, "\Downarrow \rho_{\mathsf{C}}"] \arrow[d, "\mathsf{Ff}"'] \arrow[dr, phantom, "\cong \Downarrow \eta_{\mathsf{f}}"] & \mathsf{GC} \arrow[d, "\mathsf{Gf}"] \\
\mathsf{FD} \arrow[r, bend left, "\theta_{\mathsf{D}}"] \arrow[r, bend right, "\eta_{\mathsf{D}}"'] \arrow[r, phantom, "\Downarrow \rho_{\mathsf{D}}"] & \mathsf{GD} & & \mathsf{FD} \arrow[r, "\eta_{\mathsf{D}}"'] & \mathsf{GD}
\end{tikzcd}$$

As a particular case, we have modifications *between 2-natural transformations, which are families of 2-cells as above satisfying* $\rho_{\mathsf{D}}\mathsf{Ff} = \mathsf{Gf}\rho_{\mathsf{C}}$.

1.1.18. By the theory of enriched categories, it is well known that 2-categories, 2-functors and 2-natural transformations form a 2-category (which actually underlies a 3-category) that we denote 2-$\mathcal{CAT}$. Horizontal composition of 2-functors and vertical composition of 2-natural transformations are the usual ones, and the horizontal composition of 2-natural transformations is defined by:

Given $\mathsf{C} \overset{\mathsf{F}}{\underset{\mathsf{G}}{\overset{\Downarrow \alpha}{\rightrightarrows}}} \mathsf{D} \overset{\mathsf{F}'}{\underset{\mathsf{G}'}{\overset{\Downarrow \alpha'}{\rightrightarrows}}} \mathsf{E}$, $(\alpha'\alpha)_{\mathsf{C}} = \alpha'_{\mathsf{GC}} \circ \mathsf{F}'\alpha_{\mathsf{C}} \quad (= \mathsf{G}'\alpha_{\mathsf{C}} \circ \alpha'_{\mathsf{FC}})$.

1.1.19 Definition. *Given two 2-categories* C *and* $\mathcal{D}$*, we consider three 2-categories defined as follows:*

- *$\mathcal{H}om(\mathcal{C}, \mathcal{D})$: 2-functors and 2-natural transformations.*

- *$\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$: 2-functors and pseudo-natural transformations.*

- *$p\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$: pseudo-functors and pseudo-natural transformations.*

In all cases the 2-cells are the modifications. To define compositions we draw the basic 2-category diagram:

$$\begin{tikzcd}
\mathsf{F} \arrow[r, bend left=60, "\theta"] \arrow[r, "\eta"] \arrow[r, bend right=60, "\mu"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \rho"] \arrow[r, phantom, shift right=3ex, "\Downarrow \varepsilon"] & \mathsf{G} \arrow[r, bend left=60, "\theta'"] \arrow[r, "\eta'"] \arrow[r, bend right=60, "\mu'"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \rho'"] \arrow[r, phantom, shift right=3ex, "\Downarrow \varepsilon'"] & \mathsf{H}
\end{tikzcd}$$

$$(\theta'\theta)_{\mathsf{C}} = \theta'_{\mathsf{C}}\theta_{\mathsf{C}}$$

$$(\theta'\theta)_{\mathsf{f}} = \theta'_{\mathsf{D}}\theta_{\mathsf{f}} \circ \theta'_{\mathsf{f}}\theta_{\mathsf{C}}$$

$$(\rho'\rho)_{\mathsf{C}} = \rho'_{\mathsf{C}}\rho_{\mathsf{C}}$$

$$(\epsilon \circ \rho)_{\mathsf{C}} = \epsilon_{\mathsf{C}} \circ \rho_{\mathsf{C}}$$

It is straightforward to check that these definitions determine 2-category structures. □

1.1.20 Remark. [14, 3.17] A pseudo-natural transformation $\mathsf{F} \overset{\theta}{\Longrightarrow} \mathsf{G} \in \mathcal{H}om_p(\mathcal{C}, \mathcal{D})$ (respectively $p\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$) is an equivalence iff for each $\mathsf{C} \in \mathcal{C}$, θ_{C} is an equivalence in $\mathcal{D}$. The same assertion does not hold in $\mathcal{H}om(\mathcal{C}, \mathcal{D})$ (c.f. 1.5.5). □

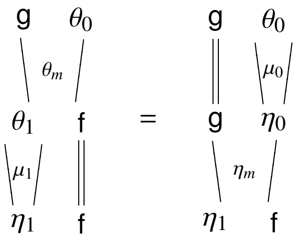
1.1.21 Remark. Since we are going to make manipulations with the 2-category $\mathcal{H}om_p(\mathsf{2}, \mathcal{C})$ (where $\mathsf{2}$ stands for the trivial 2-category with two objects, one morphism between them and no 2-cells other than identities, i.e. $\mathsf{2} = \{0 \longrightarrow 1\}$), we will give a more explicit description of it:

- An object is a morphism $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D} \in \mathcal{C}$.

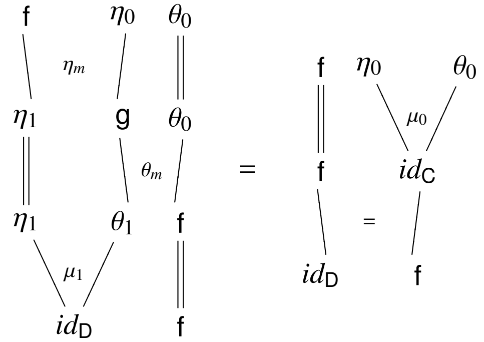
- A morphism θ between $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D}$ and $\mathsf{C}' \xrightarrow{\mathsf{g}} \mathsf{D}'$ in $\mathcal{H}om_p(\mathsf{2}, \mathcal{C})$ is given by two morphisms $\mathsf{C} \xrightarrow{\theta_0} \mathsf{C}'$, $\mathsf{D} \xrightarrow{\theta_1} \mathsf{D}' \in \mathcal{C}$ and an invertible 2-cell $\mathsf{g}\theta_0 \overset{\theta_m}{\Longrightarrow} \theta_1\mathsf{f}$ as in the following diagram:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\theta_0"] \arrow[d, "\mathsf{f}"'] \arrow[dr, phantom, "\cong \Downarrow \theta_m"] & \mathsf{C}' \arrow[d, "\mathsf{g}"] \\
\mathsf{D} \arrow[r, "\theta_1"'] & \mathsf{D}'
\end{tikzcd}$$

- A 2-cell μ in $\mathcal{H}om_p(\mathsf{2}, \mathcal{C})$ between θ and η from $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D}$ to $\mathsf{C}' \xrightarrow{\mathsf{g}} \mathsf{D}'$ is given by two 2-cells $\theta_0 \overset{\mu_0}{\Longrightarrow} \eta_0$, $\theta_1 \overset{\mu_1}{\Longrightarrow} \eta_1 \in \mathcal{C}$ such that $\mu_1\mathsf{f} \circ \theta_m = \eta_m \circ \mathsf{g}\mu_0$.



1.1.22 Definition. *Let C be a 2-category and $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D}$, $\mathsf{C}' \xrightarrow{\mathsf{g}} \mathsf{D}'$ two morphisms in C. We say that* f *is a retract of* g *in $\mathcal{H}om_p(2, C)$ if there are morphisms $\mathsf{f} \xrightarrow{\theta} \mathsf{g}$, $\mathsf{g} \xrightarrow{\eta} \mathsf{f}$ and an invertible 2-cell $\eta\theta \overset{\mu}{\Longrightarrow} id_{\mathsf{f}}$ in $\mathcal{H}om_p(2, C)$. More explicitly, the retraction consists in a tuple $(\theta_0, \theta_1, \theta_m, \eta_0, \eta_1, \eta_m, \mu_0, \mu_1)$ such that $\mathsf{g}\theta_0 \underset{\cong}{\overset{\theta_m}{\Longrightarrow}} \theta_1\mathsf{f}$, $\mathsf{f}\eta_0 \underset{\cong}{\overset{\eta_m}{\Longrightarrow}} \eta_1\mathsf{g}$, $\eta_0\theta_0 \underset{\cong}{\overset{\mu_0}{\Longrightarrow}} id_{\mathsf{C}}$, $\eta_1\theta_1 \underset{\cong}{\overset{\mu_1}{\Longrightarrow}} id_{\mathsf{D}}$ and the following equality holds:*



1.1.23. Bi-universal arrows. [14, 9.4] *Let $\mathcal{D} \xrightarrow{\mathsf{G}} C$ be a pseudo-functor, $\mathsf{C} \in C$ and $\mathsf{D} \in \mathcal{D}$. A morphism $\mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{GD} \in C$ is a bi-universal arrow from* C *to* G *if for each $\mathsf{D}' \in \mathcal{D}$, the following functor is an equivalence of categories*

$$\begin{tikzcd}
\mathcal{D}(\mathsf{D}, \mathsf{D}') \arrow[r] & C(\mathsf{C}, \mathsf{GD}') \\
\mathsf{g} \overset{\alpha}{\Longrightarrow} \mathsf{g}' \arrow[r, mapsto] & \mathsf{G(g)f} \overset{\mathsf{G}(\alpha)\mathsf{f}}{\Longrightarrow} \mathsf{G(g')f}
\end{tikzcd}$$

1.1.24. Bi-adjoint pseudo-functors. [14, 9.8] *Let $\mathsf{F} : C \rightleftarrows \mathcal{D} : \mathsf{G}$ be pseudo-functors. We say that* F *is* bi-left adjoint *to* G *(equivalently that* G *is bi-right adjoint to* F*) if for each $\mathsf{C} \in C$, $\mathsf{D} \in \mathcal{D}$, there is an equivalence of categories $\mathcal{D}(\mathsf{FC}, \mathsf{D}) \xrightarrow{\phi_{\mathsf{C,D}}} C(\mathsf{C}, \mathsf{GD})$ in a way such that ϕ is a pseudo-natural transformation in each variable. In this case, we use the notation $\mathsf{F} \dashv_b \mathsf{G}$.*

1.1.25 Remark. It is straightforward to check that F is bi-left adjoint to G iff there exist pseudo-natural transformations $\mathsf{FG} \overset{\epsilon}{\Longrightarrow} id_{\mathcal{D}}$, $id_{\mathcal{C}} \overset{\eta}{\Longrightarrow} \mathsf{GF}$ such that $\epsilon\mathsf{F} \circ \mathsf{F}\eta = \mathsf{F}$ and $\mathsf{G}\epsilon \circ \eta\mathsf{G} = \mathsf{G}$. □

1.1.26 Proposition. [14, 9.16] *Let* $\mathsf{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{G}$ *be pseudo-functors. Then* F *is bi-left adjoint to* G *iff there exists a pseudo-natural transformation* $id_{\mathcal{C}} \overset{\eta}{\Longrightarrow} \mathsf{GF}$ *such that* η_{C} *is a bi-universal arrow from* C *to* GFC $\forall\ \mathsf{C} \in \mathcal{C}$. □

1.1.27. 2-Equivalence. *A 2-functor* $\mathcal{C} \xrightarrow{\mathsf{F}} \mathcal{D}$ *is said to be a* 2-equivalence of 2-categories *if there exist a 2-functor* $\mathcal{D} \xrightarrow{\mathsf{G}} \mathcal{C}$ *and invertible 2-natural transformations* $\mathsf{FG} \overset{\alpha}{\Longrightarrow} id_{\mathcal{D}}$ *and* $\mathsf{GF} \overset{\beta}{\Longrightarrow} id_{\mathcal{C}}$*.* G *is said to be a* quasi-inverse *for* F.

1.1.28. Pseudo-equivalence. *A pseudo-functor* $\mathcal{C} \xrightarrow{\mathsf{F}} \mathcal{D}$ *is said to be a* pseudo-equivalence of 2-categories *if there exists a pseudo-functor* $\mathcal{D} \xrightarrow{\mathsf{G}} \mathcal{C}$ *and equivalence pseudo-natural transformations* $\mathsf{FG} \overset{\alpha}{\Longrightarrow} id_{\mathcal{D}}$ *and* $\mathsf{GF} \overset{\beta}{\Longrightarrow} id_{\mathcal{C}}$*.* G *is said to be a* pseudo-quasi-inverse *for* F.

Pseudo-equivalences are sometimes called bi-equivalences in the literature. See for example [21] where 1.1.30 is mentioned.

Often we have 2-functors that do not have a quasi-inverse but do have a pseudo-quasi-inverse and thus determine a pseudo-equivalence, see 2.1.5.

1.1.29 Proposition. [18, 1.11] *A 2-functor* $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *is a 2-equivalence of 2-categories if and only if it is 2-fully-faithful and essentially surjective on objects.* □

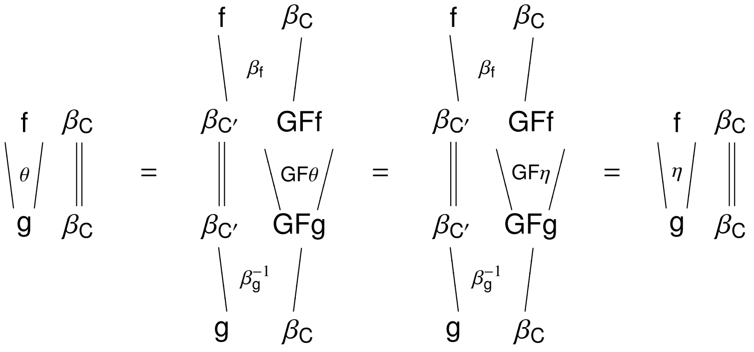
1.1.30 Proposition. *A pseudo-functor* $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{D}$ *is a pseudo-equivalence of 2-categories if and only if it is pseudo-fully-faithful and pseudo-essentially surjective on objects. Moreover,* F *is essentially surjective on objects iff the pseudo-natural transformation* α *from 1.1.28 is invertible.*

Proof. $\Rightarrow$) Let $\mathcal{D} \xrightarrow{\mathsf{G}} \mathcal{C}$ be a pseudo-quasi-inverse for F and $\mathsf{FG} \overset{\alpha}{\Longrightarrow} id_{\mathcal{D}}$, $\mathsf{GF} \overset{\beta}{\Longrightarrow} id_{\mathcal{C}}$ equivalence pseudo-natural transformations as in 1.1.28. Note that for each $\mathsf{C} \in \mathcal{C}$ and $\mathsf{D} \in \mathcal{D}$, α_{D} and β_{C} are equivalences by 1.1.20.

Let's check first that F is pseudo-essentially surjective on objects: Given $\mathsf{D} \in \mathcal{D}$, FGD is equivalent to D via α_{D}.

Let's check now that F is pseudo-fully-faithful: To do that, we need to prove that for each C, $\mathsf{C}' \in \mathcal{C}$, $\mathsf{F} : \mathcal{C}(\mathsf{C}, \mathsf{C}') \longrightarrow \mathcal{D}(\mathsf{FC}, \mathsf{FC}')$ is an equivalence of categories. Recall that this is equivalent to prove that this morphisms are essentially surjective on objects and full and faithful in the 1-dimensional sense [22].

So let $\mathsf{C},\ \mathsf{C}' \in \mathcal{C}$. To check that $\mathsf{F} : \mathcal{C}(\mathsf{C}, \mathsf{C}') \longrightarrow \mathcal{D}(\mathsf{FC}, \mathsf{FC}')$ is full and faithful, we need to prove that F induces a bijection between the set of 2-cells of $\mathcal{C}$ between two fixed morphisms $\mathsf{C} \underset{\mathsf{g}}{\overset{\mathsf{f}}{\rightrightarrows}} \mathsf{C}'$ and the set of 2-cells between Ff and Fg. We are going to see first that this induced function is injective, so suppose that we have $\begin{tikzcd} \mathsf{C} \arrow[r, shift left=1.5ex, "\mathsf{f}"] \arrow[r, shift right=1.5ex, "\mathsf{g}"'] \arrow[r, phantom, "\Downarrow\theta\ \ \Downarrow\eta"] & \mathsf{C}' \end{tikzcd} \in \mathcal{C}$ such that $\mathsf{F}\theta = \mathsf{F}\eta$. Then $\mathsf{GF}\theta = \mathsf{GF}\eta$ and so, since β is pseudo-natural, we have the following equality:



Then, since β_{C} is an equivalence, we have that $\theta = \eta$.

In the same way, one can prove that the corresponding function induced by G is also injective. This is going to be useful to prove surjectivity. So, let $\begin{tikzcd} \mathsf{FC} \arrow[r, shift left=1.5ex, "\mathsf{Ff}"] \arrow[r, shift right=1.5ex, "\mathsf{Fg}"'] \arrow[r, phantom, "\Downarrow\rho"] & \mathsf{FC}' \end{tikzcd}$. Consider the following 2-cell μ:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & f \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "="] & & \\
 & f \arrow[d, equal] & {} & id_C \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\cong"] & \\
 & f \arrow[d, dash, "\beta_f"'] & \beta_C \arrow[d, dash] & {} & \overline{\beta_C} \arrow[d, equal] \\
\mu = & \beta_{C'} \arrow[d, equal] & GFf \arrow[d, dash, "G\rho"] & & \overline{\beta_C} \arrow[d, equal] \\
 & \beta_{C'} \arrow[d, dash, "\beta_g^{-1}"'] & GFg \arrow[d, dash] & & \overline{\beta_C} \arrow[d, equal] \\
 & g \arrow[d, equal] & \beta_C \arrow[dr, dash] & {} \arrow[d, phantom, "\cong"] & \overline{\beta_C} \arrow[dl, dash] \\
 & g \arrow[dr, dash] & {} \arrow[d, phantom, "="] & id_C \arrow[dl, dash] & \\
 & & g & &
\end{tikzcd}$$

where $\overline{\beta_C}$ denotes a quasi-inverse for β_C.

Then, since β is pseudo-natural, we have the following equality:

$$\begin{tikzcd}
\beta_{C'} \arrow[d, equal] & GFf \arrow[d, dash, "GF\mu"] \\
\beta_{C'} & GFg
\end{tikzcd}
=
\begin{tikzcd}
\beta_{C'} \arrow[d, dash, "\beta_f^{-1}"'] & GFf \arrow[d, dash] \\
f \arrow[d, dash, "\mu"] & \beta_C \arrow[d, equal] \\
g \arrow[d, dash, "\beta_g"'] & \beta_C \arrow[d, dash] \\
\beta_{C'} & GFg
\end{tikzcd}
=
\begin{tikzcd}
\beta_{C'} \arrow[d, equal] & GFf \arrow[d, dash, "G\rho"] \\
\beta_{C'} & GFg
\end{tikzcd}$$

And so, since $\beta_{C'}$ is an equivalence, $GF\mu = G\rho$. This implies that $F\mu = \rho$ because of the injectivity of G that we have mentioned before.

Finally, to check that $F : \mathcal{C}(C, C') \longrightarrow \mathcal{D}(FC, FC')$ is essentially surjective on objects, let $FC \xrightarrow{f} FC' \in \mathcal{D}$. Consider $g = \beta_{C'} G f \overline{\beta_C} : C \longrightarrow C' \in \mathcal{C}$. Then, since β is pseudo-natural, we have an invertible 2-cell $GFg \Longrightarrow \overline{\beta_{C'}}g\beta_C \Longrightarrow Gf$. This, plus the fact that $G : \mathcal{D}(FC, FC') \longrightarrow \mathcal{C}(GFC, GFC')$ is full and faithful (this can be seen as we saw the equivalent assertion for F), yields that there is an invertible 2-cell $Fg \Longrightarrow f$ which concludes the proof.

$\Leftarrow$) Given $D \in \mathcal{D}$, since F is pseudo-essentially surjective on objects, there exist $GD \in \mathcal{D}$ and an equivalence $FGD \overset{\alpha_D}{\Longrightarrow} D \in \mathcal{D}$.

Given $D \xrightarrow{f} D' \in \mathcal{D}$, consider $FGD \xrightarrow{\alpha_D} D \xrightarrow{f} D' \xrightarrow{\overline{\alpha_{D'}}} FGD'$. Then, since F is pseudo-fully-faithful, there exist $GD \xrightarrow{Gf} GD' \in \mathcal{C}$ and an invertible 2-cell $FGf \overset{\tilde{\alpha}_f}{\Longrightarrow} \overline{\alpha_{D'}}f\alpha_D$.

Given $\begin{tikzcd} D \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & D' \end{tikzcd} \in \mathcal{D}$, consider $\begin{array}{ccc} & FGf & \\ \diagup & \tilde{\alpha}_f & \diagdown \\ \overline{\alpha_{D'}} & f & \alpha_D \\ \| & \diagdown\theta\diagup & \| \\ \overline{\alpha_{D'}} & g & \alpha_D \\ \diagdown & \tilde{\alpha}_g^{-1} & \diagup \\ & FGg & \end{array}$. Then, since F is pseudo-fully-faithful, there exists a unique 2-cell $\begin{tikzcd} GD \arrow[r, shift left=2, "Gf"] \arrow[r, shift right=2, "Gg"'] \arrow[r, phantom, "\Downarrow G\theta"] & GD' \end{tikzcd} \in \mathcal{C}$ such that

$$\begin{array}{c} FGf \\ \diagdown FG\theta \diagup \\ FGg \end{array} \quad = \quad \begin{array}{ccc} & FGf & \\ \diagup & \tilde{\alpha}_f & \diagdown \\ \overline{\alpha_{D'}} & f & \alpha_D \\ \| & \diagdown\theta\diagup & \| \\ \overline{\alpha_{D'}} & g & \alpha_D \\ \diagdown & \tilde{\alpha}_g^{-1} & \diagup \\ & FGg & \end{array}$$

To construct $id_{GD} \xrightarrow{\alpha^G_D} Gid_D$, consider the following invertible 2-cell $\begin{tikzcd} FGD \arrow[r, shift left=2, "FGid_D"] \arrow[r, shift right=2, "Fid_{GD}"'] \arrow[r, phantom, "\Downarrow\mu"] & FGD \end{tikzcd}$:

$$\mu = \begin{tikzcd}[column sep=small]
 & \mathsf{FG}id_{\mathsf{D}} \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\tilde{\alpha}_{id_{\mathsf{D}}}"] & \\
\overline{\alpha_{\mathsf{D}}} \arrow[dr, dash] & id_{\mathsf{D}} \arrow[d, phantom, "="] & \alpha_{\mathsf{D}} \arrow[dl, dash] \\
 & id_{\mathsf{FGD}} \arrow[d, dash, bend right] \arrow[d, dash, bend left] \arrow[d, phantom, "\alpha^{\mathsf{F}}_{\mathsf{GD}}"] & \\
 & \mathsf{F}id_{\mathsf{GD}} &
\end{tikzcd}$$

Then, since F is pseudo-fully-faithful, there exists a unique invertible 2-cell $\begin{tikzcd} \mathsf{GD} \arrow[r, shift left, "\mathsf{G}id_{\mathsf{D}}"] \arrow[r, shift right, "id_{\mathsf{GD}}"'] \arrow[r, phantom, "\Downarrow \widetilde{\mathsf{G_D}}"] & \mathsf{GD} \end{tikzcd}$ such that $\mathsf{F}\widetilde{\mathsf{G_D}} = \mu$. Take $\alpha^{\mathsf{G}}_{\mathsf{D}} = \widetilde{\mathsf{G_D}}^{-1}$.

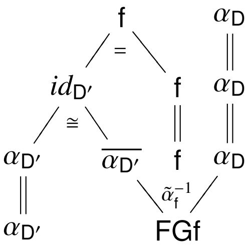
Given $\mathsf{D} \xrightarrow{\mathsf{f}} \mathsf{D}' \xrightarrow{\mathsf{g}} \mathsf{D}'' \in \mathcal{D}$, consider the following invertible 2-cell $\begin{tikzcd} \mathsf{FGD} \arrow[r, shift left, "\mathsf{F(GgGf)}"] \arrow[r, shift right, "\mathsf{FGgf}"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathsf{FGD}'' \end{tikzcd}$:

$$\eta = \begin{tikzcd}[column sep=small]
 & & & & \mathsf{F(GgGf)} \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & & \mathsf{FGg} \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\tilde{\alpha}_{\mathsf{g}}"] & & \alpha^{\mathsf{F}}_{\mathsf{Gf,Gg}} & & \mathsf{FGf} \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\tilde{\alpha}_{\mathsf{g}}"] & \\
 & \overline{\alpha_{\mathsf{D}''}} \arrow[d, dash] & \mathsf{g} & \alpha_{\mathsf{D}'} & \cong & \overline{\alpha_{\mathsf{D}'}} & \mathsf{f} & \alpha_{\mathsf{D}} \arrow[d, dash] \\
 & \overline{\alpha_{\mathsf{D}''}} \arrow[drrr, dash] & & \mathsf{g} & \arrow[d, phantom, "\tilde{\alpha}^{-1}_{\mathsf{gf}}"] & \mathsf{f} & & \alpha_{\mathsf{D}} \arrow[dlll, dash] \\
 & & & & \mathsf{FGgf} & & &
\end{tikzcd}$$

Then, since F is pseudo-fully-faithful, there exists a unique invertible 2-cell $\begin{tikzcd} \mathsf{GD} \arrow[r, shift left, "\mathsf{Gg}"] \arrow[r, shift right, "\mathsf{Ggf}"'] \arrow[r, phantom, "\Downarrow \alpha^{\mathsf{G}}_{\mathsf{f,g}}"] & \mathsf{GD}'' \end{tikzcd}$ such that $\mathsf{F}\alpha^{\mathsf{G}}_{\mathsf{f,g}} = \eta$.

It can be checked that G defined by this data is a pseudo-functor.

Define α_{f} =

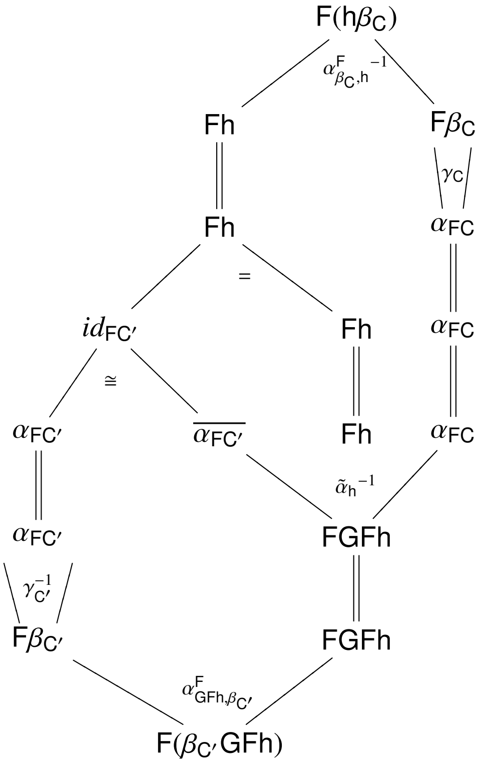


. It can be checked that α is a pseudo-natural transformation.

It only remains to define β: For $\mathsf{C} \in \mathcal{C}$, consider the equivalence $\mathsf{FGFC} \xrightarrow{\alpha_{\mathsf{FC}}} \mathsf{FC}$. Then, since F is pseudo-fully-faithful, there exist an equivalence $\mathsf{GFC} \xrightarrow{\beta_{\mathsf{C}}} \mathsf{C} \in \mathcal{C}$ and an invertible 2-cell $\mathsf{F}\beta_{\mathsf{C}} \overset{\gamma_{\mathsf{C}}}{\Longrightarrow} \alpha_{\mathsf{FC}}$. For $\mathsf{C} \xrightarrow{\mathsf{h}} \mathsf{C'} \in \mathcal{C}$, consider the following 2-cell

$$\begin{tikzcd} \mathsf{FGFC} \arrow[r, shift left=2, "\mathsf{F}(\mathsf{h}\beta_{\mathsf{C}})"] \arrow[r, shift right=2, "\mathsf{F}(\beta_{\mathsf{C'}}\mathsf{GFh})"'] \arrow[r, phantom, "\Downarrow \rho"] & \mathsf{FC'} \end{tikzcd}$$

ρ =



Then, since F is pseudo-fully-faithful, there exists a unique invertible 2-cell $\mathsf{h}\beta_{\mathsf{C}} \overset{\beta_{\mathsf{h}}}{\Longrightarrow} \beta_{\mathsf{C'}}\mathsf{GFh}$ such that $\mathsf{F}\beta_{\mathsf{h}} = \rho$.

It can be checked that β is a pseudo-natural transformation.

The remaining assertion follows immediately from the proof. □

1.1.31 Remark. The previous proof can be easily adapted to the case of 1.1.29. □

1.1.32 Remark. [15, I,4.2.] Evaluation determines a *quasifunctor* $\mathcal{H}om_p(\mathcal{C},\mathcal{D}) \times \mathcal{C} \xrightarrow{ev} \mathcal{D}$ (in the sense of [15, I,4.1.]), in particular, fixing a variable, it is a 2-functor in the other). In the strict case $\mathcal{H}om$, evaluation is actually a 2-bifunctor. In the case of $p\mathcal{H}om_p(\mathcal{C},\mathcal{D})$, it is a pseudo-functor in each variable. □

1.1.33 Remark. [15, I,4.2] Given 2-functors $\mathcal{C}' \xrightarrow{\mathsf{H}_0} \mathcal{C}$ and $\mathcal{D} \xrightarrow{\mathsf{H}_1} \mathcal{D}'$, and $\begin{tikzcd} \mathsf{F} \arrow[r, shift left=1.5ex, "\theta"] \arrow[r, shift right=1.5ex, "\eta"'] \arrow[r, phantom, "\Downarrow \rho"] & \mathsf{G} \end{tikzcd}$ in $\mathcal{H}om_\epsilon(\mathcal{C},\mathcal{D})(\mathsf{F},\mathsf{G})$, the definition

$\mathcal{H}om_\epsilon(\mathsf{H}_0,\mathsf{H}_1)(\begin{tikzcd} \mathsf{F} \arrow[r, shift left=1.5ex, "\theta"] \arrow[r, shift right=1.5ex, "\eta"'] \arrow[r, phantom, "\Downarrow \rho"] & \mathsf{G} \end{tikzcd}) = \begin{tikzcd} \mathsf{H}_1\mathsf{F}\mathsf{H}_0 \arrow[r, shift left=1.5ex, "\mathsf{H}_1\theta\mathsf{H}_0"] \arrow[r, shift right=1.5ex, "\mathsf{H}_1\eta\mathsf{H}_0"'] \arrow[r, phantom, "\Downarrow \mathsf{H}_1\rho\mathsf{H}_0"] & \mathsf{H}_1\mathsf{G}\mathsf{H}_0 \end{tikzcd}$ determines a functor $\mathcal{H}om_\epsilon(\mathcal{C},\mathcal{D})(\mathsf{F},\mathsf{G}) \longrightarrow \mathcal{H}om_\epsilon(\mathcal{C}',\mathcal{D}')(\mathsf{H}_1\mathsf{F}\mathsf{H}_0,\mathsf{H}_1\mathsf{G}\mathsf{H}_0)$, and this assignation is bifunctorial in the variable $(\mathcal{C},\mathcal{D})$ (here $\mathcal{H}om_\epsilon$ denotes either $\mathcal{H}om$ or $\mathcal{H}om_p$). Both constructions $\mathcal{H}om$ and $\mathcal{H}om_p$ determine a bifunctor $2\text{-}\mathcal{CAT}^{op} \times 2\text{-}\mathcal{CAT} \longrightarrow 2\text{-}\mathcal{CAT}$. The same assertion holds for pseudo-functors (see [15, I,4.20]).

If $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, the product 2-category $\mathcal{C} \times \mathcal{D}$ is constructed in the usual way, and this together with the 2-category $\mathcal{H}om(\mathcal{C},\mathcal{D})$ determine a symmetric cartesian closed structure as follows (see [18, chapter 2] or [15, I,2.3.]):

1.1.34 Proposition. *The usual definitions determine an isomorphism of 2-categories :*

$$\mathcal{H}om(\mathcal{C},\ \mathcal{H}om(\mathcal{D},\ \mathcal{A})) \xrightarrow{\cong} \mathcal{H}om(\mathcal{C} \times \mathcal{D},\ \mathcal{A}).$$

Composing with the symmetry $\mathcal{C} \times \mathcal{D} \xrightarrow{\cong} \mathcal{D} \times \mathcal{C}$ *yields an isomorphism:*

$$\mathcal{H}om(\mathcal{C},\ \mathcal{H}om(\mathcal{D},\ \mathcal{A})) \xrightarrow{\cong} \mathcal{H}om(\mathcal{D},\ \mathcal{H}om(\mathcal{C},\ \mathcal{A})).$$

□

1.1.35. Notation. Let $\mathcal{C}$ be a 2-category, $\mathsf{C} \in \mathcal{C}$ and $\begin{tikzcd} \mathsf{D} \arrow[r, shift left=1.5ex, "\mathsf{f}"] \arrow[r, shift right=1.5ex, "\mathsf{g}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{E} \end{tikzcd} \in \mathcal{C}$.

- f_*: $\mathcal{C}(\mathsf{C},\mathsf{D}) \xrightarrow{\mathsf{f}_*} \mathcal{C}(\mathsf{C},\mathsf{E})$, $\mathsf{f}_*(\mathsf{h} \xrightarrow{\beta} \mathsf{h}') = (\mathsf{fh} \xrightarrow{\mathsf{f}\beta} \mathsf{fh}')$.

- f^*: $\mathcal{C}(\mathsf{E},\mathsf{C}) \xrightarrow{\mathsf{f}^*} \mathcal{C}(\mathsf{D},\mathsf{C})$, $\mathsf{f}^*(\mathsf{h} \xrightarrow{\beta} \mathsf{h}') = (\mathsf{hf} \xrightarrow{\beta\mathsf{f}} \mathsf{h}'\mathsf{f})$.

- α_*: $\mathsf{f}_* \overset{\alpha_*}{\Longrightarrow} \mathsf{g}_*$, $(\alpha_*)_\mathsf{h} = \alpha\mathsf{h}$.

- α^*: $\mathsf{f}^* \overset{\alpha^*}{\Longrightarrow} \mathsf{g}^*$ $(\alpha^*)_\mathsf{h} = \mathsf{h}\alpha$.

- $\mathcal{C} \xrightarrow{\mathcal{C}(\mathsf{C},-)} Cat$: $\mathcal{C}(\mathsf{C},-)\big(\mathsf{D} \underset{\mathsf{g}}{\overset{\mathsf{f}}{\underset{\Downarrow\alpha}{\rightrightarrows}}} \mathsf{E}\big) = \big(\mathcal{C}(\mathsf{C},\mathsf{D}) \underset{\mathsf{g}_*}{\overset{\mathsf{f}_*}{\underset{\Downarrow\alpha_*}{\rightrightarrows}}} \mathcal{C}(\mathsf{C},\mathsf{E})\big)$.

- $\mathcal{C}^{op} \xrightarrow{\mathcal{C}(-,\mathsf{C})} Cat$: $\mathcal{C}(-,\mathsf{C})\big(\mathsf{D} \underset{\mathsf{g}}{\overset{\mathsf{f}}{\underset{\Downarrow\alpha}{\rightrightarrows}}} \mathsf{E}\big) = \big(\mathcal{C}(\mathsf{D},\mathsf{C}) \underset{\mathsf{g}^*}{\overset{\mathsf{f}^*}{\underset{\Downarrow\alpha^*}{\rightrightarrows}}} \mathcal{C}(\mathsf{E},\mathsf{C})\big)$.

- We will also denote by f^* the 2-natural transformation from $\mathcal{C}(\mathsf{E},-)$ to $\mathcal{C}(\mathsf{D},-)$ defined by $(\mathsf{f}^*)_{\mathsf{C}} = \mathsf{f}^*$.

- We will also denote by f_* the 2-natural transformation from $\mathcal{C}(-,\mathsf{D})$ to $\mathcal{C}(-,\mathsf{E})$ defined by $(\mathsf{f}_*)_{\mathsf{C}} = \mathsf{f}_*$.

- We will also denote by α^* the modification from f^* to g^* defined by $(\alpha^*)_{\mathsf{C}} = \alpha^*$.

- We will also denote by α_* the modification from f_* to g_* defined by $(\alpha_*)_{\mathsf{C}} = \alpha_*$. □

1.1.36. Yoneda 2-functors. *Given a locally small 2-category $\mathcal{C}$, the* Yoneda 2-functors *are the following (note that each one is the other for the dual 2-category):*

a. $\mathcal{C} \xrightarrow{\mathsf{y}^{(-)}} \mathcal{H}om(\mathcal{C},Cat)^{op}$, $\mathsf{y}^{\mathsf{C}} = \mathcal{C}(\mathsf{C},-)$, $\mathsf{y}^{\mathsf{f}} = \mathsf{f}^*$ $\mathsf{y}^{\alpha} = \alpha^*$.
b. $\mathcal{C} \xrightarrow{\mathsf{y}_{(-)}} \mathcal{H}om(\mathcal{C}^{op},Cat)$, $\mathsf{y}_{\mathsf{C}} = \mathcal{C}(-,\mathsf{C})$, $\mathsf{y}_{\mathsf{f}} = \mathsf{f}_*$ $\mathsf{y}_{\alpha} = \alpha_*$.

Recall the Yoneda Lemma for enriched categories over *Cat*. We consider explicitly only the case *a.* of 1.1.36.

1.1.37 Proposition (Yoneda lemma). *Given a locally small 2-category $\mathcal{C}$, a 2-functor $\mathsf{F} : \mathcal{C} \longrightarrow Cat$ and an object $\mathsf{C} \in \mathcal{C}$, there is an isomorphism of categories, natural in* F.

$$\begin{tikzcd} \mathcal{H}om(\mathcal{C},Cat)(\mathcal{C}(\mathsf{C},-),\mathsf{F}) \arrow[r, "h"] & \mathsf{FC} \\ \theta \xrightarrow{\rho} \eta \arrow[r, mapsto] & \theta_{\mathsf{C}}(id_{\mathsf{C}}) \xrightarrow{(\rho_{\mathsf{C}})_{id_{\mathsf{C}}}} \eta_{\mathsf{C}}(id_{\mathsf{C}}) \end{tikzcd}$$

Proof. The application h has an inverse

$$\begin{tikzcd} \mathsf{FC} \arrow[r, "\ell"] & \mathcal{H}om(\mathcal{C},Cat)(\mathcal{C}(\mathsf{C},-),\mathsf{F}) \\ C \xrightarrow{f} D \arrow[r, mapsto] & \ell C \xrightarrow{\ell f} \ell D \end{tikzcd}$$

where $(\ell C)_{\mathsf{D}}(\mathsf{f} \overset{\alpha}{\Longrightarrow} \mathsf{g}) = \mathsf{Ff}(C) \xrightarrow{(\mathsf{F}\alpha)_C} \mathsf{Fg}(C)$ and $((\ell f)_{\mathsf{D}})_{\mathsf{f}} = \mathsf{Ff}(f)$. □

1.1.38 Corollary. *The Yoneda 2-functors in 1.1.36 are 2-fully-faithful.* □

Beyond the theory of Cat-enriched categories, the lemma also holds for pseudo-functors and pseudo-natural transformations in the following way:

1.1.39 Proposition (Pseudo-Yoneda lemma). *Given a locally small 2-category* $\mathcal{C}$*, a pseudo-functor (in particular, a 2-functor)* $\mathsf{F} : \mathcal{C} \longrightarrow Cat$ *and an object* $\mathsf{C} \in \mathcal{C}$*, there is an equivalence of categories, natural in* F*.*

$$\begin{tikzcd}
p\mathcal{H}om_p(\mathcal{C}, Cat)(\mathcal{C}(\mathsf{C}, -), \mathsf{F}) \arrow[r, "\tilde{h}"] & \mathsf{FC} \\
\theta \xrightarrow{\rho} \eta \arrow[r, mapsto] & \theta_{\mathsf{C}}(id_{\mathsf{C}}) \xrightarrow{(\rho_{\mathsf{C}})_{id_{\mathsf{C}}}} \eta_{\mathsf{C}}(id_{\mathsf{C}})
\end{tikzcd}$$

Furthermore, the quasi-inverse $\tilde{\ell}$ *is a section of* $\tilde{h}$*,* $\tilde{h}\,\tilde{\ell} = id$*.*

Proof. $\tilde{h}$ and $\tilde{\ell}$ are defined as in 1.1.37, but now $\tilde{\ell}$ is only a section quasi-inverse of $\tilde{h}$. The details can be checked by the reader. One can find a guide in [25] for the case of lax functors and bi-categories. We refer to the arguing and the notation there: In our case, the unit η is an isomorphism because F is a pseudo-functor, and the counit ϵ is an isomorphism because α is pseudo-natural and the unitor r is the equality. □

1.1.40 Corollary. *For any locally small 2-category* $\mathcal{C}$*, and* $\mathsf{C} \in \mathcal{C}$*, the inclusion* $\mathcal{H}om(\mathcal{C}, Cat)(\mathcal{C}(\mathsf{C}, -), \mathsf{F}) \xrightarrow{i} \mathcal{H}om_p(\mathcal{C}, Cat)(\mathcal{C}(\mathsf{C}, -), \mathsf{F})$ *has a retraction* α*, natural in* F*,* $\alpha\, i = id$*,* $i\,\alpha \cong id$*, which determines an equivalence of categories.*

Proof. Note that $i = \tilde{\ell}\, h$, then define $\alpha = \ell\, \tilde{h}$. □

1.1.41 Corollary. *The Yoneda 2-functors in 1.1.36 can be considered as 2-functors landing in the* $\mathcal{H}om_p$ *2-functor 2-categories. In this case, they are pseudo-fully-faithful.* □

1.2 Weak limits and colimits

By *weak* we understand any of the several ways universal properties can be relaxed in 2-categories. Note that pseudo-limits and pseudo-colimits (already considered in [2]) require isomorphisms, and have many advantages over bi-limits and bi-colimits, which only require equivalences. Their universal properties are both stronger and more convenient to use. On the other hand, in many situations bi-limits and bi-colimits are unavoidable and seems to be the right concept to consider. The defining universal properties characterize bi-limits up to equivalence and pseudo-limits up to isomorphism.

1.2.1. Notation. We consider pseudo-limits $\underset{\substack{\longleftarrow \\ i\in\mathcal{I}}}{\mathsf{Lim}}\ \mathsf{F}i$, and bi-limits $\underset{\substack{\longleftarrow \\ i\in\mathcal{I}}}{\mathsf{biLim}}\ \mathsf{F}i$, of contravariant pseudo-functors, and their dual concepts, pseudo-colimits $\underset{\substack{\longrightarrow \\ i\in\mathcal{I}}}{\mathsf{Lim}}\ \mathsf{F}i$, and bi-colimits $\underset{\substack{\longrightarrow \\ i\in\mathcal{I}}}{\mathsf{biLim}}\ \mathsf{F}i$, of covariant pseudo-functors.

1.2.2. Pseudo-cone. *Let* $\mathsf{F} : \mathcal{I}^{op} \longrightarrow \mathcal{A}$ *be a pseudo-functor and* A *an object of* $\mathcal{A}$*. A* pseudo-cone *for* F *with vertex* A *is a pseudo-natural transformation from the 2-functor which is constant at* A *to* F*, i.e. it consists in a family of morphisms of* $\mathcal{A}$*,* $\left\{\mathsf{A} \xrightarrow{\theta_i} \mathsf{F}i\right\}_{i\in\mathcal{I}}$ *and a family of invertible 2-cells of* $\mathcal{A}$*,* $\left\{\mathsf{F}u\theta_j \overset{\theta_u}{\Longrightarrow} \theta_i\right\}_{i\xrightarrow{u} j\in\mathcal{I}}$ *satisfying the following equations:*

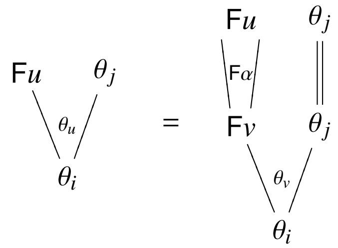
PC0. For each $i \in \mathcal{I}$*,* $\theta_{id_i} \circ \alpha_i^{\mathsf{F}}\theta_i = id_{\theta_i}$*, i.e.*

$$\begin{tikzcd}
id_{\mathsf{F}i} \arrow[d, dash, "\alpha_i^{\mathsf{F}}"] & & \theta_i \arrow[d, equal] & & & & \\
\mathsf{F}id_i \arrow[dr, dash, "\theta_{id_i}"] & & \theta_i \arrow[dl, dash] & = & id_{\mathsf{F}i} \arrow[dr, dash] & \arrow[d, phantom, "="] & \theta_i \arrow[dl, dash] \\
& \theta_i & & & & \theta_i &
\end{tikzcd}$$

PC1. For each $i \xrightarrow{u} j \xrightarrow{v} k \in \mathcal{I}$*,* $\theta_u \circ \mathsf{F}u\theta_v = \theta_{vu} \circ \alpha_{u,v}^{\mathsf{F}}\theta_k$*, i.e.*

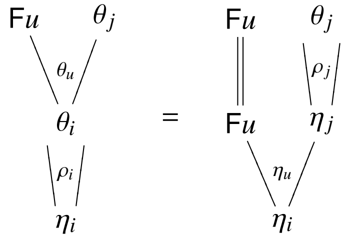
$$\begin{tikzcd}
\mathsf{F}u \arrow[d, equal] & \mathsf{F}v \arrow[d, dash, "\theta_v"] & \theta_k \arrow[dl, dash] & & \mathsf{F}u \arrow[d, dash, "\alpha_{u,v}^{\mathsf{F}}"] & \mathsf{F}v \arrow[dl, dash] & \theta_k \arrow[d, equal] \\
\mathsf{F}u \arrow[d, dash, "\theta_u"] & \theta_j \arrow[dl, dash] & & = & \mathsf{F}(vu) \arrow[d, dash, "\theta_{vu}"] & & \theta_k \arrow[dll, dash] \\
\theta_i & & & & \theta_i & &
\end{tikzcd}$$

PC2. For each $i \overset{u}{\underset{v}{\overset{\Downarrow\, \alpha}{\rightrightarrows}}} j \in \mathcal{I}$*,* $\theta_u = \theta_v \circ \mathsf{F}\alpha\theta_j$*, i.e.*



A morphism of pseudo-cones *between θ and η with the same vertex is a modification, i.e. a family of 2-cells of $\mathcal{A}$, $\left\{\theta_i \overset{\rho_i}{\Longrightarrow} \eta_i\right\}_{i\in\mathcal{I}}$ satisfying the following equation:*

PCM. For each $i \overset{u}{\to} j \in \mathcal{I}$, $\rho_i \circ \theta_u = \eta_u \circ \mathsf{F}u\rho_j$, i.e.



Pseudo-cones form a category $\mathsf{PC}_{\mathcal{A}}(\mathsf{A},\mathsf{F}) = p\mathcal{H}om_p(\mathcal{I}^{op},\mathcal{A})(\mathsf{A},\mathsf{F})$ furnished with a pseudo-cone $\mathsf{PC}_{\mathcal{A}}(\mathsf{A},\mathsf{F}) \longrightarrow \mathcal{A}(\mathsf{A},\ \mathsf{F}i)$, for the pseudo-functor $\mathcal{I}^{op} \xrightarrow{\mathcal{A}(\mathsf{A},\mathsf{F}(-))} \mathcal{CAT}$.

As a particular case, we have the notion of pseudo-cone over a 2-functor.

1.2.3 Remark. Since $p\mathcal{H}om_p(\mathcal{I}^{op},\mathcal{A})$ is a 2-category, it follows:

a. Pseudo-cones determine a 2-bifunctor $(p\mathcal{H}om_p(\mathcal{I}^{op},\mathcal{A}) \times \mathcal{A})^{op} \xrightarrow{\mathsf{PC}_{\mathcal{A}}} \mathcal{CAT}$.

From Remark 1.1.33 it follows in particular:

b. A pseudo-functor $\mathcal{A} \xrightarrow{\mathsf{H}} \mathcal{B}$ induces a functor between the categories of pseudo-cones $\mathsf{PC}_{\mathcal{A}}(\mathsf{F},\mathsf{A}) \xrightarrow{\mathsf{PC}_{\mathsf{H}}} \mathsf{PC}_{\mathcal{B}}(\mathsf{HF},\mathsf{HA})$. □

1.2.4. Pseudo-limit and bi-limit. *The* pseudo-limit *in $\mathcal{A}$ of the pseudo-functor $\mathsf{F} : \mathcal{I}^{op} \longrightarrow \mathcal{A}$ is the universal pseudo-cone, denoted $\left\{\underset{\overleftarrow{i\in\mathcal{I}}}{\mathsf{Lim}}\, \mathsf{F}i \xrightarrow{\pi_i} \mathsf{F}i\right\}_{i\in\mathcal{I}}$,*

$\left\{ \begin{matrix} \mathsf{F}u & & \pi_j \\ & \diagdown^{\pi_u} \diagup & \\ & \pi_i & \end{matrix} \right\}_{i \xrightarrow{u} j \in \mathcal{I}}$*, in the sense that for each* $\mathsf{A} \in \mathcal{A}$*, post-composition with the* π_i*'s is an isomorphism of categories*

$$\mathcal{A}(\mathsf{A}, \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathsf{F}i) \xrightarrow{\pi_*} \mathsf{PC}_{\mathcal{A}}(\mathsf{A}, \mathsf{F}) \tag{1.2.5}$$

Equivalently, there is an isomorphism of categories $\mathcal{A}(\mathsf{A}, \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathsf{F}i) \xrightarrow{\cong} \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathcal{A}(\mathsf{A}, \mathsf{F}i)$ *commuting with pseudo-cones. Remark that there is also an isomorphism of categories* $\mathsf{PC}_{\mathcal{A}}(\mathsf{A}, \mathsf{F}) \xrightarrow{\cong} \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathcal{A}(\mathsf{A}, \mathsf{F}i)$ *(note that these isomorphisms are 2-natural in the variable* A*).*

Requiring π_* *to be an equivalence (which implies that also the other two isomorphisms above are equivalences) defines the notion of* bi-limit *(note that these equivalences are pseudo-natural in the variable* A*). Clearly, pseudo-limits are bi-limits.*

We omit the explicit consideration of the dual concepts. □

As a particular case, we have pseudo-limits and bi-limits (and its dual concepts) of 2-functors.

1.2.6 Remark. As we are going to use the isomorphism (1.2.5) in the following sections (and the equivalence in case of bi-limits), we are going to make the meaning of having them explicit. In the case of pseudo-limits it means that:

- Given a pseudo-cone $\left\{\mathsf{A} \xrightarrow{\theta_i} \mathsf{F}i\right\}_{i \in \mathcal{I}}$, $\left\{\mathsf{F}u\theta_j \overset{\theta_u}{\Longrightarrow} \theta_i\right\}_{i \xrightarrow{u} j \in \mathcal{I}}$, there exists a unique morphism $\mathsf{A} \xrightarrow{\mathsf{f}} \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathsf{F}i \in \mathcal{A}$ such that $\pi_i \mathsf{f} = \theta_i \;\forall\, i \in \mathcal{I}$ and $\pi_u \mathsf{f} = \theta_u \;\forall\, i \xrightarrow{u} j \in \mathcal{I}$.
- And given a morphism of pseudo-cones $\left\{\theta_i \overset{\rho_i}{\Longrightarrow} \eta_i\right\}_{i \in \mathcal{I}}$, there exists a unique 2-cell $\mathsf{A} \underset{\mathsf{g}}{\overset{\mathsf{f}}{\underset{\Downarrow \mu}{\rightrightarrows}}} \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathsf{F}i \in \mathcal{A}$ such that $\pi_i \mu = \rho_i \;\forall\, i \in \mathcal{I}$.

In the case of bi-limits it means that:

- Given a pseudo-cone $\left\{\mathsf{A} \xrightarrow{\theta_i} \mathsf{F}i\right\}_{i \in \mathcal{I}}$, $\left\{\mathsf{F}u\theta_j \overset{\theta_u}{\Longrightarrow} \theta_i\right\}_{i \xrightarrow{u} j \in \mathcal{I}}$, there exist a morphism

$\mathsf{A} \xrightarrow{\mathsf{f}} \underset{i \in \mathcal{I}}{\underleftarrow{\mathsf{biLim}}}\, \mathsf{F}i \in \mathcal{A}$ and invertible 2-cells $\left\{ \mathsf{A} \overset{\pi_i \mathsf{f}}{\underset{\theta_i}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \alpha_i}}}} \mathsf{F}i \right\}_{i \in \mathcal{I}}$ such that

$$\begin{tikzcd}
\mathsf{F}u \arrow[dr, dash] \arrow[rr, phantom, "\pi_u"'] & & \pi_j \arrow[dl, dash] & \mathsf{f} \arrow[d, equal] & & \mathsf{F}u \arrow[d, equal] & \pi_j \arrow[dr, dash] \arrow[rr, phantom, "\alpha_j"'] & & \mathsf{f} \arrow[dl, dash] & \\
& \pi_i \arrow[dr, dash] \arrow[rr, phantom, "\alpha_i"'] & & \mathsf{f} \arrow[dl, dash] & = & \mathsf{F}u \arrow[dr, dash] \arrow[rr, phantom, "\theta_u"'] & & \theta_j \arrow[dl, dash] & & \forall\, i \xrightarrow{u} j \in \mathcal{I}. \\
& & \theta_i & & & & \theta_i & & &
\end{tikzcd}$$

- And given a morphism of pseudo-cones $\left\{ \theta_i \overset{\rho_i}{\Longrightarrow} \eta_i \right\}_{i \in \mathcal{I}}$, there exists a unique 2-cell $\mathsf{A} \overset{\mathsf{f}}{\underset{\mathsf{g}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \mu}}}} \underset{i \in \mathcal{I}}{\underleftarrow{\mathsf{biLim}}}\, \mathsf{F}i \in \mathcal{A}$ such that $\pi_i \mu = \rho_i \ \forall\, i \in \mathcal{I}$. □

It is well known that, in *Cat*-enriched theory, strict limits and colimits are performed pointwise (if they exists in the codomain category). Here we establish this fact for pseudo-limits and pseudo-colimits in both strict and pseudo 2-functor 2-categories. Abusing notation we can say that the formula $(\underset{i \in \mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{F}_i)(\mathsf{C}) = \underset{i \in \mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{F}_i(\mathsf{C})$ holds in both 2-categories. The verification of this is straightforward but requires some care. We also checked that both pseudo-limits and bi-limits (and its dual concepts) are performed pointwise in the 2-category of pseudo-functors.

1.2.7 Proposition. *Let* $\mathcal{I} \xrightarrow{\mathsf{F}} \mathcal{A}$, $i \mapsto \mathsf{F}_i$ *be a pseudo-functor where* $\mathcal{A}$ *is either* $\mathcal{H}om(\mathcal{C}, \mathcal{D})$ *or* $\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$*. For each* $\mathsf{C} \in \mathcal{C}$ *let* $\mathsf{F}_i\mathsf{C} \xrightarrow{\lambda_i^{\mathsf{C}}} \mathsf{LC}$ *be a pseudo-colimit pseudo-cone in* $\mathcal{D}$ *for the pseudo-functor* $\mathcal{I} \xrightarrow{\mathsf{F}} \mathcal{A} \xrightarrow{ev(-,\mathsf{C})} \mathcal{D}$ *(where ev is evaluation, see 1.1.32). Then* LC *is 2-functorial in* C *in such a way that* λ_i^{C} *becomes 2-natural and* $\mathsf{F}_i \xrightarrow{\lambda_i} \mathsf{L}$ *is a pseudo-colimit pseudo-cone in* $\mathcal{A}$ *in both cases. By duality the same assertion holds for pseudo-limits.*

Proof. Given $\mathsf{C} \overset{\mathsf{f}}{\underset{\mathsf{g}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \alpha}}}} \mathsf{D}$ in $\mathcal{C}$, evaluation determines a 2-cell in $p\mathcal{H}om_p(\mathcal{I}, \mathcal{D})$ $\mathsf{FC} \overset{\mathsf{Ff}}{\underset{\mathsf{Fg}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \mathsf{F}\alpha}}}} \mathsf{FD} = ev(\mathsf{F}(\text{-}), \mathsf{C} \overset{\mathsf{f}}{\underset{\mathsf{g}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \alpha}}}} \mathsf{D})$ (note that $(\mathsf{FC})_i = \mathsf{F}_i\mathsf{C}$, and similarly for f, g and α). Then, for each $\mathsf{X} \in \mathcal{D}$, it follows (from Remark 1.2.3 a.) that precomposing with this 2-cell determines a 2-cell (clearly 2-natural in the variable X) in the right leg of the diagram below. Since the rows are isomorphisms, there is a unique 2-cell (also natural in the variable X) in the left leg which makes the diagram commutative.

$$\begin{tikzcd}
\mathcal{D}(\mathsf{LD},\mathsf{X}) \arrow[r, "(\lambda^{\mathsf{D}})^*", "\cong"'] \arrow[d, shift right=4] \arrow[d, shift left=4] \arrow[d, phantom, "\Rightarrow"] & \mathsf{PC}_{\mathcal{D}}(\mathsf{FD},\mathsf{X}) \arrow[d, shift right=4] \arrow[d, shift left=4] \arrow[d, phantom, "\Rightarrow"] \\
\mathcal{D}(\mathsf{LC},\mathsf{X}) \arrow[r, "(\lambda^{\mathsf{C}})^*", "\cong"'] & \mathsf{PC}_{\mathcal{D}}(\mathsf{FC},\mathsf{X})
\end{tikzcd}$$

By the Yoneda lemma (1.1.38), the left leg is given by precomposing with a unique 2-cell in $\mathcal{D}$, that we denote $\mathsf{LC}\ \overset{\mathsf{Lf}}{\underset{\mathsf{Lg}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \mathsf{L}\alpha}}}}\ \mathsf{LD}$. It is clear by uniqueness that this determines a 2-functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{D}$.

Putting $\mathsf{X} = \mathsf{LD}$ in the upper left corner and tracing the identity down the diagram yields the following commutative diagram of pseudo-cones in $\mathcal{D}$:

$$\begin{tikzcd}
\mathsf{F}_i\mathsf{C} \arrow[r, "\lambda_i^{\mathsf{C}}"] \arrow[d, shift right=5, "\mathsf{F}_i\mathsf{f}"'] \arrow[d, shift left=5, "\mathsf{F}_i\mathsf{g}"] \arrow[d, phantom, "\overset{\mathsf{F}_i\alpha}{\Rightarrow}"] & \mathsf{LC} \arrow[d, shift right=5, "\mathsf{Lf}"'] \arrow[d, shift left=5, "\mathsf{Lg}"] \arrow[d, phantom, "\overset{\mathsf{L}\alpha}{\Rightarrow}"] \\
\mathsf{F}_i\mathsf{D} \arrow[r, "\lambda_i^{\mathsf{D}}"] & \mathsf{LD}
\end{tikzcd}$$

This shows that L is furnished with a pseudo-cone for F and that the λ_i are 2-natural. It only remains to check the universal property:

Let $\mathcal{C} \xrightarrow{\mathsf{G}} \mathcal{D}$ be a 2-functor, consider the 2-functor $\mathcal{A} \xrightarrow{ev(-,\mathsf{C})} \mathcal{D}$. We have the following diagram, where the right leg is given by Remark 1.2.3 b.:

$$\begin{tikzcd}
\mathcal{A}(\mathsf{L},\mathsf{G}) \arrow[r, "\lambda^*"] \arrow[d, "ev(-{,}\mathsf{C})"'] & \mathsf{PC}_{\mathcal{A}}(\mathsf{F},\mathsf{G}) \arrow[d, "\mathsf{PC}_{ev(-{,}\mathsf{C})}"] \\
\mathcal{D}(\mathsf{LC},\mathsf{GC}) \arrow[r, "(\lambda^{\mathsf{C}})^*", "\cong"'] & \mathsf{PC}_{\mathcal{D}}(\mathsf{FC},\mathsf{GC})
\end{tikzcd}$$

We prove now that the upper row is an isomorphism. Given $\mathsf{F}_i\ \overset{\theta_i}{\underset{\eta_i}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \rho_i}}}}\ \mathsf{G}$ in $\mathsf{PC}_{\mathcal{A}}(\mathsf{F},\mathsf{G})$, it follows there exists a unique $\mathsf{LC}\ \overset{\tilde{\theta}\mathsf{C}}{\underset{\tilde{\eta}\mathsf{C}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow \tilde{\rho}\mathsf{C}}}}}\ \mathsf{GC}$ in $\mathcal{D}(\mathsf{LC},\mathsf{GC})$ such that $\tilde{\rho}\mathsf{C}\,\lambda_i^{\mathsf{C}} = \rho_i\mathsf{C}$. It is necessary to show that this 2-cell actually lives in $\mathcal{A}$. This has to be checked for any $\mathsf{C} \underset{\mathsf{g}}{\overset{\mathsf{f}}{\underset{\Downarrow\alpha}{\rightrightarrows}}} \mathsf{D}$ in $\mathcal{C}$. In both cases it can be done considering the isomorphism

$$\mathcal{D}(\mathsf{LC},\mathsf{GD}) \xrightarrow[\cong]{(\lambda^{\mathsf{C}})^*} \mathsf{PC}_{\mathcal{D}}(\mathsf{FC},\mathsf{GD}).$$ □

1.2.8 Remark. A similar proof gives the result for $p\mathcal{H}om_p(\mathcal{C},\mathcal{D})$. It also can be checked, by changing the arguments just a little bit that bi-limits and bi-colimits are performed pointwise in $p\mathcal{H}om_p(\mathcal{C},\mathcal{D})$. We leave the details to the reader. □

1.2.9 Definition. *Let $\mathcal{A}$ be a 2-category, $\mathsf{C} \in \mathcal{A}$ and $\mathsf{E} \in \mathcal{C}at$. We define the bi-tensor $\mathsf{E}\tilde{\otimes}_{\mathcal{A}}\mathsf{C}$ as the object of $\mathcal{A}$ such that $\forall\ \mathsf{D} \in \mathcal{A}$, there is an equivalence of categories pseudo-natural in D*

$$\mathcal{C}(\mathsf{E}\tilde{\otimes}_{\mathcal{A}}\mathsf{C},\mathsf{D}) \simeq \mathcal{C}at(\mathsf{E},\mathcal{A}(\mathsf{C},\mathsf{D})).$$

If this equivalences are isomorphisms 2-natural in D, we call it pseudo-tensor and we denote it by $\otimes$ instead of $\tilde{\otimes}$. Pseudo-tensors are in fact the tensors of the 2-category seen as a $\mathcal{C}at$-enriched category.

We omit to make the dual concept explicit.

It follows from the definition and the Yoneda lemmas (1.1.37 and 1.1.39):

1.2.10 Proposition. *For each category E:*

1. $\mathsf{E} \otimes_{\mathcal{A}} (-) : \mathcal{A} \longrightarrow \mathcal{A}$ *is a 2-functor.*
2. $\mathsf{E}\tilde{\otimes}_{\mathcal{A}}(-) : \mathcal{A} \longrightarrow \mathcal{A}$ *is a pseudo-functor.* □

From 1.2.10, it can be verified the pointwise nature of pseudo-tensors and bi-tensors in the 2-functor and pseudo-functor 2-categories:

1.2.11 Proposition.

1. *Let $\mathcal{A}$ be either $\mathcal{H}om(\mathcal{C},\mathcal{D})$ or $\mathcal{H}om_p(\mathcal{C},\mathcal{D})$, $\mathsf{F} \in \mathcal{A}$ and $\mathsf{E} \in \mathcal{C}at$. Then $\mathsf{E} \otimes_{\mathcal{D}} \mathsf{FX}$ is a 2-functor in the variable X and determines a pseudo-tensor in $\mathcal{A}$. That is:*

$$(\mathsf{E} \otimes_{\mathcal{A}} \mathsf{F})(\mathsf{D}) = \mathsf{E} \otimes_{\mathcal{D}} \mathsf{FD}$$

2. *Let $\mathcal{A} = p\mathcal{H}om_p(\mathcal{C},\mathcal{D})$, $\mathsf{F} \in \mathcal{A}$ and $\mathsf{E} \in \mathcal{C}at$. Then $\mathsf{E}\tilde{\otimes}_{\mathcal{D}}\mathsf{FX}$ is a pseudo-functor in the variable X and determines a bi-tensor in $\mathcal{A}$. That is:*

$$(\mathsf{E}\tilde{\otimes}_{\mathcal{A}}\mathsf{F})(\mathsf{D}) = \mathsf{E}\tilde{\otimes}_{\mathcal{D}}\mathsf{FD}$$

□

We make now precise what we do consider as *preservation* properties of a pseudo-functor. We do it in the case of pseudo-colimits, bi-colimits, pseudo-tensors and bi-tensors, but the same clearly applies to dual concepts.

1.2.12 Definition.

1. *Let* $\mathcal{I} \xrightarrow{\mathsf{X}} \mathcal{C} \xrightarrow{\mathsf{H}} \mathcal{A}$ *be any pseudo-functors. We say that* H preserves *a pseudo-colimit (resp. bi-colimit) pseudo-cone* $\mathsf{X}_i \xrightarrow{\lambda_i} \mathsf{L}$ *in* $\mathcal{C}$*, if* $\mathsf{HX}_i \xrightarrow{\mathsf{H}\lambda_i} \mathsf{HL}$ *is a pseudo-colimit (resp. bi-colimit) pseudo-cone in* $\mathcal{A}$*. Equivalently, if the (usual) comparison arrow is an isomorphism (resp. an equivalence) in* $\mathcal{A}$*.*

2. *Let* $\mathcal{C} \xrightarrow{\mathsf{H}} \mathcal{A}$ *be any pseudo-functor. We say that* H *preserves a pseudo-tensor (respectively bi-tensor)* $\mathsf{E} \otimes_{\mathcal{C}} \mathsf{C}$ *in* $\mathcal{C}$ *(respectively* $\mathsf{E}\tilde{\otimes}_{\mathcal{C}}\mathsf{C}$*) if* $\mathsf{H}(\mathsf{E} \otimes_{\mathcal{C}} \mathsf{C})$ *(respectively* $\mathsf{H}(\mathsf{E}\tilde{\otimes}_{\mathcal{C}}\mathsf{C})$*) is the pseudo-tensor (respectively bi-tensor)* $\mathsf{E}\otimes_{\mathcal{A}}\mathsf{HC}$ *(respectively* $\mathsf{E}\tilde{\otimes}_{\mathcal{A}}\mathsf{HC}$*).*

Note that by the very definition, 2-representable 2-functors preserve pseudo-limits and bi-limits. Also, from proposition 1.2.7 it follows:

1.2.13 Proposition. *The Yoneda 2-functors in 1.1.36 preserve pseudo-limits.* ☐

Recall that small pseudo-limits and pseudo-colimits indexed by a category of locally small categories exist and are locally small, as well that the 2-category *Cat* of small categories has all small pseudo-limits and pseudo-colimits (see for example [2], [4], [19]).

1.2.14. We refer to the explicit construction of pseudo-limits of category valued 2-functors, which is similar to the construction of pseudo-limits of category-valued functors in [2, Exposé VI 6.], see full details in [9].

It is also key to our work the explicit construction of 2-filtered pseudo-colimits of category valued 2-functors developed in [11]. We recall this now.

Even though Dubuc and Street work with an alternate definition of 2-filtered 2-category that is more suitable for their calculations, we are going to use the following equivalent one (see [6]) in the following sections:

1.2.15. 2-filtered. [17] *Let* $\mathcal{C}$ *be a non-empty 2-category.* $\mathcal{C}$ *is said to be* 2-filtered *if the following axioms are satisfied:*

F0. Given two objects $\mathsf{C}, \mathsf{D} \in \mathcal{C}$*, there exist an object* $\mathsf{E} \in \mathcal{C}$ *and arrows* $\mathsf{C} \to \mathsf{E}$*,* $\mathsf{D} \to \mathsf{E}$*.*

F1. Given two arrows $\mathsf{C} \overset{\mathsf{f}}{\underset{\mathsf{g}}{\rightrightarrows}} \mathsf{D}$*, there exist an arrow* $\mathsf{D} \xrightarrow{\mathsf{h}} \mathsf{E}$ *and an invertible 2-cell* $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathsf{hf}"] \arrow[r, bend right, "\mathsf{hg}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{E} \end{tikzcd}$.

F2. Given two 2-cells $\begin{tikzcd} \mathsf{C} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha \; \Downarrow\beta"] & \mathsf{D} \end{tikzcd}$ *there exists an arrow* $\mathsf{D} \xrightarrow{\mathsf{h}} \mathsf{E}$ *such that* $\mathsf{h}\alpha = \mathsf{h}\beta$*.*

The dual notion of 2-cofiltered 2-category is given by the duals of axioms F0, F1 and F2.

1.2.16. Construction LL. [11] Let $\mathcal{I}$ be a 2-filtered 2-category and $\mathsf{F} : \mathcal{I} \to \mathcal{C}at$ a 2-functor. We define a category $\mathcal{L}(\mathsf{F})$ in two steps as follows:

First step ([11, Definition 1.5]):

Objects: (C, i) with $C \in \mathsf{F}i$.

Premorphisms: A premorphism between (C, i) and (D, j) is a triple (u, r, v) where $i \xrightarrow{u} k$, $j \xrightarrow{v} k$ in $\mathcal{I}$ and $\mathsf{F}(u)(C) \xrightarrow{r} \mathsf{F}(v)(D)$ in $\mathsf{F}k$.

Homotopies: An homotopy between two premorphisms (u_1, r_1, v_1) and (u_2, r_2, v_2) is a quadruple $(w_1, w_2, \alpha, \beta)$ where $k_1 \xrightarrow{w_1} k$, $k_2 \xrightarrow{w_2} k$ are 1-cells of $\mathcal{I}$ and $w_1 v_1 \xrightarrow{\alpha} w_2 v_2$, $w_1 u_1 \xrightarrow{\beta} w_2 u_2$ are invertible 2-cells of $\mathcal{I}$ such that the following diagram commutes in $\mathsf{F}k$:

$$\begin{tikzcd}
\mathsf{F}(w_1)\mathsf{F}(u_1)(C) = \mathsf{F}(w_1 u_1)(C) \arrow[r, "\mathsf{F}(\beta)_C"] \arrow[d, "\mathsf{F}(w_1)(r_1)"'] & \mathsf{F}(w_2 u_2)(C) = \mathsf{F}(w_2)\mathsf{F}(u_2)(C) \arrow[d, "\mathsf{F}(w_2)(r_2)"] \\
\mathsf{F}(w_1)\mathsf{F}(v_1)(D) = \mathsf{F}(w_1 v_1)(D) \arrow[r, "\mathsf{F}(\alpha)_D"'] & \mathsf{F}(w_2 v_2)(D) = \mathsf{F}(w_2)\mathsf{F}(v_2)(D)
\end{tikzcd}$$

We say that two premorphisms r_1, r_2 are equivalent if there is an homotopy between them. In that case, we write $r_1 \sim r_2$.

Equivalence is indeed an equivalence relation, and premorphisms can be (non uniquely) composed. Up to equivalence, composition is independent of the choice of representatives and of the choice of the composition between them. Since associativity holds and identities exist, the following actually does define a category.

Second step ([11, Definition 1.13]):

Objects: $(C,\ i)$ with $C \in Fi$.

Morphisms: equivalence classes of premorphisms.

Composition: defined by composing representative premorphisms.

1.2.17 Proposition. [11, Theorem 1.19] *Let $\mathcal{I}$ be a 2-filtered 2-category,* $\mathsf{F} : \mathcal{I} \to \mathcal{C}at$ *a 2-functor,* $i \xrightarrow{u} j$ *in* $\mathcal{I}$ *and* $C \xrightarrow{r} D \in \mathsf{F}i$. *The following formulas define a pseudo-cone* $\mathsf{F} \overset{\lambda}{\Longrightarrow} \mathcal{L}(\mathsf{F})$*:*

$$\lambda_i(C) = (C, i) \qquad \lambda_i(r) = [i, r, i] \qquad (\lambda_u)_C = [u, \mathsf{F}u(C), j]$$

which is a pseudo-colimit for the 2-functor F. □

1.3 2-cofinal 2-functors

Propositions 1.3.14 and 1.3.15 are key to prove reindexing properties for 2-pro-objects in section 3. In order to state and prove them, we give the following definition:

1.3.1 Definition. *Let* $\mathsf{F} : \mathcal{I} \to \mathcal{J}$ *be a pseudo-functor (as a particular case,* F *might be a 2-functor) with* $\mathcal{I}$ *a 2-filtered 2-category. We say that* F *is* 2-cofinal *if it has the following properties:*

CF0. Given $j \in \mathcal{J}$, *there exist* $i \in \mathcal{I}$ *and a morphism* $j \to \mathsf{F}i \in \mathcal{J}$.

CF1. Given $j \in \mathcal{J}$, $i \in \mathcal{I}$ *and* $\begin{tikzcd} j \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \mathsf{F}i \end{tikzcd} \in \mathcal{J}$, *there exist* $i \xrightarrow{u} i' \in \mathcal{I}$ *and an invertible 2-cell* $\begin{tikzcd} j \arrow[r, bend left, "\mathsf{F}(u)a"] \arrow[r, bend right, "\mathsf{F}(u)b"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{F}i' \end{tikzcd} \in \mathcal{J}$.

CF2. Given $j \in \mathcal{J}$, $i \in \mathcal{I}$ *and* $\begin{tikzcd} j \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\alpha \ \Downarrow\beta"] & \mathsf{F}i \end{tikzcd} \in \mathcal{J}$, *there exists* $i \xrightarrow{u} i' \in \mathcal{I}$ *such that* $\mathsf{F}(u)\alpha = \mathsf{F}(u)\beta$.

1.3.2 Remark. If $\mathsf{F} : \mathcal{I} \to \mathcal{J}$ is a 2-cofinal pseudo-functor, then $\mathcal{J}$ is also 2-filtered.

Proof. The proof is straightforward. □

1.3.3 Proposition. *Let* $\mathsf{F} : \mathcal{I} \to \mathcal{J}$ *be a 2-cofinal pseudo-functor. Then, for each* $\begin{tikzcd}[row sep=tiny] & \mathsf{F}i \\ j \arrow[ur, "a"] \arrow[dr, "b"'] & \\ & \mathsf{F}i' \end{tikzcd} \in \mathcal{J}$, *there are morphisms* $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u"] & \\ & i'' \\ i' \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$ *and an invertible 2-cell*

$$\begin{tikzcd} & \mathsf{F}i \arrow[dr, "\mathsf{F}u"] & \\ j \arrow[ur, "a"] \arrow[dr, "b"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & \mathsf{F}i'' \\ & \mathsf{F}i' \arrow[ur, "\mathsf{F}v"'] & \end{tikzcd}$$

Proof. It is straightforward from F0 and CF1. ☐

1.3.4 Corollary. *Let* $\mathsf{F} : \mathcal{I} \to \mathcal{J}$ *be a 2-cofinal pseudo-functor. Given* $\mathsf{F}i \overset{a}{\to} \mathsf{F}i' \in \mathcal{J}$, *there are morphisms*

$$\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u"] & \\ & i'' \\ i' \arrow[ur, "v"'] & \end{tikzcd}$$

$\in \mathcal{I}$ *and an invertible 2-cell*

$$\begin{tikzcd} & \mathsf{F}i \arrow[dr, "\mathsf{F}u"] \arrow[dd, phantom, "\Downarrow \alpha"] & \\ \mathsf{F}i \arrow[ur, "id"] \arrow[dr, "a"'] & & \mathsf{F}i'' \\ & \mathsf{F}i' \arrow[ur, "\mathsf{F}v"'] & \end{tikzcd}$$

(i.e. an invertible 2-cell

$$\begin{tikzcd} \mathsf{F}i \arrow[r, shift left=2, "\mathsf{F}u"] \arrow[r, shift right=2, "\mathsf{F}(v)a"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{F}i'' \end{tikzcd}$$

). ☐

1.3.5 Lemma. *Let* $\mathsf{F} : \mathcal{I} \to \mathcal{J}$ *be a 2-cofinal pseudo-functor. Then, given*

$$\begin{tikzcd} i \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & i' \end{tikzcd} \in \mathcal{I}$$

and an invertible 2-cell

$$\begin{tikzcd} \mathsf{F}i \arrow[r, shift left=2, "\mathsf{F}u"] \arrow[r, shift right=2, "\mathsf{F}v"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{F}i' \end{tikzcd} \in \mathcal{J},$$

there exist $i' \overset{w}{\to} i'' \in \mathcal{I}$ *and an invertible 2-cell*

$$\begin{tikzcd} i \arrow[r, shift left=2, "wu"] \arrow[r, shift right=2, "wv"'] \arrow[r, phantom, "\Downarrow \delta"] & i'' \end{tikzcd} \in \mathcal{I}$$

such that $\mathsf{F}(w)\alpha = \mathsf{F}\delta$.

Proof. It is straightforward from F1 and CF2. ☐

The following lemmas are used in the proof of 1.3.9.

1.3.6 Lemma. *Let* $\mathcal{J}$ *be a 2-filtered 2-category and* $\mathsf{G} : \mathcal{J} \to \mathcal{C}at$ *a 2-functor. Let* $(y, j) \overset{[a,r,b]}{\longrightarrow} (y', j')$ *be a morphism in* $\underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{G}j$, a', b', c *morphisms in* $\mathcal{J}$ *and* α, β *invertible 2-cells as in the following diagram:*

$$\begin{tikzcd} j \arrow[dr, "a"'] \arrow[drr, bend left, "a'"] & \Downarrow \alpha & \\ & j'' \arrow[r, "c"] & j''' \\ j' \arrow[ur, "b"] \arrow[urr, bend right, "b'"'] & \Downarrow \beta & \end{tikzcd}$$

Then $[a, r, b] = [a', s, b']$ *where* s *is defined as the composition* $\mathsf{G}(\beta)_{y'}\mathsf{G}(c)(r)\mathsf{G}(\alpha)_y$

Proof. It is straightforward from Lemma 1.18 of [11]. ☐

1.3.7 Corollary. *As a particular case (take* $a' = ca$, $b' = cb$ *and* α *and* β *the corresponding identities), we have that given* (a, r, b) *and* c *as in the previous proposition,* $[a, r, b] = [ca, \mathsf{G}(c)(r), cb]$. ☐

1.3.8 Lemma. *Let $\mathcal{J}$ be a 2-filtered 2-category and $\mathsf{G} : \mathcal{J} \to Cat$ a 2-functor. Let $(y, j) \xrightarrow{[a,r,b]} (y', j')$ be a morphism in $\underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{G}j$ and suppose that we have a configuration as follows:*

$$\begin{tikzcd}
j \arrow[rr, bend left, "a_0"] \arrow[dr, "a"'] \arrow[rr, phantom, "\cong \Downarrow \alpha"] & & j_0 \arrow[dr, "a_2"] & \\
& j'' \arrow[ur, "a_1"'] \arrow[dr, "b_1"] \arrow[rr, phantom, "\cong \Downarrow \gamma"] & & j_2 \\
j' \arrow[ur, "b"] \arrow[rr, bend right, "b_0"'] \arrow[rr, phantom, "\cong \Uparrow \beta"] & & j_1 \arrow[ur, "b_2"'] &
\end{tikzcd}$$

Then $[a, r, b] = [a_2a_0, s, b_2b_0]$ where s is defined as the composition $\mathsf{G}(b_2\beta^{-1})_{y'}\mathsf{G}(\gamma b)_{y'}\mathsf{G}(a_2a_1)(r)\mathsf{G}(a_2\alpha)_y$.

Proof. In 1.3.6, take $a' := a_2a_0$, $b' := b_2b_0$, $c := a_2a_1$, $\alpha := a_2\alpha$ and $\beta := b_2\beta^{-1} \circ \gamma b$. $\square$

1.3.9 Theorem. *Let $\mathsf{F} : \mathcal{I} \to \mathcal{J}$ be a 2-cofinal 2-functor and $\mathsf{G} : \mathcal{J} \to Cat$ a 2-functor. Then the canonical morphism*

$$\underset{i\in\mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{GF}i \xrightarrow{\mathsf{h}} \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{G}j$$

is an equivalence of categories.

Proof. First of all, let's note that $\mathsf{h}(x, i) = (x, \mathsf{F}i)\ \forall\ (x, i) \in \underset{i\in\mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{GF}i$ and $\mathsf{h}([u, r, v]) = [\mathsf{F}(u), r, \mathsf{F}(v)]\ \forall\ (x, i) \xrightarrow{[u,r,v]} (x', i') \in \underset{i\in\mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{GF}i$.

Now, recall that is enough to check that h is essentially surjective on objects and full and faithful ([22]).

- h is essentially surjective on objects: Let $(y, j) \in \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{G}j$. By CF0, $\exists\ j \xrightarrow{a} \mathsf{F}i \in \mathcal{J}$ and clearly $\mathsf{h}(\mathsf{G}(a)(y), i) \cong (y, j)$ in $\underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{G}j$.
- h is full: Let $(x, \mathsf{F}i) \xrightarrow{[a,r,b]} (x', \mathsf{F}i')) \in \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{G}j$ where $\begin{tikzcd}[row sep=tiny] \mathsf{F}i \arrow[dr, "a"] & \\ & j \\ \mathsf{F}i' \arrow[ur, "b"'] & \end{tikzcd} \in \mathcal{J}$ and $\mathsf{G}(a)(x) \xrightarrow{r} \mathsf{G}(b)(x')$. By CF0, $\exists\ j \xrightarrow{c} \mathsf{F}i'' \in \mathcal{J}$. Then, by 1.3.7, $[a, r, b] = [ca, \mathsf{G}(c)(r), cb]$, so without loss of generality, we can suppose $j = \mathsf{F}i''$.

By using 1.3.4 for a and b respectively and by 1.3.3, we have the following configuration:

$$\begin{tikzcd}
\mathsf{F}i \arrow[rr, bend left, "\mathsf{F}u_0"] \arrow[dr, "a"] & \cong\Downarrow\alpha & \mathsf{F}i_0 \arrow[dr, "\mathsf{F}u_2"] & \\
& \mathsf{F}i'' \arrow[ur, "\mathsf{F}u_1"] \arrow[dr, "\mathsf{F}v_1"] & \cong\Downarrow\gamma & \mathsf{F}i_2 \\
\mathsf{F}i' \arrow[ur, "b"'] \arrow[rr, bend right, "\mathsf{F}v_0"'] & \cong\Uparrow\beta & \mathsf{F}i_1 \arrow[ur, "\mathsf{F}v_2"'] &
\end{tikzcd}$$

Now, by 1.3.8, $[a,r,b] = \mathsf{h}([u_2u_0, s, v_2v_0])$ for some s.

- h is faithful: Suppose $(x,i) \underset{[u_1,r_1,v_1]}{\overset{[u_0,r_0,v_0]}{\rightrightarrows}} (x',i') \in \varinjlim_{i\in\mathcal{I}} \mathsf{GF}i$ (where $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u_0"] & \\ & i_0 \\ i' \arrow[ur, "v_0"'] & \end{tikzcd}$ and $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u_1"] & \\ & i_1 \\ i' \arrow[ur, "v_1"'] & \end{tikzcd}$ $\in \mathcal{I}$) such that $\mathsf{h}([u_0,r_0,v_0]) = \mathsf{h}([u_1,r_1,v_1])$ in $\varinjlim_{j\in\mathcal{J}} \mathsf{G}j$. Then, there exists $\begin{tikzcd}[row sep=tiny] \mathsf{F}i_0 \arrow[dr, "a"] & \\ & j \\ \mathsf{F}i_1 \arrow[ur, "b"'] & \end{tikzcd}$ and invertible 2-cells $a\mathsf{F}u_0 \overset{\alpha_0}{\Rightarrow} b\mathsf{F}u_1$, $a\mathsf{F}v_0 \overset{\beta_0}{\Rightarrow} b\mathsf{F}v_1 \in \mathcal{J}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathsf{G}(a)\mathsf{G}(\mathsf{F}(u_0))(x) \arrow[r, "\mathsf{G}(\alpha_0)_x"] \arrow[d, "\mathsf{G}(a)(r_0)"'] & \mathsf{G}(b)\mathsf{G}(\mathsf{F}(u_1))(x) \arrow[d, "\mathsf{G}(b)(r_1)"] \\
\mathsf{G}(a)\mathsf{G}(\mathsf{F}(v_0))(x') \arrow[r, "\mathsf{G}(\beta_0)_{x'}"] & \mathsf{G}(b)\alpha(\mathsf{F}(v_1))(x')
\end{tikzcd} \tag{1.3.10}$$

By CF0, $\exists\ j \overset{c}{\to} \mathsf{F}i_2 \in \mathcal{J}$. By using 1.3.4 two times for $\mathsf{F}i_0 \overset{ca}{\to} \mathsf{F}i_2$ and $\mathsf{F}i_1 \overset{cb}{\to} \mathsf{F}i_2$ respectively, we have $\begin{tikzcd}[row sep=tiny] i_0 \arrow[dr, "u_2"] & \\ & i_3 \\ i_2 \arrow[ur, "v_2"'] & \end{tikzcd}$, $\begin{tikzcd}[row sep=tiny] i_1 \arrow[dr, "u_3"] & \\ & i_4 \\ i_2 \arrow[ur, "v_3"'] & \end{tikzcd}$ $\in \mathcal{I}$ and invertible 2-cells $\mathsf{F}u_2 \overset{\alpha_1}{\Rightarrow} \mathsf{F}(v_2)ca$ and $\mathsf{F}u_3 \overset{\beta_1}{\Rightarrow} \mathsf{F}(v_3)cb \in \mathcal{J}$.

Now, consider $\begin{tikzcd}[row sep=tiny] & \mathsf{F}i_3 \\ \mathsf{F}i_2 \arrow[ur, "\mathsf{F}v_2"] \arrow[dr, "\mathsf{F}v_3"'] & \\ & \mathsf{F}i_4 \end{tikzcd}$ $\in \mathcal{J}$. By 1.3.3, we have $\begin{tikzcd}[row sep=tiny] i_3 \arrow[dr, "u_4"] & \\ & i_5 \\ i_4 \arrow[ur, "v_4"'] & \end{tikzcd}$ $\in \mathcal{I}$ and an invertible 2-cell $\mathsf{F}u_4\mathsf{F}v_2 \overset{\gamma_0}{\Rightarrow} \mathsf{F}v_4\mathsf{F}v_3 \in \mathcal{J}$.

Then we have the following configuration

$$\begin{tikzcd}
\mathsf{F}i \arrow[dr, "\mathsf{F}u_0"] & & & & & \\
& \mathsf{F}i_0 \arrow[ddr, "a"] \arrow[rrrd, bend left, "\mathsf{F}u_2"] & & & & \\
\mathsf{F}i' \arrow[ur, "\mathsf{F}v_0"'] & & & \cong\Downarrow\alpha_1 & \mathsf{F}i_3 \arrow[dr, "\mathsf{F}u_4"] & \\
& & j \arrow[r, "c"] & \mathsf{F}i_2 \arrow[ur, "\mathsf{F}v_2"] \arrow[dr, "\mathsf{F}v_3"'] & \cong\Downarrow\gamma_0 & \mathsf{F}i_5 \\
\mathsf{F}i \arrow[dr, "\mathsf{F}u_1"] & & & \cong\Downarrow\beta_1 & \mathsf{F}i_4 \arrow[ur, "\mathsf{F}v_4"'] & \\
& \mathsf{F}i_1 \arrow[uur, "b"'] \arrow[rrru, bend right, "\mathsf{F}u_3"'] & & & & \\
\mathsf{F}i' \arrow[ur, "\mathsf{F}v_1"'] & & & & &
\end{tikzcd}$$

Then we have $\begin{tikzcd}[row sep=tiny] i_0 \arrow[dr, "u_4u_2"] & \\ & i_5 \\ i_1 \arrow[ur, "v_4u_3"'] & \end{tikzcd} \in \mathcal{I}$ and invertible 2-cells $\mathsf{F}(u_5u_0) \overset{\alpha_2}{\Rightarrow} \mathsf{F}(v_5u_1)$, $\mathsf{F}(u_5v_0) \overset{\beta_2}{\Rightarrow} \mathsf{F}(v_5v_1) \in \mathcal{J}$ given by the following compositions:

$$\begin{tikzcd}
\mathsf{F}u_4 \arrow[d, equal] & & \mathsf{F}u_2 \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\alpha_1"] & & \mathsf{F}u_0 \arrow[d, equal] \\
\mathsf{F}u_4 \arrow[d, dash] \arrow[dr, phantom, "\gamma_0"] & \mathsf{F}v_2 \arrow[d, dash] & c \arrow[d, equal] & a \arrow[d, dash] \arrow[dr, phantom, "\alpha_0"] & \mathsf{F}u_0 \arrow[d, dash] \\
\mathsf{F}v_4 \arrow[d, equal] & \mathsf{F}v_3 \arrow[dr, dash] & c \arrow[d, phantom, "\beta_1{}^{-1}"] & b \arrow[dl, dash] & \mathsf{F}u_1 \arrow[d, equal] \\
\mathsf{F}v_4 & & \mathsf{F}u_3 & & \mathsf{F}u_1
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
\mathsf{F}u_4 \arrow[d, equal] & & \mathsf{F}u_2 \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\alpha_1"] & & \mathsf{F}v_0 \arrow[d, equal] \\
\mathsf{F}u_4 \arrow[d, dash] \arrow[dr, phantom, "\gamma_0"] & \mathsf{F}v_2 \arrow[d, dash] & c \arrow[d, equal] & a \arrow[d, dash] \arrow[dr, phantom, "\beta_0"] & \mathsf{F}v_0 \arrow[d, dash] \\
\mathsf{F}v_4 \arrow[d, equal] & \mathsf{F}v_3 \arrow[dr, dash] & c \arrow[d, phantom, "\beta_1{}^{-1}"] & b \arrow[dl, dash] & \mathsf{F}v_1 \arrow[d, equal] \\
\mathsf{F}v_4 & & \mathsf{F}u_3 & & \mathsf{F}v_1
\end{tikzcd}$$

It can be checked that $(\mathsf{F}u_5, \mathsf{F}(v_5)\alpha_2, \beta_2)$ is an homotopy between $(\mathsf{F}u_0, r_0, \mathsf{F}v_0)$ and $(\mathsf{F}u_1, r_1, \mathsf{F}v_1)$, so without loss of generality, we can suppose from the beginning that the homotopy between $\mathsf{h}([u_0, r_0, v_0])$ and $\mathsf{h}([u_1, r_1, v_1])$ has the form $(\mathsf{F}u, \mathsf{F}v, \alpha_0, \beta_0))$ where $\begin{tikzcd}[row sep=tiny] i_0 \arrow[dr, "u"] & \\ & i_2 \\ i_1 \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$.

Let's apply 1.3.5 to the triplets uu_0, vu_1, α_0 and uv_0, vv_1, β_0 respectively. So, we have $i_2 \overset{w_0}{\to} i_3$, $i_2 \overset{z_0}{\to} i_4$ and invertible 2-cells $\begin{tikzcd} i \arrow[r, bend left, "w_0uu_0"] \arrow[r, bend right, "w_0vu_1"'] \arrow[r, phantom, "\Downarrow\delta_0"] & i_3 \end{tikzcd}$, $\begin{tikzcd} i' \arrow[r, bend left, "z_0uv_0"] \arrow[r, bend right, "z_0vv_1"'] \arrow[r, phantom, "\Downarrow\nu_0"] & i_4 \end{tikzcd} \in \mathcal{I}$ such that $\mathsf{F}(w_0)\alpha_0 = \delta_0$ and $\mathsf{F}(z_0)\beta_0 = \nu_0$.

Then, by 1.3.3, there exist morphisms

$$\begin{tikzcd} i_3 \arrow[dr, "w_1"] & \\ & i_5 \\ i_4 \arrow[ur, "z_1"'] & \end{tikzcd}$$

$\in \mathcal{I}$ and an invertible 2-cell

$$\begin{tikzcd} \mathsf{F}(i_2) \arrow[r, bend left, "\mathsf{F}(w_1w_0)"] \arrow[r, bend right, "\mathsf{F}(z_1z_0)"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{F}(i_5) \end{tikzcd}$$

$\in \mathcal{J}$ and, by 1.3.5, we have $i_5 \overset{w_2}{\to} i''$ and an invertible 2-cell

$$\begin{tikzcd} i_2 \arrow[r, bend left, "w_2w_1w_0"] \arrow[r, bend right, "w_2z_1z_0"'] \arrow[r, phantom, "\Downarrow\delta"] & i'' \end{tikzcd}$$

$\in \mathcal{I}$ such that $\mathsf{F}(w_2)\alpha = \delta$.

It can be checked that $(w_2w_1w_0u, w_2z_1z_0v, (\delta vu_1) \circ (w_2w_1\delta_0), (w_2z_1v_0) \circ (\delta uv_0))$ is an homotopy between (u_0, r_0, v_0) and (u_1, r_1, v_1) which concludes the proof.

□

The purpose of the following is twofold. First to construct a cofinite and filtered poset with a unique initial object $\mathsf{M}(\mathcal{J})$ associated to a 2-filtered 2-category (1.3.11 and 1.3.14). And second, to prove that there is a 2-cofinal 2-functor $\mathsf{M}(\mathcal{J}) \overset{\mathsf{F}}{\to} \mathcal{J}$ (1.3.15). This is a 2-categorical version of a result of Deligne [1, Expose I,8.1.6], see also [12] Mardesick trick. This results are key to prove reindexing properties of 2-pro-objects in section 3.

1.3.11 Definition.

1. *A diagram in a 2-category $\mathcal{J}$ is a functor $\mathsf{C} \xrightarrow{f} \mathcal{J}$ from a category C to the underlying category of $\mathcal{J}$. It is said to be finite if C is a finite category.*

2. *$\mathsf{C} \xrightarrow{f} \mathcal{J}$ is a subdiagram of $\mathsf{D} \xrightarrow{g} \mathcal{J}$ if there is an injective (on objects and on morphisms) functor $\mathsf{C} \xrightarrow{h} \mathsf{D}$ such that $gh = f$. If h is an isomorphism of categories we say that the diagrams are isomorphic.*

1.3.12 Remark. Final objects $c \in \mathsf{C}$, $d \in \mathsf{D}$ correspond under isomorphism of diagrams. That is, $h(c) = d$ (thus $f(c) = g(d)$ in $\mathcal{J}$). □

1.3.13 Definition. *Let $\mathcal{J}$ be a 2-category. We denote by $\mathsf{M}(\mathcal{J})$ the poset of equivalence classes (under isomorphism) of finite diagrams over $\mathcal{J}$ ordered by the subdiagram relation (in the sense of subsets, not injections). We assume that all index categories C in $\mathsf{M}(\mathcal{J})$ have a chosen empty final object denoted $*_{\mathsf{C}}$.*

1.3.14 Proposition. *Let $\mathcal{J}$ be a 2-filtered 2-category. $\mathsf{M}(\mathcal{J})$ is cofinite, filtered and has a unique initial object.*

Proof. Clearly $\mathsf{M}(\mathcal{J})$ is cofinite and has a unique initial object. Let's check that it is filtered: Let $\mathsf{C} \xrightarrow{f} \mathcal{J}$ and $\mathsf{D} \xrightarrow{g} \mathcal{J} \in \mathsf{M}(\mathcal{J})$. Consider the category E disjoint union of C and D, and an additional object $*$ together with one morphism $c \to *$ from each object

$c \in \mathsf{C}$ or $c \in \mathsf{D}$. Clearly $* = *_{\mathsf{E}}$. Since $\mathcal{J}$ is 2-filtered, we have

$$\begin{tikzcd} f(*_{\mathsf{C}}) \arrow[dr, "a"] & \\ & j \\ g(*_{\mathsf{D}}) \arrow[ur, "b"'] & \end{tikzcd}$$

$\in \mathcal{J}$. For each $c \in \mathsf{C}$ (resp. $c \in \mathsf{D}$), $\exists\,!\; c \xrightarrow{r_c} *_{\mathsf{C}}$ (resp. $\exists\,!\, c \xrightarrow{r_c} *_{\mathsf{D}}$). Consider the diagram $\mathsf{E} \xrightarrow{h} \mathcal{J}$ defined by $h = f$ on C, $h = g$ on D, $h(*) = j$, $h(c \to *) = a \circ f(r_c)$ for $c \in \mathsf{C}$, and $h(c \to *) = b \circ g(r_c)$ for $c \in \mathsf{D}$.

It is clear that this diagram is above $\mathsf{C} \xrightarrow{f} \mathcal{J}$ and $\mathsf{D} \xrightarrow{g} \mathcal{J}$. □

1.3.15 Proposition. *Let $\mathcal{J}$ be a 2-filtered 2-category. There is a 2-cofinal 2-functor $\mathsf{M}(\mathcal{J}) \xrightarrow{\mathsf{F}} \mathcal{J}$ where $\mathsf{M}(\mathcal{J})$ is the poset defined in 1.3.11 (we are considering $\mathsf{M}(\mathcal{J})$ as a trivial 2-category).*

Proof. The 2-functor F is defined as follows:

- $\mathsf{F}(\mathsf{C} \xrightarrow{f} \mathcal{J}) = f(*_{\mathsf{C}})$.
- If $\mathsf{C} \xrightarrow{f} \mathcal{J}$ is a subdiagram of $\mathsf{D} \xrightarrow{g} \mathcal{J}$ via $\mathsf{C} \xrightarrow{\mathsf{h}} \mathsf{D}$, $\exists\,!\; h(*_{\mathsf{C}}) \xrightarrow{r} *_{\mathsf{D}}$. Then $\mathsf{F}(\mathsf{C} \xrightarrow{f} \mathcal{J} \;\leq\; \mathsf{D} \xrightarrow{g} \mathcal{J}) = f(*_{\mathsf{C}}) = g(h(*_{\mathsf{C}})) \xrightarrow{g(r)} g(*_{\mathsf{D}})$.
- The 2-cells are the identities so they go to the corresponding identities.

Let's check that F is 2-cofinal:

CF0. Let $j \in \mathcal{J}$. Then $\mathsf{F}(\{*\} \xrightarrow{j} \mathcal{J}) = j$.

CF1. Let $j \in \mathcal{J}$, $\mathsf{C} \xrightarrow{f} \mathcal{J} \in \mathsf{M}(\mathcal{J})$ and

$$\begin{tikzcd} j \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & f(*_{\mathsf{C}}) \end{tikzcd}$$

$\in \mathcal{J}$. Since $\mathcal{J}$ is 2-filtered, we have $f(*_{\mathsf{C}}) \xrightarrow{e} j' \in \mathcal{J}$ and an invertible 2-cell

$$\begin{tikzcd} j \arrow[r, shift left=1.5, "ea"] \arrow[r, shift right=1.5, "eb"'] \arrow[r, phantom, "\Downarrow\alpha"] & j' \end{tikzcd}$$

$\in \mathcal{J}$. Consider the category D disjoint union of C and $\{*\}$, with a morphism from each object of C to $*$. Clearly $* = *_{\mathsf{D}}$. Consider the diagram $\mathsf{D} \xrightarrow{g} \mathcal{J}$ where g is defined by $g = f$ in C, $g(*) = j'$, and $g(c \to *) = e \circ f(r_c)$, where r_c is the unique morphism $c \to *_{\mathsf{C}}$ in C. $\mathsf{C} \xrightarrow{f} \mathcal{J}$ is a subdiagram of $\mathsf{D} \xrightarrow{g} \mathcal{J}$ and $\mathsf{C} \xrightarrow{f} \mathcal{J} \;\leq\; \mathsf{D} \xrightarrow{g} \mathcal{J}$ is sent by F to $f(*_{\mathsf{C}}) \xrightarrow{e} g(*) = j'$.

CF2. Let $j \in \mathcal{J}$, $\mathsf{C} \xrightarrow{f} \mathcal{J}$ in $\mathsf{M}(\mathcal{J})$, and

$$\begin{tikzcd} j \arrow[r, shift left=1.5, "a"] \arrow[r, shift right=1.5, "b"'] \arrow[r, phantom, "\Downarrow\alpha\;\Downarrow\beta"] & f(*_{\mathsf{C}}) \end{tikzcd}$$

in $\mathcal{J}$. Since $\mathcal{J}$ is 2-filtered, we have $f(*_{\mathsf{C}}) \xrightarrow{e} j'$ in $\mathcal{J}$ such that $e\,\alpha = e\beta$. The proof follows in the same way that for CF1.

□

1.4 2-functor associated to a pseudo-functor

In this subsection we establish a result of independent interest and that will be needed to prove that 2-$\mathcal{P}ro_p(C)$ is a closed 2-bmodel 2-category (see 4.1.3) provided that C is (5.2.4). Our construction of $\hat{\mathsf{A}}$ and T are inspired in the constructions for the same purpose that can be found in [15] or [7]. We think the construction made in [15] has a slight mistake because the value of T in 2-cells is not considered. Our case is simpler because we are only interested in the case of filtered categories and we consider pseudo-functors instead of lax-functors. A reference to the validity of this result is made in [21]. We are going to use the results of this subsection only for cofinite filtered posets.

1.4.1 Proposition. *Any category* A *together with a class* B *of pairs of arrows* $A \underset{g}{\overset{f}{\rightrightarrows}} B \in$ A *closed under composition and containing all pairs with* $f = g$ *(note that* B *is a category) determine a 2-category* $\hat{\mathsf{A}}$ *as follows:*

Objects and arrows are those of A *and we add a 2-cell* $A \underset{g}{\overset{f}{\rightrightarrows}} B$ ($\Downarrow \theta_{g,f}$) *for each pair* $A \underset{g}{\overset{f}{\rightrightarrows}} B \in \mathsf{B}$, *subject to the equations*

- $\theta_{f,f} = id_f$
- $\theta_{h,g} \circ \theta_{g,f} = \theta_{h,f}$
- $\theta_{g',f'}\theta_{g,f} = \theta_{g'g,f'f}$ (1.4.2)

Note that $\theta^{-1}_{g,f} = \theta_{f,g}$ *(every 2-cell is invertible).* □

A lax-functor is defined by the same data that a pseudo-functor but without requiring the structural 2-cells to be invertible.

Let $\mathcal{A} \xrightarrow{\mathsf{F}} C$ be a lax-functor. Then, given any tuple of composable arrows, iterating structural 2-cells determines 2-cells from the composition of the values of F to the value of F in the composition. It easily follows from the associativity axiom that all possible iterations are equal. Thus:

1.4.3 Proposition. *Given any tuple* $\mathsf{f} = (\mathsf{f}_1, \mathsf{f}_2, ..., \mathsf{f}_n)$ *of composable arrows* $\mathsf{A}_0 \xrightarrow{\mathsf{f}_1} \mathsf{A}_1 ... \xrightarrow{\mathsf{f}_n} \mathsf{A}_n \in \mathcal{A}$, *there is a well defined (structural) 2-cell* $\mathsf{FA}_0 \underset{\mathsf{F}(\mathsf{f}_n...\mathsf{f}_1)}{\overset{\mathsf{Ff}_n...\mathsf{Ff}_1}{\rightrightarrows}} \mathsf{FA}_n$ ($\Downarrow \theta_{\mathsf{f}}$). □

1.4.4 Proposition. *Let* $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{C}$ *be a pseudo-functor and* $\mathsf{f} = (\mathsf{f}_1, \mathsf{f}_2, ..., \mathsf{f}_n)$, $\mathsf{g} = (\mathsf{g}_1, \mathsf{g}_2, ..., \mathsf{g}_m)$ *with* $\mathsf{A} \xrightarrow{\mathsf{f}_1} \mathsf{A}_1 ... \xrightarrow{\mathsf{f}_n} \mathsf{B}$, $\mathsf{A} \xrightarrow{\mathsf{g}_1} \mathsf{B}_1 ... \xrightarrow{\mathsf{g}_m} \mathsf{B} \in \mathcal{A}$ *be such that* $\mathsf{f}_n ... \mathsf{f}_2 \mathsf{f}_1 = \mathsf{g}_m ... \mathsf{g}_2 \mathsf{g}_1$. *Then there is a well defined 2-cell* $\mathsf{FA} \underset{\mathsf{Fg}_m ... \mathsf{Fg}_1}{\overset{\mathsf{Ff}_n ... \mathsf{Ff}_1}{\underset{\Downarrow \theta_{\mathsf{g},\mathsf{f}}}{\rightrightarrows}}} \mathsf{FB}$. *These 2-cells satisfy equations* (1.4.2).

Proof. Define $\theta_{\mathsf{g},\mathsf{f}} = \theta_{\mathsf{g}}^{-1} \circ \theta_{\mathsf{f}}$. □

1.4.5 Proposition. *Let* A *be a category. There exist a 2-category* $\hat{\mathsf{A}}$ *and a pseudo-functor* $\mathsf{A} \xrightarrow{\mathsf{T}} \hat{\mathsf{A}}$ *such that for each 2-category* $\mathcal{C}$*, there is an isomorphism of 2-categories*

$$\mathcal{H}om_p(\hat{\mathsf{A}}, \mathcal{C}) \xrightarrow{\mathsf{T}^*} p\mathcal{H}om_p(\mathsf{A}, \mathcal{C})$$

$$\begin{tikzcd}
\mathsf{A} \arrow[rr, "\mathsf{T}"] \arrow[dr, "\mathsf{F}"'] & & \hat{\mathsf{A}} \arrow[dl, dashed, "\exists ! \hat{\mathsf{F}}"] \\
& \mathcal{C} &
\end{tikzcd}$$

Furthermore, if A *is filtered, then* T *is 2-cofinal.*

Proof. We define $\hat{\mathsf{A}}$ as follows:

- Objects of $\hat{\mathsf{A}}$ are the objects of A.
- Morphisms of $\hat{\mathsf{A}}$ are tuples of composable morphisms of A. More explicitly, a morphism $A \xrightarrow{\mathsf{f}} B$ is a tuple $\mathsf{f} = (f_1, f_2, ..., f_n)$ with $A \xrightarrow{f_1} A_1 ... \xrightarrow{f_n} B$, $n \geq 0$.
- We consider the empty tuple $\emptyset = (-)$ corresponding to $n = 0$ as an arrow $A \xrightarrow{\emptyset_A} A$ for every object $A \in \mathsf{A}$.
- Composition is given by reverse juxtaposition (with identities $id_A = \emptyset_A$), i.e. $(g_1, g_1, ..., g_m)(f_1, f_2, ..., f_n) = (f_1, f_2, ..., f_n, g_1, g_2, ..., g_m)$.

We then apply the construction of proposition 1.4.1 with $A \overset{\mathsf{f}}{\underset{\mathsf{g}}{\rightrightarrows}} B \in \mathsf{B}$ iff $f_n ... f_1 = g_m ... g_1$, $\mathsf{f} = \emptyset_A$, $\mathsf{g} = (id_A)$ or $\mathsf{f} = (id_A)$, $\mathsf{g} = \emptyset_A$.

We define $\mathsf{A} \xrightarrow{\mathsf{T}} \hat{\mathsf{A}}$ as follows:

- $\mathsf{T}A = A$

- $\mathsf{T}(A \xrightarrow{f} B) = (f)$.

Note that since $A \underset{(id_A)}{\overset{\emptyset_A}{\rightrightarrows}} A \in \mathsf{B}$ and $A \underset{(gf)}{\overset{(f,g)}{\rightrightarrows}} C \in \mathsf{B}$. Then we have invertible 2-cells $\alpha^{\mathsf{T}}_A = \theta_{(id_A),\emptyset_A} : id_A \Longrightarrow (id_A)$ and for $A \xrightarrow{f} B \xrightarrow{g} C$, $\alpha^{\mathsf{T}}_{f,g} = \theta_{(gf),(f,g)} : (g)(f) \Longrightarrow (gf)$. It immediately follows that T given by this data is actually a pseudo-functor.

Let's check that T^* is an isomorphism of 2-categories:

- On objects: Let $\mathsf{F} \in p\mathcal{H}om_p(\mathsf{A}, \mathcal{C})$. We define $\hat{\mathsf{F}}$ as follows:

  $$\hat{\mathsf{F}}A = \mathsf{F}A, \quad \hat{\mathsf{F}}(A \xrightarrow{\mathsf{f}} B) = \mathsf{F}f_n...\mathsf{F}f_1, \quad \hat{\mathsf{F}}\theta_{\mathsf{g},\mathsf{f}} = \theta_{\mathsf{g}}^{-1} \circ \theta_{\mathsf{f}} \quad \text{(see 1.4.3).}$$

  It can be easily checked that this data defines a 2-functor $\hat{\mathsf{F}}$ which is unique such that $\hat{\mathsf{F}}\mathsf{T} = \mathsf{F}$.

- On morphisms: Let $\mathsf{F} \xrightarrow{\mu} \mathsf{G} \in p\mathcal{H}om_p(\mathsf{A}, \mathcal{C})$. We define $\hat{\mu}_A = \mu_A$ and $\hat{\mu}_{\mathsf{f}} = \mu_{f_n}\mathsf{F}f_{n-1}...\mathsf{F}f_1 \circ \mathsf{G}f_n\mu_{f_{n-1}}\mathsf{F}f_{n-1}...\mathsf{F}f_1 \circ ... \circ \mathsf{G}f_n\mathsf{G}f_{n-1}...\mathsf{G}f_2\mu_{f_1}$. It can be easily checked that this data gives the unique pseudo-natural transformation such that $\hat{\mu}\mathsf{T} = \mu$.

- On 2-cells: Let $\mathsf{F} \underset{\mu'}{\overset{\mu}{\underset{\Downarrow \rho}{\rightrightarrows}}} \mathsf{G} \in p\mathcal{H}om_p(\mathsf{A}, \mathcal{C})$. We define $\hat{\rho}_A = \rho_A$. It can be easily checked that this data gives the unique modification such that $\hat{\rho}\mathsf{T} = \rho$.

Finally, let's check that T is 2-cofinal in case A is filtered. CF0 is clear and CF2 is vacuous since A is a category. CF1: Given $A \in \hat{\mathsf{A}}$, $B \in \mathsf{A}$ and two morphisms $A \underset{\mathsf{g}=(g_1,...,g_m)}{\overset{\mathsf{f}=(f_1,...,f_n)}{\rightrightarrows}} B \in \hat{\mathsf{A}}$, since A is filtered, $\exists\ B \xrightarrow{h} C \in \mathsf{A}$ such that $hf_1...f_n = hg_1...g_m$. Then $B \underset{(g_1,...,g_m,h)}{\overset{(f_1,...,f_n,h)}{\rightrightarrows}} C \in \mathsf{B}$ and thus we have an invertible 2-cell $\theta_{(h)\mathsf{g},(h)\mathsf{f}} : (h)\mathsf{f} \Longrightarrow (h)\mathsf{g} \in \hat{\mathsf{A}}$. □

1.4.6 Remark. In particular, from 1.3.2, we have that if A is filtered, then $\hat{\mathsf{A}}$ is 2-filtered. □

1.4.7 Remark. $\widehat{\mathsf{A}^{op}} = \hat{\mathsf{A}}^{op}$. □

1.4.8 Corollary. *Let A be a category. Then the 2-category $\mathcal{H}om_p(\hat{\mathsf{A}}, \mathcal{C})$ has all bi-limits of pseudo-functors and bi-cotensors and they are computed pointwise. The dual assertion also holds.*

Proof. The proof follows immediately from 1.4.5 plus 1.2.7 and 1.2.11. □

1.5 Further results.

A. Joyal pointed to us the notion of *flexible* functors, related with some of our results on pseudo-colimits of representable 2-functors. We recall now this notion since it bears some significance for the concept of 2-pro-object developed in this thesis. Any 2-pro-object determines a 2-functor which is flexible, and some of our results find their right place stated in the context of flexible 2-functors.

1.5.1. Warning. *In this subsection 2-categories are assumed to be locally small, except the illegitimate constructions $\mathcal{H}om$ and $\mathcal{H}om_p$.*

The inclusion $\mathcal{H}om(C, Cat) \xrightarrow{i} \mathcal{H}om_p(C, Cat)$ has a left adjoint $(-)' \dashv i$, we refer the reader to [4]. The 2-natural counit of this adjunction $\mathsf{F}' \overset{\varepsilon_\mathsf{F}}{\Longrightarrow} \mathsf{F}$ is an equivalence in $\mathcal{H}om_p(C, Cat)$, with a section given by the pseudo-natural unit $\mathsf{F} \overset{\eta_\mathsf{F}}{\Longrightarrow} \mathsf{F}'$, $\varepsilon_\mathsf{F}\eta_\mathsf{F} = 1_\mathsf{F}$, $\eta_\mathsf{F}\varepsilon_\mathsf{F} \cong 1_{\mathsf{F}'}$, [4, Proposition 4.1.]

1.5.2 Definition. [4, Proposition 4.2] *A 2-functor $C \xrightarrow{\mathsf{F}} Cat$ is* flexible *if the counit $\mathsf{F}' \overset{\varepsilon_\mathsf{F}}{\Longrightarrow} \mathsf{F}$ has a 2-natural section $\mathsf{F} \overset{\lambda}{\Longrightarrow} \mathsf{F}'$, $\varepsilon_\mathsf{F}\lambda = 1_\mathsf{F}$, $\lambda\varepsilon_\mathsf{F} \cong 1_{\mathsf{F}'}$, which determines an equivalence in $\mathcal{H}om(C, Cat)$.*

We state now a useful characterization of flexible 2-functors F independent of the left adjoint $(-)'$, the proof will appear elsewhere [10].

1.5.3 Proposition. *A 2-functor $C \xrightarrow{\mathsf{F}} Cat$ is flexible $\iff$ for all 2-functors G, the inclusion $\mathcal{H}om(C, Cat)(\mathsf{F}, \mathsf{G}) \xrightarrow{i_\mathsf{G}} \mathcal{H}om_p(C, Cat)(\mathsf{F}, \mathsf{G})$ has a retraction α_G natural in G, $\alpha_\mathsf{G} i_\mathsf{G} = id$, $i_\mathsf{G}\alpha_\mathsf{G} \cong id$, which determines an equivalence of categories.* □

Let $\mathcal{H}om(C, Cat)_f$ and $\mathcal{H}om_p(C, Cat)_f$ be the subcategories whose objects are the flexible 2-functors. We have the following corollaries:

1.5.4 Corollary. *The 2-categories $\mathcal{H}om(C, Cat)_f$ and $\mathcal{H}om_p(C, Cat)_f$ are* pseudoequivalent *in the sense they have the same objects and retract equivalent hom categories.* □

By 1.1.30 the inclusion 2-functor $\mathcal{H}om(C, Cat)_f \longrightarrow \mathcal{H}om_p(C, Cat)_f$ has the identity (on objects) as a retraction pseudo-quasi-inverse, with the equality as the invertible pseudo-natural transformation $\mathsf{F} \xrightarrow{=} \mathsf{F}$ in $\mathcal{H}om_p(C, Cat)_f$.

An important property of flexible 2-functors, false in general, is the following:

1.5.5 Corollary. *Let $\theta : \mathsf{G} \Rightarrow \mathsf{F} \in \mathcal{H}om(C, Cat)_f$ be such that $\theta_\mathsf{C} : \mathsf{GC} \to \mathsf{FC}$ is an equivalence of categories for each $\mathsf{C} \in C$. Then, θ is an equivalence in $\mathcal{H}om(C, Cat)_f$.*

Proof. It is easy to check that there is a pseudo-natural transformation $\eta' : \mathsf{F} \Rightarrow \mathsf{G}$ such that $\theta\eta' \cong \mathsf{F}$ and $\eta'\theta \cong \mathsf{G}$ in $\mathcal{H}om_p(\mathsf{F},\mathsf{F})$ and $\mathcal{H}om_p(\mathsf{G},\mathsf{G})$ respectively. Now, by 1.5.3, there is a 2-natural transformation $\eta : \mathsf{F} \Rightarrow \mathsf{G}$ such that $\eta \cong \eta'$ in $\mathcal{H}om_p(\mathsf{F},\mathsf{G})$. Then, $\theta\eta \cong \mathsf{F}$ and $\eta\theta \cong \mathsf{G}$ in $\mathcal{H}om(\mathsf{F},\mathsf{F})$ and $\mathcal{H}om(\mathsf{G},\mathsf{G})$ respectively and so θ is an equivalence in $\mathcal{H}om(C, Cat)$. □

1.5.6 Proposition. *Small pseudo-colimits of flexible 2-functors are flexible.*

Proof. Let $\mathsf{F} = \underset{i\in\mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathsf{F}i$, where each $\mathsf{F}i$ is flexible, and let G be any other 2-functor. Set $\mathcal{A} = \mathcal{H}om(C, Cat)$ and $\mathcal{A}_p = \mathcal{H}om_p(C, Cat)$. Then:

$$\mathcal{A}(\mathsf{F},\mathsf{G}) \cong \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\, \mathcal{A}(\mathsf{F}i,\ \mathsf{G}) \xrightarrow{i} \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\, \mathcal{A}_p(\mathsf{F}i,\ \mathsf{G}) \cong \mathcal{A}_p(\mathsf{F},\mathsf{G}).$$

The two isomorphisms are given by definition 1.2.4. The arrow i is the pseudo-limit of the equivalences with retraction quasi-inverses corresponding to each $\mathsf{F}i$. It is not difficult to check that i is also such an equivalence. □

It follows also from 1.5.3 that the pseudo-Yoneda lemma (1.1.39, 1.1.40) says that the representable 2-functors are flexible, so we have:

1.5.7 Corollary. *Small pseudo-colimits of representable 2-functors are flexible.* □

Note that 1.5.6 and 1.5.7 hold for any pseudo-colimit that may exist.

Resumen en castellano de la sección 1

En esta sección se fija la notación que se va a usar a lo largo de toda la tesis y se enuncian las definiciones y los resultados básicos de la teoría de 2-categorías necesarios para este trabajo.

La mayoría de estos resultados son conocidos. Para aquellos que no hemos encontrado en la literatura, damos demostraciones detalladas.

En 1.2 probamos que los pseudo-límites (cónicos) en las 2-categorías de 2-funtores $\mathcal{H}om(\mathcal{C}, \mathcal{D})$, $\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$ y $p\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$ (definición 1.1.19) y los bi-límites en $p\mathcal{H}om_p(\mathcal{C}, \mathcal{D})$ se calculan punto a punto.

En 1.3 definimos la noción de pseudo-funtor 2-cofinal entre 2-categorías y probamos ciertas propiedades que usaremos en la sección 3 para demostrar las propiedades de reindexación de 2-pro-objetos. Allí construimos un poset cofinito y filtrante con un único objeto inicial $\mathsf{M}(\mathcal{J})$ asociado a una 2-categoría 2-filtrante (1.3.11 and 1.3.14) y probamos que se tiene un 2-funtor 2-cofinal $\mathsf{M}(\mathcal{J}) \overset{\mathsf{F}}{\to} \mathcal{J}$ (1.3.15).

En 1.4 construimos un 2-funtor asociado via un pseudo-funtor 2-cofinal a un pseudo-funtor dado. Este resultado tiene interés independiente y será usado en la sección 5. Nuestra construcción de $\hat{\mathsf{A}}$ y T fueron inspiradas en las construcciones hechas en [15] o [7].

A. Joyal nos señaló la noción de funtores flexibles, relacionada con algunos resultados de esta tesis acerca de pseudo-colímites de 2-funtores representables. Recordamos en 1.5 esta noción ya que tiene relevancia para el concepto de 2-pro-objeto desarrollado en esta tesis. Todo 2-pro-objeto determina un 2-funtor flexible, y algunos de nuestros resultados tienen su enunciado correcto en el contexto de 2-funtores flexibles.

2 2-Pro-objects

Warning: *In this section 2-categories are assumed to be locally small, except illegitimate constructions as* $\mathcal{H}om$, $\mathcal{H}om_p$ *for large* C *or* $2\text{-}\mathcal{CAT}$.

Some of the main results of this thesis are in this section. In 2.1 we define the 2-category of 2-pro-objects of a 2-category C and establish the basic formula for morphisms and 2-cells of this 2-category. Then, in 2.2, we develop the notion of a morphism and a 2-cell in C *representing* a morphism and a 2-cell in $2\text{-}\mathcal{P}ro(C)$ respectively, inspired in the 1-dimensional notion of an arrow representing a morphism of pro-objects found in [3]. We use this in 2.3 to construct the 2-filtered 2-category that serves as the index 2-category for the 2-cofiltered pseudo-limit of 2-pro-objects. This is also inspired in a construction for the same purpose found in [3]. We were forced to appeal to this complicated construction because the conceptual treatment of this problem found in [1] does not apply in the 2-categorical case. This is because a 2-functor is not the pseudo-colimit indexed by its 2-category of elements of 2-representable 2-functors. Finally, in 2.4 we prove the universal property of $2\text{-}\mathcal{P}ro(C)$.

2.1 Definition of the 2-category of 2-pro-objects

In this subsection we define the 2-category of 2-pro-objects of a fixed 2-category and prove its basic properties. A 2-pro-object over a 2-category C will be a small 2-cofiltered diagram in C and it will be the pseudo-limit of its own diagram in the 2-category $2\text{-}\mathcal{P}ro(C)$.

2.1.1 Definition. *Let* C *be a 2-category. We define the 2-category of 2-pro-objects of* C*, which we denote by* $2\text{-}\mathcal{P}ro(C)$*, as follows:*

1. *Its objects are the 2-functors* $\mathcal{I}^{op} \xrightarrow{\mathsf{X}} C$, $\mathsf{X} = (\mathsf{X}_i, \mathsf{X}_u, \mathsf{X}_\alpha)_{i,u,\alpha\in\mathcal{I}}$, *with* $\mathcal{I}$ *a small 2-filtered 2-category. Often we are going to abuse the notation by saying* $\mathsf{X} = (\mathsf{X}_i)_{i\in\mathcal{I}}$.

2. *If* $\mathsf{X} = (\mathsf{X}_i)_{i\in\mathcal{I}}$ *and* $\mathsf{Y} = (\mathsf{Y}_j)_{j\in\mathcal{J}}$ *are two 2-pro-objects,*

$$2\text{-}\mathcal{P}ro(C)(\mathsf{X},\mathsf{Y}) = \mathcal{H}om(C,\, Cat)^{op}(\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\, C(\mathsf{X}_i, -), \underset{j\in\mathcal{J}}{\underleftarrow{\mathsf{Lim}}}\, C(\mathsf{Y}_j, -))$$

$$= \mathcal{H}om(C,\, Cat)(\underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, C(\mathsf{Y}_j, -), \underset{i\in\mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, C(\mathsf{X}_i, -))$$

Compositions are given by the corresponding compositions in the 2-category $\mathcal{H}om(C,\, Cat)^{op}$ *so it is easy to check that* $2\text{-}\mathcal{P}ro(C)$ *is indeed a 2-category.*

2.1.2. Notation. We are going to use the subindex notation to denote the evaluation of 2-pro-objects.

2.1.3 Proposition. *By definition there is a 2-fully-faithful 2-functor* $2\text{-}\mathcal{P}ro(\mathcal{C}) \xrightarrow{\mathsf{L}} \mathcal{H}om(\mathcal{C},\, \mathcal{C}at)^{op}$. *Thus, there is a contravariant 2-equivalence of 2-categories* $2\text{-}\mathcal{P}ro(\mathcal{C}) \xrightarrow{\mathsf{L}} \mathcal{H}om(\mathcal{C},\, \mathcal{C}at)^{op}_{fc}$*, where* $\mathcal{H}om(\mathcal{C},\, \mathcal{C}at)_{fc}$ *stands for the full subcategory of* $\mathcal{H}om(\mathcal{C},\, \mathcal{C}at)$ *whose objects are those 2-functors which are small 2-filtered pseudo-colimits of representable 2-functors. However, it is important to note that this equivalence is not injective on objects.* □

From Corollary 1.5.7 it follows:

2.1.4 Proposition. *For any 2-pro-object* X*, the corresponding 2-functor* LX *is flexible.* □

2.1.5 Remark. If we use pseudo-natural transformations to define morphisms of 2-pro-objects we obtain a 2-category $2\text{-}\mathcal{P}ro_p(\mathcal{C})$, which anyway, by 2.1.4, results pseudoequivalent (see 1.5.4) to $2\text{-}\mathcal{P}ro(\mathcal{C})$, with the same objects and retract equivalent hom categories. We think our choice of morphisms, which is much more convenient to use, will prove to be the good one for the applications. Nevertheless, this other version is unavoidable to prove that $2\text{-}\mathcal{P}ro(\mathcal{C})$ has a closed 2-bmodel structure (see section 5) due to the nature of the axioms of closed 2-bmodel 2-category where commutativities are non-strict but only holds up to invertible 2-cells.

2.1.6 Remark. The assertion from 2.1.3 also holds replacing $2\text{-}\mathcal{P}ro(\mathcal{C})$ for $2\text{-}\mathcal{P}ro_p(\mathcal{C})$ and $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)$ for $\mathcal{H}om_p(\mathcal{C}, \mathcal{C}at)$. □

Next we establish the basic formula which is essential in many computations in the 2-category $2\text{-}\mathcal{P}ro(\mathcal{C})$:

2.1.7 Proposition. *There is an isomorphism of categories:*

$$2\text{-}\mathcal{P}ro(\mathcal{C})(\mathsf{X}, \mathsf{Y}) \cong \underset{j \in \mathcal{J}}{\underleftarrow{\mathsf{Lim}}}\ \underset{i \in \mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{X}_i, \mathsf{Y}_j) \tag{2.1.7}$$

Proof.

$$\begin{aligned} 2\text{-}\mathcal{P}ro(\mathcal{C})(\mathsf{X}, \mathsf{Y}) \ &= \ \mathcal{H}om(\mathcal{C},\, \mathcal{C}at)(\underset{j \in \mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{Y}_j, -),\ \underset{i \in \mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{X}_i, -)) \ \cong \\ &\underset{j \in \mathcal{J}}{\underleftarrow{\mathsf{Lim}}}\ \mathcal{H}om(\mathcal{C},\, \mathcal{C}at)(\mathcal{C}(\mathsf{Y}_j, -),\ \underset{i \in \mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{X}_i, -)) \ \cong \ \underset{j \in \mathcal{J}}{\underleftarrow{\mathsf{Lim}}}\ \underset{i \in \mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{X}_i, \mathsf{Y}_j) \end{aligned}$$

The first isomorphism is due to 1.2.4 and the second one to 1.1.37. □

2.1.8 Remark. In the case of $2\text{-}\mathcal{P}ro_p(\mathcal{C})$, formula (2.1.7) is an equivalence of categories instead of an isomorphism since the second $\cong$ is only an equivalence (see 1.1.39). □

2.1.9 Corollary. *The 2-category* $2\text{-}\mathcal{P}ro(\mathcal{C})$ *is locally small.* ☐

2.1.10 Corollary. *There is a canonical 2-fully-faithful 2-functor* $\mathcal{C} \xrightarrow{c} 2\text{-}\mathcal{P}ro(\mathcal{C})$ *which sends an object of* $\mathcal{C}$ *into the corresponding 2-pro-object with index 2-category* $\{*\}$*. Since this 2-functor is also injective on objects, we can identify* $\mathcal{C}$ *with a 2-full subcategory of* $2\text{-}\mathcal{P}ro(\mathcal{C})$*.* ☐

Where there is no risk of confusion, we will omit to indicate notationally this identification. By the very definition of $2\text{-}\mathcal{P}ro(\mathcal{C})$ it follows:

2.1.11 Proposition. *If* $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}}$ *is any 2-pro-object of* $\mathcal{C}$*, then* $\mathsf{X} = \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\ \mathsf{X}_i$ *in* $2\text{-}\mathcal{P}ro(\mathcal{C})$*.*

X *is equipped with a pseudo-cone structure,* $\left\{\mathsf{X} \xrightarrow{\pi_i} \mathsf{X}_i\right\}_{i \in \mathcal{I}}$, $\left\{\mathsf{X}_u \, \pi_j \overset{\pi_u}{\Longrightarrow} \pi_i\right\}_{i \overset{u}{\longrightarrow} j \in \mathcal{I}}$.

Under the isomorphism $2\text{-}\mathcal{P}ro(\mathcal{C})(\mathsf{X}, \mathsf{X}_i) \cong \underset{\overrightarrow{k \in \mathcal{I}}}{\mathsf{Lim}}\ \mathcal{C}(\mathsf{X}_k, \mathsf{X}_i)$ (2.1.7)*, projections* $\mathsf{X} \xrightarrow{\pi_i} \mathsf{X}_i$ *correspond to objects* (id_{X_i}, i) *in construction 1.2.16.* ☐

2.1.12 Remark. The previous proposition also holds in $2\text{-}\mathcal{P}ro_p(\mathcal{C})$. ☐

Note that from proposition 2.1.11 it follows:

2.1.13 Remark. Given any two pro-objects X, $\mathsf{Z} \in 2\text{-}\mathcal{P}ro(\mathcal{C})$, there is an isomorphism of categories $2\text{-}\mathcal{P}ro(\mathcal{C})(\mathsf{Z}, \mathsf{X}) \xrightarrow{\cong} \mathsf{PC}_{2\text{-}\mathcal{P}ro(\mathcal{C})}(\mathsf{Z}, c\mathsf{X})$, where $\mathsf{PC}_{2\text{-}\mathcal{P}ro(\mathcal{C})}$ is the category of pseudo-cones for the 2-functor $c\mathsf{X}$ with vertex Z.

It is important to note that when $\underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\ \mathsf{X}_i$ exists in $\mathcal{C}$, this pseudo-limit would not be isomorphic to X in $2\text{-}\mathcal{P}ro(\mathcal{C})$. In general, the functor c does not preserve 2-cofiltered pseudo-limits, in fact, it will preserve them only when $\mathcal{C}$ is already $2\text{-}\mathcal{P}ro(\mathcal{C})$, that is, when c is an equivalence.

2.2 Lemmas to compute with 2-pro-objects.

In this subsection, we establish technical lemmas to be used in computations with 2-pro-objects.

2.2.1 Definition.

1. *Let* $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y}$ *be an arrow in* $2\text{-}\mathcal{P}ro(\mathcal{C})$*. We say that a pair* (r, φ) represents f*, if* φ *is an invertible* 2*-cell* $\mathsf{X} \mathrel{\substack{\xrightarrow{\mathsf{r}\pi_i} \\ \Downarrow\varphi \\ \xrightarrow[\pi_j \mathsf{f}]{}}} \mathsf{Y}_j$*. That is, if we have the following diagram in*

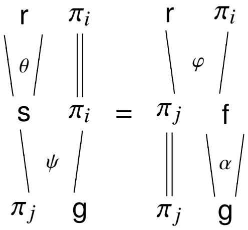
2-$\mathcal{P}ro(\mathcal{C})$:

$$\begin{tikzcd} X \arrow[r, "\pi_i"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong \Downarrow \varphi"] & X_i \arrow[d, "r"] \\ Y \arrow[r, "\pi_j"'] & Y_j \end{tikzcd}$$

2. *Let* $X \underset{g}{\overset{f}{\substack{\longrightarrow \\ \Downarrow\alpha \\ \longrightarrow}}} Y \in$ *2-$\mathcal{P}ro(\mathcal{C})$ and* $X_i \underset{s}{\overset{r}{\substack{\longrightarrow \\ \Downarrow\theta \\ \longrightarrow}}} Y_j \in \mathcal{C}$ *as in the following diagram:*

$$\begin{tikzcd} X \arrow[rr, "\pi_i"] \arrow[d, shift right=3, "f"'] \arrow[d, shift left=3, "g"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & & X_i \arrow[d, shift right=3, "r"'] \arrow[d, shift left=3, "s"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \\ Y \arrow[rr, "\pi_j"'] & & Y_j \end{tikzcd}$$

We say that $(\theta, r, \varphi, s, \psi)$ represents α *if* (r, φ) *represents* f, (s, ψ) *represents* g, *and the following equality holds in 2-$\mathcal{P}ro(\mathcal{C})$:*



That is, $\theta\pi_i = \pi_j\alpha$ *"modulo" a pair of invertible 2-cells* φ, ψ.

2.2.2 Remark. Same definitions may be given in 2-$\mathcal{P}ro_p(\mathcal{C})$. □

2.2.3 Proposition. *Let* $X = (X_i)_{i \in \mathcal{I}}$ *and* $Y = (Y_j)_{j \in \mathcal{J}}$ *be any two 2-pro-objects.*

1. *Let* $X \xrightarrow{f} Y \in$ *2-$\mathcal{P}ro(\mathcal{C})$. Then, for any* $j \in \mathcal{J}$ *there exist* $i \in \mathcal{I}$ *and* $X_i \xrightarrow{r} Y_j \in \mathcal{C}$, *such that* (r, id) *represents* f.

2. *Let* $X \underset{g}{\overset{f}{\substack{\longrightarrow \\ \Downarrow\alpha \\ \longrightarrow}}} Y \in$ *2-$\mathcal{P}ro(\mathcal{C})$. Then, for any* $j \in \mathcal{J}$ *there exist* $i \in \mathcal{I}$, $X_i \underset{s}{\overset{r}{\substack{\longrightarrow \\ \Downarrow\theta \\ \longrightarrow}}} Y_j \in \mathcal{C}$, *and appropriate invertible 2-cells* φ *and* ψ *such that* $(\theta, r, \varphi, s, \psi)$ *represents* α.

   Observe that in case α *is invertible, one can choose a representative with an invertible* θ.

Proof. Consider $X \underset{\pi_j g}{\overset{\pi_j f}{\underset{\Downarrow \pi_j\alpha}{\rightrightarrows}}} Y_j$ and use formula 2.1.7 plus the constructions of pseudo-limits (1.2.14) and 2-filtered pseudo-colimits of categories (1.2.16). ☐

From the previous proposition plus the pseudo-equivalence of 2.1.5, it follows:

2.2.4 Proposition. *Let* $X = (X_i)_{i\in\mathcal{I}}$ *and* $Y = (Y_j)_{j\in\mathcal{J}}$ *be any two 2-pro-objects.*

1. *Let* $X \xrightarrow{f} Y \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$*. Then, for any* $j \in \mathcal{J}$ *there exist* $i \in \mathcal{I}$, $X_i \xrightarrow{r} Y_j \in \mathcal{C}$ *and an invertible 2-cell* φ*, such that* (r, φ) *represents* f.

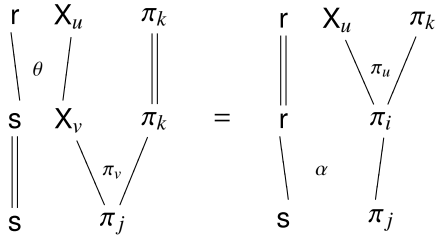
2. *Let* $X \underset{g}{\overset{f}{\underset{\Downarrow \alpha}{\rightrightarrows}}} Y \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$*. Then, for any* $j \in \mathcal{J}$ *there exist* $i \in \mathcal{I}$,

   $X_i \underset{s}{\overset{r}{\underset{\Downarrow \theta}{\rightrightarrows}}} Y_j \in \mathcal{C}$*, and appropriate invertible 2-cells* φ *and* ψ *such that* $(\theta, r, \varphi, s, \psi)$ *represents* α.

☐

2.2.5 Lemma. *Let* $X = (X_i)_{i\in\mathcal{I}}$ *be a 2-pro-object, let* $X_i \xrightarrow{r} C$, $X_j \xrightarrow{s} C \in \mathcal{C}$*, and* $X \underset{s\pi_j}{\overset{r\pi_i}{\underset{\Downarrow \alpha}{\rightrightarrows}}} C \in 2\text{-}\mathcal{P}ro(\mathcal{C})$*. Then,* $\exists$ $\begin{smallmatrix} i \xrightarrow{u} \\ j \xrightarrow[v]{} \end{smallmatrix} k \in \mathcal{I}$ *and* $X_k \underset{sX_v}{\overset{rX_u}{\underset{\Downarrow \theta}{\rightrightarrows}}} C \in \mathcal{C}$ *such that* $\alpha \circ r\pi_u = s\pi_v \circ \theta\pi_k$ *in* $2\text{-}\mathcal{P}ro(\mathcal{C})$:

$$\begin{tikzcd}
& X_k \arrow[r, "X_u"] \arrow[dd, "X_v"'] & X_i \arrow[dd, "r"] \\
X \arrow[ur, "\pi_k"] \arrow[dr, "\pi_j"'] \arrow[r, phantom, "\Downarrow \pi_v"] & {} \arrow[r, phantom, "\Downarrow \theta"] & {} \\
& X_j \arrow[r, "s"'] & C
\end{tikzcd}
=
\begin{tikzcd}
& X_k \arrow[dl, bend right, "\pi_k"'] \arrow[d, "X_u"] \\
X \arrow[ur, phantom, "\overset{\pi_u}{\Rightarrow}"] \arrow[r, "\pi_i"] \arrow[d, "\pi_j"'] \arrow[dr, phantom, "\Downarrow \alpha"] & X_i \arrow[d, "r"] \\
X_j \arrow[r, "s"'] & C
\end{tikzcd}
\qquad i.e.$$



Observe that in case α *is invertible, one can choose* θ *to be invertible.*

Proof. By formula 2.1.7 and the construction of 2-filtered pseudo-colimits (1.2.16), α is given by $(\mathsf{r}, i) \xrightarrow{[u,\theta,v]} (\mathsf{s}, j) \in \underset{\overrightarrow{i\in\mathcal{I}}}{\mathrm{Lim}}\; \mathcal{C}(\mathsf{X}_i, \mathsf{C})$. Thus, $\exists$ $\begin{tikzcd}[row sep=small] i \arrow[dr, "u"] & \\ & k \\ j \arrow[ur, "v"'] & \end{tikzcd}$ $\in \mathcal{I}$ and $\begin{tikzcd} \mathsf{X}_k \arrow[r, shift left=1.5, "\mathsf{rX}_u"] \arrow[r, shift right=1.5, "\mathsf{sX}_v"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathsf{C} \end{tikzcd} \in \mathcal{C}$ such that $\alpha \circ \mathsf{r}\pi_u = \mathsf{s}\pi_v \circ \theta\pi_k$, as we wanted to prove.

□

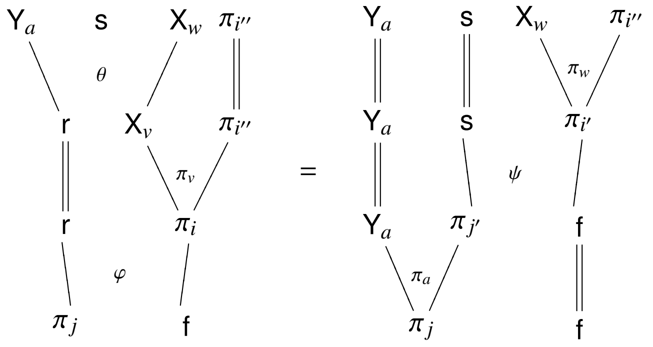
The following is an immediate consequence of [11, Lemma 2.2.]

2.2.6 Remark. If $i = j$, then one can choose $u = v$. □

2.2.7 Remark. From the previous proposition plus the pseudo-equivalence of 2.1.5, it follows that the previous lemma also holds in 2-$\mathcal{P}ro_p(\mathcal{C})$, and so also 2.2.8 and 2.2.16. □

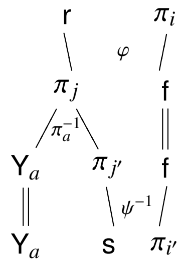
The following two lemmas will be used to prove reindexing properties of 2-pro-objects in section 3 and will be also needed in section 5:

2.2.8 Lemma. *Let* $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y}$ *be a morphism in* 2-$\mathcal{P}ro(\mathcal{C})$, $\mathsf{X}_i \xrightarrow{\mathsf{r}} \mathsf{Y}_j$, $\mathsf{X}_{i'} \xrightarrow{\mathsf{s}} \mathsf{Y}_{j'} \in \mathcal{C}$ *and* φ, ψ *invertible 2-cells in* $\mathcal{C}$ *such that* (r, φ) *and* (s, ψ) *both represent* f *and there are morphisms* $i \xrightarrow{u} i' \in \mathcal{I}$, $j \xrightarrow{a} j' \in \mathcal{J}$. *Then there are morphisms* $\begin{tikzcd}[row sep=small] i \arrow[dr, "v"] & \\ & i'' \\ i' \arrow[ur, "w"'] & \end{tikzcd}$ $\in \mathcal{I}$ *and an invertible 2-cell* $\begin{array}{c} \mathsf{Y}_a\ \mathsf{s}\quad \mathsf{X}_w \\ \backslash\ \theta\ / \\ \mathsf{r}\ \ \mathsf{X}_v \end{array}$ $\in \mathcal{C}$ *such that* $\varphi \circ \mathsf{r}\pi_v \circ \theta\pi_{i''} = \pi_a\mathsf{f} \circ \mathsf{Y}_a\psi \circ \mathsf{Y}_a\mathsf{s}\pi_w$, *i.e.*



Proof. In 2.2.5, take $\mathsf{X} := \mathsf{X}$, $\mathsf{X}_i := \mathsf{X}_i$, $\mathsf{X}_j := \mathsf{X}_{i'}$, $\mathsf{C} := \mathsf{Y}_j$, $\mathsf{r} := \mathsf{r}$, $\mathsf{r} := \mathsf{Y}_a\mathsf{s}$ and

$\alpha :=$



. □

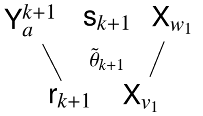
2.2.9 Lemma. *Let* $\left\{\mathsf{X} \xrightarrow{\mathsf{f}_l} \mathsf{Y}^l\right\}_{l=1,\dots,k}$ *be a finite family of morphisms in* 2-$\mathcal{P}ro(\mathcal{C})$ *with* X *indexed by* $\mathcal{I}$ *and for every* $l = 1,\dots,k$, Y^l *indexed by* $\mathcal{J}$. *Consider* $\left\{\mathsf{X}_i \xrightarrow{\mathsf{r}_l} \mathsf{Y}^l_j\right\}_{l=1,\dots,k}$, $\left\{\mathsf{X}_{i'} \xrightarrow{\mathsf{s}_l} \mathsf{Y}^l_{j'}\right\}_{l=1,\dots,k}$ $\in \mathcal{C}$ *and invertible 2-cells* $\{\varphi_l\}_{l=1,\dots k}$, $\{\psi_l\}_{l=1,\dots k}$ *in* $\mathcal{C}$ *such that* $(\mathsf{r}_l, \varphi_l)$ *and* (s_l, ψ_l) *both represent* f_l $\forall\ l = 1,\dots,k$ *and there are morphisms* $i \xrightarrow{u} i' \in \mathcal{I}$, $j \xrightarrow{a} j' \in \mathcal{J}$. *Then there are morphisms* $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "v"] & \\ & i'' \\ i' \arrow[ur, "w"'] & \end{tikzcd}$ $\in \mathcal{I}$ *and invertible 2-cells* $\begin{array}{ccc} \mathsf{Y}^l_a & \mathsf{s}_l & \mathsf{X}_w \\ \backslash & \theta_l & / \\ \mathsf{r}_l & \mathsf{X}_v & \end{array}$ $\in \mathcal{C}$ $\forall\ l = 1,\dots k$ *such that for each* $l = 1,\dots k$, $\varphi_l \circ \mathsf{r}_l\pi_v \circ \theta_l\pi_{i''} = \pi_a\mathsf{f} \circ \mathsf{Y}^l_a\psi_l \circ \mathsf{Y}^l_a\mathsf{s}_l\pi_w$, *i.e.*

$$
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{Y}^l_a \arrow[dd, dash] & \mathsf{s}_l & \mathsf{X}_w \arrow[dd, dash] & & \pi_{i''} \arrow[dd, equal] \\
& \theta_l & & & \\
\mathsf{r}_l \arrow[dd, equal] & & \mathsf{X}_v \arrow[dr, dash] & & \pi_{i''} \arrow[dl, dash] \\
& & & \pi_v \arrow[d, phantom] & \\
\mathsf{r}_l \arrow[dd, dash] & & & \pi_i \arrow[dd, dash] & \\
& \varphi_l & & & \\
\pi_j & & & \mathsf{f}_l &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{Y}^l_a \arrow[dd, equal] & & \mathsf{s}_l \arrow[dd, equal] & \mathsf{X}_w \arrow[ddr, dash] & & \pi_{i''} \arrow[ddl, dash] \\
& & & & \pi_w & \\
\mathsf{Y}^l_a \arrow[dd, equal] & & \mathsf{s}_l \arrow[dd, dash] & & \pi_{i'} \arrow[dd, dash] & \\
& & & \psi_l & & \\
\mathsf{Y}^l_a \arrow[ddr, dash] & & \pi_{j'} \arrow[ddl, dash] & & \mathsf{f}_l \arrow[dd, equal] & \\
& \pi_a & & & & \\
& \pi_j & & & \mathsf{f}_l &
\end{tikzcd}
$$

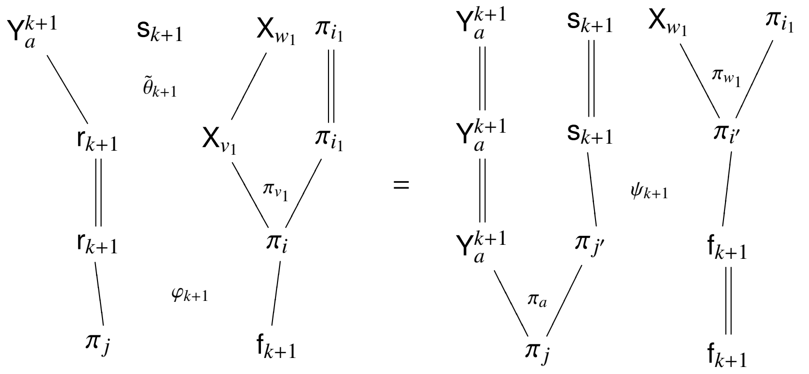
Proof. We are going to proceed by induction in k. For $k = 1$, use 2.2.8.
$k \Rightarrow k+1$: by inductive hypothesis, $\exists$ $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "v_0"] & \\ & i_0 \\ i' \arrow[ur, "w_0"'] & \end{tikzcd}$ $\in \mathcal{I}$ and, for each $l = 1,\dots k$, an invertible 2-cell $\begin{array}{ccc} \mathsf{Y}^l_a & \mathsf{s}_l & \mathsf{X}_{w_0} \\ \backslash & \tilde{\theta}_l & / \\ \mathsf{r}_l & \mathsf{X}_{v_0} & \end{array}$ $\in \mathcal{C}$ such that

$$
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{Y}^l_a \arrow[dd, dash] & \mathsf{s}_l & \mathsf{X}_{w_0} \arrow[dd, dash] & & \pi_{i_0} \arrow[dd, equal] \\
& \tilde{\theta}_l & & & \\
\mathsf{r}_l \arrow[dd, equal] & & \mathsf{X}_{v_0} \arrow[dr, dash] & & \pi_{i_0} \arrow[dl, dash] \\
& & & \pi_{v_0} \arrow[d, phantom] & \\
\mathsf{r}_l \arrow[dd, dash] & & & \pi_i \arrow[dd, dash] & \\
& \varphi_l & & & \\
\pi_j & & & \mathsf{f}_l &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{Y}^l_a \arrow[dd, equal] & & \mathsf{s}_l \arrow[dd, equal] & \mathsf{X}_{w_0} \arrow[ddr, dash] & & \pi_{i_0} \arrow[ddl, dash] \\
& & & & \pi_{w_0} & \\
\mathsf{Y}^l_a \arrow[dd, equal] & & \mathsf{s}_l \arrow[dd, dash] & & \pi_{i'} \arrow[dd, dash] & \\
& & & \psi_l & & \\
\mathsf{Y}^l_a \arrow[ddr, dash] & & \pi_{j'} \arrow[ddl, dash] & & \mathsf{f}_l \arrow[dd, equal] & \\
& \pi_a & & & & \\
& \pi_j & & & \mathsf{f}_l &
\end{tikzcd}
\tag{2.2.10}
$$

Also, $\exists$ $\begin{tikzcd} i \arrow[dr, "v_1"] & \\ & i_1 \\ i' \arrow[ur, "w_1"'] & \end{tikzcd}$ $\in \mathcal{I}$ and an invertible 2-cell



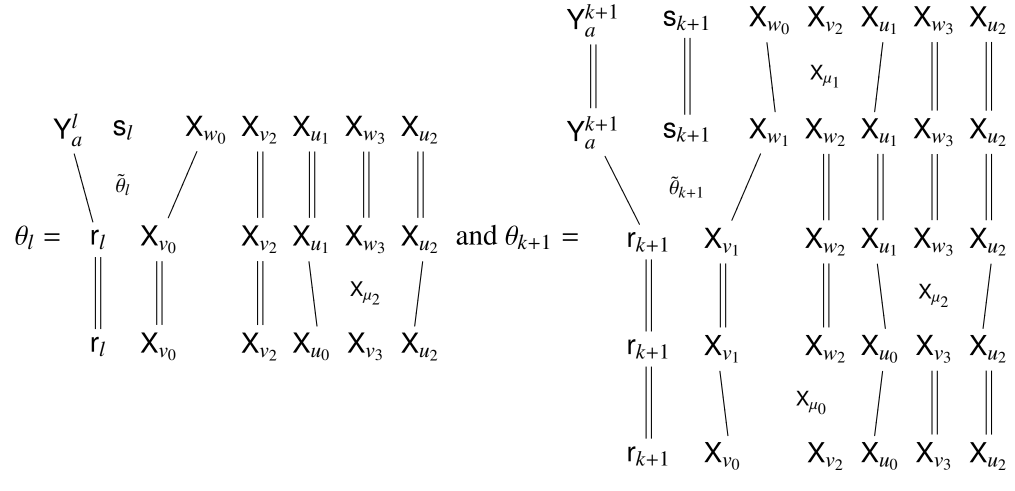
$\in \mathcal{C}$ such that



(2.2.11)

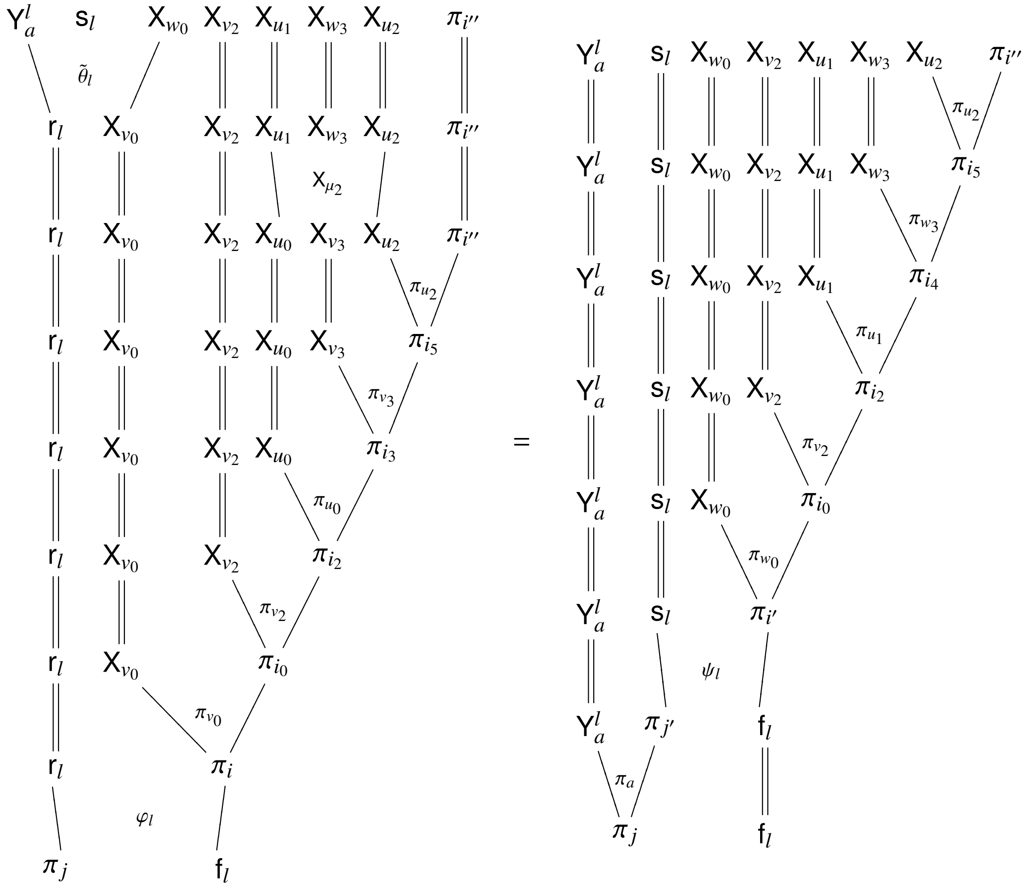
Since $\mathcal{I}$ is 2-filtered, $\exists$ $\begin{tikzcd} i_0 \arrow[dr, "v_2"] & \\ & i_2 \\ i_1 \arrow[ur, "w_2"'] & \end{tikzcd}$, morphisms $i_2 \xrightarrow{u_0} i_3$, $i_2 \xrightarrow{u_1} i_4$ and invertible 2-cells $\begin{tikzcd} i \arrow[r, shift left=2, "w_2v_1"] \arrow[r, shift right=2, "u_0v_2v_0"'] \arrow[r, phantom, "\Downarrow\mu_0"] & i_3 \end{tikzcd}$, $\begin{tikzcd} i' \arrow[r, shift left=2, "v_2w_0"] \arrow[r, shift right=2, "u_1w_2w_1"'] \arrow[r, phantom, "\Downarrow\mu_1"] & i_4 \end{tikzcd}$ $\in \mathcal{I}$, $\begin{tikzcd} i_3 \arrow[dr, "v_3"] & \\ & i_5 \\ i_4 \arrow[ur, "w_3"'] & \end{tikzcd}$, a morphism $i_5 \xrightarrow{u_2} i''$ and an invertible 2-cell $\begin{tikzcd} i_2 \arrow[r, shift left=2, "w_3u_1"] \arrow[r, shift right=2, "u_2v_3u_0"'] \arrow[r, phantom, "\Downarrow\mu_2"] & i'' \end{tikzcd}$ $\in \mathcal{I}$.

Consider $v = u_2v_3u_0v_2v_0$ and $w = u_2w_3u_1v_2w_0$, for each $l = 1, .., k$



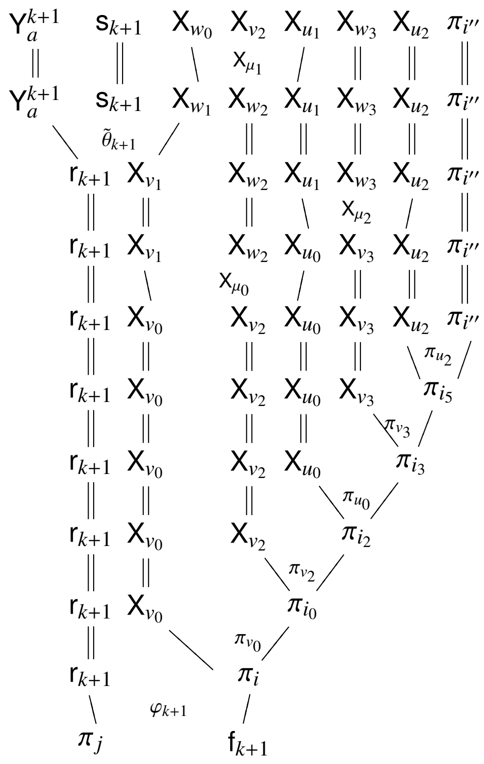
Let's check that this data satisfies the desired property:

For $l = 1, ..., k$:

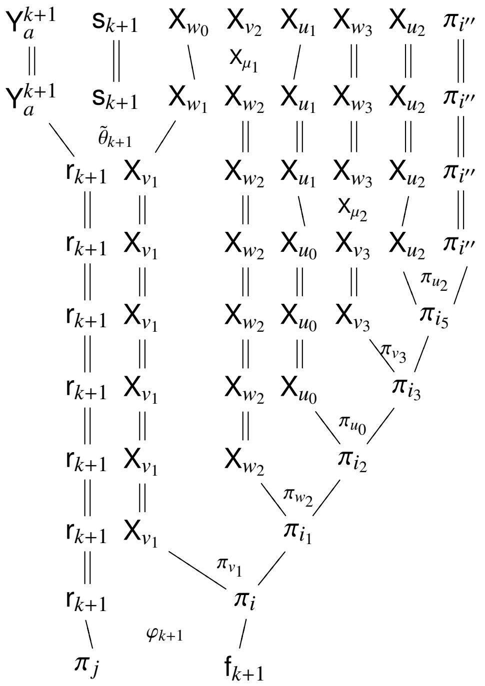


Where the equality is due to (2.2.10) plus axiom PC2 of pseudo-cones.

For $l = k + 1$:

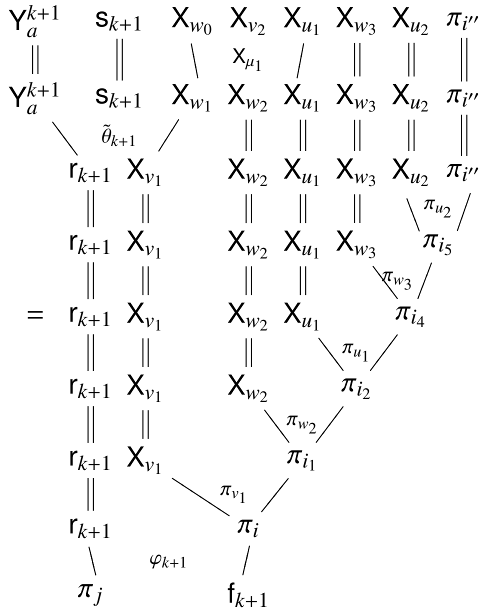

Y_a^{k+1} s_{k+1} X_{w_0} X_{v_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
X_{μ_1}
Y_a^{k+1} s_{k+1} X_{w_1} X_{w_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
$\tilde{\theta}_{k+1}$
r_{k+1} X_{v_1} X_{w_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
X_{μ_2}
r_{k+1} X_{v_1} X_{w_2} X_{u_0} X_{v_3} X_{u_2} $\pi_{i''}$
X_{μ_0}
r_{k+1} X_{v_0} X_{v_2} X_{u_0} X_{v_3} X_{u_2} $\pi_{i''}$
π_{u_2}
r_{k+1} X_{v_0} X_{v_2} X_{u_0} X_{v_3} π_{i_5}
π_{v_3}
r_{k+1} X_{v_0} X_{v_2} X_{u_0} π_{i_3}
π_{u_0}
r_{k+1} X_{v_0} X_{v_2} π_{i_2}
π_{v_2}
r_{k+1} X_{v_0} π_{i_0}
π_{v_0}
r_{k+1} π_i
φ_{k+1}
π_j f_{k+1}

=

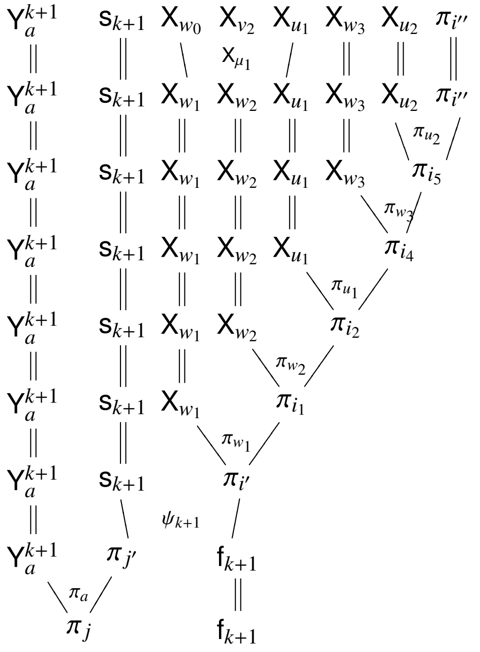

Y_a^{k+1} s_{k+1} X_{w_0} X_{v_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
X_{μ_1}
Y_a^{k+1} s_{k+1} X_{w_1} X_{w_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
$\tilde{\theta}_{k+1}$
r_{k+1} X_{v_1} X_{w_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
X_{μ_2}
r_{k+1} X_{v_1} X_{w_2} X_{u_0} X_{v_3} X_{u_2} $\pi_{i''}$
π_{u_2}
r_{k+1} X_{v_1} X_{w_2} X_{u_0} X_{v_3} π_{i_5}
π_{v_3}
r_{k+1} X_{v_1} X_{w_2} X_{u_0} π_{i_3}
π_{u_0}
r_{k+1} X_{v_1} X_{w_2} π_{i_2}
π_{w_2}
r_{k+1} X_{v_1} π_{i_1}
π_{v_1}
r_{k+1} π_i
φ_{k+1}
π_j f_{k+1}

=

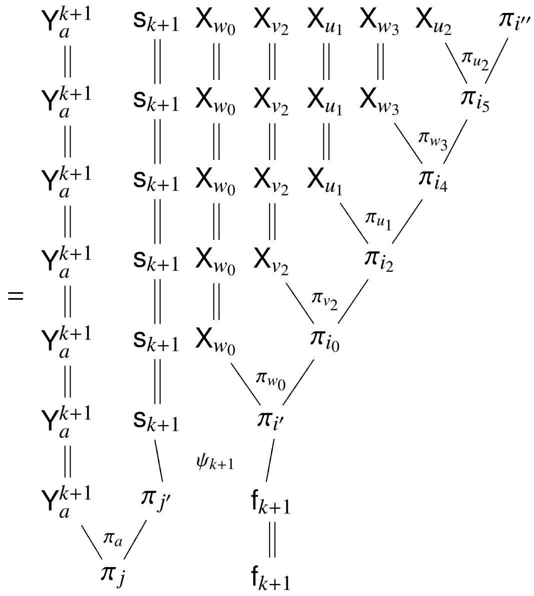
=


Y_a^{k+1} s_{k+1} X_{w_0} X_{v_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
X_{μ_1}
Y_a^{k+1} s_{k+1} X_{w_1} X_{w_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
$\tilde{\theta}_{k+1}$
r_{k+1} X_{v_1} X_{w_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
π_{u_2}
r_{k+1} X_{v_1} X_{w_2} X_{u_1} X_{w_3} π_{i_5}
π_{w_3}
r_{k+1} X_{v_1} X_{w_2} X_{u_1} π_{i_4}
π_{u_1}
r_{k+1} X_{v_1} X_{w_2} π_{i_2}
π_{w_2}
r_{k+1} X_{v_1} π_{i_1}
π_{v_1}
r_{k+1} π_i
φ_{k+1}
π_j f_{k+1}

=


Y_a^{k+1} s_{k+1} X_{w_0} X_{v_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
X_{μ_1}
Y_a^{k+1} s_{k+1} X_{w_1} X_{w_2} X_{u_1} X_{w_3} X_{u_2} $\pi_{i''}$
π_{u_2}
Y_a^{k+1} s_{k+1} X_{w_1} X_{w_2} X_{u_1} X_{w_3} π_{i_5}
π_{w_3}
Y_a^{k+1} s_{k+1} X_{w_1} X_{w_2} X_{u_1} π_{i_4}
π_{u_1}
Y_a^{k+1} s_{k+1} X_{w_1} X_{w_2} π_{i_2}
π_{w_2}
Y_a^{k+1} s_{k+1} X_{w_1} π_{i_1}
π_{w_1}
Y_a^{k+1} s_{k+1} $\pi_{i'}$
ψ_{k+1}
Y_a^{k+1} $\pi_{j'}$ f_{k+1}
π_a
π_j f_{k+1}

=



Where the first, the second and the last equalities are due to elevators calculus plus axiom PC2 of pseudo-cones and the third one holds by elevators calculus plus (2.2.11). □

2.2.12 Lemma. *Let* $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}}$ *be a 2-pro-object and* $\mathsf{X}_i \underset{\mathsf{g}}{\overset{\mathsf{f}}{\underset{\Downarrow\theta\ \Downarrow\theta'}{\rightrightarrows}}} \mathsf{C} \in \mathcal{C}$ *such that* $\theta\pi_i = \theta'\pi_i$ *in* 2-$\mathcal{P}ro(\mathcal{C})$. *Then* $\exists\ i \xrightarrow{u} i' \in \mathcal{I}$ *such that* $\theta\mathsf{X}_u = \theta'\mathsf{X}_u$.

Proof. It follows from 2.1.7 and [11, Lemma 1.20.] □

The following lemma will also be used in section 3:

2.2.13 Lemma. *Let* $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}}$ *be a 2-pro-object and* $\left\{\mathsf{X}_i \underset{\mathsf{g}_l}{\overset{\mathsf{f}_l}{\underset{\Downarrow\theta_l\ \Downarrow\theta'_l}{\rightrightarrows}}} \mathsf{C}\right\}_{l=1,...k} \in \mathcal{C}$ *be such that* $\theta_l\pi_i = \theta'_l\pi_i\ \forall\ l = 1, ...k$ *in* 2-$\mathcal{P}ro(\mathcal{C})$. *Then* $\exists\ i \xrightarrow{u} i' \in \mathcal{I}$ *such that* $\theta_l\mathsf{X}_u = \theta'_l\mathsf{X}_u$ $\forall\ l = 1, ...k$.

Proof. We are going to proceed by induction in k. For $k = 1$, use 2.2.12.

$k \Rightarrow k+1$: By inductive hypothesis, $\exists\ i \xrightarrow{u_0} i_0 \in \mathcal{I}$ such that $\theta_l\mathsf{X}_{u_0} = \theta'_l\mathsf{X}_{u_0}$ $\forall\ l = 1, ..., k$ and $\exists\ i \xrightarrow{u_1} i_1 \in \mathcal{I}$ such that $\theta_{k+1}\mathsf{X}_{u_1} = \theta'_{k+1}\mathsf{X}_{u_1}$.

Since $\mathcal{I}$ is 2-filtered, we have morphisms $i_0 \xrightarrow{v_0} i'$, $i_1 \xrightarrow{v_1} i'$ and an invertible 2-cell $i \underset{v_1u_1}{\overset{v_0u_0}{\underset{\Downarrow\mu}{\rightrightarrows}}} i' \in \mathcal{I}$.

It is easy to check that $u = v_0 u_0$ satisfies the desired property. ☐

2.2.14 Remark. The previous two lemmas (and so the following one) also hold in 2-$\mathcal{P}ro_p(C)$ (have in mind the pseudo-equivalence of 2.1.5). ☐

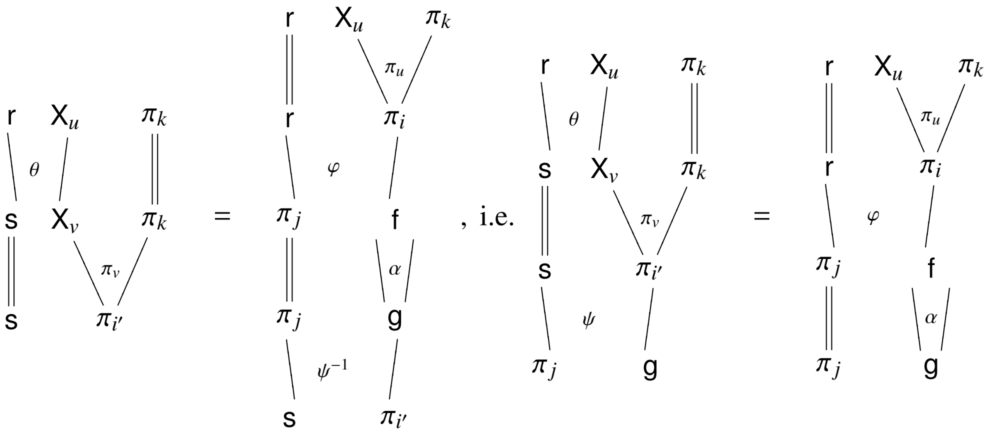
2.2.15 Lemma. *Let* $\begin{tikzcd} \mathsf{X} \arrow[r, bend left, "\mathsf{f}"] \arrow[r, bend right, "\mathsf{g}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{Y} \end{tikzcd}$ *in* 2-$\mathcal{P}ro(C)$ *and* $\begin{tikzcd} \mathsf{X}_i \arrow[r, bend left, "\mathsf{r}"] \arrow[r, bend right, "\mathsf{s}"'] \arrow[r, phantom, "\Downarrow\theta\ \Downarrow\theta'"] & \mathsf{Y}_j \end{tikzcd}$ *in* C *such that* $(\theta, \mathsf{r}, \varphi, \mathsf{s}, \psi)$ *and* $(\theta', \mathsf{r}, \varphi, \mathsf{s}, \psi)$ *both represent* α*. Then, there exists* $i \xrightarrow{u} i' \in \mathcal{I}$ *such that* $\theta\mathsf{X}_u = \theta'\mathsf{X}_u$.

Proof. Since both $(\theta, \mathsf{r}, \varphi, \mathsf{s}, \psi)$ and $(\theta', \mathsf{r}, \varphi, \mathsf{s}, \psi)$ represents α, and φ, ψ are invertible, it follows that $\theta\pi_i = \theta'\pi_i$. Then, by 2.2.12, there exists $i \xrightarrow{u} i' \in \mathcal{I}$ such that $\theta\mathsf{X}_u = \theta'\mathsf{X}_u$. ☐

2.2.16 Lemma. *Let* $\begin{tikzcd} \mathsf{X} \arrow[r, bend left, "\mathsf{f}"] \arrow[r, bend right, "\mathsf{g}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{Y} \end{tikzcd} \in$ 2-$\mathcal{P}ro(C)$*,* (r, φ) *representing* f, $\mathsf{X}_i \xrightarrow{\mathsf{r}} \mathsf{Y}_j$ *and* (s, ψ) *representing* g, $\mathsf{X}_{i'} \xrightarrow{\mathsf{s}} \mathsf{Y}_j$*. Then,* $\exists$ $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u"] & \\ & k \\ i' \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$ *and* $\begin{tikzcd} \mathsf{X}_k \arrow[r, bend left, "\mathsf{r}\mathsf{X}_u"] \arrow[r, bend right, "\mathsf{s}\mathsf{X}_v"'] \arrow[r, phantom, "\Downarrow \theta"] & \mathsf{Y}_j \end{tikzcd} \in C$ *such that* $(\theta,\ \mathsf{r}\mathsf{X}_u,\ \mathsf{r}\pi_u \circ \varphi,\ \mathsf{s}\mathsf{X}_v,\ \mathsf{s}\pi_v \circ \psi)$ *represents* α*. Observe that in case* α *is invertible, one can choose* θ *to be invertible.*

Proof. In lemma 2.2.5, take $\mathsf{C} = \mathsf{Y}_j$, and $\alpha = \begin{array}{ccc} \mathsf{r} & & \pi_i \\ & \backslash\ \varphi\ / & \\ \pi_j & & \mathsf{f} \\ \| & & \backslash\alpha/ \\ \pi_j & & \mathsf{g} \\ & \backslash\ \psi^{-1}\ / & \\ \mathsf{s} & & \pi_{i'} \end{array}$. Then, $\exists$ $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u"] & \\ & k \\ i' \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$ and

$\mathsf{X}_k \overset{\mathsf{rX}_u}{\underset{\mathsf{sX}_v}{\overset{\Downarrow\,\theta}{\rightrightarrows}}} \mathsf{Y}_j \in \mathcal{C}$ such that



This proves that $(\theta,\ \mathsf{rX}_u,\ \mathsf{r}\pi_u \circ \varphi,\ \mathsf{sX}_v,\ \mathsf{s}\pi_v \circ \psi)$ represents α. □

From remark 2.2.6 we have:

2.2.17 Remark. If $i = i'$, then one can choose $u = v$. □

2.3 2-cofiltered pseudo-limits in 2-$\mathcal{P}ro(\mathcal{C})$.

Let $\mathcal{J}$ be a small 2-filtered 2-category and $\mathcal{J}^{op} \xrightarrow{\mathsf{X}}$ 2-$\mathcal{P}ro(\mathcal{C})$ a 2-functor, $\mathsf{X}^j = (\mathsf{X}^j_i)_{i \in \mathcal{I}_j}$, $\mathcal{I}_j^{op} \xrightarrow{\mathsf{X}^j} \mathcal{C}$. Recall (2.1.11) that for each j in $\mathcal{J}$, X^j is equipped with a pseudo-limit pseudo-cone $\left\{\pi^j_i\right\}_{i \in \mathcal{I}_j}$, $\left\{\pi^j_u\right\}_{i \xrightarrow{u} i' \in \mathcal{I}_j}$ for the 2-functor X^j. We are using the supra-index notation to denote the evaluation of X.

We are going to construct a 2-pro-object which is going to be the pseudo-limit of X in 2-$\mathcal{P}ro(\mathcal{C})$. First we construct its index category.

2.3.1 Definition. *Let $\mathcal{K}_{\mathsf{X}}$ be the 2-category consisting on:*

0-cells of $\mathcal{K}_{\mathsf{X}}$: (i, j), where $j \in \mathcal{J}$, $i \in \mathcal{I}_j$.

1-cells of $\mathcal{K}_{\mathsf{X}}$: $(i, j) \xrightarrow{(a,\mathsf{r},\varphi)} (i', j')$, where $j \xrightarrow{a} j' \in \mathcal{J}$, $\mathsf{X}^{j'}_{i'} \xrightarrow{\mathsf{r}} \mathsf{X}^j_i \in \mathcal{C}$ are such that (r, φ) represents X^a.

2-cells of $\mathcal{K}_{\mathsf{X}}$: $(a, \mathsf{r}, \varphi) \overset{(\alpha,\theta)}{\Longrightarrow} (b, \mathsf{s}, \psi)$, where $a \overset{\alpha}{\Longrightarrow} b \in \mathcal{J}$ and $(\theta, \mathsf{r}, \varphi, \mathsf{s}, \psi)$ represents X^{α}.

The 2-category structure is given as follows:

$$\begin{tikzcd}
(i,j) \arrow[r, shift left=6, "(a{,}\mathsf{r}{,}\varphi)"] \arrow[r, "(b{,}\mathsf{s}{,}\psi)"] \arrow[r, shift right=6, "(c{,}\mathsf{t}{,}\phi)"] \arrow[r, phantom, shift left=3, "\Downarrow(\alpha{,}\theta)"] \arrow[r, phantom, shift right=3, "\Downarrow(\beta{,}\eta)"] & (i',j') \arrow[r, shift left=6, "(a'{,}\mathsf{r}'{,}\varphi')"] \arrow[r, "(b'{,}\mathsf{s}'{,}\psi')"] \arrow[r, shift right=6, "(c'{,}\mathsf{t}'{,}\phi')"] \arrow[r, phantom, shift left=3, "\Downarrow(\alpha'{,}\theta')"] \arrow[r, phantom, shift right=3, "\Downarrow(\beta'{,}\eta')"] & (i'',j'')
\end{tikzcd}$$

$$-\ \ (a',\mathsf{r}',\varphi')(a,\mathsf{r},\varphi) = (a'a, \mathsf{r}\mathsf{r}', \begin{array}{cccc} \mathsf{r} & \mathsf{r}' & & \pi_{i''} \\ \| & \backslash & \varphi' & / \\ \mathsf{r} & \pi_{i'} & & \mathsf{X}^{a'} \\ \backslash\ \varphi & / & & \| \\ \pi_i & \mathsf{X}^{a} & & \mathsf{X}^{a'} \\ \| & \backslash & = & / \\ \pi_i & & \mathsf{X}^{a'a} & \end{array})$$

- $(\alpha',\theta')(\alpha,\theta) = (\alpha'\alpha, \theta\theta')$

- $(\beta,\eta)\circ(\alpha,\theta) = (\beta\circ\alpha, \eta\circ\theta)$

One can easily check that the structure so defined is indeed a 2-category, which is clearly small.

2.3.2 Proposition. *The 2-category $\mathcal{K}_{\mathsf{X}}$ is 2-filtered.*

Proof. F0. Let (i,j),$(i',j') \in \mathcal{K}_{\mathsf{X}}$. Since $\mathcal{J}$ is 2-filtered, $\exists$

$$\begin{tikzcd}
j \arrow[dr, "a"] & \\
& j'' \\
j' \arrow[ur, "b"'] &
\end{tikzcd}$$

$\in \mathcal{J}$. By 2.2.3, $\exists\ \mathsf{X}^{j''}_{i_1} \xrightarrow{\mathsf{r}_1} \mathsf{X}^{j}_{i}$ and $\mathsf{X}^{j''}_{i_2} \xrightarrow{\mathsf{r}_2} \mathsf{X}^{j'}_{i'} \in \mathcal{C}$ such that (r_1, id) represents X^a and (r_2, id) represents X^b. Since $\mathcal{I}_{j''}$ is 2-filtered, $\exists$

$$\begin{tikzcd}
i_1 \arrow[dr, "u"] & \\
& i'' \\
i_2 \arrow[ur, "v"'] &
\end{tikzcd}$$

$\in \mathcal{I}_{j''}$. Then, we have the following situation in $\mathcal{K}_{\mathsf{X}}$ which concludes the proof of axiom $F0$:

$$\begin{tikzcd}
(i,j) \arrow[dr, "(a{,}\,\mathsf{r}_1\mathsf{X}^{j''}_u{,}\,\mathsf{r}_1\pi^{j''}_u)"] & \\
& (i'',j'') \\
(i',j') \arrow[ur, "(b{,}\,\mathsf{r}_2\mathsf{X}^{j''}_v{,}\,\mathsf{r}_2\pi^{j''}_v)"'] &
\end{tikzcd}$$

Note that $\mathsf{r}_1\pi^{j''}_u = \begin{array}{cccc} \mathsf{r}_1 & \mathsf{X}^{j''}_u & & \pi^{j''}_{i''} \\ \Big\| & \backslash & \pi^{j''}_u & / \\ \mathsf{r}_1 & & \pi^{j''}_{i_1} & \\ \backslash & = & / & \\ \pi^{j}_{i} & & \mathsf{X}^a & \end{array}$ and $\mathsf{r}_2\pi^{j''}_v = \begin{array}{cccc} \mathsf{r}_2 & \mathsf{X}^{j''}_v & & \pi^{j''}_{i''} \\ \Big\| & \backslash & \pi^{j''}_v & / \\ \mathsf{r}_2 & & \pi^{j''}_{i_2} & \\ \backslash & = & / & \\ \pi^{j'}_{i'} & & \mathsf{X}^b & \end{array}$

F1. Let $(i,j) \overset{(a,\mathsf{r},\varphi)}{\underset{(b,\mathsf{s},\psi)}{\rightrightarrows}} (i',j') \in \mathcal{K}_{\mathsf{X}}$. Since $\mathcal{J}$ is 2-filtered, $\exists\, j' \xrightarrow{c} j''$ and an invertible 2-cell $j \overset{ca}{\underset{cb}{\Downarrow \alpha}} j'' \in \mathcal{J}$. By 2.2.3, $\exists\, \mathsf{X}_k^{j''} \xrightarrow{\mathsf{t}} \mathsf{X}_{i'}^{j'} \in \mathcal{C}$ such that (t, id) represents X^c. Then $\left(\mathsf{rt},\ \begin{array}{ccccc} \mathsf{r} & \mathsf{t} & & \pi_k^{j''} & \\ \| & \backslash & = & / & \\ \mathsf{r} & \pi_{i'}^{j'} & & \mathsf{X}^c & \\ \backslash\ \varphi & / & & \| & \\ \pi_i^j & \mathsf{X}^a & & \mathsf{X}^c & \\ \| & \backslash & = & / & \\ \pi_i^j & & \mathsf{X}^{ca} & & \end{array}\right)$ represents X^{ca} and $\left(\mathsf{st},\ \begin{array}{ccccc} \mathsf{s} & \mathsf{t} & & \pi_k^{j''} & \\ \| & \backslash & = & / & \\ \mathsf{s} & \pi_{i'}^{j'} & & \mathsf{X}^c & \\ \backslash\ \psi & / & & \| & \\ \pi_i^j & \mathsf{X}^b & & \mathsf{X}^c & \\ \| & \backslash & = & / & \\ \pi_i^j & & \mathsf{X}^{ca} & & \end{array}\right)$ represents X^{cb}, so, by 2.2.16, there exists $k \xrightarrow{w} i'' \in \mathcal{I}_{j''}$ and an invertible 2-cell $\mathsf{X}_{i''}^{j''} \overset{\mathsf{rtX}_w^{j''}}{\underset{\mathsf{stX}_w^{j''}}{\Downarrow \theta}} \mathsf{X}_i^j \in \mathcal{C}$ such that

$$\left(\theta,\ \mathsf{rtX}_w^{j''},\ \begin{array}{ccccc} \mathsf{r} & \mathsf{t} & \mathsf{X}_w^{j''} & & \pi_{i''}^{j''} \\ \| & \| & \backslash & \pi_w^{j''} & / \\ \mathsf{r} & \mathsf{t} & & \pi_k^{j''} & \\ \| & \backslash & = & / & \\ \mathsf{r} & \pi_{i'}^{j'} & & \mathsf{X}^c & \\ \backslash\ \varphi & / & & \| & \\ \pi_i^j & \mathsf{X}^a & & \mathsf{X}^c & \\ \| & \backslash & = & / & \\ \pi_i^j & & \mathsf{X}^{ca} & & \end{array},\ \mathsf{stX}_w^{j''},\ \begin{array}{ccccc} \mathsf{s} & \mathsf{t} & \mathsf{X}_w^{j''} & & \pi_{i''}^{j''} \\ \| & \| & \backslash & \pi_w^{j''} & / \\ \mathsf{s} & \mathsf{t} & & \pi_k^{j''} & \\ \| & \backslash & = & / & \\ \mathsf{s} & \pi_{i'}^{j'} & & \mathsf{X}^c & \\ \backslash\ \psi & / & & \| & \\ \pi_i^j & \mathsf{X}^b & & \mathsf{X}^c & \\ \| & \backslash & = & / & \\ \pi_i^j & & \mathsf{X}^{cb} & & \end{array}\right) \text{ represents } \mathsf{X}^{\alpha}.$$

Then we have an invertible 2-cell in $\mathcal{K}_{\mathsf{X}}$ $(i,j) \overset{(c,\mathsf{tX}_w^{j''},\mathsf{t}\pi_w^{j''})(a,\mathsf{r},\varphi)}{\underset{(c,\mathsf{tX}_w^{j''},\mathsf{t}\pi_w^{j''})(b,\mathsf{s},\psi)}{\Downarrow (\alpha,\theta)}} (i'',j'')$ which concludes the proof of axiom $F1$.

F2. Let $(i,j) \underset{(b,\mathsf{s},\psi)}{\overset{(a,\mathsf{r},\varphi)}{\underset{\Downarrow(\alpha,\theta)\ \Downarrow(\alpha',\theta')}{\rightrightarrows}}} (i',j') \ \in \mathcal{K}_{\mathsf{X}}$. Since $\mathcal{J}$ is 2-filtered, $\exists\, j' \xrightarrow{c} j'' \in \mathcal{J}$ such that $c\alpha = c\alpha'$. Also, by 2.2.3, $\exists\ \mathsf{X}_k^{j''} \xrightarrow{\mathsf{t}} \mathsf{X}_{i'}^{j'} \in \mathcal{C}$ such that (t, id) represents X^c. Then, it is easy to check that $(\mathsf{t}, \mathsf{t}, id, \mathsf{t}, id)$ represents X^c and therefore we have that $\Bigg(\theta\mathsf{t}, \mathsf{rt},$
$$\begin{array}{cccc}
\mathsf{r} & \mathsf{t} & & \pi_k^{j''} \\
\| & \backslash & = & / \\
\mathsf{r} & \pi_{i'}^{j'} & & \mathsf{X}^c \\
\backslash\ \varphi & / & & \| \\
\pi_i^j & \mathsf{X}^a & & \mathsf{X}^c \\
\| & \backslash & = & / \\
\pi_i^j & & \mathsf{X}^{ca} &
\end{array}
, \mathsf{st},
\begin{array}{cccc}
\mathsf{s} & \mathsf{t} & & \pi_k^{j''} \\
\| & \backslash & = & / \\
\mathsf{s} & \pi_{i'}^{j'} & & \mathsf{X}^c \\
\backslash\ \psi & / & & \| \\
\pi_i^j & \mathsf{X}^b & & \mathsf{X}^c \\
\| & \backslash & = & / \\
\pi_i^j & & \mathsf{X}^{ca} &
\end{array}\Bigg)$$
and
$$\Bigg(\theta'\mathsf{t}, \mathsf{rt},
\begin{array}{cccc}
\mathsf{r} & \mathsf{t} & & \pi_k^{j''} \\
\| & \backslash & = & / \\
\mathsf{r} & \pi_{i'}^{j'} & & \mathsf{X}^c \\
\backslash\ \varphi & / & & \| \\
\pi_i^j & \mathsf{X}^a & & \mathsf{X}^c \\
\| & \backslash & = & / \\
\pi_i^j & & \mathsf{X}^{ca} &
\end{array}
, \mathsf{st},
\begin{array}{cccc}
\mathsf{s} & \mathsf{t} & & \pi_k^{j''} \\
\| & \backslash & = & / \\
\mathsf{s} & \pi_{i'}^{j'} & & \mathsf{X}^c \\
\backslash\ \psi & / & & \| \\
\pi_i^j & \mathsf{X}^b & & \mathsf{X}^c \\
\| & \backslash & = & / \\
\pi_i^j & & \mathsf{X}^{ca} &
\end{array}\Bigg)$$
both represent $\mathsf{X}^{c\alpha}$:

$$\begin{array}{cccc}
\mathsf{r} & \mathsf{t} & & \pi_k^{j''} \\
\backslash\ \theta\ / & \| & & \| \\
\mathsf{s} & \mathsf{t} & & \pi_k^{j''} \\
\| & \backslash & = & / \\
\mathsf{s} & \pi_{i'}^{j'} & & \mathsf{X}^c \\
\backslash\ \psi & / & & \| \\
\pi_i^j & \mathsf{X}^b & & \mathsf{X}^c \\
\| & \backslash & = & / \\
\pi_i^j & & \mathsf{X}^{cb} &
\end{array}
\ = \
\begin{array}{cccc}
\mathsf{r} & \mathsf{t} & & \pi_k^{j''} \\
\| & \backslash & = & / \\
\mathsf{r} & \pi_{i'}^{j'} & & \mathsf{X}^c \\
\backslash\ \varphi & / & & \| \\
\pi_i^j & \mathsf{X}^a & & \mathsf{X}^c \\
\| & \backslash\ \mathsf{X}^{\alpha}\ / & & \| \\
\pi_i^j & \mathsf{X}^b & & \mathsf{X}^c \\
\| & \backslash & = & / \\
\pi_i^j & & \mathsf{X}^{cb} &
\end{array}
\ = \
\begin{array}{cccc}
\mathsf{r} & \mathsf{t} & & \pi_k^{j''} \\
\| & \backslash & = & / \\
\mathsf{r} & \pi_{i'}^{j'} & & \mathsf{X}^c \\
\backslash\ \varphi & / & & \| \\
\pi_i^j & \mathsf{X}^a & & \mathsf{X}^c \\
\| & \backslash & = & / \\
\pi_i^j & & \mathsf{X}^{ca} & \\
\| & & \backslash\ \mathsf{X}^{c\alpha}\ / & \\
\pi_i^j & & \mathsf{X}^{cb} &
\end{array}$$

where the first equality is due to elevators calculus plus the fact that $(\theta, \mathsf{r}, \varphi, \mathsf{s}, \psi)$ represents X^{α}.

Then, by 2.2.15, $\exists\, k \xrightarrow{w} i'' \in \mathcal{I}_{j''}$ such that $\theta\mathsf{t}\mathsf{X}_w^{j''} = \theta'\mathsf{t}\mathsf{X}_w^{j''}$, so $(c, \mathsf{t}\mathsf{X}_w^{j''}, \mathsf{t}\pi_w)(\alpha,\theta) = (c, \mathsf{t}\mathsf{X}_w^{j''}, t\pi_w)(\alpha',\theta')$, which concludes the proof of axiom $F2$.

□

2.3.3 Proposition. *Let $\widetilde{\mathsf{X}}$ be the 2-pro-object $\mathcal{K}_{\mathsf{X}}^{op} \xrightarrow{\widetilde{\mathsf{X}}} C$ defined by $\widetilde{\mathsf{X}}_{(i,j)} = \mathsf{X}_i^j$, $\widetilde{\mathsf{X}}_{(a,\mathsf{r},\varphi)} = \mathsf{r}$, and $\widetilde{\mathsf{X}}_{(\alpha,\theta)} = \theta$. Then the following equation holds in 2-$\mathcal{P}ro(C)$:*

$$\widetilde{\mathsf{X}} = \underset{j\in\mathcal{J}}{\underleftarrow{\mathsf{Lim}}}\, \mathsf{X}^j$$

Proof. Let $\mathsf{Z} \in$ 2-$\mathcal{P}ro(C)$, and $\left\{\mathsf{Z} \xrightarrow{\mathsf{h}_j} \mathsf{X}^j\right\}_{j\in\mathcal{J}}$, $\left\{\mathsf{X}^a\mathsf{h}_{j'} \overset{\mathsf{h}_a}{\Longrightarrow} \mathsf{h}_j\right\}_{j\xrightarrow{a} j'\in\mathcal{J}}$ be a pseudo-cone for X with vertex Z (1.2.2). Given $(i,j) \xrightarrow{(a,\mathsf{r},\varphi)} (i',j') \in \mathcal{K}_{\mathsf{X}}$, the definitions $\mathsf{h}_{(i,j)} = \pi_i^j \mathsf{h}_j$ and $\mathsf{h}_{(a,\mathsf{r},\varphi)} = \begin{array}{ccccc} \mathsf{r} & & \pi_{i'}^{j'} & & \mathsf{h}_{j'} \\ \backslash & \varphi & / & & \| \\ \pi_i^j & & \mathsf{X}^a & & \mathsf{h}_{j'} \\ \| & & \backslash & \mathsf{h}_a & / \\ \pi_i^j & & & \mathsf{h}_j & \end{array}$ determine a pseudo-cone for $c\widetilde{\mathsf{X}}$ (where c is the morphism of 2.1.10) with vertex Z:

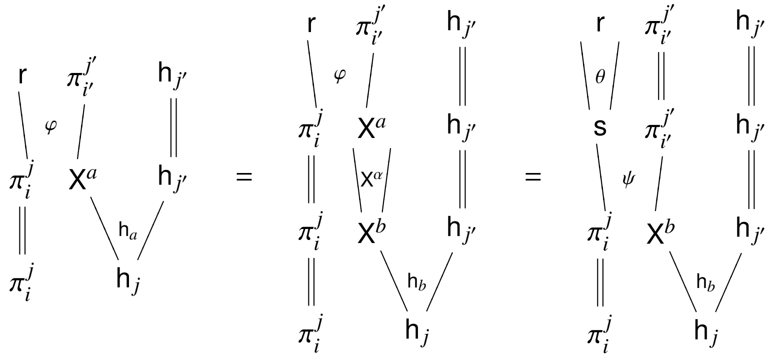
PC0. It is straightforward.

PC1. Given $(i,j) \xrightarrow{(a,\mathsf{r},\varphi)} (i',j') \xrightarrow{(b,\mathsf{s},\psi)} (i'',j'') \in \mathcal{K}_{\mathsf{X}}$,

$$\begin{array}{ccccccc} \mathsf{r} & & \mathsf{s} & & \pi_{i''}^{j''} & & \mathsf{h}_{j''} \\ \| & & \backslash & \psi & / & & \| \\ \mathsf{r} & & \pi_{i'}^{j'} & & \mathsf{X}^b & & \mathsf{h}_{j''} \\ \| & & \| & & \backslash & \mathsf{h}_b & / \\ \mathsf{r} & & \pi_{i'}^{j'} & & & \mathsf{h}_{j'} & \\ \backslash & \varphi & / & & & \| & \\ \pi_i^j & & \mathsf{X}^a & & & \mathsf{h}_{j'} & \\ \| & & \backslash & \mathsf{h}_a & / & & \\ \pi_i^j & & & \mathsf{h}_j & & & \end{array} \quad = \quad \begin{array}{ccccccc} \mathsf{r} & & \mathsf{s} & & \pi_{i''}^{j''} & & \mathsf{h}_{j''} \\ \| & & \backslash & \psi & / & & \| \\ \mathsf{r} & & \pi_{i'}^{j'} & & \mathsf{X}^b & & \mathsf{h}_{j''} \\ \backslash & \varphi & / & & \| & & \| \\ \pi_i^j & & \mathsf{X}^a & & \mathsf{X}^b & & \mathsf{h}_{j''} \\ \| & & \backslash & = & / & & \| \\ \pi_i^j & & & \mathsf{X}^{ba} & & & \mathsf{h}_{j''} \\ \| & & & \backslash & \mathsf{h}_{ba} & / & \\ \pi_i^j & & & & \mathsf{h}_j & & \end{array}$$

where the equality is due to elevators calculus plus the fact that h is a pseudo-cone.

PC2. Given $(i,j) \underset{(b,\mathsf{s},\psi)}{\overset{(a,\mathsf{r},\varphi)}{\underset{\Downarrow\,(\alpha,\theta)}{\longrightarrow}}} (i',j') \in \mathcal{K}_{\mathsf{X}}$,



where the first equality is due to the fact that h is a pseudo-cone and the second one is valid because $(\theta, r, \varphi, s, \psi)$ represents X^α.

It is straightforward to check that this extends to a functor, that we denote p (for the isomorphism below see 2.1.13):

$$PC_{2\text{-}\mathcal{P}ro(C)}(Z, X) \xrightarrow{p} PC_{2\text{-}\mathcal{P}ro(C)}(Z, c\widetilde{X}) \cong 2\text{-}\mathcal{P}ro(C)(Z, \widetilde{X})$$

The proposition follows if p is an isomorphism. In the sequel we prove that this is the case.

1. p is bijective on objects: Let

$$\left\{Z \xrightarrow{h_{(i,j)}} X^j_i\right\}_{(i,j)\in\mathcal{K}_X}, \quad \left\{\widetilde{X}_{(a,r,\varphi)} h_{(i',j')} = r\, h_{(i',j')} \overset{h_{(a,r,\varphi)}}{\Longrightarrow} h_{(i,j)}\right\}_{(i,j)\xrightarrow{(a,r,\varphi)}(i',j')\in\mathcal{K}_X}$$

be a pseudo-cone for $c\widetilde{X}$ with vertex Z (1.2.2).

Check that for each $j \in \mathcal{J}$, $\left\{Z \xrightarrow{h_{(i,j)}} X^j_i\right\}_{i\in\mathcal{I}_j}$ together with $\left\{h_u = h_{(j,X^j_u,\pi^j_u)} : X^j_u h_{(i',j')} \Longrightarrow h_{(i,j)}\right\}_{i\xrightarrow{u}i'\in\mathcal{I}_j}$ is a pseudo-cone for X^j. Then, since $\left\{X^j \xrightarrow{\pi^j_i} X^j_i\right\}_{i\in\mathcal{I}_j}, \left\{X^j_u \pi^j_{i'} \overset{\pi^j_u}{\Longrightarrow} \pi^j_i\right\}_{i\xrightarrow{u}i'\in\mathcal{I}_j}$ is a pseudo-limit pseudo-cone, it follows that there exists a unique $Z \xrightarrow{h_j} X^j$ such that

$$\forall i \in \mathcal{I}_j \;\; \pi^j_i h_j = h_{(i,j)} \;\; and \;\; \forall\; i \xrightarrow{u} i' \in \mathcal{I}_j \;\; \pi^j_u h_j = h_u. \tag{2.3.4}$$

It only remains to define the 2-cells of the pseudo-cone structure. That is, for each $j \xrightarrow{a} j' \in \mathcal{J}$, we need invertible 2-cells $X^a h_{j'} \overset{h_a}{\Longrightarrow} h_j$, such that $\{h_j\}_{j\in\mathcal{J}}$ together with $\{h_a\}_{j\xrightarrow{a}j'\in\mathcal{J}}$ form a pseudo-cone for X with vertex Z.

Consider the pseudo-cone $\left\{X^j \xrightarrow{\pi_i^j} X_i^j\right\}_{i\in\mathcal{I}_j}$. Then the compositions $\left\{Z \ \overset{\pi_i^j h_j}{\underset{\pi_i^j X^a h_{j'}}{\substack{\longrightarrow\\ \longrightarrow}}}\ X_i^j\right\}_{i\in\mathcal{I}_j}$ determine two pseudo-cones for X^j with vertex Z.

Claim 1 *Let* (r,φ) *and* (s,ψ) *be two pairs representing* X^a *as follows:*

$$\begin{tikzcd}
X^{j'} \arrow[r, "\pi_{i'}^{j'}"] \arrow[d, "X^a"'] \arrow[dr, phantom, "\cong \Downarrow \varphi"] & X_{i'}^{j'} \arrow[d, "r"] \\
X^j \arrow[r, "\pi_i^j"'] & X_i^j
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{j'} \arrow[r, "\pi_{i''}^{j'}"] \arrow[d, "X^a"'] \arrow[dr, phantom, "\cong \Downarrow \psi"] & X_{i''}^{j'} \arrow[d, "s"] \\
X^j \arrow[r, "\pi_i^j"'] & X_i^j
\end{tikzcd}$$

Then, $h_{(a,r,\varphi)} \circ \varphi^{-1} h_{j'} = h_{(a,s,\psi)} \circ \psi^{-1} h_{j'}$, *i.e.*

$$\begin{array}{ccc}
\pi_i^j & X^a & h_{j'} \\
\backslash & \varphi^{-1} \quad / & \| \\
r & \pi_{i'}^{j'} & h_{j'} \\
\backslash & h_{(a,r,\varphi)} & / \\
 & \pi_i^j \qquad h_j &
\end{array}
\quad = \quad
\begin{array}{ccc}
\pi_i^j & X^a & h_{j'} \\
\backslash & \psi^{-1} \quad / & \| \\
s & \pi_{i''}^{j'} & h_{j'} \\
\backslash & h_{(a,s,\psi)} & / \\
 & \pi_i^j \qquad h_j &
\end{array}$$

(proof below).

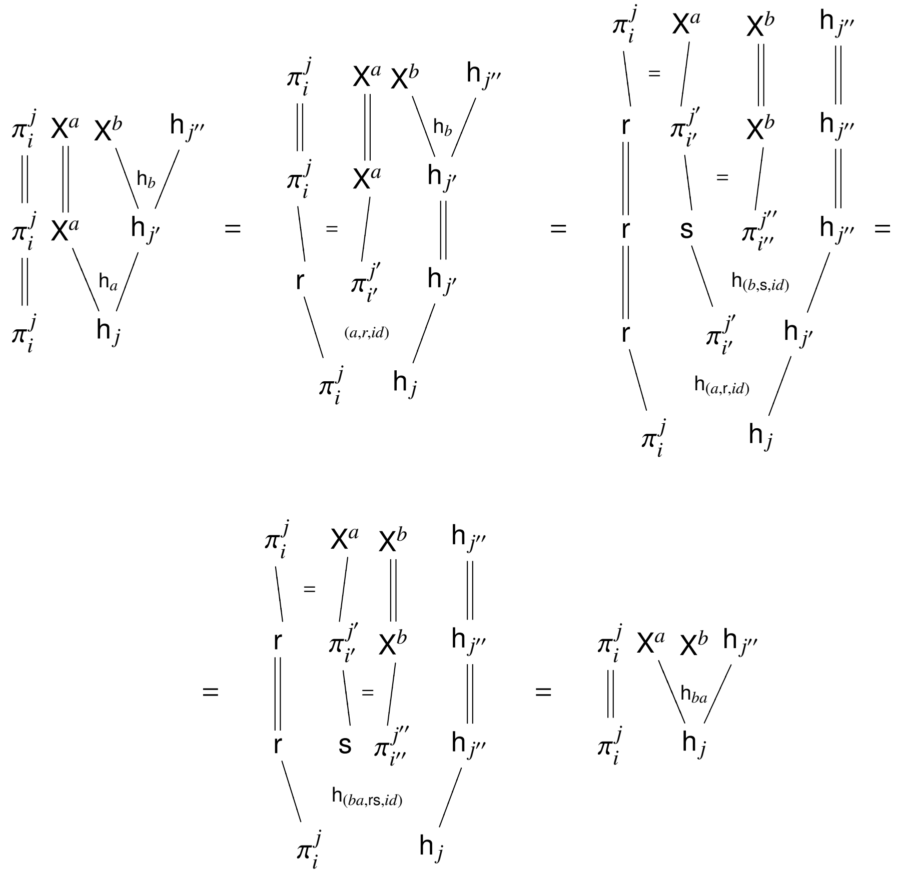
Claim 2 *For each* $i \in \mathcal{I}_j$, *let* (r,φ) *be a pair representing* X^a, *and set* $\rho_i = h_{(a,r,\varphi)} \circ \varphi^{-1} h_{j'} = \begin{array}{ccc} \pi_i^j & X^a & h_{j'} \\ \backslash\ \varphi^{-1} & / & \| \\ r & \pi_{i'}^{j'} & h_{j'} \\ \backslash & h_{(a,r,\varphi)} & / \\ & \pi_i^j \quad h_j & \end{array}$. *Then,* $\{\rho_i\}_{i\in\mathcal{I}_j}$ *determines an isomorphism of pseudo-cones* $\left\{Z \ \overset{\pi_i^j h_j}{\underset{\pi_i^j X^a h_{j'}}{\substack{\longrightarrow\\ \Downarrow \rho_i \\ \longrightarrow}}}\ X_i^j\right\}_{i\in\mathcal{I}_j}$ (proof below).

Since $\left\{X^j \xrightarrow{\pi_i^j} X_i^j\right\}_{i\in\mathcal{I}_j}$, $\left\{X_u^j \pi_{i'}^j \overset{\pi_u^j}{\Longrightarrow} \pi_i^j\right\}_{i \xrightarrow{u} i' \in \mathcal{I}_j}$ is a pseudo-limit pseudo-cone, the functor $2\text{-}\mathcal{P}ro(\mathcal{C})(Z, X^j) \xrightarrow{(\pi^j)_*} PC_{2\text{-}\mathcal{P}ro(\mathcal{C})}(Z, X^j)$ is an isomorphism of categories $\forall\ j \in \mathcal{J}$. Then, from Claim 2 it follows that there are invertible 2-cells $Z \ \overset{h_j}{\underset{X^a h_{j'}}{\substack{\longrightarrow\\ \Downarrow h_a \\ \longrightarrow}}}\ X^j \in 2\text{-}\mathcal{P}ro(\mathcal{C})$

such that $\rho_i = \pi^j_i \mathsf{h}_a \ \forall\ i \in \mathcal{I}_j \ \forall\ j \in \mathcal{J}$. Let's check that $\left\{\mathsf{Z} \xrightarrow{\mathsf{h}_j} \mathsf{X}^j\right\}_{j\in\mathcal{J}}$ together with $\left\{\mathsf{X}^a \mathsf{h}_{j'} \overset{\mathsf{h}_a}{\Longrightarrow} \mathsf{h}_j\right\}_{j \xrightarrow{a} j' \in \mathcal{J}}$ is a pseudo-cone over X:

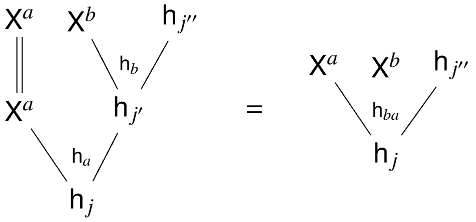
PC0. By Claim 1, in Claim 2 we can take $\mathsf{r} = id$ and $\varphi = id$, so $\rho_i = id$ and therefore $\mathsf{h}_{id} = id$.

PC1. Given $j \xrightarrow{a} j' \xrightarrow{b} j'' \in \mathcal{J}$ and $i \in \mathcal{I}_j$, by Claim 1, in Claim 2 we can take (r, id) representing X^a, $\mathsf{X}^{j'}_{i'} \xrightarrow{\mathsf{r}} \mathsf{X}^j_i$, (s, id) representing X^b, $\mathsf{X}^{j''}_{i''} \xrightarrow{\mathsf{s}} \mathsf{X}^{j'}_{i'}$ and (rs, id) representing X^{ba}. Then

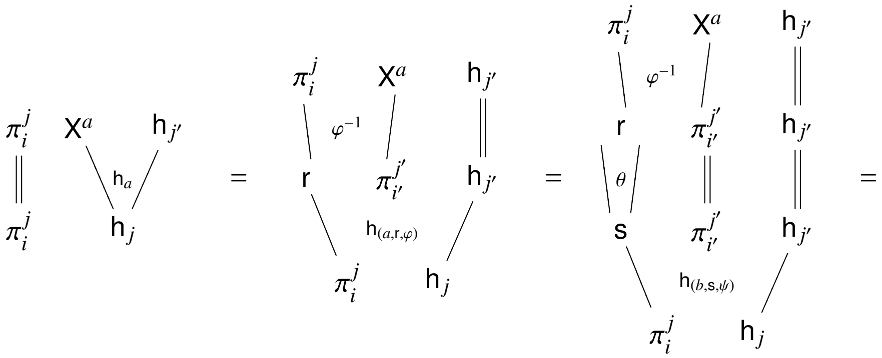


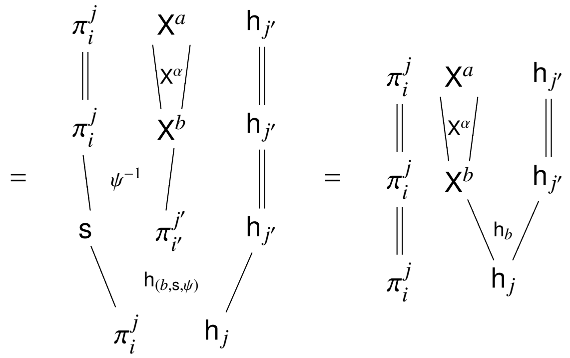
where the first, the second and the last equalities hold by definition of $\mathsf{h}_{(a,\mathsf{r},id)}$, $\mathsf{h}_{(b,\mathsf{s},id)}$ and $\mathsf{h}_{(ba,\mathsf{rs},id)}$ respectively plus some elevators calculus; and the third equality is due to the fact that h is a pseudo-cone.

Since we checked this for any $i \in \mathcal{I}_j$, it follows:



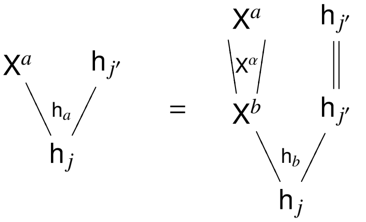
PC2. Given $j \mathrel{\substack{\xrightarrow{a} \\ \Downarrow\alpha \\ \xrightarrow[b]{}}} j' \in \mathcal{J}$ and $i \in \mathcal{I}_j$, there is $\mathsf{X}^{j'}_{i'} \mathrel{\substack{\xrightarrow{\mathsf{r}} \\ \Downarrow\theta \\ \xrightarrow[\mathsf{s}]{}}} \mathsf{X}^j_i$ and appropriate invertible 2-cells φ, ψ such that $(\theta, \mathsf{r}, \varphi, \mathsf{s}, \psi)$ represents X^α. By Claim 1, in Claim 2 we can take those representatives of X^a and X^b and then:





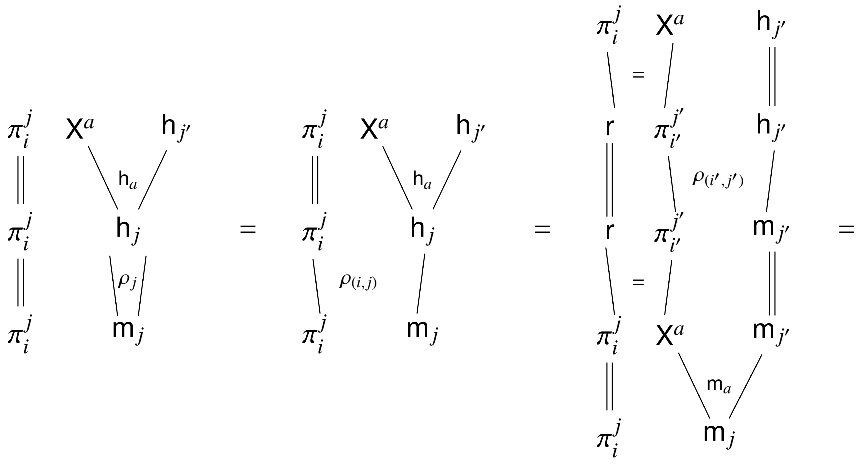
where the first and last equalities hold by definition of $\mathsf{h}_{(a,\mathsf{r},\varphi)}$ and $\mathsf{h}_{(b,\mathsf{s},\psi)}$ respectively, the second equality is due to the fact that h is a pseudo-cone and the third one is valid because $(\theta, \mathsf{r}, \varphi, \mathsf{s}, \psi)$ represents X^α.

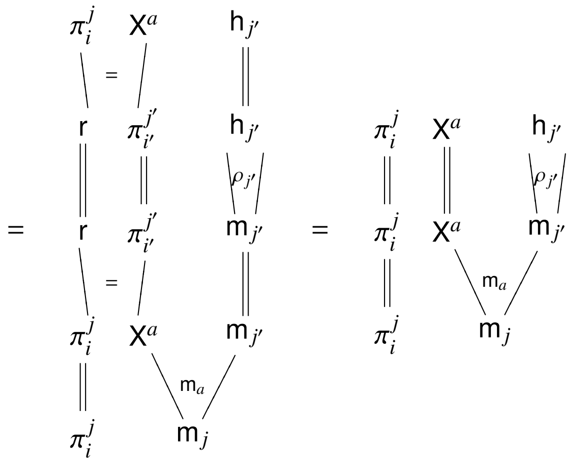
Since we checked this for any $i \in \mathcal{I}_j$, it follows:



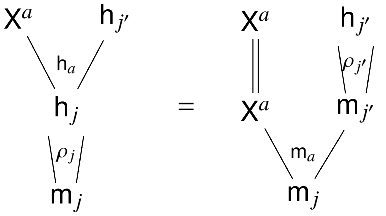
2. p is full and faithful: Let $\left\{ \mathsf{Z} \underset{\pi^j_i \mathsf{m}_j}{\overset{\pi^j_i \mathsf{h}_j}{\underset{\Downarrow \rho_{(i,j)}}{\rightrightarrows}}} \mathsf{X}^j_i \right\}_{(i,j)\in\mathcal{K}_\mathsf{X}}$ be a morphism of pseudo-cones for $\tilde{\mathsf{X}}$. It is easy to check that for each $j \in \mathcal{J}$, $\left\{ \mathsf{Z} \underset{\pi^j_i \mathsf{m}_j}{\overset{\pi^j_i \mathsf{h}_j}{\underset{\Downarrow \rho_{(i,j)}}{\rightrightarrows}}} \mathsf{X}^j_i \right\}_{i\in\mathcal{I}_j}$ is a morphism of pseudo-cones for X^j. Then arguing as above, there exists a unique morphism $\mathsf{Z} \underset{\mathsf{m}_j}{\overset{\mathsf{h}_j}{\underset{\Downarrow \rho_j}{\rightrightarrows}}} \mathsf{X}^j \in 2\text{-}\mathcal{P}ro(\mathcal{C})$ such that for each $i \in \mathcal{I}_j$, $\pi^j_i \rho_j = \rho_{(i,j)}$. It only remains to prove that $\{\rho_j\}_{j\in\mathcal{J}}$ is a morphism of pseudo-cones:

PCM. Given $j \xrightarrow{a} j' \in \mathcal{J}$ and $i \in \mathcal{I}_j$, by Claim 1, in Claim 2 we can take (r, id) representing X^a, $\mathsf{X}^{j'}_{i'} \xrightarrow{\mathsf{r}} \mathsf{X}^j_i$ and then:



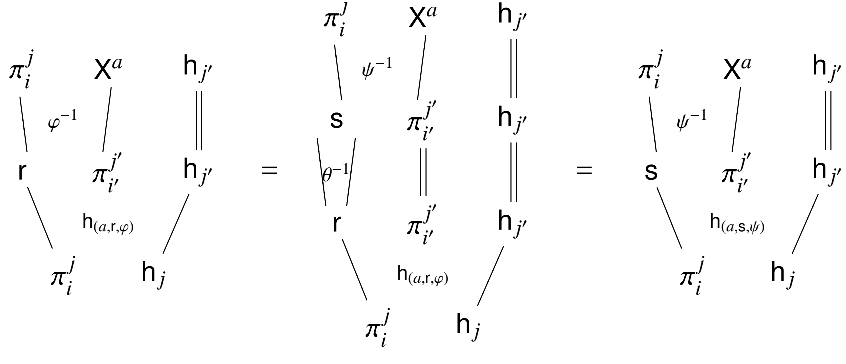


where the second equality is valid because ρ is a morphism of pseudo-cones. Since we checked this for any $i \in \mathcal{I}_j$, it follows:



□

Proof of Claim 1. First assume that $i' = i''$ and (r, φ), (s, ψ) are related by a 2-cell $(i, j) \underset{(a,\mathsf{s},\psi)}{\overset{(a,\mathsf{r},\varphi)}{\underset{\Downarrow\,(a,\theta)}{\rightrightarrows}}} (i', j')$ in $\mathcal{K}_{\mathsf{X}}$. Then:



where the first equality holds because θ represents *id* (the identity of X^a), and the second one is valid because h is a pseudo-cone.

The general case reduces to this one as follows: we have

$$\begin{tikzcd}[row sep=tiny] & (i', j') \\ (i, j) \arrow[ur, "{(a,\mathsf{r},\varphi)}"] \arrow[dr, "{(a,\mathsf{s},\psi)}"'] & \\ & (i'', j') \end{tikzcd} \in \mathcal{K}_{\mathsf{X}}.$$

Take $\begin{array}{l} i' \xrightarrow{u} k \\ i'' \xrightarrow[v]{} k \end{array}$ in $\mathcal{I}_j$. This yields a particular instance of lemma 2.2.16:

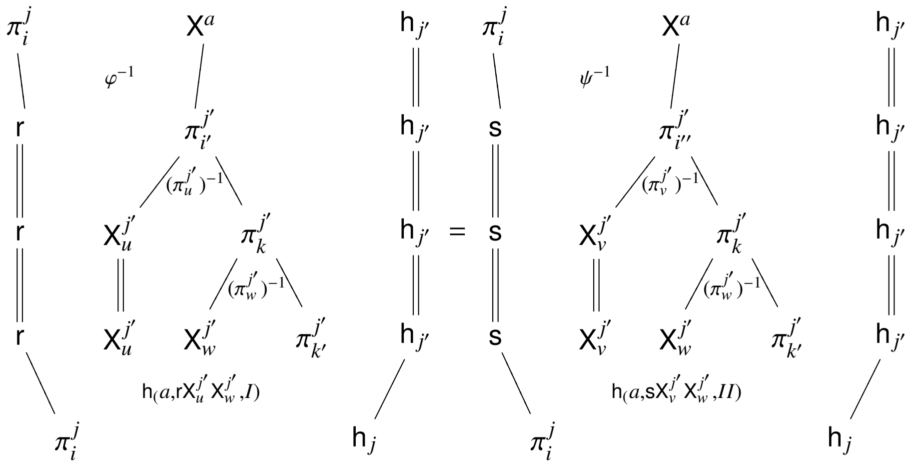
$$\begin{tikzcd}
\mathsf{X}^{j'} \arrow[r, "\pi^{j'}_k"] \arrow[d, shift right=3, "\mathsf{X}^a"'] \arrow[d, shift left=3, "\mathsf{X}^a"] \arrow[d, phantom, "\overset{id}{\Rightarrow}"] & \mathsf{X}^{j'}_k \arrow[d, shift right=3, "\mathsf{rX}^{j'}_u"'] \arrow[d, shift left=3, "\mathsf{sX}^{j'}_v"] \\
\mathsf{X}^j \arrow[r, "\pi^j_i"'] & \mathsf{X}^j_i
\end{tikzcd}$$

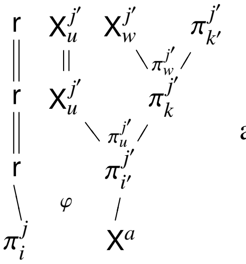
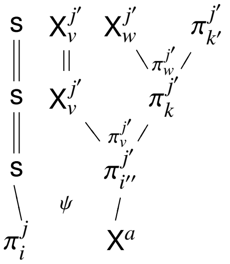
with $\left(\mathsf{rX}^{j'}_u,\ \begin{array}{cccc} \mathsf{r} & \mathsf{X}^{j'}_u & & \mathsf{X}^{j'}_k \\ \| & & \backslash\,\pi^{j'}_u\,/ & \\ \mathsf{r} & & \pi^{j'}_{i'} & \\ \backslash & \varphi & / & \\ \pi^j_i & & \mathsf{X}^a & \end{array}\right)$ and $\left(\mathsf{sX}^{j'}_v,\ \begin{array}{cccc} \mathsf{s} & \mathsf{X}^{j'}_v & & \mathsf{X}^{j'}_k \\ \| & & \backslash\,\pi^{j'}_v\,/ & \\ \mathsf{s} & & \pi^{j'}_{i''} & \\ \backslash & \psi & / & \\ \pi^j_i & & \mathsf{X}^a & \end{array}\right)$ both representing X^a. It follows that there exists $k \xrightarrow{w} k' \in \mathcal{I}_{j'}$ and $\begin{tikzcd} \mathsf{X}^{j'}_{k'} \arrow[r, shift left=2, "\mathsf{rX}^{j'}_u\mathsf{X}^{j'}_w"] \arrow[r, shift right=2, "\mathsf{sX}^{j'}_v\mathsf{X}^{j'}_w"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathsf{X}^j_i \end{tikzcd} \in \mathcal{C}$ such that

$$\left(\theta,\ \mathsf{rX}^{j'}_u\mathsf{X}^{j'}_w,\ \begin{array}{ccccc} \mathsf{r} & \mathsf{X}^{j'}_u & \mathsf{X}^{j'}_w & & \pi^{j'}_{k'} \\ \| & \| & & \backslash\,\pi^{j'}_w\,/ & \\ \mathsf{r} & \mathsf{X}^{j'}_u & & \pi^{j'}_k & \\ \| & & \backslash\,\pi^{j'}_u\,/ & & \\ \mathsf{r} & & \pi^{j'}_{i'} & & \\ \backslash & \varphi & / & & \\ \pi^j_i & & \mathsf{X}^a & & \end{array},\ \mathsf{sX}^{j'}_v\mathsf{X}^{j'}_w,\ \begin{array}{ccccc} \mathsf{s} & \mathsf{X}^{j'}_v & \mathsf{X}^{j'}_w & & \pi^{j'}_{k'} \\ \| & \| & & \backslash\,\pi^{j'}_w\,/ & \\ \mathsf{s} & \mathsf{X}^{j'}_v & & \pi^{j'}_k & \\ \| & & \backslash\,\pi^{j'}_v\,/ & & \\ \mathsf{s} & & \pi^{j'}_{i''} & & \\ \backslash & \psi & / & & \\ \pi^j_i & & \mathsf{X}^a & & \end{array}\right)$$

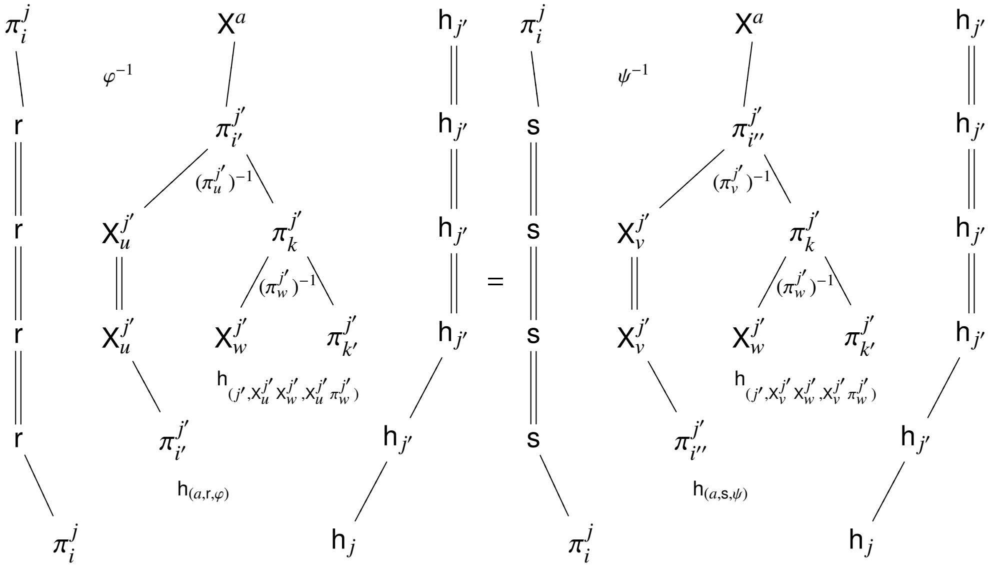
represents *id* (the identity of X^a).

Considering $\left(\mathsf{rX}^{j'}_u\mathsf{X}^{j'}_w,\ \begin{array}{ccccc} \mathsf{r} & \mathsf{X}^{j'}_u & \mathsf{X}^{j'}_w & & \pi^{j'}_{k'} \\ \| & \| & & \backslash\,\pi^{j'}_w\,/ & \\ \mathsf{r} & \mathsf{X}^{j'}_u & & \pi^{j'}_k & \\ \| & & \backslash\,\pi^{j'}_u\,/ & & \\ \mathsf{r} & & \pi^{j'}_{i'} & & \\ \backslash & \varphi & / & & \\ \pi^j_i & & \mathsf{X}^a & & \end{array}\right)$ and $\left(\mathsf{sX}^{j'}_v\mathsf{X}^{j'}_w,\ \begin{array}{ccccc} \mathsf{s} & \mathsf{X}^{j'}_v & \mathsf{X}^{j'}_w & & \pi^{j'}_{k'} \\ \| & \| & & \backslash\,\pi^{j'}_w\,/ & \\ \mathsf{s} & \mathsf{X}^{j'}_v & & \pi^{j'}_k & \\ \| & & \backslash\,\pi^{j'}_v\,/ & & \\ \mathsf{s} & & \pi^{j'}_{i''} & & \\ \backslash & \psi & / & & \\ \pi^j_i & & \mathsf{X}^a & & \end{array}\right)$ both representing X^a, we have a situation that corresponds to the previous case. Thus:

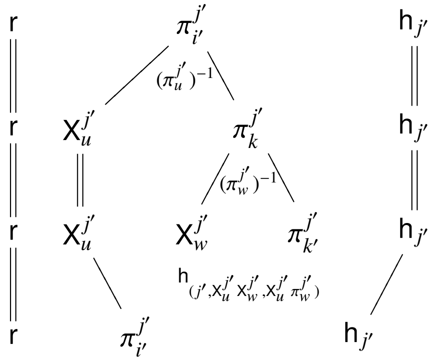


where $I =$  and $II =$ 

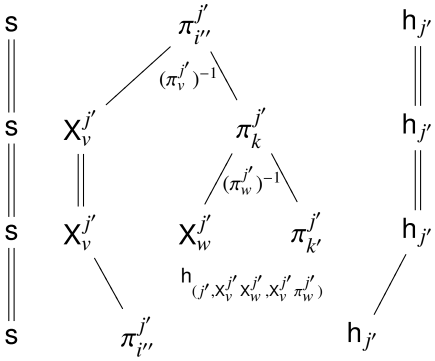
Then, since h is a pseudo-cone, we have that



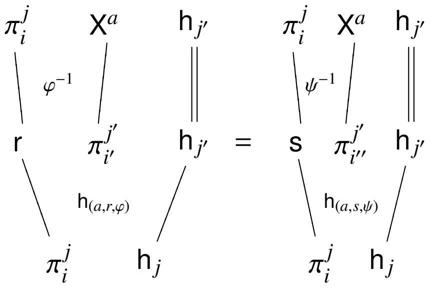
From 2.3.4 and the fact that X^j is a pseudo-cone, it follows that



and



are identities. So



as we wanted to prove. □

Proof of Claim 2. Given any $i \xrightarrow{u} k \in \mathcal{I}_j$, we have to check the PCM equation in 1.2.2. Given the pair (s, ψ) used to define ρ_k, it is possible to choose a pair (r, φ) to define ρ_i in such a way that the equation holds. This arguing is justified by Claim 1. □

2.3.5 Remark. A similar proof can be done in case X is only a pseudo-functor. Replace the equality 2-cell $\begin{array}{c} \mathsf{X}^a \quad \mathsf{X}^b \\ \diagdown = \diagup \\ \mathsf{X}^{ba} \end{array}$ by the structure 2-cell $\begin{array}{c} \mathsf{X}^a \quad \mathsf{X}^b \\ \diagdown\, \alpha^{\mathsf{X}}_{a,b} \diagup \\ \mathsf{X}^{ba} \end{array}$ in the elevators. □

2.3.6 Theorem. *2-$\mathcal{P}ro(C)$ is closed under small 2-cofiltered pseudo-limits. Considering the equivalence in 2.1.3, it follows that the inclusion $\mathcal{H}om(C, Cat)_{fc} \subset \mathcal{H}om(C, Cat)$ is closed under small 2-filtered pseudo-colimits* □

Proof. It is immediate from 2.3.3. □

Having 2.3.5 in mind, from the fact that 2-$\mathcal{P}ro_p(C)$ is pseudo-equivalent to 2-$\mathcal{P}ro(C)$ it follows easily that:

2.3.7 Corollary. *2-$\mathcal{P}ro_p(C)$ is closed under small 2-cofiltered bi-limits of pseudo-functors.* □

2.4 Universal property of $2\text{-}\mathcal{P}ro(\mathcal{C})$

In this subsection we prove for 2-pro-objects the universal property established for pro-objects in [1, Ex. I, Prop. 8.7.3.]. Consider the 2-functor $\mathcal{C} \xrightarrow{c} 2\text{-}\mathcal{P}ro(\mathcal{C})$ of Corollary 2.1.10 and a 2-pro-object $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}}$. Given a 2-functor $\mathcal{C} \xrightarrow{\mathsf{F}} \mathcal{E}$ into a 2-category closed under small 2-cofiltered pseudo-limits, we can naively extend F into a 2-cofiltered pseudo-limit preserving 2-functor $2\text{-}\mathcal{P}ro(\mathcal{C}) \xrightarrow{\widehat{\mathsf{F}}} \mathcal{E}$ by defining $\widehat{\mathsf{F}}\mathsf{X} = \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathsf{F}\mathsf{X}_i$. This is just part of a 2-equivalence of 2-categories that we develop with the necessary precision in this subsection. First the universal property should be wholly established for $\mathcal{E} = \mathcal{C}at$, and only afterwards can be lifted to any 2-category $\mathcal{E}$ closed under small 2-cofiltered pseudo-limits.

2.4.1 Lemma. *Let $\mathcal{C}$ be a 2-category and $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{C}at$ a 2-functor. Then, there exist a 2-functor $\widehat{\mathsf{F}} : 2\text{-}\mathcal{P}ro(\mathcal{C}) \longrightarrow \mathcal{C}at$ that preserves small 2-cofiltered pseudo-limits, and an isomorphism $\widehat{\mathsf{F}}c \xrightarrow{\cong} \mathsf{F}$ in $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)$.*

Proof. Let $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}} \in 2\text{-}\mathcal{P}ro(\mathcal{C})$ be a 2-pro-object. Define:

$$\widehat{\mathsf{F}}\mathsf{X} \;=\; (\mathcal{H}om(\mathcal{C}, \mathcal{C}at)(-, \mathsf{F}) \circ \mathsf{L})\mathsf{X} \;=\; \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(\underset{\overrightarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathcal{C}(\mathsf{X}_i, -), \mathsf{F}) \xrightarrow{\cong}$$

$$\xrightarrow{\cong} \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathcal{H}om(\mathcal{C}, \mathcal{C}at)(\mathcal{C}(\mathsf{X}_i, -), \mathsf{F}) \xrightarrow{\cong} \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\, \mathsf{F}\mathsf{X}_i.$$

Where L is the 2-functor of 2.1.3, the first isomorphism is by definition of pseudo-colimit 1.2.4, and the second is due to the Yoneda isomorphism 1.1.37. Since it is a 2-equivalence, the 2-functor L preserves any pseudo-limit. Then by Corollary 2.3.6 it follows that the composite $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)(-, \mathsf{F}) \circ \mathsf{L}$ preserves small 2-cofiltered pseudo-limits □

2.4.2 Theorem. *Let $\mathcal{C}$ be any 2-category. Then, pre-composition with $\mathcal{C} \xrightarrow{c} 2\text{-}\mathcal{P}ro(\mathcal{C})$ is a 2-equivalence of 2-categories:*

$$\mathcal{H}om(2\text{-}\mathcal{P}ro(\mathcal{C}), \mathcal{C}at)_+ \xrightarrow{\quad c^* \quad} \mathcal{H}om(\mathcal{C}, \mathcal{C}at)$$

(where $\mathcal{H}om(2\text{-}\mathcal{P}ro(\mathcal{C}), \mathcal{C}at)_+$ stands for the full subcategory whose objects are those 2-functors that preserve small 2-cofiltered pseudo-limits).

Proof. We will check that the 2-functor c^* is essentially surjective on objects and 2-fully-faithful (see 1.1.29):

- *Essentially surjective on objects*: It follows from lemma 2.4.1.

- *2-fully-faithful*: We will check that if F and G are 2-functors from 2-$\mathcal{P}ro(\mathcal{C})$ to $\mathcal{C}at$ that preserve small 2-cofiltered pseudo-limits, then

$$\mathcal{H}om(2\text{-}\mathcal{P}ro(\mathcal{C}),\mathcal{C}at)_+(\mathsf{F},\mathsf{G}) \xrightarrow{c^*} \mathcal{H}om(\mathcal{C},\mathcal{C}at)(\mathsf{F}c,\mathsf{G}c) \tag{2.4.3}$$

  is an isomorphism of categories.

  Let $\mathsf{F}c \; \substack{\xrightarrow{\theta c} \\ \Downarrow \mu \\ \xrightarrow[\eta c]{}} \; \mathsf{G}c \in \mathcal{H}om(\mathcal{C},\mathcal{C}at)(\mathsf{F}c,\mathsf{G}c)$. It can be easily checked that composites $\left\{\mathsf{FX} \xrightarrow{\mathsf{F}\pi_i} \mathsf{FX}_i \; \substack{\xrightarrow{\theta \mathsf{X}_i} \\ \Downarrow \mu \mathsf{X}_i \\ \xrightarrow[\eta \mathsf{X}_i]{}} \; \mathsf{GX}_i\right\}_{i\in\mathcal{I}}$ determine two pseudo-cones for GX together with a morphism of pseudo-cones. Since G preserves small 2-cofiltered pseudo-limits, post-composing with $\mathsf{GX} \xrightarrow{\mathsf{G}\pi_i} \mathsf{GX}_i$ is an isomorphism of categories $\mathcal{C}at(\mathsf{FX},\mathsf{GX}) \xrightarrow{(\mathsf{G}\pi)_*} \mathsf{PC}_{\mathcal{C}at}(\mathsf{FX},\mathsf{GX})$. It follows that there exists a unique 2-cell in $\mathcal{C}at$, $\mathsf{FX} \; \substack{\xrightarrow{\theta'_{\mathsf{X}}} \\ \Downarrow \mu'_{\mathsf{X}} \\ \xrightarrow[\eta'_{\mathsf{X}}]{}} \; \mathsf{GX}$, such that $\mathsf{G}\pi_i\theta'_{\mathsf{X}} = \theta_{\mathsf{X}_i}\mathsf{F}\pi_i$, $\mathsf{G}\pi_i\eta'_{\mathsf{X}} = \eta_{\mathsf{X}_i}\mathsf{F}\pi_i$, and $\mathsf{G}\pi_i\mu'_{\mathsf{X}} = \mu_{\mathsf{X}_i}\mathsf{F}\pi_i$, $\forall i \in \mathcal{I}$. It is not difficult to check that θ'_{X}, η'_{X} are in fact 2-natural on X, and that μ'_{X} is a modification. Clearly $\theta' c = \theta$, $\eta' c = \eta$, and $\mu' c = \mu$. Thus 2.4.3 is an isomorphism of categories.

□

2.4.4 Lemma. *Let $\mathcal{C}$ be a 2-category, $\mathcal{E}$ a 2-category closed under small 2-cofiltered pseudo-limits and* $\mathsf{F} : \mathcal{C} \longrightarrow \mathcal{E}$ *a 2-functor. Then, there exists a 2-functor* $\widehat{\mathsf{F}} : 2\text{-}\mathcal{P}ro(\mathcal{C}) \longrightarrow \mathcal{E}$ *that preserves small 2-cofiltered pseudo-limits, and an isomorphism* $\widehat{\mathsf{F}}c \xrightarrow{\cong} \mathsf{F}$ *in* $\mathcal{H}om(\mathcal{C},\mathcal{E})$.

Proof. If $\mathsf{X} = (\mathsf{X}_i)_{i\in\mathcal{I}} \in 2\text{-}\mathcal{P}ro(\mathcal{C})$, define $\widehat{\mathsf{F}}\mathsf{X} = \underset{\overleftarrow{i\in\mathcal{I}}}{\mathsf{Lim}}\, \mathsf{FX}_i$. We will prove that this is the object function part of a 2-functor, and that this 2-functor has the rest of the properties asserted in the proposition.

Consider the composition $\mathsf{y}_{(-)}\mathsf{F} : \mathcal{C} \xrightarrow{\mathsf{F}} \mathcal{E} \xrightarrow{\mathsf{y}_{(-)}} \mathcal{H}om(\mathcal{E}^{op},\mathcal{C}at)$, where $\mathsf{y}_{(-)}$ is the Yoneda 2-functor (1.1.36). Under the isomorphism 1.1.34 this corresponds to a 2-functor $\mathcal{E}^{op} \longrightarrow \mathcal{H}om(\mathcal{C},\mathcal{C}at)$. Composing this 2-functor with a quasi-inverse $\widehat{(-)}$ for the 2-equivalence in 2.4.2, we obtain a 2-functor $\mathcal{E}^{op} \longrightarrow \mathcal{H}om(2\text{-}\mathcal{P}ro(\mathcal{C}),\mathcal{C}at)_+$, which in turn corresponds to a 2-functor $2\text{-}\mathcal{P}ro(\mathcal{C}) \xrightarrow{\widetilde{\mathsf{F}}} \mathcal{H}om(\mathcal{E}^{op},\mathcal{C}at)$. The 2-functor $\widetilde{\mathsf{F}}$ preserves small 2-cofiltered pseudo-limits because they are computed pointwise in $\mathcal{H}om(\mathcal{E}^{op},\mathcal{C}at)$

(1.2.7). By chasing the isomorphisms one can check that we have the following diagram:

$$\widetilde{\mathsf{F}}c \xrightarrow{\cong} \mathsf{y}_{(-)}\mathsf{F}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "c"] \arrow[d, "\mathsf{F}"'] \arrow[dr, phantom, "\Downarrow\cong"] & 2\text{-}\mathcal{P}ro(\mathcal{C}) \arrow[d, "\widetilde{\mathsf{F}}"] \\ \mathcal{E} \arrow[r, "\mathsf{y}_{(-)}"'] & \mathcal{H}om(\mathcal{E}^{op}, \mathcal{C}at) \end{tikzcd} \tag{2.4.5}$$

Consider the following chain of isomorphisms (the first and the third because $\widetilde{\mathsf{F}}$ and $\mathsf{y}_{(-)}$ preserve pseudo-limits (1.2.13), and the middle one given by 2.4.5):

$$\widetilde{\mathsf{F}}\mathsf{X} = \widetilde{\mathsf{F}}\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\,\mathsf{X}_i \xrightarrow{\cong} \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\,\widetilde{\mathsf{F}}c\mathsf{X}_i \xrightarrow{\cong} \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\,\mathsf{y}_{(-)}\mathsf{F}\mathsf{X}_i \xleftarrow{\cong} \mathsf{y}_{(-)}\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\,\mathsf{F}\mathsf{X}_i.$$

This shows that $\widetilde{\mathsf{F}}\mathsf{X}$ is in the essential image of $\mathsf{y}_{(-)}$. Since $\mathsf{y}_{(-)}$ is 2-fully-faithful (1.1.38), it follows there is a factorization $\mathsf{y}_{(-)}\widehat{\mathsf{F}} \xrightarrow{\cong} \widetilde{\mathsf{F}}$, given by a 2-functor $2\text{-}\mathcal{P}ro(\mathcal{C}) \xrightarrow{\widehat{\mathsf{F}}} \mathsf{E}$. Clearly $\widehat{\mathsf{F}}$ preserves small 2-cofiltered pseudo-limits. We have $\mathsf{y}_{(-)}\widehat{\mathsf{F}}c \xrightarrow{\cong} \widetilde{\mathsf{F}}c \xrightarrow{\cong} \mathsf{y}_{(-)}\mathsf{F}$. Finally, the fully-faithfulness of $\mathsf{y}_{(-)}$ provides an isomorphism $\widehat{\mathsf{F}}c \xrightarrow{\cong} \mathsf{F}$. This finishes the proof. □

Exactly the same proof of theorem 2.4.2 applies with an arbitrary 2-category $\mathcal{E}$ in place of $\mathcal{C}at$, and we have:

2.4.6 Theorem. *Let $\mathcal{C}$ be any 2-category, and $\mathcal{E}$ a 2-category closed under small 2-cofiltered pseudo-limits. Then, pre-composition with $\mathsf{C} \xrightarrow{c} 2\text{-}\mathcal{P}ro(\mathcal{C})$ is a 2-equivalence of 2-categories:*

$$\mathcal{H}om(2\text{-}\mathcal{P}ro(\mathcal{C}), \mathcal{E})_+ \xrightarrow{c^*} \mathcal{H}om(\mathcal{C}, \mathcal{E})$$

Where $\mathcal{H}om(2\text{-}\mathcal{P}ro(\mathcal{C}), \mathcal{E})_+$ stands for the full subcategory whose objects are those 2-functors that preserve small 2-cofiltered pseudo-limits. □

From theorem 2.4.6 it follows automatically the pseudo-functoriality of the assignment of the 2-category $2\text{-}\mathcal{P}ro(\mathcal{C})$ to each 2-category $\mathcal{C}$, and in such a way that c becomes a pseudo-natural transformation. But we can do better:

If we put $\mathcal{E} = 2\text{-}\mathcal{P}ro(\mathcal{D})$ in 2.4.6 it follows there is a 2-functor (post-composing with c followed by a quasi-inverse in 2.4.6)

$$\mathcal{H}om(\mathcal{C}, \mathcal{D}) \xrightarrow{\widehat{(-)}} \mathcal{H}om(2\text{-}\mathcal{P}ro(\mathcal{C}),\ 2\text{-}\mathcal{P}ro(\mathcal{D}))_+\,, \tag{2.4.7}$$

and for each 2-functor $\mathcal{C} \xrightarrow{\mathsf{F}} \mathcal{D}$, a diagram:

$$\begin{tikzcd}
2\text{-}\mathcal{P}ro(\mathcal{C}) \arrow[r, "\widehat{\mathsf{F}}"] & 2\text{-}\mathcal{P}ro(\mathcal{D}) \\
\mathcal{C} \arrow[u, "c"] \arrow[r, "\mathsf{F}"'] \arrow[ur, phantom, "\cong \Downarrow"] & \mathcal{D} \arrow[u, "c"']
\end{tikzcd} \tag{2.4.8}$$

Given any 2-pro-object $\mathsf{X} \in 2\text{-}\mathcal{P}ro(\mathcal{C})$, set $2\text{-}\mathcal{P}ro(\mathsf{F})(\mathsf{X}) = \widehat{\mathsf{F}}\mathsf{X}$. It is straightforward to check that this determines a 2-functor

$$2\text{-}\mathcal{P}ro(\mathcal{C}) \xrightarrow{2\text{-}\mathcal{P}ro(\mathsf{F})} 2\text{-}\mathcal{P}ro(\mathcal{D})$$

making diagram 2.4.8 commutative. It follows we have an isomorphism $\widehat{\mathsf{F}}\mathsf{X} \xrightarrow{\cong} 2\text{-}\mathcal{P}ro(\mathsf{F})(\mathsf{X})$ 2-natural in X. This shows that the 2-functor $2\text{-}\mathcal{P}ro(\mathsf{F})$ preserves small 2-cofiltered pseudo-limits because $\widehat{\mathsf{F}}$ does. Also, it follows that $2\text{-}\mathcal{P}ro(\mathsf{F})$ determines a 2-functor as in 2.4.7. In conclusion, denoting now by $2\text{-}\mathcal{CAT}$ the 2-category of locally small 2-categories (see 1.1.18) we have:

2.4.9 Theorem. *The definition* $2\text{-}\mathcal{P}ro(\mathsf{F})(\mathsf{X}) = \widehat{\mathsf{F}}\mathsf{X}$ *determines a 2-functor*

$$2\text{-}\mathcal{P}ro(\text{-}) : 2\text{-}\mathcal{CAT} \longrightarrow 2\text{-}\mathcal{CAT}_+$$

in such a way that c becomes a 2-natural transformation (where $2\text{-}\mathcal{CAT}_+$ *is the full sub 2-category of locally small 2-categories closed under small 2-cofiltered pseudo limits and small pseudo-limit preserving 2-functors).* □

Resumen en castellano de la sección 2

En esta sección se encuentran algunos de los resultados claves de este trabajo. En 2.1, dada una 2-categoría $\mathcal{C}$ definimos la 2-categoría 2-$\mathcal{P}ro(\mathcal{C})$ cuyos objetos llamamos 2-pro-objetos. Un 2-pro-objeto de $\mathcal{C}$ es un 2-funtor a valores en $\mathcal{C}$ (o diagrama en $\mathcal{C}$) indexado por una 2-categoría 2-cofiltrante y será el pseudo-límite de su propio diagrama en la 2-categoría 2-$\mathcal{P}ro(\mathcal{C})$. También en 2.1, establecemos la fórmula básica que describe los morfismos y las 2-celdas entre 2-pro-objetos en términos de pseudo-límites y pseudo-colímites de las categorías de morfismos de $\mathcal{C}$.

En 2.2, establecemos ciertos lemas técnicos que nos permiten operar con 2-pro-objetos en las secciones siguientes.

En 2.3, dada $\mathcal{J}$ una 2-categoría 2-filtrante y un funtor $\mathcal{J}^{op} \xrightarrow{\mathsf{X}}$ 2-$\mathcal{P}ro(\mathcal{C})$, construimos un 2-pro-objeto que será el pseudo-límite de X en 2-$\mathcal{P}ro(\mathcal{C})$. Para esto, primero construimos una 2-categoría 2-filtrante que sirve como 2-categoría de índices para el pseudo-límite (Definición 2.3.1 y proposición 2.3.3).

Finalmente, en 2.4, enunciamos y demostramos la propiedad universal de 2-$\mathcal{P}ro(\mathcal{C})$ (Teorema 2.4.6) establecida para pro-objetos en [1, Ex. I, Prop. 8.7.3.]. Considerar el 2-funtor $\mathcal{C} \xrightarrow{c}$ 2-$\mathcal{P}ro(\mathcal{C})$ del corolario 2.1.10 y un 2-pro-objeto $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}}$. Dado un 2-funtor $\mathcal{C} \xrightarrow{\mathsf{F}} \mathcal{E}$ a una 2-categoría cerrada por pseudo-límites 2-cofiltrantes, podemos extender F a un 2-funtor que preserva pseudo-límites 2-cofiltrantes 2-$\mathcal{P}ro(\mathcal{C}) \xrightarrow{\widehat{\mathsf{F}}} \mathcal{E}$ definiendo $\widehat{\mathsf{F}}\mathsf{X} = \underset{\overleftarrow{i \in \mathcal{I}}}{\mathsf{Lim}}\ \mathsf{F}\mathsf{X}_i$. Esto es solo una parte de una 2-equivalencia de 2-categorías que desarrollamos aquí. Primero debemos desarrollar completamente la propiedad universal para $\mathcal{E} = Cat$, y solo después de esto, puede ser traspasada a una 2-categoría cualquiera $\mathcal{E}$ cerrada por pseudo-límites 2-cofiltrantes.

También consideramos en esta sección la 2-categoría 2-$\mathcal{P}ro_p(\mathcal{C})$ que es "retract pseudo-equivalent" a 2-$\mathcal{P}ro(\mathcal{C})$, 2.1.5, hecho que se sigue de que los 2-funtores a valores en Cat asociados a 2-pro-objetos son flexibles. Esta 2-categoría será esencial en la sección 5 y probará ser interesante en sí misma.

3 Reindexing properties for 2-pro-objects

In this section we prove some reindexing properties for the 2-categories 2-$\mathcal{P}ro(\mathcal{C})$ and 2-$\mathcal{P}ro_p(\mathcal{C})$ that will be used to determine closed 2-bmodel structures on them (see 4.1.3) as Edwards and Hastings do in [12] for $\mathsf{Pro(C)}$ in the 1-dimensional case. The reindexing properties for $\mathsf{Pro(C)}$ can be found in [3] or [1].

3.1 Reindexing for objects

3.1.1 Proposition. *Let* $\mathsf{X} = (\mathsf{X}_j)_{j\in\mathcal{J}}$ *be a 2-pro-object and* $\mathsf{F} : \mathcal{I} \to \mathcal{J}$ *be a 2-cofinal 2-functor with* $\mathcal{I}$ *a 2-filtered 2-category. Then, the 2-pro-object* $\mathsf{X}_\mathsf{F} = (\mathsf{X}_{\mathsf{F}(i)})_{i\in\mathcal{I}}$ *is equivalent to* X *in* 2-$\mathcal{P}ro(\mathcal{C})$.

Proof. First note that the 2-pro-objects X and X_F are equivalent if the canonical 2-natural transformation $\underset{i\in\mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathcal{C}(\mathsf{X}_{\mathsf{F}(i)}, -) \overset{\theta}{\Rightarrow} \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathcal{C}(\mathsf{X}_j, -)$ is an equivalence in $\mathcal{H}om(\mathcal{C}, \mathcal{C}at)$.

Now, for each $\mathsf{C} \in \mathcal{C}$, consider the 2-functor

$$\begin{tikzcd} \mathcal{J} \arrow[r] & \mathcal{C}at \\ j \arrow[r, mapsto] & \mathcal{C}(\mathsf{X}_j, \mathsf{C}) \end{tikzcd}$$

. Then, by 1.3.9, $\underset{i\in\mathcal{I}}{\underrightarrow{\mathsf{Lim}}}\, \mathcal{C}(\mathsf{X}_{\mathsf{F}(i)}, \mathsf{C}) \overset{\theta_\mathsf{C}}{\to} \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\, \mathcal{C}(\mathsf{X}_j, \mathsf{C})$ is an equivalence of categories $\forall \mathsf{C} \in \mathcal{C}$ and so, by 1.5.6 and 1.5.5, θ is an equivalence. □

3.1.2 Remark. If we denote the equivalence given by the previous proposition $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{X}_\mathsf{F}$ and its quasi-inverse $\bar{\mathsf{f}}$, then $\bar{\mathsf{f}}\mathsf{f} = id_\mathsf{X}$.

3.1.3 Corollary. *It follows from 1.3.15 that every 2-pro-object* $\mathsf{X} = (\mathsf{X}_j)_{j\in\mathcal{J}}$ *is equivalent in* 2-$\mathcal{P}ro(\mathcal{C})$ *to a 2-pro-object indexed by the cofinite filtered poset with a unique initial object* $\mathsf{M}(\mathcal{J})$ *via a 2-cofinal 2-functor* $\mathsf{M}(\mathcal{J}) \xrightarrow{\mathsf{F}} \mathcal{J}$.

Proof. It is immediate from 1.3.15 and 3.1.1. □

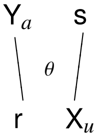
Since every morphism in 2-$\mathcal{P}ro(\mathcal{C})$ is a morphism in 2-$\mathcal{P}ro_p(\mathcal{C})$, 3.1.1 and 3.1.3 also hold in 2-$\mathcal{P}ro_p(\mathcal{C})$. However, it is worth mentioning that a proof similar to the one for 3.1.1 can be done for 2-$\mathcal{P}ro_p(\mathcal{C})$ and 1.5.6, 1.5.5 wouldn't be needed because of 1.1.20.

3.2 Reindexing for morphisms

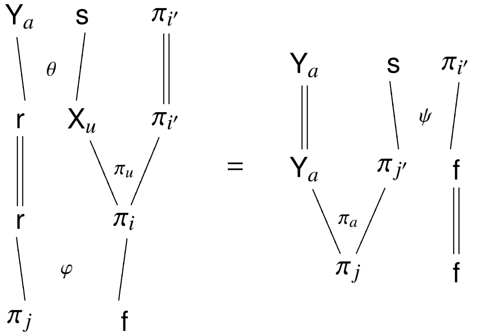
3.2.1 Definition. *Let* f *be a morphism from* $\mathsf{X} = (\mathsf{X}_i)_{i\in\mathcal{I}}$ *to* $\mathsf{Y} = (\mathsf{Y}_j)_{j\in\mathcal{J}}$ *in* 2-$\mathcal{P}ro(\mathcal{C})$. *We are going to denote by* $\mathcal{M}_\mathsf{f}$ *the following 2-category:*

Objects are the pairs (r, φ) *that represent* f.

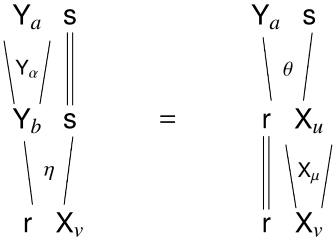
Morphisms $(\mathsf{r},\varphi) \to (\mathsf{s},\psi)$ *(*$\mathsf{r} : \mathsf{X}_i \to \mathsf{Y}_j$*,* $\mathsf{s} : \mathsf{X}_{i'} \to \mathsf{Y}_{j'}$*) are triplex* (u,a,θ) *where* $i \xrightarrow{u} i' \in \mathcal{I}$*,* $j \xrightarrow{a} j' \in \mathcal{J}$ *and* θ *is an invertible 2-cell*



such that



A 2-cell $(\mathsf{r},\varphi) \underset{(v,b,\eta)}{\overset{(u,a,\theta)}{\Downarrow}} (\mathsf{s},\psi)$ *is a pair* (μ,α) *where* $i \underset{v}{\overset{u}{\Downarrow \mu}} i' \in \mathcal{I}$ *and* $j \underset{b}{\overset{a}{\Downarrow \alpha}} j' \in \mathcal{J}$ *are such that*



Identities and compositions are defined in the obvious way.

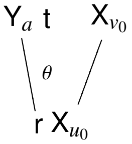
3.2.2 Lemma. *Let* f *be a morphism from* $\mathsf{X} = (\mathsf{X}_i)_{i\in\mathcal{I}}$ *to* $\mathsf{Y} = (\mathsf{Y}_j)_{j\in\mathcal{J}}$ *in* $2\text{-}\mathcal{P}ro(\mathcal{C})$*. The 2-category* $\mathcal{M}_\mathsf{f}$ *is 2-filtered and the 2-functors*

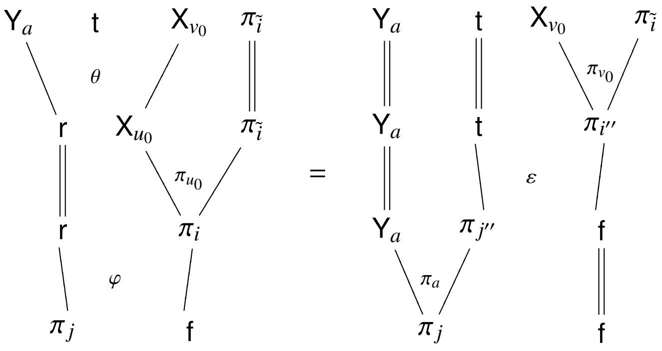
$$\begin{tikzcd} \mathcal{M}_\mathsf{f} \arrow[r] & \mathcal{I} \\ (\mathsf{r},\varphi) \arrow[r, mapsto] & i \end{tikzcd}, \quad \begin{tikzcd} \mathcal{M}_\mathsf{f} \arrow[r] & \mathcal{J} \\ (\mathsf{r},\varphi) \arrow[r, mapsto] & j \end{tikzcd}$$

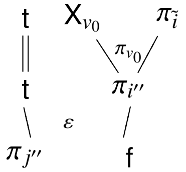
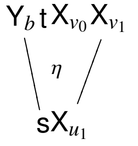
are 2-cofinal.

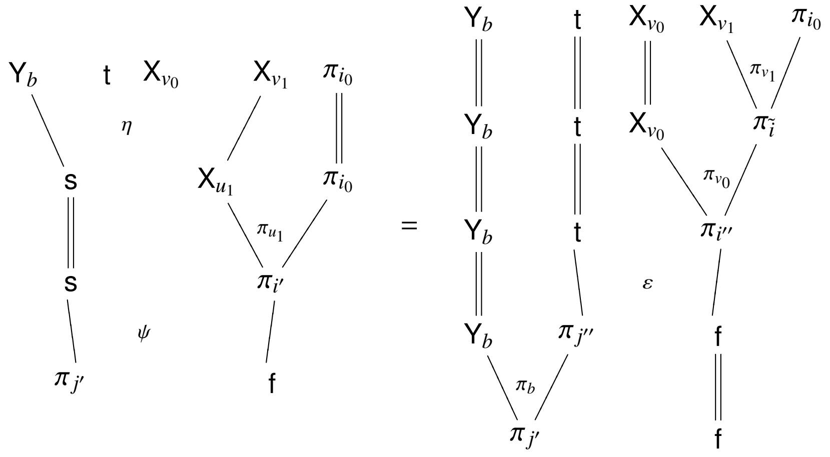
Proof. $\mathcal{M}_\mathsf{f}$ is 2-filtered:

F0: Let (r,φ), $(\mathsf{s},\psi) \in \mathcal{M}_\mathsf{f}$ ($\mathsf{r} : \mathsf{X}_i \to \mathsf{Y}_j$, $\mathsf{s} : \mathsf{X}_{i'} \to \mathsf{Y}_{j'}$). Since $\mathcal{J}$ is 2-filtered, we have $\begin{tikzcd}[row sep=small] j \arrow[dr, "a"] & \\ & j'' \\ j' \arrow[ur, "b"'] & \end{tikzcd} \in \mathcal{J}$ and, by 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have

$\mathsf{X}_{i''} \xrightarrow{\mathsf{t}} \mathsf{Y}_{j''} \in \mathcal{C}$ and an invertible 2-cell ϵ such that (t, ϵ) represents f and there are morphisms $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u"] & \\ & i'' \\ i' \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$. Then we have (r, φ) and (t, ϵ) both representing f equipped with morphisms $i \xrightarrow{u} i'' \in \mathcal{I}$, $j \xrightarrow{a} j'' \in \mathcal{J}$. So, by 2.2.8, there are morphisms $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u_0"] & \\ & \tilde{i} \\ i'' \arrow[ur, "v_0"'] & \end{tikzcd} \in \mathcal{I}$ and an invertible 2-cell  $\in \mathcal{C}$ such that



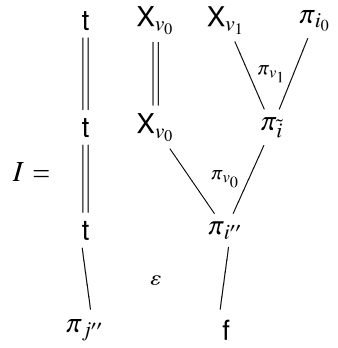
Then we have (s, ψ) and $(\mathsf{t}\mathsf{X}_{v_0},$  $)$ both representing f equipped with morphisms $i' \xrightarrow{v_0 v} \tilde{i} \in \mathcal{I}$, $j' \xrightarrow{b} j'' \in \mathcal{J}$. So, by 2.2.8, there are morphisms $\begin{tikzcd}[row sep=tiny] i' \arrow[dr, "u_1"] & \\ & i_0 \\ \tilde{i} \arrow[ur, "v_1"'] & \end{tikzcd} \in \mathcal{I}$ and an invertible 2-cell  $\in \mathcal{C}$ such that



It can be checked that there are morphisms in $\mathcal{M}_{\mathsf{f}}$

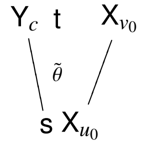
$$\begin{tikzcd}
(\mathsf{r},\varphi) \arrow[dr, "(v_1u_0,a,\theta\mathsf{X}_{v_1})"] & \\
& (\mathsf{t}\mathsf{X}_{v_0}\mathsf{X}_{v_1}, I) \ . \\
(\mathsf{s},\psi) \arrow[ur, "(u_1,b,\eta)"'] &
\end{tikzcd}$$

where

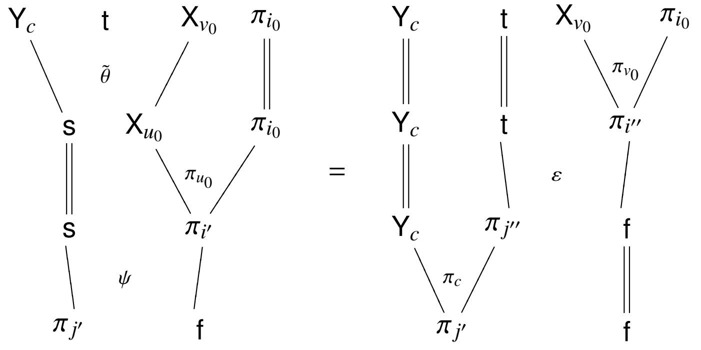


F1: Let $(\mathsf{r},\varphi) \underset{(v,b,\eta)}{\overset{(u,a,\theta)}{\rightrightarrows}} (\mathsf{s},\psi) \in \mathcal{M}_{\mathsf{f}}$. Since $\mathcal{J}$ is 2-filtered, we have $j' \overset{c}{\to} j''$ and an invertible 2-cell $ca \overset{\alpha}{\Rightarrow} cb \in \mathcal{J}$ and, by 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have $\mathsf{X}_{i''} \xrightarrow{\mathsf{t}} \mathsf{Y}_{j''} \in \mathcal{C}$ and an invertible 2-cell ϵ such that (t,ϵ) represents f and there are a morphism $i' \xrightarrow{w} i''$ and an invertible 2-cell $wu \overset{\mu}{\Longrightarrow} wv \in \mathcal{I}$.

Then we have (s,ψ) and (t,ϵ) both representing f equipped with morphisms $i' \xrightarrow{w} i'' \in \mathcal{I}$, $j' \xrightarrow{c} j'' \in \mathcal{J}$. So, by 2.2.8, there are morphisms
$$\begin{tikzcd}
i' \arrow[dr, "u_0"] & \\
& i_0 \ \in \mathcal{I} \\
i'' \arrow[ur, "v_0"'] &
\end{tikzcd}$$

and an invertible 2-cell 

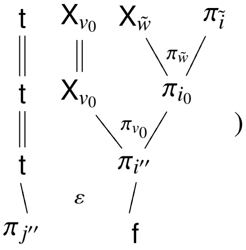
$\in \mathcal{C}$ such that



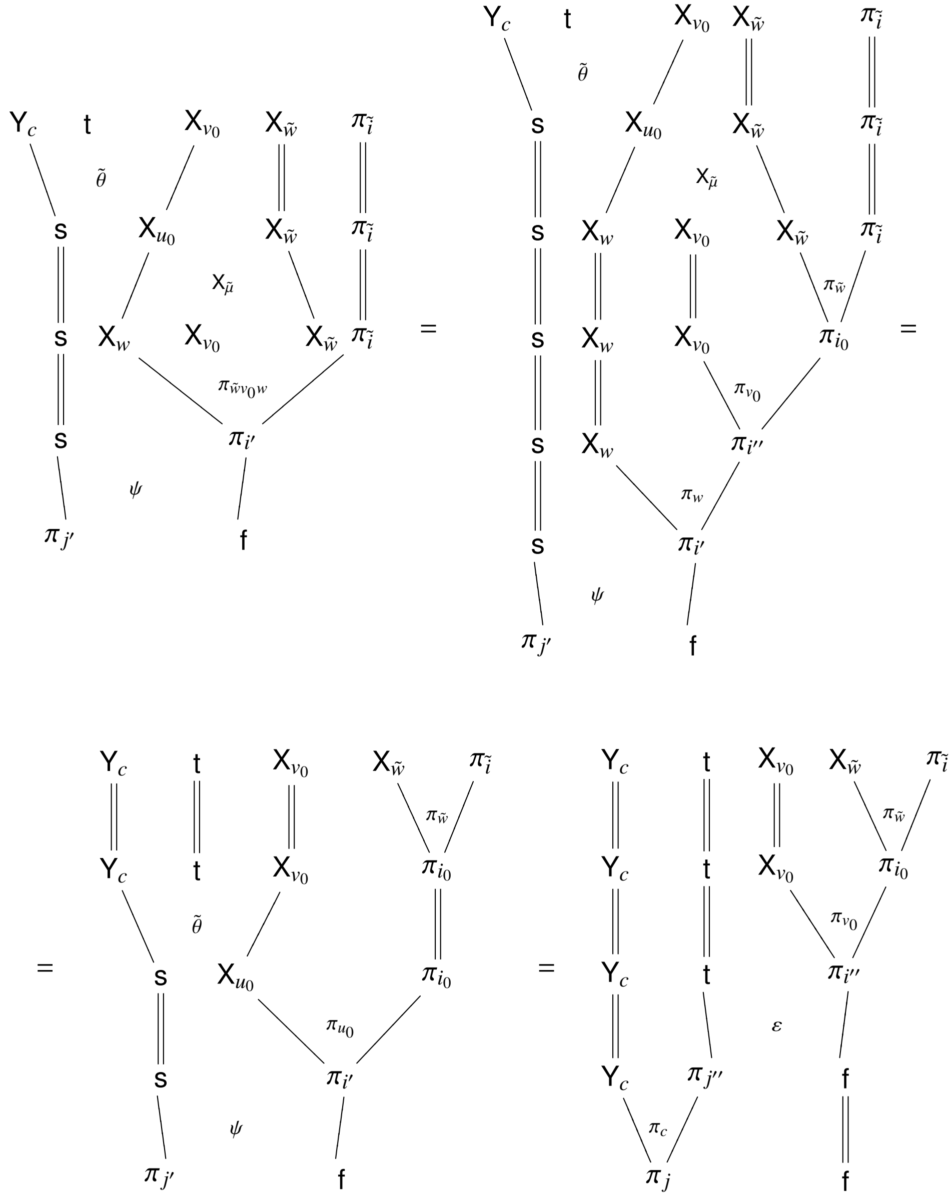
(3.2.3)

Plus, since $\mathcal{I}$ is 2-filtered, there is a morphism $i_0 \xrightarrow{\tilde{w}} \tilde{i}$ and an invertible 2-cell $\tilde{w}u_0 \overset{\tilde{\mu}}{\Longrightarrow} \tilde{w}v_0w \in \mathcal{I}$.

Let's check that we a have a morphism in $\mathcal{M}_{\mathsf{f}}$

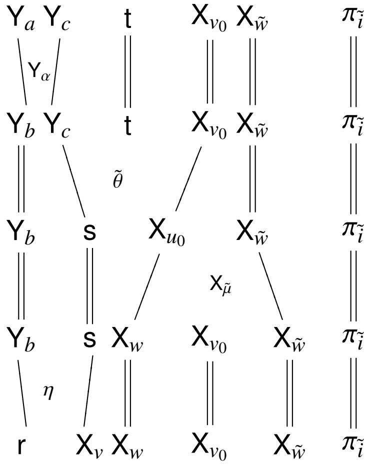
$(\mathsf{s}, \psi) \xrightarrow{(\tilde{w}v_0w, c, \mathsf{s}\mathsf{X}_{\tilde{\mu}} \circ \tilde{\theta}\mathsf{X}_{\tilde{w}})} (\mathsf{t}\mathsf{X}_{v_0}\mathsf{X}_{\tilde{w}},$ 

$)$:



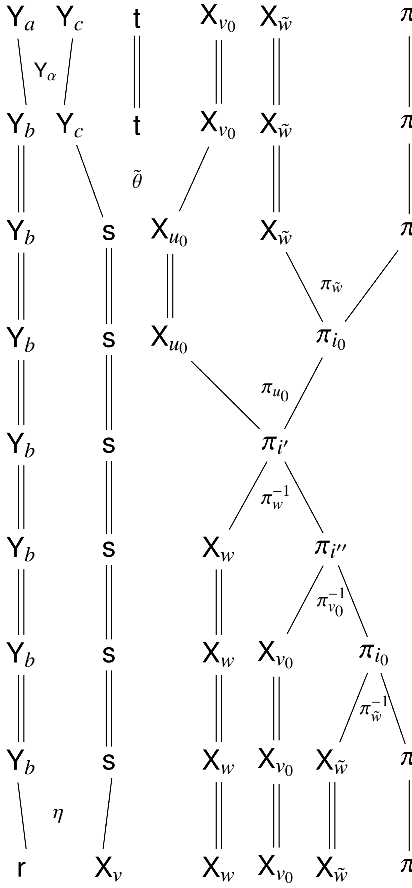
where the first equality follows from axiom PC1, the second one holds by elevators calculus plus axiom PC2 and the last one is due to (3.2.3).

Now observe that



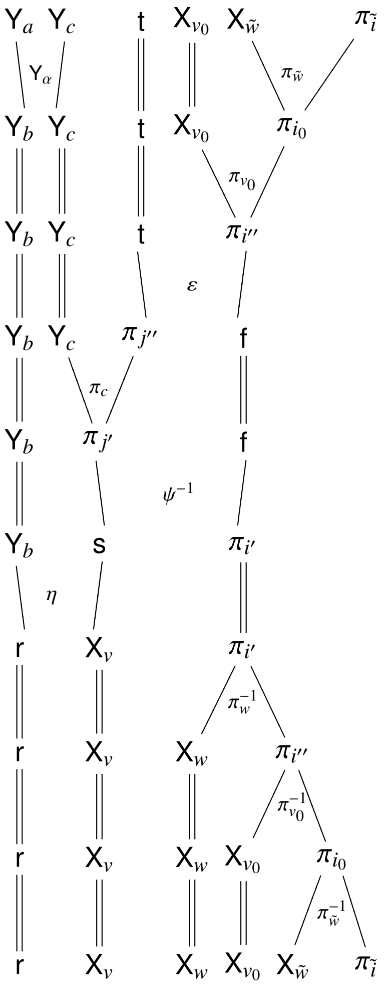
Y_a Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
Y_α
Y_b Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\tilde{\theta}$
Y_b s X_{u_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\mathsf{X}_{\tilde{\mu}}$
Y_b s X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
η
r X_v X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$

$=$



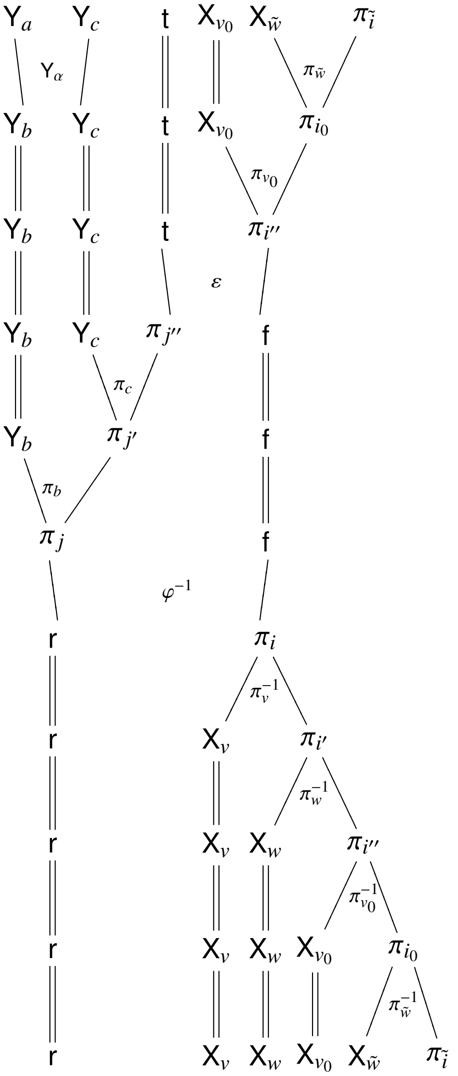
Y_a Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
Y_α
Y_b Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\tilde{\theta}$
Y_b s X_{u_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\pi_{\tilde{w}}$
Y_b s X_{u_0} π_{i_0}
π_{u_0}
Y_b s $\pi_{i'}$
π_w^{-1}
Y_b s X_w $\pi_{i''}$
$\pi_{v_0}^{-1}$
Y_b s X_w X_{v_0} π_{i_0}
$\pi_{\tilde{w}}^{-1}$
Y_b s X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
η
r X_v X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$

$=$



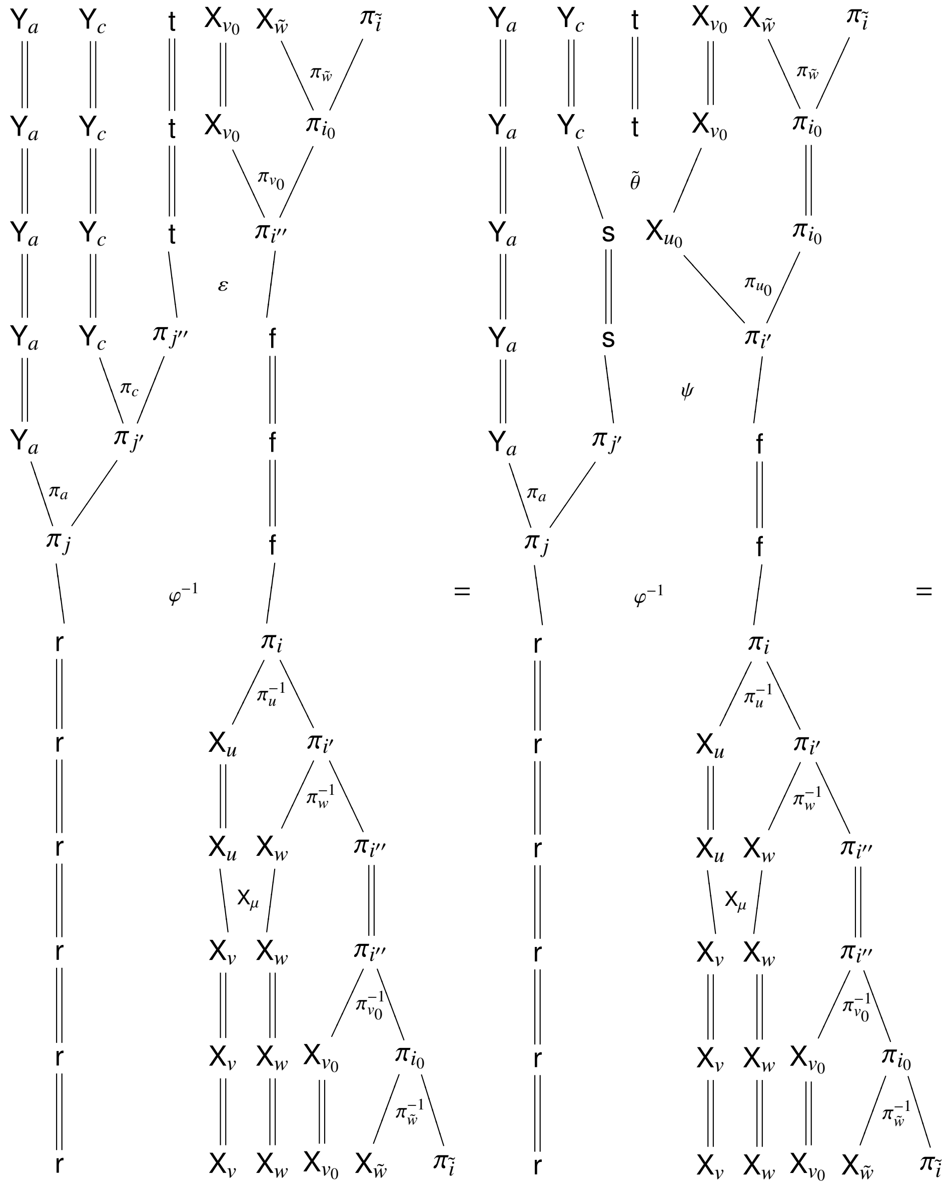
Y_a Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
Y_α
$\pi_{\tilde{w}}$
Y_b Y_c t X_{v_0} π_{i_0}
π_{v_0}
Y_b Y_c t $\pi_{i''}$
ε
Y_b Y_c $\pi_{j''}$ f
π_c
Y_b $\pi_{j'}$ f
ψ^{-1}
Y_b s $\pi_{i'}$
η
r X_v $\pi_{i'}$
π_w^{-1}
r X_v X_w $\pi_{i''}$
$\pi_{v_0}^{-1}$
r X_v X_w X_{v_0} π_{i_0}
$\pi_{\tilde{w}}^{-1}$
r X_v X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$

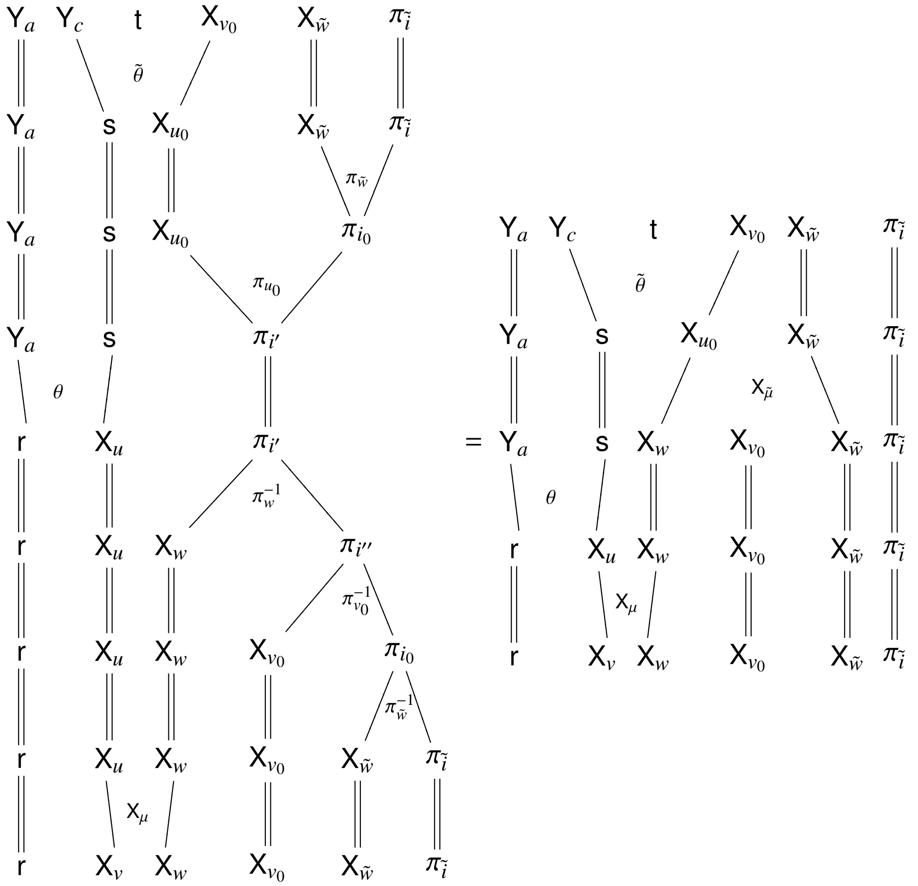
=



Y_a Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
Y_α
$\pi_{\tilde{w}}$
Y_b Y_c t X_{v_0} π_{i_0}
π_{v_0}
Y_b Y_c t $\pi_{i''}$
ε
Y_b Y_c $\pi_{j''}$ f
π_c
Y_b $\pi_{j'}$ f
π_b
π_j f
φ^{-1}
r π_i
π_v^{-1}
r X_v $\pi_{i'}$
π_w^{-1}
r X_v X_w $\pi_{i''}$
$\pi_{v_0}^{-1}$
r X_v X_w X_{v_0} π_{i_0}
$\pi_{\tilde{w}}^{-1}$
r X_v X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$

=


Y_a Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$ Y_a Y_c t X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\pi_{\tilde{w}}$ $\pi_{\tilde{w}}$
Y_a Y_c t X_{v_0} π_{i_0} Y_a Y_c t X_{v_0} π_{i_0}
π_{v_0} $\tilde{\theta}$
Y_a Y_c t $\pi_{i''}$ Y_a s X_{u_0} π_{i_0}
ε π_{u_0}
Y_a Y_c $\pi_{j''}$ f Y_a s $\pi_{i'}$
π_c ψ
Y_a $\pi_{j'}$ f Y_a $\pi_{j'}$ f
π_a π_a
π_j f π_j f
φ^{-1} = φ^{-1} =
r π_i r π_i
π_u^{-1} π_u^{-1}
r X_u $\pi_{i'}$ r X_u $\pi_{i'}$
π_w^{-1} π_w^{-1}
r X_u X_w $\pi_{i''}$ r X_u X_w $\pi_{i''}$
X_μ X_μ
r X_v X_w $\pi_{i''}$ r X_v X_w $\pi_{i''}$
$\pi_{v_0}^{-1}$ $\pi_{v_0}^{-1}$
r X_v X_w X_{v_0} π_{i_0} r X_v X_w X_{v_0} π_{i_0}
$\pi_{\tilde{w}}^{-1}$ $\pi_{\tilde{w}}^{-1}$
r X_v X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$ r X_v X_w X_{v_0} $\mathsf{X}_{\tilde{w}}$ $\pi_{\tilde{i}}$



where the first equality is due to axiom PC2, the second one and the fifth one require some elevators calculus plus (3.2.3), the third holds because (v, b, η) is a morphism in $\mathcal{M}_{\mathsf{f}}$, the fourth one and the last one are valid by elevators calculus plus axiom PC2 and the sixth one is due to elevators calculus plus the fact that (u, a, θ) is a morphism in $\mathcal{M}_{\mathsf{f}}$.

Then, by 2.2.12, there exist a morphism $\tilde{i} \xrightarrow{w_0} i_1 \in \mathcal{I}$ such that

$$
\begin{array}{cccccc}
\mathsf{Y}_a\mathsf{Y}_c & \mathsf{t} & & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\backslash \mathsf{Y}_\alpha / & \| & & \| & \| & \| \\
\mathsf{Y}_b\mathsf{Y}_c & \mathsf{t} & & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\| \ \backslash & & \tilde{\theta} & / & \| & \| \\
\mathsf{Y}_b & \mathsf{s} & \mathsf{X}_{u_0} & & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\| & \| & / & \mathsf{X}_{\tilde{\mu}} & \backslash & \| \\
\mathsf{Y}_b & \mathsf{s} & \mathsf{X}_w & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\backslash & \eta \ / & \| & \| & \| & \| \\
\mathsf{r} & \mathsf{X}_v & \mathsf{X}_w & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0}
\end{array}
\quad = \quad
\begin{array}{ccccccc}
\mathsf{Y}_a & \mathsf{Y}_c & \mathsf{t} & & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\| & \backslash & \tilde{\theta} & / & & \| & \| \\
\mathsf{Y}_a & \mathsf{s} & \mathsf{X}_{u_0} & & & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\| & \| & / & \mathsf{X}_{\tilde{\mu}} & & \backslash & \| \\
\mathsf{Y}_a & \mathsf{s} & \mathsf{X}_w & \mathsf{X}_{v_0} & & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\backslash \ \theta & / & \| & \| & & \| & \| \\
\mathsf{r} & \mathsf{X}_u & \mathsf{X}_w & \mathsf{X}_{v_0} & & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} \\
\| & \backslash \mathsf{X}_\mu / & & \| & & \| & \| \\
\mathsf{r} & \mathsf{X}_v & \mathsf{X}_w & \mathsf{X}_{v_0} & & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0}
\end{array}
$$

We can conclude that we a have a morphism in $\mathcal{M}_{\mathsf{f}}$

$$
(\mathsf{s},\psi) \overset{(w_0\tilde{w}v_0w,c,\mathsf{s}\mathsf{X}_{\tilde{\mu}}\mathsf{X}_{w_0}\circ\tilde{\theta}\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0})}{\longrightarrow} \left(\mathsf{t}\mathsf{X}_{v_0}\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0},\
\begin{array}{ccccc}
\mathsf{t} & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0} & \pi_{i_1} \\
\| & \| & \| & \backslash \pi_{w_0} / & \\
\mathsf{t} & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \pi_{\tilde{i}} & \\
\| & \| & \backslash \pi_{\tilde{w}} / & & \\
\mathsf{t} & \mathsf{X}_{v_0} & \pi_{i_0} & & \\
\| & \backslash \pi_{v_0} / & & & \\
\mathsf{t} & \pi_{i''} & & & \\
\backslash & \varepsilon & / & & \\
\pi_{j''} & & \mathsf{f} & &
\end{array}
\right)
$$

and an invertible 2-cell in $\mathcal{M}_{\mathsf{f}}$,

$$(w_0\tilde{w}v_0w, c, \mathsf{s}\mathsf{X}_{\tilde{\mu}}\mathsf{X}_{w_0} \circ \tilde{\theta}\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0})(u,a,\theta) \overset{(w_0\tilde{w}v_0\mu,\alpha)}{\Longrightarrow} (w_0\tilde{w}v_0w, c, \mathsf{s}\mathsf{X}_{\tilde{\mu}}\mathsf{X}_{w_0} \circ \tilde{\theta}\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0})(v,b,\eta).$$

F2: Let $(\mathsf{r},\varphi) \underset{(v,b,\eta)}{\overset{(u,a,\theta)}{\rightrightarrows}} (\mathsf{s},\psi) \in \mathcal{M}_{\mathsf{f}}$, with 2-cells $\Downarrow(\mu,\alpha)\ \Downarrow(\rho,\beta)$. Since $\mathcal{J}$ is 2-filtered, we have $j' \xrightarrow{c} j'' \in \mathcal{J}$ such that $c\alpha = c\beta$ and, by 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have $\mathsf{t} : \mathsf{X}_{i''} \to \mathsf{Y}_{j''} \in \mathcal{C}$ and an invertible 2-cell ϵ such that (t,ϵ) represents f and there is a morphism $i' \xrightarrow{w} \tilde{i} \in \mathcal{I}$ such that $w\mu = w\rho$.

Now, since $\mathcal{I}$ is 2-filtered, we have morphisms $\begin{array}{c}\tilde{i} \xrightarrow{u_0} \\ i'' \xrightarrow[v_0]{} \end{array} i_0 \in \mathcal{I}$.

Then we have (s, ψ) and $\left(\mathsf{t}\mathsf{X}_{v_0}, \begin{array}{ccccc} \mathsf{t} & & \mathsf{X}_{v_0} & & \pi_{\tilde{i}} \\ \| & & \backslash & {\scriptstyle \pi_{v_0}} & / \\ \mathsf{t} & & & \pi_{i''} & \\ \backslash & \varepsilon & & / & \\ \pi_{j''} & & & \mathsf{f} & \end{array}\right)$ both representing f equipped with morphisms $i' \xrightarrow{u_0 w} i_0 \in \mathcal{I}$, $j' \xrightarrow{c} j'' \in \mathcal{J}$. Then, by 2.2.8, we have morphisms $\begin{array}{l} i' \xrightarrow{u_1} \\ i_0 \xrightarrow[v_1]{} \end{array} i_1 \in \mathcal{I}$ and an invertible 2-cell $\begin{array}{ccc} & \mathsf{Y}_c \, \mathsf{t} \, \mathsf{X}_{v_0} \mathsf{X}_{v_1} & \\ \backslash & \tilde{\theta} & / \\ & \mathsf{s}\mathsf{X}_{u_1} & \end{array} \in \mathcal{C}$ such that

$$\begin{array}{ccccc} \mathsf{Y}_c & \mathsf{t} & \mathsf{X}_{v_0} & \mathsf{X}_{v_1} & \pi_{i_1} \\ \backslash & & \tilde{\theta} & / & \| \\ \mathsf{s} & & & \mathsf{X}_{u_1} & \pi_{i_1} \\ \| & & & \backslash \; {\scriptstyle \pi_{u_1}} & / \\ \mathsf{s} & & & \pi_{i'} & \\ \backslash & & \psi & / & \\ \pi_{j'} & & & \mathsf{f} & \end{array} \quad = \quad \begin{array}{ccccc} \mathsf{Y}_c & \mathsf{t} & \mathsf{X}_{v_0} & \mathsf{X}_{v_1} & \pi_{i_1} \\ \| & \| & \| & \backslash \; {\scriptstyle \pi_{v_1}} & / \\ \mathsf{Y}_c & \mathsf{t} & \mathsf{X}_{v_0} & & \pi_{i_0} \\ \| & \| & \backslash & {\scriptstyle \pi_{v_0}} & / \\ \mathsf{Y}_c & \mathsf{t} & & \pi_{i''} & \\ \| & \backslash & \varepsilon & / & \\ \mathsf{Y}_c & \pi_{j''} & & \mathsf{f} & \\ \backslash \; {\scriptstyle \pi_c} & / & & \| & \\ \pi_{j'} & & & \mathsf{f} & \end{array}$$

Plus, since $\mathcal{I}$ is 2-filtered, we have a morphism $i_1 \xrightarrow{\tilde{w}} i_2$ and an invertible 2-cell $\tilde{w}u_1 \overset{\tilde{\mu}}{\Longrightarrow} \tilde{w}v_1u_0w \in \mathcal{I}$.

It can be checked that there is a morphism in $\mathcal{M}_{\mathsf{f}}$ $(\mathsf{s}, \psi) \xrightarrow{(\tilde{w}v_1u_0w, c, \mathsf{s}\mathsf{X}_{\tilde{\mu}} \circ \tilde{\theta}\mathsf{X}_{\tilde{w}})} \left(\mathsf{t}\mathsf{X}_{v_0}\mathsf{X}_{v_1}\mathsf{X}_{\tilde{w}}, \begin{array}{cccccc} \mathsf{t} & \mathsf{X}_{v_0} & \mathsf{X}_{v_1} & \mathsf{X}_{\tilde{w}} & & \pi_{i_1} \\ \| & \| & \| & \backslash & {\scriptstyle \pi_{\tilde{w}}} & / \\ \mathsf{t} & \mathsf{X}_{v_0} & \mathsf{X}_{v_1} & & \pi_{i_1} & \\ \| & \| & \backslash & {\scriptstyle \pi_{v_1}} & / & \\ \mathsf{t} & \mathsf{X}_{v_0} & & \pi_{i_0} & & \\ \| & \backslash & {\scriptstyle \pi_{v_0}} & / & & \\ \mathsf{t} & & \pi_{i''} & & & \\ \backslash & \varepsilon & / & & & \\ \pi_{j''} & & \mathsf{f} & & & \end{array}\right)$ such that $(\tilde{w}v_1u_0w, c, \mathsf{s}\mathsf{X}_{\tilde{\mu}} \circ \tilde{\theta}\mathsf{X}_{\tilde{w}})(u, a, \theta) = (\tilde{w}v_1u_0w, c, \mathsf{s}\mathsf{X}_{\tilde{\mu}} \circ \tilde{\theta}\mathsf{X}_{\tilde{w}})(v, b, \eta)$.

$$\begin{tikzcd} \mathcal{M}_{\mathsf{f}} \arrow[r] & \mathcal{I} \\ (\mathsf{r},\varphi) \arrow[r, mapsto] & i \end{tikzcd}$$ is 2-cofinal:

CF0: Let $i \in \mathcal{I}$ and let $\mathsf{X}_{i'} \xrightarrow{\mathsf{r}} \mathsf{Y}_j \in \mathcal{C}$ such that (r, id) represents f. Since $\mathcal{I}$ is 2-filtered, we have $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u"] & \\ & i'' \\ i' \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$. It is straightforward to check that $(\mathsf{r}\mathsf{X}_v, \mathsf{r}\pi_v)$ together with $i \xrightarrow{u} i'' \in \mathcal{I}$ proves CF0.

CF1: Let $i \in \mathcal{I}$, $(\mathsf{r}, \varphi) \in \mathcal{M}_{\mathsf{f}}$ $(\mathsf{r} : \mathsf{X}_{i'} \to \mathsf{Y}_j)$ and $i \underset{v}{\overset{u}{\rightrightarrows}} i' \in \mathcal{I}$. Since $\mathcal{I}$ is 2-filtered, we have $i' \overset{w}{\to} i''$ and an invertible 2-cell $wu \overset{\mu}{\Rightarrow} wv \in \mathcal{I}$. It is straightforward to check that $(\mathsf{r}, \varphi) \overset{(w,id,id)}{\longrightarrow} \left(\mathsf{r}\mathsf{X}_w, \begin{array}{ccc} \mathsf{r} & \mathsf{X}_w & \pi_{i''} \\ \| & & \diagdown^{\pi_w}\diagup \\ \mathsf{r} & & \pi_{i'} \\ \diagdown & \varphi & \diagup \\ \pi_{j'} & & \mathsf{f} \end{array}\right)$ proves CF1.

CF2: Let $i \in \mathcal{I}$, $(\mathsf{r}, \varphi) \in \mathcal{M}_{\mathsf{f}}$ $(\mathsf{r} : \mathsf{X}_{i'} \to \mathsf{Y}_j)$ and $i \underset{v}{\overset{u}{\rightrightarrows}} i'$ with $\Downarrow\mu\ \Downarrow\rho$, $i' \in \mathcal{I}$. Since $\mathcal{I}$ is 2-filtered, we have $i' \overset{w}{\to} i'' \in \mathcal{I}$ such that $w\mu = w\rho$. It is straightforward to check that $(\mathsf{r}, \varphi) \overset{(w,id,id)}{\longrightarrow} \left(\mathsf{r}\mathsf{X}_w, \begin{array}{ccc} \mathsf{r} & \mathsf{X}_w & \pi_{i''} \\ \| & & \diagdown^{\pi_w}\diagup \\ \mathsf{r} & & \pi_{i'} \\ \diagdown & \varphi & \diagup \\ \pi_{j'} & & \mathsf{f} \end{array}\right)$ proves CF2.

$$\begin{tikzcd} \mathcal{M}_{\mathsf{f}} \arrow[r] & \mathcal{J} \\ (\mathsf{r},\varphi) \arrow[r, mapsto] & j \end{tikzcd}$$ is 2-cofinal:

CF0: Let $j \in \mathcal{J}$. By 2.2.3, we have $\mathsf{r} : \mathsf{X}_i \to \mathsf{Y}_j \in \mathcal{C}$ such that (r, id) represents f. This clearly proves CF0.

CF1: Let $j_0 \in \mathcal{J}$, $(\mathsf{r}, \varphi) \in \mathcal{M}_{\mathsf{f}}$ $(\mathsf{r} : \mathsf{X}_i \to \mathsf{Y}_j)$ and $j_0 \underset{b}{\overset{a}{\rightrightarrows}} j \in \mathcal{J}$. Since $\mathcal{J}$ is 2-filtered, we have $j \overset{c}{\to} j'$ and an invertible 2-cell $ca \overset{\alpha}{\Rightarrow} cb \in \mathcal{J}$. Now, by 2.2.3, we have $\mathsf{s} : \mathsf{X}_{i'} \to \mathsf{Y}_{j'} \in \mathcal{C}$ such that (s, id) represents f.

From the proof of the fact that $\mathcal{M}_{\mathsf{f}}$ is 2-filtered, we have morphisms $\begin{tikzcd}[row sep=tiny] (\mathsf{r},\varphi) \arrow[dr, "{(u,c,\theta)}"] & \\ & (\mathsf{t},\psi) \\ (\mathsf{s}, id) \arrow[ur, "{(v,id,\eta)}"'] & \end{tikzcd}$. It is straightforward to check that $(\mathsf{r}, \varphi) \overset{(u,c,\theta)}{\to} (\mathsf{t}, \psi)$ proves CF1.

CF2: Let $j_0 \in \mathcal{J}$, $(\mathsf{r},\varphi) \in \mathcal{M}_{\mathsf{f}}$ ($\mathsf{r} : \mathsf{X}_i \to \mathsf{Y}_j$) and $\begin{tikzcd} j_0 \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "b"'] \arrow[r, phantom, "\Downarrow\alpha\ \Downarrow\beta"] & j \end{tikzcd} \in \mathcal{J}$. Since $\mathcal{J}$ is 2-filtered, we have $j \overset{c}{\to} j' \in \mathcal{J}$ such that $c\alpha = c\beta$. Now, by 2.2.3, we have $\mathsf{s} : \mathsf{X}_{i'} \to \mathsf{Y}_{j'} \in \mathcal{C}$ such that (s, id) represents f.

From the proof of the fact that $\mathcal{M}_{\mathsf{f}}$ is 2-filtered, we have morphisms $\begin{tikzcd}[row sep=small] (\mathsf{r},\varphi) \arrow[dr, "{(u,c,\theta)}"] & \\ & (\mathsf{t},\psi) \\ (\mathsf{s},id) \arrow[ur, "{(v,id,\eta)}"'] & \end{tikzcd}$. It is straightforward to check that $(\mathsf{r},\varphi) \overset{(u,c,\theta)}{\to} (\mathsf{t},\psi)$ proves CF2.

□

3.2.4 Proposition. *Every morphism of 2-pro-objects* $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}} \overset{\mathsf{f}}{\to} \mathsf{Y} = (\mathsf{Y}_j)_{j \in \mathcal{J}}$ *can be represented up to equivalence by a 2-pro-object* $\{\mathsf{X}'_m \overset{\mathsf{f}_m}{\to} \mathsf{Y}'_m\}_{m \in \mathcal{M}}$ *in* 2-$\mathcal{P}ro(\mathcal{H}om_p(2, \mathcal{C}))$, *i.e.* $\exists$ *a 2-filtered 2-category* $\mathcal{M}$, *2-pro-objects* $\mathsf{X}' = (\mathsf{X}'_m)_{m \in \mathcal{M}}$, $\mathsf{Y}' = (\mathsf{Y}'_m)_{m \in \mathcal{M}}$ *and a morphism* $\mathsf{X}' \overset{\mathsf{f}'}{\longrightarrow} \mathsf{Y}'$ *such that the following diagram commutes in* 2-$\mathcal{P}ro(\mathcal{C})$ *up to isomorphism:*

$$\begin{tikzcd} \mathsf{X} \arrow[r, "\mathsf{f}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\cong"] & \mathsf{Y} \arrow[d, "\simeq"] \\ \mathsf{X}' \arrow[r, "\mathsf{f}'"'] & \mathsf{Y}' \end{tikzcd} \tag{3.2.5}$$

Proof. Take $\mathcal{M} = \mathcal{M}_{\mathsf{f}}$ as defined in 3.2.1, $\mathsf{X}'_{(\mathsf{r},\varphi)} = \mathsf{X}_i$ and $\mathsf{Y}'_{(\mathsf{r},\varphi)} = \mathsf{Y}_j$ (if $\mathsf{r} : \mathsf{X}_i \to \mathsf{Y}_j$) and $\mathsf{f}_{(\mathsf{r},\varphi)} = \mathsf{r}$.

Since $\begin{tikzcd}[row sep=small] \mathcal{M}_{\mathsf{f}} \arrow[r] & \mathcal{I} \\ (\mathsf{r},\varphi) \arrow[r, mapsto] & i \end{tikzcd}$ and $\begin{tikzcd}[row sep=small] \mathcal{M}_{\mathsf{f}} \arrow[r] & \mathcal{J} \\ (\mathsf{r},\varphi) \arrow[r, mapsto] & j \end{tikzcd}$ are 2-cofinal, by 3.1.1, X' is equivalent to X and Y' is equivalent to Y. It is straightforward to check that diagram (3.2.5) commutes up to the isomorphism given by φ and the universal property of Y'. □

The previous proposition can be also stated as follows:

3.2.6 Remark. Every object $\mathsf{f} \in \mathcal{H}om(2, 2\text{-}\mathcal{P}ro(\mathcal{C}))$ have a lifting to $\mathcal{H}om(\mathcal{M}^{op}, \mathcal{C})$ up to equivalence for some 2-filtered 2-category $\mathcal{M}$.

$$\begin{tikzcd} & \mathcal{H}om(\mathcal{M}^{op}, \mathcal{C}) \arrow[d, "inc"] \\ 2 \arrow[ur, "\mathsf{f}'"] \arrow[r, "\mathsf{f}"'] \arrow[ur, phantom, shift right=3, "\simeq"] & 2\text{-}\mathcal{P}ro(\mathcal{C}) \end{tikzcd}$$

□

3.2.7 Corollary. *Let* $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}} \xrightarrow{\mathsf{f}} \mathsf{Y} = (\mathsf{Y}_j)_{j \in \mathcal{J}} \in 2\text{-}\mathcal{P}ro(\mathcal{C})$*. There exists a cofinite filtered poset with a unique initial object* J*, and a morphism* $\mathsf{X}' \xrightarrow{\mathsf{f}'} \mathsf{Y}' \in \mathcal{H}om(\mathsf{J}^{op}, \mathcal{C})$ *such that the following diagram commutes in* $2\text{-}\mathcal{P}ro(\mathcal{C})$ *up to isomorphism:*

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{f}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\cong"] & \mathsf{Y} \arrow[d, "\simeq"] \\
\mathsf{X}' \arrow[r, "\mathsf{f}'"'] & \mathsf{Y}'
\end{tikzcd} \tag{3.2.8}$$

Equivalently every object $\mathsf{f} \in \mathcal{H}om(\mathsf{2}, 2\text{-}\mathcal{P}ro(\mathcal{C}))$ *have a lifting to* $\mathcal{H}om(\mathsf{J}^{op}, \mathcal{C})$ *up to equivalence for some cofinite and filtered poset with a unique initial object* J.

$$\begin{tikzcd}
 & \mathcal{H}om(\mathsf{J}^{op}, \mathcal{C}) \arrow[d, "inc"] \\
\mathsf{2} \arrow[ur, "\mathsf{f}'"] \arrow[r, "\mathsf{f}"'] \arrow[ur, phantom, "\simeq", shift right=3] & 2\text{-}\mathcal{P}ro(\mathcal{C})
\end{tikzcd}$$

Proof. Consider $\tilde{\mathsf{f}}'$ given by 3.2.4 and consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{H}om(\mathcal{M}^{op}, \mathcal{C}) \arrow[dd, "inc"] \arrow[dr, "(\mathsf{F}^{op})^*"] & \\
\mathsf{2} \arrow[ur, "\tilde{\mathsf{f}}'"] \arrow[dr, "\mathsf{f}"'] \arrow[r, phantom, "\simeq"] & \arrow[r, phantom, "\simeq"] & \mathcal{H}om(\mathsf{M}(\mathcal{M})^{op}, \mathcal{C}) \arrow[dl, "inc"] \\
 & 2\text{-}\mathcal{P}ro(\mathcal{C}) &
\end{tikzcd}$$

where $\mathsf{M}(\mathcal{M}) \xrightarrow{\mathsf{F}} \mathcal{M}$ is the one given by 1.3.15. Note that the equivalence in the right triangle is because F is 2-cofinal.

Then take $\mathsf{J} = \mathsf{M}(\mathcal{M})$ and $\mathsf{f}' = (\mathsf{F}^{op})^* \tilde{\mathsf{f}}'$.

□

The lifting property 3.2.7 also holds for $2\text{-}\mathcal{P}ro_p(\mathcal{C})$:

3.2.9 Corollary. *Let* $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}} \xrightarrow{\mathsf{f}} \mathsf{Y} = (\mathsf{Y}_j)_{j \in \mathcal{J}} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$*. There exist a cofinite filtered poset with a unique initial object* J*, and a morphism* $\mathsf{X}' \xrightarrow{\mathsf{f}'} \mathsf{Y}' \in \mathcal{H}om(\mathsf{J}, \mathcal{C})$ *such* *that the following diagram commutes in $2\text{-}\mathcal{P}ro_p(\mathcal{C})$ up to isomorphism:*

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{f}"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\cong"] & \mathsf{Y} \arrow[d, "\cong"] \\
\mathsf{X}' \arrow[r, "\mathsf{f}'"'] & \mathsf{Y}'
\end{tikzcd} \tag{3.2.10}$$

Equivalently every object $\mathsf{f} \in \mathcal{H}om(2, 2\text{-}\mathcal{P}ro_p(\mathcal{C}))$ *have a lifting to* $\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ *up to equivalence for some cofinite and filtered poset with a unique initial object* J.

$$\begin{tikzcd}
 & \mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C}) \arrow[d, "inc"] \\
2 \arrow[ur, "\mathsf{f}'"] \arrow[r, "\mathsf{f}"'] \arrow[ur, phantom, "\simeq", shift left=4] & 2\text{-}\mathcal{P}ro_p(\mathcal{C})
\end{tikzcd}$$

Proof. By 2.1.5, there exist $\mathsf{X} \overset{\tilde{\mathsf{f}}}{\longrightarrow} \mathsf{Y} \in 2\text{-}\mathcal{P}ro(\mathcal{C})$ and an invertible 2-cell $\mathsf{f} \overset{\alpha}{\Longrightarrow} \tilde{\mathsf{f}} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$. Apply 3.2.7 to $\tilde{\mathsf{f}}$ to obtain a cofinite filtered poset with a unique initial object J, and a morphism $\mathsf{X}' \overset{\mathsf{f}'}{\to} \mathsf{Y}' \in \mathcal{H}om(\mathsf{J}^{op}, \mathcal{C})$ such that the following diagram commutes in $2\text{-}\mathcal{P}ro(\mathcal{C})$ up to an invertible 2-cell γ:

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\tilde{\mathsf{f}}"] \arrow[d, "\mathsf{a}\simeq"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Y} \arrow[d, "\mathsf{b}\simeq"] \\
\mathsf{X}' \arrow[r, "\mathsf{f}'"'] & \mathsf{Y}'
\end{tikzcd} \tag{3.2.11}$$

Then we have

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{f}"] \arrow[d, "\mathsf{a}\simeq"'] \arrow[dr, phantom, "\cong \Downarrow \gamma \circ \mathsf{b}\alpha"] & \mathsf{Y} \arrow[d, "\mathsf{b}\simeq"] \\
\mathsf{X}' \arrow[r, "\mathsf{f}'"'] & \mathsf{Y}'
\end{tikzcd} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$$

as we wanted to prove. □

3.3 Reindexing for diagrams

The following proposition is a generalization of 3.2.4.

3.3.1 Proposition. *Let* $\Delta \xrightarrow{\mathsf{D}} 2\text{-}\mathcal{P}ro(\mathcal{C})$ *be a finite diagram with commutation relations and no loops in* $2\text{-}\mathcal{P}ro(\mathcal{C})$*. Then* D *can be represented up to equivalence by a 2-pro-object over* $\mathcal{H}om_p(\Delta, \mathcal{C})$*, i.e. there exists an inverse 2-filtered system of diagrams* $\{\Delta \xrightarrow{\mathsf{D}_k} \mathcal{C}\}_{k \in \mathcal{M}}$ *in* $\mathcal{C}$ *such that the diagram induced by the* D_k*'s in* $2\text{-}\mathcal{P}ro(\mathcal{C})$ *is equivalent to* D *up to isomorphism.*

Equivalently, every object $\mathsf{D} \in \mathcal{H}om(\Delta, 2\text{-}\mathcal{P}ro(\mathcal{C}))$ *have a lifting to* $\mathcal{H}om(\mathcal{M}^{op}, \mathcal{C})$ *up to equivalence for some 2-filtered 2-category* $\mathcal{M}$*.*

$$\begin{tikzcd}
 & \mathcal{H}om(\mathcal{M}^{op}, \mathcal{C}) \arrow[d, "inc"] \\
\Delta \arrow[ur, "\mathsf{D}'"] \arrow[r, "\mathsf{D}"'] \arrow[r, phantom, shift left=3, "\simeq"] & 2\text{-}\mathcal{P}ro(\mathcal{C})
\end{tikzcd}$$

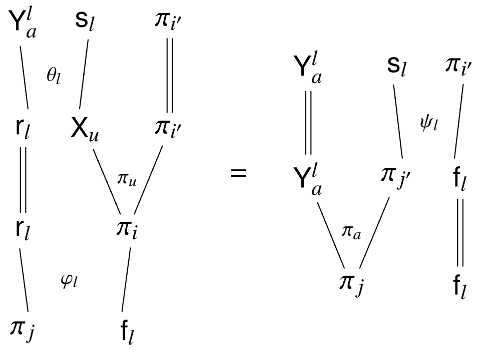
Proof. We are going to proceed by induction in the amount of vertices of Δ. The initial case is trivial.

Now, suppose that we have proved the proposition for diagrams with $n-1$ vertices and let $\Delta \xrightarrow{\mathsf{D}} 2\text{-}\mathcal{P}ro(\mathcal{C})$ be a diagram with n vertices. Let x be an initial vertex of Δ and let Δ' be the diagram resulting by taking x out of Δ and D' the induced diagram in $2\text{-}\mathcal{P}ro(\mathcal{C})$. By inductive hypothesis, there exists an inverse 2-filtered system of diagrams $\{\mathsf{D}'_j\}_{j \in \mathcal{J}}$ such that the diagram induced by the D'_j's in $2\text{-}\mathcal{P}ro(\mathcal{C})$ is equivalent to D'. Let $\mathsf{X} = (\mathsf{X}_i)_{i \in \mathcal{I}}$ be the object of $2\text{-}\mathcal{P}ro(\mathcal{C})$ corresponding to x by D and let $\{\mathsf{X} \xrightarrow{f_l} \mathsf{Y}^l\}_{l=1,\dots,m}$ be all the morphisms in $2\text{-}\mathcal{P}ro(\mathcal{C})$ corresponding to morphisms from x to some other vertex of Δ when we apply D. In what follows, we are going to abuse the notation by using Y^l for the corresponding objects via the equivalence between D' and the diagram induced by the D'_j's and f_l for the composition of the previous f_l with the corresponding equivalence.

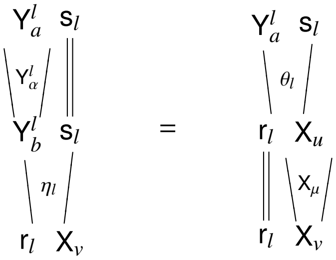
Define $\mathcal{M}$ as the following 2-category:

Objects are m-tuples of pairs $(\mathsf{r}_l, \varphi_l)$ with $\mathsf{X}_i \xrightarrow{\mathsf{r}_l} \mathsf{Y}^l_j \in \mathcal{C}$ such that $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $(\mathsf{r}_l, \varphi_l)$ represents f_l $\forall\, l = 1, \dots, m$.

Morphisms $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,\dots,m} \longrightarrow \{(\mathsf{s}_l, \psi_l)\}_{l=1,\dots,m}$ ($\mathsf{r}_l : \mathsf{X}_i \to \mathsf{Y}^l_j$, $\mathsf{s}_l : \mathsf{X}_{i'} \to \mathsf{Y}^l_{j'}$) are triplex $(u, a, \{\theta_l\}_{l=1,\dots,m})$ where $i \overset{u}{\to} i' \in \mathcal{I}$, $j \overset{a}{\to} j' \in \mathcal{J}$ and $\forall\, l = 1, \dots, m$ θ_l is an invertible 2-cell $\begin{array}{ccc} \mathsf{Y}^l_a & & \mathsf{s}_l \\ \backslash & \theta_l & / \\ \mathsf{r}_l & & \mathsf{X}_u \end{array}$ such that



A 2-cell $\{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \underset{(v,b,\{\eta_l\}_{l=1,\dots,m})}{\overset{(u,a,\{\theta_l\}_{l=1,\dots,m})}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}}}} \{(\mathsf{s}_l,\psi_l)\}_{l=1,\dots,m}$ is a pair (μ,α) where $i \underset{v}{\overset{u}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow\mu}}}} i' \in \mathcal{I}$, $j \underset{b}{\overset{a}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow\alpha}}}} j' \in \mathcal{J}$ and $\forall\ l = 1,\dots,m$



Identities and compositions are defined in the obvious way.

In the following, we are going to prove that $\mathcal{M}$ is 2-filtered and the 2-functors

$$\begin{tikzcd} \mathcal{M} \arrow[r] & \mathcal{I} \\ \{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \arrow[r, mapsto] & i \end{tikzcd} \text{ and } \begin{tikzcd} \mathcal{M} \arrow[r] & \mathcal{J} \\ \{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \arrow[r, mapsto] & j \end{tikzcd}$$

are 2-cofinal.

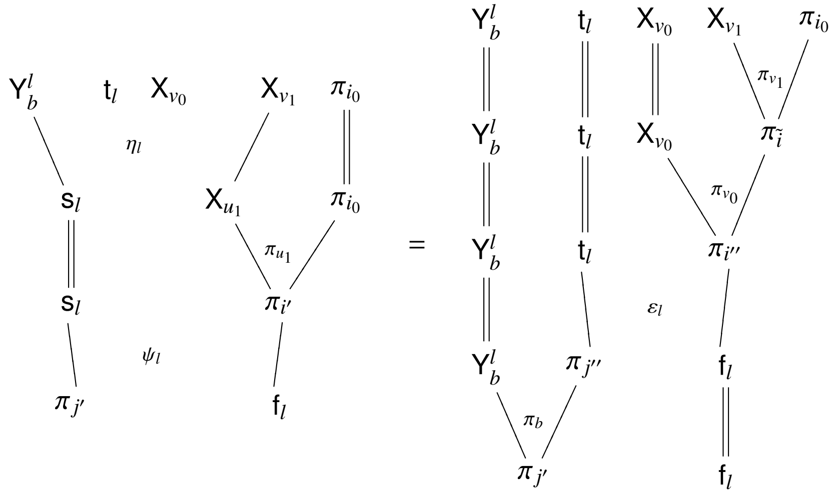
$\mathcal{M}$ is 2-filtered:

F0: Let $\{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m}$, $\{(\mathsf{s}_l,\psi_l)\}_{l=1,\dots,m} \in \mathcal{M}$ ($\mathsf{r}_l : \mathsf{X}_i \to \mathsf{Y}^l_j$, $\mathsf{s}_l : \mathsf{X}_{i'} \to \mathsf{Y}^l_{j'}$). Since $\mathcal{J}$ is 2-filtered, we have $\begin{tikzcd} j \arrow[dr, "a"] & \\ & j'' \\ j' \arrow[ur, "b"'] & \end{tikzcd} \in \mathcal{J}$ and, by 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have $\mathsf{X}_{i''} \xrightarrow{\mathsf{t}_l} \mathsf{Y}^l_{j''} \in \mathcal{C}$ and invertible 2-cells ϵ_l such that $\forall\ l = 1,..,m$ $(\mathsf{t}_l,\epsilon_l)$ represents f_l and there are morphisms $\begin{tikzcd} i \arrow[dr, "u"] & \\ & i'' \\ i' \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$. Observe that, in this case, we are using the fact that $\mathcal{I}$ is 2-filtered as in 3.2.2 but also to achieve that all t_l have the same source.

Then we have $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m}$ and $\{(\mathsf{t}_l, \epsilon_l)\}_{l=1,...,m}$ such that $\forall\ l = 1, ..., m$, $(\mathsf{r}_l, \varphi_l)$ and $(\mathsf{t}_l, \epsilon_l)$ both represent f_l and there are morphisms $i \xrightarrow{u} i'' \in \mathcal{I}$, $j \xrightarrow{a} j'' \in \mathcal{J}$. So, by 2.2.9, there are morphisms $\begin{tikzcd}[row sep=tiny] i \arrow[dr, "u_0"] & \\ & \tilde{i} \\ i'' \arrow[ur, "v_0"'] & \end{tikzcd}$ $\in \mathcal{I}$ and invertible 2-cells $\begin{tikzcd}[row sep=small, column sep=small] \mathsf{Y}^l_a \arrow[dd, dash] & \mathsf{t}_l & \mathsf{X}_{v_0} \arrow[ddl, dash] \\ & \theta_l & \\ \mathsf{r}_l & \mathsf{X}_{u_0} & \end{tikzcd}$ $\in \mathcal{C}$ such that

$$\begin{tikzcd}[row sep=small, column sep=small]
\mathsf{Y}^l_a \arrow[dd, dash] & \mathsf{t}_l & \mathsf{X}_{v_0} \arrow[ddl, dash] & \pi_{\tilde{i}} \arrow[dd, equal] \\
& \theta_l & & \\
\mathsf{r}_l \arrow[dd, equal] & \mathsf{X}_{u_0} \arrow[ddr, dash] & & \pi_{\tilde{i}} \arrow[ddl, dash] \\
& & \pi_{u_0} & \\
\mathsf{r}_l \arrow[dd, dash] & & \pi_i \arrow[dd, dash] & \\
& \varphi_l & & \\
\pi_j & & \mathsf{f}_l &
\end{tikzcd}
=
\begin{tikzcd}[row sep=small, column sep=small]
\mathsf{Y}^l_a \arrow[dd, equal] & & \mathsf{t}_l \arrow[dd, equal] & \mathsf{X}_{v_0} \arrow[ddr, dash] & & \pi_{\tilde{i}} \arrow[ddl, dash] \\
& & & & \pi_{v_0} & \\
\mathsf{Y}^l_a \arrow[dd, equal] & & \mathsf{t}_l \arrow[dd, dash] & & \pi_{i''} \arrow[dd, dash] & \\
& & & \varepsilon_l & & \\
\mathsf{Y}^l_a \arrow[ddr, dash] & & \pi_{j''} \arrow[ddl, dash] & & \mathsf{f}_l \arrow[dd, equal] & \\
& \pi_a & & & & \\
& \pi_j & & & \mathsf{f}_l &
\end{tikzcd}$$

Then we have $\{(\mathsf{s}_l, \psi_l)\}_{l=1,...,m}$ and $\{(\mathsf{t}_l\mathsf{X}_{v_0},$ $\begin{tikzcd}[row sep=tiny, column sep=tiny] \mathsf{t}_l \arrow[dd, equal] & \mathsf{X}_{v_0} \arrow[ddr, dash] & & \pi_{\tilde{i}} \arrow[ddl, dash] \\ & & \pi_{v_0} & \\ \mathsf{t}_l \arrow[dd, dash] & & \pi_{i''} \arrow[dd, dash] & \\ & \varepsilon_l & & \\ \pi_j & & \mathsf{f}_l & \end{tikzcd}$ $)\}_{l=1,...,m}$ such that $\forall\ l = 1, ..., m$ (s_l, ψ_l) and $(\mathsf{t}_l\mathsf{X}_{v_0},$ $\begin{tikzcd}[row sep=tiny, column sep=tiny] \mathsf{t}_l \arrow[dd, equal] & \mathsf{X}_{v_0} \arrow[ddr, dash] & & \pi_{\tilde{i}} \arrow[ddl, dash] \\ & & \pi_{v_0} & \\ \mathsf{t}_l \arrow[dd, dash] & & \pi_{i''} \arrow[dd, dash] & \\ & \varepsilon_l & & \\ \pi_j & & \mathsf{f}_l & \end{tikzcd}$ $)$ both represent f_l and there are morphisms $i' \xrightarrow{v_0 v} \tilde{i} \in \mathcal{I}$, $j' \xrightarrow{b} j'' \in \mathcal{J}$. So, by 2.2.9, there are morphisms $\begin{tikzcd}[row sep=tiny] i' \arrow[dr, "u_1"] & \\ & i_0 \\ \tilde{i} \arrow[ur, "v_1"'] & \end{tikzcd}$ $\in \mathcal{I}$ and invertible 2-cells $\begin{tikzcd}[row sep=small, column sep=tiny] \mathsf{Y}^l_b \arrow[dd, dash] & \mathsf{t}_l\mathsf{X}_{v_0} & \mathsf{X}_{v_1} \arrow[ddl, dash] \\ & \eta_l & \\ \mathsf{s}_l & \mathsf{X}_{u_1} & \end{tikzcd}$ $\in \mathcal{C}$ such that

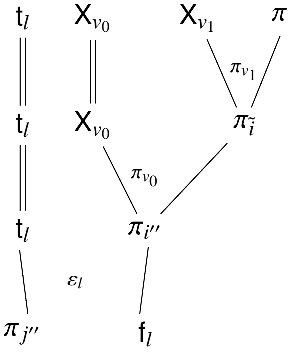


It can be checked that there are morphisms in $\mathcal{M}$

$$\begin{tikzcd}
\{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \arrow[dr, "{(v_1u_0,a,\{\theta_l\mathsf{X}_{v_1}\}_{l=1,\dots,m})}"] & \\
& \{(\mathsf{t}_l\mathsf{X}_{v_0}\mathsf{X}_{v_1}, I)\}_{l=1,\dots,m} \ . \\
\{(\mathsf{s}_l,\psi_l)\}_{l=1,\dots,m} \arrow[ur, swap, "{(u_1,b,\{\eta_l\}_{l=1,\dots,m})}"] &
\end{tikzcd}$$

where

$I =$



F1: Let $\{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \overset{(u,a,\{\theta_l\}_{l=1,\dots,m})}{\underset{(v,b,\{\eta_l\}_{l=1,\dots,m})}{\rightrightarrows}} \{(\mathsf{s}_l,\psi_l)\}_{l=1,\dots,m} \in \mathcal{M}$. Since $\mathcal{J}$ is 2-filtered, we have $j' \overset{c}{\to} j''$ and an invertible 2-cell $ca \overset{\alpha}{\Rightarrow} cb \in \mathcal{J}$ and, by 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have morphisms $\mathsf{X}_{i''} \xrightarrow{\mathsf{t}_l} \mathsf{Y}^l_{j''} \in \mathcal{C}$ and invertible 2-cells ϵ_l such that $\forall\ l = 1,\dots,m$ $(\mathsf{t}_l,\epsilon_l)$ represents f_l and there is a morphism $i' \xrightarrow{w} i''$ and an invertible 2-cell $wu \overset{\mu}{\Longrightarrow} wv \in \mathcal{I}$. Have in mind the same observation made in the proof of axiom F0.

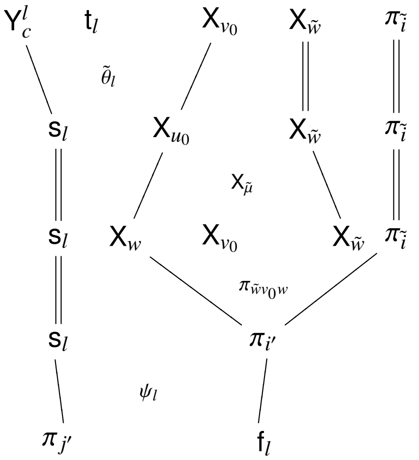
Then we have $\{(\mathsf{s}_l,\psi_l)\}_{l=1,...,m}$ and $\{(\mathsf{t}_l,\epsilon_l)\}_{l=1,...,m}$ such that $\forall\ l = 1,..,m$ (s_l,ψ_l) and $(\mathsf{t}_l,\epsilon_l)$ both represent f_l and there are morphisms $i' \xrightarrow{w} i'' \in \mathcal{I}$, $j' \xrightarrow{c} j'' \in \mathcal{J}$. So, by 2.2.9, there are morphisms $\begin{tikzcd} i' \arrow[rd, "u_0"] & \\ & i_0 \\ i'' \arrow[ru, "v_0"'] & \end{tikzcd}$ $\in \mathcal{I}$ and invertible 2-cells $\begin{array}{ccc} \mathsf{Y}^l_c & \mathsf{t}_l & \mathsf{X}_{v_0} \\ \backslash & \tilde{\theta}_l & / \\ \mathsf{s}_l & & \mathsf{X}_{u_0} \end{array}$ $\in \mathcal{C}$ such that

$$\begin{array}{ccccc} \mathsf{Y}^l_c & \mathsf{t}_l & \mathsf{X}_{v_0} & & \pi_{i_0} \\ \backslash & \tilde{\theta}_l & / & & \| \\ \mathsf{s}_l & \mathsf{X}_{u_0} & & & \pi_{i_0} \\ \| & \backslash & \pi_{u_0} & / & \\ \mathsf{s}_l & & \pi_{i'} & & \\ \backslash & \psi_l & / & & \\ \pi_{j'} & & \mathsf{f}_l & & \end{array} = \begin{array}{ccccc} \mathsf{Y}^l_c & \mathsf{t}_l & \mathsf{X}_{v_0} & & \pi_{i_0} \\ \| & \| & \backslash & \pi_{v_0} & / \\ \mathsf{Y}^l_c & \mathsf{t}_l & & \pi_{i''} & \\ \| & \backslash & \varepsilon_l & / & \\ \mathsf{Y}^l_c & \pi_{j''} & & \mathsf{f}_l & \\ \backslash & \pi_a \quad / & & \| & \\ & \pi_{j'} & & \mathsf{f}_l & \end{array} \tag{3.3.2}$$

Plus, since $\mathcal{I}$ is 2-filtered, there is a morphism $i_0 \xrightarrow{\tilde{w}} \tilde{i}$ and an invertible 2-cell $\tilde{w}u_0 \overset{\tilde{\mu}}{\Longrightarrow} \tilde{w}v_0w \in \mathcal{I}$.

Let's check that we a have a morphism in $\mathcal{M}$

$$\{(\mathsf{s}_l,\psi_l)\}_{l=1,...,m} \xrightarrow{(\tilde{w}v_0w,c,\{\mathsf{s}_l\mathsf{X}_{\tilde{\mu}}\circ\tilde{\theta}_l\mathsf{X}_{\tilde{w}}\}_{l=1,...,m})} \{(\mathsf{t}_l\mathsf{X}_{v_0}\mathsf{X}_{\tilde{w}}, \begin{array}{ccccc} \mathsf{t}_l & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & & \pi_{\tilde{i}} \\ \| & \| & \backslash & \pi_{\tilde{w}} & / \\ \mathsf{t}_l & \mathsf{X}_{v_0} & & \pi_{i_0} & \\ \| & \backslash & \pi_{v_0} & / & \\ \mathsf{t}_l & & \pi_{i''} & & \\ \backslash & \varepsilon_l & / & & \\ \pi_{j''} & & \mathsf{f}_l & & \end{array})\}_{l=1,...,m}:$$



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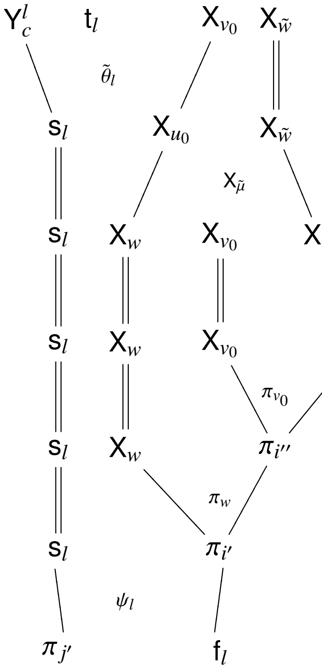

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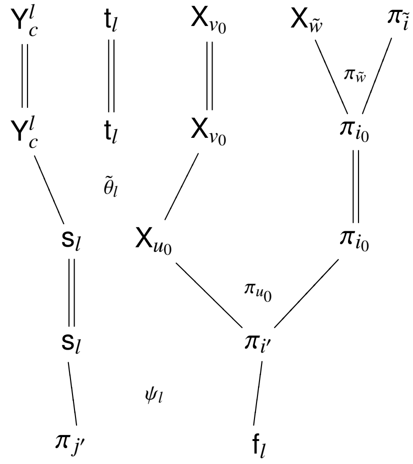

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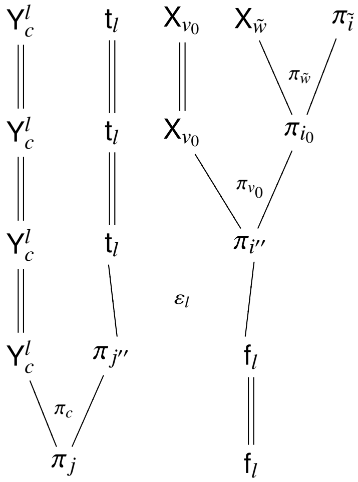
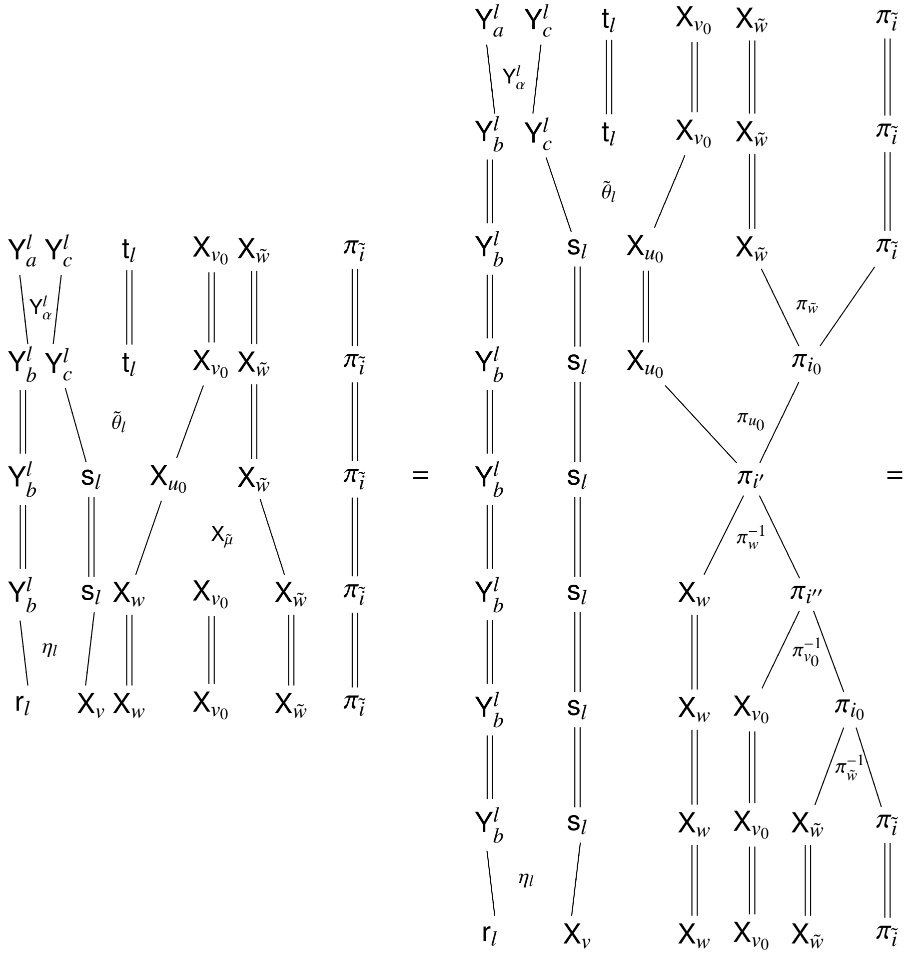
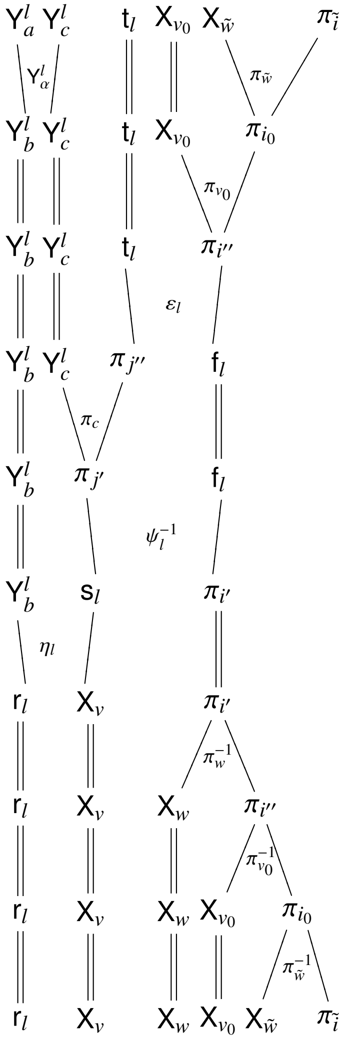


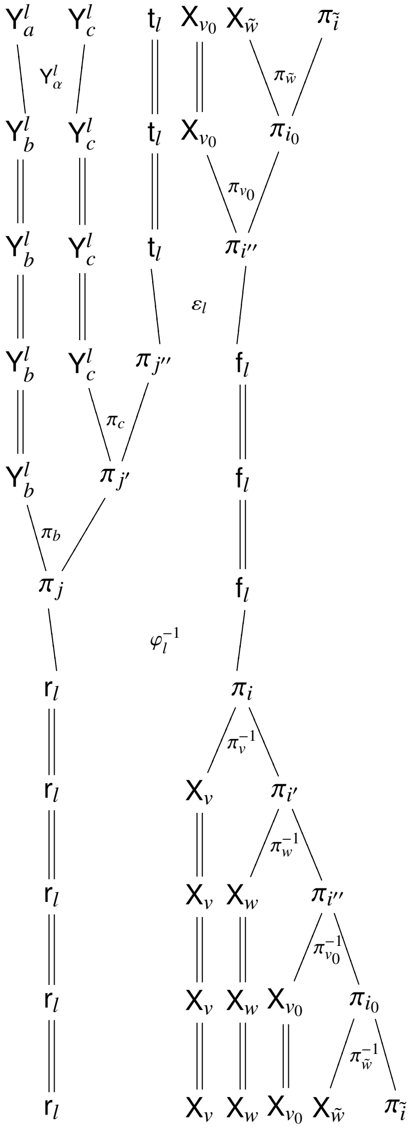
where the first equality follows from axiom PC1, the second one holds by elevators calculus plus axiom PC2 and the last one is due to (3.3.2).

Now observe that

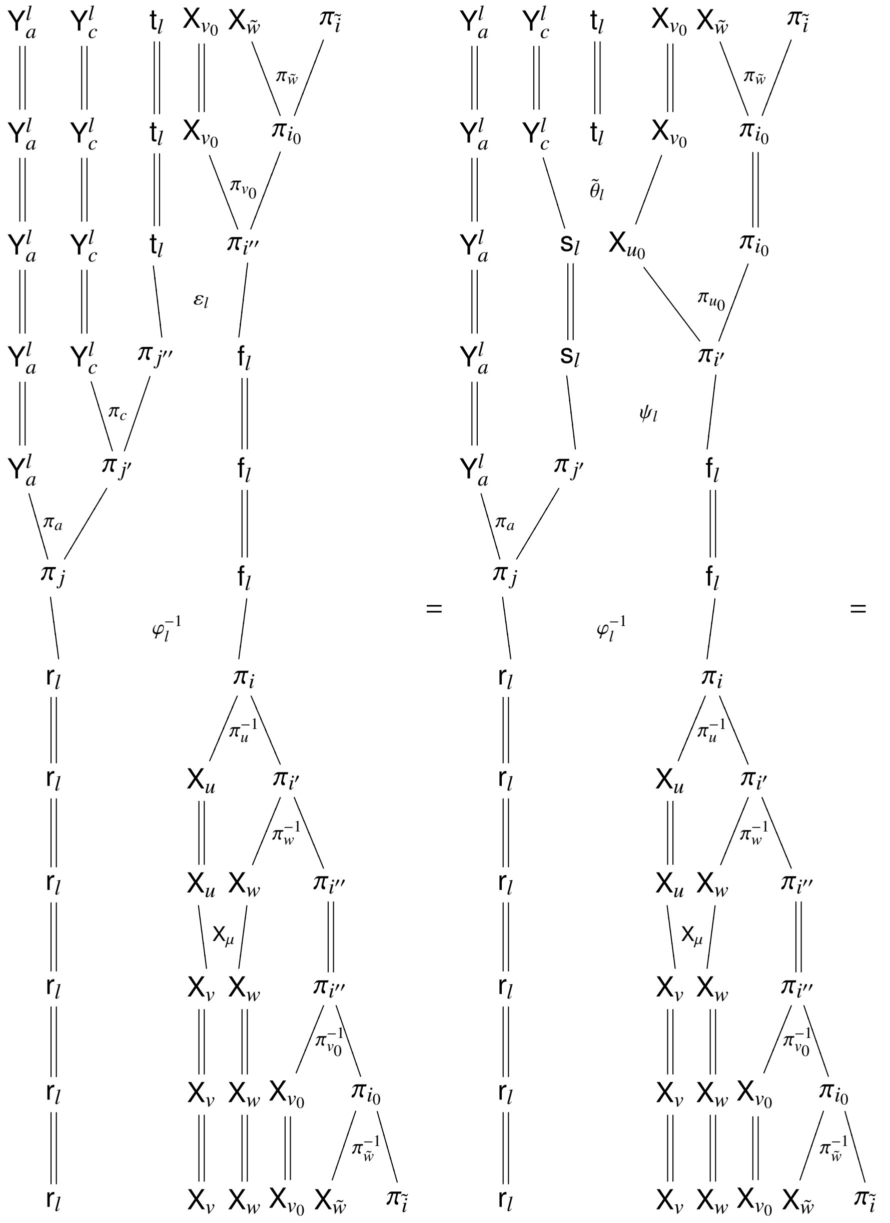

Y^l_a Y^l_c t_l X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
Y^l_α
Y^l_b Y^l_c t_l X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\tilde{\theta}_l$
Y^l_b s_l X_{u_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
$X_{\tilde{\mu}}$
Y^l_b s_l X_w X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
η_l
r_l X_v X_w X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
=
Y^l_a Y^l_c t_l X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
Y^l_α
Y^l_b Y^l_c t_l X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\tilde{\theta}_l$
Y^l_b s_l X_{u_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
$\pi_{\tilde{w}}$
Y^l_b s_l X_{u_0} π_{i_0}
π_{u_0}
Y^l_b s_l $\pi_{i'}$
π_w^{-1}
Y^l_b s_l X_w $\pi_{i''}$
$\pi_{v_0}^{-1}$
Y^l_b s_l X_w X_{v_0} π_{i_0}
$\pi_{\tilde{w}}^{-1}$
Y^l_b s_l X_w X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
η_l
r_l X_v X_w X_{v_0} $X_{\tilde{w}}$ $\pi_{\tilde{i}}$
=

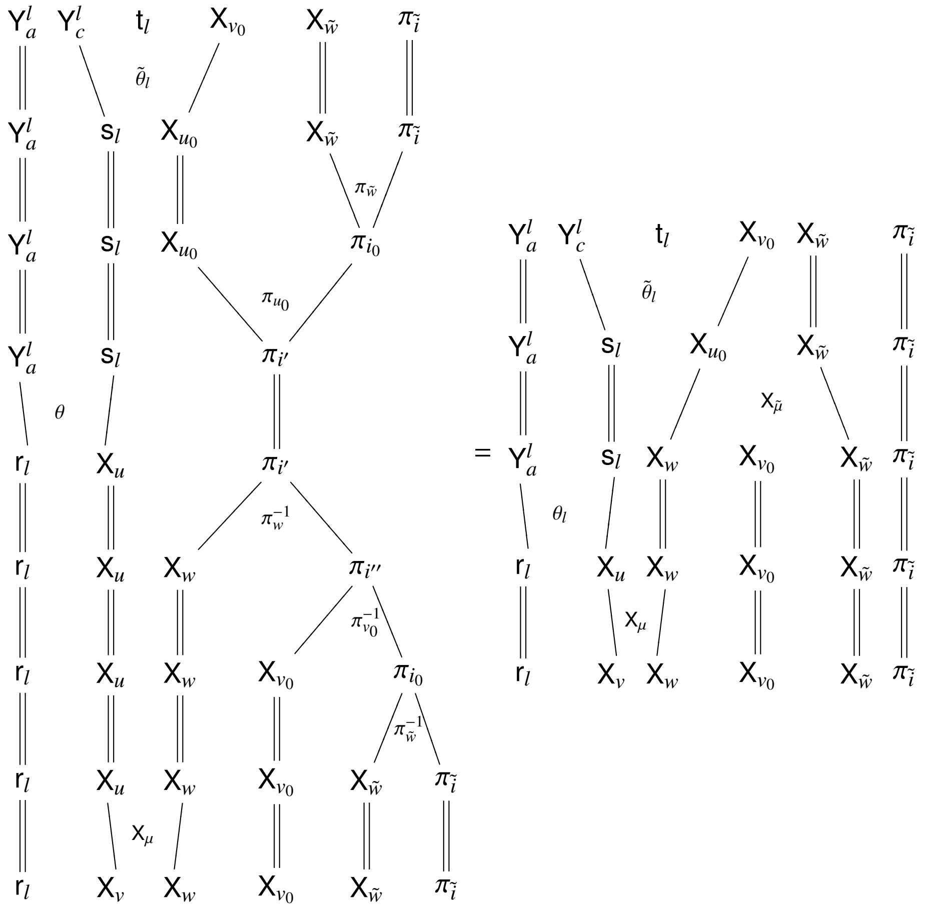

Y_a^l Y_c^l t_l X_{v_0} X_{\tilde{w}} π_{\tilde{i}}
Y_α^l
π_{\tilde{w}}
Y_b^l Y_c^l t_l X_{v_0} π_{i_0}
π_{v_0}
Y_b^l Y_c^l t_l π_{i''}
ε_l
Y_b^l Y_c^l π_{j''} f_l
π_c
Y_b^l π_{j'} f_l
ψ_l^{-1}
Y_b^l s_l π_{i'}
η_l
r_l X_v π_{i'}
π_w^{-1}
r_l X_v X_w π_{i''}
π_{v_0}^{-1}
r_l X_v X_w X_{v_0} π_{i_0}
π_{\tilde{w}}^{-1}
r_l X_v X_w X_{v_0} X_{\tilde{w}} π_{\tilde{i}}

=


Y_a^l Y_c^l t_l X_{v_0} X_{\tilde{w}} π_{\tilde{i}}
Y_α^l
π_{\tilde{w}}
Y_b^l Y_c^l t_l X_{v_0} π_{i_0}
π_{v_0}
Y_b^l Y_c^l t_l π_{i''}
ε_l
Y_b^l Y_c^l π_{j''} f_l
π_c
Y_b^l π_{j'} f_l
π_b
π_j f_l
φ_l^{-1}
r_l π_i
π_v^{-1}
r_l X_v π_{i'}
π_w^{-1}
r_l X_v X_w π_{i''}
π_{v_0}^{-1}
r_l X_v X_w X_{v_0} π_{i_0}
π_{\tilde{w}}^{-1}
r_l X_v X_w X_{v_0} X_{\tilde{w}} π_{\tilde{i}}

=


Y_a^l Y_c^l t_l X_{v_0} $\mathsf{X}_{\tilde w}$ $\pi_{\tilde i}$
$\pi_{\tilde w}$
Y_a^l Y_c^l t_l X_{v_0} π_{i_0}
π_{v_0}
Y_a^l Y_c^l t_l $\pi_{i''}$
ε_l
Y_a^l Y_c^l $\pi_{j''}$ f_l
π_c
Y_a^l $\pi_{j'}$ f_l
π_a
π_j f_l
φ_l^{-1}
r_l π_i
π_u^{-1}
r_l X_u $\pi_{i'}$
π_w^{-1}
r_l X_u X_w $\pi_{i''}$
X_μ
r_l X_v X_w $\pi_{i''}$
$\pi_{v_0}^{-1}$
r_l X_v X_w X_{v_0} π_{i_0}
$\pi_{\tilde w}^{-1}$
r_l X_v X_w X_{v_0} $\mathsf{X}_{\tilde w}$ $\pi_{\tilde i}$
=
Y_a^l Y_c^l t_l X_{v_0} $\mathsf{X}_{\tilde w}$ $\pi_{\tilde i}$
$\pi_{\tilde w}$
Y_a^l Y_c^l t_l X_{v_0} π_{i_0}
$\tilde\theta_l$
Y_a^l s_l X_{u_0} π_{i_0}
π_{u_0}
Y_a^l s_l $\pi_{i'}$
ψ_l
Y_a^l $\pi_{j'}$ f_l
π_a
π_j f_l
φ_l^{-1}
r_l π_i
π_u^{-1}
r_l X_u $\pi_{i'}$
π_w^{-1}
r_l X_u X_w $\pi_{i''}$
X_μ
r_l X_v X_w $\pi_{i''}$
$\pi_{v_0}^{-1}$
r_l X_v X_w X_{v_0} π_{i_0}
$\pi_{\tilde w}^{-1}$
r_l X_v X_w X_{v_0} $\mathsf{X}_{\tilde w}$ $\pi_{\tilde i}$
=



where the first equality is due to axiom PC2, the second one and the fifth one require some elevators calculus plus (3.3.2), the third holds because (v, b, η_l) is a morphism in $\mathcal{M}$, the fourth one and the last one are valid by elevators calculus plus axiom PC2 and the sixth one is due to elevators calculus plus the fact that $(u, a\theta_l)$ is a morphism in $\mathcal{M}$.

Then, by 2.2.13, there exist a morphism $\tilde{i} \xrightarrow{w_0} i_1 \in \mathcal{I}$ such that

$$\begin{tikzcd}[column sep=tiny]
\mathsf{Y}^l_a \arrow[d, dash, "\mathsf{Y}^l_\alpha"] & \mathsf{Y}^l_c \arrow[d, dash] & \mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{\tilde{w}} \arrow[d, equal] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{Y}^l_b \arrow[d, equal] & \mathsf{Y}^l_c \arrow[d, dash] & \mathsf{t}_l \arrow[d, phantom, "\tilde{\theta}_l"] & \mathsf{X}_{v_0} \arrow[dl, dash] & \mathsf{X}_{\tilde{w}} \arrow[d, equal] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{Y}^l_b \arrow[d, equal] & \mathsf{s}_l \arrow[d, equal] & \mathsf{X}_{u_0} \arrow[d, dash] \arrow[dr, phantom, "\mathsf{X}_{\tilde{\mu}}"] & & \mathsf{X}_{\tilde{w}} \arrow[d, dash] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{Y}^l_b \arrow[d, dash, "\eta_l"] & \mathsf{s}_l \arrow[dl, dash] & \mathsf{X}_w \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{\tilde{w}} \arrow[d, equal] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{r}_l & \mathsf{X}_v & \mathsf{X}_w & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0}
\end{tikzcd}
=
\begin{tikzcd}[column sep=tiny]
\mathsf{Y}^l_a \arrow[d, equal] & \mathsf{Y}^l_c \arrow[d, dash] \arrow[dr, phantom, "\tilde{\theta}_l"] & \mathsf{t}_l & \mathsf{X}_{v_0} \arrow[dl, dash] & \mathsf{X}_{\tilde{w}} \arrow[d, equal] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{Y}^l_a \arrow[d, equal] & \mathsf{s}_l \arrow[d, equal] & \mathsf{X}_{u_0} \arrow[d, dash] \arrow[dr, phantom, "\mathsf{X}_{\tilde{\mu}}"] & & \mathsf{X}_{\tilde{w}} \arrow[d, dash] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{Y}^l_a \arrow[d, dash, "\theta_l"] & \mathsf{s}_l \arrow[d, dash] & \mathsf{X}_w \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{\tilde{w}} \arrow[d, equal] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{r}_l \arrow[d, equal] & \mathsf{X}_u \arrow[d, dash, "\mathsf{X}_\mu"] & \mathsf{X}_w \arrow[dl, dash] \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{\tilde{w}} \arrow[d, equal] & \mathsf{X}_{w_0} \arrow[d, equal] \\
\mathsf{r}_l & \mathsf{X}_v & \mathsf{X}_w & \mathsf{X}_{v_0} & \mathsf{X}_{\tilde{w}} & \mathsf{X}_{w_0}
\end{tikzcd}$$

We can conclude that we a have a morphism in $\mathcal{M}$

$$\{(\mathsf{s}_l,\psi_l)\}_{l=1,\dots,m} \xrightarrow{(w_0\tilde{w}v_0w,c,\{\mathsf{s}_l\mathsf{X}_{\tilde{\mu}}\mathsf{X}_{w_0}\circ\tilde{\theta}_l\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0}\}_{l=1,\dots,m})} \{(\mathsf{t}_l\mathsf{X}_{v_0}\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0},\ \begin{tikzcd}[column sep=tiny]
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{\tilde{w}} \arrow[d, equal] & \mathsf{X}_{w_0} \arrow[d, dash, "\pi_{w_0}"] & \pi_{i_1} \arrow[dl, dash] \\
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{\tilde{w}} \arrow[d, dash, "\pi_{\tilde{w}}"] & \pi_{\tilde{i}} \arrow[dl, dash] & \\
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, dash, "\pi_{v_0}"] & \pi_{i_0} \arrow[dl, dash] & & \\
\mathsf{t}_l \arrow[d, dash, "\varepsilon_l"] & \pi_{i''} \arrow[d, dash] & & & \\
\pi_{j''} & \mathsf{f}_l & & &
\end{tikzcd})\}_{l=1,\dots,m}$$

and an invertible 2-cell in $\mathcal{M}$

$$((w_0\tilde{w}v_0w,c,\{\mathsf{s}_l\mathsf{X}_{\tilde{\mu}}\mathsf{X}_{w_0}\circ\tilde{\theta}_l\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0}\}_{l=1,\dots,m})(u,a,\{\theta_l\}_{l=1,\dots,m}) \overset{(w_0\tilde{w}v_0\mu,\alpha)}{\Longrightarrow}$$
$$((w_0\tilde{w}v_0w,c,\{\mathsf{s}_l\mathsf{X}_{\tilde{\mu}}\mathsf{X}_{w_0}\circ\tilde{\theta}_l\mathsf{X}_{\tilde{w}}\mathsf{X}_{w_0}\}_{l=1,\dots,m})(v,b,\{\eta\}_{l=1,\dots,m}).$$

F2: Let $\begin{tikzcd}[column sep=large] \{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \arrow[r, bend left, "{(u,a,\{\theta_l\}_{l=1,\dots,m})}"] \arrow[r, bend right, "{(v,b,\{\eta_l\}_{l=1,\dots,m})}"'] \arrow[r, phantom, "\Downarrow(\mu,\alpha)\ \Downarrow(\rho,\beta)"] & \{(\mathsf{s}_l,\psi_l)\}_{l=1,\dots,m} \end{tikzcd} \in \mathcal{M}$. Since $\mathcal{J}$ is 2-filtered, we have $j' \xrightarrow{c} j'' \in \mathcal{J}$ such that $c\alpha = c\beta$ and, by 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have $\{\mathsf{X}_{i''} \xrightarrow{\mathsf{t}_l} \mathsf{Y}^l_{j''}\}_{l=1,\dots,m}$ and invertible 2-cells ϵ_l such that $\forall\ l = 1,\dots,m$ $(\mathsf{t}_l,\epsilon_l)$ represents f_l and there is a morphism $i' \xrightarrow{w} \tilde{i} \in \mathcal{I}$ such that $w\mu = w\rho$.

Now, since $\mathcal{I}$ is 2-filtered, we have morphisms $\begin{tikzcd}[row sep=tiny] \tilde{i} \arrow[dr, "u_0"] & \\ & i_0 \\ i'' \arrow[ur, "v_0"'] & \end{tikzcd} \in \mathcal{I}$.

Then we have $\{(\mathsf{s}_l, \psi_l)\}_{l=1,...,m}$ and $\{(\mathsf{t}_l\mathsf{X}_{v_0},$

$$\begin{tikzcd}
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[dr, dash] & & \pi_{i_0} \arrow[dl, dash, "\pi_{v_0}"'] \\
\mathsf{t}_l \arrow[d, dash] & \varepsilon_l & \pi_{i''} \arrow[d, dash] & \\
\pi_{j''} & & \mathsf{f}_l &
\end{tikzcd}$$

$)\}_{l=1,...,m}$ such that $\forall\ l = 1, ..., m$ (s_l, ψ_l) and $(\mathsf{t}_l\mathsf{X}_{v_0},$

$$\begin{tikzcd}
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[dr, dash] & & \pi_{\tilde{i}} \arrow[dl, dash, "\pi_{v_0}"'] \\
\mathsf{t}_l \arrow[d, dash] & \varepsilon_l & \pi_{i''} \arrow[d, dash] & \\
\pi_{j} & & \mathsf{f}_l &
\end{tikzcd}$$

$)$ both represent f_l and there are morphisms $i' \xrightarrow{u_0 w} i'' \in \mathcal{I}$, $j' \xrightarrow{c} j'' \in \mathcal{J}$. Then, by 2.2.9, we have morphisms

$$\begin{tikzcd}
i' \arrow[dr, "u_1"] & \\
& i_1 \\
i_0 \arrow[ur, "v_1"'] &
\end{tikzcd}$$

$\in \mathcal{I}$ and invertible 2-cells

$$\begin{tikzcd}
\mathsf{Y}^l_c\mathsf{t}_l\mathsf{X}_{v_0}\mathsf{X}_{v_1} \arrow[d, dash, "\tilde{\theta}_l"] \\
\mathsf{s}_l\mathsf{X}_{u_1}
\end{tikzcd}$$

$\in \mathcal{C}$ such that

$$\begin{tikzcd}
\mathsf{Y}^l_c \arrow[d, dash] & \mathsf{t}_l & \mathsf{X}_{v_0} & & \mathsf{X}_{v_1} \arrow[dl, dash] & \pi_{i_1} \arrow[d, equal] \\
\mathsf{s}_l \arrow[d, equal] & & \tilde{\theta}_l & \mathsf{X}_{u_1} \arrow[dr, dash, "\pi_{u_1}"] & & \pi_{i_1} \arrow[dl, dash] \\
\mathsf{s}_l \arrow[d, dash] & & & & \pi_{i'} \arrow[d, dash] & \\
\pi_{j'} & & \psi_l & & \mathsf{f}_l &
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{Y}^l_c \arrow[d, equal] & & \mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & & \mathsf{X}_{v_1} \arrow[dr, dash, "\pi_{v_1}"] & & \pi_{i_1} \arrow[dl, dash] \\
\mathsf{Y}^l_c \arrow[d, equal] & & \mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[dr, dash, "\pi_{v_0}"] & & & \pi_{i_0} \arrow[dll, dash] & \\
\mathsf{Y}^l_c \arrow[d, equal] & & \mathsf{t}_l \arrow[d, dash] & \varepsilon_l & \pi_{i''} \arrow[d, dash] & & & \\
\mathsf{Y}^l_c \arrow[dr, dash, "\pi_c"] & & \pi_{j''} \arrow[dl, dash] & & \mathsf{f}_l \arrow[d, equal] & & & \\
& \pi_{j'} & & & \mathsf{f}_l & & &
\end{tikzcd}$$

Plus, since $\mathcal{I}$ is 2-filtered, we have a morphism $i_1 \xrightarrow{\tilde{w}} i_2$ and an invertible 2-cell $\tilde{w}u_1 \overset{\tilde{\mu}}{\Longrightarrow} \tilde{w}v_1u_0w \in \mathcal{I}$.

It can be checked that there is a morphism in $\mathcal{M}$

$$\{(\mathsf{s}_l,\psi_l)\}_{l=1,\dots,m} \xrightarrow{(\tilde{w}v_1u_0w,c,\{\mathsf{s}_l\mathsf{X}_{\tilde{\mu}}\circ\tilde{\theta}_l\mathsf{X}_{\tilde{w}}\}_{l=1,\dots,m})} \{(\mathsf{t}_l\mathsf{X}_{v_0}\mathsf{X}_{v_1}\mathsf{X}_{\tilde{w}},\ \begin{tikzcd}
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{v_1} \arrow[d, equal] & \mathsf{X}_{\tilde{w}} \arrow[dr, dash, "\pi_{\tilde{w}}"] & & \pi_{i_2} \arrow[dl, dash] \\
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[d, equal] & \mathsf{X}_{v_1} \arrow[dr, dash, "\pi_{v_1}"] & & \pi_{i_1} \arrow[dl, dash] & \\
\mathsf{t}_l \arrow[d, equal] & \mathsf{X}_{v_0} \arrow[dr, dash, "\pi_{v_0}"] & & \pi_{i_0} \arrow[dl, dash] & & \\
\mathsf{t}_l \arrow[d, dash] & & \pi_{i''} \arrow[d, dash] & & & \\
\pi_{j''} & \varepsilon_l & \mathsf{f}_l & & &
\end{tikzcd})\}_{l=1,\dots,m}$$

such that

$$(\tilde{w}v_1u_0w,c,\{\mathsf{s}_l\mathsf{X}_{\tilde{\mu}}\circ\tilde{\theta}_l\mathsf{X}_{\tilde{w}}\}_{l=1,\dots,m})(u,a,\{\theta_l\}_{l=1,\dots,m}) =$$

$$(\tilde{w}v_1u_0w,c,\{\mathsf{s}_l\mathsf{X}_{\tilde{\mu}}\circ\tilde{\theta}_l\mathsf{X}_{\tilde{w}}\}_{l=1,\dots,m})(v,b,\{\eta_l\}_{l=1,\dots,m}).$$

$$\begin{tikzcd}
\mathcal{M} \arrow[r] & \mathcal{I} \\
\{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \arrow[r, mapsto] & i
\end{tikzcd}$$ is 2-cofinal:

CF0: Let $i \in \mathcal{I}$ and let $\forall\ l = 1,\dots,m\ \mathsf{X}_{i'} \xrightarrow{\mathsf{r}_l} \mathsf{Y}^l_j \in \mathcal{C}$ and invertible 2-cells φ_l such that (r_l,φ_l) represents f_l. Note that we are using the fact that $\mathcal{I}$ is 2-filtered to make all r_l's have the same source. Also because $\mathcal{I}$ is 2-filtered, we have $\begin{tikzcd}[row sep=tiny] i \arrow[r, "u"] & i'' \\ i' \arrow[ur, "v"'] & \end{tikzcd} \in \mathcal{I}$. It is straightforward to check that $\{(\mathsf{r}_l\mathsf{X}_v,\ \begin{tikzcd}[row sep=small, column sep=small]
\mathsf{r}_l \arrow[d, equal] & \mathsf{X}_v \arrow[dr, dash, "\pi_v"] & & \pi_{i''} \arrow[dl, dash] \\
\mathsf{r}_l \arrow[d, dash] & & \pi_{i'} \arrow[d, dash] & \\
\pi_j & \varphi_l & \mathsf{f}_l &
\end{tikzcd})\}_{l=1,\dots,m}$ together with $i \xrightarrow{u} i''$ proves CF0.

CF1: Let $i \in \mathcal{I}$, $\{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \in \mathcal{M}$ ($\mathsf{r}_l : \mathsf{X}_{i'} \to \mathsf{Y}^l_j$) and $i \overset{u}{\underset{v}{\rightrightarrows}} i' \in \mathcal{I}$. Since $\mathcal{I}$ is 2-filtered, we have $i' \overset{w}{\to} i''$ and an invertible 2-cell $wu \overset{\mu}{\Rightarrow} wv \in \mathcal{I}$. It is straightforward to check that $\{(\mathsf{r}_l,\varphi_l)\}_{l=1,\dots,m} \xrightarrow{(w,id,\{id\}_{l=1,\dots,m})} \{(\mathsf{r}_l\mathsf{X}_w,\ \begin{tikzcd}[row sep=small, column sep=small]
\mathsf{r}_l \arrow[d, equal] & \mathsf{X}_w \arrow[dr, dash, "\pi_w"] & & \pi_{i''} \arrow[dl, dash] \\
\mathsf{r}_l \arrow[d, dash] & & \pi_{i'} \arrow[d, dash] & \\
\pi_{j'} & \varphi_l & \mathsf{f}_l &
\end{tikzcd})\}_{l=1,\dots,m}$ proves CF1.

CF2: Let $i \in \mathcal{I}$, $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \in \mathcal{M}\,(\mathsf{r}_l : \mathsf{X}_{i'} \to \mathsf{Y}^l_j)$ and $i \underset{v}{\overset{u}{\underset{\Downarrow\mu \;\; \Downarrow\rho}{\rightrightarrows}}} i' \in \mathcal{I}$. Since $\mathcal{I}$ is 2-filtered, we have $i' \overset{w}{\to} i'' \in \mathcal{I}$ such that $w\mu = w\rho$. It is straightforward to check that

$$\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \overset{(w,id,\{id\}_{l=1,...,m})}{\longrightarrow} \{(\mathsf{r}_l\mathsf{X}_w, \begin{tikzcd}[row sep=small, column sep=small] \mathsf{r}_l \arrow[d, equal] & \mathsf{X}_w \arrow[dr, dash] & & \pi_{i''} \arrow[dl, dash, "\pi_w"'] \\ \mathsf{r}_l \arrow[d, dash] & \varphi_l & \pi_{i'} \arrow[d, dash] & \\ \pi_{j'} & & \mathsf{f}_l & \end{tikzcd})\}_{l=1,...,m}$$

proves CF2.

$$\begin{tikzcd} \mathcal{M} \arrow[r] & \mathcal{J} \\ \{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \arrow[r, mapsto] & j \end{tikzcd}$$

is 2-cofinal:

CF0: Let $j \in \mathcal{J}$. By 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have $\forall\ l = 1, ..., m\ \mathsf{r}_l : \mathsf{X}_i \to \mathsf{Y}^l_j \in \mathcal{C}$ and invertible 2-cells φ_l such that $(\mathsf{r}_l, \varphi_l)$ represents f_l. $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m}$ clearly proves CF0.

CF1: Let $j_0 \in \mathcal{J}$, $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \in \mathcal{M}\,(\mathsf{r}_l : \mathsf{X}_i \to \mathsf{Y}^l_j)$ and $j_0 \underset{b}{\overset{a}{\rightrightarrows}} j \in \mathcal{J}$. Since $\mathcal{J}$ is 2-filtered, we have $j \overset{c}{\to} j' \in \mathcal{J}$ and an invertible 2-cell $ca \overset{\alpha}{\Rightarrow} cb \in \mathcal{J}$. Now, by 2.2.3, we have $\mathsf{s} : \mathsf{X}_{i'} \to \mathsf{Y}_{j'} \in \mathcal{C}$ such that (s, id) represents f.

From the proof of the fact that $\mathcal{M}$ is 2-filtered, we have morphisms

$$\begin{tikzcd}[column sep=small] \{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \arrow[dr, "{(u,c,\{\theta_l\}_{l=1,...,m})}"] & \\ & \{(\mathsf{t}_l, \psi_l)\}_{l=1,...,m} \\ \{(\mathsf{s}_l, id)\}_{l=1,...,m} \arrow[ur, "{(v,id,\{\eta_l\}_{l=1,...,m})}"'] & \end{tikzcd}$$

. It is straightforward to check that $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \overset{(u,c,\{\theta_l\}_{l=1,...,m})}{\longrightarrow} \{(\mathsf{t}_l, \psi_l)\}_{l=1,...,m}$ proves CF1.

CF2: Let $j_0 \in \mathcal{J}$, $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \in \mathcal{M}\,(\mathsf{r}_l : \mathsf{X}_i \to \mathsf{Y}^l_j)$ and $j_0 \underset{b}{\overset{a}{\underset{\Downarrow\alpha \;\; \Downarrow\beta}{\rightrightarrows}}} j \in \mathcal{J}$. Since $\mathcal{J}$ is 2-filtered, we have $j \overset{c}{\to} j' \in \mathcal{J}$ such that $c\alpha = c\beta$. Now, by 2.2.3 and the fact that $\mathcal{I}$ is 2-filtered, we have $\forall\ l = 1, ..., m\ \ \mathsf{s}_l : \mathsf{X}_{i'} \to \mathsf{Y}^l_{j'} \in \mathcal{C}$ and invertible 2-cells ψ_l such that (s_l, ψ_l) represents f_l.

From the proof of the fact that $\mathcal{M}$ is 2-filtered, we have morphisms

$$\begin{tikzcd}[column sep=small] \{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \arrow[dr, "{(u,c,\{\theta_l\}_{l=1,...,m})}"] & \\ & \{(\mathsf{t}_l, \epsilon_l)\}_{l=1,...,m} \\ \{(\mathsf{s}_l, id)\}_{l=1,...,m} \arrow[ur, "{(v,id,\{\eta_l\}_{l=1,...,m})}"'] & \end{tikzcd}$$

. It is straightforward to check that $\{(\mathsf{r}_l, \varphi_l)\}_{l=1,...,m} \overset{(u,c,\{\theta_l\}_{l=1,...,m})}{\longrightarrow} \{(\mathsf{t}_l, \epsilon_l)\}_{l=1,...,m}$ proves CF2.

Finally, take $\mathsf{D}_{(\mathsf{r}_l,\varphi_l)_{l=1,\dots,k}}$ with $\mathsf{X}_i \xrightarrow{\mathsf{r}_l} \mathsf{Y}^l_j$ given by the following:
Send x to X_i, Δ' to D'_j and the morphisms that link both parts to the corresponding r_l's.

□

3.3.3 Corollary. *Let* $\Delta \xrightarrow{\mathsf{D}} 2\text{-}\mathcal{P}ro(\mathcal{C})$ *be a finite diagram with commutation relations and no loops in* $2\text{-}\mathcal{P}ro(\mathcal{C})$*. Then there exists a cofinite and filtered poset with a unique initial object* J *and a diagram* $\Delta \xrightarrow{\mathsf{D}'} \mathcal{H}om(\mathsf{J}^{op}, \mathcal{C})$ *equivalent to* D *up to isomorphism.*

Equivalently every object $\mathsf{D} \in \mathcal{H}om(\Delta, 2\text{-}\mathcal{P}ro(\mathcal{C}))$ *have a lifting to* $\mathcal{H}om(\mathsf{J}^{op}, \mathcal{C})$ *up to equivalence for some cofinite and filtered poset with a unique initial object* J.

$$\begin{tikzcd}
 & \mathcal{H}om(\mathsf{J}^{op}, \mathcal{C}) \arrow[d, "inc"] \\
\Delta \arrow[ur, "\mathsf{f}'"] \arrow[r, "\mathsf{f}"'] \arrow[ur, phantom, "\simeq", shift right=3] & 2\text{-}\mathcal{P}ro(\mathcal{C})
\end{tikzcd}$$

Proof. It follows from 3.3.1 as 3.2.7 follows from 3.2.4. □

3.3.4 Corollary. *3.3.3 also holds in* $2\text{-}\mathcal{P}ro_p(\mathcal{C})$*.*

Proof. It follows from 3.3.3 as 3.2.9 follows from 3.2.7. □

Resumen en castellano de la sección 3

En esta sección probamos ciertas propiedades de reindexación para las 2-categorías 2-$\mathcal{P}ro(C)$ y 2-$\mathcal{P}ro_p(C)$ que serán usadas para probar que son "closed 2-bmodel 2-categories" (4.1.3) así como Edwards-Hastings lo hacen para $\mathsf{Pro}(\mathsf{C})$ en [12] en el caso 1-dimensional. Las propiedades de reindexación para $\mathsf{Pro}(\mathsf{C})$ pueden hallarse en [3] o [1].

El primer resultado es una versión 2-categórica de un resultado debido a Deligne [1, Expose I, 8.1.6] que es clave en el caso 1-dimensional en el desarrollo de la estructura de modelos de la categoría $\mathsf{Pro}(\mathsf{C})$ [12]. El enunciado 1-dimensional establece que todo pro-objeto es isomorfo a uno indexado por un poset cofinito y filtrante. Nuestra versión establece que todo 2-pro-objeto es equivalente a uno indexado por un poset cofinito y filtrante. El segundo resultado establece que todo morfismo de 2-pro-objetos puede ser levantado salvo equivalencia a un morfismo entre 2-pro-objetos indexados por un poset cofinito y filtrante. Esto es un caso particular de un tercer resultado que establece que todo diagrama finito en 2-$\mathcal{P}ro(C)$ puede ser levantado salvo equivalencia a un diagrama finito de 2-pro-objetos indexados por un poset cofinito y filtrante. Es clave para estos resultados la noción de pseudo-funtor 2-cofinal dada en la sección 1. Toda esta sección será usada para probar el teorema central de la sección 5.

4 Closed 2-model 2-categories

In this section we introduce original notions of closed 2-model and closed 2-bmodel 2-category and state some lemmas and propositions that we are going to use later. Our notion is stronger than Pronk's "fibration structures" ([26]) since it is a 2-dimensional version of the full Quillen's axioms for closed model structures. It also differs in the important fact that we do not assume the choice of a privileged global factorization given in a pseudo-functorial way but stipulates, as Quillen does, only the existence of factorizations for each arrow.

Most of the results of this section are generalizations to the context of 2-categories of well known statements about closed model categories. For definitions and results in the 1-dimensional case, check for example [27] or [13].

4.1 Definitions and basic lemmas

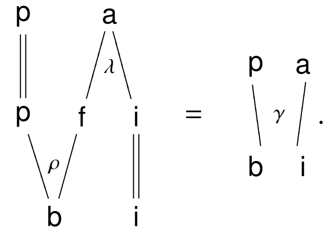
4.1.1 Definition. *Let C be a 2-category and* $\mathsf{A} \xrightarrow{\mathsf{i}} \mathsf{X}$, $\mathsf{Y} \xrightarrow{\mathsf{p}} \mathsf{B}$ *two morphisms in C. We say that the pair* (i, p) *has the* lifting property *(or equivalently that* i *has the left lifting property with respect to* p *or equivalently that* p *has the right lifting property with respect to* i*) if for each diagram in C of the form*

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{B}
\end{tikzcd} \tag{4.1.2}$$

there exist a morphism f *and invertible 2-cells λ, ρ as in the following diagram*

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] \arrow[ur, dashed, "\mathsf{f}" description] & \mathsf{B}
\end{tikzcd}
\quad \cong \Downarrow \lambda, \ \cong \Downarrow \rho$$

such that



In this case, we say that $(\mathsf{f}, \lambda, \rho)$ *is a filler for diagram* (4.1.2).

4.1.3 Definition. *We say that a 2-category* $\mathcal{C}$ *is a* closed 2-model 2-category (respectively a closed 2-bmodel 2-category) *if it is equipped with three classes of morphisms called* fibrations, cofibrations *and* weak equivalences *satisfying the following properties:*

2-M0: $\mathcal{C}$ is closed under finite weighted pseudo-limits and pseudo-colimits of pseudo-functors $\mathsf{F} : \mathcal{P} \longrightarrow \mathcal{C}$ with finite weights $\mathsf{W} : \mathcal{P} \longrightarrow \mathcal{C}at$ (see [19]).
(Respectively:

2-M0b: $\mathcal{C}$ is closed under finite weighted bi-limits and bi-colimits of pseudo-functors $\mathsf{F} : \mathcal{P} \longrightarrow \mathcal{C}$ with finite weights $\mathsf{W} : \mathcal{P} \longrightarrow \mathcal{C}at$ (see [19]).)
To simplify, by finite we mean that $\mathcal{P}$ is finite and $\mathsf{W}(\mathsf{P})$ is finite for all $\mathsf{P} \in \mathcal{P}$.

2-M2: Every morphism $\mathsf{f} \in \mathcal{C}$ can be factored up to isomorphism as $\mathsf{f} \cong \mathsf{pi}$ with i a cofibration which is also a weak equivalence and p a fibration or i a cofibration and p a fibration which is also a weak equivalence.

2-M5: Given a diagram of the form

$$\begin{tikzcd}
\mathsf{X} \arrow[rr, "h"] \arrow[dr, "\mathsf{f}"'] & \cong & \mathsf{Y} \\
& \mathsf{Z} \arrow[ur, "\mathsf{g}"'] &
\end{tikzcd}$$

If two of the three f, g, h are weak equivalences, then so is the third one. Every isomorphism is a weak equivalence.

2-M6a): A morphism $\mathsf{p} \in \mathcal{C}$ is a fibration iff the pair (i, p) has the lifting property for every i that is both a cofibration and a weak equivalence.

2-M6b): A morphism $\mathsf{i} \in \mathcal{C}$ is a cofibration iff the pair (i, p) has the lifting property for every p that is both a fibration and a weak equivalence.

2-M6c): A morphism $\mathsf{f} \in \mathcal{C}$ is a weak equivalence iff it can be factored up to isomorphism as $\mathsf{f} \cong \mathsf{uv}$ where u has the right lifting property with respect to all cofibrations and v has the left lifting property with respect to all fibrations.

For some of the proofs of section 5 we are going to assume that our 2-category $\mathcal{C}$ satisfies the following “2-niceness conditions”:

2-N1 Every cofibration is a bi-pushout of a cofibration between cofibrant objects.
2-N2 Every fibration is a bi-pullback of a fibration between fibrant objects.
2-N3 At least one of the following is satisfied:
2-N3a) Every object is cofibrant. *2-N3b) Every object is fibrant.*

4.1.4 Remark. A fourth niceness condition is considered in [12] in the 1-dimensional case: *N4: There exist functorial cylinder objects.* Though it is not mentioned, we think that this condition is only needed in the proof of the analogous of 5.2.12, but we also believe that this condition is not necessary as we proved in the 2-dimensional case that locally pseudo-functorial cylinder objects can be chosen (see 4.2.8) which is enough to prove 5.2.12. Clearly, from the proof of 4.2.8 we see that it follows from Quillen's axioms that is possible to choose locally functorial cylinder objects, result that we have not found in the literature.

4.1.5 Remark. Any closed 2-model 2-category is a closed 2-bmodel 2-category. Note also that the two notions differ only in the first axiom. □

4.1.6 Remark. To check axiom 2-M0b, it is enough to check the existence of bi-equalizers, finite bi-products (binary plus bi-1) and bi-cotensors with a finite category (see [28], [19], [5]). □

4.1.7 Remark. If $\mathcal{C}$ is a closed 2-bmodel 2-category, in particular $\mathcal{C}$ has finite bi-limits (that is finite conical weighted bi-limits) indexed by a poset, and more in particular bi-pullbacks and bi-1. □

4.1.8 Lemma. *Let $\mathcal{C}$ be a 2-category with three classes of morphisms satisfying 2-M6a), 2-M6b) and 2-M6c). Then a morphism $p \in \mathcal{C}$ is both a fibration (respectively cofibration) and a weak equivalence iff it has the right lifting property (respectively left lifting property) with respect to all cofibrations (respectively fibrations).*

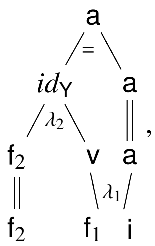
Proof. We will prove the case where p is a fibration. The other case is similar and we omit it.

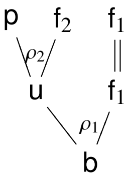
⇒) Let $Y \xrightarrow{p} B \in \mathcal{C}$ be a morphism that is both a fibration and a weak equivalence and $A \xrightarrow{i} X \in \mathcal{C}$ a cofibration and suppose that we have the following situation:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[rd, phantom, "\cong \Downarrow \gamma"] & Y \arrow[d, "p"] \\
X \arrow[r, "b"'] & B
\end{tikzcd}$$

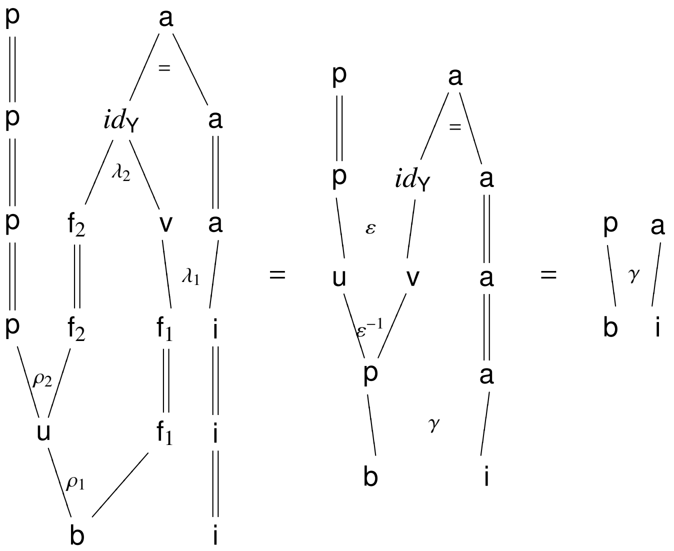
Since $\mathcal{C}$ satisfies axiom 2-M6c) and p is a weak equivalence, there exist morphisms u, v and an invertible 2-cell $p \overset{\epsilon}{\Longrightarrow} uv$ such that u has the right lifting property with respect to all cofibrations and v has the left lifting property with respect to all fibrations. Then there exist fillers (f_1, λ_1, ρ_1), (f_2, λ_2, ρ_2) for the following diagrams

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[rd, phantom, "\cong \Downarrow \gamma"] & Y \arrow[r, "v"] \arrow[d, "p"] \arrow[rd, phantom, "\overset{\epsilon}{\underset{\cong}{\Rightarrow}}"] & Z \arrow[ld, bend left, "u"] \\
X \arrow[r, "b"'] & B &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[rrd, phantom, "\cong \Downarrow \gamma \circ \epsilon^{-1} a"] & Y \arrow[r, "v"] & Z \arrow[d, "u"] \\
X \arrow[rr, "b"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "id_Y"] \arrow[d, "v"'] \arrow[rd, phantom, "\cong \Downarrow \epsilon"] & Y \arrow[d, "p"] \\
Z \arrow[r, "u"'] & B
\end{tikzcd}$$

Let's check that ($\mathsf{f}_2\mathsf{f}_1$, 

, 

) is the filler that we were looking for:



The first equality holds by elevators calculus plus the fact that $(\mathsf{f}_1, \lambda_1, \rho_1)$ and $(\mathsf{f}_2, \lambda_2, \rho_2)$ are fillers for the corresponding diagrams.

$\Leftarrow$) Since C satisfies axioms 2-M6a) and 2-M6c), it is clear that p is a fibration and it is also a weak equivalence because it can be factored as $\mathsf{p} = \mathsf{p}id_{\mathsf{Y}}$.

□

4.1.9 Proposition. *Let C be a 2-category with three classes of morphisms satisfying axioms 2-M6a), 2-M6b) and 2-M6c). Then the following hold:*

2-M1: Given i a cofibration and p a fibration, if one of them is a weak equivalence, then the pair (i, p) has the lifting property.

2-M3b: Fibrations (respectively cofibrations) are closed under composition and bi-pullbacks (respectively bi-pushouts). Every isomorphism is a fibration and a cofibration.

In particular:

2-M3: Fibrations (respectively cofibrations) are closed under composition and pseudo-pullbacks (respectively pseudo-pushouts). Every isomorphism is a fibration and a cofibration.

2-M4b: *If* $\mathsf{f} \in \mathcal{C}$ *is the bi-pullback (respectively bi-pushout) of a fibration (respectively cofibration) which is also a weak equivalence, then* f *is a weak equivalence.*

In particular:

2-M4: *If* $\mathsf{f} \in \mathcal{C}$ *is the pseudo-pullback (respectively pseudo-pushout) of a fibration (respectively cofibration) which is also a weak equivalence, then* f *is a weak equivalence.*

2-M7: *Fibrations, cofibrations and weak equivalences are closed under isomorphisms, i.e. if there is an invertible 2-cell* $\mathsf{f} \Longrightarrow \mathsf{g}$ *and* f *is a fibration (respectively a cofibration or a weak equivalence), then* g *is also a fibration (respectively a cofibration or a weak equivalence).*

Proof. 2-M1: is clear by 4.1.8.

2-M3b: We are going to prove the case of fibrations, the case of cofibrations is similar and we leave it to the reader.

- Suppose that p and q are two fibrations in $\mathcal{C}$. By axiom 2-M6a), to prove that qp is a fibration, we only need to check that it has the right lifting property with respect to all morphisms that are both cofibrations and weak equivalences. So let i be a cofibration which is also a weak equivalence and suppose that we have a diagram of the form

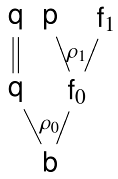
$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Y} \arrow[d, "\mathsf{qp}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{B}
\end{tikzcd}$$

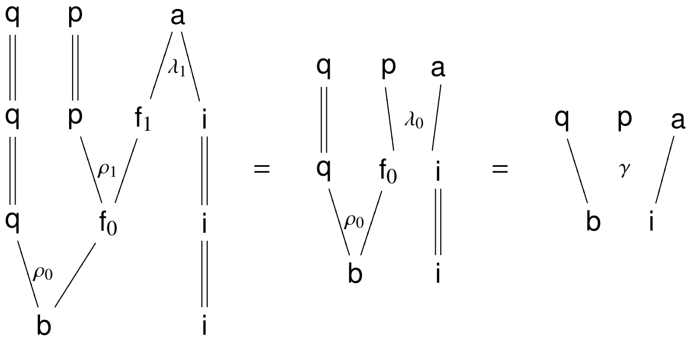
  Since q is a fibration, there exists a filler $(\mathsf{f}_0, \lambda_0, \rho_0)$ for the following diagram

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] & \mathsf{Y} \arrow[r, "\mathsf{p}"] \arrow[d, phantom, "\cong \Downarrow \gamma"] & \mathsf{Z} \arrow[d, "\mathsf{q}"] \\
\mathsf{X} \arrow[rr, "\mathsf{b}"'] & & \mathsf{B}
\end{tikzcd}$$

  Now, since p is also a fibration, there exists a filler $(\mathsf{f}_1, \lambda_1, \rho_1)$ for the following diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong \Downarrow \lambda_0"] & Y \arrow[d, "p"] \\
X \arrow[r, "f_0"'] & Z
\end{tikzcd}$$

Let's check that $(f_1, \lambda_1,$  $)$ is the filler that we were looking for:



- Now suppose that p is a fibration and we have a bi-pullback

$$\begin{tikzcd}
P \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\cong \Downarrow \alpha"] & Y \arrow[d, "p"] \\
Z \arrow[r, "f"'] & B
\end{tikzcd} \tag{4.1.10}$$

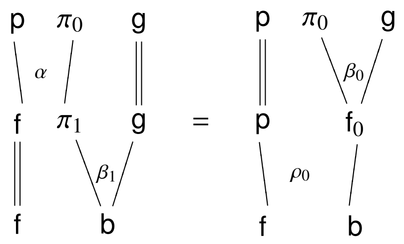
We need to prove that π_1 has the right lifting property with respect to all morphisms that are both cofibrations and weak equivalences. So let i be a cofibration which is also a weak equivalence and suppose that we have a diagram of the form

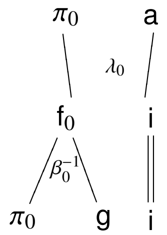
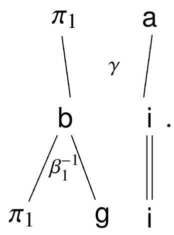
$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & P \arrow[d, "\pi_1"] \\
X \arrow[r, "b"'] & Z
\end{tikzcd}$$

Since p is a fibration, there exists a filler $(\mathsf{f}_0, \lambda_0, \rho_0)$ for the following diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong\Downarrow\gamma"] & P \arrow[r, "\pi_0"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\cong\Downarrow\alpha"] & Y \arrow[d, "p"] \arrow[dr, phantom, "="] & A \arrow[r, "\pi_0 a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong\Downarrow f\gamma\circ\alpha a"] & Y \arrow[d, "p"] \\ X \arrow[r, "b"'] & Z \arrow[r, "f"'] & B & X \arrow[r, "fb"'] & B \end{tikzcd}$$

Since (4.1.10) is a bi-pullback, there exists a morphism $\mathsf{X} \xrightarrow{\mathsf{g}} \mathsf{P}$ and invertible 2-cells $\pi_0\mathsf{g} \overset{\beta_0}{\Longrightarrow} \mathsf{f}_0$, $\pi_1\mathsf{g} \overset{\beta_1}{\Longrightarrow} \mathsf{b}$ satisfying the following equality:



It is straightforward to check that $(\mathsf{g}, \lambda, \beta_1)$ is the filler that we were looking for, where λ is such that $\pi_0\lambda =$  and $\pi_1\lambda =$  .

- To conclude with axiom 2-M3b, suppose that f is an isomorphism, i is a morphism that is both a cofibration and a weak equivalence and we have a diagram of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong\Downarrow\gamma"] & Y \arrow[d, "f"] \\ X \arrow[r, "b"'] & B \end{tikzcd}$$

Since f is an isomorphism, there exists $\mathsf{B} \xrightarrow{\mathsf{g}} \mathsf{Y}$ such that $\mathsf{fg} = id_{\mathsf{B}}$ and $\mathsf{gf} = id_{\mathsf{Y}}$. It is clear that $(\mathsf{gb}, \gamma, id)$ is the filler that we were looking for.

2-M4b: Suppose that p is both a fibration and a weak equivalence and we have a bi-pullback

$$\begin{tikzcd}
\mathsf{P} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\cong\Downarrow\alpha"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\
\mathsf{Z} \arrow[r, "\mathsf{f}"'] & \mathsf{B}
\end{tikzcd} \tag{4.1.11}$$

Since we have already proved axiom 2-M3b, we know that π_1 is a fibration. By 4.1.8, we have to check that π_1 has the right lifting property with respect to all cofibrations. The proof follows as the proof of axiom 2-M3b.

2-M7: Suppose that f is a fibration (the case of a cofibration is similar and we leave it to the reader) and there is an isomorphism $\mathsf{f} \overset{\alpha}{\Longrightarrow} \mathsf{g}$. We want to check that g has the right lifting property with respect to all morphisms that are both cofibrations and weak equivalences. So, suppose that i is a morphism that is both a cofibration and a weak equivalence and we have a diagram of the form

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong\Downarrow\gamma"] & \mathsf{Y} \arrow[d, "\mathsf{g}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{B}
\end{tikzcd}$$

Since f is a fibration, there is a filler $(\mathsf{f}_0, \lambda_0, \rho_0)$ for the following diagram

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong\Downarrow\gamma"] & \mathsf{Y} \arrow[d, shift right=3, "\mathsf{g}"'] \arrow[d, shift left=3, "\mathsf{f}"] \arrow[d, phantom, "\overset{\alpha}{\underset{\cong}{\Leftarrow}}"] & = & \mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong\Downarrow\gamma\circ\alpha\mathsf{a}"] & \mathsf{Y} \arrow[d, "\mathsf{f}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{B} & & \mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{B}
\end{tikzcd}$$

It is straightforward to check that $\left(\mathsf{f}_0, \lambda_0, \begin{array}{c} \mathsf{g} \quad \mathsf{f}_0 \\ \backslash{\scriptstyle\alpha^{-1}}/ \quad \| \\ \mathsf{f} \quad \mathsf{f}_0 \\ \backslash{\scriptstyle\rho_0}/ \\ \mathsf{b} \end{array}\right)$ is the filler that we were looking for.

To conclude, suppose that f is a weak equivalence and there is an isomorphism $\mathsf{f} \overset{\alpha}{\Longrightarrow} \mathsf{g}$. Then $\mathsf{g} \cong \mathsf{f} \cong \mathsf{uv}$ as in 2-M6c) and so is also a weak equivalence. □

Although we do not use it in this work, we set (for the record) the following definition:

4.1.12 Definition. *We say that a 2-category C is a* 2-model 2-category *(respectively a* 2-bmodel 2-category*) if it is equipped with three classes of morphisms called fibrations, cofibrations and weak equivalences satisfying 2-M0 (respectively 2-M0b), 2-M1, 2-M2, 2-M3 (respectively 2-M3b), 2-M4 (respectively 2-M4b), 2-M5 and 2-M7.*

4.1.13 Corollary (of 4.1.9)**.**

1. *If C is a closed 2-bmodel 2-category, then C is a 2-bmodel 2-category.*
2. *If C is a closed 2-model 2-category, then C is a 2-model 2-category.* □

4.1.14 Remark. By axiom 2-M7, axiom 2-M5 can be replaced in each definition by its following weaker version

2-M5w: Given two composable morphisms $\mathsf{f}, \mathsf{g} \in C$, if two of the three $\mathsf{f}, \mathsf{g}, \mathsf{gf}$ are weak equivalences, then so is the third one. Every isomorphism is a weak equivalence. □

The following is the 2-dimensional version of "the retract argument" [16, 7.2.2].

4.1.15 Proposition. *Let C be a 2-category and let $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y} \in C$.*

1. *If f is factorized as $\mathsf{f} \cong \mathsf{pi}$ and the pair (f, p) has the lifting property, then f is a retract of i in $\mathcal{M}aps_p(C)$.*
2. *If f is factorized as $\mathsf{f} \cong \mathsf{pi}$ and the pair (i, f) has the lifting property, then f is a retract of p in $\mathcal{M}aps_p(C)$.*

Proof. 1. Let $(\mathsf{g}, \lambda, \rho)$ be a filler for the following diagram

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{i}"] \arrow[d, "\mathsf{f}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Z} \arrow[d, "\mathsf{p}"] \\
\mathsf{Y} \arrow[r, "id_{\mathsf{Y}}"'] & \mathsf{Y}
\end{tikzcd}$$

Then f is a retract of i via $(id_{\mathsf{X}}, \mathsf{g}, \lambda, id_{\mathsf{X}}, \mathsf{p}, \gamma^{-1}, id_{id_{\mathsf{X}}}, \rho)$.

2. The proof is similar to the previous one and we leave it to the reader.

□

4.1.16 Proposition. *Let C be a 2-category, p' a retract of p and i' a retract of i. If the pair (i, p) has the lifting property, then the pair $(\mathsf{i}', \mathsf{p}')$ also does.*

Proof. We know that p' is a retract of p via $(\theta_0, \theta_1, \theta_m, \eta_0, \eta_1, \eta_m, \mu_0, \mu_1)$ and i' is a retract of i via $(\theta'_0, \theta'_1, \theta'_m, \eta'_0, \eta'_1, \eta'_m, \mu'_0, \mu'_1)$. Suppose that we have a diagram of the form

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "i'"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & Y' \arrow[d, "p'"] \\ X' \arrow[r, "b"'] & B' \end{tikzcd} \qquad (4.1.17)$$

By hypothesis, there is a filler $(\mathsf{f}, \lambda, \rho)$ for diagram

$$\begin{tikzcd} A \arrow[r, "\theta_0 a \eta'_0"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong \Downarrow \gamma'"] & Y \arrow[d, "p"] \\ X \arrow[r, "\theta_1 b \eta'_1"'] & B \end{tikzcd}$$

where $\gamma = \begin{array}{cccccc} p & & \theta_0 & a & & \eta'_0 \\ & \backslash \;\theta_m\; / & & \| & & \| \\ \theta_1 & & p' & a & & \eta'_0 \\ \| & & \backslash & \gamma & / & \| \\ \theta_1 & & b & i' & & \eta'_0 \\ \| & & \| & \backslash & \eta'_m \; / & \\ \theta_1 & & b & \eta'_1 & & i \end{array}$.

It can be checked that $\left(\eta_0 \mathsf{f} \theta'_1,\ \begin{array}{ccccccc} & & & a & & & \\ & / & & = & & \backslash & \\ & id_{Y'} & & a & & id_{A'} & \\ / & \mu_0{}^{-1} & \backslash & \| & / & \mu'_0{}^{-1} & \backslash \\ \eta_0 & & \theta_0 & a & \eta'_0 & & \theta'_0 \\ \| & & \backslash & \lambda & / & & \| \\ \eta_0 & & f & & i & & \theta'_0 \\ \| & & \| & & \backslash & \theta_m & / \\ \eta_0 & & f & & \theta'_1 & & i' \end{array},\ \begin{array}{ccccccc} p' & & \eta_0 & f & & & \theta'_1 \\ \backslash & \eta_m & / & \| & & & \| \\ \eta_1 & & p & f & & & \theta'_1 \\ \| & & / & \rho & \backslash & & \| \\ \eta_1 & \theta_1 & & b & \eta'_1 & & \theta'_1 \\ \backslash & \mu_1 \; / & & \| & \backslash & \mu'_1 & / \\ & id_{B'} & & b & & id_{X'} & \\ & & \backslash & = & / & & \\ & & & b & & & \end{array}\right)$

is the filler that we were looking for. ◻

4.1.18 Proposition. *Let C be a 2-category with three classes of morphisms satisfying axioms 2-M6a), 2-M6b) and 2-M6c). Then fibrations, cofibrations, morphisms that are* *both fibrations and weak equivalences and morphisms that are both cofibrations and weak equivalences are closed under the formation of retracts.*

Proof. It is straightforward from 4.1.16 plus 4.1.8. ☐

We believe the previous statement is also true for weak equivalences but its proof is a deeper result. Classically (in the 1-dimensional case, for closed model categories) it follows immediately from Quillen's theorem "$\gamma(\mathsf{f})$ *invertible* iff f *is a weak equivalence*" (where γ is the universal functor inverting weak equivalences). We expect to finish a 2-dimensional version of this theorem in future work.

4.2 Locally functorial factorizations and cylinder objects

4.2.1 Definition. *A factorization up to invertible 2-cell for arrows in a 2-category C is said to be locally pseudo-functorial (on the sequel we will just say functorial) if given* $\mathsf{f} \xrightarrow{\theta} \mathsf{f}' \xrightarrow{\eta} \mathsf{f}'' \in \mathcal{H}om_p(2, C)$ *as in the following diagram*

$$\begin{tikzcd}
X \arrow[r, "\theta_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong \Downarrow \theta_m"] & X' \arrow[r, "\eta_0"] \arrow[d, "f'"'] \arrow[dr, phantom, "\cong \Downarrow \eta_m"] & X'' \arrow[d, "f''"'] \\
Y \arrow[r, "\theta_1"'] & Y' \arrow[r, "\eta_1"'] & Y''
\end{tikzcd}$$

there are suitable factorizations

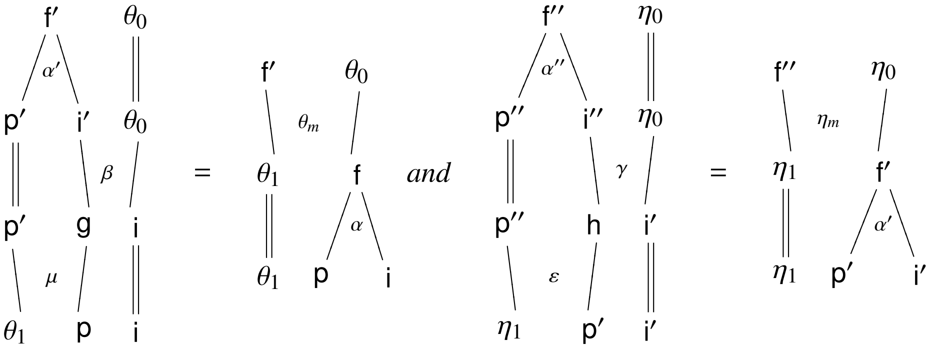
$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & \arrow[d, phantom, "\cong \Downarrow \alpha"] & Y , \\
& Z \arrow[ur, "p"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "i'"'] & \arrow[d, phantom, "\cong \Downarrow \alpha'"] & Y' , \\
& Z' \arrow[ur, "p'"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
X'' \arrow[rr, "f''"] \arrow[dr, "i''"'] & \arrow[d, phantom, "\cong \Downarrow \alpha''"] & Y'' \\
& Z'' \arrow[ur, "p''"'] &
\end{tikzcd}$$

such that there exist morphisms $\mathsf{Z} \xrightarrow{g} \mathsf{Z}' \xrightarrow{h} \mathsf{Z}''$ *and invertible 2-cells* $\begin{array}{ccc} i' & & \theta_0 \\ \backslash & \beta & / \\ g & & i \end{array}$, $\begin{array}{ccc} i' & & \eta_0 \\ \backslash & \gamma & / \\ h & & i' \end{array}$,

$\begin{array}{ccc} p' & & g \\ \backslash & \mu & / \\ \theta_1 & & p \end{array}$, $\begin{array}{ccc} p'' & & h \\ \backslash & \epsilon & / \\ \eta_1 & & p' \end{array}$ *fitting in the following diagram:*

$$\begin{tikzcd}
X \arrow[rr, "\theta_0"] \arrow[dr, "i"] \arrow[dd, "f"'] & & X' \arrow[rr, "\eta_0"] \arrow[dr, "i'"] \arrow[dd, "f'"'] & & X'' \arrow[dr, "i''"] \arrow[dd, "f''"'] & \\
& Z \arrow[rr, "g"] \arrow[dl, "p"] & & Z' \arrow[rr, "h"] \arrow[dl, "p'"] & & Z'' \arrow[dl, "p''"] \\
Y \arrow[rr, "\theta_1"'] & & Y' \arrow[rr, "\eta_1"'] & & Y'' &
\end{tikzcd}$$

and satisfying the following equations:



4.2.2 Proposition. *If C is a closed 2-bmodel 2-category, then the factorization of axiom 2-M2 is locally functorial.*

Proof. In the situation of 4.2.1, by axiom 2-M2, we have a factorization $f \overset{\alpha\cong}{\Longrightarrow} pi$ where i is a cofibration and p is both a fibration that is also a weak equivalence. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "\theta_0"] \arrow[d, "i"'] \arrow[dr, phantom, "\overset{\cong\Downarrow\delta}{bi\text{-}p.o.}"] & X' \arrow[rrr, "\eta_0"] \arrow[d, "\lambda_0"] \arrow[drrr, phantom, "\overset{\cong\Downarrow\delta'}{bi\text{-}p.o.}"] & & & X'' \arrow[d, "\lambda'_0"] \\
Z \arrow[r, "\lambda_1"] \arrow[dd, "p"'] \arrow[ddr, phantom, "\cong\Downarrow\beta_1"] & P \arrow[rrr, "\lambda'_1"] \arrow[dr, dashed, "\tilde{i}"] \arrow[dd, dashed, "a"'] \arrow[ddrrr, phantom, "\cong\Downarrow\lambda"] & & & Q \arrow[dl, dashed, "\tilde{i}'"'] \arrow[dd, dashed, "b"] \\
& \cong\Rightarrow\tilde{\alpha} & Z' \arrow[r, dashed, "h"] \arrow[dl, dashed, "p'"] & Z'' \arrow[dr, dashed, "p''"'] & \cong\Rightarrow\tilde{\alpha}' \\
Y \arrow[r, "\theta_1"'] & Y' \arrow[rrr, "\eta_1"'] \arrow[rrr, phantom, bend left, "\cong\Downarrow\rho"] & & & Y''
\end{tikzcd}$$

Both bi-pushouts exist by axiom 2-M0b. $P \xrightarrow{a} Y'$ together with invertible 2-cells $\begin{array}{ccc} a & & \lambda_0 \\ \backslash & \beta_0 & / \\ & f' & \end{array}$, $\begin{array}{ccc} a & & \lambda_1 \\ \backslash & \beta_1 & / \\ \theta_1 & & p \end{array}$ such that $\begin{array}{cccccc} a & & \lambda_0 & & \theta_0 & \\ \backslash & \beta_0 & / & & \| & \\ & f' & & & \theta_0 & \\ & \backslash & \theta_m & / & & \\ & \theta_1 & & f & & \\ & \| & / & \alpha & \backslash & \\ & \theta_1 & p & & i & \end{array} = \begin{array}{ccccc} a & & \lambda_0 & & \theta_0 \\ \| & & \backslash & \delta & / \\ a & & \lambda_1 & & i \\ \backslash & \beta_1 & / & & \| \\ \theta_1 & & p & & i \end{array}$ exist by universal property of P. By axiom 2-M2, we have a factorization $a \overset{\tilde{\alpha}}{\Longrightarrow} p'\tilde{i}$ with $\tilde{i}$ a cofibration and p' a fibration and a weak equivalence. $Q \xrightarrow{b} Y''$ together with invertible 2-cells $\begin{array}{ccc} b & & \lambda'_0 \\ \backslash & \beta'_0 & / \\ & f'' & \end{array}$,

$\begin{array}{ccccc} & b & & \lambda'_1 & \\ / & & \beta'_1 & & \backslash \\ \eta_1 & & p' & & \tilde{i} \end{array}$ such that $\begin{array}{ccccccc} b & & \lambda'_0 & & & \eta_0 & \\ \backslash & \beta'_0 & / & & & \| & \\ & f'' & & & & \eta_0 & \\ & \backslash & & \eta_m & & / & \\ & \eta_1 & & & & f' & \\ & \| & & & / & \beta_0^{-1} & \backslash \\ & \eta_1 & & & a & & \lambda_0 \\ & \| & & / & \tilde{\alpha} & \backslash & \| \\ & \eta_1 & & p' & & \tilde{i} & \lambda_0 \end{array} = \begin{array}{ccccccc} & b & & \lambda'_0 & & & \eta_0 \\ & \| & & \backslash & & \delta' & / \\ & b & & \lambda'_1 & & & \lambda_0 \\ / & & \beta'_1 & & \backslash & & \| \\ \eta_1 & & p' & & \tilde{i} & & \lambda_0 \end{array}$ exist by universal property of Q. By axiom 2-M2, we have a factorization $b \overset{\tilde{\alpha}'}{\Longrightarrow} p''\tilde{i}'$ with $\tilde{i}'$ a cofibration and p'' a fibration and a weak equivalence. Finally, (h, λ, ρ) is a filler given by axiom 2-M6. Thus we have the following equality:

$$\begin{array}{ccccc} p'' & & \tilde{i}' & & \lambda'_1 \\ \| & & \backslash & \lambda & / \\ p'' & & h & & \tilde{i} \\ \backslash & \rho & / & & \| \\ \eta_1 & & p' & & \tilde{i} \end{array} = \begin{array}{ccccc} p'' & & \tilde{i}' & \lambda'_1 & \\ \backslash & \tilde{\alpha}'^{-1} & / & \| & \\ & b & & \lambda'_1 & \\ / & & \beta'_1 & & \backslash \\ \eta_1 & & p' & & \tilde{i} \end{array}$$

Take $i' = \tilde{i}\lambda_0$ and $i'' = \tilde{i}'\lambda'_0$ which are cofibrations because they are compositions of cofibrations (λ_0 and λ'_0 are cofibrations by axiom 2-M3). It is straightforward to check that

$$\begin{array}{ccc} & \mathsf{f} & \\ / & \alpha & \backslash \\ \mathsf{p} & & \mathsf{i} \end{array}\ , \quad \begin{array}{ccccc} & & & \mathsf{f}' & \\ & & / & \beta_0^{-1} & \backslash \\ & & \mathsf{a} & & \lambda_0 \\ & / & \tilde{\alpha} & \backslash & \| \\ \mathsf{p}' & & & \tilde{\mathsf{i}} & \lambda_0 \end{array}\ , \quad \begin{array}{ccccc} & & & \mathsf{f}'' & \\ & & / & \beta_0'^{-1} & \backslash \\ & & \mathsf{b} & & \lambda_0' \\ & / & \tilde{\alpha}' & \backslash & \| \\ \mathsf{p}'' & & & \tilde{\mathsf{i}}' & \lambda_0' \end{array}\ , \mathsf{g} = \tilde{\mathsf{i}}\lambda_1, \mathsf{h}, \beta = \begin{array}{ccccc} \tilde{\mathsf{i}} & \lambda_0 & & \theta_0 \\ \| & \backslash & \delta & / \\ \tilde{\mathsf{i}} & \lambda_1 & & \mathsf{i} \end{array}\ , \gamma = \begin{array}{ccccc} \tilde{\mathsf{i}}' & \lambda_0' & & \eta_0 \\ \| & \backslash & \delta' & / \\ \tilde{\mathsf{i}}' & \lambda_1' & & \lambda_0 \\ \backslash & \lambda & / & \| \\ \mathsf{h} & \tilde{\mathsf{i}} & & \lambda_0 \end{array}\ ,$$

$$\mu = \begin{array}{ccccc} \mathsf{p}' & & \tilde{\mathsf{i}} & \lambda_1 \\ \backslash & \tilde{\alpha}^{-1} & / & \| \\ & \mathsf{a} & & \lambda_1 \\ & \backslash & \beta_1 & / \\ & \theta_1 & & \mathsf{p} \end{array} \text{ and } \epsilon = \begin{array}{ccc} \mathsf{p}'' & & \mathsf{h} \\ \backslash & \rho' & / \\ \eta_1 & & \mathsf{p}' \end{array} \text{ satisfy the desired property.}$$

□

4.2.3 Remark. In the situation of 4.2.1, if $\mathsf{f} = \mathsf{f}''$ and θ, η are part of a retraction from f to f', then factorizations for f and f' given by 4.2.2 can be chosen in such a way that i is a retract of i' and p is a retract of p'. □

4.2.4 Definition. *Let C be a closed 2-bmodel 2-category and X an object of C.*

1. *We say that X is a fibrant object if the only morphism $\mathsf{X} \longrightarrow *$ is a fibration.*
2. *We say that X is a cofibrant object if the only morphism $0 \longrightarrow \mathsf{X}$ is a cofibration.*

4.2.5 Remark. Note that 0 and $*$ are denoting the bi-initial and the bi-terminal object respectively given by axiom M0b. More explicitly, 0 satisfies that for each $\mathsf{X} \in C$, there exists a morphism $0 \longrightarrow \mathsf{X} \in C$ up to unique invertible 2-cell. And $*$ satisfies that for each $\mathsf{X} \in C$, there exists a morphism $\mathsf{X} \longrightarrow * \in C$ up to unique invertible 2-cell.

In the previous definition the abuse of saying "the only morphism" is justified by axiom 2-M7. □

4.2.6 Definition. *Let C be a closed 2-bmodel 2-category and X an object of C.*

1. *A cylinder object for X consists of a diagram*

$$\begin{tikzcd} \mathsf{X} \amalg \mathsf{X} \arrow[d, "\mathsf{i}^{\mathsf{X}}"'] \arrow[dr, bend left, "\nabla^{\mathsf{X}}"] \arrow[dr, phantom, "\cong \Uparrow \gamma^{\mathsf{X}}"] & \\ \widetilde{\mathsf{X}} \arrow[r, "\sigma^{\mathsf{X}}"'] & \mathsf{X} \end{tikzcd} \tag{4.2.7}$$

where $\nabla^{\mathsf{X}} = \begin{pmatrix} id_{\mathsf{X}} \\ id_{\mathsf{X}} \end{pmatrix}$, $\mathsf{i}^{\mathsf{X}} = \begin{pmatrix} \mathsf{i}_0^{\mathsf{X}} \\ \mathsf{i}_1^{\mathsf{X}} \end{pmatrix}$ *is a cofibration and σ^{X} is a weak equivalence. By* $\mathsf{X} \amalg \mathsf{X}$ *we denote the bi-coproduct.*

2. *A path object* X *consists of a diagram*

$$\begin{tikzcd}
 & \widehat{\mathsf{X}} \arrow[d, "\mathsf{p}^{\mathsf{X}}"] \\
\mathsf{X} \arrow[ur, bend left, "\mathsf{s}^{\mathsf{X}}"] \arrow[r, "\Delta^{\mathsf{X}}"'] \arrow[ur, phantom, shift left=2ex, "\cong \Downarrow \gamma^{\mathsf{X}}"] & \mathsf{X}\times\mathsf{X}
\end{tikzcd}$$

where $\Delta^{\mathsf{X}} = (id_{\mathsf{X}}, id_{\mathsf{X}})$*,* $\mathsf{p}^{\mathsf{X}} = (\mathsf{p}_0^{\mathsf{X}}, \mathsf{p}_1^{\mathsf{X}})$ *is a fibration and* s^{X} *is a weak equivalence. By* $\mathsf{X}\times\mathsf{X}$ *we denote the bi-product.*

The following corollary says that in a closed 2-bmodel 2-category there are locally functorial cylinder objects (c.f. 4.1.4).

4.2.8 Corollary (of 4.2.2)**.** *Let* $\mathcal{C}$ *be a closed 2-bmodel 2-category. Given* $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{X}' \xrightarrow{\mathsf{f}'} \mathsf{X}'' \in \mathcal{C}$*. There are suitable cylinder objects for* X*,* X' *and* X''

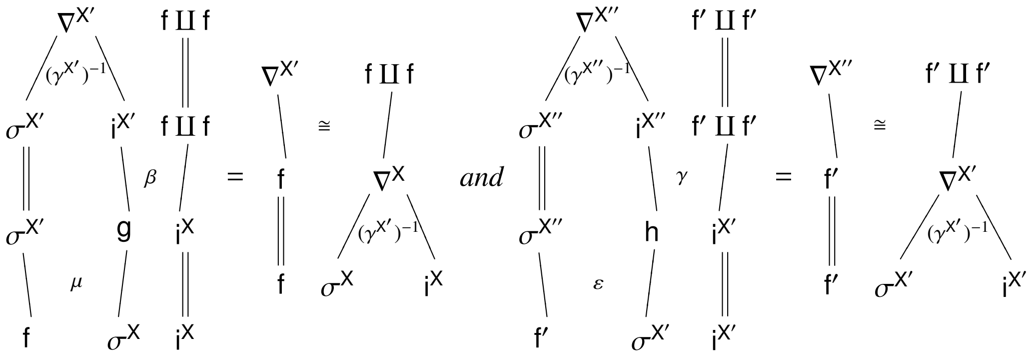
$$\begin{tikzcd}
\mathsf{X}\amalg\mathsf{X} \arrow[d, "\mathsf{i}^{\mathsf{X}}"'] \arrow[dr, bend left, "\nabla^{\mathsf{X}}"] \arrow[dr, phantom, "\cong \Uparrow \gamma^{\mathsf{X}}"] & \\
\widetilde{\mathsf{X}} \arrow[r, "\sigma^{\mathsf{X}}"'] & \mathsf{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{X}'\amalg\mathsf{X}' \arrow[d, "\mathsf{i}^{\mathsf{X}'}"'] \arrow[dr, bend left, "\nabla^{\mathsf{X}'}"] \arrow[dr, phantom, "\cong \Uparrow \gamma^{\mathsf{X}'}"] & \\
\widetilde{\mathsf{X}'} \arrow[r, "\sigma^{\mathsf{X}'}"'] & \mathsf{X}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{X}''\amalg\mathsf{X}'' \arrow[d, "\mathsf{i}^{\mathsf{X}''}"'] \arrow[dr, bend left, "\nabla^{\mathsf{X}''}"] \arrow[dr, phantom, "\cong \Uparrow \gamma^{\mathsf{X}''}"] & \\
\widetilde{\mathsf{X}''} \arrow[r, "\sigma^{\mathsf{X}''}"'] & \mathsf{X}''
\end{tikzcd}$$

morphisms $\widetilde{\mathsf{X}} \xrightarrow{\mathsf{g}} \widetilde{\mathsf{X}'} \xrightarrow{\mathsf{h}} \widetilde{\mathsf{X}''}$ *and invertible 2-cells* $\begin{array}{ccc} \mathsf{i}^{\mathsf{X}'} & & \mathsf{f}\amalg\mathsf{f} \\ \backslash & \beta & / \\ \mathsf{g} & & \mathsf{i}^{\mathsf{X}} \end{array}$, $\begin{array}{ccc} \mathsf{i}^{\mathsf{X}''} & & \mathsf{f}'\amalg\mathsf{f}' \\ \backslash & \gamma & / \\ \mathsf{h} & & \mathsf{i}^{\mathsf{X}'} \end{array}$, $\begin{array}{ccc} \sigma^{\mathsf{X}'} & & \mathsf{g} \\ \backslash & \mu & / \\ \mathsf{f} & & \sigma^{\mathsf{X}} \end{array}$,

$\begin{array}{ccc} \sigma^{\mathsf{X}''} & & \mathsf{h} \\ \backslash & \epsilon & / \\ \mathsf{f}' & & \sigma^{\mathsf{X}'} \end{array}$ *fitting in the following diagram:*

$$\begin{tikzcd}
\mathsf{X}\amalg\mathsf{X} \arrow[rr, "\mathsf{f}\amalg\mathsf{f}"] \arrow[dr, "i^{\mathsf{X}}"] \arrow[dd, "\nabla^{\mathsf{X}}"'] & & \mathsf{X}'\amalg\mathsf{X}' \arrow[rr, "\mathsf{f}'\amalg\mathsf{f}'"] \arrow[dr, "i^{\mathsf{X}'}"] \arrow[dd, "\nabla^{\mathsf{X}'}"' near end] & & \mathsf{X}''\amalg\mathsf{X}'' \arrow[dr, "i^{\mathsf{X}''}"] \arrow[dd, "\nabla^{\mathsf{X}''}"' near end] & \\
 & \widetilde{\mathsf{X}} \arrow[rr, crossing over, "\mathsf{g}" near end] \arrow[dl, "\sigma^{X}"] & & \widetilde{\mathsf{X}'} \arrow[rr, crossing over, "\mathsf{h}" near end] \arrow[dl, "\sigma^{X'}"] & & \widetilde{\mathsf{X}''} \arrow[dl, "\sigma^{X''}"] \\
\mathsf{X} \arrow[rr, "\mathsf{f}"'] & & \mathsf{X}' \arrow[rr, "\mathsf{f}'"'] & & \mathsf{X}'' &
\end{tikzcd}$$

satisfying the following equations:



□

4.2.9 Proposition. *Let C be a closed 2-bmodel 2-category and $X \xrightarrow{f} Y \in C$ a cofibration. There exist suitable cylinder objects for X and Y such that f induces a morphism that is both a cofibration and a weak equivalence*

$$Y \vartriangle \widetilde{X} \xrightarrow{k'_f} \widetilde{Y}$$

and a cofibration

$$Y \vartriangle \widetilde{X} \triangledown Y \xrightarrow{k_f} \widetilde{Y}$$

where $Y \vartriangle \widetilde{X}$ is the following bi-pushout

$$\begin{tikzcd}
X \arrow[r, "i_0^X"] \arrow[d, "f"'] \arrow[dr, phantom, "\substack{\cong \Downarrow \delta'_f \\ bi-p.o}"] & \widetilde{X} \arrow[d, "a'_f"] \\
Y \arrow[r, "b'_f"'] & Y \vartriangle \widetilde{X}
\end{tikzcd}$$

and $Y \vartriangle \widetilde{X} \triangledown Y$ is the following bi-pushout

$$\begin{tikzcd}
X \amalg X \arrow[r, "i^X"] \arrow[d, "f \amalg f"'] \arrow[dr, phantom, "\substack{\cong \Downarrow \delta'_f \\ bi-p.o}"] & \widetilde{X} \arrow[d, "a_f"] \\
Y \amalg Y \arrow[r, "b_f"'] & Y \vartriangle \widetilde{X} \triangledown Y
\end{tikzcd}$$

Plus, if f is both a cofibration and a weak equivalence, so is k_f.

For a geometric intuition, one can think of $\mathsf{Y} \vartriangle \widetilde{\mathsf{X}}$ *as the cylinder of* f *which is obtained by pasting a copy of* Y *on the bottom part of the cylinder of* X *via* f*. And* $\mathsf{Y} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Y}$ *can be seen as the double cylinder of* f*, which is obtained by pasting a copy of* Y *on the top and another one on the bottom of the cylinder of* X *via* f*.*

Proof. By axiom 2-M2, we can choose a cylinder object for X with σ_{X} a fibration

$$\begin{tikzcd}
\mathsf{X} \amalg \mathsf{X} \arrow[d, "\mathsf{i}^{\mathsf{X}}"'] \arrow[dr, bend left, "\nabla^{\mathsf{X}}"] \arrow[d, phantom, shift left=6ex, "\cong \Uparrow \gamma^{\mathsf{X}}"] & \\
\widetilde{\mathsf{X}} \arrow[r, "\sigma^{\mathsf{X}}"'] & \mathsf{X}
\end{tikzcd} \tag{4.2.10}$$

Now consider the following diagram

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathsf{X} \amalg \mathsf{X} \arrow[r, "\mathsf{i}^{\mathsf{X}}"] \arrow[rr, bend left=40, "\nabla^{\mathsf{X}}"] \arrow[d, "\mathsf{f} \amalg \mathsf{f}"'] \arrow[dr, phantom, "\substack{\cong \Downarrow \delta_{\mathsf{f}} \\ bi-p.o.}"] & \widetilde{\mathsf{X}} \arrow[r, "\sigma^{\mathsf{X}}"] \arrow[d, "\mathsf{a}_{\mathsf{f}}"] \arrow[u, phantom, "\cong \Uparrow \gamma^{\mathsf{X}}"] \arrow[dr, phantom, "\cong \Uparrow \beta_{\mathsf{f}}"] & \mathsf{X} \arrow[d, "\mathsf{f}"] \\
\mathsf{Y} \amalg \mathsf{Y} \arrow[r, "\mathsf{b}_{\mathsf{f}}"] \arrow[rr, bend right=60, "\nabla^{\mathsf{Y}}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma_{\mathsf{f}}"] & \mathsf{Y} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Y} \arrow[r, dashed, "\nabla_{\mathsf{f}}"] \arrow[dr, dashed, "\mathsf{k}_{\mathsf{f}}"'] & \widetilde{\mathsf{Y}} \\
 & & \mathsf{Y} \arrow[u, dashed, "\sigma^{\mathsf{Y}}"'] \arrow[ul, phantom, "\cong \Downarrow \theta_{\mathsf{f}}"]
\end{tikzcd} \tag{4.2.11}$$

The upper left bi-pushout exists by axiom 2-M0b. ∇_{f} and invertible 2-cells $\begin{array}{ccc} \nabla_{\mathsf{f}} & & \mathsf{a}_{\mathsf{f}} \\ \backslash & \beta_{\mathsf{f}} & / \\ \mathsf{f} & & \sigma^{\mathsf{X}} \end{array}$,

$\begin{array}{ccc} \nabla_{\mathsf{f}} & & \mathsf{b}_{\mathsf{f}} \\ \backslash & \gamma_{\mathsf{f}} & / \\ & \nabla^{\mathsf{Y}} & \end{array}$ such that $\begin{array}{ccccc} \nabla_{\mathsf{f}} & & \mathsf{a}_{\mathsf{f}} & & \mathsf{i}^{\mathsf{X}} \\ \backslash & \beta_{\mathsf{f}} & / & & \| \\ \mathsf{f} & & \sigma^{\mathsf{X}} & & \mathsf{i}^{\mathsf{X}} \\ \| & & \backslash & \gamma^{\mathsf{X}} & / \\ \mathsf{f} & & & \nabla^{\mathsf{X}} & \\ \backslash & & \cong & / & \\ \nabla^{\mathsf{Y}} & & & \mathsf{f} \amalg \mathsf{f} & \end{array} = \begin{array}{ccccc} \nabla_{\mathsf{f}} & & \mathsf{a}_{\mathsf{f}} & & \mathsf{i}^{\mathsf{X}} \\ \| & & \backslash & \delta_{\mathsf{f}} & / \\ \nabla_{\mathsf{f}} & & \mathsf{b}_{\mathsf{f}} & & \mathsf{f} \amalg \mathsf{f} \\ \backslash & \gamma_{\mathsf{f}} & / & & \| \\ & \nabla^{\mathsf{Y}} & & & \mathsf{f} \amalg \mathsf{f} \end{array}$ are given by the universal property of $\mathsf{Y} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Y}$.

By axiom 2-M2, we can factorize ∇_f with σ_Y a fibration that is also a weak equivalence and k_f a cofibration.

Consider the following cylinder object for Y:

$$\begin{tikzcd}
Y \amalg Y \arrow[d, "k_f b_f"'] \arrow[dr, bend left, "\nabla^Y"] \arrow[d, phantom, shift left=4ex, "\cong \Uparrow \gamma^Y"] & \\
\widetilde{Y} \arrow[r, "\sigma^Y"'] & Y
\end{tikzcd} \tag{4.2.12}$$

where $\gamma^Y = \gamma_f b_f \circ \theta_f^{-1}$ Then, k_f is the wanted cofibration.

To construct $Y \vartriangle \widetilde{X}$ and k'_f, consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "i_0^X"] \arrow[d, "f"'] \arrow[dr, phantom, "\substack{\cong \Downarrow \delta'_f \\ bi-p.o}"] & \widetilde{X} \arrow[d, "a'_f"] \arrow[ddr, bend left, "a_f"] & \\
Y \arrow[r, "b'_f"'] \arrow[drr, bend right, "b_f \lambda_0"'] & Y \vartriangle \widetilde{X} \arrow[dr, dashed, "\tilde{k}_f"] \arrow[r, phantom, "\cong \Uparrow \alpha_f"] \arrow[d, phantom, "\cong \Downarrow \mu_f"] & \\
& & Y \vartriangle \widetilde{X} \triangledown Y
\end{tikzcd}$$

The upper left bi-pushout exists by axiom 2-M0b and then, by universal property, we have $\tilde{k}_f$ and invertible 2-cells $\begin{array}{ccc} \tilde{k}_f & & a'_f \\ \backslash & \alpha_f & / \\ & a_f & \end{array}$, $\begin{array}{ccc} \tilde{k}_f & & b'_f \\ \backslash & \mu_f & / \\ b_f & & \lambda_0 \end{array}$ such that

$$\begin{array}{ccccc} \tilde{k}_f & & a'_f & & i_0^X \\ \backslash & \alpha_f & / & & \| \\ & a_f & & & i_0^X \\ / & & (\delta_f)_0 & & \backslash \\ b_f & & \lambda_0 & & f \end{array} \quad = \quad \begin{array}{ccccc} \tilde{k}_f & & a'_f & & i_0^X \\ \| & & \backslash & \delta'_f & / \\ \tilde{k}_f & & b'_f & & f \\ \backslash & \mu_f & / & & \| \\ b_f & & \lambda_0 & & f \end{array} \; .$$

Take $k'_f = k_f \tilde{k}_f$.

Recall that the upper left square of (4.2.11) is a bi-pushout. Then, it can be easily checked that the following diagram is also a bi-pushout:

$$\begin{tikzcd}
X \arrow[r, "i_1^X"] \arrow[d, "f"'] & \widetilde{X} \arrow[r, "a'_f"] \arrow[dr, "a_f"'] \arrow[d, phantom, "\cong \Downarrow (\delta_f)_1"] \arrow[r, phantom, shift right=3ex, "\cong \Downarrow \alpha_f"] & Y \triangle \widetilde{X} \arrow[d, "\tilde{k}_f"] \\
Y \arrow[r, "\lambda_1"'] & Y \amalg Y \arrow[r, "b_f"'] & Y \triangle \widetilde{X} \triangledown Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "a'_f i_1^X"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong \Downarrow (\delta_f)_1 \circ \alpha_f i_1^X"] & Y \triangle \widetilde{X} \arrow[d, "\tilde{k}_f"] \\
Y \arrow[r, "b_f \lambda_1"'] & Y \triangle \widetilde{X} \triangledown Y
\end{tikzcd}$$

Thus, since f is a cofibration it follows from axiom 2-M3b that $\tilde{k}_f$ is also a cofibration. We can conclude then that k'_f is a cofibration.

It only remains to check that k'_f is a weak equivalence (then, in case f is a weak equivalence, $\tilde{k}_f$ is also a weak equivalence by axiom 2-M3b and so, by axiom 2-M5, k_f is a weak equivalence):

Also by axiom 2-M5, it is enough to check that b'_f is a weak equivalence because σ_Y is a weak equivalence and $\sigma_Y k'_f b'_f \cong id_Y$. To check that, consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "i_0^X"] \arrow[rr, bend left, "id_X"] \arrow[rr, phantom, bend left=20, "\cong \Uparrow \gamma_0^X"] \arrow[d, "f"'] & \widetilde{X} \arrow[r, "\sigma^X"] \arrow[d, "a'_f"] \arrow[l, phantom, shift right=0ex, ""] & X \arrow[d, "f"] \\
Y \arrow[r, "b'_f"'] \arrow[rr, bend right, "id_Y"'] \arrow[rr, phantom, bend right=20, "\cong \Downarrow \nu_f"] \arrow[ur, phantom, "\cong \Downarrow \delta'_f"] & Y \triangle \widetilde{X} \arrow[r, dashed, "h"'] \arrow[ur, phantom, "\cong \Uparrow \epsilon_f"] & Y
\end{tikzcd} \tag{4.2.13}$$

By universal property of $Y \triangle \widetilde{X}$, there exist h and invertible 2-cells $\begin{array}{ccc} h & & a'_f \\ \backslash & \epsilon_f & / \\ f & & \sigma^X \end{array}$, $\begin{array}{ccc} h & & b'_f \\ & \backslash \nu_f / & \\ & id_Y & \end{array}$

such that $\begin{array}{ccccc} h & & a'_f & & i_0^X \\ \backslash & \epsilon_f & / & & \| \\ f & & \sigma^X & & i_0^X \\ \| & & & \backslash \gamma_0^X / & \\ f & & & id_X & \\ \backslash & & = & / & \\ id_Y & & & f & \end{array} \quad = \quad \begin{array}{cccc} h & a'_f & & i_0^X \\ \| & & \backslash \delta'_f / & \\ h & b'_f & & f \\ \backslash \nu_f / & & & \| \\ id_Y & & & f \end{array}$.

Again, by axiom 2-M5, it is enough to check that h is a weak equivalence and this is the case because σ^X is a weak equivalence and the right square of diagram (4.2.13) is a bi-pushout since both the left square and the outside square are bi-pushouts.

□

4.2.14 Remark. If $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y}$ is a retract of $\mathsf{X}' \xrightarrow{\mathsf{f}'} \mathsf{Y}'$ via $(\theta_0, \theta_1, \theta_m, \eta_0, \eta_1, \eta_m, \mu_0, \mu_1)$ and the constructions of the previous proposition are performed for both morphisms, then $\mathsf{k_f}$ is a retract of $\mathsf{k_{f'}}$ and $\mathsf{k'_f}$ is a retract of $\mathsf{k'_{f'}}$.

Proof. We give a sketch of the proof, leaving the details to the reader. First observe that ∇^{X} is a retract of $\nabla^{\mathsf{X}'}$. Then, from 4.2.3 plus 4.2.8, one can choose "retract cylinder objects" for X and X' as in the following diagram

$$\begin{tikzcd}
\mathsf{X} \amalg \mathsf{X} \arrow[rr, "\mathsf{f} \amalg \mathsf{f}"] \arrow[dr, "\mathsf{i}^{\mathsf{X}}"] \arrow[dd, "\nabla^{\mathsf{X}}"'] & & \mathsf{X}' \amalg \mathsf{X}' \arrow[rr, "\mathsf{f}' \amalg \mathsf{f}'"] \arrow[dr, "\mathsf{i}^{\mathsf{X}'}"] \arrow[dd, "\nabla^{\mathsf{X}'}"'] & & \mathsf{X} \amalg \mathsf{X} \arrow[dr, "\mathsf{i}^{\mathsf{X}}"] \arrow[dd, "\nabla^{\mathsf{X}}"'] & \\
& \widetilde{\mathsf{X}} \arrow[rr, "\mathsf{g}"] \arrow[dl, "\sigma^X"] & & \widetilde{\mathsf{X}'} \arrow[rr, "\mathsf{h}"] \arrow[dl, "\sigma^{X'}"] & & \widetilde{\mathsf{X}} \arrow[dl, "\sigma^X"] \\
\mathsf{X} \arrow[rr, "\mathsf{f}"'] & & \mathsf{X}' \arrow[rr, "\mathsf{f}'"'] & & \mathsf{X}'' &
\end{tikzcd}$$

By functoriality of the bi-pushout, one can construct a retraction $\mathsf{Y} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Y} \xrightarrow{\tilde{\mathsf{g}}} \mathsf{Y}' \vartriangle \widetilde{\mathsf{X}'} \triangledown \mathsf{Y}' \xrightarrow{\tilde{\mathsf{h}}} \mathsf{Y} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Y}$ and it can be checked that this is part of a retraction from ∇_{f} to $\nabla_{\mathsf{f}'}$. Then, by 4.2.3, one can factorize ∇_{f} and $\nabla_{\mathsf{f}'}$ in such way that $\mathsf{k_f}$ is a retract of $\mathsf{k_{f'}}$

$$\begin{tikzcd}
\mathsf{Y} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Y} \arrow[r, "\tilde{\mathsf{g}}"] \arrow[d, "\mathsf{k_f}"'] & \mathsf{Y}' \vartriangle \widetilde{\mathsf{X}'} \triangledown \mathsf{Y}' \arrow[r, "\tilde{\mathsf{h}}"] \arrow[d, "\mathsf{k_{f'}}"'] & \mathsf{Y} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Y} \arrow[d, "\mathsf{k_f}"'] \\
\widetilde{\mathsf{Y}} \arrow[r, "\tilde{\tilde{\mathsf{g}}}"] \arrow[d, "\sigma^{\mathsf{Y}}"'] & \widetilde{\mathsf{Y}'} \arrow[r, "\tilde{\tilde{\mathsf{h}}}"] \arrow[d, "\sigma^{\mathsf{Y}'}"'] & \widetilde{\mathsf{Y}} \arrow[d, "\sigma^{\mathsf{Y}}"'] \\
\mathsf{Y} \arrow[r, "\theta_1"'] & \mathsf{Y}' \arrow[r, "\eta_1"'] & \mathsf{Y}
\end{tikzcd}$$

Similar arguments can be used to prove that $\mathsf{k'_f}$ is a retract of $\mathsf{k'_{f'}}$. □

4.3 Some transfer properties

4.3.1 Lemma. *Let* $\mathsf{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{G}$ *be pseudo-functors such that* $\mathsf{F} \dashv_b \mathsf{G}$ *via* ϵ, η *(see 1.1.25) and let* $\mathsf{A} \xrightarrow{\mathsf{i}} \mathsf{X} \in \mathcal{C}$ *and* $\mathsf{Y} \xrightarrow{\mathsf{p}} \mathsf{B} \in \mathcal{D}$*. Then the pair* $(\mathsf{Fi}, \mathsf{p})$ *has the lifting property iff the pair* $(\mathsf{i}, \mathsf{Gp})$ *does.*

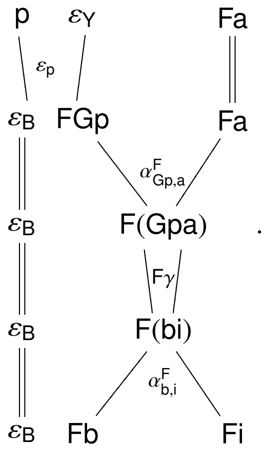
Proof. $\Rightarrow$) Suppose that we have a diagram of the form

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{GY} \arrow[d, "\mathsf{Gp}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{GB}
\end{tikzcd}$$

By hypothesis, we have a filler $(\mathsf{f}, \lambda, \rho)$ for the following diagram:

$$\begin{tikzcd}
\mathsf{FA} \arrow[r, "\mathsf{Fa}"] \arrow[d, "\mathsf{Fi}"'] & \mathsf{FGY} \arrow[r, "\epsilon_{\mathsf{Y}}"] \arrow[d, phantom, "\cong \Downarrow \gamma'"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\
\mathsf{FX} \arrow[r, "\mathsf{Fb}"'] & \mathsf{FGB} \arrow[r, "\epsilon_{\mathsf{B}}"'] & \mathsf{B}
\end{tikzcd}$$

where $\gamma' =$



.

It is straightforward to check that

$$\left(\mathsf{Gf}\eta_{\mathsf{X}},\ \begin{tikzcd}[column sep=small, row sep=small]
 & & \mathsf{a} \arrow[dl, -] \arrow[dr, -] \arrow[d, phantom, "="] & \\
 & id_{\mathsf{GY}} \arrow[dl, -] \arrow[dr, -] \arrow[d, phantom, "="] & {} & \mathsf{a} \arrow[d, equal] \\
\mathsf{G}\varepsilon_{\mathsf{Y}} \arrow[d, equal] & {} & \eta_{\mathsf{GY}} \arrow[d, -] \arrow[r, phantom, "\eta_{\mathsf{a}}"] & \mathsf{a} \arrow[d, -] \\
\mathsf{G}\varepsilon_{\mathsf{Y}} \arrow[dr, -] & & \mathsf{GFa} \arrow[dl, -, "\alpha^{\mathsf{G}}_{\varepsilon_{\mathsf{Y}},\mathsf{Fa}}"'] & \eta_{\mathsf{A}} \arrow[d, equal] \\
 & \mathsf{G}(\varepsilon_{\mathsf{Y}}\mathsf{Fa}) \arrow[d, -, "\mathsf{G}\lambda"] & & \eta_{\mathsf{A}} \arrow[d, equal] \\
 & \mathsf{G}(\mathsf{fFi}) \arrow[dl, -] \arrow[dr, -, "\alpha^{\mathsf{G}}_{\mathsf{f},\mathsf{Fi}}"'] & & \eta_{\mathsf{A}} \arrow[d, equal] \\
\mathsf{Gf} \arrow[d, equal] & & \mathsf{GFi} \arrow[d, -] \arrow[r, phantom, "\eta_{\mathsf{i}}"] & \eta_{\mathsf{A}} \arrow[d, -] \\
\mathsf{Gf} & & \eta_{\mathsf{X}} & \mathsf{Fi}
\end{tikzcd},\ \begin{tikzcd}[column sep=small, row sep=small]
\mathsf{Gp} \arrow[dr, -, "\alpha^{\mathsf{G}}_{\mathsf{p},\mathsf{f}}"] & & \mathsf{Gf} \arrow[dl, -] & \eta_{\mathsf{X}} \arrow[d, equal] \\
 & \mathsf{G}(\mathsf{pf}) \arrow[d, -, "\mathsf{G}\rho"] & & \eta_{\mathsf{X}} \arrow[d, equal] \\
 & \mathsf{G}(\varepsilon_{\mathsf{B}}\mathsf{Fb}) \arrow[dl, -] \arrow[dr, -, "\alpha^{\mathsf{G}}_{\varepsilon_{\mathsf{B}},\mathsf{Fb}}"] & & \eta_{\mathsf{X}} \arrow[d, equal] \\
\mathsf{G}\varepsilon_{\mathsf{B}} \arrow[d, equal] & & \mathsf{GFb} \arrow[d, -] \arrow[r, phantom, "\eta_{\mathsf{b}}"] & \eta_{\mathsf{X}} \arrow[d, -] \\
\mathsf{G}\varepsilon_{\mathsf{B}} \arrow[dr, -] & {} & \eta_{\mathsf{GB}} \arrow[dl, -] & \mathsf{b} \arrow[d, equal] \\
 & id_{\mathsf{GB}} \arrow[dr, -] \arrow[u, phantom, "="] & {} & \mathsf{b} \arrow[dl, -] \\
 & & \mathsf{b} \arrow[u, phantom, "="] &
\end{tikzcd}\right) \text{ is}$$

the filler that we were looking for.

⇐) The proof is similar to the previous one. □

4.3.2 Lemma. *Let* $\mathsf{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathsf{G}$ *be pseudo-functors between closed 2-bmodel 2-categories such that* $\mathsf{F} \dashv_b \mathsf{G}$. *The following properties hold:*

1. F *preserves cofibrations iff* G *preserves morphisms that are both fibrations and weak equivalences.*

2. F *preserves morphisms that are both cofibrations and weak equivalences iff* G *preserves fibrations.*

Proof. 1. ⇒) Let p be a morphism in $\mathcal{D}$ that is both a fibration and a weak equivalence. By 4.1.8, we only need to check that Gp has the right lifting property with respect to all cofibrations. So let i be a cofibration in $\mathcal{C}$, then Fi is a cofibration in $\mathcal{D}$ and so it has the left lifting property with respect to all morphisms that are both fibrations and weak equivalence. Then the pair $(\mathsf{Fi}, \mathsf{p})$ has the lifting property and, by 4.3.1, so does the pair $(\mathsf{i}, \mathsf{Gp})$.

⇐) The proof is similar.

2. The proof is similar.

□

4.3.3 Proposition. *Let* $\mathcal{C} \xrightarrow{\mathsf{F}} \mathcal{D}$ *be a 2-functor that is a retract pseudo-equivalence of 2-categories (1.1.28). If* $\mathcal{D}$ *is a closed 2-bmodel 2-category and we define the corresponding structure in* $\mathcal{C}$*, then* $\mathcal{C}$ *is also a closed 2-bmodel 2-category.*

Proof. Let G be a pseudo-quasi-inverse to F and $FG \overset{\alpha}{\Longrightarrow} id_{\mathcal{D}}$ an equivalence as in 1.1.28 such that $GF = id_{\mathcal{C}}$.

2-M0b: The proof is straightforward.

2-M2: Let $f \in \mathcal{C}$. Since $\mathcal{D}$ satisfies 2-M2, Ff can be factorized as $Ff \cong \tilde{p}\tilde{i}$ where $\tilde{p}$ is a fibration and $\tilde{i}$ is both a cofibration and a weak equivalence. Let $p = G\tilde{p}$ and $i = G\tilde{i}$. Then, it can be easily checked that $Fp = FG\tilde{p}$ is a retract of $\tilde{p}$. Then, by 4.1.18, Fp is a fibration in $\mathcal{D}$ and so p is a fibration in $\mathcal{C}$. With a similar argument, one can check that i is both a cofibration and a weak equivalence in $\mathcal{C}$. Plus $f = GFf \cong G\tilde{p}\tilde{i} \cong G\tilde{p}G\tilde{i} = pi$ as we wanted to prove. The case in which p is a weak equivalence is similar and we leave it to the reader.

2-M5w: Let f and g be two composable arrows in $\mathcal{C}$ such that two out of the three f, g and gf are weak equivalences. Then two out of the three Ff, Fg and $Fgf = FgFf$ are weak equivalences and so is the third because 2-M5w is satisfied in $\mathcal{D}$. But this implies that f, g and gf are all weak equivalences as we wanted to prove.

2-M6a): ⇒) Let p be a fibration and i a cofibration which is also a weak equivalence in $\mathcal{C}$ and suppose that we have a diagram of the form:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & Y \arrow[d, "p"] \\
X \arrow[r, "b"'] & B
\end{tikzcd}$$

Since Fp is a fibration and Fi is both a cofibration and a weak equivalence in $\mathcal{D}$, there exists a filler (f_0, λ_0, ρ_0) for the following diagram

$$\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "Fi"'] \arrow[dr, phantom, "\cong \Downarrow F\gamma"] & FY \arrow[d, "Fp"] \\
FX \arrow[r, "Fb"'] & FB
\end{tikzcd}$$

It is straightforward to check that $(Gf_0,$

$$\begin{tikzcd}
& a \arrow[d, dash, "="] & \\
& GFa \arrow[d, dash, "G\lambda_0"] & \\
& G(f_0Fi) \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "(\alpha^G_{f_0,Fi})^{-1}"] & \\
Gf_0 \arrow[d, equal] & {} & GFi \arrow[d, dash, "="] \\
Gf_0 & & i
\end{tikzcd}$$

$,$

$$\begin{tikzcd}
p \arrow[d, dash, "="] & & Gf_0 \arrow[d, equal] \\
GFp \arrow[dr, dash] & \alpha^G_{Fp,f_0} & Gf_0 \arrow[dl, dash] \\
& G(Fpf_0) \arrow[d, dash, "G\rho_0"] & \\
& GFb \arrow[d, dash, "="] & \\
& b &
\end{tikzcd}$$

$)$ is the filler

that we were looking for.

$\Leftarrow$) Let $\mathsf{Y} \overset{\mathsf{p}}{\longrightarrow} \mathsf{B} \in \mathcal{C}$ such that it has the right lifting property with respect to all morphisms that are both a cofibration and a weak equivalence. To check that it is a fibration, we need to check that Fp is a fibration in $\mathcal{D}$ where we have axiom 2-M6a). So suppose that we have a morphism $\mathsf{i} \in \mathcal{D}$ which is both a cofibration and a weak equivalence and a diagram of the form:

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{FY} \arrow[d, "\mathsf{Fp}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{FB}
\end{tikzcd}$$

Then we have a diagram in $\mathcal{C}$ as follows:

$$\begin{tikzcd}
\mathsf{GA} \arrow[r, "\mathsf{Ga}"] \arrow[d, "\mathsf{Gi}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma'"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\
\mathsf{GX} \arrow[r, "\mathsf{Gb}"'] & \mathsf{B}
\end{tikzcd} \tag{4.3.4}$$

where $\gamma' =$

$$\begin{tikzcd}
\mathsf{p} \arrow[d, dash, phantom, "="] & & \mathsf{Ga} \arrow[d, equal] \\
\mathsf{GFp} \arrow[dr, dash] & \alpha^{\mathsf{G}}_{\mathsf{Fp,a}} & \mathsf{Ga} \arrow[dl, dash] \\
& \mathsf{G(Fpa)} \arrow[d, dash, "\mathsf{G}\gamma"] & \\
& \mathsf{G(bi)} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathsf{Gb} & (\alpha^{\mathsf{G}}_{\mathsf{b,i}})^{-1} & \mathsf{Gi}
\end{tikzcd}$$

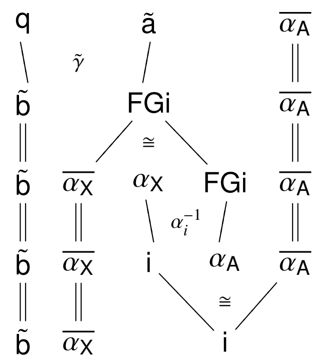
.

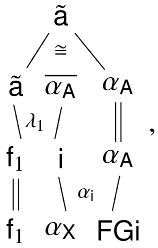
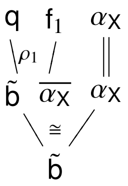
We are going to prove that Gi is both a cofibration and a weak equivalence: In order to do that, by 4.1.8, we only need to check that FGi has the left lifting property with respect to all fibrations. So suppose that we have a fibration q and a diagram of the form

$$\begin{tikzcd}
\mathsf{FGA} \arrow[r, "\tilde{\mathsf{a}}"] \arrow[d, "\mathsf{FGi}"'] \arrow[dr, phantom, "\cong \Downarrow \tilde{\gamma}"] & \tilde{\mathsf{Y}} \arrow[d, "\mathsf{q}"] \\
\mathsf{FGX} \arrow[r, "\tilde{\mathsf{b}}"'] & \tilde{\mathsf{B}}
\end{tikzcd}$$

Since i is both a cofibration and a weak equivalence and q is a fibration, the following diagram admits a filler $(\mathsf{f}_1, \lambda_1, \rho_1)$

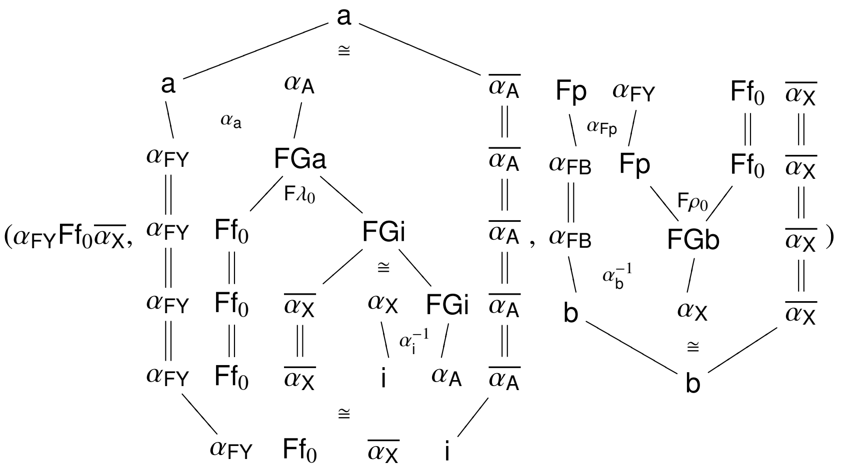
$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\overline{\alpha_{\mathsf{A}}}"] \arrow[d, "\mathsf{i}"'] & \mathsf{FGA} \arrow[r, "\tilde{\mathsf{a}}"] \arrow[d, phantom, "\cong \Downarrow \tilde{\gamma'}"] & \tilde{\mathsf{Y}} \arrow[d, "\mathsf{q}"] \\
\mathsf{X} \arrow[r, "\overline{\alpha_{\mathsf{X}}}"'] & \mathsf{FGX} \arrow[r, "\tilde{\mathsf{b}}"'] & \tilde{\mathsf{B}}
\end{tikzcd}$$

where $\tilde{\gamma'} =$ 

It is straightforward to check that $(\mathsf{f}_1\alpha_{\mathsf{X}},$  , ) is the filler that we were looking for.

Then, by hypothesis, diagram (4.3.4) admits a filler $(\mathsf{f}_0, \lambda_0, \rho_0)$.

It is straightforward to check that



is the filler that we were looking for.

2-M6b): The proof of this axiom is similar to the previous one and we leave it to the reader.

2-M6c): ⇒) Let $\mathsf{f} \in \mathcal{C}$ be a weak equivalence. Then $\mathsf{Ff} \in \mathcal{D}$ is a weak equivalence and therefore we can factorize it as $\mathsf{Ff} \cong \mathsf{uv}$ where u has the right lifting property with respect to all cofibrations and v has the left lifting property with respect to all fibrations. Consider $\tilde{\mathsf{u}} = \mathsf{Gu}$ and $\tilde{\mathsf{v}} = \mathsf{Gv}$. Then $\mathsf{F}\tilde{\mathsf{u}} = \mathsf{FGu}$ is a retract of u. Then, by 4.1.18, $\tilde{\mathsf{u}}$ has the right lifting property with respect to all cofibrations.

By a similar argument, we can prove that $\tilde{\mathsf{v}}$ has the left lifting property with respect to all fibrations.

So we factorized f as we wanted because $\mathsf{f} = \mathsf{GFf} \cong \mathsf{Guv} \cong \mathsf{GuGv} = \tilde{\mathsf{u}}\tilde{\mathsf{v}}$.

⇐) Let $\mathsf{f} \cong \mathsf{uv} \in \mathcal{C}$ with u having the right lifting property with respect to all cofibrations and v having the left lifting property with respect to all fibrations. We want to check that Ff is a weak equivalence in $\mathcal{D}$: since $\mathsf{Ff} \cong \mathsf{FuFv}$, we only need to check that Fu has the right lifting property with respect to all cofibrations and Fv has the left lifting property with respect to all fibrations. This can be checked by working as before. □

Resumen en castellano de la sección 4

En esta sección introducimos las nociones inéditas de “closed 2-model 2-category” y “closed 2-bmodel 2-category” y enunciamos y demostramos algunos lemas y proposiciones que usaremos más adelante. Nuestra noción es más fuerte que las “fibration structures” de Pronk ([26]) pues es una versión 2-dimensional de los axiomas de Quillen completos para “closed model categories”. También difiere en el hecho importante de que no asumimos la elección de una factorización global privilegiada dada de forma pseudofuntorial sino que estipulamos, como Quillen, solo la existencia de factorizaciones para cada flecha.

La mayoría de los resultados de esta sección son generalizaciones al contexto de 2-categorías de enunciados bien conocidos de la teoría de “closed model categories”. Para ver las definiciones y resultados en el caso 1-dimensional, se puede consultar por ejemplo [27] o [13].

5 Closed 2-bmodel structure for 2-$\mathcal{P}ro(C)$

In this section, we give 2-$\mathcal{P}ro(C)$ a closed 2-bmodel structure provided that C has one. This section is inspired in the proof given in [12] of the fact that $\mathsf{Pro(C)}$ is a closed model category in the 1-dimensional case. The proof in our context turned out to be more complicated due to the fact that diagrams doesn't strictly commute but only commute up to an invertible 2-cell. This is the reason why we were forced to work with pseudo-functors and pseudo-natural transformations even though objects and morphisms in 2-$\mathcal{P}ro(C)$ are 2-functors and 2-natural transformations. We proceed in three steps. First, in 5.1 we define a closed 2-bmodel structure for the 2-category $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ (see 1.1.19) out of a closed 2-bmodel structure for C, where J is a cofinite and filtered poset with a unique initial object. Second, in 5.2 we use the closed 2-bmodel structure in $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ to define such an structure in the 2-category 2-$\mathcal{P}ro_p(C)$. Finally, we transfer this structure into 2-$\mathcal{P}ro(C)$ using that this 2-category is retract pseudo-equivalent to 2-$\mathcal{P}ro_p(C)$ (see 2.1.5).

5.1 Closed 2-bmodel structure for $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$

The aim of this subsection is to prove that given a closed 2-bmodel 2-category C and a cofinite and filtered poset J with a unique initial object, the 2-category $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ (see 1.1.19) is a closed 2-bmodel 2-category. The proof is inspired in the 1-dimensional case treated in [12]. For the 2-categorical setting, things become more complicated. So is that we were forced to work with pseudo-functors and pseudo-natural transformations instead of 2-functors and 2-natural transformations because of the non-strict commutativity of diagrams. One would think (and we did for a while) that 2-functors and pseudo-natural transformations should be enough but they are not. The reason for taking pseudo-functors evidences itself in the proof of axiom 2-M2 where Z turns out to be a pseudo-functor that is not necessarily a 2-functor even if all the others are.

All along this subsection, J will be a cofinite and filtered poset with a unique initial object 0 and C will be a closed 2-bmodel 2-category. We comment that in [12] J is not supposed to have a unique initial object, which for us is an essential requirement, also in the 1-dimensional case.

5.1.1. Notation. Since there are at most one morphism between any pair of objects of J, we will write $\alpha_{k,l,j}$ instead of $\alpha_{k \leq l, l \leq j}$ (see 1.1.10). Also, we will use the subindex notation for the evaluation of 2-functors.

5.1.2 Definition. *We define fibrations, cofibrations and weak equivalences in* $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ *as follows:*

- *A morphism* $\mathsf{f} \in p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ *is a cofibration if the morphism* f_j *is a cofibration in* C $\forall\ j \in \mathsf{J}$. *We say "pointwise cofibration".*

- *A morphism* $\mathsf{f} \in p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ *is a weak equivalence if the morphism* f_j *is a weak equivalence in* C $\forall$ $j \in \mathsf{J}$. *We say "pointwise weak equivalence".*

- *A morphism* $\mathsf{f} \in p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ *is a fibration if it has the right lifting property with respect to all the morphisms that are both cofibrations and weak equivalences.*

5.1.3 Lemma.

1. *The 2-functor constant diagram from* C *to* $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ *preserves cofibrations, fibrations and weak equivalences.*

2. *The pseudo-functor inverse bi-limit from* $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ *to* $\mathcal{C}$ *preserves fibrations and morphisms that are both fibrations and weak equivalences.*

Proof. 1. It is clear that the 2-functor constant diagram preserves cofibrations and weak equivalences. We will check now that it also preserves fibrations:

Let $\mathsf{C} \xrightarrow{\mathsf{p}} \mathsf{D} \in \mathcal{C}$ be a fibration. We need to check that if we embed p in $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$, then it has the right lifting property with respect to all morphisms that are both cofibrations and weak equivalences. So let $\mathsf{A} \xrightarrow{\mathsf{i}} \mathsf{X} \in p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ be a cofibration which is also a weak equivalence and suppose that we have the following situation:

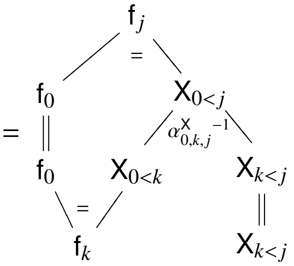
$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{C} \arrow[d, "\mathsf{p}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{D}
\end{tikzcd}$$

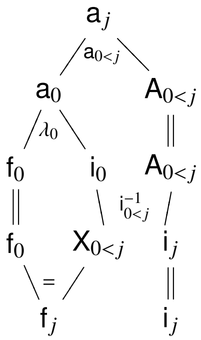
We are going to define the filler $(\mathsf{f}, \lambda, \rho)$:

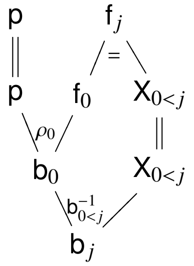
$\mathcal{C}$ is a closed 2-bmodel 2-category, so, since p is a fibration and i_0 is both a cofibration and a weak equivalence, there exists a filler $(\mathsf{f}_0, \lambda_0, \rho_0)$ for the following diagram

$$\begin{tikzcd}
\mathsf{A}_0 \arrow[r, "\mathsf{a}_0"] \arrow[d, "\mathsf{i}_0"'] \arrow[dr, phantom, "\cong \Downarrow \gamma_0"] & \mathsf{C} \arrow[d, "\mathsf{p}"] \\
\mathsf{X}_0 \arrow[r, "\mathsf{b}_0"'] & \mathsf{D}
\end{tikzcd} \tag{5.1.4}$$

For $j \neq 0 \in \mathsf{J}$, take $\mathsf{f}_j = \mathsf{f}_0\mathsf{X}_{0<j}$, $\mathsf{f}_{0<j} = id_{\mathsf{f}_j}$, $\mathsf{f}_{id_j} = \mathsf{f}_j\alpha^{\mathsf{X}}_j$,

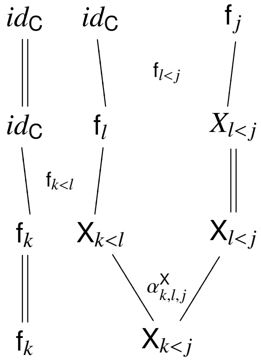
$\mathsf{f}_{k<j} =$ 

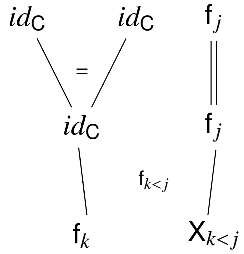
$, \lambda_j =$ 

and $\rho_j =$ 

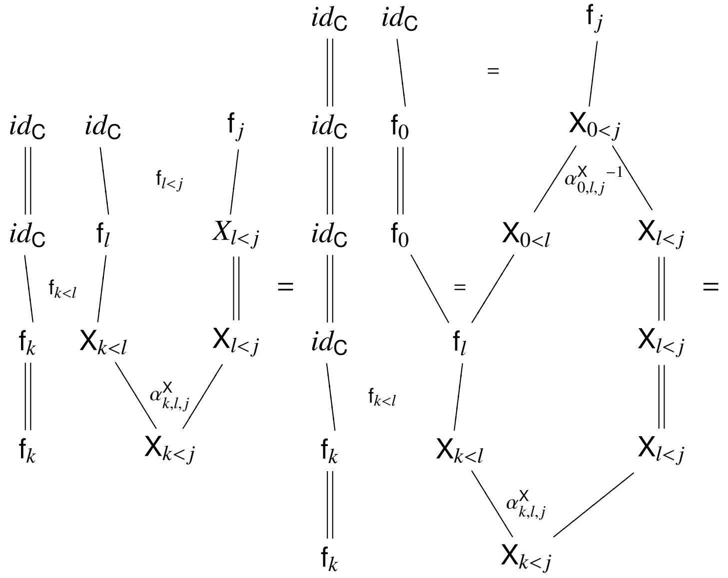
.

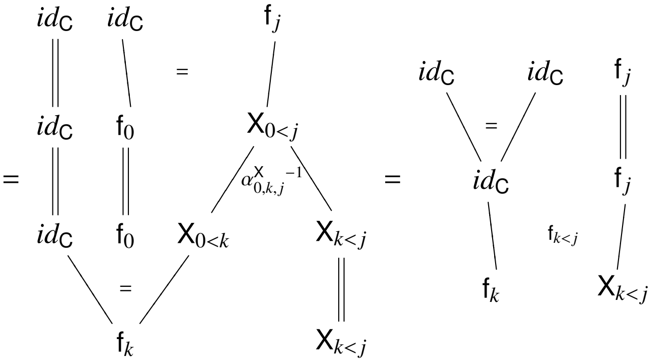
Let's check that f defined this way is a pseudo-natural transformation: PNO is straightforward and PN2 is vacuous since J doesn't have any non trivial 2-cells. For axiom PN1, we need to verify that $\forall\ k < l < j \in \mathsf{J}$ the following equality holds:



=


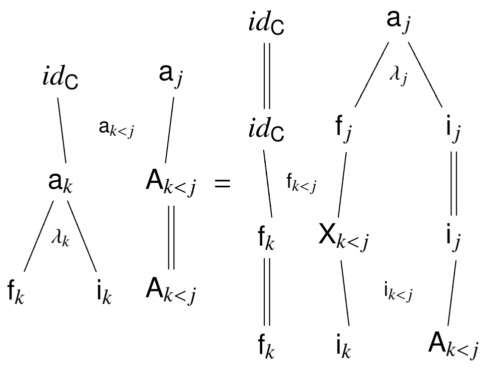
But, if $0 < k < l < j$,





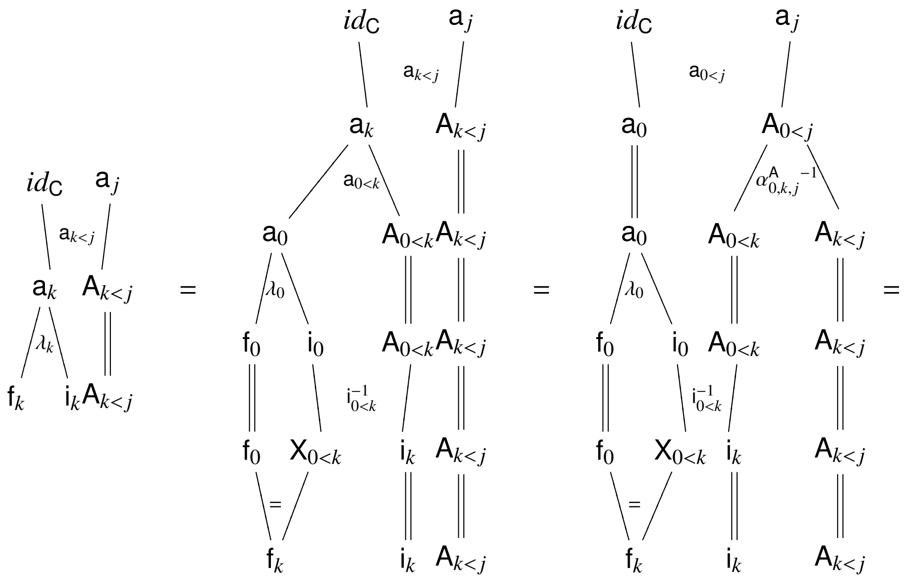
where the second equality is satisfied by inductive hypothesis and the third one is due to the fact that X is a pseudo-functor (cases where $k = 0$ or there are equalities are straightforward).

To check that λ is a modification, we need to verify that $\forall\ k < j \in \mathsf{J}$ the following equality holds:



idC
aj
ak<j
ak
Ak<j
λk
fk
ik
Ak<j
=
idC
aj
λj
idC
fj
ij
fk<j
fk
Xk<j
ij
ik<j
fk
ik
Ak<j

But



idC
aj
ak<j
ak
Ak<j
λk
fk
ik
Ak<j
=
idC
aj
ak<j
ak
Ak<j
a0<k
a0
A0<k
Ak<j
λ0
f0
i0
A0<k
Ak<j
i−1 0<k
f0
X0<k
ik
Ak<j
=
fk
ik
Ak<j
=
idC
aj
a0<j
a0
A0<j
αA 0,k,j −1
a0
A0<k
Ak<j
λ0
f0
i0
A0<k
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i−1 0<k
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=
fk
ik
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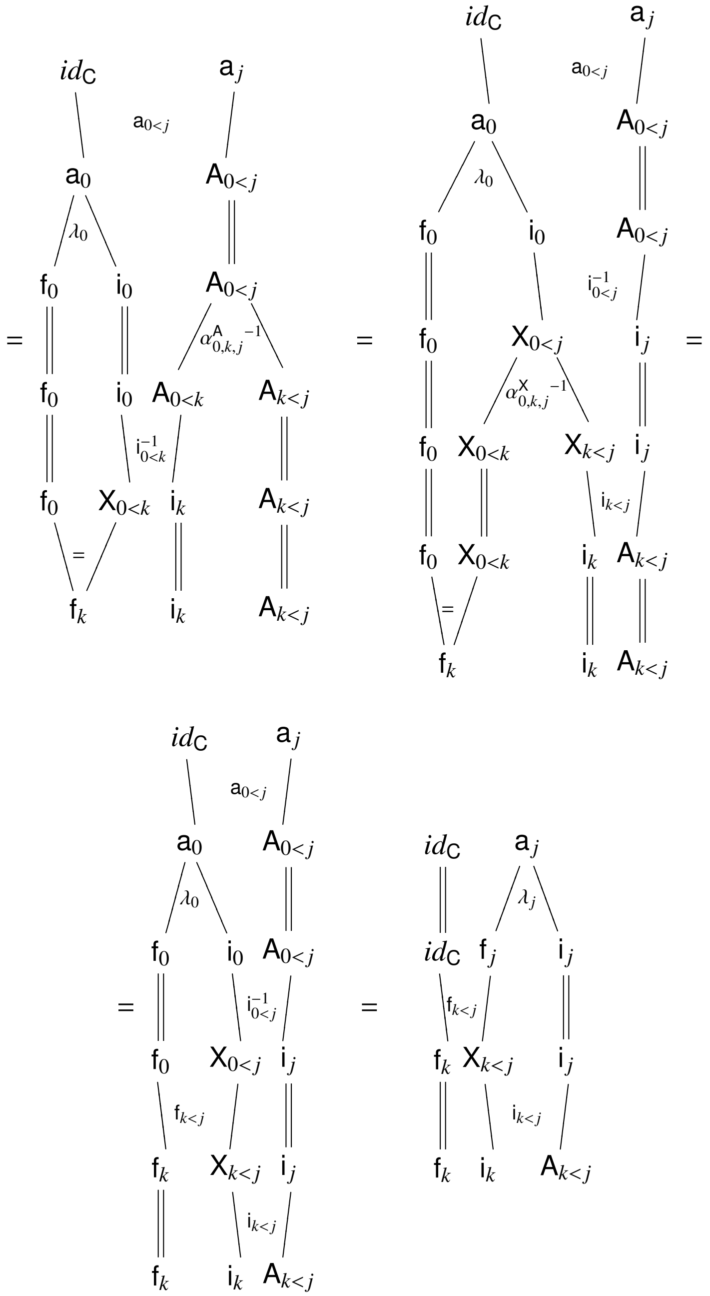
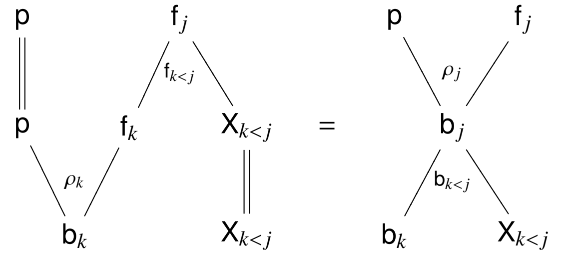
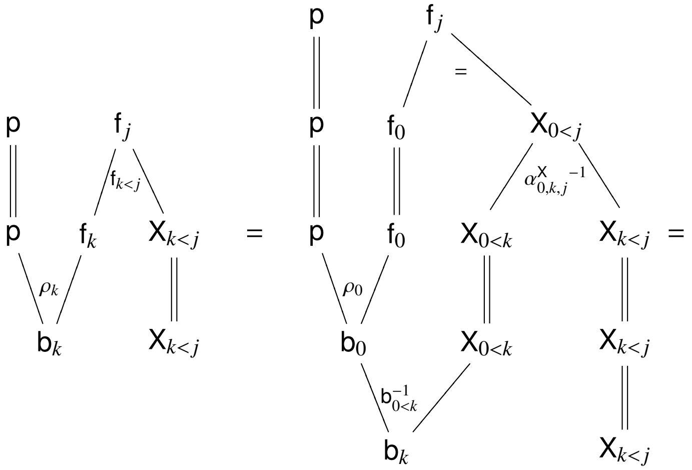


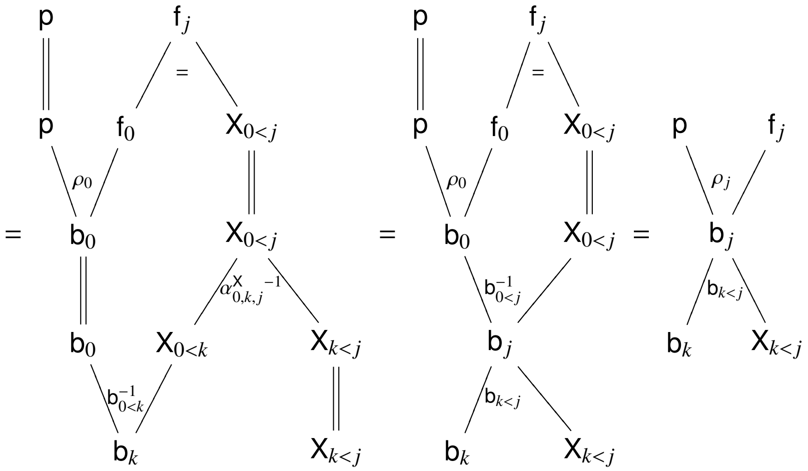
where the second equality is due to the fact that a is a pseudo-natural transformation, the third one is due to elevators calculus, the fourth one is due to the fact that i is a pseudo-natural transformation, the fifth one is due to elevators calculus plus the definition of $\mathsf{f}_{k<j}$ and the last one is due to the definition of λ_j. Again, simpler cases are omitted.

To check that ρ is a modification, we need to verify that $\forall\ k < j \in \mathsf{J}$ the following equality holds:



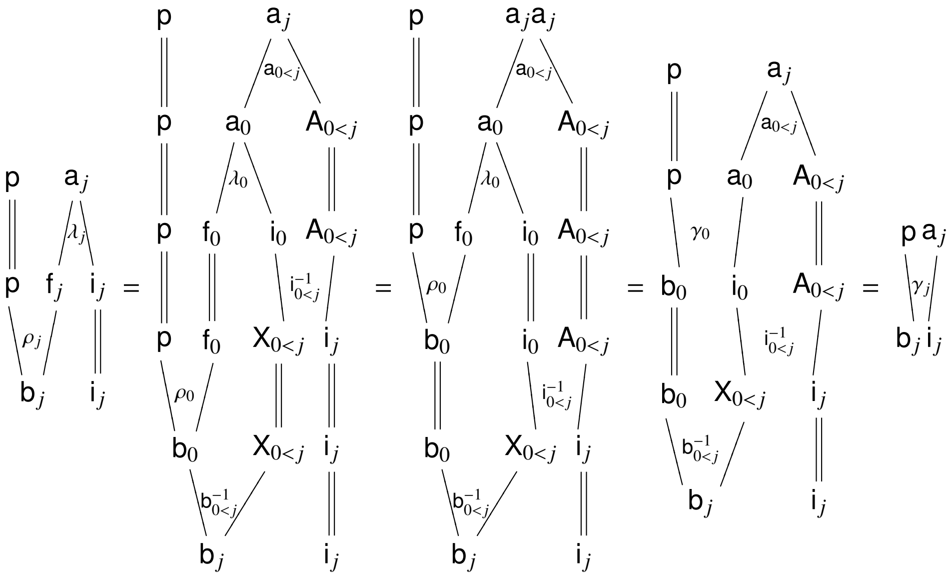
But





where the second equality is due to elevators calculus and the third one is due to the fact that b is a pseudo-natural transformation. Simpler cases are omitted.

Finally, let's check that $(\mathsf{f}, \lambda, \rho)$ is the filler that we were looking for:



The second equality is due to elevators calculus, the third one is due to the fact that $(\mathsf{f}_0, \lambda_0, \rho_0)$ is a filler for diagram (5.1.4) and the last one is due to the fact that γ is a modification.

2. By 1.1.26, it is straightforward that the 2-functor constant diagram is left bi-adjoint to the pseudo-functor inverse bi-limit. Then, by 4.3.2 and the previous item, we have what we wanted.

□

The following characterization of fibrations, similar but stronger than pointwise fibrations, is key to manipulate fibrations and prove that 5.1.2 determines a closed 2-bmolel structure (Theorem 5.1.14).

5.1.5 Lemma. *A morphism* $\mathsf{Y} \xrightarrow{\mathsf{p}} \mathsf{B} \in p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ *is a fibration iff* $\forall\ j \in \mathsf{J}$ *the morphism* q_j *of the following diagram is a fibration in* $\mathcal{C}$*:*

$$\begin{tikzcd}
\mathsf{Y}_j \arrow[rrd, bend left, "\mathsf{a}^j_{\mathsf{Y}}"] \arrow[rd, dashed, "\mathsf{q}_j"'] \arrow[rdd, bend right, "\mathsf{p}_j"'] & \cong \Uparrow \beta^j_0 & \\
\cong \Downarrow \beta^j_1 & \mathsf{P}_j \arrow[r, "\pi^j_0"] \arrow[d, "\pi^j_1"'] \arrow[rd, phantom, "\substack{bipb \\ \cong \Downarrow \alpha_j}"] & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{Y}_k \arrow[d, "\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k"] \\
& \mathsf{B}_j \arrow[r, "\mathsf{a}^j_{\mathsf{B}}"'] & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{B}_k
\end{tikzcd}$$

where $\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k$ *is induced by the pseudo-cone* $\{\mathsf{p}_k\pi_k\}_{k<j}$,

$$\left\{ \begin{array}{ccccc} \mathsf{B}_{k<l} & & \mathsf{p}_l & & \pi_l \\ \backslash & \mathsf{p}_{k<l} & / & & \| \\ \mathsf{p}_k & & \mathsf{Y}_{k<l} & & \pi_l \\ \| & & \backslash & \pi_{k<l} & / \\ \mathsf{p}_k & & & \pi_k & \end{array} \right\}_{k<l<j} \cup \left\{ \begin{array}{ccccc} \backslash & \mathsf{B}_{id_k} & / & \mathsf{p}_k & \pi_k \\ & (\alpha^{\mathsf{B}}_k)^{-1} & & \| & \| \\ & id_{\mathsf{B}_k} & & \mathsf{p}_k & \pi_k \\ & \backslash & & = & / \\ & & \mathsf{p}_k & & \pi_k \end{array} \right\}_{k<j},$$

$\mathsf{a}^j_{\mathsf{Y}}$ *is induced by the pseudo-cone* $\{\mathsf{Y}_{k<j}\}_{k<j}$,

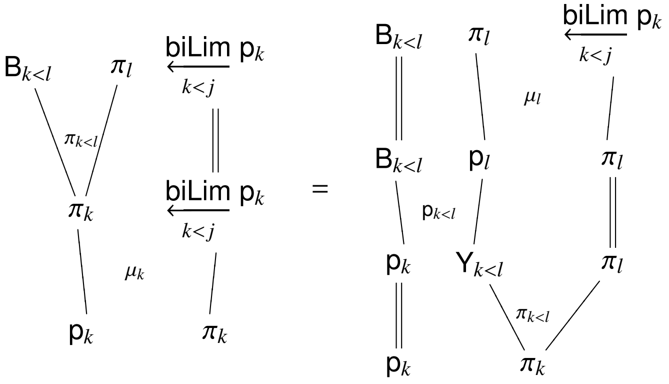
$$\left\{ \begin{array}{ccc} \mathsf{Y}_{k<l} & & \mathsf{Y}_{l<j} \\ \backslash & \mathsf{Y}_{k,l,j} & / \\ & \mathsf{Y}_{k<j} & \end{array} \right\}_{k<l<j} \cup \left\{ \begin{array}{ccccc} \backslash & \mathsf{Y}_{id_k} & / & & \mathsf{Y}_{k<j} \\ & (\alpha^{\mathsf{Y}}_k)^{-1} & & & \| \\ & id_{\mathsf{Y}_k} & & & \mathsf{Y}_{k<j} \\ & \backslash & = & / & \\ & & \mathsf{Y}_{k<j} & & \end{array} \right\}_{k<j},$$

$\mathsf{a}^j_{\mathsf{B}}$ *is induced by the pseudo-cone* $\{\mathsf{B}_{k<j}\}_{k<j}$,

$$\left\{ \begin{array}{ccc} \mathsf{B}_{k<l} & & \mathsf{B}_{l<j} \\ \backslash & \mathsf{B}_{k,l,j} & / \\ & \mathsf{B}_{k<j} & \end{array} \right\}_{k<l<j} \cup \left\{ \begin{array}{ccccc} \backslash & \mathsf{B}_{id_k} & / & & \mathsf{B}_{k<j} \\ & (\alpha^{\mathsf{B}}_k)^{-1} & & & \| \\ & id_{\mathsf{B}_k} & & & \mathsf{B}_{k<j} \\ & \backslash & = & / & \\ & & \mathsf{B}_{k<j} & & \end{array} \right\}_{k<j}$$

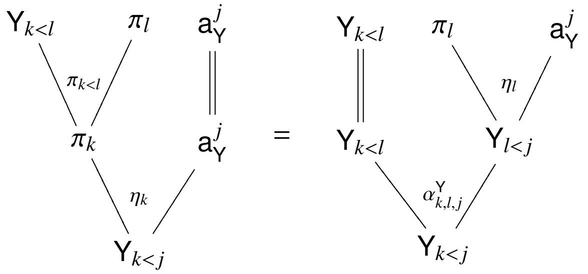
and so we have:

a) invertible 2-cells $\pi_k \underset{\overleftarrow{k<j}}{\mathsf{biLim}}\, \mathsf{p}_k \overset{\mu_k}{\Longrightarrow} \mathsf{p}_k \pi_k\ \forall\ k < j$ *such that*



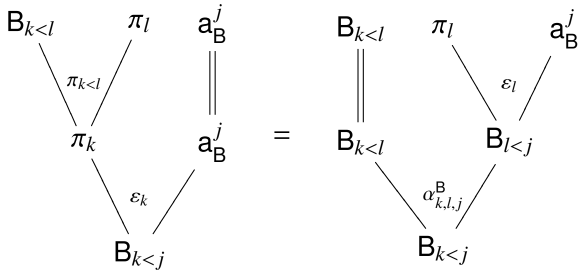
$\forall\ k < l < j$

b) invertible 2-cells $\pi_k \mathsf{a}^j_{\mathsf{Y}} \overset{\eta_k}{\Longrightarrow} \mathsf{Y}_{k<j}\ \forall\ k < j$ *such that*



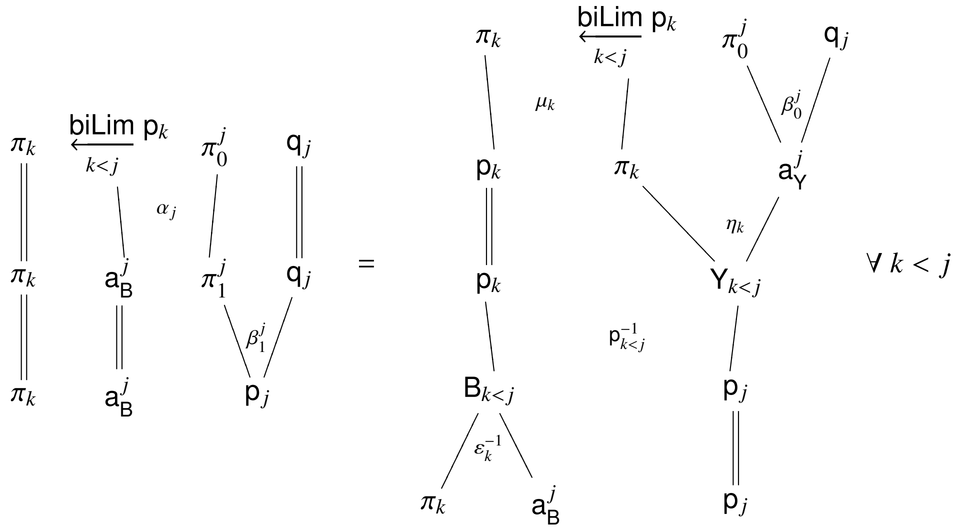
$\forall\ k < l < j$

c) invertible 2-cells $\pi_k \mathsf{a}^j_{\mathsf{B}} \overset{\epsilon_k}{\Longrightarrow} \mathsf{B}_{k<j}\ \forall\ k < j$ *such that*



$\forall\ k < l < j$

d)



Proof. ⇒) Since $\mathcal{C}$ is a closed 2-bmodel 2-category, it is enough to check that $\forall\ j \in \mathsf{J}$, q_j has the right lifting property with respect to all morphisms that are both cofibrations and weak equivalences. So, let $A \xrightarrow{i} X \in \mathcal{C}$ be a cofibration which is also a weak equivalence and suppose that we have the following situation:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & Y_j \arrow[d, "q_j"] \\
X \arrow[r, "b"'] & P_j
\end{tikzcd}$$

Let's define K, L, K $\xrightarrow{i'}$ L, K $\xrightarrow{a'}$ Y and L $\xrightarrow{b'}$ B in $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$

$$\text{by } K_k = \begin{cases} X & \text{if } k < j \\ A & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}, \quad K_{id_k} = id_{K_k}, \quad K_{k<l} = \begin{cases} id_X & \text{if } k < l < j \\ i & \text{if } k < j \text{ and } l = j \\ 0 \longrightarrow X & \text{if } k < j \text{ and } l \nleq j, \\ 0 \longrightarrow A & \text{if } k = j \text{ and } l \nleq j \\ id_0 & \text{otherwise} \end{cases}$$

$$\mathsf{L}_k = \begin{cases} \mathsf{X} & \text{if } k \leq j \\ 0 & \text{otherwise} \end{cases}, \quad \mathsf{L}_{id_k} = id_{\mathsf{L}_k}, \quad \mathsf{L}_{k<l} = \begin{cases} id_{\mathsf{X}} & \text{if } k < l \leq j \\ 0 \longrightarrow \mathsf{X} & \text{if } k \leq j \text{ and } l \nleq j \\ id_0 & \text{otherwise} \end{cases},$$

$$\mathsf{i}'_k = \begin{cases} id_{\mathsf{X}} & \text{if } k < j \\ \mathsf{i} & \text{if } k = j \\ id_0 & \text{otherwise} \end{cases}, \quad \mathsf{a}'_k = \begin{cases} \pi_k \pi_0^j \mathsf{b} & \text{if } k < j \\ \mathsf{a} & \text{if } k = j \\ 0 \longrightarrow \mathsf{Y}_k & \text{otherwise} \end{cases}, \quad \mathsf{a}'_{id_k} = (\alpha_k^{\mathsf{Y}})^{-1} \mathsf{a}'_k,$$

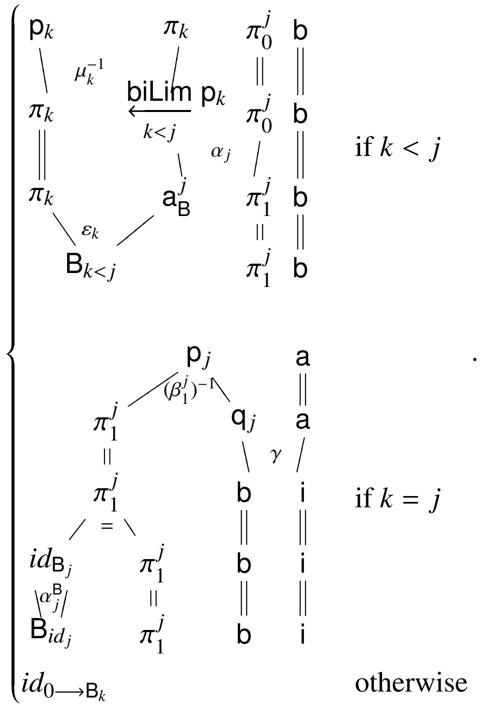
$$\mathsf{a}'_{k<l} = \begin{cases} \pi_{k<l} \pi_0^j \mathsf{b} & \text{if } k < l < j \\ \pi_k \pi_0^j \gamma \circ \pi_k (\beta_0^j)^{-1} \mathsf{a} \circ \eta_k^{-1} \mathsf{a} & \text{if } k < j \text{ and } l = j \\ id_{0 \longrightarrow \mathsf{Y}_k} & \text{otherwise} \end{cases}, \quad \mathsf{b}'_k = \begin{cases} \mathsf{B}_{k \leq j} \pi_1^j \mathsf{b} & \text{if } k \leq j \\ 0 \longrightarrow \mathsf{B}_k & \text{otherwise} \end{cases},$$

$\mathsf{b}'_{id_k} = (\alpha_k^{\mathsf{B}})^{-1} \mathsf{b}'_k$ and $\mathsf{b}'_{k<l} = \begin{cases} \alpha^{\mathsf{B}}_{k,l,j} \pi_1^j \mathsf{b} & \text{if } k < l < j \\ \mathsf{B}_{k<j} (\alpha_j^{\mathsf{B}})^{-1} \pi_1^j \mathsf{b} & \text{if } k < j \text{ and } l = j \\ id_{0 \longrightarrow \mathsf{B}_k} & \text{otherwise} \end{cases}$. It is straightforward to check that K and L are 2-functors, i' is a 2-natural transformation and a', b' are pseudo-natural transformations.

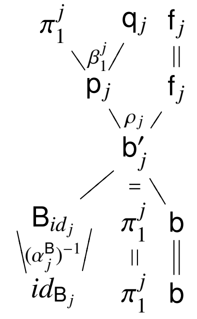
Then, since p is a fibration and i' is both a cofibration and a weak equivalence, there exists a filler $(\mathsf{f}, \lambda, \rho)$ for the following diagram

$$\begin{tikzcd} \mathsf{K} \arrow[r, "\mathsf{a}'"] \arrow[d, "\mathsf{i}'"'] \arrow[dr, phantom, "\cong \Downarrow \tilde{\gamma}"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\ \mathsf{L} \arrow[r, "\mathsf{b}'"'] & \mathsf{B} \end{tikzcd} \tag{5.1.6}$$

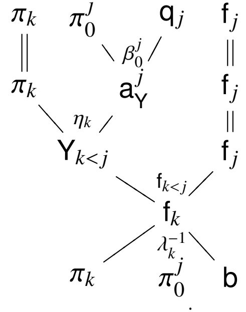
where $\tilde{\gamma}_k =$



Let's check that (f_j, λ_j, ρ') (where ρ' is such that $\pi_1\rho' =$

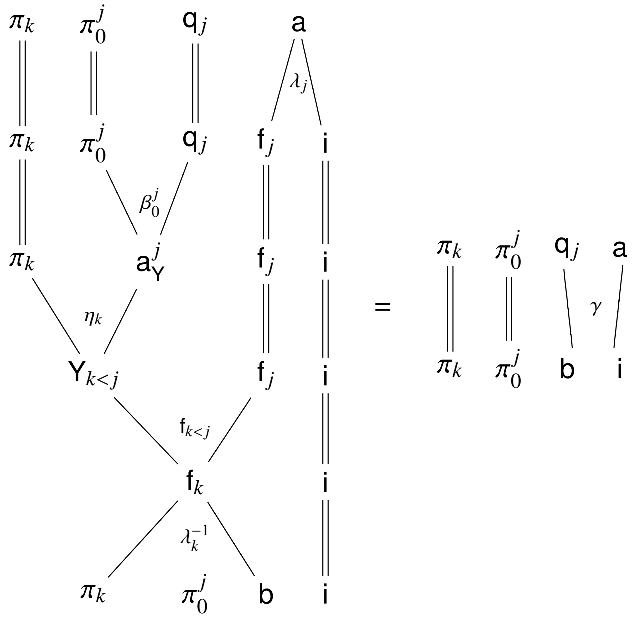


and $\pi_k\pi_0\rho' =$

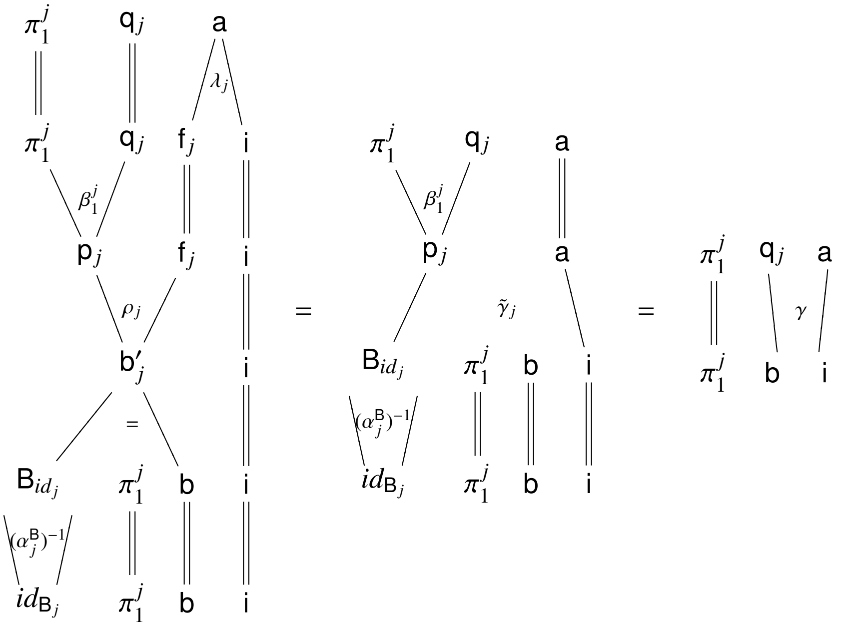


) is the filler that we were looking for: To do that, it is enough to check that $\pi_k\pi_0^j(\rho' i \circ q_j\lambda_j) = \pi_k\pi_0^j\gamma\ \forall\ k < j$ and $\pi_1^j(\rho' i \circ q_j\lambda_j) = \pi_1^j\gamma$:

By elevators calculus plus the fact that λ is a modification, we have the following equality:



And, by elevators calculus plus the fact that $(\mathsf{f}, \lambda, \rho)$ is a filler for diagram (5.1.6), we have the following equality:



$\Leftarrow$) Let $\mathsf{A} \xrightarrow{\mathsf{i}} \mathsf{X} \in p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ be a cofibration that is also a weak equivalence and suppose that we have the following situation:

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\ \mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{B} \end{tikzcd}$$

We are going to construct the filler $(\mathsf{f}, \lambda, \rho)$ inductively:
$\mathsf{p}_0 \cong \mathsf{q}_0$ and therefore is a fibration. So, since $\mathcal{C}$ is a closed 2-bmodel 2-category, there exists a filler $(\mathsf{f}_0, \lambda_0, \rho_0)$ for the following diagram

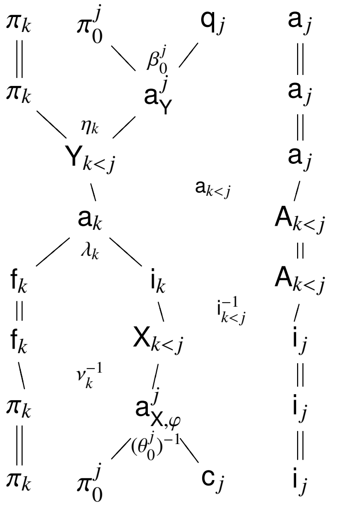
$$\begin{tikzcd} \mathsf{A}_0 \arrow[r, "\mathsf{a}_0"] \arrow[d, "\mathsf{i}_0"'] \arrow[dr, phantom, "\cong \Downarrow \gamma_0"] & \mathsf{Y}_0 \arrow[d, "\mathsf{p}_0"] \\ \mathsf{X}_0 \arrow[r, "\mathsf{b}_0"'] & \mathsf{B}_0 \end{tikzcd}$$

Suppose that we have already constructed $(\mathsf{f}_k, \lambda_k, \rho_k)$ $\forall k < j$. Then, since q_j is a fibration and i_j is both a cofibration and a weak equivalence, there exists a filler $(\mathsf{f}_j, \lambda_j, \tilde{\rho_j})$ for the following diagram

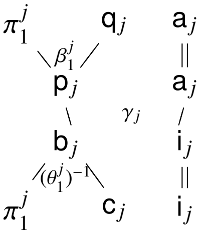
$$\begin{tikzcd} \mathsf{A}_j \arrow[r, "\mathsf{a}_j"] \arrow[d, "\mathsf{i}_j"'] \arrow[dr, phantom, "\cong \Downarrow \tilde{\gamma}_j"] & \mathsf{Y}_j \arrow[d, "\mathsf{q}_j"] \\ \mathsf{X}_j \arrow[r, "\mathsf{c}_j"'] & \mathsf{P}_j \end{tikzcd} \tag{5.1.7}$$

where c_j is given by diagram (5.1.8) and $\tilde{\gamma}_j$ is such that

$\pi_k \pi_0^j \tilde{\gamma}_j =$



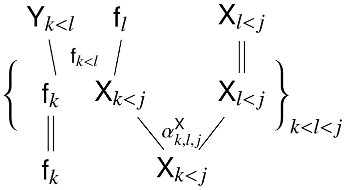
and $\pi_1^j \tilde{\gamma}_j =$



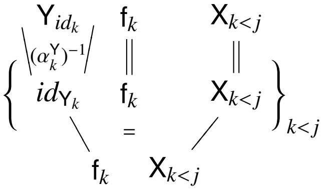
:

$$\begin{tikzcd}
X_j \arrow[rr, bend left, "a^j_{X,Y}"] \arrow[dr, dashed, "c_j"] \arrow[ddr, bend right, "b_j"'] \arrow[dr, phantom, "\cong \Uparrow \theta_0^j", near end, shift left=4] \arrow[dr, phantom, "\cong \Downarrow \theta_1^j", shift right=4] & & \\
& P_j \arrow[r, "\pi_0^j"] \arrow[d, "\pi_1^j"'] \arrow[dr, phantom, "\begin{array}{c} bipb \\ \cong \Downarrow \alpha_j \end{array}"] & \underleftarrow{\mathrm{biLim}}_{k<j}\, Y_k \arrow[d, "\underleftarrow{\mathrm{biLim}}_{k<j}\, p_k"] \\
& B_j \arrow[r, "a_B^j"'] & \underleftarrow{\mathrm{biLim}}_{k<j}\, B_k
\end{tikzcd} \tag{5.1.8}$$

where $a^j_{X,Y}$ is induced by the pseudo-cone $\{f_k X_{k<j}\}_{k<j}$,

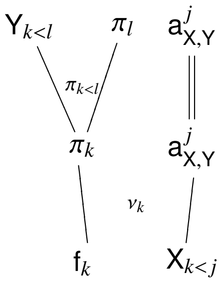


$\cup$

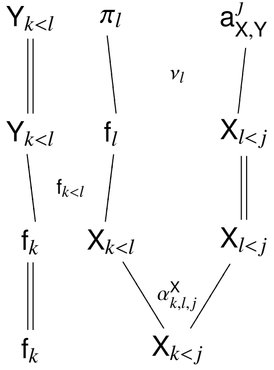


and so we have:

e) invertible 2-cells $\pi_k a^j_{X,Y} \overset{\nu_k}{\Longrightarrow} f_k X_{k<j}$ ∀ $k < j$ such that

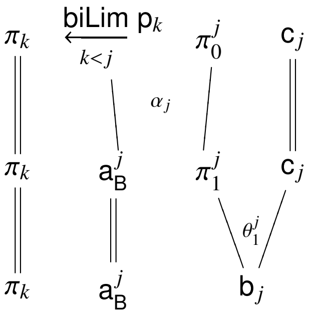


$=$

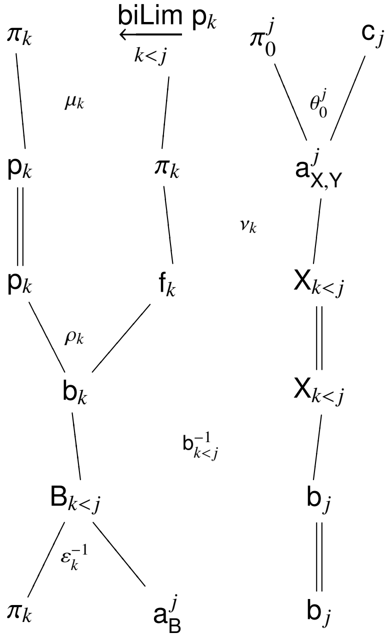


$\forall\ k < l < j$

f)

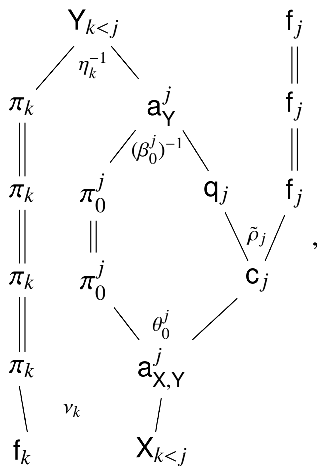


$=$

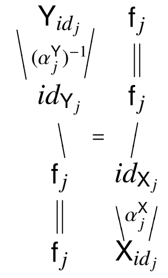


$\forall\ k < j$

Take $\mathsf{f}_{k<j}$ =

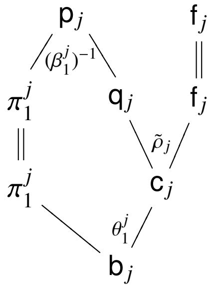


, f_{id_j} =



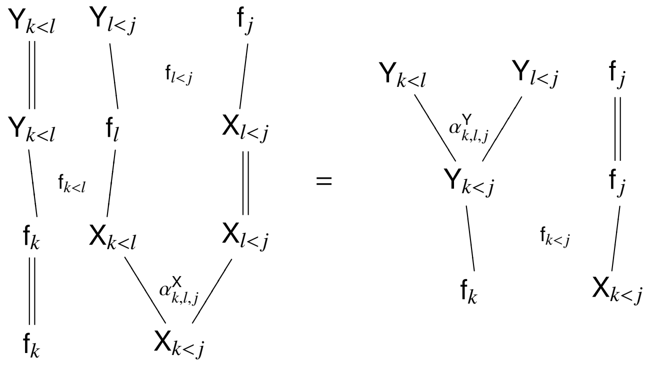
and

ρ_j =

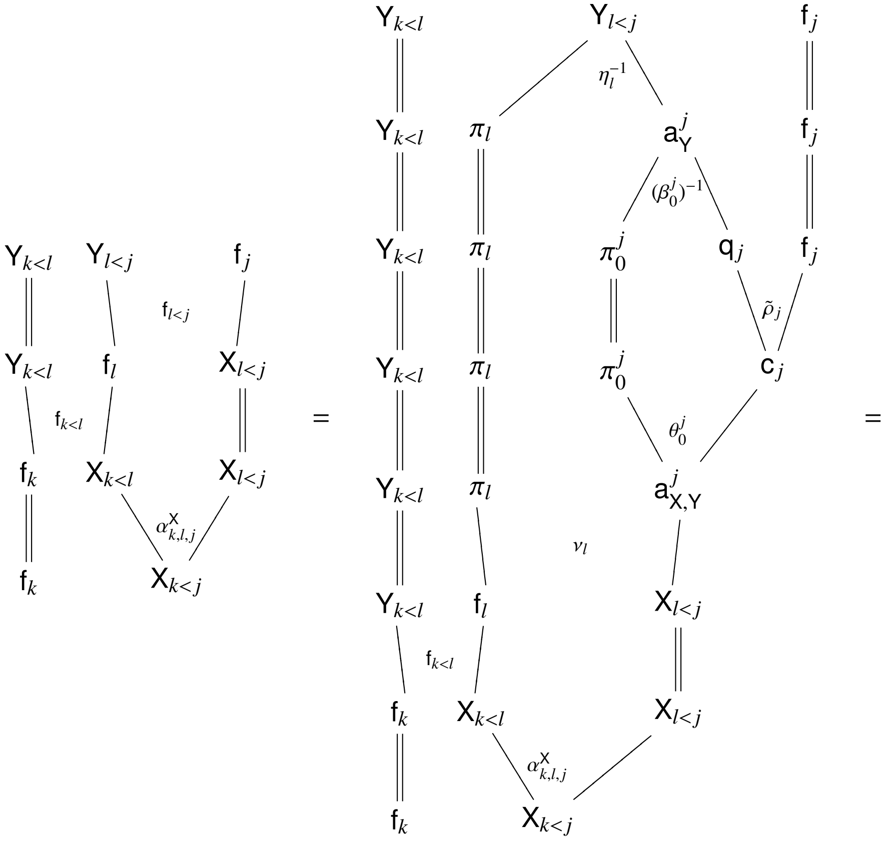


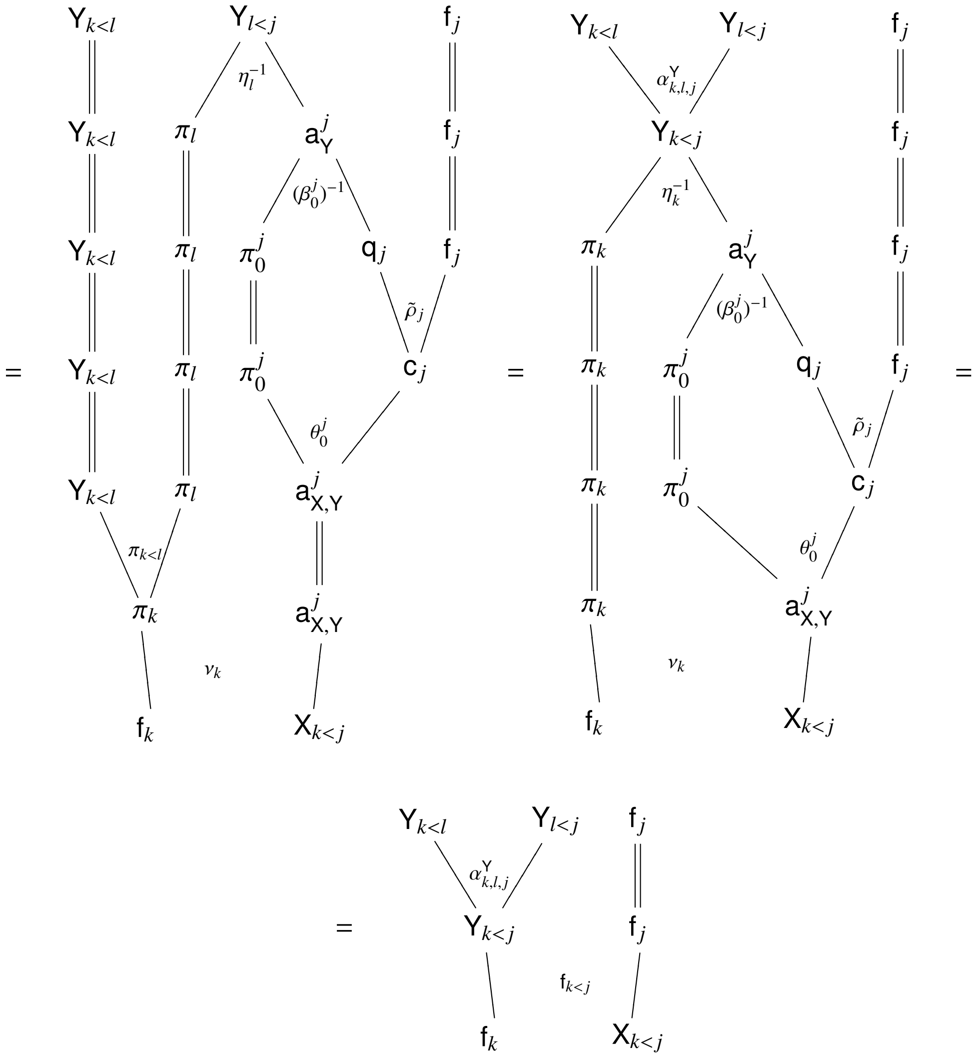
.

Now we are going to check that f constructed this way is a pseudo-natural transformation: PNO is satisfied by construction and PN2 is vacuous since there are no 2-cells in J. To check axiom PN1, we need to check that the following equality holds $\forall\ k < l < j$:



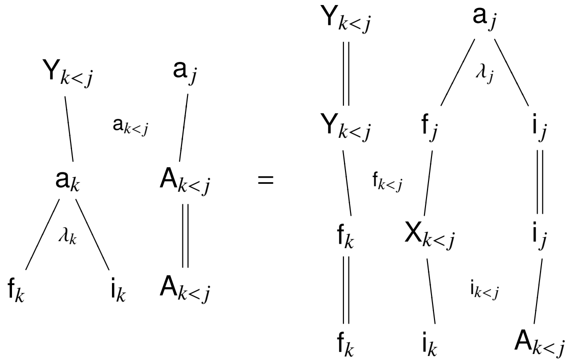
But


$\mathsf{Y}_{k<l}$
$\mathsf{Y}_{l<j}$
f_j
$\mathsf{f}_{l<j}$
$\mathsf{Y}_{k<l}$
f_l
$\mathsf{X}_{l<j}$
$\mathsf{f}_{k<l}$
f_k
$\mathsf{X}_{k<l}$
$\mathsf{X}_{l<j}$
$\alpha^{\mathsf{X}}_{k,l,j}$
f_k
$\mathsf{X}_{k<j}$
=
$\mathsf{Y}_{k<l}$
$\mathsf{Y}_{l<j}$
f_j
η_l^{-1}
$\mathsf{Y}_{k<l}$
π_l
$\mathsf{a}^j_{\mathsf{Y}}$
f_j
$(\beta^j_0)^{-1}$
$\mathsf{Y}_{k<l}$
π_l
π^j_0
q_j
f_j
$\tilde{\rho}_j$
$\mathsf{Y}_{k<l}$
π_l
π^j_0
c_j
θ^j_0
$\mathsf{Y}_{k<l}$
π_l
$\mathsf{a}^j_{\mathsf{X},\mathsf{Y}}$
ν_l
$\mathsf{Y}_{k<l}$
f_l
$\mathsf{X}_{l<j}$
$\mathsf{f}_{k<l}$
f_k
$\mathsf{X}_{k<l}$
$\mathsf{X}_{l<j}$
$\alpha^{\mathsf{X}}_{k,l,j}$
f_k
$\mathsf{X}_{k<j}$
=



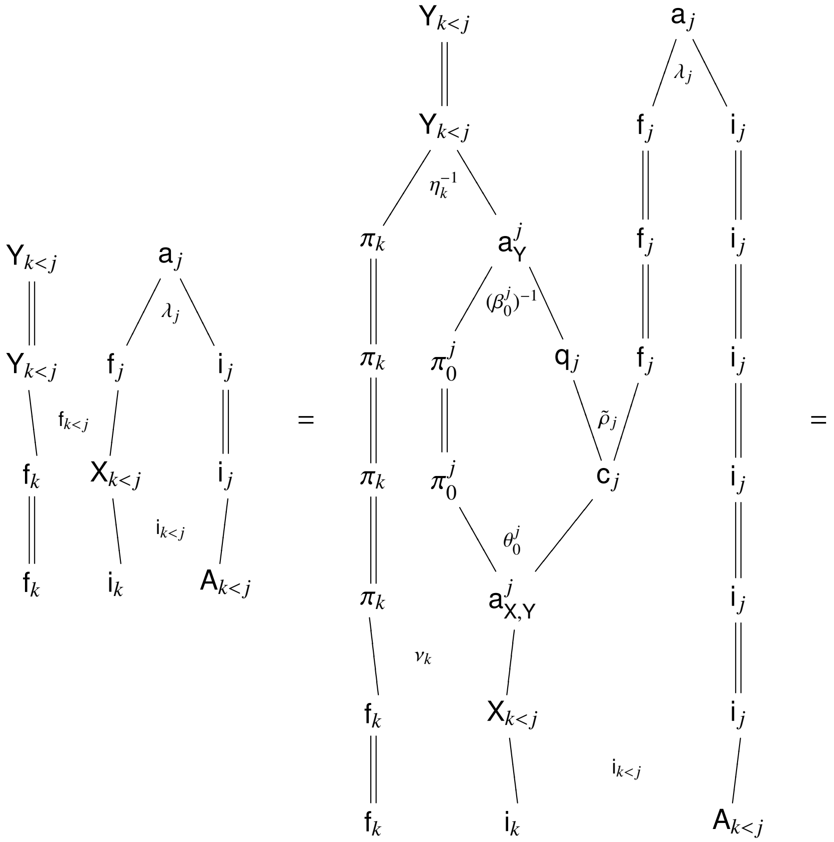
where the second equality is due to *e*) and the third one is due to elevators calculus plus *b*).

To check that λ is a modification, we need to verify that the following equality holds $\forall\ k < j$:

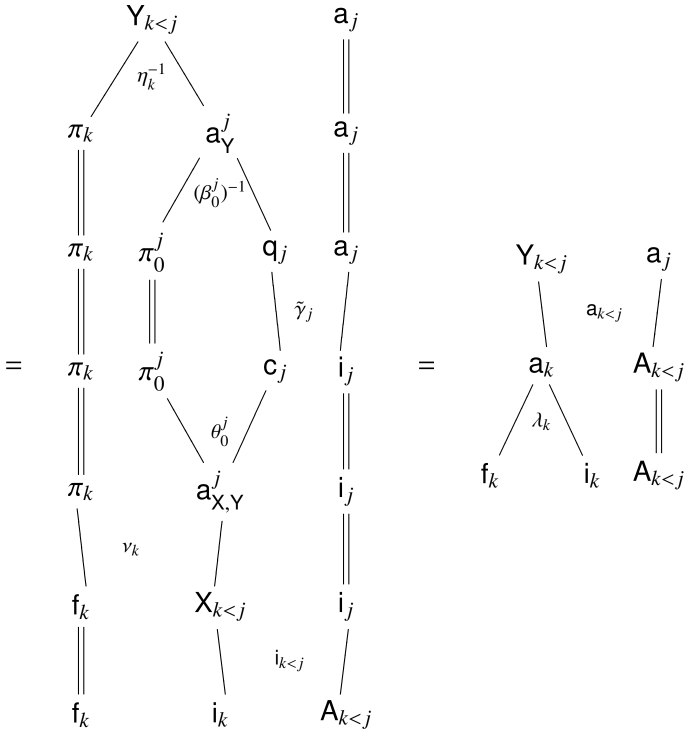


$Y_{k<j}$
a_j
$a_{k<j}$
a_k
$A_{k<j}$
λ_k
f_k
i_k
$A_{k<j}$
=
$Y_{k<j}$
a_j
λ_j
$Y_{k<j}$
f_j
i_j
$f_{k<j}$
f_k
$X_{k<j}$
i_j
$i_{k<j}$
f_k
i_k
$A_{k<j}$

But

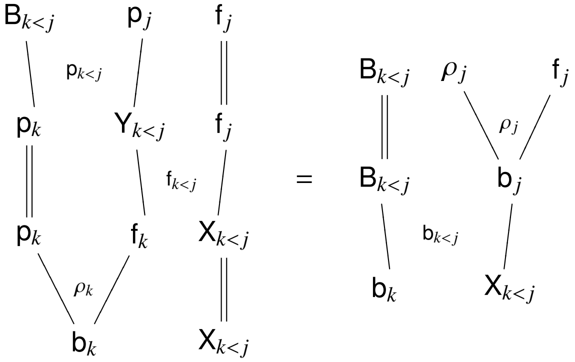


$Y_{k<j}$
a_j
λ_j
$Y_{k<j}$
f_j
i_j
$f_{k<j}$
f_k
$X_{k<j}$
i_j
$i_{k<j}$
f_k
i_k
$A_{k<j}$
=
$Y_{k<j}$
a_j
λ_j
$Y_{k<j}$
f_j
i_j
η_k^{-1}
π_k
a^j_Y
f_j
i_j
$(\beta_0^j)^{-1}$
π_k
π_0^j
q_j
f_j
i_j
$\tilde{\rho}_j$
π_k
π_0^j
c_j
i_j
θ_0^j
π_k
$a^j_{X,Y}$
i_j
ν_k
f_k
$X_{k<j}$
i_j
$i_{k<j}$
f_k
i_k
$A_{k<j}$
=

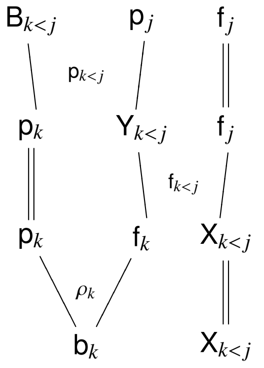


where the second equality is due to elevators calculus plus the fact that $(\mathsf{f}_j, \lambda_j, \tilde{\rho}_j)$ is a filler for diagram (5.1.7) and the last one is due to the definition of $\tilde{\gamma}_j$ plus elevators calculus.

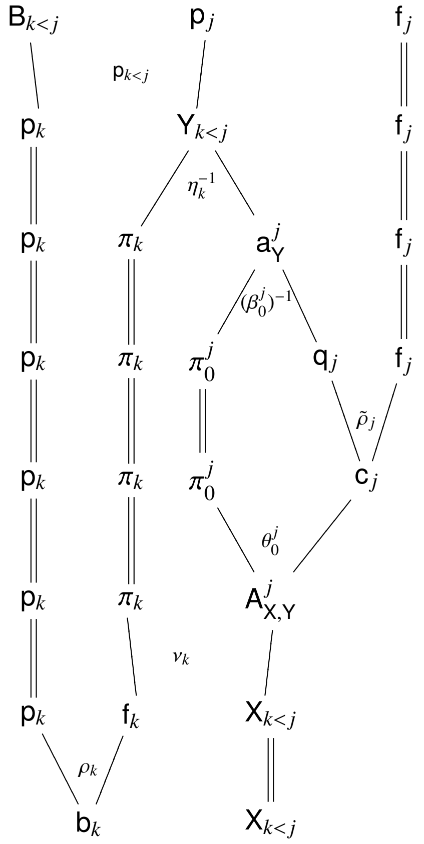
To check that ρ is a modification, we need to verify that the following equality holds $\forall\ k < j$:



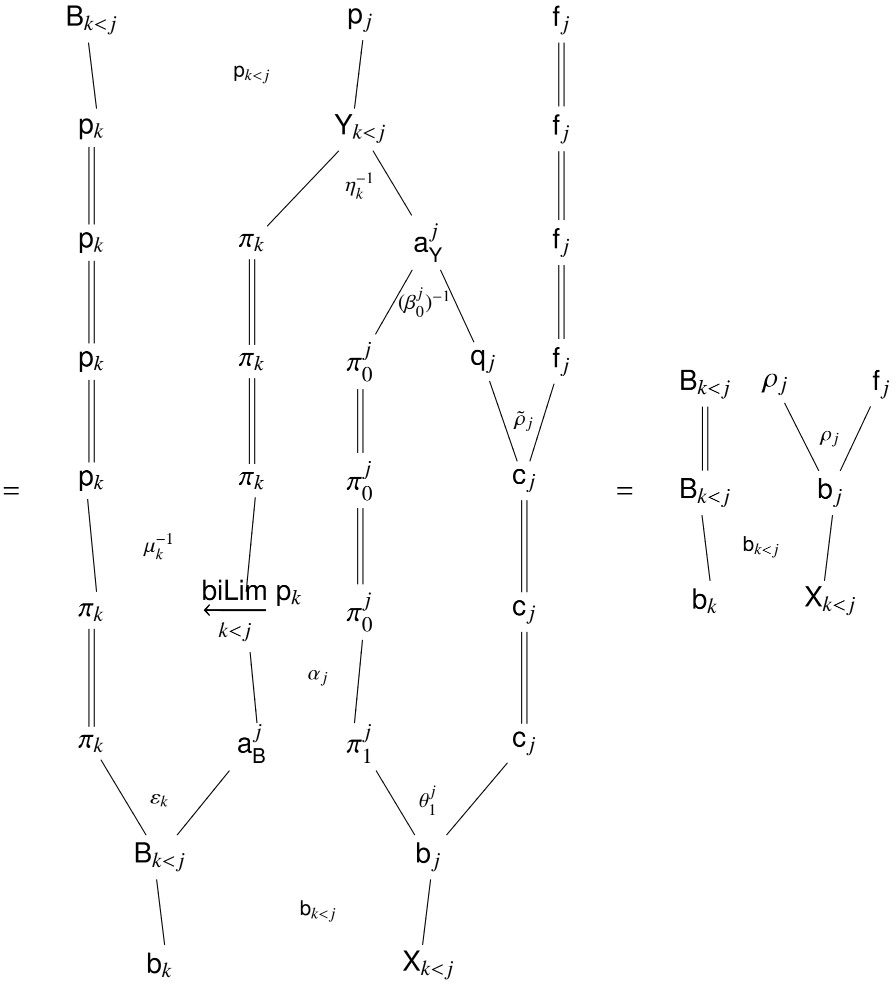
But


$B_{k<j}$
p_j
f_j
$p_{k<j}$
p_k
$Y_{k<j}$
f_j
$f_{k<j}$
p_k
f_k
$X_{k<j}$
ρ_k
b_k
$X_{k<j}$

=

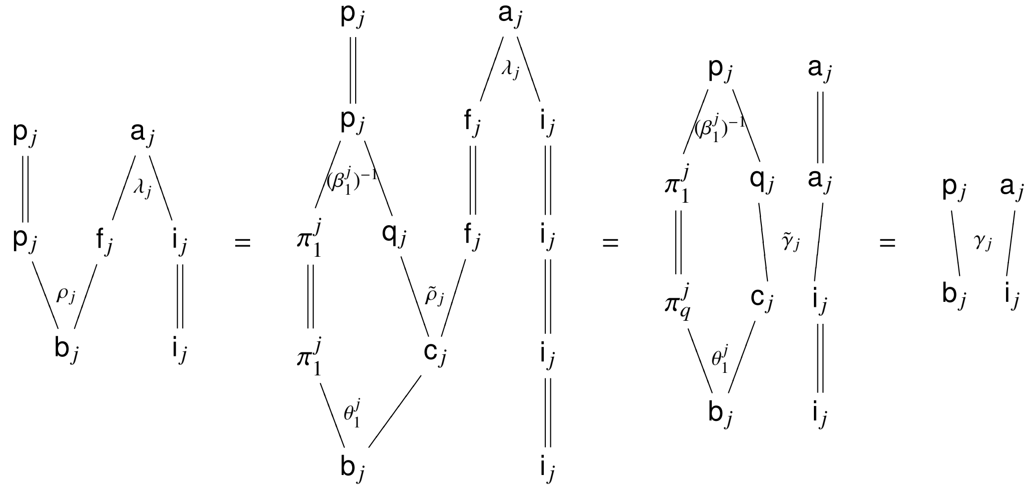

$B_{k<j}$
p_j
f_j
$p_{k<j}$
p_k
$Y_{k<j}$
f_j
η_k^{-1}
p_k
π_k
a_Y^j
f_j
$(\beta_0^j)^{-1}$
p_k
π_k
π_0^j
q_j
f_j
$\tilde{\rho}_j$
p_k
π_k
π_0^j
c_j
θ_0^j
p_k
π_k
$A_{X,Y}^j$
ν_k
p_k
f_k
$X_{k<j}$
ρ_k
b_k
$X_{k<j}$

=



where the second equality is due to item *f*) and the last one is due to item *d*).

Finally, let's check that (f, λ, ρ) constructed this way is the filler that we were looking for:



where the second equality is due to elevators calculus plus the fact that $(f_j, \lambda_j, \tilde{\rho}_j)$ is a filler for diagram (5.1.7) and the last one is due to the definition of $\tilde{\gamma}_j$. □

5.1.9 Lemma. *If* $Y \xrightarrow{p} B \in p\mathcal{H}om_p(J^{op}, C)$ *is a fibration, then* p_j *is a fibration in* $C\ \forall\ j \in J$.

Proof. $p_0 \cong q_0$ which is a fibration by 5.1.5.

If $j \in J$ is not the initial object, consider p as an object in $p\mathcal{H}om_p(\{k \in J \mid k < j\}^{op}, C)$. Since $p \in p\mathcal{H}om_p(J^{op}, C)$ is a fibration, by 5.1.5, $p \in p\mathcal{H}om_p(\{k \in J \mid k < j\}^{op}, C)$ is a fibration and then, by 5.1.3 $\underset{\underset{k<j}{\longleftarrow}}{\text{biLim}}\ p_k \in C$ is a fibration. Then, since C is a closed 2-bmodel 2-category, π_1^j is a fibration. We also know that q_j is a fibration by 5.1.5. Then, $p_j \cong \pi_1^j q_j \in C$ is also a fibration. □

5.1.10 Lemma. *A morphism* $Y \xrightarrow{p} B \in p\mathcal{H}om_p(J^{op}, C)$ *is both a fibration and a weak equivalence iff* q_j *is both a fibration and a weak equivalence in* $C\ \forall j \in J$ *where* q_j *is defined as in 5.1.5.*

Proof. ⇒) By 5.1.5, it only remains to check that q_j is a weak equivalence $\forall\ j \in J$. We are going to prove this inductively:

$q_0 \cong p_0$ and therefore is a weak equivalence.

Suppose that q_k is a weak equivalence $\forall\ k < m$. Let's check that, in that case, $\underset{\underset{k<m}{\longleftarrow}}{\text{biLim}}\ p_k$ is both a fibration and a weak equivalence:

By 4.1.8, it is enough to check that it has the right lifting property with respect to all cofibrations. So let $\mathsf{A} \xrightarrow{\mathsf{i}} \mathsf{X} \in \mathcal{C}$ be a cofibration and suppose that we have the following situation:

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \underset{k<m}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{Y}_k \arrow[d, "\underset{k<m}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \underset{k<m}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{B}_k
\end{tikzcd}$$

We are going to define a pseudo-cone $\left\{\mathsf{X} \xrightarrow{\mathsf{f}_k} \mathsf{Y}_k\right\}_{k<m}$, $\left\{\mathsf{Y}_{k<j}\mathsf{f}_j \overset{\mathsf{f}_{k<j}}{\Longrightarrow} \mathsf{f}_k\right\}_{k<j<m}$ and invertible morphisms of pseudo-cones $\left\{\pi_k\mathsf{a} \overset{\tilde{\lambda}_k}{\Longrightarrow} \mathsf{f}_k\mathsf{i}\right\}_{k<m}$, $\left\{\mathsf{p}_k\mathsf{f}_k \overset{\tilde{\rho}_k}{\Longrightarrow} \pi_k\mathsf{b}\right\}_{k<m}$ as follows:

For the initial object, use 4.1.8 to construct a filler $(\mathsf{f}_0, \tilde{\lambda}_0, \tilde{\rho}_0)$ for the following diagram:

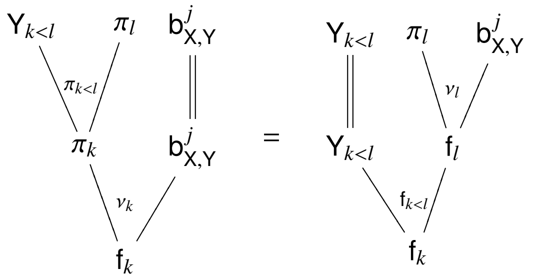
$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\pi_0\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \pi_0\gamma"] & \mathsf{Y}_0 \arrow[d, "\mathsf{p}_0"] \\
\mathsf{X} \arrow[r, "\pi_0\mathsf{b}"'] & \mathsf{B}_0
\end{tikzcd}$$

If j is not the initial object, suppose that we have already defined f_k, $\tilde{\lambda}_k$, $\tilde{\rho}_k$ $\forall\ k < j$ and consider the following diagram:

$$\begin{tikzcd}
\mathsf{X} \arrow[dr, dashed, "\mathsf{c}_j"] \arrow[drr, bend left, "\mathsf{b}^j_{\mathsf{X,Y}}"] \arrow[ddr, bend right, "\pi_j\mathsf{b}"'] & \cong \Uparrow \theta^j_0 & \\
\cong \Downarrow \theta^j_1 & \mathsf{P}_j \arrow[r, "\pi^j_0"] \arrow[d, "\pi^j_1"'] \arrow[dr, phantom, "\substack{bipb \\ \cong \Downarrow \alpha_j}"] & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{Y}_k \arrow[d, "\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k"] \\
& \mathsf{B}_j \arrow[r, "\mathsf{a}^j_{\mathsf{B}}"'] & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{B}_k
\end{tikzcd}$$

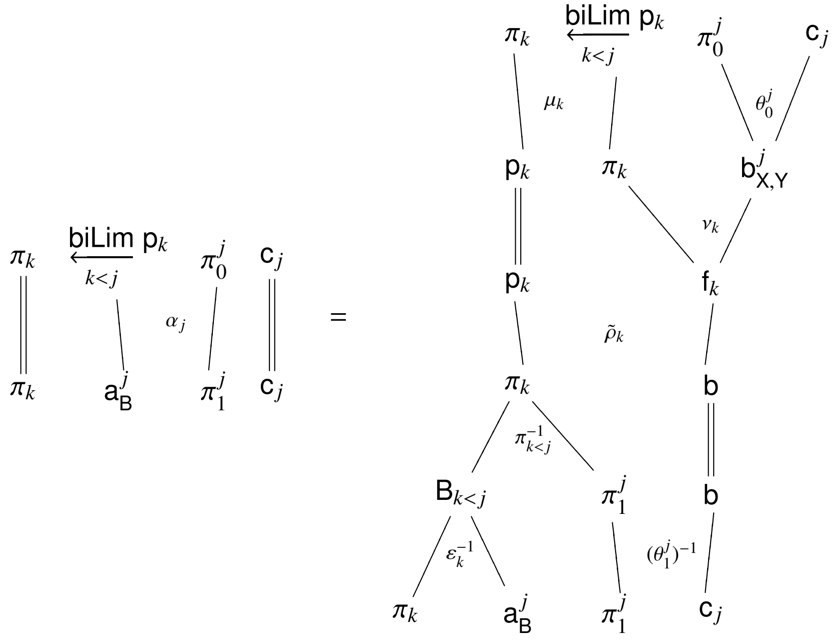
where $\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k$ and $\mathsf{a}^j_{\mathsf{B}}$ are defined as in 5.1.5 and $\mathsf{b}^j_{\mathsf{X,Y}}$ is induced by the pseudo-cone

$\{\mathsf{f}_k\}_{k<j}$, $\{\mathsf{f}_{k<l}\}_{k<l<j}$. Then we have invertible 2-cells $\pi_k \mathsf{b}^j_{\mathsf{X},\mathsf{Y}} \overset{\nu_k}{\Longrightarrow} \mathsf{f}_k$ $\forall\ k < j$ such that



$\forall\ k < l < j$ (5.1.11)

and we also have the following equality



$\forall\ k < j$

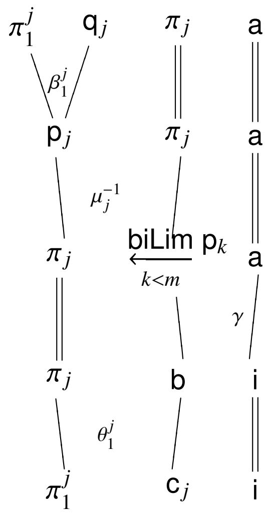
(5.1.12)

Then there exists a filler $(\mathsf{f}_j, \tilde{\lambda}_j, \rho'_j)$ for the following diagram

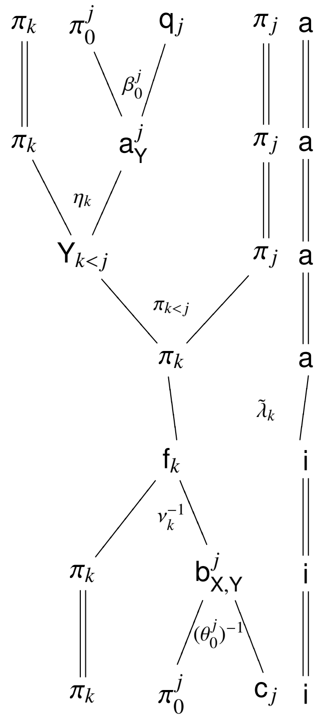
$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\pi_j \mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \tilde{\gamma}_j"] & \mathsf{Y}_j \arrow[d, "\mathsf{q}_j"] \\
\mathsf{X} \arrow[r, "\mathsf{c}_j"'] & \mathsf{P}_j
\end{tikzcd} \tag{5.1.13}$$

where q_j corresponds to a diagram as the one in 5.1.5, β_j is given by the formulas

$\pi_1^j\tilde{\gamma}_j =$

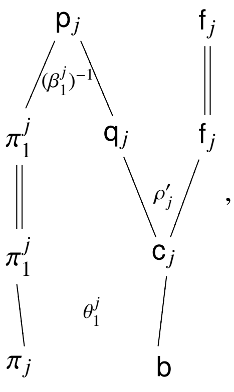


and $\pi_k\pi_0^j\tilde{\gamma}_j =$

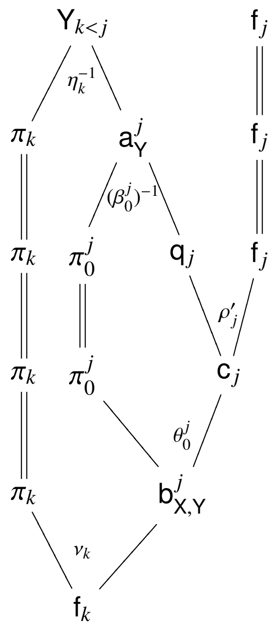


$\forall\, k < j$.

Take $\tilde{\rho}_j =$

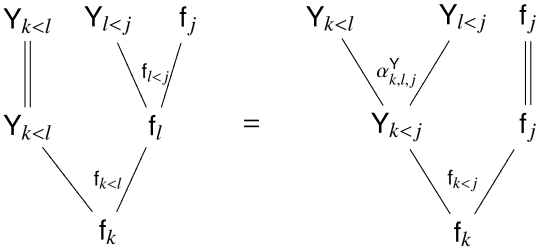


, $f_{id_j} = (\alpha_j^Y)^{-1} f_j$ and $f_{k<j} =$

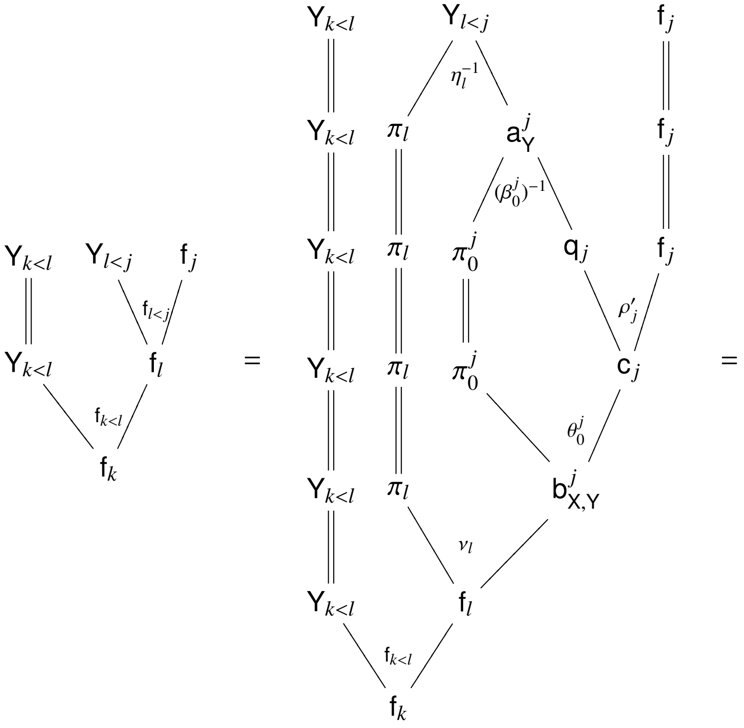


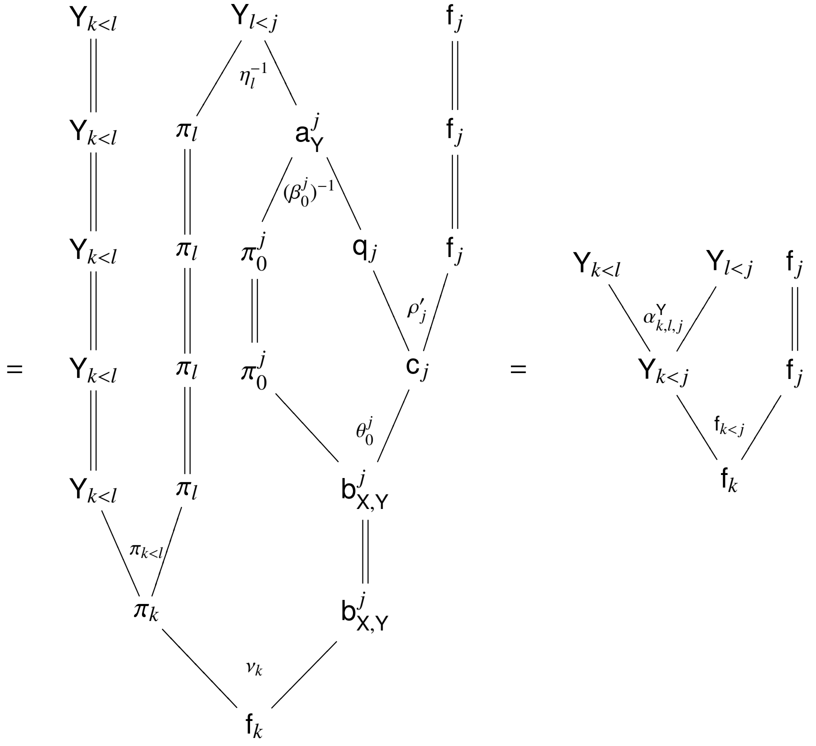
.

To verify that $\left\{\mathsf{X} \xrightarrow{\mathsf{f}_k} \mathsf{Y}_k\right\}_{k<m}$, $\left\{\mathsf{Y}_{k<j}\mathsf{f}_j \xRightarrow{\mathsf{f}_{k<j}} \mathsf{f}_k\right\}_{k<j<m}$ is a pseudo-cone, observe that axiom PC0 is satisfied by definition and axiom PC2 is vacuous because there are no 2-cells in J. So, we only need to check that axiom PC1 holds. To do that, we need to prove that the following equality holds $\forall\ k < l < j < m$:



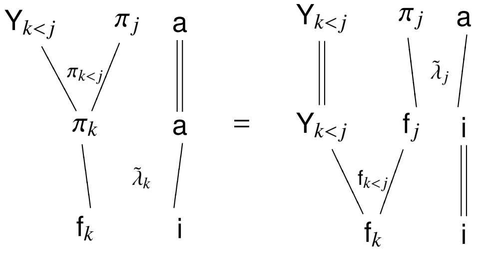
But



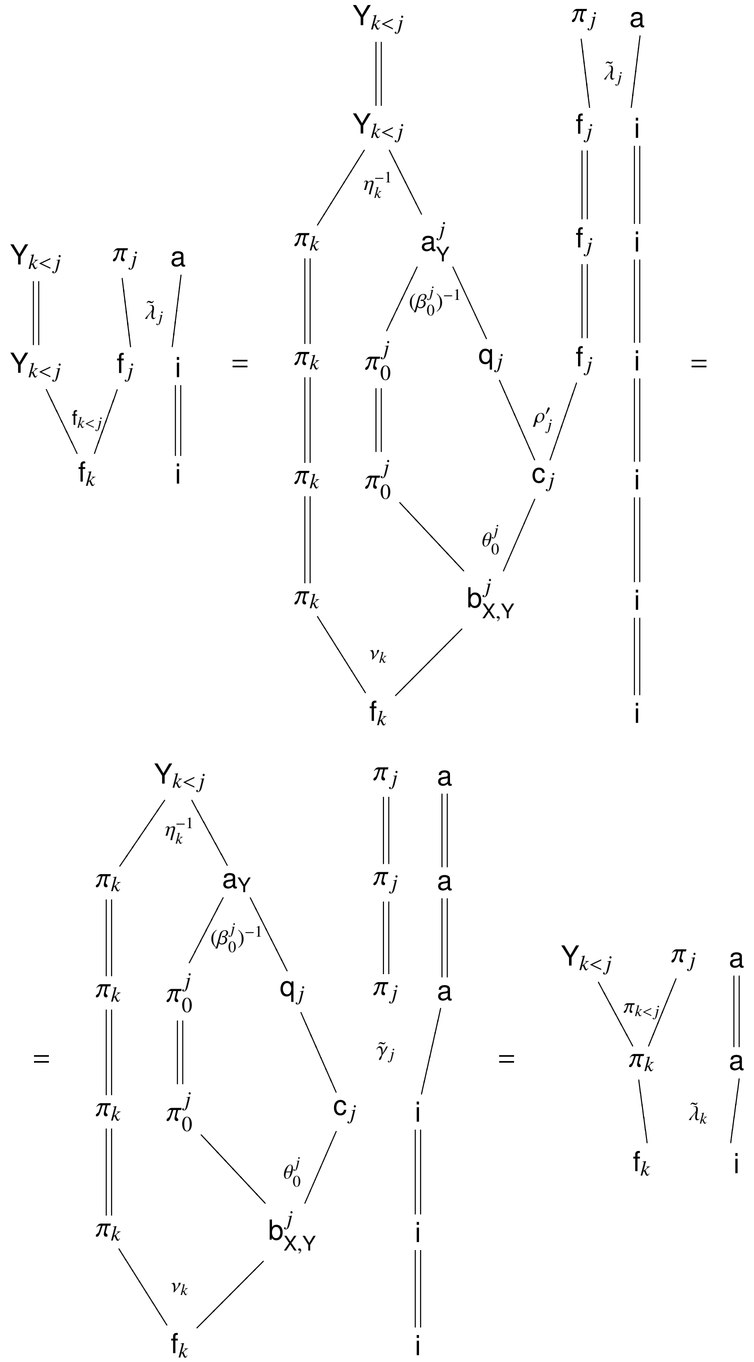


where the second equality is due to the definition of $b_{X,Y}^j$ and the last one to the definition a_Y^j.

To check that $\left\{ \pi_k a \overset{\tilde{\lambda}_k}{\Longrightarrow} f_k i \right\}_{k<m}$ is a morphism of pseudo-cones, we need to verify that the following equality holds $\forall\ k < j < m$:



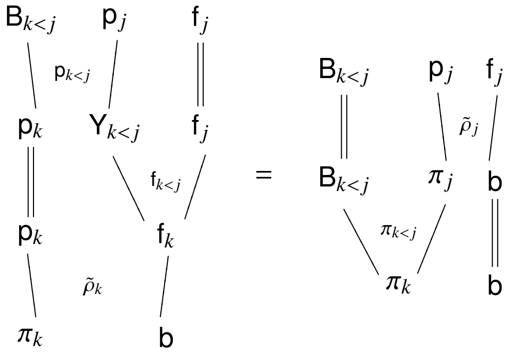
But



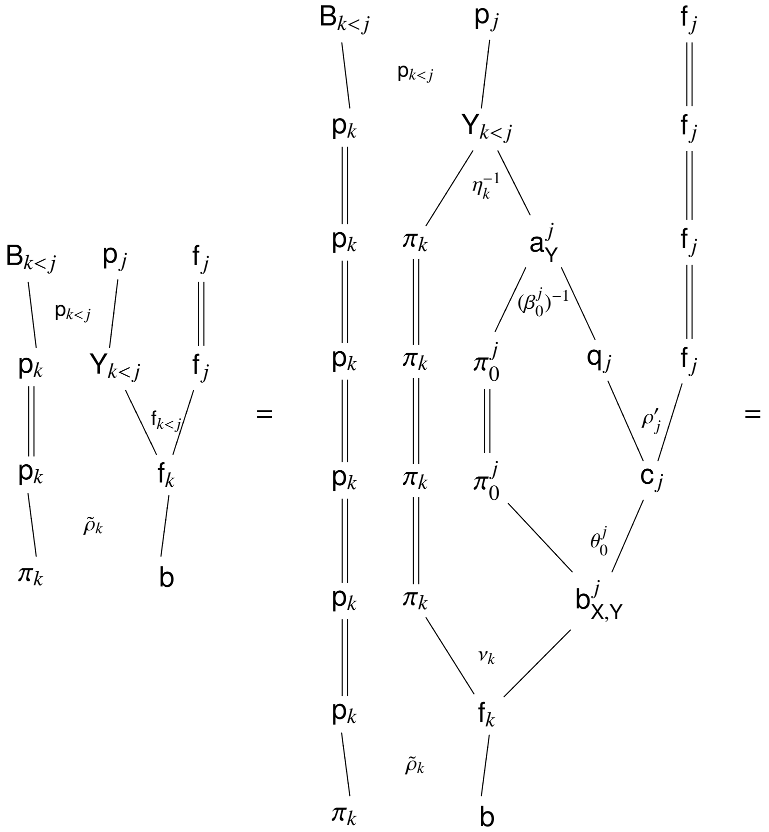
where the first equality is due to the definition of $f_{k<j}$, the second one is due to elevators

calculus plus the fact that $(\mathsf{f}_j, \tilde{\lambda}_j, \rho'_j)$ is a filler for diagram (5.1.13) and the last one is due to the definition of $\tilde{\gamma}_j$.

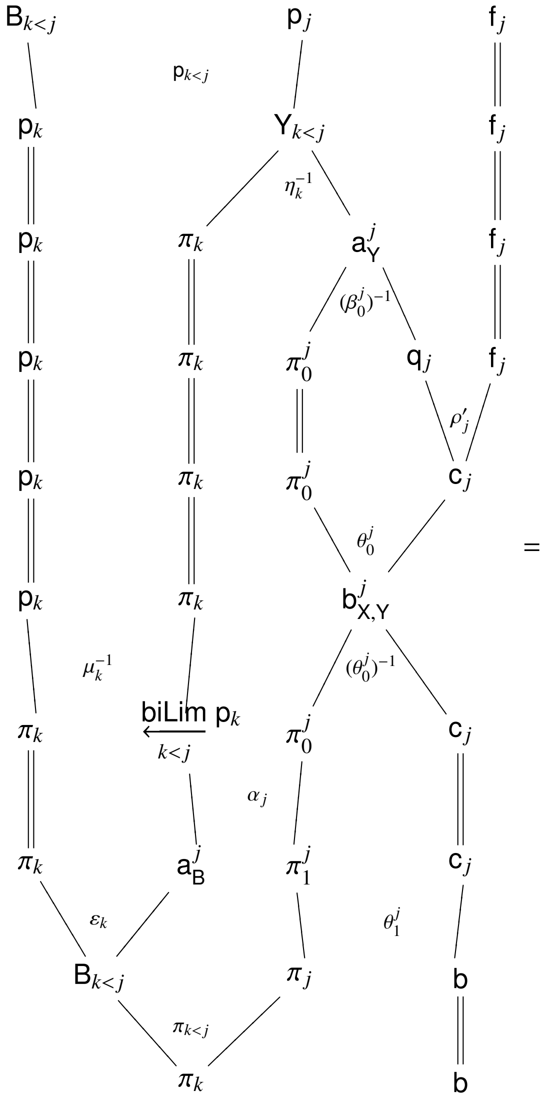
To check that $\left\{\mathsf{p}_k\mathsf{f}_k \overset{\tilde{\rho}_k}{\Longrightarrow} \pi_k\mathsf{b}\right\}_{k<m}$ is a morphism of pseudo-cones, we need to verify that the following equality holds $\forall\ k < j < m$:



But

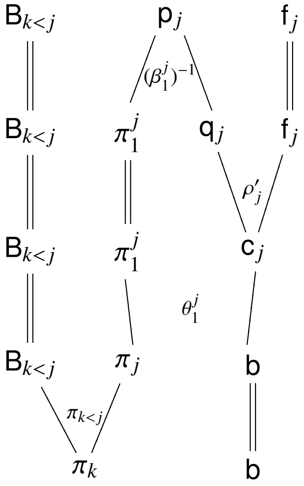

B_{k<j}
p_j
f_j
p_{k<j}
p_k
Y_{k<j}
f_{k<j}
f_k
ρ̃_k
π_k
b
=
η_k^{-1}
a_Y^j
(β_0^j)^{-1}
π_0^j
q_j
ρ'_j
c_j
θ_0^j
b_{X,Y}^j
ν_k
=

=



$B_{k<j}$
p_j
f_j
$p_{k<j}$
p_k
$Y_{k<j}$
η_k^{-1}
π_k
a^j_Y
$(\beta^j_0)^{-1}$
π^j_0
q_j
ρ'_j
c_j
θ^j_0
$b^j_{X,Y}$
μ_k^{-1}
$(\theta^j_0)^{-1}$
$\operatorname{biLim}_{\overleftarrow{k<j}} p_k$
α_j
a^j_B
π^j_1
θ^j_1
ε_k
$B_{k<j}$
π_j
b
$\pi_{k<j}$
π_k

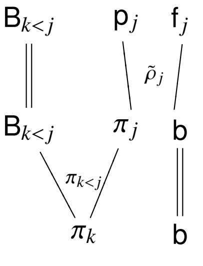
=



$B_{k<j}$
p_j
f_j
$(\beta^j_1)^{-1}$
π^j_1
q_j
ρ'_j
c_j
θ^j_1
π_j
b
$\pi_{k<j}$
π_k

=

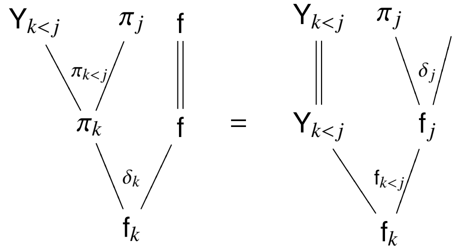
=



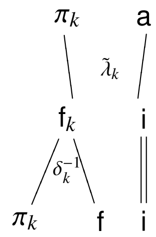
$B_{k<j}$
p_j
f_j
$\tilde\rho_j$
π_j
b
$\pi_{k<j}$
π_k

where the first equality is due to the definition of $\mathsf{f}_{k<j}$, the second one is due to (5.1.12), the third one is due to item *d*) from 5.1.5 and the last one is due to the definition of $\tilde{\rho}_j$.

Then, by the universal property of $\underset{k<m}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{Y}_k$, there exist a morphism $\mathsf{X} \xrightarrow{\mathsf{f}} \underset{k<m}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{Y}_k$ and invertible 2-cells $\pi_k\mathsf{f} \overset{\delta_k}{\Longrightarrow} \mathsf{f}_k \ \forall\ k < m$ such that

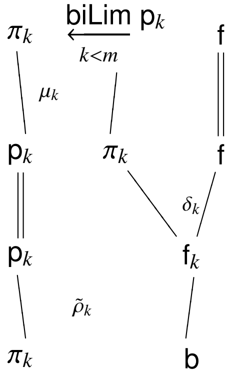


$\forall\ k < j < m$; and there also exist invertible 2-cells $\mathsf{a} \overset{\lambda}{\Longrightarrow} \mathsf{fi}$, $\mathsf{b} \overset{\rho}{\Longrightarrow} \underset{k<m}{\underleftarrow{\mathsf{biLim}}}\ p_k\mathsf{f}$ such that $\pi_k\lambda =$

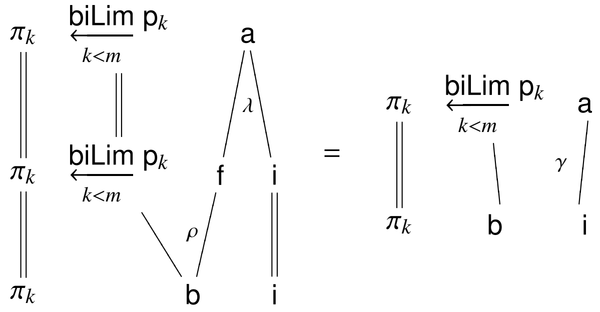


and

$\pi_k\rho =$

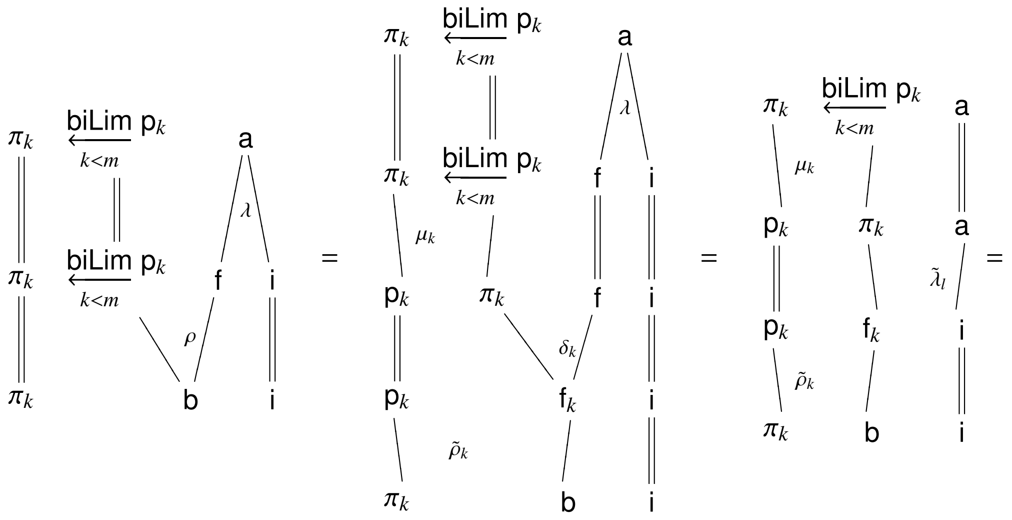


$\forall\ k < m$. To check that $(\mathsf{f}, \lambda, \rho)$ is the filler that we were looking for, it is enough to check that the following equality holds $\forall\ k < m$:

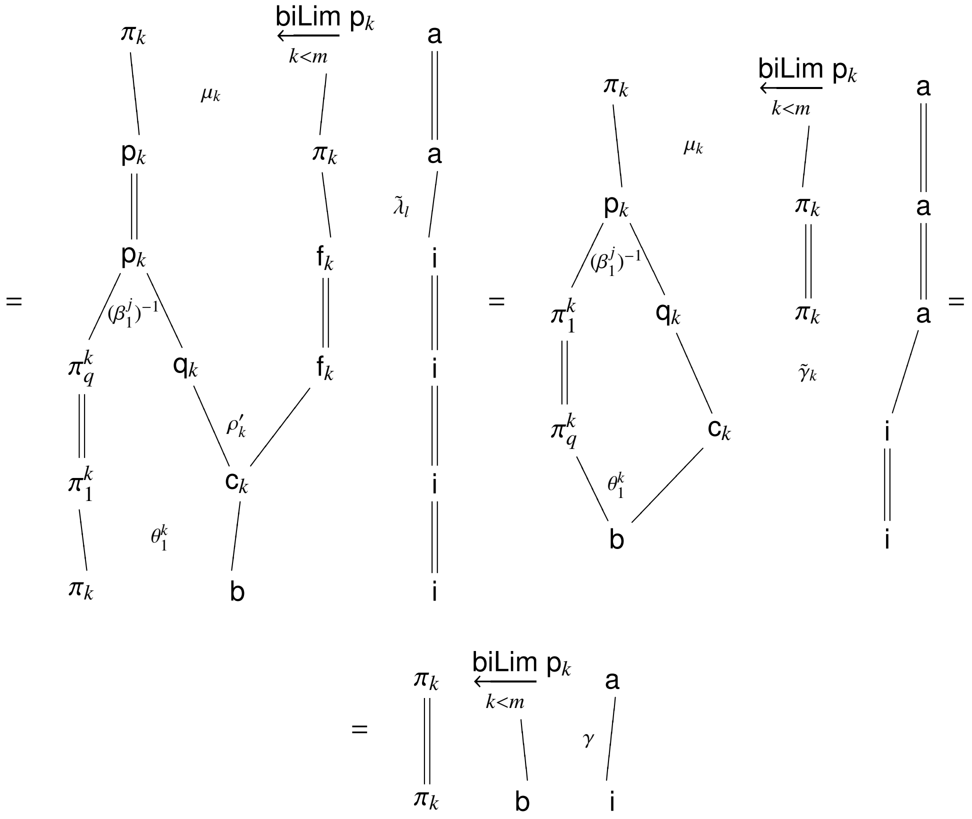


π_k
$\underset{k<m}{\overleftarrow{\text{biLim}}}\ \mathsf{p}_k$
a
λ
f
i
ρ
b
=
γ

But



π_k
$\underset{k<m}{\overleftarrow{\text{biLim}}}\ \mathsf{p}_k$
a
λ
f
i
ρ
b
=
μ_k
p_k
δ_k
f_k
$\tilde{\rho}_k$
$\tilde{\lambda}_l$



where the first equality is due to the definition of ρ, the second one is due to elevators calculus plus the definition of λ, the third one is due to the definition of $\tilde{\rho}_k$, the fourth one is due to elevators calculus plus the fact that $(\mathsf{f}_k, \tilde{\lambda}_k, \rho'_k)$ is a filler for diagram (5.1.13) and the last one is due to the definition of $\tilde{\gamma}_k$.

Finally, since $\mathcal{C}$ is a closed 2-bmodel 2-category, $\mathsf{P}_j \xrightarrow{\pi_1^j} \mathsf{B}_j$ is both a fibration and a weak equivalence. Also, by definition of weak equivalences, p_j is a weak equivalence. So, q_j is also a weak equivalence by axiom 2-M5.

$\Leftarrow$) By 5.1.5, it is clear that p is a fibration. Let's check inductively that p_j is a weak equivalence: If $j = 0$, $\mathsf{p}_0 \cong \mathsf{q}_0$ and so is a weak equivalence. Now suppose that p_k is a weak equivalence $\forall\ k < j$ and consider p as an object of $p\mathcal{H}om_p(\{k \in \mathsf{J} \mid k < j\}^{op}, \mathcal{C})$. Since $\mathsf{p} \in p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ is a fibration, by 5.1.5, $\mathsf{p} \in p\mathcal{H}om_p(\{k \in \mathsf{J} \mid k < j\}^{op}, \mathcal{C})$ is also a fibration. Plus, we know that p is a weak equivalence as an object of $p\mathcal{H}om_p(\{k \in \mathsf{J} \mid k < j\}^{op}, \mathcal{C})$ and then, by 5.1.3 $\underleftarrow{\text{biLim}}_{k<j}\ \mathsf{p}_k \in \mathcal{C}$ is both a fibration and a weak equivalence. Then, since $\mathcal{C}$ is a closed 2-bmodel 2-category, π_1^j is a weak equiv-

alence. We also know that q_j is a weak equivalence and so $\mathsf{p}_j \cong \pi_1^j \mathsf{q}_j \in \mathcal{C}$ is a weak equivalence as we wanted to prove. □

5.1.14 Theorem. *$p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ with the structure provided in 5.1.2 is a closed 2-bmodel 2-category.*

Proof.

Axiom 2-M0b: It is clear from 4.1.6 since bi-limits, bi-colimits, bi-tensors and bi-cotensors in $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ are computed pointwise and therefore exist (see 1.2.8 and 1.2.11). □

Axiom 2-M2: We will first do the case where p is a fibration and i is both a cofibration and a weak equivalence:

It is enough to check that we can factor $\mathsf{p}_j \mathsf{i}_j \overset{\gamma_j \cong}{\Longrightarrow} \mathsf{f}_j \ \forall\ j \in \mathsf{J}$, $\mathsf{i} = \left\{\mathsf{i}_j\right\}_{j \in \mathsf{J}}$ and $\mathsf{p} = \left\{\mathsf{p}_j\right\}_{j \in \mathsf{J}}$ are pseudo-natural transformations, i_j is both a cofibration and a weak equivalence $\forall\ j \in \mathsf{J}$, q_j (associated to p as in 5.1.5) is a fibration $\forall\ j \in \mathsf{J}$ and $\gamma = \left\{\gamma_j\right\}_{j \in \mathsf{J}}$ is a modification. We are going to do this by induction in j:

If j is the initial object of J, since $\mathcal{C}$ is a closed 2-bmodel 2-category, we can factorize f_0 as $\mathsf{p}_0 \mathsf{i}_0 \overset{\gamma_0 \cong}{\Longrightarrow} \mathsf{f}_0$ where i_0 is both a cofibration and a weak equivalence and p_0 is a fibration. As $\mathsf{q}_0 \cong \mathsf{p}_0$, p_0 is a fibration.

Now, suppose that we have already defined p_k, i_k and $\gamma_k \ \forall\ k < j$ and let's define p_j, i_j and γ_j:

Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
\mathsf{X}_j \arrow[rr, "\mathsf{a}_{\mathsf{X}}^j"] \arrow[dr, dashed, "\mathsf{a}_j"] \arrow[ddr, bend right, "\mathsf{f}_j"'] & & \underset{k<j}{\underleftarrow{\mathrm{biLim}}}\ \mathsf{X}_k \arrow[d, "\underset{k<j}{\underleftarrow{\mathrm{biLim}}}\ \mathsf{i}_k"] \arrow[dl, phantom, "\cong \Uparrow \theta_0^j"] \\
\arrow[r, phantom, "\cong \Downarrow \theta_1^j"] & \mathsf{Q}_j \arrow[r, "\mathsf{h}_0^j"] \arrow[d, "\mathsf{h}_1^j"] \arrow[dr, phantom, "\overset{bipb}{\cong \Downarrow \delta_j}"] & \underset{k<j}{\underleftarrow{\mathrm{biLim}}}\ \mathsf{Z}_k \arrow[d, "\underset{k<j}{\underleftarrow{\mathrm{biLim}}}\ \mathsf{p}_k"] \\
& \mathsf{Y}_j \arrow[r, "\mathsf{a}_{\mathsf{Y}}^j"'] & \underset{k<j}{\underleftarrow{\mathrm{biLim}}}\ \mathsf{Y}_k
\end{tikzcd} \tag{5.1.15}$$

where $\underset{k<j}{\underleftarrow{\mathrm{biLim}}}\ \mathsf{p}_k$ is induced by the pseudo-cone $\{\mathsf{p}_k \pi_k\}_{k<j}$, $\left\{ \begin{array}{ccccc} \mathsf{Y}_{k<l} & & \mathsf{p}_l & & \pi_l \\ \backslash & \mathsf{p}_{k<l} & / & & \| \\ \mathsf{p}_k & & \mathsf{Z}_{k<l} & & \pi_l \\ \| & & \backslash & \pi_{k<l} & / \\ \mathsf{p}_k & & & \pi_k & \end{array} \right\}_{k<l<j} \cup$

$\left\{ \begin{array}{ccccc} & \mathsf{Y}_{id_k} & & \mathsf{p}_k & \pi_k \\ \backslash & {\scriptstyle (\alpha_k^{\mathsf{Y}})^{-1}} & / & \| & \| \\ & id_{\mathsf{Y}_k} & & \mathsf{p}_k & \pi_k \\ & \backslash & & = & / \\ & & \mathsf{p}_k & & \pi_k \end{array} \right\}_{k<j}$, $\underset{k<j}{\underleftarrow{\mathsf{biLim}}}$ i_k is induced by the pseudo-cone $\{\mathsf{i}_k\pi_k\}_{k<j}$,

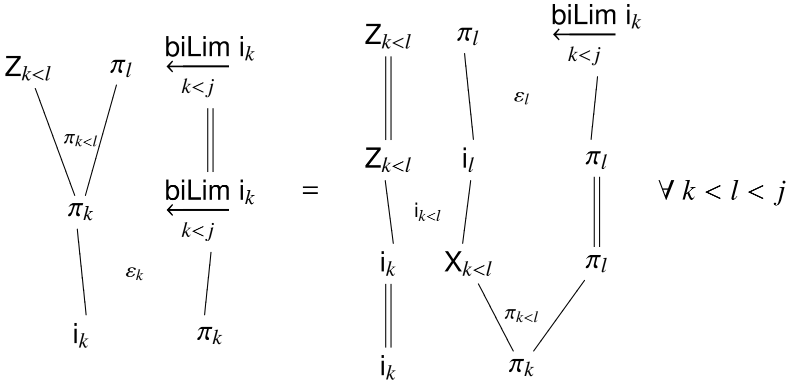
$\left\{ \begin{array}{cccc} \mathsf{Z}_{k<l} & \mathsf{i}_l & & \pi_l \\ \backslash & {\scriptstyle \mathsf{i}_{k<l}} & / & \| \\ \mathsf{i}_k & \mathsf{X}_{k<l} & & \pi_l \\ \| & \backslash & {\scriptstyle \pi_{k<l}} & / \\ \mathsf{i}_k & & \pi_k & \end{array} \right\}_{k<l<j} \cup \left\{ \begin{array}{ccccc} & \mathsf{Z}_{id_k} & & \mathsf{i}_k & \pi_k \\ \backslash & {\scriptstyle (\alpha_k^{\mathsf{Z}})^{-1}} & / & \| & \| \\ & id_{\mathsf{Z}_k} & & \mathsf{i}_k & \pi_k \\ & \backslash & & = & / \\ & & \mathsf{i}_k & & \pi_k \end{array} \right\}_{k<j}$, $\mathsf{a}_{\mathsf{X}}^{j}$ is induced by the pseudo-cone

$\left\{\mathsf{X}_{k<j}\right\}_{k<j}$, $\left\{ \begin{array}{ccc} \mathsf{X}_{k<l} & & \mathsf{X}_{l<j} \\ \backslash & {\scriptstyle \alpha_{k,l,j}^{\mathsf{X}}} & / \\ & \mathsf{X}_{k<j} & \end{array} \right\}_{k<l<j} \cup \left\{ \begin{array}{cccc} & \mathsf{X}_{id_k} & & \mathsf{X}_{k<j} \\ \backslash & {\scriptstyle (\alpha_k^{\mathsf{X}})^{-1}} & / & \| \\ & id_{\mathsf{X}_k} & & \mathsf{X}_{k<j} \\ & \backslash & = & / \\ & & \mathsf{X}_{k<j} & \end{array} \right\}_{k<j}$, $\mathsf{a}_{\mathsf{Y}}^{j}$ is induced by the pseudo-cone $\left\{\mathsf{Y}_{k<j}\right\}_{k<j}$, $\left\{ \begin{array}{ccc} \mathsf{Y}_{k<l} & & \mathsf{Y}_{l<j} \\ \backslash & {\scriptstyle \alpha_{k,l,j}^{\mathsf{Y}}} & / \\ & \mathsf{Y}_{k<j} & \end{array} \right\}_{k<l<j} \cup \left\{ \begin{array}{cccc} & \mathsf{Y}_{id_k} & & \mathsf{Y}_{k<j} \\ \backslash & {\scriptstyle (\alpha_k^{\mathsf{Y}})^{-1}} & / & \| \\ & id_{\mathsf{Y}_k} & & \mathsf{Y}_{k<j} \\ & \backslash & = & / \\ & & \mathsf{Y}_{k<j} & \end{array} \right\}_{k<j}$, and so we have:

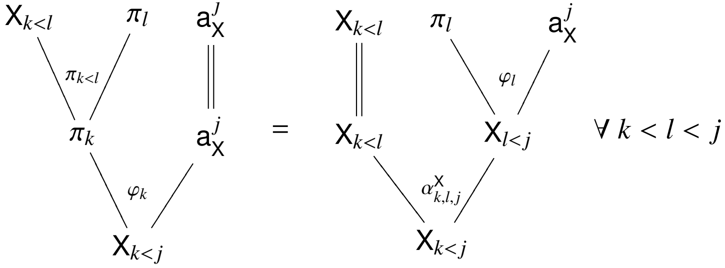
a) invertible 2-cells $\pi_k \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k \overset{\mu_k}{\Longrightarrow} \mathsf{p}_k\pi_k\ \forall\ k < j$ such that

$$\begin{array}{ccc|c} \mathsf{Y}_{k<l} & & \pi_l & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k \\ \backslash & {\scriptstyle \pi_{k<l}} & / & \| \\ & \pi_k & & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k \\ & | & {\scriptstyle \mu_k} & | \\ & \mathsf{p}_k & & \pi_k \end{array} \quad = \quad \begin{array}{ccccc} \mathsf{Y}_{k<l} & & \pi_l & & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k \\ \| & & | & {\scriptstyle \mu_l} & | \\ \mathsf{Y}_{k<l} & & \mathsf{p}_l & & \pi_l \\ | & {\scriptstyle \mathsf{p}_{k<l}} & | & & \| \\ \mathsf{p}_k & & \mathsf{Z}_{k<l} & & \pi_l \\ \| & & \backslash & {\scriptstyle \pi_{k<l}} & / \\ \mathsf{p}_k & & & \pi_k & \end{array} \qquad \forall\ k < l < j$$

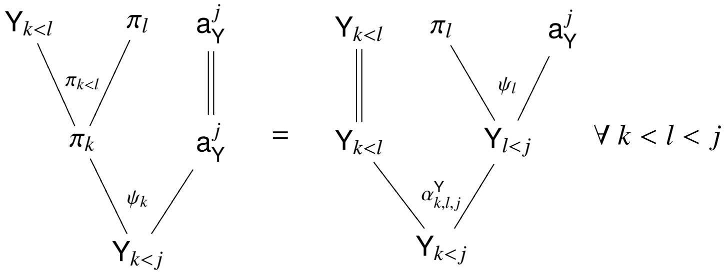
b) invertible 2-cells $\pi_k \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{i}_k \overset{\epsilon_k}{\Longrightarrow} \mathsf{i}_k\pi_k\ \forall\ k < j$ such that



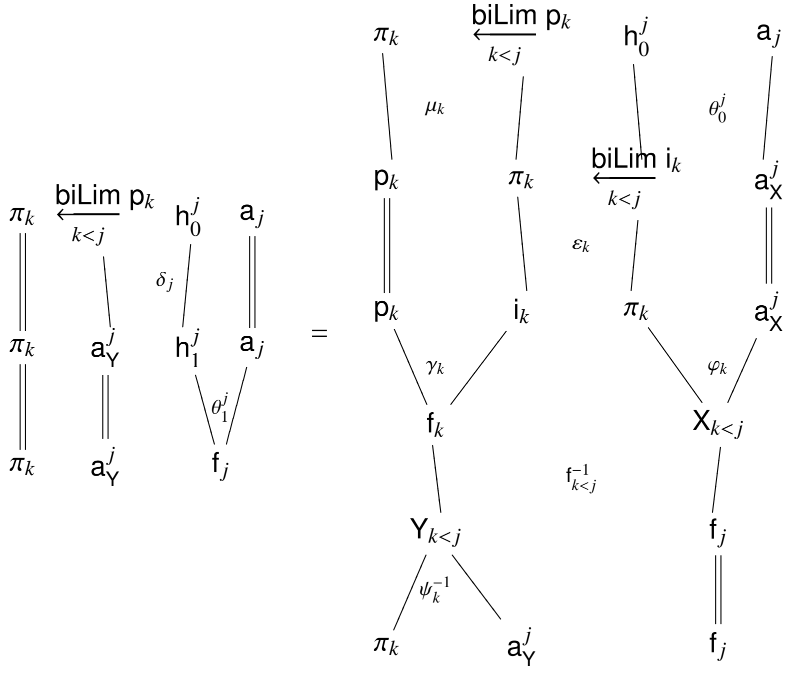
c) invertible 2-cells $\pi_k a^j_X \overset{\varphi_k}{\Longrightarrow} X_{k<j}$ $\forall$ $k < j$ such that



d) invertible 2-cells $\pi_k a^j_Y \overset{\psi_k}{\Longrightarrow} Y_{k<j}$ $\forall$ $k < j$ such that



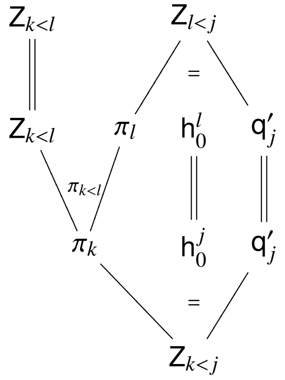
e)



$\forall\, k < j$

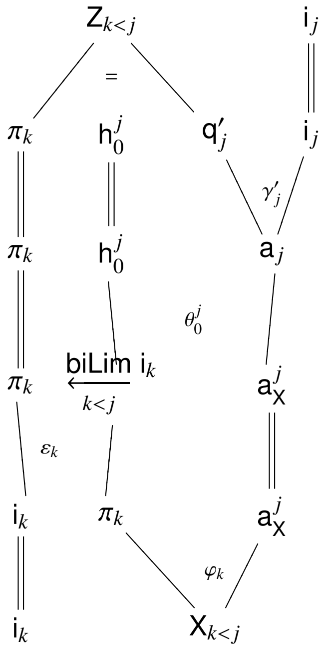
Since C is a closed 2-bmodel 2-category, a_j can be factored as $q'_j i_j \overset{\gamma'_j \cong}{\Longrightarrow} a_j$ where i_j is both a cofibration and a weak equivalence and q'_j is a fibration. Consider $p_j = h_1^j q'_j$ and $\gamma_j = h_1^j \gamma'_j$. Let's check that this data satisfies the desired properties:

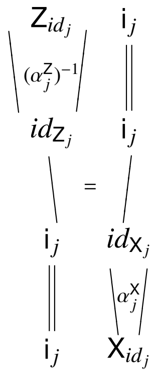
It can be easily checked that Z is a pseudo-functor ($Z_{k<j} = \pi_k h_0^j q'_j$, $\alpha_j^Z = id_{Z_j}$, $\alpha_{k,l,j}^Z =$



).

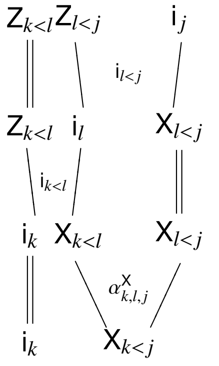
We are now going to prove that i is pseudo-natural:

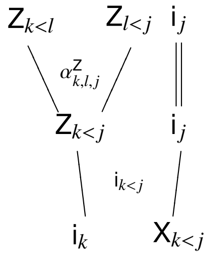
We define $\mathsf{i}_{k<j}$ = 

and i_{id_j} = 

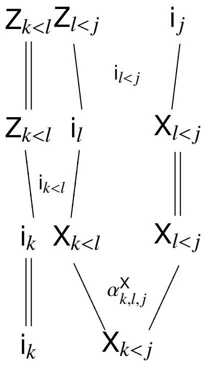
.

PN0 is satisfied by definition and PN2 is vacuous because there are no 2-cells in J, so we only need to check PN1: consider $k < l < j$, we want to check that the following equality holds:



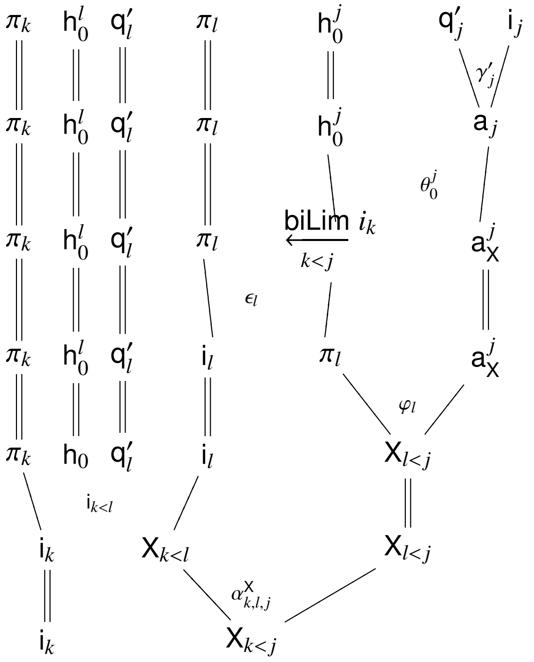
=


But



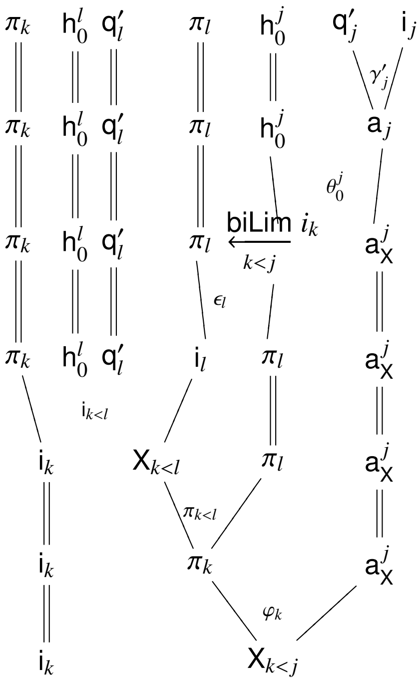
Z_{k<l} Z_{l<j} i_j
i_{l<j}
Z_{k<l} i_l X_{l<j}
i_{k<l}
i_k X_{k<l} X_{l<j}
α^X_{k,l,j}
i_k X_{k<j}

=



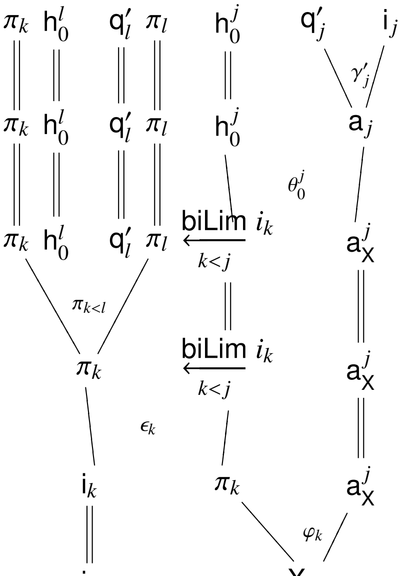
π_k h_0^l q′_l π_l h_0^j q′_j i_j
γ′_j
π_k h_0^l q′_l π_l h_0^j a_j
θ_0^j
π_k h_0^l q′_l π_l biLim i_k k<j a_X^j
ε_l
π_k h_0^l q′_l i_l π_l a_X^j
φ_l
π_k h_0 q′_l i_l X_{l<j}
i_{k<l}
i_k X_{k<l} X_{l<j}
α^X_{k,l,j}
i_k X_{k<j}

=

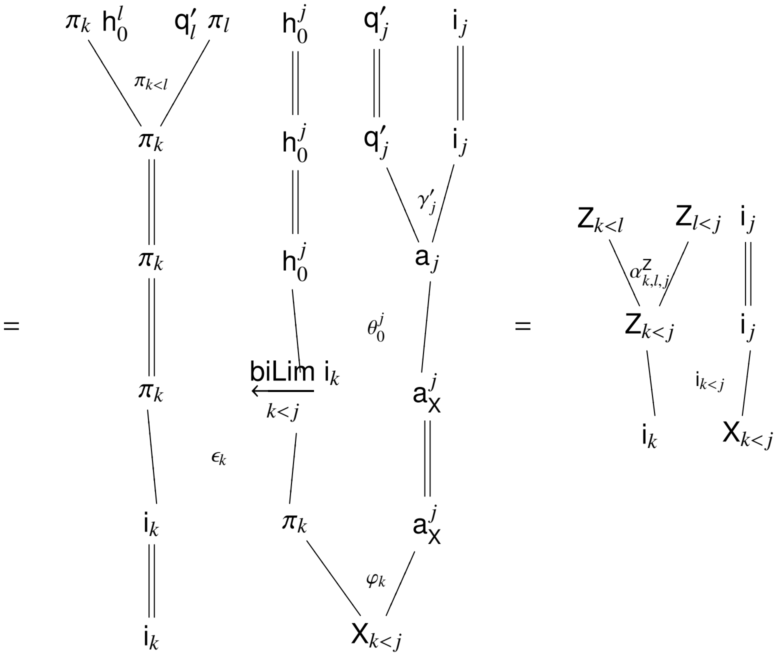


π_k h_0^l q′_l π_l h_0^j q′_j i_j
γ′_j
π_k h_0^l q′_l π_l h_0^j a_j
θ_0^j
π_k h_0^l q′_l π_l biLim i_k k<j a_X^j
ε_l
π_k h_0^l q′_l i_l π_l a_X^j
i_{k<l}
i_k X_{k<l} π_l a_X^j
π_{k<l}
i_k π_k a_X^j
φ_k
i_k X_{k<j}

=

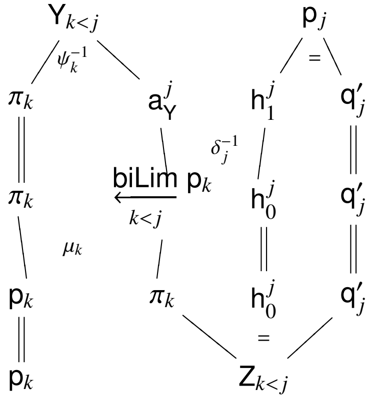


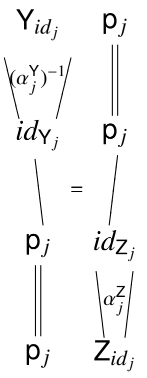
π_k h_0^l q′_l π_l h_0^j q′_j i_j
γ′_j
π_k h_0^l q′_l π_l h_0^j a_j
θ_0^j
π_k h_0^l q′_l π_l biLim i_k k<j a_X^j
π_{k<l}
π_k biLim i_k k<j a_X^j
ε_k
i_k π_k a_X^j
φ_k
i_k X_{k<j}



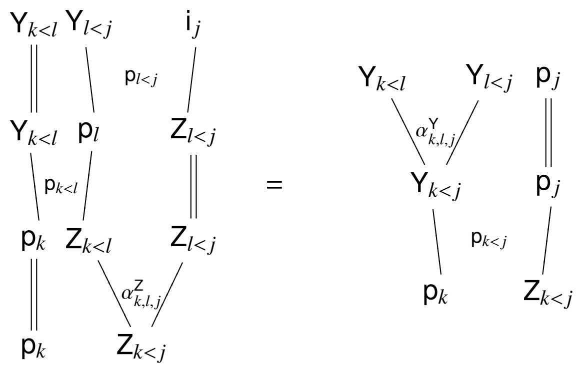
where the second equality is due to the elevators calculus plus c), the third one is due to b) and the fourth one is due to elevators calculus again.

Now we are going to prove that p is pseudo-natural:

We define $\mathsf{p}_{k<j}$ = 

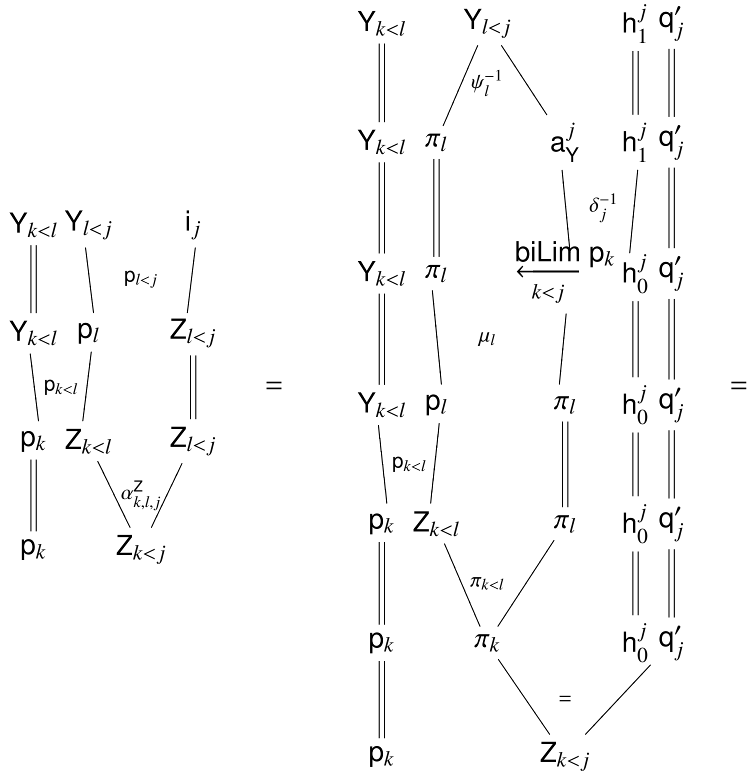
and p_{id_j} = 

. PN0 is satisfied by definition and PN2 is vacuous because there are no 2-cells in J, so we only need to check PN1: consider $k < l < j$, we want to check that the following equality holds:

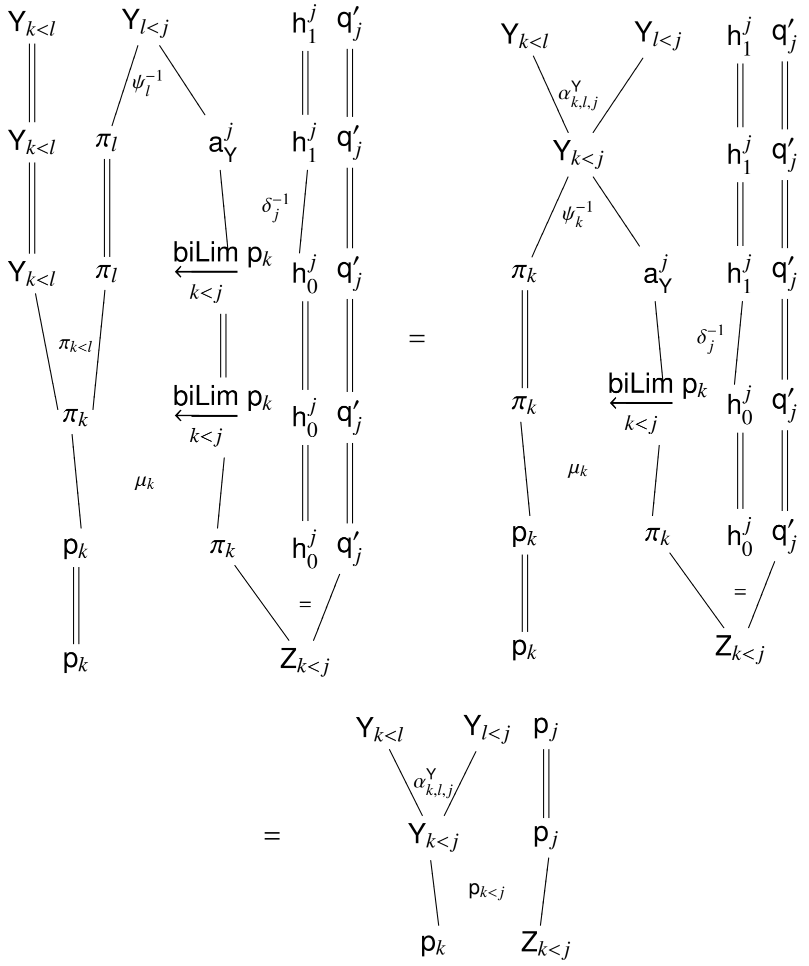


$\mathsf{Y}_{k<l}$
$\mathsf{Y}_{l<j}$
i_j
$\mathsf{p}_{l<j}$
$\mathsf{Y}_{k<l}$
p_l
$\mathsf{Z}_{l<j}$
$\mathsf{p}_{k<l}$
p_k
$\mathsf{Z}_{k<l}$
$\mathsf{Z}_{l<j}$
$\alpha^{\mathsf{Z}}_{k,l,j}$
p_k
$\mathsf{Z}_{k<j}$
=
$\mathsf{Y}_{k<l}$
$\mathsf{Y}_{l<j}$
p_j
$\alpha^{\mathsf{Y}}_{k,l,j}$
$\mathsf{Y}_{k<j}$
p_j
$\mathsf{p}_{k<j}$
p_k
$\mathsf{Z}_{k<j}$

But

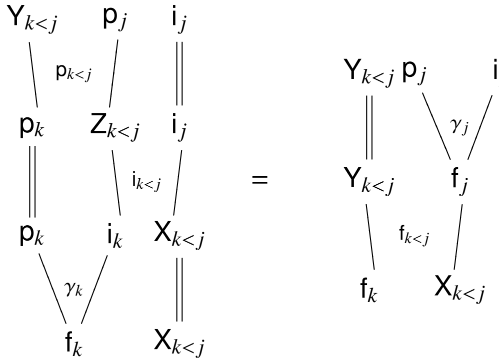


$\mathsf{Y}_{k<l}$
$\mathsf{Y}_{l<j}$
i_j
$\mathsf{p}_{l<j}$
$\mathsf{Y}_{k<l}$
p_l
$\mathsf{Z}_{l<j}$
$\mathsf{p}_{k<l}$
p_k
$\mathsf{Z}_{k<l}$
$\mathsf{Z}_{l<j}$
$\alpha^{\mathsf{Z}}_{k,l,j}$
p_k
$\mathsf{Z}_{k<j}$
=
$\mathsf{Y}_{k<l}$
$\mathsf{Y}_{l<j}$
h^j_1
q'_j
ψ_l^{-1}
$\mathsf{Y}_{k<l}$
π_l
$\mathsf{a}^j_{\mathsf{Y}}$
h^j_1
q'_j
δ_j^{-1}
$\mathsf{Y}_{k<l}$
π_l
$\underset{k<j}{\xleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k$
h^j_0
q'_j
μ_l
$\mathsf{Y}_{k<l}$
p_l
π_l
h^j_0
q'_j
$\mathsf{p}_{k<l}$
p_k
$\mathsf{Z}_{k<l}$
π_l
h^j_0
q'_j
$\pi_{k<l}$
p_k
π_k
h^j_0
q'_j
=
p_k
$\mathsf{Z}_{k<j}$
=

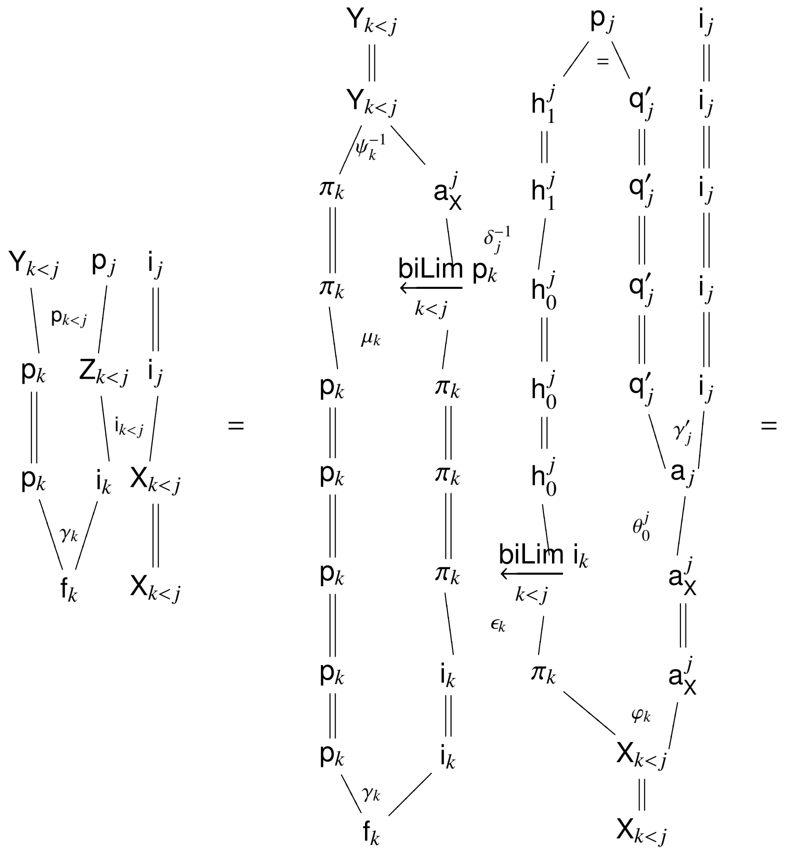


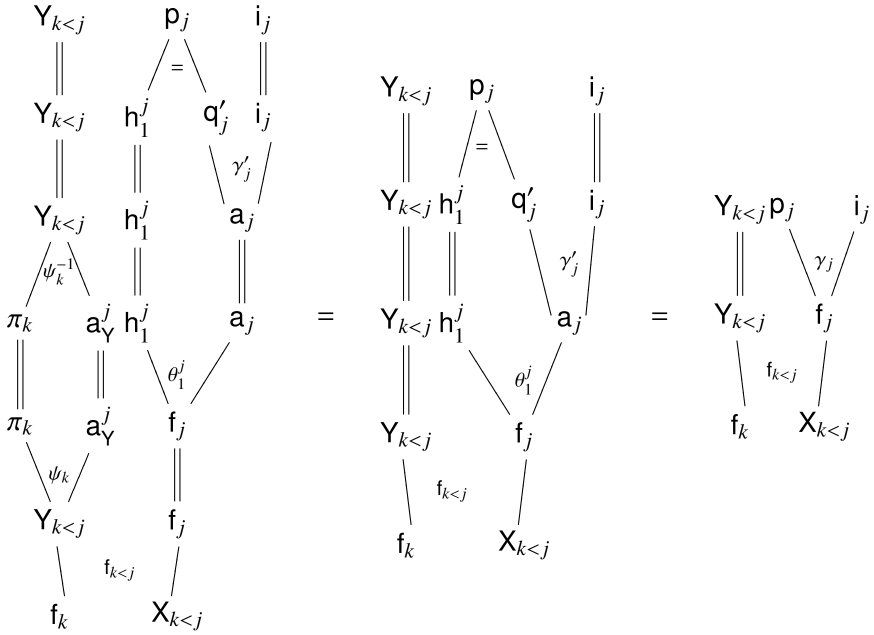
where the second equality is due to *a*) and the third one is due to elevators calculus plus *d*).

Now we are going to prove that γ is a modification: Consider $k < j$, we want to check that the following equality holds:


Y_{k<j}
p_j
i_j
p_{k<j}
p_k
Z_{k<j}
i_j
i_{k<j}
p_k
i_k
X_{k<j}
γ_k
f_k
X_{k<j}
=
Y_{k<j}
p_j
i_j
γ_j
Y_{k<j}
f_j
f_{k<j}
f_k
X_{k<j}

But


Y_{k<j}
p_j
i_j
p_{k<j}
p_k
Z_{k<j}
i_j
i_{k<j}
p_k
i_k
X_{k<j}
γ_k
f_k
X_{k<j}
=
Y_{k<j}
Y_{k<j}
ψ_k^{-1}
π_k
a_X^j
p_j
=
h_1^j
q'_j
i_j
h_1^j
q'_j
i_j
δ_j^{-1}
π_k
biLim k<j
p_k
h_0^j
q'_j
i_j
μ_k
p_k
π_k
h_0^j
q'_j
i_j
γ'_j
p_k
π_k
h_0^j
a_j
θ_0^j
p_k
π_k
biLim k<j
i_k
a_X^j
ε_k
p_k
i_k
π_k
a_X^j
φ_k
p_k
i_k
X_{k<j}
γ_k
f_k
X_{k<j}
=



where the second equality is due to elevators calculus plus *e*) and the third one is due to elevators calculus.

i_j is both a cofibration and a weak equivalence by construction.

It can be easily checked that the q_j associated to p is q'_j and so is a fibration.

Now we will focus on the case where p is both a fibration and a weak equivalence and i is a cofibration:

It is enough to check that we can factor $p_j i_j \overset{\gamma_j \cong}{\Longrightarrow} f_j$ $\forall\ j \in J$, $i = \left\{i_j\right\}_{j \in J}$ and $p = \left\{p_j\right\}_{j \in J}$ are pseudo-natural transformations, i_j is a cofibration $\forall\ j \in J$, q_j (associated to p as in 5.1.5) is a fibration $\forall\ j \in J$, p_j is a weak equivalence $\forall\ j \in J$ and $\gamma = \left\{\gamma_j\right\}_{j \in J}$ is a modification. We are going to do this by induction in j:

If j is the initial object of J, since C is a closed 2-bmodel 2-category, we have $p_0 i_0 \overset{\gamma_0 \cong}{\Longrightarrow} f_0$ where i_0 is a cofibration and p_0 is a both a fibration and a weak equivalence. As $q_0 \cong p_0$, q_0 is a fibration.

Now, suppose that we have already defined p_k, i_k and γ_k $\forall\ k < j$. In order to define p_j, i_j and γ_j, consider diagram (5.1.15) as before:

Since C is a closed 2-bmodel 2-category, a_j may be factored as $q'_j i_j \overset{\gamma'_j \cong}{\Longrightarrow} a_j$ where i_j is a cofibration and q'_j is both a fibration and a weak equivalence. Consider $p_j = h_1^j q'_j$ and

$$\gamma_j = \begin{tikzcd}
 & \mathsf{p}_j \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "="] & & \mathsf{i}_j \arrow[d, equal] \\
\mathsf{h}_1^j \arrow[d, equal] & {} & \mathsf{q}'_j \arrow[d, dash] \arrow[d, phantom, shift left=3, "\gamma'_j"] & \mathsf{i}_j \arrow[dl, dash] \\
\mathsf{h}_1^j \arrow[dr, dash] & {} \arrow[d, phantom, "\theta_1^j"] & \mathsf{a}_j \arrow[dl, dash] & \\
 & \mathsf{f}_j & &
\end{tikzcd}$$

. Let's check that this data satisfies the desired properties:

One can check as before that Z is a pseudo-functor, i is pseudo-natural, p is pseudo-natural and γ is a modification.

i_j is a cofibration by construction.

As before, q'_j is the q_j associated to p and it is a fibration.

It only remains to check that p_j is a weak equivalence: Since p is a fibration, by 5.1.9, p_j is a fibration $\forall\ j \in \mathsf{J}$. Then p_k is both a fibration and a weak equivalence $\forall\ k < j$ and so, by 5.1.3, $\underset{\underset{k<j}{\longleftarrow}}{\mathsf{biLim}}\ \mathsf{p}_k$ is both a fibration and a weak equivalence. Therefore, since 2-M4 is satisfied in C, h_1^j is a weak equivalence. Then p_j is a weak equivalence by axiom 2-M5.

Axiom 2-M5: It follows from the fact that weak equivalences in $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ are defined pointwise and C is a closed 2-bmodel 2-category.

Axiom 2-M6a): It is tautological from the definition of fibrations in $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$.

Axiom 2-M6b): $\Rightarrow$) Suppose that we have a cofibration i, a fibration p which is also a weak equivalence and they fit in a diagram

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{B}
\end{tikzcd}$$

The filler can be constructed inductively exactly as in the proof of 5.1.5 $\Leftarrow$.

$\Leftarrow$) Suppose that $\mathsf{A} \xrightarrow{\mathsf{i}} \mathsf{X}$ has the left lifting property with respect to all morphisms that are both fibrations and weak equivalences. We have to check that i_j is a cofibration $\forall\ j \in \mathsf{J}$ but, since C is a closed 2-bmodel 2-category, it is enough to check that i_j has the left lifting property with respect to all morphisms that are both fibrations and weak equivalences. So take a morphism $\mathsf{Y} \xrightarrow{\tilde{\mathsf{p}}} \mathsf{B} \in C$ which is both a fibration and a weak equivalence and suppose that we have a diagram of the form

$$\begin{tikzcd}
\mathsf{A}_j \arrow[r, "\tilde{\mathsf{a}}"] \arrow[d, "\mathsf{i}_j"'] \arrow[dr, phantom, "\cong \Downarrow \tilde{\gamma}"] & \mathsf{Y} \arrow[d, "\tilde{\mathsf{p}}"] \\
\mathsf{X}_j \arrow[r, "\tilde{\mathsf{b}}"'] & \mathsf{B}
\end{tikzcd}$$

Let's define $\mathsf{E}, \mathsf{D}, \mathsf{E} \xrightarrow{\mathsf{p}} \mathsf{D}, \mathsf{A} \xrightarrow{\mathsf{a}} \mathsf{E}, \mathsf{X} \xrightarrow{\mathsf{b}} \mathsf{D} \in p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ by:

$$\mathsf{E}_k = \begin{cases} \mathsf{Y} & \text{if } k \geq j \\ * & \text{otherwise} \end{cases}, \qquad \mathsf{E}_{id_k} = id_{\mathsf{E}_k}, \qquad \mathsf{E}_{k<l} = \begin{cases} id_{\mathsf{Y}} & \text{if } k \geq j \\ \mathsf{Y} \longrightarrow * & \text{if } k \ngeq j \text{ and } l \geq j \\ id_* & \text{otherwise} \end{cases},$$

$$\mathsf{D}_k = \begin{cases} \mathsf{B} & \text{if } k \geq j \\ * & \text{otherwise} \end{cases}, \qquad \mathsf{D}_{id_k} = id_{\mathsf{D}_k}, \qquad \mathsf{D}_{k<l} = \begin{cases} id_{\mathsf{B}} & \text{if } k \geq j \\ \mathsf{B} \longrightarrow * & \text{if } k \ngeq j \text{ and } l \geq j \\ id_* & \text{otherwise} \end{cases},$$

$$\mathsf{p}_k = \begin{cases} \tilde{\mathsf{p}} & \text{if } k \geq j \\ id_* & \text{otherwise} \end{cases}, \quad \mathsf{p}_{id_k} = \begin{cases} id_{\tilde{\mathsf{p}}} & \text{if } k \geq j \\ id_{id_*} & \text{otherwise} \end{cases}, \quad \mathsf{p}_{k<l} = \begin{cases} id_{\tilde{\mathsf{p}}} & \text{if } k \geq j \\ id_{\mathsf{Y} \longrightarrow *} & \text{if } k \ngeq j \text{ and } l \geq j \\ id_{id_*} & \text{otherwise} \end{cases},$$

$$\mathsf{a}_k = \begin{cases} \tilde{\mathsf{a}}\mathsf{A}_{j\leq k} & \text{if } k \geq j \\ \mathsf{A}_k \longrightarrow * & \text{otherwise} \end{cases}, \qquad \mathsf{a}_{id_k} = \begin{cases} \tilde{\mathsf{a}}\mathsf{A}_{j\leq k}\alpha^{\mathsf{A}}_k & \text{if } k \geq j \\ id_{\mathsf{A}_k \longrightarrow *} & \text{otherwise} \end{cases},$$

$$\mathsf{a}_{k<l} = \begin{cases} \tilde{\mathsf{a}}(\alpha^{\mathsf{A}}_{j,k,l})^{-1} & \text{if } k \geq j \\ id_{\mathsf{A}_l \longrightarrow *} & \text{otherwise} \end{cases}, \qquad \mathsf{b}_k = \begin{cases} \tilde{\mathsf{b}}\mathsf{X}_{j\leq k} & \text{if } k \geq j \\ \mathsf{X}_k \longrightarrow * & \text{otherwise} \end{cases}$$

$$\mathsf{b}_{id_k} = \begin{cases} \tilde{\mathsf{b}}\mathsf{X}_{j\leq k}\alpha^{\mathsf{X}}_k & \text{if } k \geq j \\ id_{\mathsf{X}_k \longrightarrow *} & \text{otherwise} \end{cases}$$

and $\mathsf{b}_{k<l} = \begin{cases} \tilde{\mathsf{b}}(\alpha^{\mathsf{X}}_{j,k,l})^{-1} & \text{if } k \geq j \\ id_{\mathsf{X}_l \longrightarrow *} & \text{otherwise} \end{cases}.$

It is straightforward to check that E and D are 2-functors and that p, a and b are pseudo-natural transformations.

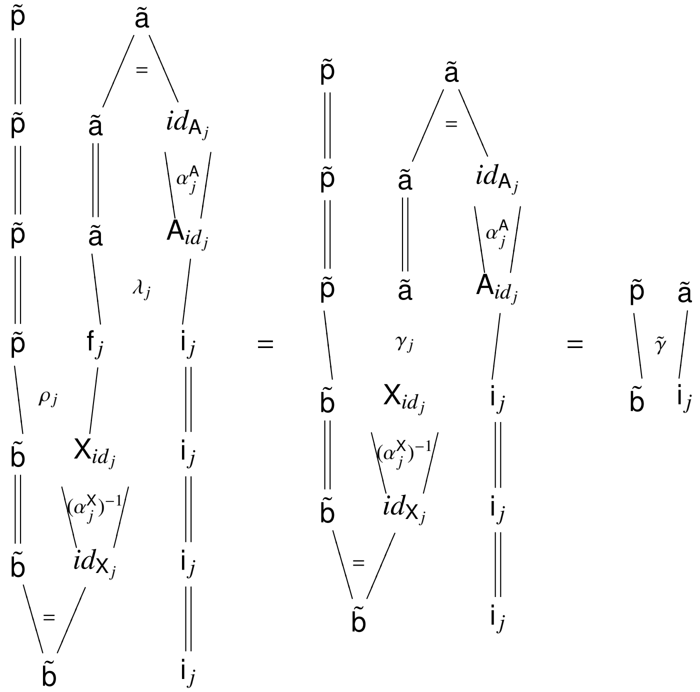
By using 5.1.10, p is both a fibration and a weak equivalence and then there exists a filler $(\mathsf{f},\lambda,\rho)$ for the following diagram

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{E} \arrow[d, "\mathsf{p}"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{D}
\end{tikzcd} \tag{5.1.16}$$

$$\text{where } \gamma_k = \begin{cases} \begin{tikzcd}[row sep=small]
\tilde{\mathsf{p}} \arrow[d, dash] \arrow[r, phantom, "\tilde{\gamma}"] & \tilde{\mathsf{a}} \arrow[d, dash] & \mathsf{A}_{j\leq k} \arrow[d, equal] \\
\tilde{\mathsf{b}} \arrow[d, equal] & \mathsf{i}_j \arrow[d, dash] \arrow[r, phantom, "\mathsf{i}^{-1}_{j\leq k}"] & \mathsf{A}_{j\leq k} \arrow[d, dash] \\
\tilde{\mathsf{b}} & \mathsf{X}_{j\leq k} & \mathsf{i}_k
\end{tikzcd} & \text{if } k \geq j \\ id_{\mathsf{A}_k \longrightarrow *} & \text{otherwise} \end{cases} .$$

$$\text{Consider } \tilde{\lambda} = \begin{tikzcd}[row sep=small]
& \tilde{\mathsf{a}} \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "="] & \\
\tilde{\mathsf{a}} \arrow[d, equal] & {} & id_{\mathsf{A}_j} \arrow[d, dash, "\alpha^{\mathsf{A}}_j"'] \\
\tilde{\mathsf{a}} \arrow[d, dash] \arrow[rr, phantom, "\lambda_j"] & & \mathsf{A}_{id_j} \arrow[d, dash] \\
\mathsf{f}_j & & \mathsf{i}_j
\end{tikzcd} \text{ and } \tilde{\rho} = \begin{tikzcd}[row sep=small]
\tilde{\mathsf{p}} \arrow[d, dash] \arrow[rr, phantom, "\rho_j"] & & \mathsf{f}_j \arrow[d, dash] \\
\tilde{\mathsf{b}} \arrow[d, equal] & & \mathsf{X}_{id_j} \arrow[d, dash, "(\alpha^{\mathsf{X}}_j)^{-1}"'] \\
\tilde{\mathsf{b}} \arrow[dr, dash] & {} \arrow[u, phantom] & id_{\mathsf{X}_j} \arrow[dl, dash] \\
& \tilde{\mathsf{b}} \arrow[u, phantom, "="] &
\end{tikzcd}$$

Let's check that $(\mathsf{f}_j, \lambda_j, \rho_j)$ is the filler that we were looking for:



where the first equality is due to the fact that $(\mathsf{f}, \lambda, \rho)$ is a filler for diagram (5.1.16) and the last one is due to elevators calculus plus the definition of γ_j and the fact that i is a pseudo-natural transformation.

Axiom 2-M6c): ⇒) Let f be a weak equivalence. By axiom 2-M2, f can be factored as $\mathsf{f} \cong \mathsf{uv}$ where u is both a fibration and a weak equivalence and v is a cofibration. Since f_j is a weak equivalence $\forall\ j \in \mathsf{J}$ and C is a closed 2-bmodel 2-category, $\mathsf{u}_j\mathsf{v}_j$ is a weak equivalence $\forall\ j \in \mathsf{J}$ and so uv is a weak equivalence. Then, by axiom 2-M5, v is also a weak equivalence. This plus axioms 2-M6a) and 2-M6b) conclude the proof.

⇐) By an argument similar to the one used in the proof of ⇒), it is enough to check that uv is a weak equivalence. And to do that, it is enough to check that u and v are both weak equivalences. We are going to do the proof for v (the proof for u is analogous but easier): By definition, we want to check that v_j is a weak equivalence $\forall\ j \in \mathsf{J}$ and, by 4.1.8 and the fact that C is a closed 2-bmodel 2-category, it is enough to check that it has the left lifting property with respect to all fibrations. So suppose that we have a diagram of the form

$$\begin{tikzcd}
\mathsf{X}_j \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{v}_j"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{A} \arrow[d, "\mathsf{p}"] \\
\mathsf{Y}_j \arrow[r, "\mathsf{b}"'] & \mathsf{B}
\end{tikzcd}$$

where p is a fibration.

The proof follows by an argument exactly as the one used in the proof of axiom 2-M6b) $\Leftarrow$.

□

5.1.17 Corollary. *Let C be a closed 2-bmodel 2-category. Then $\mathcal{H}om_p(\hat{\mathsf{J}}^{op}, C)$ is a closed 2-bmodel 2-category.*

Proof. It follows immediately from 5.1.14 and 1.4.5. □

It is worth mention that the proof given in this subsection can be easily adapted to the case of closed 2-model 2-categories giving the following also interesting result:

5.1.18 Theorem. *$p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ with the structure provided in 5.1.2 is a closed 2-model 2-category if C is.* □

5.2 Closed 2-bmodel structure in 2-$\mathcal{P}ro(C)$

In order to prove that 2-$\mathcal{P}ro(C)$ is a closed 2-bmodel 2-category, we are going to give first a closed 2-bmodel structure to its retract pseudo-equivalent 2-category 2-$\mathcal{P}ro_p(C)$ (see 2.1.5). As we have already said, we were forced to work with pseudo-natural transformations instead of 2-natural transformations due to the non-strict commutativity of diagrams. We start with the finite completeness and finite cocompleteness aspects of the 2-category 2-$\mathcal{P}ro_p(C)$.

5.2.1 Proposition. *Let J be a filtered category and C a 2-category with finite bi-colimits (respectively bi-limits, bi-tensors and bi-cotensors). Consider $\mathcal{J} = \hat{\mathsf{J}}$ (see 1.4.5). Then the inclusion 2-functor $\mathcal{H}om_p(\mathcal{J}^{op}, C) \xrightarrow{inc}$ 2-$\mathcal{P}ro_p(C)$ preserves finite bi-colimits (respectively bi-limits, bi-tensors and bi-cotensors).*

Proof. We are going to prove the assertion for bi-colimits and bi-tensors, other cases are dual to these ones.

- Let $\{\mathsf{D}_\alpha\}_{\alpha\in\Gamma}$ be a finite diagram in $\mathcal{H}om_p(\mathcal{J}^{op},\mathcal{C})$. We have to check that $inc(\underset{\alpha\in\Gamma}{\underrightarrow{\mathsf{biLim}}}\ \mathsf{D}_\alpha)$ is the bi-colimit $\underset{\alpha\in\Gamma}{\underrightarrow{\mathsf{biLim}}}\ inc(\mathsf{D}_\alpha)$ in $2\text{-}\mathcal{P}ro_p(\mathcal{C})$ i.e. that $\forall\ \mathsf{Y} = \{\mathsf{Y}_i\}_{i\in\mathcal{I}} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$,

$$2\text{-}\mathcal{P}ro_p(\mathcal{C})(inc(\underset{\alpha\in\Gamma}{\underrightarrow{\mathsf{biLim}}}\ \mathsf{D}_\alpha),\mathsf{Y}) \simeq \underset{\alpha\in\Gamma}{\underleftarrow{\mathsf{biLim}}}\ 2\text{-}\mathcal{P}ro_p(\mathcal{C})(inc(\mathsf{D}_\alpha),\mathsf{Y})(\text{see 1.2.4}):$$

We do as follows:

$$2\text{-}\mathcal{P}ro_p(\mathcal{C})(inc(\underset{\alpha\in\Gamma}{\underrightarrow{\mathsf{biLim}}}\ \mathsf{D}_\alpha),\mathsf{Y}) \overset{I}{\simeq} \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}((\underset{\alpha\in\Gamma}{\underrightarrow{\mathsf{biLim}}}\ \mathsf{D}_\alpha)(j),\mathsf{Y}_i) \overset{II}{\simeq}$$

$$\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\underset{\alpha\in\Gamma}{\underrightarrow{\mathsf{biLim}}}\ \mathsf{D}_\alpha j,\mathsf{Y}_i) \overset{III}{\simeq} \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \underset{\alpha\in\Gamma}{\underleftarrow{\mathsf{biLim}}}\ \mathcal{C}(\mathsf{D}_\alpha j,\mathsf{Y}_i) \overset{IV}{\simeq}$$

$$\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{\alpha\in\Gamma}{\underleftarrow{\mathsf{biLim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{D}_\alpha j,\mathsf{Y}_i) \overset{V}{\simeq} \underset{\alpha\in\Gamma}{\underleftarrow{\mathsf{biLim}}}\ \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{D}_\alpha j,\mathsf{Y}_i) \overset{VI}{\simeq}$$

$$\underset{\alpha\in\Gamma}{\underleftarrow{\mathsf{biLim}}}\ 2\text{-}\mathcal{P}ro_p(\mathcal{C})(inc(\mathsf{D}_\alpha),\mathsf{Y})$$

where I is due to 2.1.8, II is due to the fact that bi-colimits in $\mathcal{H}om_p(\mathcal{J}^{op},\mathcal{C})$ are computed pointwise (see 1.4.8), III holds by 1.2.4, IV is true because the 2-filtered bi-colimit and the finite bi-limit of categories can be replaced by equivalent pseudo-colimit and pseudo-limit, and these commute ([5]), V holds because bi-limits are associative up to equivalence and VI is due to 2.1.8 again.

- Let E be a finite category and $\mathsf{F} \in \mathcal{H}om_p(\mathcal{J}^{op},\mathcal{C})$. We denote for simplicity $\tilde{\otimes}_{\mathcal{H}} = \tilde{\otimes}_{\mathcal{H}om_p(\mathcal{J}^{op},\mathcal{C})}$, $\tilde{\otimes}_{\mathcal{P}} = \tilde{\otimes}_{2\text{-}\mathcal{P}ro_p(\mathcal{C})}$. We have to check that $inc(\mathsf{E}\tilde{\otimes}_{\mathcal{H}}\mathsf{F})$ is the bi-tensor $\mathsf{E}\tilde{\otimes}_{\mathcal{P}}inc(\mathsf{F})$, i.e. that $\forall\ \mathsf{Y} = \{\mathsf{Y}_i\}_{i\in\mathcal{I}} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$,

$$2\text{-}\mathcal{P}ro_p(\mathcal{C})(inc(\mathsf{E}\tilde{\otimes}_{\mathcal{H}}\mathsf{F}),\mathsf{Y}) \simeq \mathcal{C}at(\mathsf{E},2\text{-}\mathcal{P}ro_p(\mathcal{C})(inc(\mathsf{F}),\mathsf{Y}))(\text{see 1.2.9}):$$

We do as follows:

$$2\text{-}\mathcal{P}ro_p(\mathcal{C})(inc(\mathsf{E}\tilde{\otimes}_{\mathcal{H}}\mathsf{F}),\mathsf{Y}) \overset{I}{\simeq} \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}((\mathsf{E}\tilde{\otimes}_{\mathcal{H}}\mathsf{F})(j),\mathsf{Y}_i) \overset{II}{\simeq}$$

$$\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{E}\tilde{\otimes}_{\mathcal{C}}\mathsf{F}j,\mathsf{Y}_i) \overset{III}{\simeq} \underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}at(\mathsf{E},\mathcal{C}(\mathsf{F}j,\mathsf{Y}_i)) \overset{IV}{\simeq}$$

$$\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \mathcal{C}at(\mathsf{E},\underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{F}j,\mathsf{Y}_i)) \overset{V}{\simeq} \mathcal{C}at(\mathsf{E},\underset{i\in\mathcal{I}}{\underleftarrow{\mathsf{Lim}}}\ \underset{j\in\mathcal{J}}{\underrightarrow{\mathsf{Lim}}}\ \mathcal{C}(\mathsf{F}j,\mathsf{Y}_i)) \overset{VI}{\simeq}$$

$$Cat(\mathsf{E}, 2\text{-}\mathcal{P}ro_p(C)(inc(\mathsf{F}), \mathsf{Y}))$$

where I is due to 2.1.8, II is due to the fact that bi-tensors in $\mathcal{H}om_p(\mathcal{J}^{op}, C)$ are computed pointwise (see 1.4.8, 1.2.11), III holds by definition 1.2.9, IV is true by [11, 2.4], V is due to definition 1.2.4 and VI holds by 2.1.8 again.

□

5.2.2 Proposition. *If C has finite weighted bi-limits and bi-colimits of pseudo-functors $\mathsf{F} : \mathcal{P} \longrightarrow C$ with finite weights $\mathsf{W} : \mathcal{P} \longrightarrow Cat$ (see [19]), then so does $2\text{-}\mathcal{P}ro_p(C)$ (to simplify, by finite we mean that $\mathcal{P}$ is finite and $\mathsf{W}(\mathsf{P})$ is finite for all $\mathsf{P} \in \mathcal{P}$).*

Proof. We are going to prove only the case of bi-colimits. The case of bilimits is analogous and we leave it to the reader. We are going to check that $2\text{-}\mathcal{P}ro_p(C)$ has bi-colimits of pseudo-functors indexed by a finite category Δ with no loops. As a particular case, we will have bi-coequalizers, binary bi-coproducts and 0 which, by 4.1.6, is enough to prove the statement in the proposition.

Let $\Delta \overset{\mathsf{D}}{\longrightarrow} 2\text{-}\mathcal{P}ro_p(C)$ be a pseudo-functor. Then, by 3.3.4, we have $\Delta \overset{\mathsf{D}'}{\longrightarrow} \mathcal{H}om_p(\mathsf{J}^{op}, C)$ equivalent to D in $2\text{-}\mathcal{P}ro_p(C)$ as in the following diagram with J a cofinite and filtered poset with a unique initial object:

$$\begin{tikzcd}
 & \mathcal{H}om_p(\mathsf{J}^{op}, C) \arrow[d, "inc"] \\
\Delta \arrow[ur, "\mathsf{D}'"] \arrow[r, "\mathsf{D}"'] \arrow[ur, phantom, "\simeq", shift right=3] & 2\text{-}\mathcal{P}ro_p(C)
\end{tikzcd}$$

If we apply the construction of 1.4.5 to J, we obtain $\Delta \overset{\mathsf{D}''}{\longrightarrow} \mathcal{H}om_p(\hat{\mathsf{J}}^{op}, C)$ which by 3.1.1 is equivalent to D in $2\text{-}\mathcal{P}ro_p(C)$.

It suffices to show that the bi-colimit of $inc(\mathsf{D}'')$ exists in $2\text{-}\mathcal{P}ro_p(C)$ and this follows from 5.2.1 plus the fact that C is a closed 2-bmodel 2-category.

To conclude the proof, we have to check that $2\text{-}\mathcal{P}ro_p(C)$ has bi-tensors $\mathsf{E}\tilde{\otimes}_{2\text{-}\mathcal{P}ro_p(C)}\mathsf{X}$ with E a finite category:

By 3.1.3, we obtain a 2-pro-object X' equivalent to X and indexed by a cofinite and filtered poset with a unique initial object J. Then, by 1.4.5 plus 3.1.1, we can construct a 2-pro-object X'' equivalent to X and indexed by $\mathcal{J} = \hat{\mathsf{J}}$. Then, by 5.2.1, there exists the bi-tensor $\mathsf{E}\tilde{\otimes}_{2\text{-}\mathcal{P}ro_p(C)}\mathsf{X}''$ and so there exists the bi-tensor $\mathsf{E}\tilde{\otimes}_{2\text{-}\mathcal{P}ro_p(C)}\mathsf{X}$. □

In all what it follows we assume that C is a closed 2-bmodel 2-category.

5.2.3 Definition. *We define strong fibrations, strong cofibrations, strong trivial fibrations, strong trivial cofibrations in $2\text{-}\mathcal{P}ro_p(C)$ as the image of fibrations, cofibrations, fibrations that are also weak equivalences and cofibrations that are also weak equivalences respectively in some $\mathcal{H}om_p(\hat{\mathsf{J}}^{op}, C)$ with J a cofinite, filtered poset with a unique initial object.*

- *A morphism* $\mathsf{f} \in 2\text{-}\mathcal{P}ro_p(C)$ *is a cofibration if is the retract in* $\mathcal{H}om_p(\mathsf{2}, 2\text{-}\mathcal{P}ro_p(C))$ *of a strong cofibration.*
- *A morphism* $\mathsf{f} \in 2\text{-}\mathcal{P}ro_p(C)$ *is a fibration if is the retract in* $\mathcal{H}om_p(\mathsf{2}, 2\text{-}\mathcal{P}ro_p(C))$ *of a strong fibration.*
- *A morphism* $\mathsf{f} \in 2\text{-}\mathcal{P}ro_p(C)$ *is a trivial cofibration if is the retract in* $\mathcal{H}om_p(\mathsf{2}, 2\text{-}\mathcal{P}ro_p(C))$ *of a strong trivial cofibration.*
- *A morphism* $\mathsf{f} \in 2\text{-}\mathcal{P}ro_p(C)$ *is a trivial fibration if is the retract in* $\mathcal{H}om_p(\mathsf{2}, 2\text{-}\mathcal{P}ro_p(C))$ *of a strong trivial fibration.*
- *A morphism* $\mathsf{f} \in 2\text{-}\mathcal{P}ro_p(C)$ *is a weak equivalence if it can be factored up to isomorphism as* $\mathsf{f} \cong \mathsf{pi}$ *where* p *is a trivial fibration and* i *is a trivial cofibration.*

All the rest of this section is devoted to prove the following theorem:

5.2.4 Theorem. *If C is a closed 2-bmodel 2-category, then* $2\text{-}\mathcal{P}ro_p(C)$ *with the structure given in 5.2.3 is a closed 2-bmodel 2-category.* □

But before we state and prove the theorem we wanted in the first place:

5.2.5 Theorem. *If C is a closed 2-bmodel 2-category, then* $2\text{-}\mathcal{P}ro(C)$ *is a closed 2-bmodel 2-category.*

Proof. Recall that the inclusion $2\text{-}\mathcal{P}ro(C) \longrightarrow 2\text{-}\mathcal{P}ro_p(C)$ is a retract pseudo-equivalence (see 2.1.5). Then, the result follows immediately from 4.3.3 and 5.2.4. □

Proof of theorem 5.2.4

Axiom 2-M0b: It holds by proposition 5.2.2.

Axiom 2-M2: We are going to give the proof for the case where p is both a fibration and a weak equivalence and i is a cofibration. The other case is analogous and we leave it to the reader.

Let $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y} \in 2\text{-}\mathcal{P}ro_p(C)$. By 3.2.9, we have a diagram of the form

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{f}"] \arrow[d, "\mathsf{a}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Y} \arrow[d, "\mathsf{b}"] \\
\mathsf{X}' \arrow[r, "\mathsf{f}'"'] & \mathsf{Y}'
\end{tikzcd}$$

where a and b are equivalences with quasi inverses $\overline{a}$ and $\overline{b}$ and $X' \xrightarrow{f'} Y' \in \mathcal{H}om_p(J^{op}, \mathcal{C})$ for some J cofinite, filtered poset with a unique initial object.

Consider $J^{op} \xrightarrow{T} \widehat{J^{op}} = \hat{J}^{op}$ as in 1.4.5. Then there are 2-functors $\widehat{X'}, \widehat{Y'} : \hat{J}^{op} \longrightarrow \mathcal{C}$ such that $\widehat{X'}T = X'$ and $\widehat{Y'}T = Y'$. Then, by 1.4.5 plus 3.1.1, $\widehat{X'}$ and $\widehat{Y'}$ are equivalent to X' and Y' respectively in 2-$\mathcal{P}ro_p(\mathcal{C})$ via some equivalences a', b' with quasi inverses $\overline{a'}$, $\overline{b'}$. Then we have the following diagram in 2-$\mathcal{P}ro_p(\mathcal{C})$:

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[d, "a"'] \arrow[rrrd, phantom, "\cong \Downarrow \gamma"] & & & Y \arrow[d, "b"] \\
X' \arrow[rrr, "f'"] \arrow[d, "a'"'] \arrow[rrrd, phantom, "= \Downarrow id"] & & & Y' \arrow[d, "b'"] \\
\widehat{X'} \arrow[r, "\overline{a'}"'] & X' \arrow[r, "f'"'] & Y' \arrow[r, "b'"'] & \widehat{Y'}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a'a"'] \arrow[rd, phantom, "\cong \Downarrow b'\gamma \circ b'f'"] & Y \arrow[d, "b'b"] \\
\widehat{X'} \arrow[r, "b'f'\overline{a'}"'] & \widehat{Y'}
\end{tikzcd}$$

where the bottom row of this diagram belongs to $\mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$.

Since $\mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$ is a closed 2-bmodel 2-category, $b'f'\overline{a'}$ can be factored as $b'f'\overline{a'} \overset{\alpha\cong}{\Longrightarrow} p'i'$ where p' is both a fibration and a weak equivalence and i' is a cofibration in $\mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$. Consider $i = i'a'a$ and $p = \overline{b}\,\overline{b'}p'$. Then $f \cong \overline{b}\,\overline{b'}b'bf \cong \overline{b}\,\overline{b'}b'f'\overline{a'}a'a \cong \overline{b}\,\overline{b'}p'i'a'a = pi$. It can be easily checked that p is a retract of p' and so, by definition of the structure in 2-$\mathcal{P}ro_p(\mathcal{C})$, it is a fibration and i is a retract of i' and so it is a cofibration. It only remains to check that p is a weak equivalence: We know that p' is a weak equivalence in $\mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$, then $p' \cong uv$ where u is both a fibration and a weak equivalence and v is both a cofibration and a weak equivalence in $\mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$ and so, $p \cong \overline{b}\,\overline{b'}uv$. It can be checked that $\overline{b}\,\overline{b'}u$ is a retract of u and v is a retract of v which concludes the proof of axiom 2-M2.

In order to prove axioms 2-M5 and 2-M6, we state and prove some previous lemmas:

5.2.6 Lemma. *Given a diagram in* 2-$\mathcal{P}ro_p(\mathcal{C})$ *of the form*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[rd, phantom, "\cong \Downarrow \gamma"] & Y \arrow[d, "p"] \\
X \arrow[r, "b"'] & B
\end{tikzcd}$$

where p *is a fibration and* i *is a trivial cofibration (or* p *is a trivial fibration and* i *is a cofibration), there exists a filler* (f, λ, ρ) *for that diagram.*

Proof. We will focus on the case where i is a trivial cofibration and p is a fibration. The other one is analogous and we omit it. Also, since the lifting property is preserved by the formation of retracts (see 4.1.16), it is enough to check this lemma for i a strong trivial cofibration in some $\mathcal{H}om_p(\hat{\mathsf{K}}^{op}, \mathcal{C})$ and p a strong trivial fibration in some $\mathcal{H}om_p(\hat{\mathsf{J}}^{op}, \mathcal{C})$:

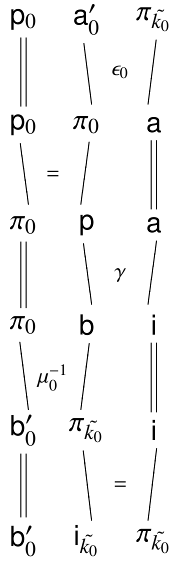
We are going to define an order preserving morphism $\hat{\mathsf{J}}^{op} \xrightarrow{\mathsf{F}} \hat{\mathsf{K}}^{op}$ and we are going to construct a diagram

$$\begin{tikzcd} \mathsf{A}' \arrow[r, "\tilde{\mathsf{a}}"] \arrow[d, "\mathsf{i}'"'] \arrow[dr, phantom, "\cong \Downarrow \tilde{\gamma}"] & \mathsf{Y} \arrow[d, "\mathsf{p}"] \\ \mathsf{X}' \arrow[r, "\tilde{\mathsf{b}}"'] & \mathsf{B} \end{tikzcd} \quad \in \mathcal{H}om_p(\hat{\mathsf{J}}^{op}, \mathcal{C}) \tag{5.2.7}$$

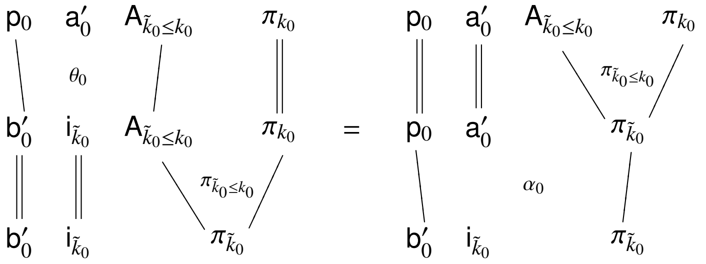
where $\mathsf{A}' = \mathsf{AF}$, $\mathsf{X}' = \mathsf{XF}$, $(\tilde{\mathsf{a}}_j, \epsilon_j)$ represents a and $(\tilde{\mathsf{b}}_j, \mu_j)$ represents b $\forall j \in \mathsf{J}$ for some invertible 2-cells ϵ_j, μ_j:

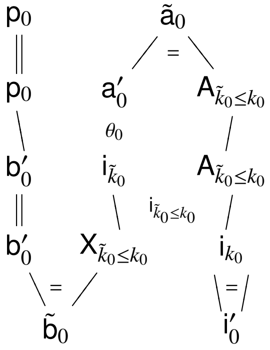
For $j = 0$, consider $\tilde{k_0} \in \mathsf{K}$ and morphisms $\mathsf{A}_{\tilde{k_0}} \xrightarrow{\mathsf{a}'_0} \mathsf{Y}_0$, $\mathsf{X}_{\tilde{k_0}} \xrightarrow{\mathsf{b}'_0} \mathsf{B}_0$ and appropriate invertible 2-cells ϵ_0, μ_0 such that $(\mathsf{a}'_0, \epsilon_0)$ represents a and (b'_0, μ_0) represents b (see 2.2.4).

Consider $\mathsf{A}_{\tilde{k_0}} \underset{\mathsf{b}'_0 \mathsf{i}_{\tilde{k_0}}}{\overset{\mathsf{p}_0 \mathsf{a}'_0}{\rightrightarrows}} \mathsf{B}_0$ and α_0 given by the following composition



Then, by 2.2.7, there exists $\tilde{k_0} \leq k_0 \in \mathsf{K}$ and an invertible 2-cell $\mathsf{A}_{k_0} \underset{\mathsf{b}'_0 \mathsf{i}_{\tilde{k_0}} \mathsf{A}_{\tilde{k_0} \leq k_0}}{\overset{\mathsf{p}_0 \mathsf{a}'_0 \mathsf{A}_{\tilde{k_0} \leq k_0}}{\underset{\Downarrow \theta_0}{\rightrightarrows}}} \mathsf{B}_0 \in \mathcal{C}$ such that

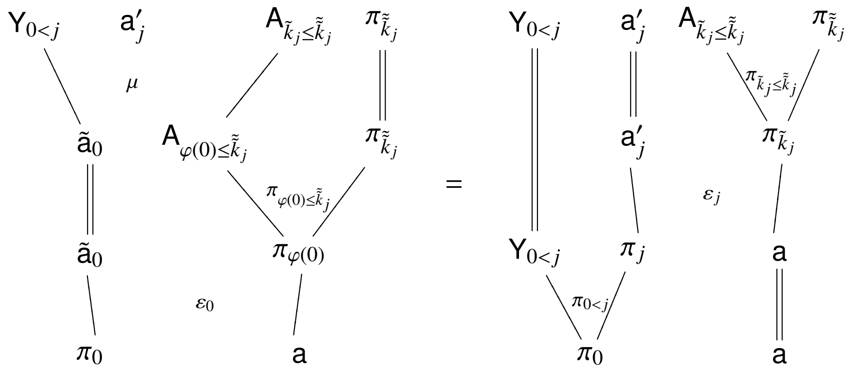


Take $F(0) = k_0$, $i'_0 = i_{k_0}$, $\tilde{a}_0 = a'_0A_{\tilde{k}_0\leq k_0}$, $\tilde{b}_0 = b'_0X_{\tilde{k}_0\leq k_0}$, $\tilde{\gamma}_0 =$ 

and we redefine ϵ_0 and μ_0 so that $(\tilde{a}_0, \epsilon_0)$ represents a and $(\tilde{b}_0, \mu_0)$ represents b.

For $j \neq 0$, suppose that we have already defined all the data $\forall\ j' < j$. By using 2.2.4, consider $\tilde{k}_j \in K$, morphisms $A_{\tilde{k}_j} \xrightarrow{a'_j} Y_j$, $X_{\tilde{k}_j} \xrightarrow{b'_j} B_j$ and appropriate invertible 2-cells ϵ_j, μ_j such that (a'_j, ϵ_j) represents a, (b'_j, μ_j) represents b and $\tilde{k}_j \geq F(j')\ \forall\ j' < j$ (this can be done because K is cofinite and filtered).

Suppose $\{j'|j' < j\} = \{0, j_0, ..., j_n\}$. By applying 2.2.7 to $\tilde{a}_0$ and a'_j we obtain $\tilde{\tilde{k}}_j \geq \tilde{k}_j, F(0)$ and an invertible 2-cell $A_{\tilde{\tilde{k}}_j} \underset{\tilde{b}_0A_{F(0)\leq\tilde{\tilde{k}}_j}}{\overset{Y_{0<j}a'_jA_{\tilde{k}_j\leq\tilde{\tilde{k}}_j}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow\mu}}}} Y_0$ such that



We rename $\mathsf{a}'_j = \mathsf{a}'_j\mathsf{A}_{\tilde{k}_j \leq \tilde{\tilde{k}}_j}$ and ϵ_j the corresponding invertible 2-cell. We repeat the procedure with each j_i from j_0 to j_n instead of 0. Then we do the same thing for b and consider a new k_j above both the obtained ones. Then we repeat for j the procedure we did in the beginning for 0 and we take $\mathsf{i}'_j = \mathsf{i}_{\mathsf{F}(j)}$ and $\tilde{\gamma}_j =$

$$\begin{array}{ccccc}
 & \mathsf{p}_j & & \tilde{\mathsf{a}}_j & \\
/ & & \theta_j & & \backslash \\
\mathsf{b}'_j & & \mathsf{i}_{\tilde{k}_j} & & \mathsf{A}_{\tilde{k}_j \leq \varphi(j)} \\
\| & & \backslash & \mathsf{i}_{\tilde{k}_j \leq \varphi(j)} & / \\
\mathsf{b}'_j & & \mathsf{X}_{\tilde{k}_j \leq \varphi(j)} & & \mathsf{i}_{\varphi(j)} \\
\backslash & = & / & & \| \\
 & \tilde{\mathsf{b}}_j & & & \mathsf{i}'_j
\end{array} .$$

Since F is an order preserving morphism, we define it in morphisms and 2-cells in the obvious way.

It is straightforward to check that A' and X' are 2-functors. It also can be easily checked that $\tilde{\mathsf{a}}$, $\tilde{\mathsf{b}}$ and i' are pseudo-natural transformations.

Then we have a filler $(\mathsf{f}', \lambda', \rho')$ for diagram (5.2.7).

Since $\{\mathsf{X} \xrightarrow{\pi_{\mathsf{F}(j)}} \mathsf{X}'_j\}_{j \in \mathsf{J}}$, $\left\{ \begin{array}{ccc}
\mathsf{X}'_{j'<j} & & \pi_{\varphi(j)} \\
\backslash = / & & \| \\
\mathsf{X}_{\mathsf{F}(j')<\mathsf{F}(j)} & & \pi_{\varphi(j)} \\
 & \backslash \; \pi_{\mathsf{F}(j')<\mathsf{F}(j)} \; / & \\
 & \pi_{\varphi(j')} &
\end{array} \right\}_{j'<j}$ is a pseudo-cone, there exists a morphism $\mathsf{X} \xrightarrow{\mathsf{g}} \mathsf{X}' \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$ such that $\pi_j\mathsf{g} = \pi_{\mathsf{F}(j)}\ \forall\ j \in \mathsf{J}$ and $\pi_{j'<j}\mathsf{g} = \pi_{\mathsf{F}(j')<\mathsf{F}(j)}\ \forall\ j' < j \in \mathsf{J}$. In a similar way, we can define a morphism $\mathsf{A} \xrightarrow{\mathsf{f}} \mathsf{A}' \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$ such that $\pi_j\mathsf{f} = \pi_{\mathsf{F}(j)}\ \forall\ j \in \mathsf{J}$ and $\pi_{j'<j}\mathsf{f} = \pi_{\mathsf{F}(j')<\mathsf{F}(j)}\ \forall\ j' < j \in \mathsf{J}$.

We also have invertible 2-cells $\mathsf{A} \underset{\tilde{\mathsf{a}}\mathsf{f}}{\overset{\mathsf{a}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \alpha}}}} \mathsf{Y}$, $\mathsf{X} \underset{\mathsf{b}}{\overset{\tilde{\mathsf{b}}\mathsf{g}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \beta}}}} \mathsf{Y}$ induced by the morphisms of pseudo-cones $\left\{ \begin{array}{ccc}
\pi_j & & \mathsf{a} \\
\backslash \; \varepsilon_j^{-1} & & / \\
\tilde{a}_j & & \pi_{\varphi(j)} \\
\| & & / = \backslash \\
\tilde{a}_j \;\; \pi_j & & \mathsf{f} \\
\backslash = / & & \| \\
\pi_j \;\; \tilde{\mathsf{a}} & & \mathsf{f}
\end{array} \right\}_{j \in \mathsf{J}}$, $\left\{ \begin{array}{ccc}
\pi_j \;\; \tilde{b} & & \mathsf{g} \\
\backslash = / & & \| \\
\tilde{b}_j \;\; \pi_j & & \mathsf{g} \\
\| & \backslash = / & \\
\tilde{b}_j & \pi_{\varphi(j)} & \\
\backslash \;\; \mu_j & / & \\
\pi_j & \mathsf{b} &
\end{array} \right\}_{j \in \mathsf{J}}$ respectively.

Note that $\mathsf{i}'\mathsf{f} = \mathsf{g}\mathsf{i}$.

$$\text{Then, }\left(f'g,\ \begin{array}{ccccc} & & & a & \\ & & / & \alpha & \backslash \\ & \tilde{a} & & & f \\ / & \lambda' & \backslash & & \| \\ f' & & i' & & f \\ \| & & \backslash & = & / \\ f' & & g & & i \end{array},\ \begin{array}{cccc} p & & f' & g \\ \backslash & \rho' & / & \| \\ & \tilde{b} & & g \\ & & \backslash\beta/ & \\ & & b & \end{array}\right)\text{ is the filler that we were looking for.}$$

$\square$

5.2.8 Lemma.

1. *A morphism* $\mathsf{i} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$ *is a trivial cofibration iff is both a cofibration and a weak equivalence.*
2. *A morphism* $\mathsf{p} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$ *is a trivial fibration iff is both a fibration and a weak equivalence.*

Proof.

1. ⇒) Let $\mathsf{i} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$ be a trivial cofibration. Since strong trivial cofibrations are cofibrations, i is a cofibration by definition. Plus, we can factorize $\mathsf{i} \cong id\mathsf{i}$ and so i is a weak equivalence.

   ⇐) Let $\mathsf{i} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$ be a morphism that is both a cofibration and a weak equivalence. Then we can factorize $\mathsf{i} \overset{\gamma}{\cong} \mathsf{pj}$ where p is a trivial fibration and j is a trivial cofibration. It is enough to check that i is a retract of j but this is true by 4.1.15 plus 5.2.6.
2. The proof is analogous to the previous one and we leave it to the reader.

$\square$

Axiom 2-M6a): ⇒) If p is a fibration and i is both a cofibration and a weak equivalence, then, by 5.2.8, i is a trivial cofibration and, by 5.2.6, the pair (i, p) has the lifting property.

⇐) Suppose that we have a morphism $\mathsf{X} \xrightarrow{\mathsf{p}} \mathsf{Y}$ that has the right lifting property with respect to all morphisms that are both a cofibration and a weak equivalence. By 2-M2, p can be factorized as $\mathsf{p} \overset{\gamma}{\cong} \mathsf{qi}$ where q is a fibration and i is both a cofibration and a weak equivalence. Then the pair (i, p) has the lifting property and so, by 4.1.15, p is a retract of q. Then p is a fibration.

Axiom 2-M6b): The proof of this axiom is analogous to the previous one and we leave it to the reader.

5.2.9 Lemma.

1. *A morphism* $\mathsf{i} \in 2\text{-}\mathcal{P}ro_p(C)$ *is a trivial cofibration iff for every fibration* p*, the pair* (p, i) *has the lifting property.*
2. *A morphism* $\mathsf{p} \in 2\text{-}\mathcal{P}ro_p(C)$ *is a trivial fibration iff for every cofibration* i*, the pair* (p, i) *has the lifting property.*

Proof.

1. $\Rightarrow$) It is immediate from 5.2.6.

   $\Leftarrow$) By axiom 2-M2 plus 5.2.8, we can factorize $\mathsf{i} \overset{\gamma}{\cong} \mathsf{uv}$ where u is a fibration and v is a trivial cofibration. By 4.1.15, i is a retract of v which concludes the proof.
2. The proof is analogous to the previous one and we leave it to the reader.

□

Axiom 2-M6c): It follows immediately from 5.2.9 and the definition of weak equivalences in $2\text{-}\mathcal{P}ro_p(C)$.

5.2.10 Lemma.

1. *If a morphism* $\mathsf{i} \in 2\text{-}\mathcal{P}ro_p(C)$ *has the left lifting property with respect to all fibrations between fibrant objects, then* i *is a trivial cofibration.*
2. *If a morphism* $\mathsf{p} \in 2\text{-}\mathcal{P}ro_p(C)$ *has the right lifting property with respect to all cofibrations between cofibrant objects, then* p *is a trivial fibration.*

Proof.

1. By 5.2.9, it is enough to check that the set $L = \{\mathsf{p} \mid \text{ the pair } (\mathsf{i}, \mathsf{p}) \text{ has the lifting property }\}$ contains all fibrations. In order to do that, first, we are going to prove some properties about this set:

   (a) L is closed under bi-pullbacks: The proof is the same as the one in 4.1.9 of the fact that fibrations are closed under bi-pushouts.

   (b) Given an inverse bi-limit pseudo-cone $\{\mathsf{Y} \xrightarrow{\pi_j} \mathsf{E}(j)\}_{j \in \mathsf{J}}$ where $\{\mathsf{E}(j)\}_{j \in \mathsf{J}}$ is an inverse pseudo-system in $2\text{-}\mathcal{P}ro_p(C)$ with J a cofinite filtered poset with a unique initial object 0 such that for each $j \neq 0 \in \mathsf{J}$, the morphism $\mathsf{E}(j) \xrightarrow{\mathsf{e}_j} \underset{\substack{\longleftarrow \\ k<j}}{\mathsf{biLim}}\ \mathsf{E}(k)$ induced by the pseudo-cone $\{\mathsf{E}(j) \xrightarrow{\mathsf{E}(k<j)} \mathsf{E}(k)\}_{k<j}$,

$$\left\{\begin{tikzcd}[column sep=small] \mathsf{E}(k<l) \arrow[dr, dash] & & \mathsf{E}(l<j) \arrow[dl, dash] \\ & \mathsf{E}(k<j) \arrow[u, phantom, "\alpha^{\mathsf{E}}_{k,l,j}"] & \end{tikzcd}\right\}_{k<l<j}$$ belongs to L, then the induced morphism $\mathsf{Y} \xrightarrow{\pi_0} E(0)$ also belongs to L:

Suppose that we have a diagram as follows in 2-$\mathcal{P}ro_p(C)$:

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{Y} \arrow[d, "\pi_0"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \mathsf{E}(0)
\end{tikzcd}$$

We are going to define inductively a pseudo-cone $\{\mathsf{X} \xrightarrow{\mathsf{f}_j} \mathsf{E}(j)\}_{j\in\mathsf{J}}, \{\mathsf{f}_{k<j}\}_{k<j}$ and an isomorphism of pseudo-cones $\{\pi_j\mathsf{a} \overset{\alpha_j}{\Longrightarrow} \mathsf{f}_j\mathsf{i}\}_{j\in\mathsf{J}}$: $\mathsf{f}_0 = \mathsf{b}$, $\alpha_0 = \gamma$. Suppose that we have already defined f_k, α_k $\forall\ k < j$:

From definition of e_j, we have invertible 2-cells $\pi_k\mathsf{e}_j \overset{\beta_k}{\Longrightarrow} \mathsf{E}(k < j)$ $\forall\ k < j$

$$\text{such that} \quad \begin{tikzcd}[column sep=tiny]
\mathsf{E}(k<l) \arrow[dr, dash] & & \pi_l \arrow[dl, dash] & \mathsf{e}_j \arrow[d, equal] \\
& \pi_k \arrow[dr, dash] \arrow[u, phantom, "\pi_{k<l}"] & & \mathsf{e}_j \arrow[dl, dash] \\
& & \mathsf{E}(k<j) \arrow[u, phantom, "\beta_k"] &
\end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=tiny]
\mathsf{E}(k<l) \arrow[d, equal] & \pi_l \arrow[dr, dash] & & \mathsf{e}_j \arrow[dl, dash] \\
\mathsf{E}(k<l) \arrow[dr, dash] & & \mathsf{E}(l<j) \arrow[dl, dash] \arrow[u, phantom, "\beta_l"] & \\
& \mathsf{E}(k<j) \arrow[u, phantom, "\alpha^{\mathsf{E}}_{k,l,j}"] & &
\end{tikzcd} \quad \forall\ k < l < j.$$

Since $\mathsf{e}_j \in L$, there exists a filler $(\mathsf{f}_j, \tilde{\lambda}_j, \tilde{\rho}_j)$ for the following diagram:

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\pi_j\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \tilde{\gamma}"] & \mathsf{E}(j) \arrow[d, "\mathsf{e}_j"] \\
\mathsf{X} \arrow[r, "\tilde{\mathsf{b}}"'] & \underset{\underset{k<j}{\longleftarrow}}{\mathsf{biLim}}\ \mathsf{E}(k)
\end{tikzcd}$$

where $\tilde{b}$ is induced by the pseudo-cone $\{\mathsf{f}_k\}_{k<j}$, $\{\mathsf{f}_{k<l}\}_{k<l<j}$ and so we have invertible 2-cells $\pi_k\tilde{\mathsf{b}} \overset{\delta_k}{\Longrightarrow} \mathsf{f}_k$ $\forall\ k < j$ such that

$$\begin{tikzcd}[column sep=small]
\mathsf{E}(k<l) \arrow[dr, dash] & & \pi_l \arrow[dl, dash, "\pi_{k<l}"'] & \tilde{\mathsf{b}} \arrow[d, equal] \\
& \pi_k \arrow[dr, dash, "\delta_k"'] & & \tilde{\mathsf{b}} \arrow[dl, dash] \\
& & \mathsf{f}_k &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
\mathsf{E}(k<l) \arrow[d, equal] & & \pi_l \arrow[dr, dash, "\delta_l"] & & \tilde{\mathsf{b}} \arrow[dl, dash] \\
\mathsf{E}(k<l) \arrow[dr, dash, "\mathsf{f}_{k<l}"'] & & & \mathsf{f}_l \arrow[dll, dash] & \\
& \mathsf{f}_k & & &
\end{tikzcd}
\quad \forall\ k < l < j.$$

And $\tilde{\gamma}$ is such that

$$\pi_k\gamma = \begin{tikzcd}[column sep=small]
\pi_k \arrow[dr, dash, "\beta_k"'] & & \mathsf{e}_j \arrow[dl, dash] & \pi_j \arrow[d, equal] & \mathsf{a} \arrow[d, equal] \\
& \mathsf{E}(k<j) \arrow[dr, dash, "\pi_{k<j}"'] & & \pi_j \arrow[dl, dash] & \mathsf{a} \arrow[d, equal] \\
& & \pi_k \arrow[d, dash] & \alpha_k & \mathsf{a} \arrow[d, dash] \\
& & \mathsf{f}_k \arrow[dl, dash] \arrow[dr, dash, "\delta_k^{-1}"'] & & \mathsf{i} \arrow[d, equal] \\
& \pi_k & & \tilde{\mathsf{b}} & \mathsf{i}
\end{tikzcd}
\quad \forall k < j.$$

$$\text{Take } \mathsf{f}_{k<j} = \begin{tikzcd}[column sep=small]
& \mathsf{E}(k<j) \arrow[dl, dash] \arrow[dr, dash, "\beta_k^{-1}"'] & & \mathsf{f}_j \arrow[d, equal] \\
\pi_k \arrow[d, equal] & & \mathsf{e}_j \arrow[dr, dash, "\tilde{\rho}_j"'] & \mathsf{f}_j \arrow[d, dash] \\
\pi_k \arrow[dr, dash, "\delta_k"'] & & & \tilde{\mathsf{b}} \arrow[dll, dash] \\
& \mathsf{f}_k & &
\end{tikzcd}
\text{ and } \alpha_j = \tilde{\lambda}_j.$$

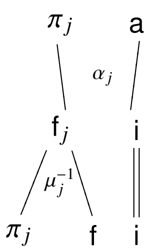
It can be easily checked that this data defines a pseudo-cone and an isomorphism of pseudo-cones as we wanted.

This pseudo-cone induces a morphism $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$ and so we have invertible 2-cells $\pi_j\mathsf{f} \overset{\mu_j}{\Longrightarrow} \mathsf{f}_j\ \forall\ j \in \mathsf{J}$ such that

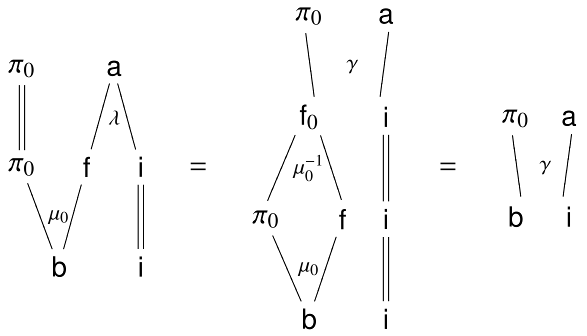
$$\begin{tikzcd}[column sep=small]
\mathsf{E}(k<l) \arrow[dr, dash] & & \pi_l \arrow[dl, dash, "\pi_{k<l}"'] & \mathsf{f} \arrow[d, equal] \\
& \pi_k \arrow[dr, dash, "\mu_k"'] & & \mathsf{f} \arrow[dl, dash] \\
& & \mathsf{f}_k &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
\mathsf{E}(k<l) \arrow[d, equal] & & \pi_l \arrow[dr, dash, "\mu_l"] & & \mathsf{f} \arrow[dl, dash] \\
\mathsf{E}(k<l) \arrow[dr, dash, "\mathsf{f}_{k<l}"'] & & & \mathsf{f}_l \arrow[dll, dash] & \\
& \mathsf{f}_k & & &
\end{tikzcd}
\quad \forall\ k < l < j.$$

Let's check that $(\mathsf{f}, \lambda, \rho)$ is the filler that we were looking for, where λ is such that

$\pi_j\lambda =$



$\forall\ j \in \mathsf{J}$ and $\rho = \mu_0$:



(c) L is closed under the formation of retracts: It follows immediately from 4.1.16.

Now we are going to prove that L contains all fibrations in 2-$\mathcal{P}ro_p(C)$:

Let p be a fibration in 2-$\mathcal{P}ro_p(C)$. If p is a fibration between fibrant objects of C, by 5.1.3, p is a fibration between fibrant objects in $p\mathcal{H}om_p(\mathsf{J}^{op}, C)$ and $\hat{\mathsf{p}}$ is a fibration between fibrant objects in $\mathcal{H}om_p(\hat{\mathsf{J}}^{op}, C)$. But $\hat{\mathsf{p}}$ is p seen as a morphism in $\mathcal{H}om_p(\hat{\mathsf{J}}^{op}, C)$ and so p is a fibration between fibrant objects in 2-$\mathcal{P}ro_p(C)$. Then, by hypothesis, $\mathsf{p} \in L$.

Since L is closed under bi-pullbacks and C satisfies axiom 2-N2, L contains every fibration in C.

Let $\mathsf{Y} \xrightarrow{\mathsf{p}} \mathsf{B} \in \mathcal{H}om_p(\hat{\mathsf{J}}^{op}, C)$ be a fibration for some cofinite and filtered poset J with a unique initial object.

We are going to construct a system $\{\mathsf{E}(j)\}_{j\in\mathsf{J}}$ satisfying the hypothesis of b):

Take $\mathsf{E}(0) = \mathsf{B}$ and for each $j \neq 0$, let $\mathsf{E}(j)$ be the following bi-pullback:

$$\begin{tikzcd}
\mathsf{E}(j) \arrow[r, "\mathsf{h}_0^j"] \arrow[d, "\mathsf{h}_1^j"'] \arrow[dr, phantom, "\cong \Downarrow \delta_j"] & \mathsf{Y}_j \arrow[d, "\mathsf{p}_j"] \\
\mathsf{B} \arrow[r, "\pi_j^{\mathsf{B}}"'] & \mathsf{B}_j
\end{tikzcd}$$

$E(0<j) = h_1^j$ and if $0 \neq k < j$, $E(k<j)$ is given by the following diagram

$$\begin{tikzcd}
E(j) \arrow[dr, dashed, "E(k<j)"] \arrow[drr, bend left, "Y_{k<j}h_0^j"] \arrow[ddr, bend right, "h_1^j"'] & \cong \Uparrow \beta_0^{k,j} & \\
\cong \Downarrow \beta_1^{k,j} & E(k) \arrow[r, "h_0^k"] \arrow[d, "h_1^k"'] \arrow[dr, phantom, "\cong \Downarrow \delta_k"] & Y_k \arrow[d, "p_k"] \\
& B \arrow[r, "\pi_k^B"'] & B_k
\end{tikzcd}$$

and so
$$\begin{tikzcd}[column sep=small]
p_k \arrow[d, dash] \arrow[dr, phantom, "\delta_k"] & h_0^k \arrow[d, dash] & & E(k<j) \arrow[d, equal] \\
\pi_k^B \arrow[d, equal] & h_1^k \arrow[dr, dash] \arrow[rr, phantom, "\beta_1^{k,j}"] & & E(k<j) \arrow[dl, dash] \\
\pi_k^B & & h_1^j &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
p_k \arrow[d, equal] & & h_0^k \arrow[d, dash] \arrow[dr, phantom, "\beta_0^{k,j}"] & E(k<j) \arrow[d, dash] \\
p_k \arrow[d, dash] \arrow[drr, phantom, "p_{k<j}^{-1}"] & & Y_{k<j} \arrow[d, dash] & h_0^j \arrow[d, equal] \\
B_{k<j} \arrow[d, equal] & & p_j \arrow[d, dash] \arrow[dr, phantom, "\delta_j"] & h_0^j \arrow[d, dash] \\
B_{k<j} \arrow[dr, dash] \arrow[rr, phantom, "\pi_{k<j}"] & & \pi_j^B \arrow[dl, dash] & h_1^j \arrow[d, equal] \\
& \pi_k^B & & h_1^j
\end{tikzcd}$$
. It can be checked that E is a pseudo-functor.

Let's check that Y satisfies the universal property of $\underset{j\in J}{\underleftarrow{\mathrm{biLim}}}\ E(j)$ with projection $h_0 = p$:

For $j \neq 0$, take $Y \xrightarrow{\pi_j} E(j)$ as in the following diagram:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\pi_j"] \arrow[drr, bend left, "\pi_j^Y"] \arrow[ddr, bend right, "p"'] & \cong \Uparrow \theta_0^j & \\
\cong \Downarrow \theta_1^j & E(j) \arrow[r, "h_0^j"] \arrow[d, "h_1^j"'] \arrow[dr, phantom, "\cong \Downarrow \delta_j"] & Y_j \arrow[d, "p_j"] \\
& B \arrow[r, "\pi_j^B"'] & B_j
\end{tikzcd}$$

and so we have

$$\begin{tikzcd}[column sep=small]
\mathsf{p}_j \arrow[d, dash, "\delta_j"] & \mathsf{h}_0^j \arrow[d, dash] & & \pi_j \arrow[d, equal] & & \mathsf{p}_j \arrow[d, equal] & \mathsf{h}_0^j \arrow[dr, dash, "\theta_0^j"] & & \pi_j \arrow[dl, dash] \\
\pi_j^{\mathsf{B}} \arrow[d, equal] & \mathsf{h}_1^j \arrow[dr, dash, "\theta_1^j"] & & \pi_j \arrow[dl, dash] & = & \mathsf{p}_j \arrow[d, dash] & = & \pi_j^{\mathsf{Y}} \arrow[d, dash] & \\
\pi_j^{\mathsf{B}} & & \mathsf{p} & & & \pi_j^{\mathsf{B}} & & \mathsf{p} &
\end{tikzcd}$$

Then take $\pi_{0<j} = \theta_1^j$ and if $0 \neq k < j$, $\pi_{k<j}$ such that

$$\mathsf{h}_0^k\pi_{k<j} = \begin{tikzcd}[column sep=small]
\mathsf{h}_0^k \arrow[d, dash, "\beta_0^{k,j}"] & \mathsf{E}(k<j) \arrow[d, dash] & & \pi_j \arrow[d, equal] \\
\mathsf{Y}_{k<j} \arrow[d, equal] & \mathsf{h}_0^j \arrow[dr, dash, "\theta_0^j"] & & \pi_j \arrow[dl, dash] \\
\mathsf{Y}_{k<j} \arrow[dr, dash, "\pi^{\mathsf{Y}}_{k<j}"] & & \pi_j^{\mathsf{Y}} \arrow[dl, dash] & \\
& \pi_k^{\mathsf{Y}} \arrow[dl, dash, "(\theta_0^k)^{-1}"] \arrow[dr, dash] & & \\
\mathsf{h}_0^k & & \pi_k &
\end{tikzcd} \quad \text{and} \quad \mathsf{h}_1^k\pi_{k<j} = \begin{tikzcd}[column sep=small]
\mathsf{h}_1^k \arrow[dr, dash, "\beta_1^{k,j}"] & & \mathsf{E}(k<j) \arrow[dl, dash] & \pi_j \arrow[d, equal] \\
& \mathsf{h}_1^j \arrow[dr, dash, "\theta_1^j"] & & \pi_j \arrow[dl, dash] \\
& & \mathsf{p} \arrow[dl, dash, "(\theta_1^k)^{-1}"] \arrow[dr, dash] & \\
& \mathsf{h}_1^k & & \pi_k
\end{tikzcd}$$

It can be checked that this data defines a pseudo-cone.

Now suppose that we have another pseudo-cone $\{\mathsf{Z} \xrightarrow{\varphi_j} \mathsf{E}(j)\}_{j\in\mathsf{J}}$, $\{\mathsf{E}(k < j)\varphi_j \overset{\varphi_{k<j}}{\Longrightarrow} \varphi_k\}_{k<j}$. Consider the pseudo-cone $\{\mathsf{Z} \xrightarrow{\mathsf{h}_0^j\varphi_j} \mathsf{Y}_j\}_{j\in\mathsf{J}}$,
$\left\{ \begin{tikzcd}[column sep=small, row sep=small]
\mathsf{Y}_{k<j} \arrow[d, dash, "(\beta_0^{k,j})^{-1}"] & \mathsf{h}_0^j \arrow[d, dash] & & \varphi_j \arrow[d, equal] \\
\mathsf{h}_0^k \arrow[d, equal] & \mathsf{E}(k<j) \arrow[dr, dash, "\varphi_{k<j}"] & & \varphi_j \arrow[dl, dash] \\
\mathsf{h}_0^k & & \varphi_k &
\end{tikzcd} \right\}_{k<j}$. Then there exists a morphism $\mathsf{Z} \xrightarrow{\varphi} \mathsf{Y} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$

such that $\pi^{\mathsf{Y}}_j\varphi = \mathsf{h}^j_0\varphi_j$ and
$$\begin{array}{ccccc} \mathsf{Y}_{k<j} & & \pi^{\mathsf{Y}}_j & & \varphi \\ & \backslash \; {\scriptstyle \pi^{\mathsf{Y}}_{k<j}} \; / & & & \| \\ & \pi^{\mathsf{Y}}_k & & & \varphi \end{array} \quad = \quad \begin{array}{ccccc} \mathsf{Y}_{k<j} & & \pi^{\mathsf{Y}}_j & & \varphi \\ \| & & \backslash & = & / \\ \mathsf{Y}_{k<j} & & \mathsf{h}^j_0 & & \varphi_j \\ \backslash \; {\scriptstyle (\beta^{k,j}_0)^{-1}} & & / & & \| \\ \mathsf{h}^k_0 & & \mathsf{E}(k<j) & & \varphi_j \\ \| & & \backslash & {\scriptstyle \varphi_{k<j}} & / \\ \mathsf{h}^k_0 & & & \varphi_k & \\ \backslash & & = & / & \\ \pi^{\mathsf{Y}}_k & & & \varphi & \end{array} .$$

It can be checked that $\{\pi_j\varphi \overset{\rho_j}{\Longrightarrow} \varphi_j\}_{j\in\mathsf{J}}$ is an isomorphism of pseudo-cones where

ρ_j is such that $\mathsf{h}^j_0\rho_j =$
$$\begin{array}{ccccc} \mathsf{h}^j_0 & & \pi_j & & \varphi \\ & \backslash \; {\scriptstyle \theta^j_0} \; / & & & \| \\ & \pi_j & & & \varphi \\ & \backslash & & = & / \\ & \mathsf{h}^j_0 & & & \varphi_j \end{array}$$
and

$$\pi^{\mathsf{B}}_k\mathsf{h}^j_1\rho_j = \begin{array}{cccccc} \pi^{\mathsf{B}}_k & \mathsf{h}^j_1 & & \pi_j & & \varphi \\ \| & & \backslash \; {\scriptstyle \theta^j_1} \; / & & & \| \\ \pi^{\mathsf{B}}_k & & \mathsf{p} & & & \varphi \\ \backslash & = & / & & & \| \\ \mathsf{p}_k & & \pi^{\mathsf{Y}}_k & & & \varphi \\ \| & & \backslash & = & & / \\ \mathsf{p}_k & & \mathsf{h}^k_0 & & & \varphi_k \\ \backslash & \delta_k & / & & & \| \\ \pi^{\mathsf{B}}_k & & \mathsf{h}^k_1 & & & \varphi_k \\ \| & & \| & & / \; {\scriptstyle \varphi^{-1}_{k<j'}} \; \backslash & \\ \pi^{\mathsf{B}}_k & & \mathsf{h}^k_1 & \mathsf{E}(k<j') & & \varphi_{j'} \\ \| & & \backslash \; {\scriptstyle \beta^{k,j'}_1} \; / & & & \| \\ \pi^{\mathsf{B}}_k & & \mathsf{h}^{j'}_1 & & & \varphi_{j'} \\ \| & & / \; {\scriptstyle (\beta^{j,j'}_1)^{-1}} \; \backslash & & & \| \\ \pi^{\mathsf{B}}_k & & \mathsf{h}^j_1 & \mathsf{E}(j<j') & & \varphi_{j'} \\ \| & & \| & & \backslash \; {\scriptstyle \varphi_{j<j'}} \; / & \\ \pi^{\mathsf{B}}_k & & \mathsf{h}^j_1 & & \varphi_j & \end{array} \quad \forall\, k \in \mathsf{J} \text{ where } j' \text{ is such that } k, j \leq j'.$$

By similar arguments, one can check the full and faithfulness of the equivalence.

Now we are going to check that $\mathsf{E}(j) \xrightarrow{\mathsf{e}_j} \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{E}(k) \in L$:

Suppose that we have a diagram of the form

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & \mathsf{E}(j) \arrow[d, "\mathsf{e}_j"] \\
\mathsf{X} \arrow[r, "\mathsf{b}"'] & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{E}(k)
\end{tikzcd} \tag{5.2.11}$$

It can be checked that the following square is a bi-pullback

$$\begin{tikzcd}
\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{E}(k) \arrow[r, "\mathsf{a}^j_{\mathsf{E},\mathsf{Y}}"] \arrow[d, "\pi^{\mathsf{E}}_0"'] \arrow[dr, phantom, "\cong \Downarrow \nu_j"] & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{Y}_k \arrow[d, "\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k"] \\
\mathsf{B} \arrow[r, "\mathsf{b}^j_{\mathsf{B}}"'] & \underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{B}_k
\end{tikzcd}$$

where $\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{p}_k$ is defined as in 5.1.5, $\mathsf{b}^j_{\mathsf{B}}$ is induced by the pseudo-cone $\{\mathsf{B} \xrightarrow{\pi^{\mathsf{B}}_k} \mathsf{B}_k\}_{k<j}$, $\{\pi^{\mathsf{B}}_{k<l}\}_{k<l<j}$ and so we have an isomorphism of pseudo-cones $\{\pi^{\mathsf{B}}_k \mathsf{b}^j_{\mathsf{B}} \overset{\beta'_k}{\Longrightarrow} \pi^{\mathsf{B}}_k\}_{k<j}$; and $\mathsf{a}^j_{\mathsf{E},\mathsf{Y}}$ is induced by the pseudo-cone $\{\underset{k<j}{\underleftarrow{\mathsf{biLim}}}\ \mathsf{E}(k) \xrightarrow{\mathsf{h}^k_0 \pi_k} \mathsf{Y}_k\}_{k<j}$,

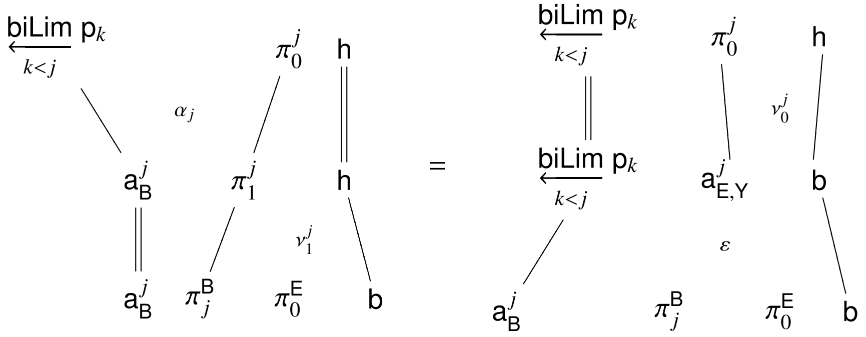
$$\left\{\begin{tikzcd}
\mathsf{Y}_{k<j} \arrow[d, dash, "(\beta^{k,l}_0)^{-1}"] & & \mathsf{h}^l_0 \arrow[d, dash] & & \pi_l \arrow[d, equal] \\
\mathsf{h}^k_0 \arrow[d, equal] & & \mathsf{E}(k<l) \arrow[dr, dash] & & \pi_l \arrow[dl, dash, "\pi_{k<l}"'] \\
\mathsf{h}^k_0 & & & \pi_k &
\end{tikzcd}\right\}_{k<l<j}$$

and so we have an isomorphism of pseudo-cones $\{\pi^{\mathsf{Y}}_k \mathsf{a}^j_{\mathsf{E},\mathsf{Y}} \overset{\theta_k}{\Longrightarrow} \mathsf{h}^k_0 \pi_k\}_{k<j}$.

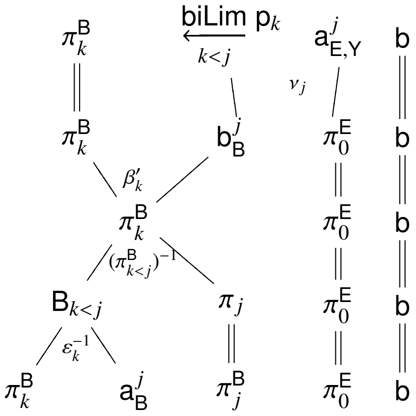
Consider the following diagram

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{b}"] \arrow[d, "\mathsf{b}"'] \arrow[dr, dashed, "\mathsf{h}"] & \underset{k<j}{\underleftarrow{\mathrm{biLim}}}\, \mathsf{E}(k) \arrow[dr, bend left, "\mathsf{a}^j_{\mathsf{E},\mathsf{Y}}"] \arrow[d, phantom, "\cong \Uparrow \nu^j_0"] & \\
\underset{k<j}{\underleftarrow{\mathrm{biLim}}}\, \mathsf{E}(k) \arrow[d, "\pi^{\mathsf{E}}_0"'] \arrow[dr, phantom, "\cong \Downarrow \nu^j_1"] & \mathsf{P}_j \arrow[r, "\pi^j_0"] \arrow[d, "\pi^j_1"'] \arrow[dr, phantom, "bipb \; \cong \Downarrow \alpha_j"] & \underset{k<j}{\underleftarrow{\mathrm{biLim}}}\, \mathsf{Y}_k \arrow[d, "\underset{k<j}{\underleftarrow{\mathrm{biLim}}}\, \mathsf{p}_k"] \\
\mathsf{B} \arrow[r, "\pi^{\mathsf{B}}_j"'] & \mathsf{B}_j \arrow[r, "\mathsf{a}^j_{\mathsf{B}}"'] & \underset{k<j}{\underleftarrow{\mathrm{biLim}}}\, \mathsf{B}_k
\end{tikzcd}$$

where $\mathsf{a}^j_{\mathsf{B}}$ is defined as in 5.1.5 and the following equality holds



where ϵ is such that $\pi^{\mathsf{B}}_k \epsilon =$



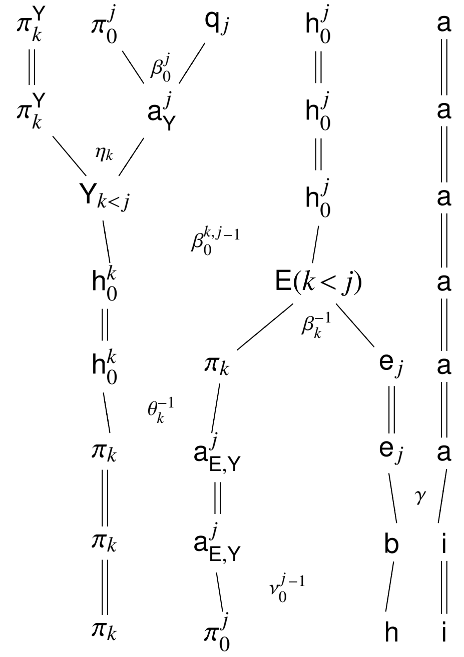
$\forall\, k < j.$

Consider also q_j as in 5.1.5.

Now consider $(\mathsf{h}', \lambda', \rho')$ a filler for the following diagram

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{h}_0^j\mathsf{a}"] \arrow[d, "\mathsf{i}"'] \arrow[dr, phantom, "\cong \Downarrow \gamma'"] & \mathsf{Y}_j \arrow[d, "\mathsf{q}_j"] \\
\mathsf{X} \arrow[r, "\mathsf{h}"'] & \mathsf{P}_j
\end{tikzcd}$$

where γ' is such that $\pi_k^{\mathsf{Y}}\pi_0^j\gamma$ =



and

$$\pi_1^j\gamma = \begin{tikzcd}
\pi_1^j \arrow[dr, dash, "\beta_1^j"] & & \mathsf{q}_j \arrow[dl, dash] & & \mathsf{h}_0^j \arrow[d, equal] & & \mathsf{a} \arrow[d, equal] \\
& \mathsf{p}_j \arrow[d, dash] \arrow[rrr, phantom, "\delta_j"] & & & \mathsf{h}_0^j \arrow[d, dash] & & \mathsf{a} \arrow[d, equal] \\
& \pi_j^{\mathsf{B}} \arrow[d, equal] & & & \mathsf{h}_1^j \arrow[d, dash, "(\beta_1^{0,j})^{-1}" description] & & \mathsf{a} \arrow[d, equal] \\
& \pi_j^{\mathsf{B}} \arrow[d, equal] & & & \mathsf{E}(0<j) \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\beta_0^{-1}"] & & \mathsf{a}\,. \arrow[d, equal] \\
& \pi_j^{\mathsf{B}} \arrow[d, equal] & & \pi_0 \arrow[d, equal] & & \mathsf{e}_j \arrow[d, dash] \arrow[r, phantom, "\gamma"] & \mathsf{a} \arrow[d, dash] \\
& \pi_j^{\mathsf{B}} \arrow[dr, dash] & & \pi_0 \arrow[d, phantom, "\nu_1^{j-1}"] & & \mathsf{b} \arrow[dl, dash] & \mathsf{i} \arrow[d, equal] \\
& & \pi_1^j & & \mathsf{h} & & \mathsf{i}
\end{tikzcd}$$

Finally, consider g as in the following diagram

$$\begin{tikzcd}
\mathsf{X} \arrow[drr, bend left, "\mathsf{h}'"] \arrow[dr, dashed, "\mathsf{g}"] \arrow[ddr, bend right, "\pi_0^{\mathsf{E}}\mathsf{b}"'] \arrow[drr, phantom, "\cong\Uparrow\rho_0^j"] \arrow[ddr, phantom, "\cong\Downarrow\rho_1^j"] & & \\
& \mathsf{E}(j) \arrow[r, "\mathsf{h}_0^j"] \arrow[d, "\mathsf{h}_1^j"'] \arrow[dr, phantom, "\substack{bipb \\ \cong\Downarrow\delta_j}"] & \mathsf{Y}_j \arrow[d, "\mathsf{p}_j"] \\
& \mathsf{B} \arrow[r, "\pi_j^{\mathsf{B}}"'] & \mathsf{B}_j
\end{tikzcd}$$

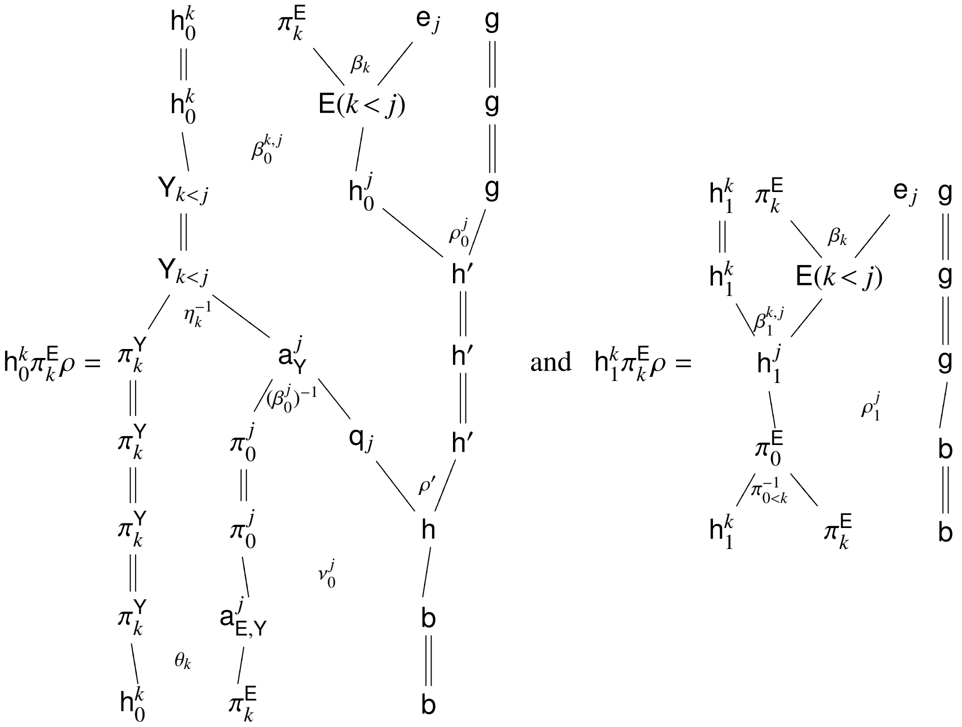
It can be checked that $(\mathsf{g}, \lambda, \rho)$ is a filler for diagram (5.2.11) where λ is such that $\mathsf{h}_0^j\lambda =$

$$\begin{tikzcd}
& \mathsf{h}_0^j \arrow[d, dash] \arrow[rr, phantom, "\lambda'"] & & \mathsf{a} \arrow[d, dash] \\
& \mathsf{h}' \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\rho_0^{j-1}"] & & \mathsf{i} \arrow[d, equal] \\
\mathsf{h}_0^j & & \mathsf{g} & \mathsf{i}
\end{tikzcd}$$

, $\mathsf{h}_1^j\lambda =$

$$\begin{tikzcd}
& \mathsf{h}_1^j \arrow[d, dash, "(\beta_1^{0,j})^{-1}" description] & & \mathsf{a} \arrow[d, equal] \\
& \mathsf{E}(0<j) \arrow[dl, dash] \arrow[dr, dash] \arrow[d, phantom, "\beta_0^{-1}"] & & \mathsf{a} \arrow[d, equal] \\
\pi_0 \arrow[d, equal] & & \mathsf{e}_j \arrow[d, dash] \arrow[r, phantom, "\gamma"] & \mathsf{a} \arrow[d, dash] \\
\pi_0 \arrow[d, dash] \arrow[rr, phantom, "\rho_1^{j-1}"] & & \mathsf{b} \arrow[d, dash] & \mathsf{i} \arrow[d, equal] \\
\mathsf{h}_1^j & & \mathsf{g} & \mathsf{i}
\end{tikzcd}$$

and ρ is such that



$\forall\ k < j$.

Then, by b), $Y \xrightarrow{p} B \in L$.

We have proved that L contains all strong fibrations. Then, by c), L contains all fibrations as we wanted to prove.

2. The proof is analogous to the previous one and we leave it to the reader.

□

5.2.12 Lemma. *If* f *and* g *are two composable morphisms in* 2-$\mathcal{P}ro_p(\mathcal{C})$ *such that* f *and* g *are both fibrations or both cofibrations, then if two out of the three morphisms* f, g *and* gf *are weak equivalences, so is the third one.*

Proof. We will do the case where f and g are both cofibrations. The other case is analogous and we omit it.

- Case I: Suppose that f and g are both weak equivalences. By 5.2.8, it is enough to check that gf is a trivial cofibration and, by 5.2.9, this can be checked by proving that gf has the left lifting property with respect to all fibrations. The proof is dual to the proof of the fact that fibrations are closed under composition from 4.1.9.

- Case II: Suppose that f and gf are both weak equivalences. We want to check that g has the right lifting property with respect to all fibrations. So suppose that we have a fibration p and a diagram of the form

  $$\begin{tikzcd} Y \arrow[r, "a"] \arrow[d, "g"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & E \arrow[d, "p"] \\ Z \arrow[r, "b"'] & B \end{tikzcd}$$

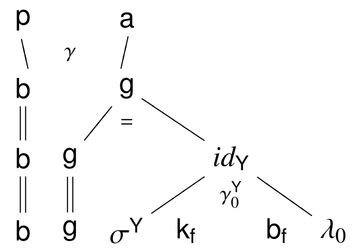
  Since gf is a trivial cofibration, there exists a filler (h', λ', ρ') for the following diagram

  $$\begin{tikzcd} X \arrow[r, "af"] \arrow[d, "f"'] & E \arrow[dd, "p"] \\ Y \arrow[d, "g"'] \arrow[r, phantom, "\cong \Downarrow \gamma f"] & {} \\ Z \arrow[r, "b"'] & B \end{tikzcd}$$

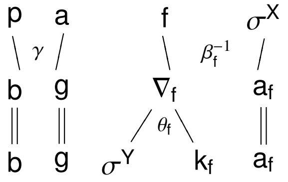
  Since f is a trivial cofibration, there exists $X' \xrightarrow{f'} Y' \in \mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$ for some cofinite filtered poset J with a unique initial object such that f is a retract of f'. From the fact that $\mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$ is a closed 2-bmodel 2-category and 4.2.9, we have that f' induces a trivial cofibration $Y' \vartriangle \widetilde{X'} \triangledown Y' \xrightarrow{k_{f'}} \widetilde{Y'} \in \mathcal{H}om_p(\hat{J}^{op}, \mathcal{C})$ and so, by 4.2.14 f induces a trivial cofibration $Y \vartriangle \widetilde{X} \triangledown Y \xrightarrow{k_f} \widetilde{Y} \in 2\text{-}\mathcal{P}ro_p(\mathcal{C})$. Then there exists a filler (h_0, λ_0, ρ_0) for the following diagram

  $$\begin{tikzcd} Y \vartriangle \widetilde{X} \triangledown Y \arrow[r, "a \vartriangle af\sigma^X \triangledown h'g"] \arrow[d, "k_f"'] \arrow[dr, phantom, "\cong \Downarrow \alpha \vartriangle \beta \triangledown \delta"] & E \arrow[d, "p"] \\ \widetilde{Y} \arrow[r, "bg\sigma^Y"'] & B \end{tikzcd}$$

where α =

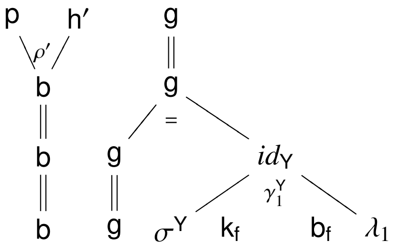


, β =



and

δ =

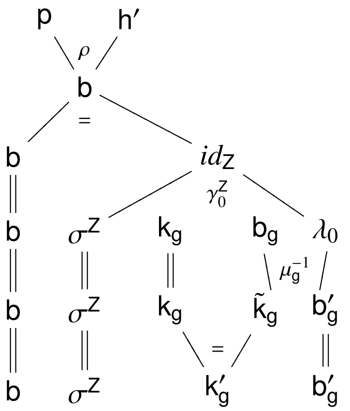


.

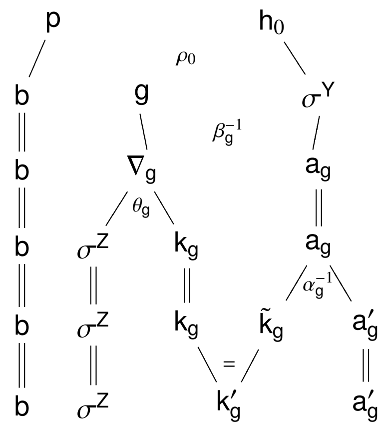
By similar arguments, g induces a trivial cofibration $Z \vartriangle \widetilde{Y} \xrightarrow{k'_g} \widetilde{Z}$. Then there exists a filler (h_1, λ_1, ρ_1) for the following diagram

$$\begin{tikzcd}
Z \vartriangle \widetilde{Y} \arrow[r, "h' \vartriangle h_0"] \arrow[d, "k'_g"'] \arrow[dr, phantom, "\cong \Downarrow \alpha' \vartriangle \beta'"] & E \arrow[d, "p"] \\
\widetilde{Z} \arrow[r, "b\sigma^Z"'] & B
\end{tikzcd}$$

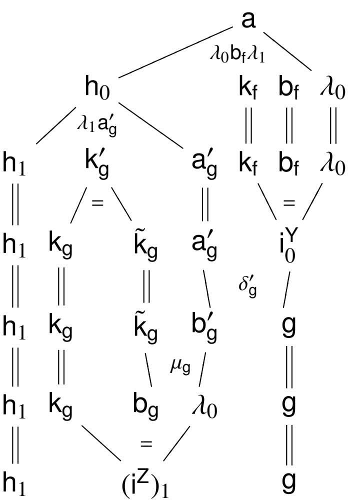
where α' =

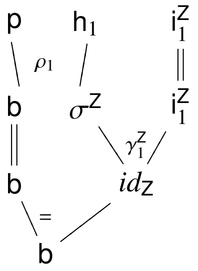


and

$\beta' =$ 

.

It can be checked that ($h_1 \circ i^Z_1$, 

, 

) is the filler that we were looking for.

- Case III: Suppose that g and gf are weak equivalences. By 5.2.10, it is enough to check that f has the left lifting property with respect to all fibrations between fibrant objects. So suppose that we have a fibration $E \xrightarrow{p} B$ between fibrant objects and a diagram of the form

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & E \arrow[d, "p"] \\
Y \arrow[r, "b"'] & B
\end{tikzcd}$$

Since g is a trivial cofibration, there exists a filler (h', λ', ρ') for the following diagram

$$\begin{tikzcd}
Y \arrow[r, "b"] \arrow[d, "g"'] \arrow[dr, phantom, "\cong"] & B \arrow[d] \\
Z \arrow[r] & *
\end{tikzcd}$$

Since gf is a trivial cofibration there exists a filler (h, λ, ρ) for the following diagram

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong \Downarrow \gamma"] & E \arrow[dd, "p"] \\
Y \arrow[d, "g"'] \arrow[dr, bend left=15, "b"] \arrow[dr, phantom, "\cong \Downarrow \lambda'"'] & \\
Z \arrow[r, "h'"'] & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & E \arrow[dd, "p"] \\
Y \arrow[d, "g"'] \arrow[r, phantom, "\cong \Downarrow \lambda' f \circ \gamma"] & \\
Z \arrow[r, "h'"'] & B
\end{tikzcd}$$

It is straightforward to check that $\left(hg, \lambda, \begin{tikzcd}[column sep=tiny, row sep=small]
p \arrow[dr, dash] \arrow[rr, phantom, "\rho"'] & & h \arrow[dl, dash] & g \arrow[d, equal] \\
& h' \arrow[dr, dash] \arrow[rr, phantom, "\lambda'^{-1}"'] & & g \arrow[dl, dash] \\
& & b &
\end{tikzcd}\right)$ is the filler that we were looking for.

□

5.2.13 Lemma. *If* $X \xrightarrow{f} Y$ *is a weak equivalence and* $f \overset{\alpha\cong}{\Longrightarrow} pi$ *with* i *a trivial cofibration and* p *a fibration, then* p *is a trivial fibration.*

Proof. Since f is a weak equivalence, we can factorize $f \overset{\beta\cong}{\Longrightarrow} p'i'$ where p' is a trivial fibration and i' is a trivial cofibration. Then there exist fillers (g, λ, ρ), (g', λ', ρ') for the following diagrams

$$\begin{tikzcd}
X \arrow[r, "i'"] \arrow[d, "i"'] \arrow[dr, "f"] \arrow[dr, phantom, "\cong \Uparrow \beta", near start, shift left=3ex] \arrow[dr, phantom, "\cong \Downarrow \alpha"', near end, shift right=3ex] & Z' \arrow[d, "p'"] \\
Z \arrow[r, "p"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "i'"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong \Downarrow \alpha \circ \beta^{-1}"] & Z' \arrow[d, "p'"] \\
Z \arrow[r, "p"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "i'"'] \arrow[dr, phantom, "\cong \Downarrow \beta^{-1} \circ \alpha"] & Z \arrow[d, "p"] \\
Z' \arrow[r, "p'"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "i'"'] \arrow[dr, "f"] \arrow[dr, phantom, "\cong \Uparrow \alpha", near start, shift left=3ex] \arrow[dr, phantom, "\cong \Downarrow \beta"', near end, shift right=3ex] & Z \arrow[d, "p"] \\
Z' \arrow[r, "p'"'] & Y
\end{tikzcd}$$

By 4.2.9, there exist fillers (h_0, λ_0, ρ_0), (h_1, λ_1, ρ_1) for the following diagrams

$$\begin{tikzcd}
\mathsf{Z} \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Z} \arrow[r, "id_{\mathsf{Z}} \vartriangle \mathsf{i}\sigma^{\mathsf{X}} \triangledown \mathsf{g}'\mathsf{g}"] \arrow[d, "\mathsf{k}_{\mathsf{i}}"'] \arrow[dr, phantom, "\cong \Downarrow id_{\mathsf{p}} \vartriangle id_{\mathsf{pi}} \triangledown \rho\circ\rho'\mathsf{g}"] & \mathsf{Z} \arrow[d, "\mathsf{p}"] \\
\widetilde{\mathsf{Z}} \arrow[r, "\mathsf{p}\sigma^{\mathsf{Z}}"'] & \mathsf{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Z}' \vartriangle \widetilde{\mathsf{X}} \triangledown \mathsf{Z}' \arrow[r, "id_{\mathsf{Z}'} \vartriangle \mathsf{i}'\sigma^{\mathsf{X}} \triangledown \mathsf{g}\mathsf{g}'"] \arrow[d, "\mathsf{k}_{\mathsf{i}'}"'] \arrow[dr, phantom, "\cong \Downarrow id_{\mathsf{p}'} \vartriangle id_{\mathsf{p}'\mathsf{i}'} \triangledown \rho'\circ\rho\mathsf{g}'"] & \mathsf{Z}' \arrow[d, "\mathsf{p}'"] \\
\widetilde{\mathsf{Z}}' \arrow[r, "\mathsf{p}'\sigma^{\mathsf{Z}'}"'] & \mathsf{Y}
\end{tikzcd}$$

By working as in the proof of 5.2.12 Case II, one can check that p and p' have similar lifting properties and so, by 5.2.9, p is a trivial fibration. □

5.2.14 Lemma. *If* $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y}$ *is a weak equivalence in* 2-$\mathcal{P}ro_p(\mathcal{C})$ *and* $\mathsf{f} \overset{\alpha\cong}{\Longrightarrow} \mathsf{pi}$ *with* i *a cofibration and* p *a trivial fibration, then* i *is a trivial cofibration.*

Proof. The proof is analogous to the proof of 5.2.13 and is omitted. □

The following lemmas assume that 2-N3a) holds in $\mathcal{C}$ but they would be analogous in case 2-N3b) holds instead of 2-N3a).

5.2.15 Lemma. *If* $\mathsf{E} \xrightarrow{\mathsf{p}} \mathsf{B}$ *is a trivial fibration in* 2-$\mathcal{P}ro_p(\mathcal{C})$*, then there exists a trivial cofibration* $\mathsf{B} \xrightarrow{\mathsf{s}} \mathsf{E}$ *such that* $\mathsf{ps} \cong id$*. Furthermore, every quasi-section of* p *is a trivial cofibration.*

Proof. By axiom 2-N3, B is cofibrant. Then, by 5.2.6, we have a filler $(\mathsf{s}, \lambda, \rho)$ for the following diagram:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathsf{E} \arrow[d, "\mathsf{p}"] \\
\mathsf{B} \arrow[r, "id_{\mathsf{B}}"'] & \mathsf{B}
\end{tikzcd}$$

Then $\mathsf{ps} \cong id$.

Let s' be a quasi-section of p. One can proceed as in the proof of 5.2.12 Case II to prove that s' has the left lifting property with respect to all fibrations and so is a trivial cofibration. □

5.2.16 Lemma. *If* $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y}$ *is a trivial fibration,* $\mathsf{Y} \xrightarrow{\mathsf{g}} \mathsf{Z}$ *is a trivial cofibration in* 2-$\mathcal{P}ro_p(\mathcal{C})$ *and* $\mathsf{h} \cong \mathsf{gf}$*, then* h *is a weak equivalence.*

Proof. First observe that since we have already proved axioms 2-M6a), 2-M6b) and 2-M6c), we can assume that axiom 2-M7 holds in 2-$\mathcal{P}ro_p(C)$. Then it is enough to prove the lemma only for the case where $\mathsf{h} = \mathsf{gf}$.

By axiom 2-M2, gf can be factored as $\mathsf{gf} \overset{\cong\alpha}{\Longrightarrow} \mathsf{pi}$ where p is a trivial fibration and i is a cofibration. Also, by 5.2.15, there exists a trivial cofibration $\mathsf{Y} \xrightarrow{\mathsf{s}} \mathsf{X}$ such that $\mathsf{fs} \overset{\cong\beta}{\Longrightarrow} 1$.

Since p is a fibration and g is a trivial cofibration, there exists a filler $(\mathsf{h}, \lambda, \rho)$ for the following diagram

$$\begin{tikzcd}
\mathsf{Y} \arrow[r, "\mathsf{is}"] \arrow[d, "\mathsf{g}"'] \arrow[dr, phantom, "\cong \Downarrow \mathsf{g}\beta\circ\alpha\mathsf{s}"] & \mathsf{W} \arrow[d, "\mathsf{p}"] \\
\mathsf{Z} \arrow[r, "id_{\mathsf{Z}}"'] & \mathsf{Z}
\end{tikzcd}$$

Observe that, by 5.2.15, h is a trivial cofibration. Then, since g is also a trivial cofibration, by axiom 2-M3 (this axiom is satisfied by 4.1.9 and the fact that we have already proved axioms 2-M6a),b) and c)) plus 5.2.12, hg is a trivial cofibration. Then, since $\mathsf{is} \cong \mathsf{hg}$, is is also a trivial cofibration and so, by 5.2.12, i is a trivial cofibration which concludes the proof. □

Axiom 2-M5w: Case I: Suppose that f and g are weak equivalences: In this case, we can factorize $\mathsf{f} \cong \mathsf{pi}$ and $\mathsf{g} \cong \mathsf{qj}$ where p, q are trivial fibrations and i, j are trivial cofibrations. By 5.2.16, jp is a weak equivalence and so it can be factorized as $\mathsf{jp} \cong \mathsf{rk}$ where r is a trivial fibration and k is a trivial cofibration.
By 5.2.12 plus axiom 2-M3, qr is a trivial fibration and ki is a trivial cofibration. Then $\mathsf{gf} \cong \mathsf{qjpi} \cong \mathsf{qrki}$ and so is a weak equivalence as we wanted to prove.

Case II: Suppose that f and gf are weak equivalences: In this case, we can factorize $\mathsf{f} \cong \mathsf{pi}$ where p is a trivial fibration and i is a trivial cofibration. By axiom 2-M2, we can also factorize $\mathsf{g} \cong \mathsf{qj}$ where q is a fibration and j is a trivial cofibration. Then $\mathsf{gf} \cong \mathsf{qjpi}$ where jp is a weak equivalence by 5.2.16. Then we can factorize $\mathsf{jp} \cong \mathsf{rk}$ where r is a trivial fibration and k is a trivial cofibration and so $\mathsf{gf} \cong \mathsf{qrki}$ where qr is a fibration and ki is a trivial cofibration by 5.2.12. Therefore, by 5.2.13, qr is a trivial fibration and so, by 5.2.12, q is a trivial fibration which concludes the proof that g is a weak equivalence.

Case III: Suppose that g and gf are weak equivalences: The proof is analogous to the proof of Case II but using 5.2.14 instead of 5.2.13.

To conclude the proof, suppose that f is an isomorphism. Then it can be easily checked that f has the left lifting property with respect to all fibrations and so, by 5.2.9, f is a trivial cofibration. Besides, f can be factored as $\mathsf{f} \cong id\mathsf{f}$ and so is a weak equivalence.

This was the only remaining axiom to conclude that 2-$\mathcal{P}ro_p(C)$ is a closed 2-bmodel 2-category as we wanted to prove. □

Resumen en castellano de la sección 5

En esta sección dotamos a 2-$\mathcal{P}ro(\mathcal{C})$ de una estructura de "closed 2-bmodel 2-category" cuando $\mathcal{C}$ la posee. Esta sección está inspirada en la prueba dada en [12] del hecho de que $\mathsf{Pro}(\mathsf{C})$ es una "closed model category" en el caso 1-dimensional. La demostración en nuestro contexto resultó ser mucho más complicada debido a que los diagramas no conmutan estrictamente sino que conmutan salvo isomorfismo. Por esta razón, nos vimos obligados a trabajar con pseudo-funtores y transformaciones pseudo-naturales si bien los objetos y los morfismos en 2-$\mathcal{P}ro(\mathcal{C})$ son 2-funtores y transformaciones 2-naturales. Se podría pensar (y nosotros lo hicimos por un tiempo) que trabajar con 2-funtores y transformaciones pseudo-naturales sería suficiente pero no lo es. La razón para tomar pseudo-funtores queda evidenciada en la prueba del axioma 2-M2 para $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ donde Z resulta un pseudo-funtor que no es necesariamente un 2-funtor aún cuando todos los demás lo son.

La demostración del teorema principal que establece que 2-$\mathcal{P}ro(\mathcal{C})$ es una "closed 2-bmodel 2-category" tiene tres pasos. El primero, 5.1, consiste en definir una estructura de "closed 2-bmodel 2-category" para la 2-categoría $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ (1.1.19) a partir de una estructura para $\mathcal{C}$, donde J es un poset cofinito y filtrante con un único objeto inicial. Cabe comentar que en [12] no se pide que J tenga un único objeto inicial, lo cual para nosotros es un requisito esencial, incluso en el caso 1-dimensional. El segundo paso, 5.2, consiste en usar la estructura en $p\mathcal{H}om_p(\mathsf{J}^{op}, \mathcal{C})$ para definir una en la 2-categoría 2-$\mathcal{P}ro_p(\mathcal{C})$. Para esto se prueban primero los aspectos de completitud y co-completitud finita que resultarán en la demostración del axioma 2-M0b y luego se demuestran de manera encadenada el resto de los axiomas que requieren la demostración de varias propiedades que destacamos como lemas pues tienen interés en sí mismas. Finalmente, el tercer paso consiste en transferir esta estructura a 2-$\mathcal{P}ro(\mathcal{C})$ usando que esta 2-categoría es "retract pseudo-equivalente" a 2-$\mathcal{P}ro_p(\mathcal{C})$ (2.1.5) mediante el resultado probado en 4.3.3.

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