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Sobre el espacio proyectivo relativo y el caso octoniónico

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Sobre el espacio proyectivo relativo y el caso octoniónico

En esta tesis utilizamos la teoría de la geometría algebraica relativa dada por B. Toen y M. Vaquie, para construir el espacio proyectivo relativo a una categoría monoidal simétrica. Como ejemplo de la construcción tenemos un esquema que llamamos el espacio proyectivo octoniónico.

S. Majid y H. Alburquerque construyen una categoría monoidal simétrica donde el álgebra de los octoniones, $\mathbb{O}$ es un objeto álgebra conmutativa. Esta es la categoría de base para la definición del espacio proyectivo octoniónico, que da lugar a un esquema relativo a dicha categoría.

La construcción del espacio proyectivo es via su funtor de puntos, es decir, un funtor contravariante de la categoría de esquemas afines relativos a una categoría monoidal simétrica, en la categoría de conjuntos. Demostramos que dicho pre-haz es representable por un esquema relativo cuando la categoría de base posee ciertas propiedades de exactitud.

Una vez construido este esquema, le asociamos su categoría de haces quasi-coherentes. Se construyen los haces torcidos $\mathcal{O}_{\mathbb{P}^n_{\mathcal{C}}}(m)$ en analogía al caso clásico del espacio proyectivo.

On the Relative Projective Space and the Octonionic Case

In this thesis we work in the context of relative algebraic geometry introduced by B. Toen y M. Vaquie, to construct a projective space relative to any symmetric monoidal category. As an example of this construction we have a scheme that we call the octonionic projective space.

S. Majid y H. Alburquerque construct a symmetric monoidal category where the algebra of the octonions $\mathbb{O}$ is a commutative algebra. This is the base category for the definition of the octonionic projective space, which gives rise to a scheme relative to such category. The construction of the projective space is given through its functor of points, that is, a contravariant functor from the category of affine schemes relative to any symmetric monoidal category to the category of sets. We prove that if the base category has some exactness properties, this pre-sheaf is representable by some relative scheme.

Once we construct this scheme, we associate its category of quasi-coherent sheaves. We construct the twisted sheaves $\mathcal{O}_{\mathbb{P}^n_{\mathcal{C}}}(m)$ in analogy with the classical case of the projective space.

Contents

Introducción

Desde la segunda parte de los años 80 la geometría usual ha sido extendida en diferentes direcciones. Por una lado ha cobrado importancia la geometría no conmutativa. Una razón por la cual se ha extendido es que la geometría sobre anillos de escalares más generales que los usuales – los reales y los complejos – tiene interés en matemática pura y en aplicaciones, especialmente en la física. Una primera generalización dada por el estudio de geometría sobre los cuaterniones, llamada geometría hiperkaehleriana, cobró impulso en la década de los años 80 por su relación con la física y la geometría de variedades de Calabi-Yau. El interés desde el punto de vista geométrico de las variedades hiperkahlerianas es que en ellas se combinan dos áreas de interés clásico de la geometría. En física muchos ejemplos de variedades hiperkaehlerianas aparecen como espacios de moduli de soluciones de ecuaciones de teorías de Gauge, por ejemplo métricas de Einstein, instantones gravitacionales, espacios de moduli de monopolos, espacios de moduli de soluciones de Hitchin de métricas auto-duales. En la matemática pura aparecen en el estudio del esquema de Hilbert de una variedad hyperkaehleriana y en las variedades de quivers de Nakajima que están relacionadas con teoría de representación de grupos.

Recientemente la geometría sobre los octoniones ha hecho su aparición, por ejemplo en las Variedades de Joyce [17]. Estas variedades tienen gran importancia en Teoría M [3] ya que la teoría de supercuerdas compactificada en una

de estas variedades proporciona una teoría realista en cuatro dimensiones con supersimetría. Uno de los principales problemas es el estudio de compactificaciones convenientes de variedades octoniónicas.
Actualmente no se conocen modelos de Espacios Proyectivos Octoniónicos (salvo en dimensiones uno y dos). Las estrategias naturales para la construcción de estos espacios presentan problemas. La construcción como cociente de un espacio afín por la acción de un grupo de elementos inversibles presenta gran dificultad debido a la no-asociatividad de los octoniones ya que los elementos no nulos no forman un grupo. Otra posible estrategia emulando la geometría clásica es usar pegado de espacios afines lo que corresponde al desarrollo de geometría tórica octonionica [23]. Los problemas son similares a la estrategia anterior.

Los esquemas conmutativos fueron primero pensados como espacios geométricos, después A. Grothendieck observó que para la construcción de nuevos esquemas y establecer sus propiedades categóricas es más conveniente reemplazar los esquemas por haces de conjuntos en la categoría de esquemas afines (el sitio Zariski por ejemplo) que ellos representan [15]. Más explícitamente, la categoría de esqemas está incluida en una categoría más grande, la categoría de funtores contravariantes de la categoría de esquemas a las categoría de conjuntos. Vía esta inclusión, un esquema X ahora es visto como el funtor representado por dicho esquema i.e., $h_X = Hom(-, X)$. A este funtor se le conoce como el funtor de puntos del esquema X. Esta forma de ver a los esquemas es útil por al menos las siguientes dos razones:

1. El producto de esquemas es mucho más fácil de describir cuando se ven como funtores que como esquemas.

2. Cuando se quiere construir un esquema, es más simple construir el funtor

de puntos de dicho esquema. Este problema de construcción de esquemas se reduce entonces a probar que un funtor es representable.

Las ideas generales de la geometría algebraica relativa se remontan a [16] donde se define la noción de esquema relativo sobre un topos anillado. En [7, 8] se considera el caso de los esquemas sobre una categoría tannakiana. El contexto relativo que en esta tesis se trabaja es el presentado en [26] que está altamente inspirado en estas últimas dos referencias pero en donde no es necesario que la categoría de base sea aditiva. Sea $\mathcal{C}$ una categoría monoidal simétrica cerrada, completa y cocompleta. Gracias a la estructura monoidal trenzada (no necesariamente simétrica) se tiene en $\mathcal{C}$ la noción de objeto álgebra conmutativa, asociativa y unitaria. Para una tal álgebra A se considera su categoría $\mathcal{M}od_{\mathcal{C}}(A)$ de A-modulos a izquierda y para un morfismo $A \to B$ entre objetos álgebra se tiene el funtor $B \otimes_A -$ de cambio de base. A la categoría formada por las álgebras conmutativas asociativas unitarias la denotamos como $Comm(\mathcal{C})$. Se define formalmente la categoría de los esquemas afines relativos a $\mathcal{C}$, $Aff_{\mathcal{C}}$ como la categoría opuesta a $Comm(\mathcal{C})$. Se tienen las siguientes construcciones en $Aff_{\mathcal{C}}$

- Existe una topología de Grothendieck canónica en $Aff_{\mathcal{C}}$, llamada la topología playa cuyas familias cubrientes corresponden a familias finitas de morfismos $\{A \to A_i\}$ en $Comm(\mathcal{C})$ tales que el funtor de cambio de base
$$\prod_i A_i \otimes_A - : \mathcal{M}od_{\mathcal{C}}(A) \to \prod_i \mathcal{M}od_{\mathcal{C}}(A_i)$$
es conservativo.

- Los prehaces representables son haces para esta topología.

- Existe una noción de abierto Zariski en $Aff_{\mathcal{C}}$ que por definición es un morfismo $f : Spec(B) = X \to Spec(A) = Y$ tal que el morfismo correspondiente $A \to B$ en $Comm(\mathcal{C})$ es un epimorfismo playo de presentación finita.

- La noción de abierto Zariski se extiende a morfismo entre haces.
- Los abierto Zariski son estables por composición, isomorfismos y cambio de base.
- La noción de abierto Zariski da lugar a una topología, llamada la topología Zariski y para esta topología los prehaces representables también son haces.

Estas propiedades son necesarias para definir una categoría de esquemas relativa a $\mathcal{C}$. Así, un esquema relativo es por definición un haz en el sitio $Aff_{\mathcal{C}}$ con la topología Zariski y que posee un cubrimiento Zariski por esquemas afines. A la categoría de esquemas relativos a $\mathcal{C}$ la denotamos $Sch(\mathcal{C})$ y se prueba que ésta es una subcategoría plena de la categoría de haces sobre $Aff_{\mathcal{C}}$, estable por productos fibrados y sumas disjuntas.

La definición del esquema proyectivo relativo a una categoría $\mathcal{C}$ está dada por un funtor en $Aff_{\mathcal{C}}$, $\mathbb{P}^n_{\mathcal{C}} : Aff_{\mathcal{C}} \to Ens$ que a cada "anillo" de $\mathcal{C}$ le asigna el conjunto de los sumandos directos L de A^{n+1} con L un objeto de linea de $\mathcal{M}od_{\mathcal{C}}(A)$. Los objetos de linea son en el caso clásico equivalentes a los modulos inversibles de rango 1.
El resultado más importante de este trabajo es que si $\mathcal{C}$ es una categoría monoidal simétrica cerrada, abeliana, completa y cocompleta y tal que su objeto unidad es un generador proyectivo de presentación finita entonces $\mathbb{P}^n_{\mathcal{C}}$ es en efecto un esquema relativo a $\mathcal{C}$, Teorema 4.2.4. A este conjunto de hipótesis sobre $\mathcal{C}$ lo llamamos un *contexto abeliano relativo fuerte.*

Damos ahora un detalle de cada capítulo de esta tesis.

El capítulo 1 es una introducción a las categorías monoidales trenzadas y contiene algunos resultados de coherencia. El objetivo de este capítulo es el de fijar notación y de referencia para los capítulos subsiguientes. Presentamos algunas construcciones del álgebra conmutativa en el contexto de categorías simétricas. Es un compendio de todas las construcciones clásicas que se tienen en la categoría de grupos abelianos, trasladadas a categorías monoidales simétricas. Las construcciones más importantes de éste capítulo son la categoría de módulos sobre un objeto álgebra conmutativa $\mathcal{M}od(A)$ y las propiedades que heredan de $\mathcal{C}$ esta categoría y la categoría de los objetos álgebra conmutativas $Comm(\mathcal{C})$. También están las construcciones del álgebra tensorial y simétrica y sus respectivas propiedades universales. Si bien muchos de los resultados que figuran en este capítulo son parte del folklore de las categorías monoidales, nos tomamos el trabajo de hacer algunas de las demostraciones debido a su importancia para resultados posteriores y también porque en la literatura no es fácil encontrar un compendio de dichos resultados, más aún algunos de estos no parecen encontrarse en la literatura. Estos resultados son los Lemas 1.2.8, 1.2.9 y las Proposiciones 1.2.30 y 1.3.19.

El capítulo 2 se ocupa de la noción de álgebra de Hopf y sus representaciones. Se consideran versiones torcidas de éstas álgebras, llamadas *transformaciones de gauge* y la equivalencia monoidal entre la categoría de representaciones de un álgebra de Hopf y la de sus deformaciones. Es acá donde se presenta la categoría para la cual los octoniones son un objeto álgebra conmutativa, asociativa y unitaria y será la categoría de base para la definición del espacio projectivo octoniónico $\mathbb{P}^n_{\mathbb{O}}$. Probamos en este capítulo además, que la categoría de base para la construcción de $\mathbb{P}^n_{\mathbb{O}}$ satisface las condiciones requeridas por el teorema 4.2.4, mas precisamente las Proposiciones 2.4.4 y 2.4.6.

Tenemos entonces como resultado que $\mathbb{P}^n_{\mathbb{O}}$ es un esquema relativo (Teorema 4.2.7). Este último resultado corresponde al objetivo fundamental de esta tesis.

En el capítulo 3 se hace una revisión las ideas de Toen y Vaquie de la geometría algebraica relativa. Incluyen las definiciones del sitio $Zariski$ asociado a una categoría monoidal simétrica cerrada y cocompleta y la de los esquemas relativos a la categoría $\mathcal{C}$.

El capítulo 4 se concentra en la definicion del funtor de puntos del espacio proyectivo relativo y se tienen los resultados mas relevantes de esta tesis, en particular los Teoremas 4.2.4 y 4.2.7. Damos también una definición del proyectivo (Definición 4.2.9) que involucra los cocientes $A^{n+1} \twoheadrightarrow L$ con L un objeto de línea y probamos que estas dos definiciones dan lugar a esquemas isomorfos (Teorema 4.2.11). Esta segunda definición del proyectivo es en cierto sentido mas simple pues es independiente de los Lemas 4.1.6 y 4.1.7. Para la demostración de los Teoremas 4.2.4, 4.2.10 fue fundamental demostrar los Lemas 4.1.3, 4.1.4 y 4.1.17. El Lema 4.1.3 dice que asociado a un ideal I de un objeto álgebra A, se tiene un sub esquema llamado el abierto complementario. El lema 4.1.4 da una condición suficiente para que un morfismo de haces sea un epimorfismo.
En la última parte de este capítulo nos concentramos en la categoría de haces quasi-coherentes sobre un esquema relativo. Probamos una propiedad de pegado de haces quasi-coherentes (Proposición 4.3.3). Definimos los haces torcidos $\mathcal{O}_{\mathbb{P}^n_{\mathcal{C}}}(m)$ para m un número entero. La motivación para definir estos haces es la de tener un teorema del estilo de Serre que caracteriza los haces quasi-coherentes sobre el espacio proyectivo como módulos graduados. Si bien tenemos resultados parciales, éstos no están incluidos en esta tesis y corresponden a trabajos futuros por desarrollar.

En el apéndice hacemos una revisión de las herramientas necesarias para el enfoque algebraico de la construccion de espacios proyectivos, esto es, la de definir un espacio via una categoría de haces. Necesitamos introducir la teoría de categorías enriquecidas pues en este caso el enriquecimiento es fundamental para las construcciones, así como en el caso clásico la categoría de haces quasi-coherentes sobre un esquema está enriquecida en el anillo de secciones globales, lo mismo ocurre en el caso relativo, y es solo luego de este enriquecimiento que se podría comparar ambas construcciones (geométrica y algebraica). Esta comparación también es parte de trabajos futuros a desarrollarse.
La versión enriquecida está justificada en el siguiente argumento. En el caso de los octoniones, la categoria algebraica no enriquecida es degenerada pues sus morfismos no son los correctos, son sólo los de grado cero, cuando en realidad se necesitan los de todos los grados, esto se debe a que el functor V_0 denominado comúnmente como "conjunto subyacente" no es plenamente fiel, aunque si es conservativo.

Originalmente se pensó en la definición del espacio proyectivo octoniónico desde el punto de vista algebraico, esto es, definir este espacio como aquel cuya categoría de haces quasi-coherentes es un cálculo de fracciones de la categoría de módulos graduados sobre el álgebra graduada $\mathbb{O}[x_0, \ldots, x_n] := Sym(\mathbb{O}^{n+1})$. Sin embargo no encontramos cómo contrastar con ejemplos conocidos, que en efecto esta construcción es la correcta. Una gran virtud de la definición del proyectivo que damos en el capítulo 4, es que cuando se reemplaza la categoría de base $\mathcal{C}$ por categorías clásicas como la de los grupos abelianos $\mathcal{Ab}$ o la de las $\mathbb{C}$- álgebras se tienen las definiciones de $\mathbb{P}^n_{\mathbb{Z}}$ y $\mathbb{P}^n_{\mathbb{C}}$ respectivamente.

Chapter 1

Categorical preliminaries

Resumen

En este capítulo hacemos una introducción a las categorías monoidales trenzadas. El objetivo es el de fijar notación y de referencia para los capítulos subsiguientes. Presentamos algunas construcciones del álgebra conmutativa en el contexto de categorías simétricas. Es un compendio de todas las construcciones clásicas que se tienen en la categoría de grupos abelianos, trasladada a categorías monoidales simétricas. Las construcciones más importantes son la categoría de módulos sobre un objeto álgebra conmutativa $\mathcal{M}od(A)$ y las propiedades que heredan de $\mathcal{C}$ esta categoría y la categoría de los objetos álgebra conmutativas $Comm(\mathcal{C})$. También están las construcciones del álgebra tensorial y simétrica y sus respectivas propiedades universales.

En la sección 1.1 se da la definición de categoría monoidal, de funtor monoidal laxo y fuerte y la noción de categoría trenzada.

En la sección 1.2 se define categoría monoidal cerrada, es decir una categría monoidal para la cual el funtor $X \otimes -$ posee un adjunto a derecha, denotado $hom_{\mathcal{C}}(X, -)$ y que llamamos *hom interno*. Se da la noción de dual y se prueban

una propiedad para la suma directa de objetos dualizables, lema 1.2.7, además se prueba que los duales de los morfismos canónicos $\lambda_j : \mathbb{1} \to \mathbb{1}^n$ son las proyecciones canónicas $\pi_j : \mathbb{1}^n \to \mathbb{1}$.

En la sección 1.2 se define para cada objeto álgebra A de $\mathcal{C}$, su categoría de módulos $\mathcal{M}od_{\mathcal{C}}(A)$. Se prueba que si la categoría de base $\mathcal{C}$ es cerrada y bicompleta entonces también lo es $\mathcal{M}od_{\mathcal{C}}(A)$, proposición 1.2.16. Un resultado análgo se tiene para la categoría de objetos ágebra conmutativas, proposición 1.2.18. Si además $\mathcal{C}$ es abeliana, se tiene que $\mathcal{M}od_{\mathcal{C}}(A)$ es abeliana. Finalmente se construye una estructura monoidal para $\mathcal{M}od_{\mathcal{C}}(A)$ a partir de la dada en $\mathcal{C}$ y se prueba que el funtor $A \otimes - : \mathcal{C} \to \mathcal{M}od_{\mathcal{C}}(A)$ es monoidal fuerte, proposición 1.2.30. Se presentan también el álgebra tensorial y simétrica y sus porpiedades universales.

En la sección 1.3 se dan las nociones de ideal de un objeto algebra $A \in Comm(\mathcal{C})$ y la existencia de ideales primos de A. Se da la definición de objeto cuerpo y una caracteirzación en términos de elementos no nulos inversibles, proposición 1.3.19. Finalmente se hace un breve compendio de la localización en objetos de $Comm(\mathcal{C})$ y sus módulos.

Summary

In this chapter we introduce the theory of monoidal categories and monoidal functors. We present some constructions of commutative algebra in the context of monoidal categories. The most important construction is the category of modules over an algebra A in $\mathcal{C}$ and its adjunctions. When the category $\mathcal{C}$ is symmetric and A is a commutative algebra with respect to this symmetry, we won't make any difference between left and right modules and we will call them simply modules. When the category $\mathcal{C}$ is abelian symmetric monoidal closed and bicomplete, the category of A-modules inherits the same properties. Even more, the category $Comm(\mathcal{C})$ of commutative algebras is also bicomplete. In this thesis we use elevator calculus introduced by E. Dubuc to prove that certain diagrams commute.

1.1 Monoidal Categories

Definition 1.1.1. A monoidal category consists of the following data $(\mathcal{C}, \otimes, \mathbb{1}, \Phi, l, r)$, with $\mathcal{C}$ any category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a bifunctor called the tensor product, $\mathbb{1}$ the identity object and natural isomorphisms

$$(X \otimes Y) \otimes Z \xrightarrow{\Phi_{X,Y,Z}} X \otimes (Y \otimes Z), \qquad X \otimes \mathbb{1} \xrightarrow{l_X} X \qquad \mathbb{1} \otimes X \xrightarrow{r_X} X,$$

called the associativity, left unity and right unity constraints, respectively, and such that the pentagon and triangle axioms are fulfilled

$$\begin{tikzcd}[column sep=small]
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\Phi_{W\otimes X,Y,Z}"'] \arrow[dr, "\Phi_{W,X,Y}\otimes 1_Z"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\Phi_{W,X,Y\otimes Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\Phi_{W,X\otimes Y,Z}"] \\
W \otimes (X \otimes (Y \otimes Z)) & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ll, "1_W\otimes\Phi_{X,Y,Z}"']
\end{tikzcd} \tag{1.1.1}$$

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\Phi_{X,\mathbb{1},Y}"] \arrow[dr, "r_X\otimes 1_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "1_X\otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd} \tag{1.1.2}$$

A monoidal category is called *strict* if all constraints $\Phi_{X,Y,Z}$, l_X, r_X are the identities.

Examples 1.

1. A category $\mathcal{C}$ with finite products and $\mathbb{1}$ the terminal object. The constraints of unity and associativity are the natural isomorphisms

$$(X \times Y) \times Z \cong X \times (Y \times Z) \cong X \times Y \times Z$$
$$X \times \mathbb{1} \cong \mathbb{1} \times X \cong X \quad \forall \;\; X, Y, Z \in \mathcal{C}$$

2. Let k be a field. The categories $(\mathbf{Vect}(k), k, \otimes_k, a, l, r)$, $(\mathbf{Vect}^{fin}(k), k, \otimes_k, a, l, r)$ of k-vector spaces and finite k-vector spaces, respectively, with the constraints a, l, r being the canonical isomorphisms.
3. The category $\mathbf{Vect}_G(k)$ whose objects are G-graded vector spaces with tensor product given by $V \otimes W := \bigoplus_{g \cdot h} V_g \otimes_k W_h$ with $\cdot$ the product in G, $\mathbb{1} := k$ considered as trivially G- graded, the associativity and unit constraints being the canonical isomorphisms. This category is equivalent to the category of comodules over the group algebra kG.
4. Let $\phi : G \times G \times G \to k^*$ be a 3-cocycle. Let $\mathbf{Vect}_G^{\phi}(k)$ be as before but with associator Φ defined using the function ϕ. For homogeneous elements of degree $|x|, |y|$ and $|z|$, respectively, we have $\Phi((x \otimes y) \otimes z) = \phi(|x|, |y|, |z|)(x \otimes (y \otimes z))$. This example is widely explained in Chapter 2 since this category with $G = \mathbb{Z}_2^3$ is the base category for the octonions.
5. Let H be a Hopf algebra. The category of its modules, as well as its comodules are also examples of monoidal categories. In Chapter 2 we will see these examples in more detail.

We will denote monoidal categories by letters $\mathcal{C}, \mathcal{D}$... instead of all the data their structure involve.

Definition 1.1.2. Let $\mathcal{C}, \mathcal{D}$ be monoidal categories, a lax monoidal functor between them consists of the following:

1. A functor $F : \mathcal{C} \to \mathcal{D}$.

2. For every pair of objects $X, Y \in \mathcal{C}$ a morphism $\theta_{X,Y} : FX \otimes_{\mathcal{D}} FY \to F(X \otimes_{\mathcal{C}} Y)$.
3. A morphism $\varepsilon : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$.

and such that for every $X, Y, Z \in \mathcal{C}$ the following diagrams commute

$$
\begin{tikzcd}
(F(X) \underset{\mathcal{D}}{\otimes} F(Y)) \underset{\mathcal{D}}{\otimes} F(Z) \arrow[r] \arrow[d] & F(X \underset{\mathcal{C}}{\otimes} Y) \underset{\mathcal{D}}{\otimes} F(Z) \arrow[r] & F((X \underset{\mathcal{C}}{\otimes} Y) \underset{\mathcal{C}}{\otimes} Z) \arrow[d] \\
F(X) \underset{\mathcal{D}}{\otimes} (F(Y) \underset{\mathcal{D}}{\otimes} F(Z)) \arrow[r] & F(X) \underset{\mathcal{D}}{\otimes} F(Y \underset{\mathcal{C}}{\otimes} Z) \arrow[r] & F(X \underset{\mathcal{C}}{\otimes} (Y \underset{\mathcal{C}}{\otimes} Z))
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{1}_{\mathcal{D}} \underset{\mathcal{D}}{\otimes} F(X) \arrow[r] \arrow[d] & F(X) \\
F(\mathbb{1}_{\mathcal{C}}) \underset{\mathcal{D}}{\otimes} F(X) \arrow[r] & F(\mathbb{1}_{\mathcal{C}} \underset{\mathcal{C}}{\otimes} X) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \underset{\mathcal{D}}{\otimes} \mathbb{1}_{\mathcal{D}} \arrow[r] \arrow[d] & F(X) \\
F(X) \underset{\mathcal{D}}{\otimes} F(\mathbb{1}_{\mathcal{C}}) \arrow[r] & F(X \underset{\mathcal{C}}{\otimes} \mathbb{1}_{\mathcal{C}}) \arrow[u]
\end{tikzcd}.
$$

Where all the arrows are the obvious ones involving the constraints of associativity, right and left unity.

Definition 1.1.3. A strong monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is given by a lax monoidal functor with the morphisms $\theta_{X,Y} : FX \otimes_{\mathcal{D}} FY \to F(X \otimes_{\mathcal{C}} Y)$ and $\varepsilon : 1_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ being isomorphisms.

A monoidal equivalence is a monoidal functor either lax or strong, that is an equivalence of the underlying categories. The following Theorem, due to S. Mac Lane allows us to drop off the restrictions of associativity and unity to show some properties of monoidal categories.

Theorem 1.1.4. *[Mac Lane's coherence Theorem] Every monoidal category is monoidal equivalent, via a strong monoidal functor, to a strict one, i.e., a category where the associator and unitors are the identity arrows.*

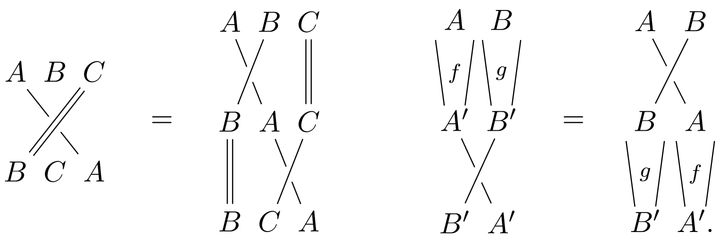
Braided and Symmetric Categories.

A braiding for a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \Phi, l, r)$ is a natural family of isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ such that

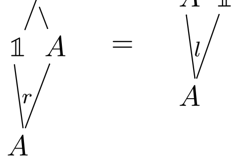
$$\begin{tikzcd}
 & A\otimes(B\otimes C) \arrow[r, "\sigma_{A,B\otimes C}"] & (B\otimes C)\otimes A \arrow[dr, "\Phi_{B,C,A}"] & \\
(A\otimes B)\otimes C \arrow[ur, "\Phi_{A,B,C}"] \arrow[dr, "\sigma_{A,B}\otimes 1"'] & & & B\otimes(C\otimes A) \\
 & (B\otimes A)\otimes C \arrow[r, "\Phi_{B,A,C}"'] & B\otimes(A\otimes C) \arrow[ur, "1\otimes\sigma_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "\sigma_{A\otimes B,C}"] & C\otimes(A\otimes B) \arrow[dr, "\Phi^{-1}_{C,A,B}"] & \\
A\otimes(B\otimes C) \arrow[ur, "\Phi^{-1}_{A,B,C}"] \arrow[dr, "1\otimes\sigma_{B,C}"'] & & & (C\otimes A)\otimes B \\
 & A\otimes(C\otimes B) \arrow[r, "\Phi^{-1}_{A,C,B}"'] & (A\otimes C)\otimes B \arrow[ur, "\sigma_{A,C}\otimes 1"'] &
\end{tikzcd}$$

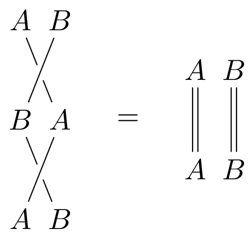
In this thesis, we will use the elevator calculus, introduced by E. Dubuc and can be seen in more detail in [25] to prove the commutativity of diagrams involving the naturality of tensor product and braidings of the category. As an example of the elevator calculus, we will depict the coherence and natural condition for the braiding as follows:



The braiding also satisfies a coherence with respect to the unity isomorphisms, we depict them in elevator diagrams:



Definition 1.1.5. A braided monoidal category is a pair $(\mathcal{C}, \sigma)$ with σ a chosen braiding. The category is said to be symmetric if the braiding satisfies $\sigma^2 = 1$.



Example 1.1.6. All examples in 1 are braided, in fact symmetric, with the braidings being the canonical isomorphisms.

1.2 Adjunctions in a Monoidal Category.

Through this section $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \sigma, \Phi, l, r)$ is a symmetric monoidal category. By Theorem 1.1.4 one can throw away the associativity constraint from all diagrams.

Closed Monoidal Categories

For monoidal categories there is a canonical forgetful functor given by $Hom_{\mathcal{C}}(\mathbb{1}, -) : \mathcal{C} \to \mathcal{E}ns$ which we call the underlying set functor.

In this work we are interested in monoidal categories $\mathcal{C}$ such that for every object X, the functor $- \otimes_{\mathcal{C}} X : \mathcal{C} \to \mathcal{C}$ possess a right adjoint, these categories are called closed. We denote the right adjoint $hom_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{C}$ and the adjunction is precisely the bijection between the Hom sets

$$Hom_{\mathcal{C}}(Y \otimes_{\mathcal{C}} X, Z) \cong Hom_{\mathcal{C}}(Y, hom_{\mathcal{C}}(X, Z)), \quad \forall Y, Z \in \mathcal{C}, \tag{1.2.1}$$

with unit and counit (valuation and co evaluation maps)

$$\eta : Y \to hom_{\mathcal{C}}(X, Y \otimes_{\mathcal{C}} X), \quad \epsilon : hom_{\mathcal{C}}(X, Y) \otimes_{\mathcal{C}} X \to Y. \tag{1.2.2}$$

Putting $Y = \mathbb{1}$ in (1.2.1) we get $Hom_{\mathcal{C}}(X,Z) \cong Hom_{\mathcal{C}}(\mathbb{1}, hom_{\mathcal{C}}(X,Z))$. This says that the underlying set of the object $hom_{\mathcal{C}}(X,Z)$ is isomorphic to the hom set $Hom_{\mathcal{C}}(X,Z)$. The right adjoint functor is called the internal hom.
Taking $X = \mathbb{1}$ in (1.2.1) we get $Hom_{\mathcal{C}}(Y,Z) \cong Hom_{\mathcal{C}}(Y, hom_{\mathcal{C}}(\mathbb{1},Z))$ for all $Y, Z \in \mathcal{C}$, then by Yoneda's Lemma we have a natural isomorphism

$$Z \cong hom_{\mathcal{C}}(\mathbb{1}, Z).$$

Remark 1.2.1. The monoidal category $\mathcal{C}$ is said to be *biclosed* if not only does every $- \otimes_{\mathcal{C}} X$ have a right adjoint $hom_{\mathcal{C}}(X,-)$, but also every $X \otimes_{\mathcal{C}} -$ has a right adjoint $\mathfrak{h}om_{\mathcal{C}}(X,-)$. When $\mathcal{C}$ is symmetric, it is biclosed if closed, with $hom_{\mathcal{C}}(X,-) = \mathfrak{h}om_{\mathcal{C}}(X,-)$.

Definition 1.2.2 (Evaluation map)**.** For X, Y two objects in $\mathcal{C}$, the evaluation map $\varepsilon_{X,Y} : hom_{\mathcal{C}}(X,Y) \otimes_{\mathcal{C}} X \to Y$ is defined as the adjoint arrow to the identity morphism $1 : hom_{\mathcal{C}}(X,Y) \to hom_{\mathcal{C}}(X,Y)$.

Definition 1.2.3 (Composition law)**.** For $X, Y, , Z$ three objects in $\mathcal{C}$, the composition morphism

$$\circ_{X,Y,Z} : hom_{\mathcal{C}}(Y,Z) \otimes hom_{\mathcal{C}}(X,Y) \to hom_{\mathcal{C}}(X,Z)$$

is the adjoint morphism to the composition of the evaluation maps:

$$hom_{\mathcal{C}}(Y,Z) \otimes hom_{\mathcal{C}}(X,Y) \otimes X \xrightarrow{1 \otimes \varepsilon_{M,L}} hom_{\mathcal{C}}(Y,Z) \otimes Y \xrightarrow{\varepsilon_{Y,Z}} Z$$

Duals

Definition 1.2.4. Let $\mathcal{C}$ be a symmetric monoidal category. A dual for an object M is a triple $(M^\vee, \epsilon, \eta)$ with $M^\vee$ an object in $\mathcal{C}$ and

$$M^\vee \otimes M \xrightarrow{\epsilon} \mathbb{1}, \qquad \mathbb{1} \xrightarrow{\eta} M \otimes M^\vee,$$

such that the two diagrams (triangular identities)

$$\begin{tikzcd}
M \cong \mathbb{1} \otimes M \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & M \otimes M^\vee \otimes M \arrow[d, "1\otimes\epsilon"] \\
& M \otimes \mathbb{1} \cong M
\end{tikzcd}
\qquad
\begin{tikzcd}
M^\vee \otimes M \otimes M^\vee \arrow[d, "\epsilon\otimes 1"'] & M^\vee \cong M^\vee \otimes \mathbb{1} \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\
\mathbb{1} \otimes M^\vee \cong M^\vee &
\end{tikzcd}$$

commute. $(M, M^\vee, \epsilon, \eta)$ is called a duality in $\mathcal{C}$. If M has a dual we call M dualizable.

Remark 1.2.5.

1. $\mathbb{1}$ is dualizable with dual $(\mathbb{1}, l_{\mathbb{1}}, l_{\mathbb{1}}^{-1})$ with l the unity constraint in $\mathcal{C}$.

2. M is dualizable if and only if there exist an object $M^\vee$ in $\mathcal{C}$ such that $M - \otimes_{\mathcal{C}}$ is left adjoint to $- \otimes M^\vee$. In this case $- \otimes_{\mathcal{C}} M^\vee = hom_{\mathcal{C}}(M, -)$ and $M^\vee \cong hom_{\mathcal{C}}(M, \mathbb{1})$.

Definition 1.2.6 (Dual morphism)**.** Let M, N dualizable objects in $\mathcal{C}$ and $f : M \to N$. The dual morphism $f^\vee : N^\vee \to M^\vee$ is defined by the composition

$$N^\vee \cong N^\vee \otimes 1 \xrightarrow{1\otimes\eta_M} N^\vee \otimes M \otimes M^\vee \xrightarrow{1\otimes f\otimes 1} N^\vee \otimes N \otimes M^\vee \xrightarrow{\epsilon_N\otimes 1} \mathbb{1} \otimes M^\vee \cong M^\vee$$

and satisfies the diagrams

$$\begin{tikzcd}
N^\vee \otimes M \arrow[r, "f^\vee\otimes 1"] \arrow[d, "1\otimes f"'] & M^\vee \otimes M \arrow[d, "\epsilon_M"] \\
N^\vee \otimes N \arrow[r, "\epsilon_N"'] & \mathbb{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta_N"] \arrow[d, "\eta_M"'] & N \otimes N^\vee \arrow[d, "1\otimes f^\vee"] \\
M \otimes M^\vee \arrow[r, "f\otimes 1"'] & N \otimes M^\vee.
\end{tikzcd}$$

Lemma 1.2.7. *Let $\mathcal{C}$ be abelian. If M_i are dualizable objects for $i = 1, \dots n$, then $\bigoplus_i M_i$ is dualizable.*

Proof. let us check that the functor $- \otimes (\bigoplus M_i)$ has a right adjoint. For every $L, K \in \mathcal{C}$ we have

$$\begin{aligned}
Hom_{\mathcal{C}}\left(L \otimes \oplus_i M_i, K\right) &\cong Hom_{\mathcal{C}}\left(\oplus_i (L \otimes M_i), K\right) \\
&\cong \prod_i Hom_{\mathcal{C}}(L \otimes M_i, K) \cong \prod_i Hom_{\mathcal{C}}(L, K \otimes M_i^\vee) \\
&\cong Hom_{\mathcal{C}}\left(L, \oplus_i (K \otimes M_i^\vee)\right) \cong Hom_{\mathcal{C}}\left(L, K \otimes (\oplus_i M_i^\vee)\right)
\end{aligned}$$

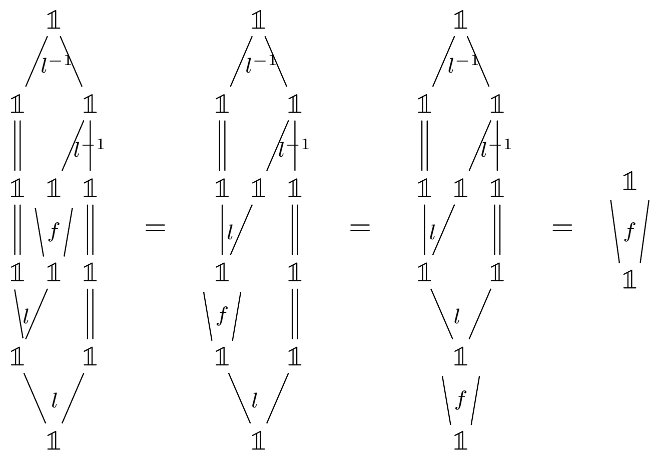
this means that $(\bigoplus_i M_i)^\vee = \bigoplus_i M_i^\vee$. The morphisms ϵ, η are defined via the unit and counit of the adjunction.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta"] \arrow[dr, "\eta_{ij}"'] & \bigoplus_i M_i \otimes \bigoplus_i M_i^\vee \arrow[d, "\pi_{ij}=\pi_i\otimes\pi_j"] \\
& M_i \otimes M_j^\vee
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigoplus_i M_i^\vee \otimes \bigoplus_i M_i \arrow[r, "\epsilon"] & \mathbb{1} \\
M_i^\vee \otimes M_j \arrow[u, "\lambda_{ij}=\lambda_i\otimes\lambda_j"] \arrow[ur, "\epsilon_{ij}"']
\end{tikzcd}$$

with $\eta_{ii} = \eta_{M_i},\ \epsilon_{ii} = \epsilon_{M_i}$ and $\eta_{ij},\ \epsilon_{ij} = 0$ for $i \neq j$. □

Lemma 1.2.8. *For every $f : \mathbb{1} \to \mathbb{1}$, $f^\vee = f$.*

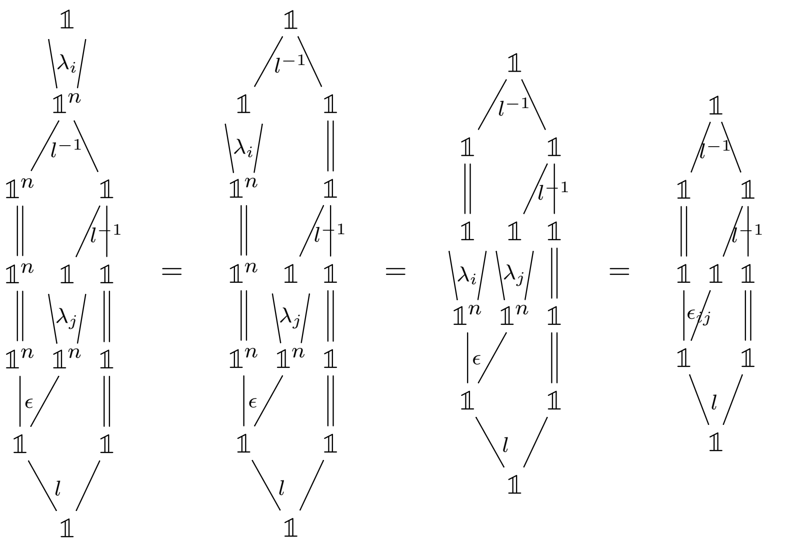
Proof. Since in this case we have $\mathbb{1}^\vee = \mathbb{1}$, $\epsilon = l = r$, $\eta = l^{-1}$ the claim follows from the naturality of l.



□

Lemma 1.2.9. *Let $\mathcal{C}$ be abelian. For $\mathbb{1} \xrightarrow{\lambda_j} \mathbb{1}^n$, its dual coincides with the projection $\mathbb{1}^n \xrightarrow{\pi_j} \mathbb{1}$ for every j.*

Proof. we will prove that for j fixed and all i, $\lambda_j^\vee \lambda_i = \pi_j \lambda_i$ then as $(\lambda_i)_i$ is an epimorphic family then $\lambda_j^\vee = \pi_j$. Recall $\epsilon_{\mathbb{1}} = l$ and $\eta_{\mathbb{1}} = l^{-1}$. We use elevator calculus to show the equality:



the last elevator is the identity arrow $1 : \mathbb{1} \to \mathbb{1}$ when $i = j$ and is the zero arrow when $i \neq j$. Since $\pi_j \lambda_i = \delta_{ij}$, we get the result. $\square$

The Category $\mathcal{M}od_{\mathcal{C}}(A)$

Definition 1.2.10. An algebra object, or simply an algebra in $\mathcal{C}$ is a triple (A, m, η) with A an object in $\mathcal{C}$ and morphisms $m : A \otimes A \to A$, $\eta : 1 \to A$ morphisms in $\mathcal{C}$ called the multiplication and unity and such that

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1\otimes m"] \arrow[d, "m\otimes 1"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "\eta\otimes 1_A"] \arrow[dr, "r"'] & A \otimes A \arrow[d, "m"] & A \otimes \mathbb{1} \arrow[l, "1_A\otimes\eta"'] \arrow[dl, "l"] \\
& A &
\end{tikzcd}
. \tag{1.2.3}
$$

A is said to be commutative if the product m commutes with σ, that is to say that

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "\sigma_{A,A}"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
$$

Definition 1.2.11. Let A, B be algebras in $\mathcal{C}$. An algebra morphism is an arrow $f : A \to B$ in $\mathcal{C}$ such that the following diagrams (depicting the compatibility with both products) commute:

$$\begin{tikzcd} \mathbb{1} \arrow[r, equal] \arrow[d, "\eta_A"'] & \mathbb{1} \arrow[d, "\eta_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m_A"'] & B \otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The category of commutative algebras in $\mathcal{C}$ is denoted by $Comm(\mathcal{C})$.

Examples 2. Let k be a field.

1. If $\mathcal{C}$ is the monoidal category $\mathbf{Vect}(k)$ we get the usual definition of commutative associative unital k-algebra.

2. In $\mathbf{Vect}_G(k)$ an algebra is just a G graded k-algebra, for instance kG is an algebra in $\mathbf{Vect}_G(k)$

3. An algebra A in $\mathcal{Ab}$ is just a ring with unity.

4. If $\phi : G \times G \times G \to k^*$ is a 3-cocycle and F a 2-cocycle in G such that $\partial F = \phi$ then the twisted group algebra $k_F G$ is an algebra in $\mathbf{Vect}_G^{\phi}(k)$. If ϕ is trivial, i.e., F is a 2-cocycle then $k_F G$ is associative in the usual sense. If ϕ is not trivial then the twisted group algebra is not associative in the usual sense but in the category $\mathbf{Vect}_G^{\phi}(k)$. When the 2-cocycle F is symmetric $k_F G$ is a commutative algebra in the usual sense.

As in the classical setting, given a monoidal category (not necessarily symmetric) $\mathcal{C}$ and (A, m, η) an algebra, one can consider its category of left A-modules $\mathcal{M}od_{\mathcal{C}}(A)$.

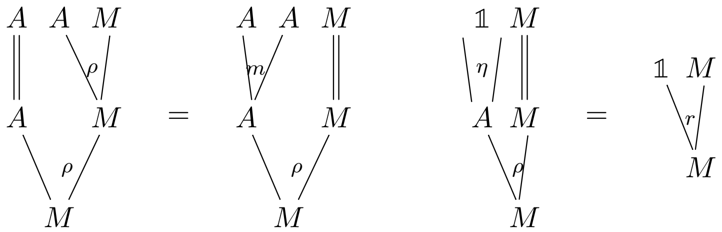
Definition 1.2.12. $\mathcal{M}od_{\mathcal{C}}(A)$ is the category whose objects are pairs (M, ρ) where M is an object in $\mathcal{C}$ and $\rho : A \otimes M \to M$ is a morphism in $\mathcal{C}$ subject to the following conditions:

$$
\begin{tikzcd}
\mathbb{1}\otimes M \arrow[r, "\eta\otimes 1"] \arrow[dr, "r"'] & A\otimes M \arrow[d, "\rho"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
& (A\otimes A)\otimes M \arrow[dl, "\Phi"'] \arrow[dr, "m\otimes 1"] & \\
A\otimes (A\otimes M) \arrow[d, "1\otimes\rho"'] & & A\otimes M \arrow[d, "\rho"] \\
A\otimes M \arrow[rr, "\rho"'] & & M
\end{tikzcd}
$$

A morphism between two left A-modules (M, ρ_M), (N, ρ_N) is a morphism f in $\mathcal{C}$ such that

$$
\begin{tikzcd}
A\otimes M \arrow[r, "\rho_M"] \arrow[d, "1\otimes f"'] & M \arrow[d, "f"] \\
A\otimes N \arrow[r, "\rho_N"'] & N.
\end{tikzcd}
\tag{1.2.4}
$$

Dropping off the associator, the module axioms in elevator diagrams are:



The dual notion of an algebra is called a co-algebraalgebra:

Definition 1.2.13. A coalgebra in $\mathcal{C}$ is a triple (C, Δ, ϵ) with C an object in $\mathcal{C}$ and Δ, ϵ the morphisms $\Delta : C \to C\otimes C$, $\epsilon : C \to \mathbb{1}$ such that the following diagrams commute

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "1\otimes\Delta"] \\
C\otimes C \arrow[r, "\Delta\otimes 1"'] & C\otimes C\otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}\otimes C \arrow[dr, "r"'] & C\otimes C \arrow[l, "\epsilon\otimes 1"'] \arrow[r, "1\otimes\epsilon"] & C\otimes \mathbb{1} \arrow[dl, "l"] \\
& C \arrow[u, "\Delta"] &
\end{tikzcd}
\tag{1.2.5}
$$

Note that a coalgebra is an algebra in the opposite (with respect to the arrows) category $\mathcal{C}^{op}$.

Similarly we can define C-comodules in $\mathcal{C}$:

Definition 1.2.14. Let C be a coalgebra in $\mathcal{C}$. A right C-comodule is an object M in $\mathcal{C}$ together with $\rho : M \to M \otimes C$ s.t

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "1\otimes\Delta"] \\ M \otimes C \arrow[r, "\rho\otimes 1"'] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\cong"'] & M \otimes C \arrow[d, "1\otimes\epsilon"] \\ & M \otimes k \end{tikzcd} \tag{1.2.6}$$

If M, N are $C-$comodules, a morphism $f : M \to N \in$ is a morphism in $\mathcal{C}$ such that the following square is commutative

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "f"] & M \otimes C \arrow[d, "f\otimes 1"] \\ N \arrow[r, "\rho_N"] & N \otimes C \end{tikzcd} \tag{1.2.7}$$

If A is an algebra, a left A-module is the dual notion of a right C-comodule in the sense that if we change C by A and reflex horizontally and vertically the diagrams in 1.2.6 we get the usual definition of a left A-module.

The rest of the chapter is devoted to prove some exactness properties, together with the abelian and monoidal structure that can be built in $\mathcal{M}od_{\mathcal{C}}(A)$.

Definition 1.2.15. A category $\mathcal{C}$ is said to be complete if every diagram $F : J \to \mathcal{C}$ with J small, has a limit in $\mathcal{C}$. Dually, $\mathcal{C}$ is said to be cocomplete if $\mathcal{C}^{op}$ is complete. $\mathcal{C}$ is said to be bicomplete if it is both complete and cocomplete.

Proposition 1.2.16.

Let $\mathcal{C}$ be closed monoidal. If $\mathcal{C}$ is bicomplete then for every $A \in Comm(\mathcal{C})$, $\mathcal{M}od_{\mathcal{C}}(A)$ is bicomplete.

Proof. Let $(M_i, \rho_i)_{i\in I}$ be a small diagram of objects in $\mathcal{M}od_{\mathcal{C}}(A)$ of type I. Seen as objects in $\mathcal{C}$, there exist the limit, $M = \lim_i M_i$ with canonical arrows $\lambda_i : M \to M_i$. Since each M_i is an A-module, we have a family of morphisms

$A\otimes M \xrightarrow{\rho_i(1\otimes\lambda_i)} M_i$, so by the universal property of M as a limit in $\mathcal{C}$, there exists a unique ρ making the following diagram commute

$$\begin{tikzcd}
 & & M_i \\
A\otimes M \arrow[rru, bend left, "\rho_i(1\otimes\lambda_i)"] \arrow[r, dashed, "\rho"'] & M \arrow[ru, "\lambda_i"'] &
\end{tikzcd}$$

we will prove that (M,ρ) is an A-module. For this, consider the following diagram

$$\begin{tikzcd}
A\otimes A\otimes M \arrow[dr, "1\otimes 1\otimes\lambda_i"'] \arrow[rrrd, bend left, "m\otimes 1"] \arrow[rrdd, bend right, "1\otimes\rho"'] & & & \\
 & A\otimes A\otimes M_i \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes\rho_i"'] & A\otimes M_i \arrow[d, "\rho_i"'] & A\otimes M \arrow[l, "1\otimes\lambda_i"'] \arrow[d, "\rho"] \\
 & A\otimes M_i \arrow[r, "\rho_i"] & M_i & M \arrow[l, "\lambda_i"'] \\
 & & A\otimes M \arrow[ul, "1\otimes\lambda_i"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

Each of the interior squares and the exterior smaller diagrams commute:

$$(m\otimes 1)(1\otimes 1\otimes\lambda_i) = (1\otimes\lambda_i)(m\otimes 1) \quad \text{naturality of tensor product}$$

$$(1\otimes\rho_i)(1\otimes 1\otimes\lambda_i) = 1\otimes(\rho_i\circ(1\otimes\lambda_i)) = 1\otimes\lambda_i\rho = (1\otimes\lambda_i)(1\otimes\rho),$$

on the other hand,

$$\begin{aligned}
\lambda_i\rho(m\otimes 1) &= \rho_i(1\otimes\lambda_i) = \rho_i(m\otimes 1)(1\otimes 1\otimes\lambda_i) \\
&= \rho_i(1\otimes\rho_i)(1\otimes 1\otimes\lambda_i) = \rho_i(1\otimes\lambda_i)(1\otimes\rho) \\
&= \lambda_i\rho(1\otimes\rho) \quad \text{for all } i\in I
\end{aligned}$$

therefore $\rho(m\otimes 1)\rho(1\otimes\rho)$, which is precisely the commutativity of the outer diagram. For the unity axiom we consider the following diagram:

$$\begin{tikzcd}
\mathbb{1}\otimes M \arrow[rrr, "\eta\otimes 1"] \arrow[dr, "1\otimes\lambda_i"] \arrow[dddrrr, bend right, "r"'] & & & A\otimes M \arrow[dl, "1\otimes\lambda_i"'] \arrow[ddd, "\rho"] \\
& \mathbb{1}\otimes M_i \arrow[r, "\eta\otimes 1"] \arrow[dr, "r"] & A\otimes M_i \arrow[d, "\rho_i"] & \\
& & M_i & \\
& & & M \arrow[ul, "\lambda_i"']
\end{tikzcd}$$

and we have that all the interior diagrams commute:

$$(1\otimes\lambda_i)(\eta\otimes 1) = (\eta\otimes 1)(1\otimes\lambda_i) \text{ naturality of tensor product}$$
$$r(1\otimes\lambda_i) = \lambda_i\circ r \ \text{ naturality of } r,$$

therefore we have that:

$$\lambda_i\rho(\eta\otimes 1) = \rho_i(1\otimes\lambda_i)(\eta\otimes 1) = \rho_i(\eta\otimes 1)(1\otimes\lambda_i) = r(1\otimes\lambda_i) = \lambda_i r.$$

Now, for the colimit of a small diagram of type I, $(M_i,\rho_i)_{i\in I}$ in $\mathcal{M}od_{\mathcal{C}}(A)$, again consider its colimit as objects in $\mathcal{C}$. Since $\mathcal{A}\otimes -$ is left adjoint, then it commutes with all colimits, therefore we have that $A\otimes\varinjlim M_i \cong \varinjlim A\otimes M_i$. For every $i\in I$ we have morphisms $A\otimes M_i \xrightarrow{\rho_i} M_i \xrightarrow{\lambda_i} M$, then by the universal property of the colimit, there exists a unique $\rho : A\otimes M \to M$ making the diagram below commute

$$\begin{tikzcd}
A\otimes M_i \arrow[r, "\rho_i"] \arrow[dr, "1\otimes\lambda_i"] & M_i \arrow[dr, "\lambda_i"] & \\
& A\otimes M \arrow[r, dashed, "\rho"] & M \\
A\otimes M_j \arrow[ur, "1\otimes\lambda_j"] \arrow[r, "\rho_j"'] & M_j. \arrow[ur, "\lambda_i"'] &
\end{tikzcd}$$

Arguing as we did for limits we have the diagram with commutative sub-diagrams

$$
\begin{tikzcd}
A\otimes A\otimes M \arrow[drrr, bend left, "m\otimes 1"] \arrow[dddrr, bend right, "1\otimes\rho"'] & & & \\
& A\otimes A\otimes M_i \arrow[ul, "1\otimes 1\otimes\lambda_i"'] \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes\rho_i"'] & A\otimes M_i \arrow[r, "1\otimes\lambda_i"'] \arrow[d, "\rho_i"'] & A\otimes M \arrow[d, "\rho"] \\
& A\otimes M_i \arrow[r, "\rho_i"] \arrow[dr, "1\otimes\lambda_i"] & M_i \arrow[r, "\lambda_i"'] & M \\
& & A\otimes M \arrow[ur, "\rho"'] &
\end{tikzcd}
.
$$

For the unity axiom, we have a similar diagram we had for limits. □

Remark 1.2.17. Note that from the proof above, the limits and colimits in $\mathcal{M}od_{\mathcal{C}}(A)$ are computed in $\mathcal{C}$. This means that the forgetful functor from $\mathcal{M}od_{\mathcal{C}}(A) \to \mathcal{C}$ creates all limits and colimits.

Proposition 1.2.18. *Let $\mathcal{C}$ be closed monoidal. If $\mathcal{C}$ is bicomplete then $Comm(\mathcal{C})$ is bicomplete.*

Proof. The proof is similar to the one we made for $\mathcal{M}od_{\mathcal{C}}(A)$, one has to prove that the forgetful functor $Comm(\mathcal{C}) \to \mathcal{C}$ creates all limits and filtered colimits. For details see [12]. □

Definition 1.2.19 (Abelian Category.)**.** A category $\mathcal{C}$ is abelian if

A0. $\mathcal{C}$ has a zero object.

A1. For every pair of objects, there is a product.

A1*. For every pair of objects, there is a coproduct.

A2. Every morphism has a kernel.

A2*. Every morphism has a cokernel.

A3. Every monomorphism is a kernel of a morphism.

A3*. Every epimorphism is a cokernel of a morphism.

Definition 1.2.20 (Subobjects.). Two monomorphisms $A_1 \to B$ and $A_2 \to B$ are equivalent if there are morphisms $A_1 \to A_2$ and $A_2 \to A_1$ such that the following diagrams commute

$$\begin{tikzcd} A_1 \arrow[d] \arrow[dr] & \\ & B \\ A_2 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A_2 \arrow[d] \arrow[dr] & \\ & B \\ A_1 \arrow[ur] & \end{tikzcd}$$

A subobject of B is an equivalence class of monomorphism into B.
The subobject represented by $A_1 \to B$ is said to be contained in that represented by $A_2 \to B$ if there is a morphism $A_1 \to A_2$ such that the diagram commutes

$$\begin{tikzcd} A_1 \arrow[d] \arrow[dr] & \\ & B \\ A_2 \arrow[ur] & \end{tikzcd}$$

Remark 1.2.21. Note that $A_1 \to A_2$ must be a monomorphism and unique. The uniqueness implies that if the subobject $A_2 \to B$ is contained in $A_1 \to B$, then the subobjects are the same and A_1 and A_2 are isomorphic. The relation of containment is a partial ordering of subobjects.

Proposition 1.2.22. *If $\mathcal{C}$ is abelian and closed then for every $A \in Comm(\mathcal{C})$, $Mod_{\mathcal{C}}(A)$ is abelian.*

Proof. Since $\mathcal{C}$ is closed, $A \otimes -$ is left adjoint, therefore it is right exact, then it is an additive functor (see Theorem 3.12* [13].) therefore $A \otimes \mathbf{0} = \mathbf{0}$. The ***Zero*** object in $Mod_{\mathcal{C}}(A)$ is the zero object in $\mathcal{C}$ with the action given by $A \otimes \mathbf{0} = \mathbf{0}$

Kernels, cokernels, finite products and coproducts are examples of finite limits and colimits, we already proved this in the previous proposition. However, we explicit the A-module structure in kernels and cokernels.

Kernels:

Given $f \in \mathrm{Hom}_A(M, N)$ we would like to give an A-module structure in $Kerf$. Consider the following diagram in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & Kerf \arrow[r, "\iota"] & M \arrow[r, "f"] & N \\
& A \otimes Kerf \arrow[u, dotted, "\exists!\bar{\rho}"] \arrow[r, "1\otimes\iota"'] & A \otimes M \arrow[u, "\rho_M"'] \arrow[r, "1\otimes f"'] & A \otimes N \arrow[u, "\rho_N"']
\end{tikzcd}$$

Since the square in the right commutes, we have that

$$f(\rho_M(1_A \otimes \iota)) = \rho_N((1_A \otimes f)(1_A \otimes \iota)) = 0,$$

then by the universal property of $Kerf$ as an object in $\mathcal{C}$ there exists a unique $\bar{\rho}$ such that the square in the left commutes. let us see that this $\bar{\rho}$ makes $(Kerf, \bar{\rho})$ into an object in $\mathcal{M}od_{\mathcal{C}}(A)$. For this, let us note that the following diagram is commutative

$$\begin{tikzcd}
A \otimes A \otimes Kerf \arrow[r, "1\otimes1\otimes\iota"] \arrow[d, "1\otimes\bar{\rho}"'] \arrow[dr, phantom, "\equiv"] & A \otimes A \otimes M \arrow[r, "m\otimes1"] \arrow[d, "1\otimes\rho_M"] \arrow[dr, phantom, "\equiv"] & A \otimes M \arrow[d, "\rho_M"'] \\
A \otimes Kerf \arrow[r, "1\otimes i"'] & A \otimes M \arrow[r, "\rho_M"'] & M
\end{tikzcd} ,$$

therefore we have

$$\rho_M \circ (m \otimes 1) \circ (1 \otimes 1 \otimes \iota) = \rho_M \circ (1 \otimes \iota) \circ (1 \otimes \bar{\rho})$$

the left hand is equal to

$$\rho_M \circ (1 \otimes \iota)(m \otimes 1) = \iota \circ \bar{\rho}(m \otimes 1)$$

and the right hand is $\iota \circ \bar{\rho} \circ (1 \otimes \bar{\rho})$. Since ι is a monomorphism we have that

$$\bar{\rho} \circ (m \otimes 1) = \bar{\rho}(1 \otimes \bar{\rho}) \ . \qquad (1.2.8)$$

Now, for the unity axiom we consider a similar diagram

$$\begin{tikzcd}
\mathbb{1} \otimes Kerf \arrow[r, "1\otimes\iota"] \arrow[d, "r"'] \arrow[dr, phantom, "\equiv"] & \mathbb{1} \otimes M \arrow[r, "\eta\otimes1"] \arrow[d, "r"] & A \otimes M \arrow[dl, "\rho_M"] \\
Kerf \arrow[r, "\iota"'] & M &
\end{tikzcd}$$

so, we have

$$\rho_M(\eta \otimes 1)(1 \otimes \iota) = \rho_M(1 \otimes \iota)(\eta \otimes 1) = i \circ \bar{\rho}(\eta \otimes 1) = \iota \circ r$$

and we get that $\bar{\rho} \circ (\eta \otimes 1) = r$. This together with (1.2.8) say that $(Kerf, \bar{\rho})$ is an A-module. Also, $\iota : Kerf \to M$ is a morphism of A-modules.

Cokernels:

For $f : M \to N$ an A-module morphism, let us consider $(coKerf, \pi)$ in $\mathcal{C}$.

$$\begin{tikzcd} A \otimes M \arrow[r, "1_A \otimes f"] \arrow[d, "\rho_M"] & A \otimes N \arrow[r, "1_A \otimes \pi"] \arrow[d, "\rho_N"] & A \otimes coKerf \arrow[r] \arrow[d, dotted, "\exists ! \bar{\rho}"] & 0 \\ M \arrow[r, "f"] & N \arrow[r, "\pi"] & coKerf \arrow[r] & 0 \end{tikzcd}$$

Since $A \otimes -$ is right exact, the first arrow is exact therefore $A \otimes coKerf$ is the cokernel of $1 \otimes f$. By the commutativity of the first square we have

$$\pi \circ \rho_N(1 \otimes f) = \pi \circ f \circ \rho_M = 0$$

thus, by the universal property of $coKer(1_A \otimes f)$ there exists a unique $\bar{\rho}$ making the diagram in the right commutative. Reasoning as we did for the kernel of a map, we get that $(coKerf, \bar{\rho})$ is an A-module and π is an A-morphism.

Every (epi) monomorphism is the (co) kernel of a map:

Let $f : M \to N$ be a monomorphism of A-modules and consider its cokernel

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[dr, dashed, "\phi"'] & N \arrow[r, "\pi"] & coKer(f) \arrow[r] & 0 \\ & & Ker(\pi) \arrow[u, "\iota"'] & & \end{tikzcd}$$

If we consider this diagram in $\mathcal{C}$, we have that ϕ is an isomorphism satisfying $\iota \circ \phi = f$ since $\mathcal{C}$ is abelian. let us see that ϕ is an A-isomorphism. For this, consider the following diagram:

$$\begin{tikzcd}
A \otimes M \arrow[r, "1\otimes\phi"] \arrow[d, "\rho_M"'] & A \otimes Ker\pi \arrow[r, "1\otimes\iota"] \arrow[d, "\bar{\rho}"] \arrow[dr, phantom, "\equiv"] & A \otimes N \arrow[d, "\rho_M"] \\
M \arrow[r, "\phi"'] & Ker\pi \arrow[r, "\iota"'] & N.
\end{tikzcd}$$

Since

$$f\rho_M = \iota \circ \phi \circ \rho_M = \rho_N(1 \otimes \iota)(1 \otimes \phi) = \iota \circ \bar{\rho}(a \otimes \phi),$$

and ι is a monomorphism, we have that the square in the left commutes, therefore f is the kernel of its cokernel in $\mathcal{M}od_{\mathcal{C}}(A)$.
Similarly, we prove that given an A-epimorphism it is isomorphic to the cokernel of its kernel. □

Next, we will build a monoidal structure in $\mathcal{M}od_{\mathcal{C}}(A)$ from the one given in $\mathcal{C}$.

Definition 1.2.23. Let A be an algebra in $\mathcal{C}$. (M, p) a left A-module and (M, q) a right A-module. We say that (M, p, q) is an (A, A)-bimodule if

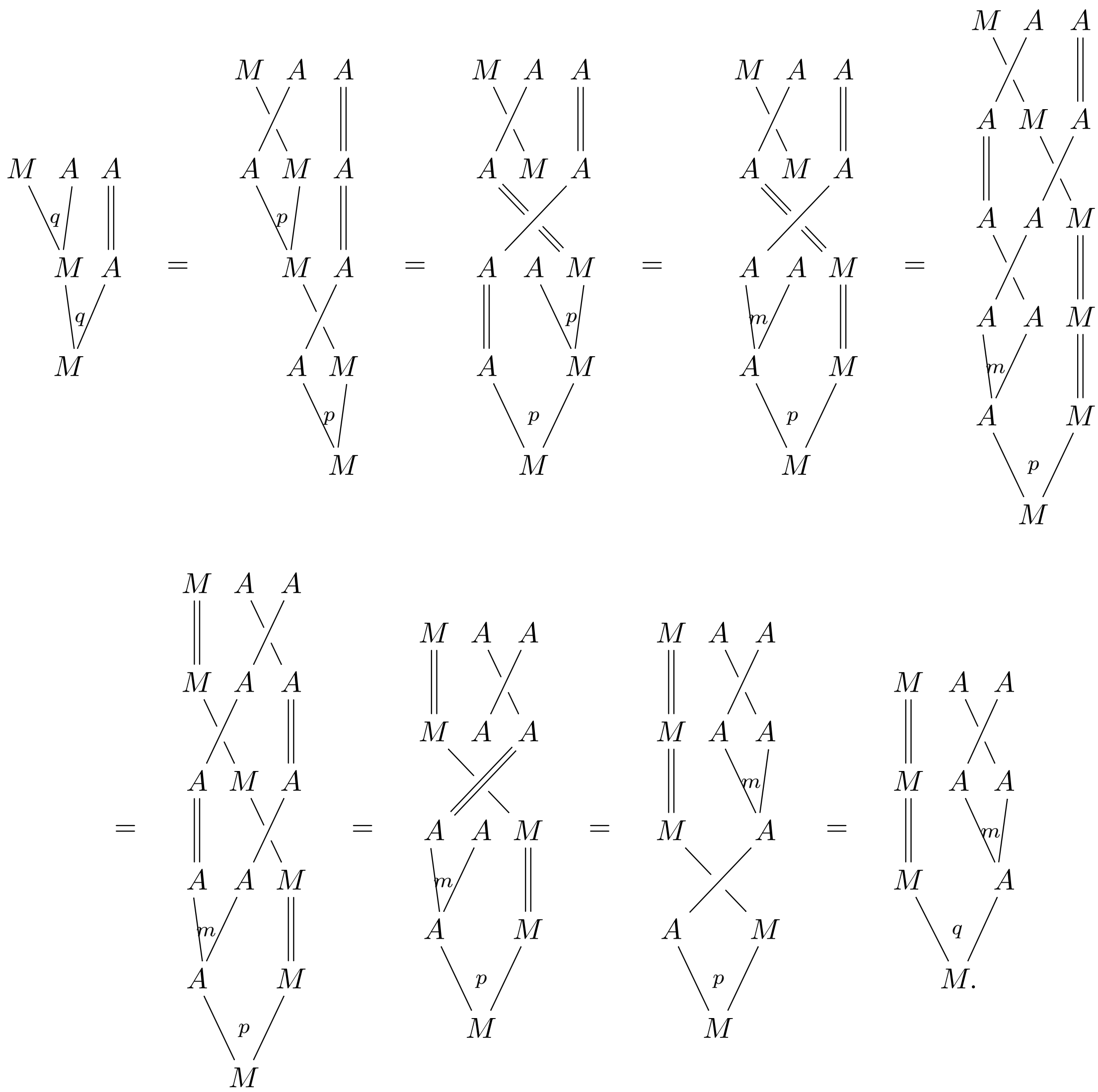
$$\begin{tikzcd}
A \otimes M \otimes A \arrow[r, "p\otimes 1"] \arrow[d, "1\otimes q"'] & M \otimes A \arrow[d, "q"] \\
A \otimes M \arrow[r, "p"] & M
\end{tikzcd}$$

Proposition 1.2.24. *If $(\mathcal{C}, \sigma)$ is braided and (M, p) is a left A-module, then we have a right A^{op} module structure in M via $q = p \circ \sigma_{M,A}$. Even more if A is σ-commutative then (M, p, q) is an A, A-bimodule.*

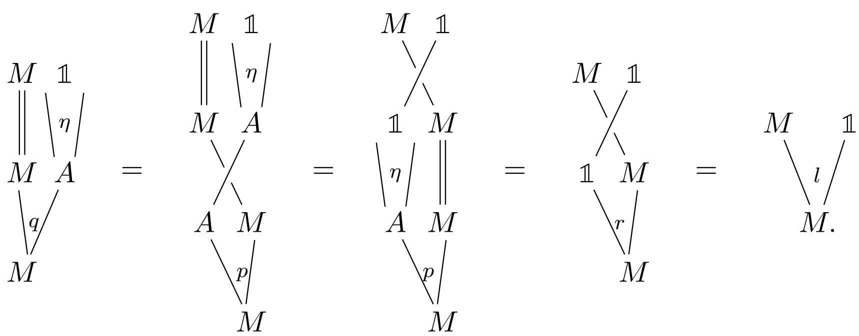
Proof. First, let us prove that (M, q) is a right A^{op}-module, i.e.,

$$q(q \otimes 1) = q(1 \otimes m^{op}), \quad q(1 \otimes \eta) = l.$$

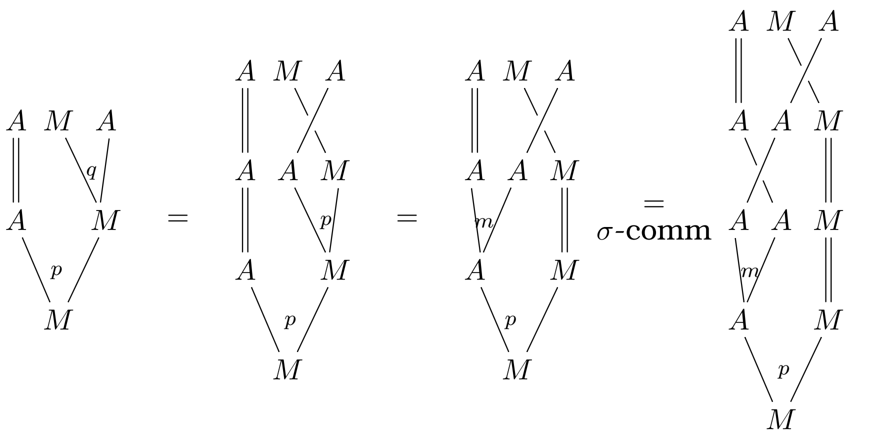
we will use the elevator calculus to prove the right module axioms:

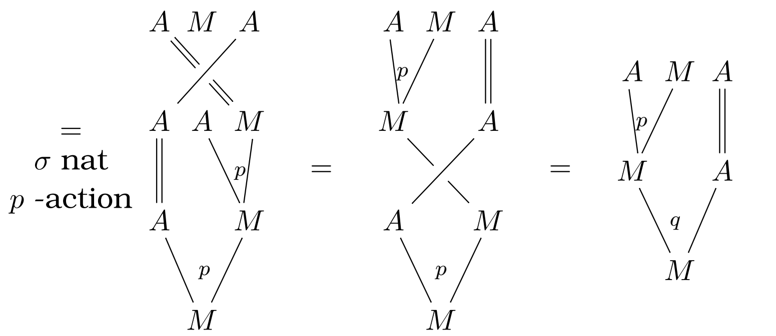


For the unity axiom:



Now, let us see the compatibility with both actions:





□

Keep the notations as in Proposition 1.2.24 with A being commutative. Let (M, q_M) be a right A-module, (N, p_N) a left A-module and consider the morphisms of left A-modules:

$$\begin{tikzcd} M \otimes A \otimes N \arrow[r, bend left, "q_M \otimes 1_N"] \arrow[r, bend right, "1_M \otimes p_N"'] & M \otimes N \end{tikzcd} \tag{1.2.9}$$

then we define a tensor product $\otimes_A$ in $\mathcal{M}od_{\mathcal{C}}(A)$ by

$$M \otimes_A N := coKer(q_M \otimes 1_N - 1_M \otimes p_N)$$

and call $\pi : M \otimes N \to M \otimes_A N$ the cokernel map.

Remark 1.2.25. If we regard $(M \otimes A \otimes N, p_M \otimes 1 \otimes 1)$ and $(M \otimes N, p_M \otimes 1)$ as left A-modules with $p_M = q_M \sigma^{-1}_{M,A}$ then $q_M \otimes 1_N,\ 1_M \otimes p_N$ are both morphisms of left A-modules. Therefore, the map π is a left A-module morphism.

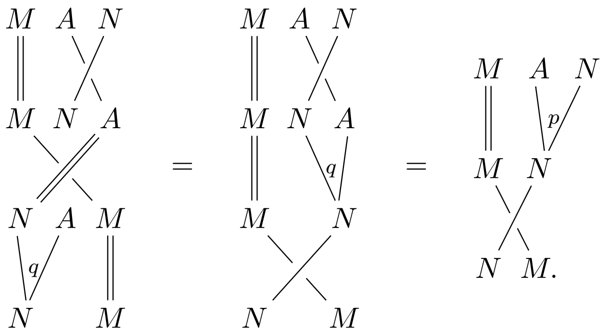
From now on, we consider, for simplicity, $\mathcal{C}$ to be symmetric and A a commutative algebra. The following lemmas will show that $\mathcal{M}od_{\mathcal{C}}(A)$ with $\otimes_A$ can be made into a symmetric monoidal category. The constraints are defined from the ones in the category $\mathcal{C}$. For the non-symmetric case see [22].

Lemma 1.2.26. *The associativity and commutative constraints in $\mathcal{C}$ induce the respective constraints $\tilde{a}, \tilde{\sigma}$ in $\mathcal{M}od_{\mathcal{C}}(A)$.*

Proof. For simplicity we will suppose that $(\mathcal{C}, \otimes)$ is strict and define the braiding $\tilde{\sigma}_{M,N}$ for $\otimes_A$ induced by the braiding in $\mathcal{C}$. In the non strict case, the definition of the associativity constraint for $\mathcal{M}od_{\mathcal{C}}(A)$ is similar.
for every M, N in $\mathcal{M}od_{\mathcal{C}}(A)$, consider the following diagram with both squares in the left (the "curved" and "straight") commutative.

$$
\begin{tikzcd}
M \otimes A \otimes N \arrow[rr, bend left, "1\otimes p_N"] \arrow[rr, "q_M\otimes 1"'] \arrow[d, "1\otimes\sigma_{A,N}"'] & & M \otimes N \arrow[r, "\pi_{M,N}"] \arrow[dd, "\sigma_{M,N}"'] & M \otimes_A N \arrow[dd, dashed, "\tilde{\sigma}_{M,N}"'] \\
M \otimes N \otimes A \arrow[d, "\sigma_{M,N\otimes A}"'] & & & \\
N \otimes A \otimes M \arrow[rr, "1\otimes p_M"] \arrow[rr, bend right, "q_N\otimes 1"'] & & N \otimes M \arrow[r, "\pi_{N,M}"] & N \otimes_A M.
\end{tikzcd}
\tag{1.2.10}
$$

In fact, for the curved diagram $(q_N \otimes 1)(\sigma_{M,N\otimes A})(1 \otimes \sigma_{A,N}) = \sigma_{M,N}(1 \otimes p_N)$



In a similar way we prove that $(1 \otimes p_M)(\sigma_{N,M\otimes A})(1 \otimes \sigma_{A,N}) = \sigma_{M,N}(q_M \otimes 1)$. Using this commutative diagram we show that $\pi_{N,M} \circ \sigma_{M,N}$ co equalizes $1 \otimes p_N$ and $q_M \otimes 1$, then by the universal property of $\pi_{M,N}$, there exists a morphism $\tilde{\sigma}$: $M \otimes_A N \to N \otimes_A M$ such that the square in the right commutes.
Analogously, $\pi_{M,N} \circ \sigma_{N,M}$ co equalizes $1 \otimes p_M$ and $q_{N\otimes 1}$, then we have the existence of the arrow $\tilde{\sigma}_{N,M}$ in the other direction. So, we have that

$$\tilde{\sigma}_{M,N}\pi_{M,N} = \pi_{N,M}\sigma_{M,N}, \quad \pi_{M,N}\sigma_{N,M} = \tilde{\sigma}_{N,M}\pi_{N,M}$$

$$\tilde{\sigma}_{N,M}\tilde{\sigma}_{M,N}\pi_{M,N} = \tilde{\sigma}_{N,M}\pi_{N,M}\sigma_{M,N} = \pi_{M,N}, \quad \tilde{\sigma}_{M,N}\pi_{M,N}\sigma_{N,M} = \tilde{\sigma}_{M,N}\tilde{\sigma}_{N,M}\pi_{N,M} = \pi_{N,M},$$

since $\pi_{M,N}$ and $\pi_{N,M}$ are epimorphisms, we have

$$\tilde{\sigma}_{M,N}\tilde{\sigma}_{N,M} = 1_{N\otimes_A M}, \quad \tilde{\sigma}_{N,M}\tilde{\sigma}_{M,N} = 1_{M\otimes_A N}$$

□

Lemma 1.2.27. *For every left A-module M we have $M \otimes_A A \cong M \cong A \otimes_A M$*

Proof. We will prove $M \otimes_A A \cong A$, the other one is entirely analogous. Consider the diagram

$$\begin{tikzcd}
M \otimes A \otimes A \arrow[r, shift left, "q_M \otimes 1"] \arrow[r, shift right, "1\otimes m"'] & M \otimes A \arrow[r, "\pi"] \arrow[d, "q_M"'] & M \underset{A}{\otimes} A \arrow[dl, dashed, "\exists!\varphi"] \\
& M &
\end{tikzcd}$$

since (M, q_M) is a right A-module, then q_M co equalizes $q_M \otimes 1$ and $1 \otimes m$, then there exists a unique φ, s.t $\varphi\pi = q_M$. let us see that φ is an A-isomorphism. The following diagram shows that φ is a left A-module morphism, since

$$\begin{tikzcd}
A \otimes M \otimes A \arrow[r, "1\otimes\pi"] \arrow[d, "p_M\otimes 1"'] & A \otimes M \otimes_A A \arrow[r, "1\otimes\varphi"] \arrow[d, "p_M\otimes 1"] & A \otimes M \arrow[d, "p_M"] \\
M \otimes A \arrow[r, "\pi"'] & M \otimes_A A \arrow[r, "\varphi"'] & M
\end{tikzcd}$$

the square in the left commutes and the bigger one does too, for

$$\varphi\pi(p_M \otimes 1) = \varphi(p_M \otimes 1)(1 \otimes \pi) = q_M(p_M \otimes 1) = p_M(1 \otimes q_M) = p_M(1 \otimes \varphi)(1 \otimes \pi).$$

Because $1 \otimes \pi$ is an epimorphism, we have $\varphi(p_M \otimes 1) = p_M(1 \otimes \varphi)$.
Next, we will define an inverse for φ:

$$\psi : \; M \xrightarrow{l^{-1}} M \otimes \mathbb{1} \xrightarrow{1\otimes\eta} M \otimes A \xrightarrow{\pi} M \otimes_A A.$$

ψ is a left A-module morphism, since it is the composition of left A-morphisms.

$$\varphi\psi = \varphi\pi(1 \otimes \eta)l_M^{-1} = q_M(1 \otimes \eta)l_M^{-1} = l_M \circ l_M^{-1} = 1_M.$$

For the other composition, we prove that $\psi\varphi\pi = \pi$, since π is an epimorphism, we have the result,

$$\begin{aligned}
\psi\varphi\pi = \psi q_M &= \pi(1 \otimes \eta)l_M^{-1} \circ q_M = \pi(1 \otimes \eta)q_M \circ l_{M\otimes A}^{-1} \\
&= \pi(q_M \otimes 1)(1 \otimes 1 \otimes \eta)l_{M\otimes A}^{-1} = \pi(1 \otimes m)(1 \otimes 1 \otimes \eta)l_{M\otimes A}^{-1} \\
&= \pi(1 \otimes l_A)l_{M\otimes A}^{-1} = \pi \circ l_{M\otimes A} \circ l_{M\otimes A}^{-1} = \pi
\end{aligned}$$

□

Lemma 1.2.28. *For every $A \in Comm(\mathcal{C})$, $M \otimes_A - : \mathcal{M}od_{\mathcal{C}}(A) \to \mathcal{M}od_{\mathcal{C}}(A)$ has a right adjoint.*

Sketch of the proof. The right adjoint $hom_A(M,-) : \mathcal{M}od_{\mathcal{C}}(A) \to \mathcal{M}od_{\mathcal{C}}(A)$ is defined from the internal hom in $\mathcal{C}$. First let us see that if M, M' are A-modules then $hom_{\mathcal{C}}(M,M')$ is an A-module via either the A-module structure on M or M'. More explicitly,

$$A \otimes hom_{\mathcal{C}}(M,M') \xrightarrow{\mu_{M'} \otimes 1} hom_{\mathcal{C}}(M',M') \otimes hom_{\mathcal{C}}(M,M') \xrightarrow{\circ} hom_{\mathcal{C}}(M,M')$$

$$hom_{\mathcal{C}}(M,M') \otimes A \xrightarrow{1 \otimes \mu_M} hom_{\mathcal{C}}(M,M') \otimes hom_{\mathcal{C}}(M,M) \xrightarrow{\circ} hom_{\mathcal{C}}(M,M')$$

where by abuse of notation we denote the adjoint arrows to the actions by the same letter and $\circ$ denotes the composition map as defined in (1.2.3). It is no hard to prove that this arrows give an A-module structure to $hom_{\mathcal{C}}(M,M')$. These two arrows correspond, by the adjunction $M \otimes - \rightleftharpoons hom_{\mathcal{C}}(M,-)$, to two arrows

$$hom_{\mathcal{C}}(M,M') \rightrightarrows hom_{\mathcal{C}}(A, hom_{\mathcal{C}}(M,M'))$$

then we define $hom_A(M,M')$ to be the equalizer of these two arrows. let us check that $hom_A(M,-) : \mathcal{M}od_{\mathcal{C}}(A) \to \mathcal{M}od_{\mathcal{C}}(A)$ is right adjoint to $M \otimes_A -$. In fact, we have the following bijections

$$\begin{aligned}
Hom_A(M \otimes_A N, M') &\cong Hom_A\Big(coeq\Big(\ M \otimes A \otimes N \rightrightarrows M \otimes N\ \Big), M'\Big) \\
&\cong eq\Big(\ Hom_A\,(M \otimes N, M') \rightrightarrows Hom_A(M \otimes A \otimes N, M')\ \Big) \\
&\cong eq\Big(\ Hom_A\,(M, hom_{\mathcal{C}}(N,M')) \rightrightarrows Hom_A(M \otimes A, hom_{\mathcal{C}}(N,M'))\ \Big) \\
&\cong eq\Big(\ Hom_A\,(M, hom_{\mathcal{C}}(N,M')) \rightrightarrows Hom_A(M, hom_{\mathcal{C}}(A \otimes N,M'))\ \Big) \\
&\cong Hom_A\Big(M, eq\Big(\ hom_{\mathcal{C}}(N,M') \rightrightarrows hom_{\mathcal{C}}(A \otimes N, M')\ \Big)\Big) \\
&= Hom_A(M, hom_A(N,M'))
\end{aligned}$$

$\square$

In summary, we have proved the following result:

Proposition 1.2.29. *If $\mathcal{C}$ is a closed symmetric monoidal category, then for every $A \in Comm(\mathcal{C})$, $(\mathcal{M}od_{\mathcal{C}}(A), \otimes_A, A)$ is also closed symmetric monoidal.*

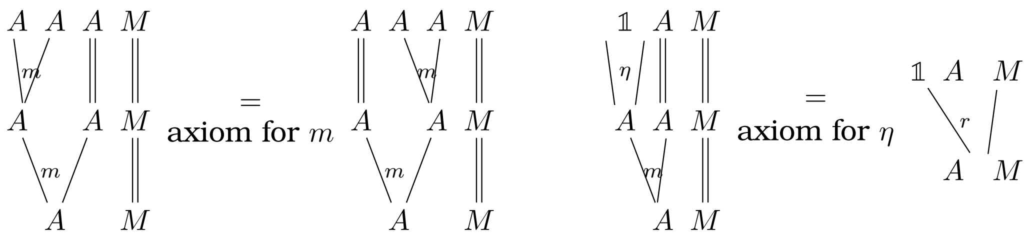
For the rest of the section we will miss out the constraint of associativity by the coherence Theorem. We are interested in the adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "A\otimes -"] \arrow[r, phantom, "\perp"] & \mathcal{M}od_{\mathcal{C}}(A). \arrow[l, shift left] \end{tikzcd}$$

For every object $M \in \mathcal{C}$, the A-module structure in $A \otimes M$ is given by the multiplication on the left, i.e., $m \otimes M : A \otimes A \otimes M \to A \otimes M$. This defines in fact an A-module since

$$(m \otimes M)(m \otimes A \otimes M) = (\, m \otimes M) \circ (A \otimes \ m \otimes M), \quad (\, m \otimes M) \circ (\eta_A \otimes A \otimes M) = r_{A \otimes M}.$$

which are the module axioms. In elevators calculus:



Thus, this correspondence defines a functor $A \otimes - : \mathcal{C} \to \mathcal{M}od_{\mathcal{C}}(A)$.

Proposition 1.2.30. *The functor $A \otimes - : \mathcal{C} \to \mathcal{M}od_{\mathcal{C}}(A)$ is a strong symmetric monoidal functor.*

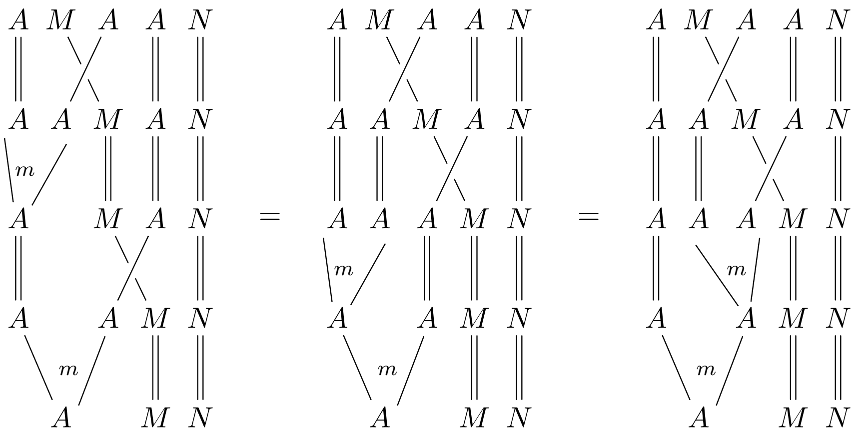
Proof. First we have to define the monoidal functor $(A \otimes -, \theta, \varepsilon)$. For every $M, N \in \mathcal{C}$, consider the family of morphisms $\theta_{M,N}$ defined as follows:

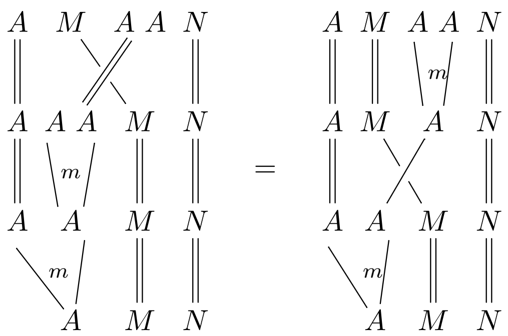
$$\begin{tikzcd} A \otimes M \otimes A \otimes A \otimes N \arrow[r, shift left, "\rho_1"] \arrow[r, shift right, "\rho_2"'] & A \otimes M \otimes A \otimes N \arrow[r, "\pi"] \arrow[d, "h"'] & A \otimes M \underset{A}{\otimes} A \otimes N \arrow[dl, dashed, "\theta_{M,N}"] \\ & A \otimes M \otimes N & \end{tikzcd}$$

the arrows ρ_1, ρ_2 being the actions on $A \otimes M$ and $A \otimes N$ respectively:

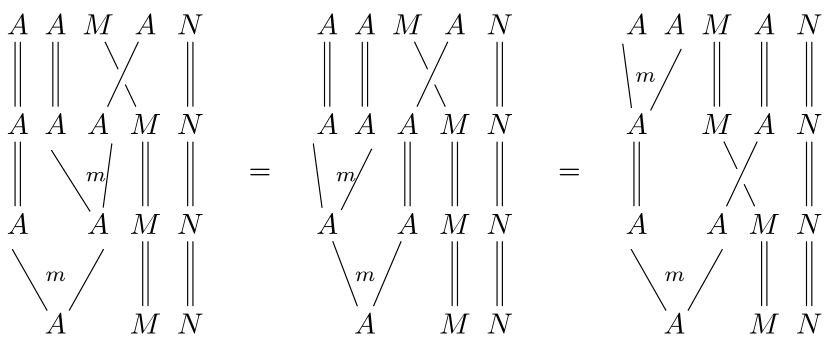
$$\rho_1 = (m \otimes 1) \circ (1 \otimes \sigma_{2,3} \otimes 1 \otimes 1), \quad \rho_2 = 1 \otimes 1 \otimes m \otimes 1, \quad h = m(1 \otimes \sigma_{2,3} \otimes 1).$$

The elevator diagram below shows that $h\rho_1 = h\rho_2$:





the first diagram is $h\rho_1$ and the last one $h\rho_2$. Then there exists the arrow $\theta_{M,N}$. The following elevator diagrams show that h is an A-module morphism between the A-modules $(A \otimes M \otimes A \otimes N, m \otimes 1 \otimes 1 \otimes 1)$ and $(A \otimes M \otimes N, m \otimes 1 \otimes 1)$:



Finally, we will see that $\theta_{M,N}$ is also an A-module morphism. Consider the following diagram:

$$\begin{tikzcd}
A\otimes A\otimes M\otimes A\otimes N \arrow[r, "1\otimes\pi"'] \arrow[d, "m\otimes 1"'] \arrow[rr, bend left, "h"] & A\otimes A\otimes M\otimes_A \otimes A\otimes N \arrow[r, "1\otimes\theta_{M,N}"'] \arrow[d, "m\otimes 1"] & A\otimes A\otimes M\otimes N \arrow[d, "m\otimes 1"] \\
A\otimes M\otimes A\otimes N \arrow[r, "\pi"] \arrow[rr, bend right, "1\otimes h"'] & A\otimes M\otimes A\otimes N \arrow[r, "\theta_{M,N}"] & A\otimes M\otimes N,
\end{tikzcd} \tag{1.2.11}$$

the square in the left and the bigger one both commute, since they say that π, h are left A-module morphisms. Since $1\otimes\pi$ is an epimorphism, the square in the right commutes, thus $\theta_{M,N}$ is also an A-module morphism.
The structure morphism ε is given by the unity isomorphism in $\mathcal{C}$

$$m(A\otimes\eta_A) = l_A : A \to A\otimes\mathbb{1}.$$

This morphism is compatible with the A-module structure, since

$$m(A\otimes\ m)(A\otimes A\eta_A) =\ m(\ m\otimes A)(A\otimes A\otimes\eta_A) =\ m(A\otimes\eta_A)(\ m\otimes\mathbb{1}),$$

then $A\cong A\otimes\mathbb{1}$ is an isomorphism in $\mathcal{M}od_{\mathcal{C}}(A)$. We have that the triple $(A\otimes -, \theta, \varepsilon)$ with $\varepsilon = l_A^{-1}$ is a lax monoidal functor.

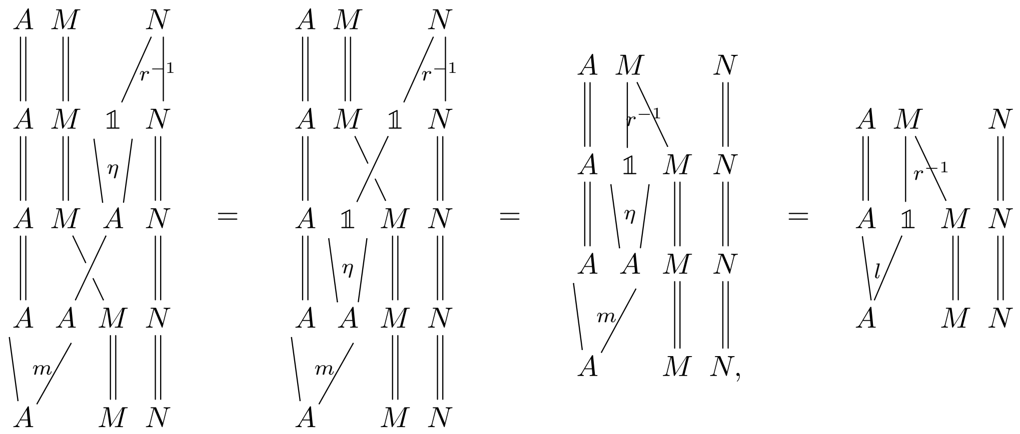
Now, let us check that $\theta_{M,N}$ is an isomorphism and the coherence with the symmetries. let us define the arrow $\bar{\theta}_{M,N}$ by the composition

$$A\otimes M\otimes N \xrightarrow{1\otimes r_N^{-1}} A\otimes M\otimes\mathbb{1}\otimes N \xrightarrow{1\otimes\eta_A\otimes 1} A\otimes M\otimes A\otimes N \xrightarrow{\pi} A\otimes M\underset{A}{\otimes} A\otimes N$$

we will prove that both compositions $\theta_{M,N}\circ\bar{\theta}_{M,N},\ \bar{\theta}_{M,N}\circ\theta_{M,N}$ are the identity morphisms.

$$\theta_{M,N}\circ\bar{\theta}_{M,N} = \theta_{M,N}\circ\pi\circ(1\otimes\eta_A\otimes 1)\circ(1r_N^{-1}) = h\circ(1\otimes\eta_A\otimes 1)\circ(1\otimes r_N^{-1}),$$

using elevator diagrams we will prove that the last member of the equality is the identity morphism:



so, we have the isomorphism $(l \otimes 1 \otimes 1)(1 \otimes r^{-1} \otimes 1) : A \otimes M \otimes N \to A \otimes M \otimes N$ which by Mac Lane's coherence Theorem must be the identity. For the other composition, we prove, in a similar way that, $\bar{\theta}_{M,N} \circ \theta_{M,N} \circ \pi = \pi$. Since π is an epimorphism and we get the result.

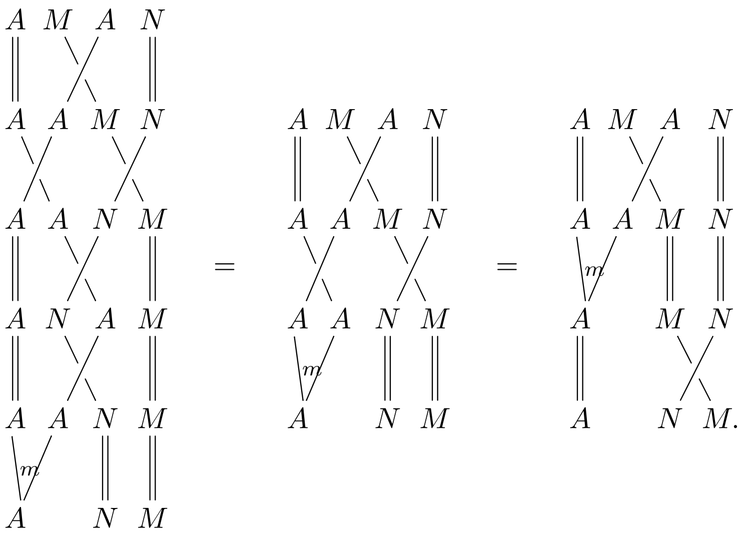
So far, we have an inverse for $\theta_{M,N}$ in the category $\mathcal{C}$, by an argument similar to the one that proves that θ is an A-module morphism, we have that its inverse is also an A-morphism.

let us check the coherence with the symmetries, i.e., we have to check that the square in the right commutes:

$$\begin{tikzcd}
A \otimes M \otimes A \otimes N \arrow[r, "\pi_{M,N}"] \arrow[d, "\text{by coherence } (1\otimes\sigma\otimes1)\circ(\sigma\otimes\sigma)\circ(1\otimes\sigma\otimes1)=" left, "\sigma_{A\otimes M, A\otimes N}"] \arrow[rr, bend left, "h_{M,N}"] & A \otimes M \underset{A}{\otimes} A \otimes N \arrow[r, "\theta_{M,N}"] \arrow[d, "\sigma^A"] & A \otimes M \otimes N \arrow[d, "1\otimes\sigma"] \\
A \otimes N \otimes A \otimes M \arrow[r, "\pi_{N,M}"'] \arrow[rr, bend right, "h_{N,M}"'] & A \otimes N \underset{A}{\otimes} A \otimes M \arrow[r, "\theta_{N,M}"'] & A \otimes N \otimes M.
\end{tikzcd}$$

For this, we prove that the bigger square commutes, that is:

$$(1 \otimes \sigma) \circ h_{M,N} = h_{N,M} \circ (1 \otimes \sigma \otimes 1) \circ (\sigma \otimes \sigma) \circ (1 \otimes \sigma \otimes 1)$$



Since the square in the left also commutes and $\pi_{M,N}$ is an epimorphism we have that the square in the right commutes. $\square$

As an example of this strong monoidal functor, we have the ”ring” base change:

Corollary 1.2.31. *Let* $A, B \in Comm(\mathcal{C})$ *and* $f : A \to B$ *be an algebra morphism. Then, the functor* $B \otimes_A - : \mathcal{M}od(A) \to \mathcal{M}od(B)$ *is symmetric strong monoidal.* $\square$

Tensor Algebra

Let $(\mathcal{C}, \sigma)$ be a cocomplete symmetric closed monoidal category, for simplicity we will assume $\mathcal{C}$ is strict. Let $V \in \mathcal{C}$, we define the *Tensor algebra* of V to be $TV = \bigoplus_{n \geq 0} V^{\otimes n}$. Via the isomorphism $V^{\otimes p} \otimes V^{\otimes q} \xrightarrow{\cong} V^{\otimes (p+q)}$, TV becomes an algebra in $\mathcal{C}$, with unity being the canonical map $V^{\otimes 0} \cong \mathbb{1} \to TV$.

TV has the following universal property: for every algebra A in $\mathcal{C}$ and for every $\mathcal{C}$-morphism $f : V \to A$ there exist a unique algebra morphism $\bar{f} : TV \to A$ such that $\bar{f} \circ i_V = f$ with $i_V : V \to TV$ the canonical map of $V^{\otimes 1}$ to the coproduct TV. In fact for every $n \in \mathbb{N}$, f induces a morphism $f^{(n)} : V^{\otimes n} \to A$ given by the composition

$$\begin{tikzcd} V^{\otimes n} \arrow[r, "f^{\otimes n}"] \arrow[dr, "i_n"'] & A^{\otimes n} \arrow[r, "m_n"] & A \\ & TV \arrow[ur, "\exists ! \bar{f}"'] & \end{tikzcd}$$

with $m_n := m \circ (m_{n-1} \otimes 1)$, $n \geq 3$, $m_2 = m$ the product in A and $i_1 = i_V$. The existence of the $\mathcal{C}$-morphism $\bar{f}$ is due to the universal property of the coproduct. But since $\bar{f}$ must be an algebra morphism, it is forced to be the coproduct of the arrows of the form $\bar{f}_n = f^{(n)}$. This means that the forgetful functor $\mathcal{A}lg(\mathcal{C}) \to \mathcal{C}$ is right adjoint to $T : \mathcal{C} \to \mathcal{A}lg(\mathcal{C})$

Symmetric Algebra

Let $\mathcal{C}$ cocomplete closed symmetric monoidal category which by simplicity we suppose is strict. For every $n \in \mathbb{N}$, there is an action of the symmetric group S_n on $V^{\otimes n}$ in $\mathcal{C}$: consider the group

$$S_n = \langle s_1, ...s_{n-1}, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \text{ if } |i-j| > 1, \quad s_i^2 = 1 \rangle$$

regarded as the category with one object and morphisms the elements of S_n then we have a functor

$$\begin{aligned} F_{n,V} : S_n &\to \mathcal{C} \\ \{*\} &\mapsto V^{\otimes n} \\ s_i &\mapsto \phi_{s_i} := 1 \otimes \sigma_i \otimes 1 : V^{\otimes n} \to V^{\otimes n}, \end{aligned} \tag{1.2.12}$$

with $\sigma_i : V \otimes V \to V \otimes V$ the braiding in $\mathcal{C}$ between positions i and $i+1$ $\forall i = 1, ..., n-1$. Since every element in S_n can be written as a product of the generators, we have that if $\sigma = s_{i_1} \cdot s_{i_2} \cdots s_{i_k}$ then $\phi_\sigma : V^{\otimes n} \to V^{\otimes n}$ is defined in the obvious way as the composition $\phi_{s_{i_1}} \circ \cdots \circ \phi_{s_{i_k}}$. As $F(1) = 1 : V^{\otimes n} \to V^{\otimes n}$ and $F(s_i s_j) = \phi_{s_i} \circ \phi_{s_j} = \phi_{s_i s_j}$, the functor F defines an action of S_n in $V^{\otimes n}$.

Remark 1.2.32. When $\mathcal{C}$ is braided non-symmetric, there is an action of the Artin braid group $\mathcal{B}_n$ with generators $s_1, \ldots, s_{n-1}$ subject to relations

$$\begin{aligned} s_i s_j =& s_j s_i \quad \text{for } |j-i|>1 \\ s_{i+1} s_i s_{i+1} =& s_i s_{i+1} s_i. \end{aligned}$$

The n-th symmetric power of V, $S^n(V)$ is defined as the colimit of the functor F.

we define the *symmetric algebra* of $V \in \mathcal{C}$ as $S(V) := \bigoplus_{n \in \mathbb{N}} S^n(V)$. We have to define morphisms $m : S(V) \otimes S(V) \to S(V),\ \eta : \mathbb{1} \to S(V)$ that make $S(V)$ into an algebra in $\mathcal{C}$. For this, consider the canonical maps

$$\begin{aligned} p_n :& V^{\otimes n} \to S^n(V) \\ \lambda_n :& S^n(V) \to S(V) \end{aligned}$$

and the following diagram

$$\begin{tikzcd} V^{\otimes n} \otimes V^{\otimes m} \arrow[r, "\cong"] \arrow[d, "p_n \otimes p_m"'] & V^{\otimes(n+m)} \arrow[d, "p_{n+m}"] \\ S^n(V) \otimes S^m(V) \arrow[r, dashed, "\exists\ \varphi_{n,m}"] \arrow[d, "\lambda_n \otimes \lambda_m"'] & S^{n+m}(V) \arrow[d, "\lambda_{n+m}"] \\ S(V) \otimes S(V) \arrow[r, dashed, "\exists\ m"'] & S(V). \end{tikzcd}$$

The existence of $\varphi_{m,n}$ is due to the universal property of the colimit $S^n(V) \otimes S^m(V)$, the existence of m is induced by $j_{n+m}\varphi_{n,m}$ an the universal property of the coproduct $S(V)$. For the unity, take $\eta = \lambda_0 : \mathbb{1} \to S(V)$. For the commutativity of m, consider the following commutative diagram

$$\begin{tikzcd}
S^n(V)\otimes S^m(V) \arrow[rr, "\sigma"] \arrow[dr, "\lambda_n\otimes\lambda_m"] \arrow[dd, "\varphi_{n,m}"'] & & S^m(V)\otimes S^n(V) \arrow[dr, "\lambda_m\otimes\lambda_n"] \arrow[dd, "\varphi_{m,n}"] & \\
& S(V)\otimes S(V) \arrow[rr, "\sigma"] \arrow[dd, "m"] & & S(V)\otimes S(V) \arrow[dd, "m"] \\
S^{n+m}(V) \arrow[rr, "1"] \arrow[dr, "\lambda_{n+m}"'] & & S^{n+m}(V) \arrow[dr, "\lambda_{n+m}"] & \\
& S(V) \arrow[rr, "1"'] & & S(V)
\end{tikzcd}$$

Example 1.2.33. Take $\mathcal{C} = (\mathbf{Vect}(k), \otimes, k)$ the category of k-vector spaces with the canonical isomorphisms for the associator and unitor constraints and consider the two braidings $\tau, -\tau$, that is, the usual transposition $\tau(v \otimes w) = w \otimes v$ and $-\tau(v \otimes w) = -w \otimes v$, both are symmetries for the category, however when considering the transposition we get that $S^n(V)$ is the usual n-th symmetric power for the vector space V. On the other hand, if we consider the braiding $-\tau$ we get that $S^n(V)$ is the n-th exterior power.

The previous construction gives a functor $S : \mathcal{C} \to CommAlg(\mathcal{C})$ from $\mathcal{C}$ to the category of commutative algebras in $\mathcal{C}$. S is actually the left adjoint functor to the forgetful functor $U : CommAlg(\mathcal{C}) \to \mathcal{C}$. This is the well-known universal property of the symmetric algebra. In [12] there is a more detailed description of the construction of the symmetric algebra as well as the proof of the adjunction $Comm(\mathcal{C}) \rightleftharpoons \mathcal{C}$.

1.3 Commutative Algebra in a Monoidal Category.

In this section $\mathcal{C}$ denotes an abelian, closed symmetric monoidal category, bicomplete such that the functor $V_0 = Hom(\mathbb{1}, -)$ is conservative i.e., reflects isomorphisms. First recall the notion of finitely presented objects in abstract categories:

Definition 1.3.1 (Finitely presented object)**.** Assume $\mathcal{C}$ has filtered colimits. An object X in $\mathcal{C}$ is said to be finitely presented if the functor $Hom_{\mathcal{C}}(X,-):\mathcal{C}\to\mathcal{E}ns$ commutes with filtered colimits.

Ideals

Definition 1.3.2. Let $A\in Comm(\mathcal{C})$, $M\in\mathcal{M}od_{\mathcal{C}}(A)$. An A-submodule of M is just a subobject $N\to M$ in $\mathcal{M}od_{\mathcal{C}}(A)$.

Definition 1.3.3 (Ideal)**.** An ideal of A is an A-submodule $I\to A$.

Remark 1.3.4. The category of subobjects of a given object M is in fact a poset with finite intersections and small unions. The intersection of two subobjects is given by the fiber product of the two monomorphisms over M. The union of a family of subobjects $(M_i)_{i\in I}$ is given by the image of the natural morphism $\coprod_{i\in I}M_i\to M$.

Definition 1.3.5. Let $f:A\to B$ in $Comm(\mathcal{C})$ and $j:I\to A$ be an ideal of A. The image ideal in B is given by the image of the morphism $B\otimes_A I\xrightarrow{1\otimes f\circ j}B\otimes_A B\xrightarrow{m_B}B$, this ideal is denoted BI.

Remark 1.3.6. This induced ideal comes from applying the functor $B\otimes_A -$ to the inclusion $I\overset{j}{\hookrightarrow}A$ and using the isomorphism $m_B(1\otimes f):B\otimes_A A\longrightarrow B$.

Proposition 1.3.7. *Let $f:A\to B$ in $Comm(\mathcal{C})$ be a flat morphism, then $B\otimes_A I\cong BI$.*

Proof. Since B is flat over A then $B\otimes_A -$ preserves monomorphism, then the composition $m_B(1\otimes f)(1\otimes j)$ is a monomorphism, therefore in the factorization diagram

$$\begin{tikzcd} B \otimes_A I \arrow[r, "1\otimes f\circ j"] \arrow[dr] & B \otimes_A B \arrow[r, "m_B"] & B \\ & BI \arrow[ur] & \end{tikzcd}$$

the epimorphism $B \otimes_A I \twoheadrightarrow BI$ is also a monomorphism, hence an isomorphism.

□

Definition 1.3.8 (Ideal Sum). Given a family of ideals $(I_i)_i$, we define the ideal sum as the image of the induced morphism $\bigoplus_i I_i \to A$, that is

$$\begin{tikzcd} \bigoplus_i I_i \arrow[rr] \arrow[dr, two heads] & & A \\ & \sum_i I_i \arrow[ur, hook] & \end{tikzcd}$$

Definition 1.3.9 (Ideal product). Let I, J be two ideals of A, the ideal product $I \cdot J$ is given by the image of the morphism

$$\begin{tikzcd} I \otimes J \arrow[r] \arrow[dr, two heads] & A \otimes A \arrow[r, "m"] & A \\ & I \cdot J. \arrow[ur, hook] & \end{tikzcd}$$

One has to check that this is in fact a sub object of A, see [6].

Proposition 1.3.10.

1. *The ideal product is associative and commutative. A is the neutral element for this operation.*
2. *The ideal product distributes over arbitrary sums.*

Proof. See [6]. □

Once one has defined the ideal product, we can define prime ideals:

Definition 1.3.11 (Prime Ideals). and ideal $\mathfrak{p}$ of A is called prime if $\mathfrak{p} \neq A$ and if for all ideals I, J we have that $I \cdot J \hookrightarrow \mathfrak{p}$, then either $I \hookrightarrow \mathfrak{p}$ or $J \hookrightarrow \mathfrak{p}$.

Proposition 1.3.12 (Existence of Prime Ideals.). *Given $A \in Comm(\mathcal{C})$ finitely presentable, then the set of prime ideals of A is not empty.*

Proof. First, let us see that there exist maximal ideals. For this, consider a non empty chain of proper ideals $(I_i)_{i\in\Gamma}$ and take its sum $\sum_i I_i \hookrightarrow A$. The sum is again a proper ideal, for if not, then since A is finitely presentable, there exists a finite set $\Lambda \subset \Gamma$ such that $\sum_{i\in\Lambda} I_i = A$, therefore, there is an index i_0 s.t $I_{i_0} = 0$, a contradiction. Then by Zorn's Lemma there exists a maximal ideal $\mathfrak{m}$ of A. We claim that $\mathfrak{m}$ is prime. In fact, if I, J are not sub ideals of $\mathfrak{m}$ but its product $I \cdot J$ is, the the sums $I + \mathfrak{m}$, $J + \mathfrak{m}$ are bigger than $\mathfrak{m}$ itself, therefore, by maximality $I + \mathfrak{m} = J + \mathfrak{m} = A$, thus

$$A = A \cdot A = (I + \mathfrak{m})(J + \mathfrak{m}) = I \cdot J + I \cdot \mathfrak{m} + J \cdot \mathfrak{m} + \mathfrak{m} \cdot \mathfrak{m} \hookrightarrow \mathfrak{m}$$

which is a contradiction. $\square$

The Monoids $Hom_{\mathcal{C}}(\mathbb{1}, A)$ and $Hom_A(A, A)$ for A in $Comm(\mathcal{C})$

For A a commutative algebra in $\mathcal{C}$, we define the operation $* : Hom_{\mathcal{C}}(\mathbb{1}, A) \times Hom_{\mathcal{C}}(\mathbb{1}, A) \to Hom_{\mathcal{C}}(\mathbb{1}, A)$ as the following composition

$$\mathbb{1} \cong \mathbb{1} \otimes \mathbb{1} \xrightarrow{f\otimes g} A \otimes A \xrightarrow{m} A.$$

Lemma 1.3.13. *$(Hom(\mathbb{1}, A), *, \eta)$ is a commutative monoid in $\mathcal{E}ns$.*

Proof. It is clear that $*$ is an associative operation. For the unity we have that

$$f * \eta = m(f \otimes \eta)(l_{\mathbb{1}}^{-1}) = m(1 \otimes \eta)(f \otimes 1)(l_{\mathbb{1}}^{-1}) = l_A(f \otimes 1)(l_{\mathbb{1}}^{-1}) = f \circ l_{\mathbb{1}} \circ l_{\mathbb{1}}^{-1} = f$$

the commutativity comes from the commutativity of A. $\square$

Lemma 1.3.14. *The operation $*$ is the adjoint of the composition operation in the monoid $Hom_{\mathcal{C}}(A, A)$. Even more, the adjunction $\varphi : Hom_{\mathcal{C}}(\mathbb{1}, A) \overset{\cong}{\to} Hom_A(A, A)$ is an isomorphism of commutative monoids.*

Proof. See [12]. □

Remark 1.3.15. In the additive case, these monoids are in fact commutative rings, since they are also abelian groups and φ is compatible with the group operation as well.

Definition 1.3.16 (Generated Ideal)**.** Let $(f_i : A \to A)_{i \in I}$ a family of arrows (possibly infinite) in $\mathcal{M}od_{\mathcal{C}}(A)$, the ideal generated by this arrows, denoted by $< f_i : i \in I >$ is defined as the image submodule of the morphism $\coprod_i f_i : \coprod_i A \to A$.

Proposition 1.3.17. *Let A in $Comm(\mathcal{C})$ and I an ideal of A containing an invertible element, that is to say, an invertible arrow $f : \mathbb{1} \to A$ factorizing through I, then $I = A$.*

Proof. Since $f : \mathbb{1} \to A$ is invertible then the adjoint $\varphi_f : A \to A$ is an isomorphism which factorizes through the ideal I, then $I \hookrightarrow A$ is also an epimorphism, therefore an isomorphism. □

Field Objects

Definition 1.3.18. $K \neq 0 \in Comm(\mathcal{C})$ is called a field object, or just a field if K has no proper non trivial ideals.

Proposition 1.3.19. *$K \neq 0$ in $Comm(\mathcal{C})$ is a field if and only if for all $0 \neq f \in Hom_{\mathcal{C}}(\mathbb{1}, K)$ there exists $g \in Hom_{\mathcal{C}}(\mathbb{1}, K)$ such that $f * g = \eta$.*

Proof. $\Leftarrow$): Let I be a proper ideal of K, then since V_0 is conservative, there exists $0 \neq f \in I$, this means that $f : \mathbb{1} \to K$ factorizes through I. On the other hand, the ideal generated by f is K, for f is invertible, therefore $< f >= K \subset I$, a contradiction.

$\Rightarrow$): Let $f : \mathbb{1} \to K$ a non zero arrow and consider its adjoint morphism $\varphi_f : K \to K$, then the image $Im(\varphi_f)$, being a subobject of K is in fact K itself. On the other hand, $Ker\varphi_f = 0$, since the kernel is a submodule, therefore, $\varphi_f : K \to K$ being an epimorphism and a monomorphism is an isomorphism with inverse $(\varphi_f)^{-1}$. Now, via the adjunction there exists the arrow $f^{-1} : \mathbb{1} \to K$ such that $(\varphi_f)^{-1} = \varphi_{f^{-1}}$, then $f * f^{-1} = \eta$. $\square$

Localizations

Definition 1.3.20. Let $A \in Comm(\mathcal{C})$ and S a multiplicatively closed subset of $Hom_{\mathcal{C}}(\mathbb{1}, A)$. The localization of A at S is given by the couple (A_S, Φ_S), with $\Phi_S : A \to A_S \in Comm(\mathcal{C})$ is an algebra morphism such that:

1. For every $f \in S$, the induced morphism $\Phi_S(f) : \mathbb{1} \to A_S$ is invertible with respect to the operation $*$,

2. Universal property of A_S: For every $u : A \to B$ in $Comm(\mathcal{C})$, such that $u(f) : \mathbb{1} \to B$ is invertible for all $f \in S$, there exists a unique algebra morphism $v : A_S \to B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\Phi_S"] \arrow[dr, "u"'] & A_S \arrow[d, dashed, "v"] \\ & B \end{tikzcd}$$

If S is generated by one element f, then the localization of A at S is denoted A_f and Φ_S just Φ.

Proposition 1.3.21 (Existence of the localization.). *Let $A \in Comm(\mathcal{C})$, $f : \mathbb{1} \to A$. Let j_0, j_1 the canonical maps from A to $S_A(A) = \bigoplus A^{\otimes_A n}/S_n$ and $\delta = j_1 \circ m(1 \otimes f) r_A^{-1}$. If ψ denotes the adjunction between the functor $S_A : \mathcal{M}od_{\mathcal{C}}(A) \to A - alg$ and the forgetful functor, then*

$$A_f \cong coeq(\ S_A(A) \underset{\psi(j_0)}{\overset{\psi(\delta)}{\rightrightarrows}} S_A(A)\ )$$

Proof. See [12, Proposition 2.2.5]. □

Proposition 1.3.22.

1. *Associated to S there is a filtered category denoted also by S, whose objects are the elements of S, and for every pair $f, g \in S$, $Hom_S(f,g) = \{h \in S \ : \ f * h = g\}$. Then*
$$A_S \cong Colim_S A_f.$$

2. *For each $f \in Hom_{\mathcal{C}}(\mathbb{1}, A)$ and φ_f the adjoint morphism we have the following isomorphism in $Comm(\mathcal{C})$*
$$Colim(\ A \xrightarrow{\varphi_f} A \xrightarrow{\varphi_f} A \xrightarrow{\varphi_f} \cdots\ ) \cong A_f.$$

Proof. See [12, Proposition 2.2.5]. □

Proposition 1.3.23. *If filtered colimits are exact in $\mathcal{C}$, then $\Phi_S : A \to A_S$ is a flat epimorphism.*

Proof. Φ_S is an epimorphism due to the universal property of the localization. A is flat as A-module and A_f is a filtered colimit of A then A_f is flat. Finally A_S is a filtered colimit of the flat modules A_f, therefore, it is a flat module. □

Definition 1.3.24. A family $(f_i : \mathbb{1} \to A)_{i \in I}$ is said to be a generating family if the adjoint family $(\varphi_{f_i} : A \to A) \subset Hom_A(A, A)$ is an epimorphic family, i.e., $\coprod_i \varphi_{f_i} : \coprod A \to A$ is an epimorphism.

Definition 1.3.25 (Partition of Unity)**.** A finite collection of $(f_i)_{i \in J} \subset Hom_{\mathcal{C}}(\mathbb{1}, A)$ is a partition of unity if there exists arrows $(s_i)_{i \in J} \in Hom_{\mathcal{C}}(\mathbb{1}, A)$ such that $\sum_{i \in J} \varphi_{s_i} \varphi_{f_i} = 1$ in $\mathcal{M}od_{\mathcal{C}}(A)$.

Proposition 1.3.26.

1. *Let* $(f_i)_{i\in I}$ *be a generating family of* A*. If* M *is an* A*-module such that* $M_{f_i} = 0$ *in* $\mathcal{M}od(A_{f_i})$ *for every* $i \in I$*, then* $M = 0$*.*

2. *Let* $(f_i)_{i\in I}$ *be a partition of unity. If* $M \xrightarrow{u} N$ *is an* A*-module morphism such that, for every* $i \in I$*,* $M_{f_i} \xrightarrow{u_{f_i}} N_{f_i}$ *is an isomorphism in* $\mathcal{M}od(A_{f_i})$*, then* $u : M \to N$ *is an isomorphism.*

Proof. The proof is similar to the case of classical commutative algebra. For details see [4]. □

Chapter 2

Hopf Algebras

Resumen

En este capítulo damos la definición de álgebra de Hopf y sus deformaciones. Nos concentramos principalmente en el álgebra de Hopf que define el álgebra de un grupo. Más precisamente nos interesa el álgebra kG con k un cuerpo de característica 0 y G un grupo abeliano finito.

En la sección 2.4 se presenta la categoría $\mathcal{U}$ para la cual la $\mathbb{R}$-álgebra de Cayley de los octoniones $\mathbb{O}$ es un objeto álgebra conmutativa asociativa y unitaria. Dicha categría es monoidalmente equivalente a la categría $Vect_G^{\phi}(\mathbb{R})$ de $\mathbb{R}$-espacios vectoriales $G = \mathbb{Z}_2^3$-graduados cuyo producto tensorial $\otimes_F$ no coincide con el canónico de **Vect**$\mathbb{R}$ y proviene de una función $F(g,h) = (-1)^{f(g,h)}$ con $f : G \times G \to G$ dada por

$$f(g,h) = \sum_{i \geq j} g_i h_j + h_1 g_2 g_3 + g_1 h_2 g_3 + g_1 g_2 h_3, \quad \text{for } g,h \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

con asociador no trivial Φ, dicho asociador proviene de una función $\phi : (kG)^{\otimes 3} \to k^*$ que satisface las condiciones de cociclo dadas en (2.4.1).

En esta sección también se presenta la categoría monoidal simétrica de los $\mathbb{O}$-módulos, $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$, se prueba que el funtor de olvido canónico $Hom_{\mathbb{O}}(\mathbb{O}, -)$

es conservativo y preserva epimorfismos y colímites filtrantes, esto dice que $\mathbb{O}$ es un generador proyectivo y de presentación finita para $\mathcal{M}od_{\mathcal{C}}(\mathbb{O})$. Como conclusión se tiene que la categoría $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$es equivalente (no en el sentido monoidal) a una categoría de módulos sobre un anillo, proposición 2.4.8.

Summary

In this chapter we give the definition of Hopf algebra and its deformations. We are most interested in the Hopf algebra kG with k a field of characteristic zero and G a finite abelian group. The category of kG-modules (or comodules) is symmetric monoidal, we study the case of the deformation of the bialgebra kG and the monoidal category obtained by this deformation.

Notation: If x is an element of a coalgebra (C, Δ, ϵ), the element $\Delta(x)$ in $C \otimes C$ is of the form

$$\Delta(x) = \sum_i x_i' \otimes x_i''.$$

In order to get rid of the subscripts, we write this sum in the form

$$\Delta(x) = \sum_{(x)} x_1 \otimes x_2.$$

2.1 Preliminaries on Hopf Algebras

Definition 2.1.1. A k- bialgebra H is an object in $Vect(k)$ which is both an algebra (H, m, η) and a coalgebra (H, Δ, ϵ) and such that either Δ, ϵ are algebra morphisms or m, η are co-algebra morphisms. This can be depicted in a commutative diagram as follows where $(m \otimes m)(1 \otimes \sigma_{2,3} \otimes 1)$ is the product map in the algebra $H \otimes H$

Before defining the object of our interest let us recall the convolution algebra.

$$\begin{tikzcd} H \otimes H \arrow[rr, "m"] \arrow[d, "\Delta\otimes\Delta"'] & & H \arrow[d, "\Delta"] \\ H \otimes H \otimes H \otimes H \arrow[rr, "(m\otimes m)(1\otimes\sigma_{2,3}\otimes 1)"'] & & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[r, "H"] \arrow[d, "\epsilon\otimes\epsilon"'] & H \arrow[d, "\epsilon"] \\ k \otimes k \arrow[r, "\simeq"] & k \end{tikzcd}$$

Definition 2.1.2. Let A be an algebra and C a coalgebra, the convolution product $f * g$ for f, g in $Hom_k(C, A)$ is defined by the following composition

$$\begin{tikzcd} C \arrow[r, "\Delta"'] \arrow[rrr, bend left, "f*g"] & C \otimes C \arrow[r, "f\otimes g"'] & A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

Proposition 2.1.3. *$(Hom_k(C, A), *, \eta \circ \epsilon)$ is a unital k-algebra.*

Proof. The associativity of $*$ is due to the associativity of m and coassociativity of Δ. We now prove that $\eta\epsilon$ is a left unit:

$$(\eta\epsilon * f)(x) = \sum_{(x)} \epsilon(x_1) f(x_2) = f(\sum_{(x)} \epsilon(x_1) x_2) = f(x)$$

the last two equalities are due to the unit and counit axioms of η and ϵ respectively. Analogously one obtains that $\eta\epsilon$ is a right unit. □

If $A = k$ then the algebra structure on $C^\vee := Hom_k(C, k)$ given by the previous proposition coincides with the one given in Proposition 2.1.4 i.

Proposition 2.1.4. *i. Let (C, Δ, ϵ) be a k-coalgebra. Then $(C^\vee, \Delta^\vee, \epsilon^\vee)$ is an algebra.*

ii. Let (A, m, η) be a finite dimensional k-algebra. Then $(A^\vee, m^\vee, \eta^\vee)$ is coalgebra.
($\vee$ means transpose of the k-linear map).

Proof. i. We obtained de algebra structure in $C^\vee$ by dualizing the diagrams 1.2.5. Explicitly $\Delta^\vee(f \otimes g)(x) = (f \otimes g)\Delta(x)$. In the case of infinite dimension we must restrict $\Delta^\vee$ to $C^\vee \otimes C^\vee \subset (C \otimes C)^\vee$.

ii. As we have a finite dimensional algebra then $A^\vee \otimes A^\vee \to (A \otimes A)^\vee$ is an isomorphism so we can define $(\Delta f)(a \otimes b) = f \circ m(a \otimes b) =: m^\vee(f)(a \otimes b)$. Again the coassociativity and count axioms are obtained by dualizing diagrams 1.2.3. □

We can put together those definitions in a compatible way and get what is called a *bialgebra*

Definition 2.1.5. A k-bialgebra $(H, m, \eta, \Delta, \epsilon)$ is a Hopf algebra if there exists an element $S \in Hom_k(H, H)$ such that S is an inverse to id_H with respect to the convolution product. More explicitly

$$S * id_H = m \circ (S \otimes id) \circ \Delta = id_H * S = m \circ (id \otimes S) \circ \Delta = \eta \circ \epsilon$$

The endomorphism S is called an *Antipode* for the bialgebra H.

The first example of a Hopf algebra one has in mind is

Example 2.1.6. Let G be a group and kG its group algebra, i.e., the k-vector space with basis the elements of G. kG is a Hopf algebra with product map the multiplication in G, the unit map is $\eta(1_k) = e$ the unit in the group and extended k-linearly. The coproduct, counit and antipode maps are given by

$$\Delta x = x \otimes x, \quad \epsilon(x) = 1, \quad Sx = x^{-1} \text{ for all } x \in G \tag{2.1.1}$$

defined in the basis elements and then extended linearly over k.

In the next sections we will be mostly interested in this example and its *Gauge transformations*.

2.2 Cocycles and twisted Hopf algebras.

Twisting or gauge transformations for quasi-Hopf algebras appeared in Drinfeld's work on deformation of Hopf algebras, see [9], [10]. In this section we will develop the theory of gauge transformations in the dual version, which roughly speaking is the deformation of the algebra structure instead of the coalgebra. We will follow [19, Chapter 2]. Gauge transformations in Hopf algebras are also known as 2-cocyle deformations

Let $(H, m, \eta, \Delta, \epsilon)$ be a Hopf algebra, a linear functional $F \in Hom_k(H^{\otimes n}, k)$ which is convolution invertible is called an *n-cocycle*. F is called a *unital cocycle* if

$$F(a_1, \cdots, 1, \cdots, a_{n-1}) = \epsilon(a_1) \cdots \epsilon(a_{n-1})$$

for 1 in any position. Its coboundary is the $n+1$ cocycle

$$\partial F = \left(\prod_{i\, even} F \circ m_i \right) \left(\prod_{i\, odd} F^{-1} \circ m_i \right)$$

where $m_i : H^{\otimes n} \to H^{\otimes n-1}$ is the multiplication in H in the $i, i+1$ positions with convention $F \circ m_0 = \epsilon \otimes F$, $F \circ m_{n+1} = F \otimes \epsilon$ and F^{-1} denotes the inverse of F with respect to the convolution product.

Thus, the first two coboundaries are:

$$\partial F(a, b) = \sum_{(a),(b)} F(b_1) F(a_1) F^{-1}(a_2 b_2) \tag{2.2.1}$$

and

$$\partial F(a, b, c) = \sum_{(a),(b),(c)} F(b_1, c_1) F(a_1, b_2 c_2) F^{-1}(a_2 b_3, c_3) F^{-1}(a_3, b_4). \tag{2.2.2}$$

The 1,2 and 3-cocycle conditions are, $\forall\ a, b, c \in H$:

$$F(ab) = F(a)F(b) \tag{2.2.3}$$

$$\sum F(b_1, c_1) F(a_1, b_2 c_2) = \sum F(a_1, b_1) F(a_2 b_2, c) \tag{2.2.4}$$

$$\sum F(b_1, c_1, d_1) F(a_1, b_2 c_2, d_2) F(a_2, b_3, c_3) = \sum F(a_1, b_1, c_1 d_1) F(a_2 b_2, c_2, d_2) \tag{2.2.5}$$

Definition 2.2.1 (Dual quasi-triangular dual quasi Hopf algebra)**.** A k-coalgebra C with k-linear morphisms $\cdot : C \otimes C \to C, \ \ \eta : k \to C$ is said to be a dual quasi-bialgebra if there exists a unital 3-cocylce ϕ s.t the *product* $\cdot$ is *associative* up to ϕ, i.e.,

$$\sum a_1 \cdot (b_1 \cdot c_1)\phi(a_2, b_2, c_2) = \sum \phi(a_1, b_1, c_1)(a_2 \cdot b_2) \cdot c_2. \tag{2.2.6}$$

A dual quasi-bialgebra is said to be dual quasi-triangular if there exists a 2-cocycle R s.t

$$R(a \cdot b, c) = \sum \phi(c_1, a_1, b_1)R(a_2, c_2)\phi^{-1}(a_3, c_3, b_2)R(b_3, c_4)\phi(a_4, b_4, c_5) \tag{2.2.7}$$

$$R(a, b \cdot c) = \sum \phi^{-1}(b_1, c_1, a_1)R(a_2, c_2)\phi(b_2, a_3, c_3)R(a_4, b_3)\phi^{-1}(a_5, b_4, c_4) \tag{2.2.8}$$

$$\sum b_1 \cdot a_1 R(a_2, b_2) = \sum R(a_1, b_1)a_2 \cdot b_2 \tag{2.2.9}$$

The last relation meaning the commutativity of $\cdot$ up to R.
Finally, a dual quasi bialgebra is said to be a dual quasi-Hopf algebra if there exists a linear map $S : C \to C$ and linear functionals (1-cocycles) $\alpha, \beta : H \to k$ s.t

$$\begin{gathered} \sum S(a_1)a_3\alpha(a_2) = \eta \circ \alpha(a), \quad \sum a_1 S(a_3)\beta(a_2) = \eta \circ \beta(a) \\ \sum \phi(a_1, S(a_3), a_5)\beta(a_2)\alpha(a_4) = \epsilon(a) \\ \sum \phi^{-1}(S(a_1), a_3, S(a_5))\alpha(a_2)\beta(a_4) = \epsilon(a) \end{gathered} \tag{2.2.10}$$

Although relations (2.2.10) look like a little too weird, we will see later that these are precisely what is needed to make the category of C-comodules a rigid category, i.e., a monoidal category where every object has a dual.

Proposition 2.2.2. *If* $(H, \Delta, \epsilon, \cdot, \eta, \phi, R)$ *is a dual quasi triangular quasi bialgebra, then the category* $(\mathcal{M}^H, \otimes, k, a, l, r)$ *is monoidal.*

Proof. Objects in $\mathcal{M}^H$ are k-vector spaces V with a coaction $\rho_V : V \to V \otimes H$.

The monoidal structure is given by:

$$\begin{aligned} \otimes : \mathcal{M}^H \times \mathcal{M}^H \to & \mathcal{M}^H \\ (V, W) \mapsto & V \otimes W := (V \otimes_k W, \rho_{V \otimes W}) \end{aligned}$$

with $\rho_{V\otimes W}$ given by the composition

$$V \otimes_k W \xrightarrow{\rho_V \otimes \rho_W} V \otimes_k H \otimes_k W \otimes_k H \xrightarrow{1\otimes\tau_{2,3}\otimes 1} V \otimes_k W \otimes_k H \otimes_k H \xrightarrow{1\otimes\cdot} V \otimes_k W \otimes_k H$$

The unit is k with the trivial action given by the unity $\eta : k \to k \otimes_k H \cong H$. the associativity constraint is given by the canonical isomorphism of associativity in **Vect**(k) followed by the action of the 3 cocycle ϕ:

$$\begin{aligned} \Phi_{V,W,Z} : (V \otimes W) \otimes Z \to & V \otimes (W \otimes Z) \\ (v \otimes w) \otimes z \mapsto & \sum \phi(v_2, w_2, z_2) v_1 \otimes (w_1 \otimes z_1) \end{aligned}$$

$\Phi_{V,W,Z}$ is a morphism of H-comodules if for every $v \otimes w \otimes z \in V \otimes W \otimes Z$

$$\begin{aligned} (\phi \otimes 1) \circ \rho_{(V\otimes W)\otimes Z}(v \otimes w \otimes z) = \sum v_1 \otimes (w_1 \otimes z_1)\phi(v_3, w_3, z_3)(v_2 \cdot w_2) \cdot z_2 = \\ \sum v_1 \otimes (w_1 \otimes z_1) v_3 \cdot (w_3 \cdot z_3)\phi(v_2, w_{2,z_2}) = \rho_{V\otimes(W\otimes Z)} \circ \phi(v \otimes w \otimes z) \end{aligned}$$

which in fact holds since ϕ satisfies relation (2.2.6). The pentagon axiom is due to the cocycle condition (2.2.5) for ϕ. $l_V : k \otimes_k V \to V$ and $r_V : V \otimes_k k \to V$ are the usual isomorphisms of k-vector spaces which are also H-comodule morphisms. The dual quasi triangular structure R allows us to have a braiding

$$\begin{aligned} \sigma_{V,W} : V \otimes W \to & W \to V \\ v \otimes w \mapsto & \sum R(v_2, w_2) w_1 \otimes v_1 \end{aligned}$$

Since R satisfy relations (2.2.7) and (2.2.8), σ satisfies the hexagon axioms. Relation (2.2.9) implies that σ is a morphism of H-comodules. $\square$

2.3 Dual quasi- Hopf Algebras and their comodules

Gauge Transformations

Let H be a dual quasi Hopf algebra, for any 2-cocycle F there exists a new dual quasi-Hopf algebra with the new product, ϕ_F, R, α, β given by

$$\begin{aligned} a \cdot_F b &= \sum F^{-1}(a_1, b_1) a_2 b_2 F(a_3, b_3) \\ \phi_F(a, b, c) &= \sum F^{-1}(b_1, c_1) F^{-1}(a_1, b_2 c_2) \phi(a_2, b_3, c_3) F(a_3 b_4, c_4) F(a_4, b_5) \\ \alpha_F(a) &= \sum F(S(a_1), a_3) \alpha(a_2), \\ \beta_F(a) &= \sum F^{-1}(a_1, S(a_3)) \alpha(a_2) \\ R_F(a, b) &= \sum F^{-1}(b_1, a_1) R(a_2, b_2) F(a_3, b_3), \qquad \forall a, b, c \in H \end{aligned} \tag{2.3.1}$$

Definition 2.3.1. Let H be a dual quasi-Hopf algebra. A right H *-comodule quasialgebra* (A, m, η) is an algebra object in the monoidal category $(\mathcal{M}^H, \otimes, \mathbb{1}, \Phi, l, r)$

Proposition 2.3.2 (Twisting proposition)**.** *If A is a right $H-$comodule algebra and F a 2-cocycle, then A_F with the new product*

$$m_F(a \otimes b) = \sum m(a_1 \otimes b_1) F(a_2, b_2) \tag{2.3.2}$$

is a right H_F-comodule quasialgebra.

Proof. The coaction of H_F over A_F is the same, as the coaction is only sensitive to the coproduct in H_F. Moreover, any right H-comodule is also a right H_F-comodule as the coproduct in H_F coincides with that of H. Given any H-comodule morphism f, it can be considered as an H_F-comodule morphism. This defines a functor $\mathcal{F} :^H \mathcal{M} \longrightarrow {}^{H_F}\mathcal{M}$. The monoidal structure in ${}^{H_F}\mathcal{M}$ is given bye the following: for $(M, \rho_M), (N, \rho_N)$ two H_F-comodules, we define $M \otimes_F N = (M \otimes N, \rho_{M,N})$, with $\rho_{M,N}$ given by the composition

$$M \otimes N \xrightarrow{\rho_M \otimes \rho_N} M \otimes H_F \otimes N \otimes H_F \xrightarrow{1 \otimes \tau \otimes 1} M \otimes N \otimes H_F \otimes H_F \xrightarrow{1 \otimes 1 \otimes \cdot_F} M \otimes N \otimes H_F.$$

The functor $\mathcal{F}$ is in fact a monoidal equivalence, the coherence maps are given by the H_F-morphisms

$$\begin{aligned} \varphi_{V,W}: \quad & \mathcal{F}(V) \otimes_F \mathcal{F}(W) \to \mathcal{F}(V \otimes W) \\ & v \otimes_F w \mapsto v_1 \otimes w_1 F^{-1}(v_2, w_2), \\ \eta: \quad & \mathbb{1}_F \to \mathcal{F}(\mathbb{1}) \quad \text{is the identity.} \end{aligned}$$

For details on this equivalence see [5].

To show that A_F is an algebra in ${}^{H_F}\mathcal{M}$, we must check the compatibility of m_F with the coaction and the associativity constraint given in the category of H_F-comodules:

$$\begin{tikzcd} A_F \otimes A_F \arrow[r, "m_F"] \arrow[d, "\rho_{A_F \otimes A_F}"'] & A_F \arrow[d, "\rho_{A_F}"] \\ A_F \otimes A_F \otimes H_F \arrow[r, "m_F \otimes 1"'] & A_F \otimes H_F \end{tikzcd} \tag{2.3.3}$$

In fact:

$$\begin{aligned} \rho_{A_F} m_F(a \otimes b) &= \sum m_F(a_1 \otimes b_1) \otimes a_2 \cdot_F b_2 \\ (m_F \otimes 1)\rho_{A_F \otimes A_F}(a \otimes b) &= (m_F \otimes 1)(\sum a_1 \otimes b_1 \otimes a_2 \cdot_F b_2) \\ &= \sum m_F(a_1 \otimes b_1) \otimes a_2 \cdot_F b_2. \end{aligned} \tag{2.3.4}$$

Associativity up to Φ_F, that is, $m_F(m_F \otimes 1) = m_F(1 \otimes m_F)\Phi_F$, in element notation we have

$$\begin{aligned} m_F(m_F(a \otimes b) \otimes c) &= \sum m_F(F(a_3, b_3)m(a_1 \otimes b_1) \otimes c_1) \\ &= \sum F(a_3, b_3)F(a_4 b_4, c_3)m(m(a_1 \otimes b_1) \otimes c_1) \\ &= \sum F(a_3, b_3)F(a_4 b_4, c_3)\phi(a_2, b_2, c_2)m(a_1 \otimes m(b_1 \otimes c_1)) \\ m_F(1 \otimes m_F)\Phi_F(a \otimes b \otimes c) &= \sum \phi_F(a_2, b_2, c_2)m_F(a_1 \otimes F(b_3, c_3)m(b_1 \otimes c_1)) \\ &= \sum \phi_F(a_2, b_2, c_2)F(b_3, c_3)F(a_3, b_4 c_4)m(a_1 \otimes m(b_1 \otimes c_1)). \end{aligned}$$

By using formula (2.3.1) for ϕ_F we obtain the desired equality. $\square$

2.4 The case of the Octonions. Majid Universe.

In this section we will see Example 2.1.6 and its deformation in more detail. Let G be a finite group, let us consider the dual quasi Hopf algebra $H = (kG, \Delta, \epsilon, m, \eta)$ with the usual product, coproduct, unit and counit given in G and extended linearly over k, this is,

$$\begin{aligned} m(x,y) =& x \cdot y \ \text{ the product in } G \\ \Delta(x) =& x \otimes x \ \ \forall x \in G \\ \eta : k \to kG, \ & \eta(1) = e, \ \text{ the identity element in } G \\ \epsilon : kG \to k, \ & \epsilon(x) = 1 \ \ \forall x \in G \end{aligned}$$

let us consider the function $\phi : (kG)^{\otimes 3} \to k^*$ satisfying that

$$\begin{aligned} \phi(y,z,w)\phi(x,y\cdot z,w)\phi(x,y,z) = \phi(x,y,z\cdot w)\phi(x\cdot y,z,w) \\ \phi(e,y,z) = \phi(x,e,z) = \phi(x,y,e) = 1, \ \ \forall x,y,z \in G. \end{aligned} \tag{2.4.1}$$

Remark 2.4.1. Examples of functions like ϕ can be obtained by considering a 2-cocyle F in H. For instance, if we take $F(g,h) = (-1)^{f(g,h)}$ with $f : G \times G \to G$ given by, let us say:

$$\begin{aligned} f(g,h) =& g_1h_1 + (g_1+g_2)h_2, \ \text{ for } g,h \in G = \mathbb{Z}_2 \times \mathbb{Z}_2 \\ f(g,h) =& \sum_{i\geq j} g_ih_j + h_1g_2g_3 + g_1h_2g_3 + g_1g_2h_3, \ \text{ for } g,h \in G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \end{aligned}$$

where we use the sum and the product in each group. These functions f given above produce the examples of the Quaternion and Octonion algebras respectively.

Now, for $H = (kG, \phi)$ we denote by $\mathfrak{U}$ the category of right H-comodules. Although it is common to denote this category by ${}^H\mathcal{M}$, we will use $\mathfrak{U}$ suggesting

that this will be "the universe" for which the octonions are a commutative associative unital algebra.
Using the product in H we can define a bifunctor $\otimes : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ given by

$$(V, \rho_V) \underset{\mathfrak{U}}{\otimes} (W, \rho_W) = (V \otimes W, \rho_{V \otimes W}),$$

with $\rho_{V\otimes W}$ given by the composition

$$V \otimes W \xrightarrow{\rho_V \otimes \rho_W} V \otimes H \otimes W \otimes H \xrightarrow{1 \otimes \tau \otimes 1} V \otimes W \otimes H \otimes H \xrightarrow{1 \otimes m} V \otimes W \otimes H.$$

In element notation this means that if v, w are homogeneous elements of degree g, h in V and W respectively, then $\rho(v, w) = v \otimes w \otimes g \cdot h$, i.e., $g \cdot h$ is the degree of the element $v \otimes w$. The unity for this tensor product is $\mathbb{1} = k$ with the gradation concentrated in the e-degree component.
For this tensor product we define the right and left unity constraints

$$r_U : U \underset{\mathfrak{U}}{\otimes} \mathbb{1} \to U, \quad l_U : \mathbb{1} \underset{\mathfrak{U}}{\otimes} U \to U$$

to be the canonical isomorphisms in the category $Vect^G(k)$ and we define the associativity constraint to be

$$\Phi_{U,V,W} : (U \underset{\mathfrak{U}}{\otimes} V) \underset{\mathfrak{U}}{\otimes} W \to U \underset{\mathfrak{U}}{\otimes} (V \underset{\mathfrak{U}}{\otimes} W)$$
$$(u \otimes v) \otimes w \mapsto \phi(|u|, |v|, |w|) u \otimes (v \otimes w),$$

with ϕ like in 2.4.1 and $|\ |$ denotes the degree of a homogenous element. Note that we have suppressed the $\mathfrak{U}$ in the tensor elements, we will keep doing this from now on.
We have to check that this vector space morphism Φ is in fact a natural isomorphism such that the pentagon and triangle axioms given in (1.1.1), (1.1.2) holds.
Φ is a morphism in $\mathfrak{U}$ since multiplying by the function ϕ is a degree preserving

morphism. It is an isomorphism with inverse given by the multiplication by ϕ^{-1}. The naturalness of Φ is expressed by the equation

$$(f \otimes (g \otimes h)) \circ \Phi_{X,Y,Z} = \Phi_{X',Y',Z'} \circ ((f \otimes g) \otimes h)$$

for every $f : X \to X'$, $g : Y \to Y'$, $h : Z \to Z'$. The equality follows from the fact that $\phi(|f(x)|, |g(y)|, |h(z)|) = \phi(|x|, |y|, |z|)$ since f, g, h are degree preserving morphisms.

The pentagon and triangle axioms say that

$$(1_X \otimes \Phi_{Y,Z,W})\Phi_{X,Y\otimes Z,W}(\Phi_{X,Y,Z} \otimes 1_W) = \Phi_{X,Y,Z\otimes W}\Phi_{X\otimes Y,Z,W}$$

$$1_X \otimes l_Y \Phi_{X,\mathbb{1},Y} = r_X \otimes 1_Y$$

in terms of the function ϕ is

$$\phi(|y|, |z|, |w|)\phi(|x|, |y| \cdot |z|, |w|)\phi(|x|, |y|, |z|)x \otimes (y \otimes (z \otimes w))$$

$$= \phi(|x|, |y|, |z| \cdot |w|)\phi(|x| \cdot |y|, |z|, |w|)x \otimes (y \otimes (z \otimes w))$$

$$\phi(|x|, e, |y|) = 1$$

and these are exactly the conditions given in (2.4.1).

So far, we have the monoidal category $(\mathfrak{U}, \underset{\mathfrak{U}}{\otimes}, k, \Phi, l, r)$, next we consider the object kG which is itself an H-comodule via Δ and consider the morphisms $m_F(x, y) = x \cdot_G yF(x, y)$, $\eta(1) = e$, for $F \in Hom_k(kG^{\otimes 2}, k)$ a convolution invertible and unital function. The next result says that (kG, m_F, η) is an algebra in the monoidal category $\mathfrak{U}$. Even more, if we define a symmetry for the tensor product, then kG is in fact a commutative algebra with respect to such symmetry. In the literature, it is common to say that it is a braided algebra.

Proposition 2.4.2. *Let $F \in Hom_k(kG^{\otimes 2}, k)$ be a convolution invertible function such that $F(x, e) = F(e, x) = 1$ for all $x \in G$ and consider the monoidal category $(\mathfrak{U}, \underset{\mathfrak{U}}{\otimes}, k, \Phi, l, r)$ as above, with the associator given by*

$$\phi(x, y, z) = \frac{F(x, y)F(xy, z)}{F(y, z)F(x, yz)}, \tag{2.4.2}$$

then (kG, m_F, η) *is an algebra in* $\mathfrak{U}$*.*

Proof. To make the notation easier we will use unadorned tensor $\otimes$ to mean $\underset{\mathfrak{U}}{\otimes}$ and we make the abuse $|x| = x$. We have to check the commutativity of the following diagrams

$$\begin{tikzcd} kG \otimes kG \arrow[r, "m_F"] \arrow[d, "\tilde{\Delta}"'] & kG \arrow[d, "\Delta"] \\ kG \otimes kG \otimes H \arrow[r, "m_F \otimes 1"'] & kG \otimes H \end{tikzcd} \qquad \begin{tikzcd} k \otimes kG \arrow[r, "\eta\otimes 1"] \arrow[dr, "l"'] & kG \otimes kG \arrow[r, "1\eta"] \arrow[d, "m_F"] & kG \otimes k \arrow[dl, "r"] \\ & kG & \end{tikzcd}$$

It is enough to check the commutativity on elements of G:

$$\begin{aligned} (m_F \otimes 1)\tilde{\Delta}(x,y) =& (m_F \otimes 1)(1 \otimes m)\tau \circ (\Delta \otimes \Delta)(x,y) \\ =& (m_F \otimes 1)(1 \otimes m)\tau(x \otimes x \otimes y \otimes y) = (m_F \otimes 1)(x \otimes y \otimes x \cdot y) \\ =& m_F(x,y) \otimes x \cdot y = F(x,y)x \cdot y \otimes x \cdot y = F(x,y)\Delta(x \cdot y) \\ \Delta \circ m_F(x,y) =& \Delta(x \cdot yF(x,y)) = F(x,y)\Delta(x \cdot y) \\ m_F \circ \eta \otimes 1(1,x) =& m_F(e,x) = e \cdot xF(e,x) = x = l(1,x), \quad \forall x,y \in G,\ 1 \in k \end{aligned}$$

The diagram for the associativity up to Φ is:

$$\begin{tikzcd} & (kG \otimes kG) \otimes kG \arrow[dl, "m_F\otimes 1"'] \arrow[dr, "\Phi"] & \\ kG \otimes kG \arrow[d, "m_F"'] & & kG \otimes (kG \otimes kG) \arrow[d, "1\otimes m_F"] \\ kG & & kG \otimes kG, \arrow[ll, "m_F"] \end{tikzcd}$$

in element notation is

$$\begin{aligned} & m_F(1 \otimes m_F)\Phi(x \otimes y \otimes z) = m_F(1 \otimes m_F)(\phi(x,y,z)x \otimes (y \otimes z)) \\ & \qquad = \phi(x,y,z)m_F(x, y \cdot zF(y,z)) = \phi(x,y,z)F(y,z)F(x, y \cdot z)x \cdot (y \cdot z) \\ & m_F(m_F \otimes 1)(x,y) = m_F(F(x,y)x \cdot y, z) = F(x,y)F(x \cdot y, z)(x \cdot y) \cdot z, \end{aligned}$$

then by the definition of ϕ we have that m_F is Φ-associative $\square$

Now, there is a symmetric braiding in $\mathfrak{U}$ making (kG, m_F, η) a commutative algebra.

Proposition 2.4.3. *Let $\sigma_F : X \otimes Y \to Y \otimes X$ be given by $\sigma_F(x,y) = \frac{F(x,y)}{F(y,x)} y \otimes x$. Then σ_F is a symmetry for $\underset{\mathfrak{U}}{\otimes}$ and with this symmetry kG is σ_F-commutative.*

Proof. Observe that this braiding is just the usual flip in $Vect(k)$ followed by multiplication of the scalar function $\frac{F(x,y)}{F(y,x)}$, thus σ_F is a natural isomorphism. The hexagon axiom expressed in equation form says that:

$$\Phi \circ \sigma_F \circ \Phi((x \otimes y) \otimes z) = (1 \otimes \sigma_F)\Phi(\sigma_F \otimes 1)((x \otimes y) \otimes z),$$

the left side of the equation is:

$$\begin{aligned}\Phi \circ \sigma_F(\phi(x,y,z) x \otimes (y \otimes z)) =& \phi(x,y,z) \frac{F(x, y \cdot z)}{F(y \cdot z, x)} \Phi(y \otimes z \otimes x) \\ =& \frac{F(x, y \cdot z)}{F(y \cdot z, x)} \phi(x,y,z)\phi(y,z,x) y \otimes (z \otimes x)\end{aligned}$$

and the right side is:

$$\begin{aligned}(1 \otimes \sigma_F)\Phi(\sigma_F \otimes 1)((x \otimes y) \otimes z) =& \frac{F(x,y)}{(y,x)}(1 \otimes \sigma_F)\Phi(y \otimes x \otimes z) \\ =& \frac{F(x,y)}{(y,x)} \phi(y,x,z)(1 \otimes \sigma_F)(y \otimes (x \otimes z)) \\ =& \frac{F(x,y)}{F(y,x)} \frac{F(x,z)}{F(z,x)} \phi(y,x,z) y \otimes (z \otimes x).\end{aligned}$$

Then left and right sides coincide if and only if $\phi(x,y,z)\phi(y,z,x)\frac{F(x,y\cdot z)}{F(y\cdot z,x)} = \phi(y,x,z)\frac{F(x,y)}{F(y,x)}\frac{F(x,z)}{F(z,x)}$. Using the formula for ϕ we obtain

$$\begin{aligned}\phi(x,y,z)\phi(y,z,x)\frac{F(x, y \cdot z)}{F(y \cdot z, x)} =& \frac{F(x,y)F(x \cdot y, z)}{F(y,z)F(x, y \cdot z)} \frac{F(y,z)F(y \cdot z, x)}{F(z,x)F(y, z \cdot x)} \frac{F(x, y \cdot z)}{F(y \cdot z, x)} \\ =& \frac{F(x,y)F(x \cdot y, z)}{F(z,x)F(y, z \cdot x)} \\ \phi(y,x,z)\frac{F(x,y)}{F(y,x)}\frac{F(x,z)}{F(z,x)} =& \frac{F(y \cdot x, z)F(y,x)}{F(x,z)F(y, x \cdot z)} \frac{F(x,y)F(x,z)}{F(y,x)F(z,x)} \\ =& \frac{F(y \cdot x, z)F(x,y)}{F(z,x)F(y, x \cdot z)}.\end{aligned}$$

Similarly we obtain the hexagon axiom involving Φ^{-1}. We also have that $\sigma_F^2 = 1$, since

$$X \otimes Y \xrightarrow{\sigma_F} Y \otimes X \xrightarrow{\quad\sigma_F\quad} X \otimes Y$$

$$x \otimes y \longmapsto \tfrac{F(x,y)}{F(y,x)} y \otimes x \longmapsto \tfrac{F(x,y)}{F(y,x)} \tfrac{F(y,x)}{F(x,y)} x \otimes y.$$

Finally, let us see that $m_F \circ \sigma_F = m_F$. In fact,

$$m_F \circ \sigma_F(x \otimes y) = m_F \left(\frac{F(x,y)}{F(y,x)} y \otimes x \right) = \frac{F(x,y)}{F(y,x)} F(y,x) y \cdot x = F(x,y) x \cdot y = m_F(x \otimes y).$$

$\square$

The Category $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$.

In this section we follow the ideas given in [1] to prove Propositions 2.4.4, 2.4.6 and 2.4.8. This results are of important use in the construction of the octonionic projective space.

In [1] the authors prove that taking $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{R}$ as the ground field and $F(g,h) = (-1)^{f(g,h)}$ with

$$f(g,h) = \sum_{i \geq j} g_i h_j + h_1 g_2 g_3 + g_1 h_2 g_3 + g_1 g_2 h_3 \quad \text{for } g, h \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

one obtains the Cayley algebra of the octonions $\mathbb{O}$ as the commutative algebra $(\mathbb{R}\mathbb{Z}_2^3, m_F, \eta)$ in the corresponding monoidal category. From now on, we will denote

Once we have a commutative algebra in a symmetric monoidal category, we can construct its category of modules and make some other constructions similar to those, one has in commutative algebra with the purpose to imitate the algebraic geometry over commutative rings.

In this section we will work on the properties of the category $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$, concerning to projective and free objects (respect to a left adjoint functor called the free functor). We will prove that in fact $\mathbb{O}$ is a projective, finitely presented generator for the category $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$, this will say by using Gabriel's Theorem that $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$ is in fact equivalent to a category of modules over a certain ring. A proof of this result can be seen in [21]. Although we are not interested in using this equivalence, it is worth to mention it.

Let us start characterizing the objects in $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$. They consist of a pair (X, ρ) with $X \in \mathcal{U}$, that is to say, a $\mathbb{Z}_2^3$-graded real vector space with a graded morphism $\rho : \mathbb{O} \otimes_{\mathfrak{U}} X \to X$ satisfying the pentagon and triangle axioms for the action. Since ρ is degree preserving, then ρ is a 8-tuple of real vector space morphisms $\rho_i : (\mathbb{O} \otimes_{\mathfrak{U}} X)_i \to X_i$, where the index denotes the i-th degree component. If we denote $\{e_i, \quad i = 0 \ldots 7\}$ a basis for $\mathbb{O}$, then the associativity of the action says that $\rho(e_i, \rho(e_i, x_k)) = -x_k$, this means that for every $i = 0, \ldots 7$, the multiplication by e_i induces an isomorphism $X_k \cong X_l$ with k, l such that $e_l = m_F(e_i, e_k)$. In summary, an $\mathbb{O}$-module is just a graded vector space with distinguished isomorphisms between the homogenous components, given by the multiplication of the basis elements of $\mathbb{O}$. Thus the data of being an $\mathbb{O}$-module is in the 0-th degree component and one obtains the rest of the components by multiplication of the $e_i's$.

Next, a morphism between objects in $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$ will be a preserving degree morphism between the graded vector spaces compatible with the actions of $\mathbb{O}$, i.e., a morphism between the degree zero components commuting with the respective isomorphisms. More explicitely, if X, Y are objects in $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$, then $f : X \to Y$ is characterized by the morphism $f_0 : X_0 \to Y_0$, since the rest of the morphisms are just conjugations of f_0 by the e_i's as is depicted in the following commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "\cdot e_i"'] & Y_0 \arrow[d, "\cdot e_i"] \\ X_i \arrow[r, "f_i"'] & Y_i. \end{tikzcd} \tag{2.4.3}$$

All this implies the following propositions:

Proposition 2.4.4. *Let $V_0 = Hom_{\mathbb{O}}(\mathbb{O}, -) : \mathcal{M}od_{\mathfrak{U}}(\mathbb{O}) \to \mathcal{E}ns$ be the "canonical" forgetful functor in monoidal categories. Then V_0 is a conservative functor.*

Remark 2.4.5. The notation for this forgetful functor comes from the enriched category setting, we use it here although we are not constructing the category of modules as an enriched category over $\mathcal{U}$. In the Appendix, however we work in the enriched context.

Proof. Let $f : (X, \rho) \to (Y, \rho')$ be a morphism in $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$, such that the induced morphism $V_0(f) : Hom_{\mathbb{O}}(\mathbb{O}, X) \to Hom_{\mathbb{O}}(\mathbb{O}, Y)$ is an isomorphism. If we denote by |-| the forgetful functor, the adjunction

$$\begin{tikzcd} \mathcal{M}od_{\mathfrak{U}}(\mathbb{O}) \arrow[r, shift right, "|-|"'] \arrow[r, phantom, "\perp"] & \mathcal{U} \arrow[l, shift right, "-\underset{\mathcal{U}}{\otimes}\mathbb{O}"'] \end{tikzcd}$$

says that we have an isomorphism $Hom_{\mathcal{U}}(\mathbb{R}, |X|) \xrightarrow{\sim} Hom_{\mathcal{U}}(\mathbb{R}, |Y|)$. Since in the category $\mathfrak{U}$ we have the isomorphisms $X_0 \cong Hom_{\mathcal{U}}(\mathbb{R}, |X|)$, $Y_0 \cong Hom_{\mathcal{U}}(\mathbb{R}, |Y|)$, hence we have the isomorphism $f_0 : X_0 \to Y_0$. Finally, by the diagram 2.4.3, we have that $f : (X, \rho) \to (Y, \rho')$ is in fact an isomorphism. $\square$

Proposition 2.4.6. *$\mathbb{O}$ is a projective finitely presented generator in $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$.*

Proof. Limits and colimits in $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$ are computed in $\mathfrak{U}$, this means in particular that |-| : $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O}) \to \mathfrak{U}$ preserves them. Now, since $\mathbb{R}$ is a projective object in $\mathfrak{U}$, then $\mathbb{O} \cong \mathbb{R} \otimes_{\mathcal{U}} \mathbb{O}$ is projective in $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$.

Now, to show that $\mathbb{O}$ is finitely presented, observe thar $Hom_{\mathfrak{U}}(\mathbb{R}, -)$ preserves them, hence by the isomorphism

$$Hom_{\mathbb{O}}(\mathbb{O}, -) \cong Hom_{\mathfrak{U}}(\mathbb{R}, |-|)$$

we get that $Hom_{\mathbb{O}}(\mathbb{O}, -)$ preserves filtered colimits, that is $\mathbb{O}$ is finitely presented. Finally, to see that $\mathbb{O}$ is a generator, we have to prove that $Hom_{\mathbb{O}}(\mathbb{O}, -)$ is faithful. The result follows by "abstract nonsense": In any category $\mathcal{C}$ with equalizers , a conservative functor $F : \mathcal{C} \to \mathcal{E}ns$ preserving them is faithful. □

Remark 2.4.7. All the previous constructions and properties are also true when $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H = (\mathbb{R}G, \Delta, \epsilon, m, \eta)$ the Hopf algebra defining the category $\mathcal{U}$. If we take $f(g, h) = g_1h_1 + (g_1 + g_2)h_2,$ for $g, h \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we obtain the category for which the Quaternion algebra is a commutative algebra.

From all stated above, we have the following result:

Proposition 2.4.8. *$\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$ is equivalent to the category of right modules over the ring $End(\mathbb{O})$.*

Remark 2.4.9. It is very important to point out that this equivalence is not monoidal. The monoidal structure for $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$ we consider here is the one that makes possible the construction of the octonionic projective space, given in Chapter 4.

Chapter 3

Relative Algebraic Geometry

Resumen

En este capítulo se presentan las ideas generales de la geometría algebraica relativa. Es una breve revisión de la geometría relativa introducida por B. Toen y M. Vaquie en [26]. Sea $\mathcal{C}$ una categoría monoidal simétrica cerrada y bicompleta, se define formalmente la categoría de los esquemas afines relativos a $\mathcal{C}$ como $Aff_{\mathcal{C}} = Comm(\mathcal{C})^{op}$. Se tienen las siguientes construcciones en $Aff_{\mathcal{C}}$

- Existe una topología de Grothendieck canónica en $Aff_{\mathcal{C}}$, llamada la topología playa cuyas familias cubrientes corresponden a familias finitas de morfismos $\{A \to A_i\}$ en $Comm(\mathcal{C})$ tales que el funtor

$$\prod_i A_i \otimes_A - : \mathcal{M}od_{\mathcal{C}}(A) \to \prod_i \mathcal{M}od_{\mathcal{C}}(A_i)$$

  es conservativo.

- Los prehaces representables son haces para esta topología.

- Existe una noción de abierto Zariski en $Aff_{\mathcal{C}}$ que por definición es un morfismo $f : X \to Y$ tal que el morfismo correspondiente $A \to B$ en $Comm(\mathcal{C})$ es un epimorfismo playo de presentación finita.

- La noción de abierto Zariski se extiende a morfismo entre haces.

- Los abierto Zariski son estables por composición, isomorfismos y cambio de base.

- La noción de abierto Zariski da lugar a una topología, llamada la topología Zariski y para esta topología los prehaces representables también son haces.

Estas propiedades son necesarias para definir una categoría de esquemas relativa a $\mathcal{C}$. Así, un esquema relativo es por definión un haz en el sitio $Aff_{\mathcal{C}}$ con la topología Zariski y que posee un cubrimiento Zariski por esquemas afines. A la categoría de esquemas relativos a $\mathcal{C}$ la denotamos $Sch(\mathcal{C})$ y se prueba que ésta es una subcategoría plena de la categoría de haces sobre $Aff_{\mathcal{C}}$, estable por productos fibrados y sumas disjuntas.

Summary

This chapter is a review of the theory developed in [26], it contains the main frame of this thesis, that is, the setting in which our object of study is defined. For a symmetric monoidal bicomplete category $\mathcal{C}$, the category of affine schemes is defined as $Aff_{\mathcal{C}} := Comm(\mathcal{C})^{op}$, associated to this category one has a Grothendieck topology, namely the Zariski topology. One has also the notion of a sheaf in the site $Aff_{\mathcal{C}}$, then a $\mathcal{C}$-scheme will be an object in $Sh(Aff_{\mathcal{C}})$ which is covered by affine schemes.

3.1 Grothendieck Topologies.

Definition 3.1.1 (Grothendieck Pretopology)**.** Let $\mathcal{C}$ be a category with pullbacks. A Grothendieck pretopology on $\mathcal{C}$ is an assignment to each object U in $\mathcal{C}$ of a collection of families $\{U_i \to U\}_{i\in I}$ of morphisms, called covering families such that the following conditions hold:

1. (Isomorphism cover.) If $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is a covering family.

2. (Stability axiom.) If $\{U_i \to U\}_{i\in I}$ is a covering family then for any morphism $f : V \to U$ in $\mathcal{C}$, the family of pullbacks $\{U_i \times_U V \to V\}_{i\in I}$ is a family covering for V.

3. (Transitivity axiom.) If $\{U_i \to U\}_{i\in I}$ is a covering family and if for each $i \in I$ $\{V_{ij} \to U_i\}_{j\in I_i}$ is a covering family for U_i, then the family of composites $\{V_{ij} \to U\}_{i\in I,\ j\in I_i}$ is a covering family for U.

In [26], the authors defined a pretopology in a category $\mathcal{C}$ through a pseudo functor $M : \mathcal{C}^{op} \to CAT$ assigning to every $X \in \mathcal{C}$ a category $M(X)$ verifying the following conditions

1. For every $X \in \mathcal{C}$, $M(X)$ has arbitrary limits and colimits.

2. For every morphism $p : X \to X'$ the functor $M(p) =: p^* : M(X) \to M(X')$ has a conservative right adjoint $p_* : M(X') \to M(X)$.

3. For every commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} Y' \arrow[r, "p'"] \arrow[d, "q'"'] & Y \arrow[d, "q"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

the natural transformation $p^*q_* \Rightarrow q'_*p'^*$ is an isomorphism. This natural transformation comes from natural isomorphisms

$$(q')^*p^* \cong (pq')^* = (qp')^* \cong (p')^*q^*$$

which composing on the right with q_* gives the natural isomorphism $(q')^*p^*q_* \cong (p')^*q^*q_*$, now composing with the counit of the adjunction $q^*q_* \Rightarrow 1$ one has the natural transformation

$$(q')^*p^*q_* \Rightarrow (p')^*$$

and again by the adjunction we have the natural transformation of base change

$$p^*q_* \Rightarrow q'_*(p')^*.$$

Definition 3.1.2 (Site)**.** A site is a category with a Grothendieck topology.

Definition 3.1.3 (M-coverings.)**.** Let $\{p_i : X_i \to X\}_{i\in I}$ be a family of morphisms in $\mathcal{C}$, we say that

1. The family is an M-covering if there exists a finite set $J \subset I$ such that the family of induced functors $\{p_i^* : M(X) \to M(X_i)\}$ are conservative.

2. the family is said to be M-flat if the functors $p_i^* : M(X) \to M(X_i)$ are left exact for every $i \in I$.

3. The family is said to be M-faithfully flat if it is an M-covering, M-flat.

Remark 3.1.4. Since $M(X)$ has all finite limits, a functor p^* which is both left exact and conservative hence faithful, thus the name M-faithfully flat is justified.

Proposition 3.1.5. *M-faithfully flat families define a pretopology in $\mathcal{C}$.*

Proof. See [26] □

The induced topology in $\mathcal{C}$ is called the M-faithfully flat topology.

Descent Data.

Given $\mathcal{V} = (U_i \to X)_{i \in I}$ a covering in the site T, one has the category $Desc(\mathcal{V}/X, M)$ of descent data, this is a category whose objects are pairs $(x_i, \theta_{i,j})_{i,j}$ with x_i an object in $M(U_i)$ and $\theta_{i,j} : (x_i)|_{U_{i,j}} \cong (x_j)|_{U_{i,j}}$ are isomorphisms in $M(U_{ij})$ satisfying the cocycle condition $\theta_{j,k} \circ \theta_{ij} = \theta_{ik}$ in $M(U_{ijk})$, where U_{ij} denotes the pullback $U_i \times_X U_j$. A morphism between two descent data $(x_i, \theta_{ij})_{ij}, (y_i, \phi_{ij})_{ij}$ is a family of morphisms $f_i : x_i \to y_i$ in $M(U_i)$ compatible with the given isomorphisms, i.e., $\phi_{i,j} f_i = f_j \theta_{ij}$ in $M(U_{ij})$. In [26, Théorèm 2.5] the authors prove that for each covering $\mathcal{V}$ the canonical functor $p^* : M(X) \to Desc(\mathcal{V}/X, M)$ is an equivalence. It is in fact an adjoint equivalence with right adjoint given by

$$p_*(x_i, \theta_{i,j}) = Lim \left(\prod_i (p_*)(x_i) \rightrightarrows \prod_{i,j} (p_{i,j})_*(x_i)|_{U_{i,j}} \right) \tag{3.1.1}$$

with $p_{i,j} : U_{i,j} \to X$.

When the site T is the category $Aff_{\mathcal{C}}$, for $\mathcal{C}$ closed symmetric monoidal and cocomplete and the functor M is the one assigning to every $A \in Comm(\mathcal{C})$ its category of modules $\mathcal{M}od_{\mathcal{C}}(A)$ and for every morphism $p : Spec(A) \to Spec(B)$ in $Aff_{\mathcal{C}}$ the functor $p^* : - \otimes_A B$, then this equivalence says that the modules are sheaves for the Zariski topology. For a detailed description of this result see [26, Corollaire 2.11].

3.2 The Zariski Site $Aff_{\mathcal{C}}$

Through this section $(\mathcal{C}, \otimes, \mathbb{1})$ is a bicomplete, closed symmetric monoidal category. Let $Comm(\mathcal{C})$ the category of commutative algebras in $\mathcal{C}$. The category of affine schemes relative to $\mathcal{C}$ is defined as $Aff_{\mathcal{C}} := Comm(\mathcal{C})^{op}$. If $A \in Comm(\mathcal{C})$ the corresponding object in $Aff_{\mathcal{C}}$ is denoted $Spec(A)$. $Aff_{\mathcal{C}}$ is bicomplete as it is $\mathcal{C}$.

Definition 3.2.1. Let $f : Spec(B) \longrightarrow Spec(A)$ a morphism in $Aff_{\mathcal{C}}$. f is called a Zariski open or Zariski open immersion if the corresponding morphism $f : A \to B$ in $Comm(\mathcal{C})$ is a flat epimorphism of finite presentation.

Definition 3.2.2 (fpqc covering)**.** The family $\{Spec(A_i) \to Spec(A)\}_{i \in I}$ in $Aff_{\mathcal{C}}$ is a flat covering (a.k.a fpqc covering) if

1. for all $i \in I$, $A \to A_i$ is flat.

2. There exists a finite set $J \subset I$ such that the family of functors

$$- \otimes_A A_j : \mathcal{M}od_{\mathcal{C}}(A) \to \mathcal{M}od_{\mathcal{C}}(A_j), \quad j \in J$$

   is jointly conservative.

Definition 3.2.3 (Zariski covering.)**.** The family $\{Spec(A_i) \to Spec(A)\}_{i \in I}$ in $Aff_{\mathcal{C}}$ is a Zariski covering if it is a fpqc covering and for all $i \in I$, $Spec(A_i) \to Spec(A)$ is a Zariski immersion.

Proposition 3.2.4. *The fpqc and the Zariski coverings define two pretopologies on $Aff_{\mathcal{C}}$. The Grothendieck topologies associated to these pretopologies are called the fpqc and the Zariski topology respectively.*

Proof. See [26]. □

Thus we have the category of presheaves $Pr(Aff_{\mathcal{C}})$, that is, the category of functors from $Aff_{\mathcal{C}}^{op}$ to the category $\mathcal{E}ns$, and its sub categories of sheaves

$$Sh^{fpqc}(Aff_{\mathcal{C}}) \subset Sh^{Zar}(Aff_{\mathcal{C}}) \subset PrSh(Aff_{\mathcal{C}}).$$

The category of sheaves with respect to the Zariski topology, $Sh^{Zar}(Aff_{\mathcal{C}})$ will be denoted simply by $Sh(Aff_{\mathcal{C}})$ and the word sheaf will mean always Zariski sheaf. Via the Yoneda functor $h_{-} : Aff_{\mathcal{C}} \to PrSh(Aff_{\mathcal{C}})$ one identifies $Aff_{\mathcal{C}}$ with the full subcategory of $PrSh(Aff_{\mathcal{C}})$ given by the image of h and one has the following result:

Proposition 3.2.5. *For every $X \in Aff_{\mathcal{C}}$, the presheaf h_X is an fpqc sheaf, hence a Zariski sheaf.*

Proof. For details see [26, Corollaire 2.11 1.]. □

This result says that modules are sheaves for the fpqc topology. This is an important fact, very useful in the construction of the projective space given in Chapter 4. Since The Zariski topology is less finer than fpqc then this is true also for the Zariski topology.

3.3 Schemes relative to $\mathcal{C}$.

In this section we give the notion of a relative scheme. As in the classical setting in algebraic geometry, a relative scheme is that of a sheaf which has a Zariski open covering by affine schemes. In order to do this one has to define the Zariski topology in $Sh(Aff_{\mathcal{C}})$.

Definition 3.3.1.

1. Let $X \in Aff_{\mathcal{C}}$ and $F \subset X$ a sub sheaf. F is said to be an open Zariski of X if there exists a family of open Zariski $\{X_i \to X\}_{i \in I}$ in $Aff_{\mathcal{C}}$ such that F is the image of the morphisms of sheaves $\coprod_{i \in I} X_i \to X$.

2. $f : F \to G$ in $Sh(Aff_{\mathcal{C}})$ is an open Zariski (open Zariski immersion, open sub functor) if for every affine scheme X and every morphism $X \to G$ the

induced morphism $F \times_G X \to X$ is a monomorphism with image a Zariski open of X, i.e., $F \times_G X$ is a Zariski open of X.

Definition 3.3.2. A sheaf $F \in Sh(Aff_{\mathcal{C}})$ is a scheme relative to $\mathcal{C}$ or a $\mathcal{C}$-scheme if there exists a family $\{X_i\}_{i \in I} \in Aff_{\mathcal{C}}$ such that for all i there exists $X_i \to F$ satisfying

1. The morphism $X_i \to F$ is a Zariski open of F for all i.
2. The induced morphism $p : \coprod_{i \in I} X_i \to F$ is an epimorphism of sheaves.

The full subcategory of relative $\mathcal{C}$-schemes will be denoted $Sch(\mathcal{C})$.

The Affine Scheme $\mathbb{A}^n_{\mathcal{C}}$

Definition 3.3.3. The Affine n-space $\mathbb{A}^n_{\mathcal{C}}$ is defined as the functor

$$\begin{aligned} \mathbb{A}^n_{\mathcal{C}} : & Comm(\mathcal{C}) \to \mathcal{E}ns \\ & A \longmapsto \mathbb{A}^n_{\mathcal{C}}(A) := Hom_{Comm(\mathcal{C})}(\mathbb{1}[x_1, \cdots, x_n], A) \end{aligned}$$

For every morphism $f : A \to B$ associates the corresponding function $f_* : \mathbb{A}^n_{\mathcal{C}}(A) \to \mathbb{A}^n_{\mathcal{C}}(B)$. Recall that the commutative algebra $\mathbb{1}[x_1, \cdots, x_n]$ is defined as the symmetric algebra of $\mathbb{1}^{\oplus n}$.

Remark 3.3.4. By Proposition 3.2.5, $\mathbb{A}^n_{\mathcal{C}} \in Sh^{fpqc}(Aff_{\mathcal{C}})$ therefore it is an affine scheme

Example 3.3.5. Taking $\mathcal{C} = \mathbb{Z}$-modules we get the usual affine scheme over $\mathbb{Z}$, $\mathbb{A}^n_{\mathbb{Z}}$. This functor is represented by $\mathbb{Z}[x_{1,\cdots}, x_n]$.

Chapter 4

The projective space in relative algebraic geometry

Resumen

Este capítulo concentra los aportes originales de esta tesis. En la sección 4.1 demostramos unos lemas auxiliares para la construcción del espacio proyectivo $\mathbb{P}^n_{\mathcal{C}}$. El lema 4.1.1 da una condición para que una familia de elementos de un objeto álgebra A sea una partición de la unidad. Este lema es importante para demostrar el lema 4.1.3 que permite asociar a un ideal de A un sub esquema de $Spec(A)$ llamado el abierto complementario. La noción de abierto complementario nos permitió probar que los esquemas afines U_i definidos en 4.2.2 son en efecto abiertos Zariski de $\mathbb{P}^n_{\mathcal{C}}$. En esta sección también presentamos los objetos que permiten definir el espacio proyecto, éstos son, los objetos de línea en una categoría monoidal cerrada. Los objetos de línea son la categorificación de los fibrados vectoriales de línea, se definen como objetos inversiones cuya signatura es la flecha identidad. Estos objetos son preservados por funtores monoidales fuertes como el cambio de base, una propiedad muy útil para la

definición del proyectivo relativo. Otra propiedad de los objetos de línea, necesaria para probar que $\mathbb{P}^n_{\mathcal{C}}$ satisface la condición de haz, es la dada en el lema 4.1.17, que dice que si A_i es una familia de A-álgebras y L_i es un objeto de línea en $\mathcal{M}od_{\mathcal{C}}(A_i)$ para cada i, se tiene que $\prod_i L_i$ es un objeto de línea en $\mathcal{M}od_{\mathcal{C}}(\prod_i A_i)$.
En la sección 4.2 definimos el funtor de puntos del espacio proyectivo relativo a una categoría $\mathcal{C}$ como el funtor $\mathbb{P}^n_{\mathcal{C}} : Aff^o_{\mathcal{C}}p \to \mathcal{E}ns$ que a cada esquema afín $Spec(A)$ le asigna el conjunto de submódulos L de A^{n+1} que satisfacen lo siguiente:

- L es un objeto de línea en $\mathcal{M}od_{\mathcal{C}}(A)$,

- L es un sumando directo of A^{n+1}.

Para cada morfismo $Spec(B) \to Spec(A)$ la función de conjuntos $\mathbb{P}^n_{\mathcal{C}}(A) \to \mathbb{P}^n_{\mathcal{C}}(B)$ asigna a un $L \in \mathbb{P}^n_{\mathcal{C}}$ el sumando directo correspondiente $B \otimes_A L$ de B^{n+1}. Las hipótesis sobre $\mathcal{C}$ para que dicho prehaz sea un esquema relativo, son las siguientes: $(\mathcal{C}, \otimes, \mathbb{1})$ es una categoría abeliana, simétrica cerrada y bicompleta tal que el objeto unidad $\mathbb{1}$ es un generador proyectivo de presentación finita. A una categoría con dichas propiedades la llamamos un contexto abeliano relativo fuerte. En el capítulo 2 probamos que la categoría $\mathcal{M}od_{\mathfrak{U}}(\mathbb{O})$ es un contexto abliano relativo fuerte, con lo cual tenemos como resultado el teorema 4.2.7 que afirma que $\mathbb{P}^n_{\mathbb{O}}$ es un esquema relativo a dicha categoría.
Damos también una definición equivalente del espacio proyectivo relativo, en términos de cocientes de A^{n+1}, a este prehaz lo denotamos $\overline{\mathbb{P}^n_{\mathcal{C}}}$. El teorema 4.2.10 afirma que si $\mathcal{C}$ es un contexto abeliano relativo fuerte entonces $\overline{\mathbb{P}^n_{\mathcal{C}}}$ es un $\mathcal{C}$-esquema. Finalmente el teorema 4.2.11 afirma que $\mathbb{P}^n_{\mathcal{C}}$ y $\overline{\mathbb{P}^n_{\mathcal{C}}}$ son isomorfos como $\mathcal{C}$-esquemas.
En la sección 4.3 nos concentramos en la categoría de haces quasi-coherentes sobre un esquema relativo. Probamos una propiedad de pegado de haces quasi

coherentes, proposición 4.3.3. Definimos los haces torcidos $\mathcal{O}_{\mathbb{P}^n_{\mathcal{C}}}(m)$ para m un número entero. La motivación para definir estos haces es la de tener un teorema del estilo de Serre que caracteriza los haces quasi-coherentes sobre el espacio proyectivo como módulos graduados. Esta última parte corresponde a trabajo en progreso.

Summary

Throughout this chapter $(\mathcal{C}, \otimes, \mathbb{1})$ is an abelian bicomplete symmetric closed monoidal category such that $\mathbb{1}$ is a projective finitely presentable generator. This condition on $\mathbb{1}$ means that the forgetful functor

$$V_0 = Hom_{\mathcal{C}}(\mathbb{1}, -) : \mathcal{C} \to \mathcal{E}ns$$

is conservative, preserves and reflects epimorphisms and filtered colimits. Although not all of these properties are needed in some of the results, these are exactly the conditions required for the relative projective space to be a scheme. We will call $\mathcal{C}$ an *abelian strong relative context*. In Chapter 1 we saw that, due to the adjunction $\mathcal{C} \rightleftharpoons \mathcal{M}od_{\mathcal{C}}(A)$, if $\mathcal{C}$ is an abelian strong relative context then $\mathcal{M}od_{\mathcal{C}}(A)$ is also an abelian strong relative context.

4.1 Preliminary Definitions and Lemmas.

In this section we prove several lemmas needed in order to prove that what we define as the projective space is in fact a $\mathcal{C}$-scheme. These lemmas are the relative version of very well-known results in algebraic geometry. Lemmas 4.1.1 and 4.1.2 are usuful to prove Lemma 4.1.3. This Lemma, together with Lemma 4.1.4 play an important role, as they are needed to prove that the projective space has a Zariski covering. In this section we also define *line objects* and give some of their properties, as well as examples in very well-known categories like A-mod and $Qcoh(X)$.

Lemma 4.1.1 (Partition of Unity). *Let* $(f_i : \mathbb{1} \to A)$ *be a generating family 1.3.24, then* $(f_i)_{i\in I}$ *form a partition of unity on* A *1.3.25.*

Proof. we will denote the adjoint arrow of each f_i as f_i. let us see that we can reduce the family to a finite one. For this, for each finite subset $J = \{i_1, \cdots i_k\} \subset I$ consider the generated ideal $I_J =< f_{i_1}, \cdots, f_{i_k} >$. Then, these ideals determine a filtered diagram as shown above

$$\begin{tikzcd}[column sep=small, row sep=small]
< f_i > \arrow[dr] & & & & \\
& <f_i, f_j> \arrow[dr] & & & \\
< f_j > \arrow[ur] \arrow[dr] & & <f_i, f_j, f_k> & & \cdots \\
& <f_j, f_k> \arrow[ur] & & & \\
< f_k > \arrow[ur] & & & &
\end{tikzcd}$$

Since the family $(f_i)_{i\in I}$ is epimorphic in $\mathcal{M}od_{\mathcal{C}}(A)$ then A is the filtered colimit of these ideals, i.e.,

$$A \cong colim < f_{i_1}, \cdots, f_{i_k} > .$$

Because A is finitely presented in $\mathcal{M}od_{\mathcal{C}}(A)$, we have the isomorphism

$$Hom_A(A, A) \cong colim_{J\subset I} Hom_A(A, I_J).$$

Then there exists and index k such that the identity arrow $1 : A \to A$ factorizes through $< f_{i_1}, \ldots, f_{i_k} >$, that is to say $A \cong< f_{i_1}, \cdots, f_{i_k} >$.

Now, let us see that the finite family indexed by J is a partition of unity. As we have an epimorphism $\coprod A \overset{(f_{i_j})}{\twoheadrightarrow} A$ and A is projective, there is a surjection

$$Hom_A(A, \textstyle\coprod_{i\in J} A) \xrightarrow{(f_{i_j})^*} Hom_A(A, A) .$$

Using the isomorphism $Hom_A(A, \coprod_{i\in J} A) \cong \prod_{i\in J} Hom_A(A, A)$, we have that for the identity arrow $1 : A \to A$ there exists a family $(s_i)_{i\in J}$ such that $\sum_{i\in J} s_i \circ f_i = 1$ $\square$

Lemma 4.1.2. *Let* $(\mathbb{1} \xrightarrow{f_i} A)_{i\in I}$ *be a generating family. Then* $(Spec(A_{f_i}) \to SpecA)_i$ *is a Zariski covering.*

Proof. By Proposition 1.3.23 each $A \to A_{f_i}$ is a flat epimorphism of finite presentation. By Lemma 4.1.1, there exists a finite subset $J \subset I$ such that the family $(f_j)_{j\in J}$ is a partition of unity and by Proposition 1.3.26 the family of functors $\mathcal{M}od(A) \to \prod_i \mathcal{M}od(A_{f_i})$ is jointly conservative.

□

Lemma 4.1.3 (Complementary open subscheme)**.** *Let* $X = SpecA$ *in* $Aff_{\mathcal{C}}$, $I \hookrightarrow A$ *an ideal. There is a subfunctor of* X *associated to the ideal* I *defined by:* $U_I(B) = \{u : A \to B \ : \ BI \cong B\}$ *where* $BI = Im(B \otimes I \xrightarrow{u\circ j_I \otimes B} B \otimes B \xrightarrow{m_B} B)$. *Moreover,* U *is a* $\mathcal{C}$*-scheme.*

Proof. First we prove that U_I is a sub sheaf. Let $(B \to B_i)_{i\in J}$ be a Zariski covering and let $(f_i)_i$ be a compatible family in $\prod_i U(B_i) \hookrightarrow \prod_i h_A(B_i)$. Since h_A is a sheaf, there exists a unique $f \in h_A(B)$ whose restrictions to every open $Spec(B_i)$ is f_i. Let us check that this f is in fact a section in $U(B)$, i.e., $f : A \to B$ induces an isomorphism $BI \cong B$. Since the B_i form an open covering for B we have that family of functors

$$- \otimes_B B_i : \mathcal{M}od(B) \to \mathcal{M}od(B_i)$$

is jointly conservative, so if we consider the inclusion $BI \hookrightarrow B$, we know that for every $i \in J$, $BI \otimes_B B_i \cong B_i I \xrightarrow{\sim} B_i$, therefore $BI \cong B$.

Now We show that if $(f_i)_i \subset Hom_A(A, A)$ is a generating family of the ideal I then $U_i = Spec(A_{f_i}) \to U$ is a Zariski open immersion and $\{U_i \to U\}_{i\in J}$ is a Zariski covering. First, note that by the universal property of localizations

$$U_i(B) = Hom_{Comm(\mathcal{C})}(A_{f_i}, B) \cong \{f : A \to B \ : \ B < f_i > \cong B\}. \qquad (4.1.1)$$

Moreover, the inclusion $U_i \to Spec(A)$ induces a morphism $U_i \to U$, by (4.1.1) this morphism is a monomorphism. We will check that this morphism is in fact a Zariski open immersion. Let $SpecB \in Aff_{\mathcal{C}}$ and $u : SpecB \to U$ and consider the pullback diagram

$$\begin{tikzcd} U_i \times_A SpecB \arrow[r] \arrow[d] & U_i \arrow[d] \arrow[dr, bend left] & \\ SpecB \arrow[r, "u"] \arrow[rr, bend right] & U \arrow[r] & SpecA, \end{tikzcd}$$

we have to prove that $U_i \times_A SpecB \to SpecB$ is a Zariski open immersion. To give the morphism $u : SpecB \to U$ is the same as giving an element in $U(B)$, that is to say, a morphism $u : A \to B$ such that $IB \cong B$, then the result follows by the isomorphism $U_i \times_A SpecB \cong SpecB_{u(f_i)}$, where $u(f_i) : A \xrightarrow{f_i} A \xrightarrow{u} B$ and $SpecB_{u(f_i)} \to SpecB$ is a Zariski open, therefore $U_i \to U$ is a Zariski open.
On the other hand, in view of $BI \cong B$, $(u(f_i))_i$ is a generating family of B. This family can be reduced to a finite family $(u(f_j))_{j \in J}$, thus by Lemma 4.1.1, $\coprod_{j \in J} SpecB_{uf_j} \to SpecB$ is an epimorphism of sheaves so is $\coprod_{j \in J} U_j \to U$ $\square$

We now give a sufficient condition for a morphism of sheaves to be an epimorphism. This result is analogous to its classical counterpart and it is very useful in order to prove that the projective space is in fact a scheme as it is covered by affine Zariski open immersions.

Lemma 4.1.4. *Let $\{U_i \to F\}$ be a finite family of affine Zariski open immersions in $Sh(Aff_{\mathcal{C}})$. If for every field object $K \in Comm(\mathcal{C})$, $\coprod_i U_i(K) \to F(K)$ is surjective then $\coprod_i U_i \to F$ is an epimorphism of sheaves.*

Proof. It is enough to prove the lemma for $F = SpecA$ since a necessary and sufficient condition for $G \to F$ to be a sheaf epimorphism is that for every affine scheme $SpecA$, $SpecA \times_F G \to SpecA$ is an epimorphism. In this case, we have to check that for each $U_j = Spec(A_j) \xrightarrow{u_j} Spec(A)$, the family of functors $\mathcal{M}od(A) \to \mathcal{M}od(A_j)$ is jointly conservative.

Let $0 \neq M \in \mathcal{M}od(A)$, we will prove that $M_j := A_j \otimes_A M \neq 0$ for all j. As $M \neq 0$, then M contains a submodule of the form A/I. In fact, there is a non zero $f : A \to M$, so we take $I = kerf$, then we have the factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & M \\ & A/I \arrow[ur, tail] & \end{tikzcd}$$

Let $\mathfrak{m}$ be a maximal ideal containing I, its existence is proven in Proposition 1.3.12, then the morphism φ from A to the field object $K = A/\mathfrak{m}$ represents an element in $F(K)$. As we have a surjective function $\coprod_i U_i(K) \to F(K)$, the element φ seen as an arrow factorizes through some $u_j : A \to A_j$, this means that there exists φ_j such that the diagram commutes

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "u_j"'] & & K \\ & A_j \arrow[ur, "\varphi_j"'] & \end{tikzcd} .$$

Now, by the universal property of $Ker\varphi_j$, there exist a unique morphism $\mathfrak{m} \to Ker\varphi_j$, then we have the pullback diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & u_j^{-1}(Ker\varphi_j) \arrow[r] \arrow[d, tail] & Ker\varphi_j \arrow[d, tail] \\ & A \arrow[r, "u"] \arrow[d, "\varphi"] & A_j \arrow[d, "\varphi_j"] \\ & K \arrow[r, equal] & K \end{tikzcd}$$

with the morphism $\mathfrak{m} \to u_j^{-1}(Ker\varphi_j)$ being a monomorphism.

Let $\mathfrak{m}_j$ be a proper maximal ideal containing $Ker(\varphi_j)$, since u_j is flat we have that $u_j^{-1}(Ker\varphi_j) \hookrightarrow u_j^{-1}(\mathfrak{m}_j)$, then $\mathfrak{m} \hookrightarrow u_j^{-1}(\mathfrak{m}_j)$. We claim that $u^{-1}(\mathfrak{m}_j)$ is a proper ideal of A. In fact, if $u^{-1}(\mathfrak{m}_j) = A$, then the morphism $u_j : A \to A_j$ factorizes through $A \twoheadrightarrow \mathfrak{m}_j$, but since $\mathfrak{m}_j$ is a proper ideal this is a contradiction.

By maximality $\mathfrak{m} = u_j^{-1}(\mathfrak{m}_j)$. Then we have the commutative diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[r] \arrow[d, tail] & \mathfrak{m}_j \arrow[d, tail] \\ A \arrow[r, "u_j"] & A_j \end{tikzcd}$$

tensoring with the A-algebra A_j we have a morphism $A_j \otimes_A \mathfrak{m}_j \cong A_j\mathfrak{m} \longrightarrow \mathfrak{m}_j$ commuting with the inclusion to A_j. Then this morphism must be a monomorphism. On the other hand, we have a monomorphism $A_jI \rightarrowtail A_j\mathfrak{m}$, it follows that $\mathfrak{m}_j$ contains the ideal A_jI then $A_j/A_jI \neq 0$ and we have a monomorphism

$$A/I \otimes_A A_j \cong A_j/A_jI \rightarrowtail A_j \otimes_A M = M_j,$$

this means that $M_j \neq 0$ for all j, therefore $\{U_i \to SpecA\}_i$ is a Zariski covering. □

The next lemma is a well known result in category theory and it will be useful to prove Lemma 4.1.6. A proof for this result can be found in [28].

Lemma 4.1.5. *Let $\mathcal{C}$ be a cocomplete category and $V_0 : Hom(\mathbb{1}, -) : \mathcal{C} \to \mathcal{E}ns$ such that:*

- *V_0 preserves filtered colimits and epimorphisms, i.e, $\mathbb{1}$ is finitely presentable and projective.*
- *Every object $X \in \mathcal{C}$ is a colimit of the diagram $F \to X$ with F a free object in $\mathcal{C}$. In the abelian case, this condition says that there exists $S \in \mathcal{E}ns$ and an epimorphism $\mathbb{1}^{(S)} \twoheadrightarrow X$.*

Then $X \in \mathcal{C}$ if finitely presented if and only if there exists $m, n \in \mathbb{N}$ and an exact diagram

$$\mathbb{1}^m \rightrightarrows \mathbb{1}^n \longrightarrow X$$

which is called a finite presentation for X.

The next tool we need in order to construct the projective space is the following two lemmas, they are the relative version of a well-known result in commutative algebra, it concerns about the stability of direct summands of a finitely presented module. We first introduce some notation. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a co-complete abelian, symmetric closed monoidal category, such that $\mathbb{1}$ is a projective finitely presentable generator and the forgetful functor $Hom_{\mathcal{C}}(\mathbb{1}, -)$ reflects epimorphisms. Let $A \to B$ be a morphism in $Comm(\mathcal{C})$ and let M, N be two A-modules, we would like to define a morphism

$$\zeta : B \otimes_A hom_A(M, N) \to hom_B(B \otimes_A M, B \otimes_A N).$$

We have the morphism $1 \otimes \varepsilon : B \otimes_A M \otimes_A hom_A(M, N) \to B \otimes_A N$ which by adjunction corresponds to a morphism

$$hom_A(M, N) \xrightarrow{\chi} hom_A(B \otimes_A M, B \otimes_A N).$$

On the other hand, as $B \otimes_A M$ and $B \otimes_A N$ are B-modules, the object $hom_A(B \otimes_A M, B \otimes_A N)$ is also a B-module, with action

$$\mu : B \otimes_A hom_A(B \otimes_A M, B \otimes_A N) \to hom_A(B \otimes_A M, B \otimes_A N)$$

by composing these two morphism, we have the morphism

$$\xi : B \otimes_A hom_A(M, N) \xrightarrow{1 \otimes \chi} B \otimes_A hom_A(B \otimes_A M, B \otimes_A N) \xrightarrow{\mu} hom_A(B \otimes_A M, B \otimes_A N).$$

It's not hard to see that ξ equalizes the two morphisms

$$hom_A(B \otimes_A M, B \otimes_A N) \rightrightarrows hom_B(B, hom_A(B \otimes_A M, B \otimes_A N))$$

and since $hom_B(B \otimes_A M, B \otimes_A N)$ is, by definition the equalizer of these two morphisms, there exists an arrow $\zeta : B \otimes_A hom_A(M, N) \to hom_B(B \otimes_A M, B \otimes_A N)$.

With notations as above we have the following results:

Lemma 4.1.6. *Let $\mathcal{C}$ be an abelian strong relative context. If $A \to B \in Comm(\mathcal{C})$ is faithfully flat and M is finitely presentable, then the induced morphism*

$$\zeta : B \underset{A}{\otimes} hom_A(M, N) \to hom_B(B \underset{A}{\otimes} M, B \underset{A}{\otimes} N)$$

is an isomorphism.

Proof. Under these hypothesis we have by Lemma 4.1.5, that M is finitely presentable if and only if M has a finite presentation.

Suppose $M = A$:

$$\begin{tikzcd} B \otimes_A hom_A(A, N) \arrow[r, "\zeta"] \arrow[d, "\cong"] & hom_B(B \otimes_A A, B \otimes_A N) \arrow[d, "\cong"'] \\ B \otimes_A N \arrow[r, "\cong"'] & hom_B(B, B \otimes_A N) \cong B \otimes_A N \end{tikzcd}$$

we have that ζ is an isomorphism.

Now suppose $M = A^m$. Since $hom_A(M, -), B \otimes_A -$ are additive functors, we have the isomorphisms

$$\begin{tikzcd} B \otimes_A hom_A(A^m, N) \arrow[r, "\zeta"] \arrow[d, "\cong"] & hom_B(B \otimes_A A^m, B \otimes_A N) \arrow[d, "\cong"'] \\ B \otimes_A N^m \arrow[r, "\cong"'] & hom_B(B^m, B \otimes_A N) \cong B \otimes_A N^m \end{tikzcd}$$

with ζ the direct sum of the isomorphisms given in the previous case.

For the general case, let $A^m \to A^n \to M \to 0$ be a finite presentation for M. Tensoring this presentation with B we obtain again an exact sequence

$$B^m \to B^n \to B \otimes_A M \to 0.$$

Applying $B \otimes_A hom_A(-, N)$ and $hom_B(-, B \otimes_A N)$ to both exact sequences respectively, we obtain the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & B \underset{A}{\otimes} hom_A(M, N) \arrow[r] \arrow[d, dashed] & B \underset{A}{\otimes} hom_A(A^n, N) \arrow[r] \arrow[d, "\cong"] & B \underset{A}{\otimes} hom_A(A^m, N) \arrow[d, "\cong"] \\ 0 \arrow[r] & hom_B(B \underset{A}{\otimes} M, N) \arrow[r] & hom_B(B \underset{A}{\otimes} A^n, N) \arrow[r] & hom_B(B \underset{A}{\otimes} A^m, N), \end{tikzcd}$$

The exactness in the left of the first row is due to the fact $A \to B$ is faithfully flat. The first column is an isomorphism since they are both the limits of isomorphic sequences. $\square$

Lemma 4.1.7. *Let $\mathcal{C}$ be an abelian strong relative context, $A \to B \in Comm(\mathcal{C})$ faithfully flat, $M \in \mathcal{M}od(A)$ of finite presentation and L an A-submodule of M. If $B \otimes_A L$ is a direct summand of $B \otimes_A M$ then L is a direct summand of M.*

Proof. L is a direct summand of M if and only if there exists a morphism $r : M \to L$ such that $r \circ i = 1_L$. This is equivalent to prove that the function between the homs is surjective, i.e., $Hom_A(M, L) \twoheadrightarrow Hom_A(L, L)$.
Since the forgetful functor $Hom_A(A, -)$ preserves epimorphisms, it is enough to prove that $\varphi : hom_A(M, L) \to hom_A(L, L)$ is an epimorphism. The result comes from the following commutative diagram

$$\begin{tikzcd} B \underset{A}{\otimes} hom_A(M,L) \arrow[r, "B\otimes\varphi"] \arrow[d, "\zeta_1\cong"'] & B \underset{A}{\otimes} hom_A(L,L) \arrow[d, "\cong\zeta_2"] \\ hom_B(B \underset{A}{\otimes} M, B \underset{A}{\otimes} L) \arrow[r, two heads, "\psi"] & hom_B(B \underset{A}{\otimes} L, B \underset{A}{\otimes} L). \end{tikzcd}$$

The previous lemma shows that ζ_1 and ζ_2 are isomorphisms. Then ψ is an epimorphisms since $B \underset{A}{\otimes} I$ is a direct summand of $B \underset{A}{\otimes} M$ and the forgetful functor $Hom_B(B, -)$ reflects epimorphisms. $\square$

Line Objects.

Next, following [6, 24] we give the definition and properties of the objects that make possible the definition of the projective space. This kind of objects are the categorification of rank one invertible sheaves over a scheme.

Definition 4.1.8 (Invertible object)**.** If $\mathcal{C}$ is a symmetric monoidal category, $L \in \mathcal{C}$ is called invertible if there exists an object $L^\vee$ and an isomorphism $\delta : \mathbb{1} \to L^\vee \otimes L$.

Note that if L is invertible then $L \otimes - : \mathcal{C} \to \mathcal{C}$ is an equivalence with inverse $L^\vee \otimes -$.

Remark 4.1.9.

1. $\mathbb{1}$ is invertible and invertible objects are closed under tensor products. Isomorphisms classes of invertible objects form a group denoted $Pic(\mathcal{C})$. For more details on $Pic(\mathcal{C})$, see [20].

2. If L is invertible, then for every isomorphism $\delta : \mathbb{1} \to L \otimes L^\vee$ there exists an isomorphism $\epsilon : L^\vee \otimes L \to \mathbb{1}$ satisfying the triangle axioms

$$\begin{tikzcd} \mathbb{1} \otimes L \arrow[r, "\delta\otimes 1"] \arrow[d, equal] & L \otimes L^\vee \otimes L \arrow[dl, "\epsilon\otimes 1"] \\ \mathbb{1} \otimes L & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \otimes L^\vee \arrow[r, "\delta\otimes 1"] \arrow[d, equal] & L \otimes L^\vee \otimes L \arrow[dl, "\epsilon\otimes 1"] \\ \mathbb{1} \otimes L^\vee & \end{tikzcd}$$

   therefore $(L, L^\vee, \epsilon, \delta)$ is a duality in $\mathcal{C}$.

3. If L is invertible and $\mathbb{1}$ is projective, then L is a projective object in $\mathcal{C}$. In fact, since L is invertible, $hom_{\mathcal{C}}(L, -)$ is left adjoint to $hom_{\mathcal{C}}(L^\vee, -)$, therefore it preserves colimits. As $Hom_{\mathcal{C}}(L, -) \cong Hom_{\mathcal{C}}(\mathbb{1}, hom_{\mathcal{C}}(L, -))$ and $Hom_{\mathcal{C}}(\mathbb{1}, -)$ preserves epimorphisms we have that $Hom_{\mathcal{C}}(L, -)$ preserves epimorphisms.

Now, for invertible objects there is a well defined signature

Definition 4.1.10 (Signature)**.** Since $L \otimes -$ is an equivalence we have bijections $End_{\mathcal{C}}(\mathbb{1}) \cong End_{\mathcal{C}}(L) \cong End_{\mathcal{C}}(L \otimes L)$ then the signature is the endomorphism of $\mathbb{1}$ corresponding to the symmetry $\sigma_{L,L} : L \otimes L \to L \otimes L$ via that bijection.

Definition 4.1.11 (Line object.)**.** $L \in \mathcal{C}$ is called a line object if it is invertible and its signature is the identity morphism.

Remark 4.1.12. An object M in $\mathcal{C}$ is said to be *symtrivial* if $\sigma_{M,M} : M \otimes M \to M \otimes M$ is the identity arrow. Since the signature of an invertible object L in $\mathcal{C}$

is the endomorphism associated to the symmetry of $L \otimes L$, then a line object is simply an invertible symtrivial object.

Proposition 4.1.13.

1. *Symtrivial objects are preserved by strong monoidal functors.*

2. *If $\mathcal{C}$ is cocomplete then $M \oplus N$ is symtrivial if and only if $M \otimes N = 0$ and M, N are symtrivial.*

3. *Let A be a faithfully flat commutative algebra in $\mathcal{C}$, $L \in \mathcal{C}$. If $A \otimes L$ is a line object in $\mathcal{M}od_{\mathcal{C}}(A)$ then L is a line object in $\mathcal{C}$*

Proof. For details see [6]. □

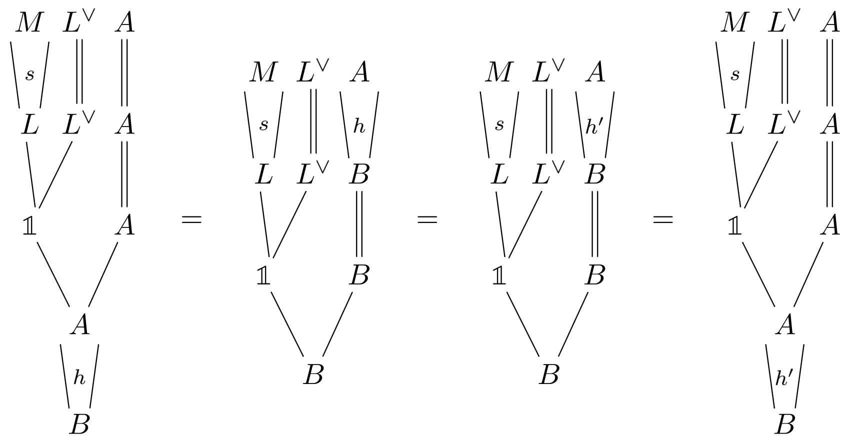
Examples 3.

1. Let R be a commutative ring. Then M is a line object in $\mathcal{M}od(R)$ if and only if M is projective module of rank one.
2. Let X be a scheme, then the line objects in $Qcoh(X)$ are precisely the invertible sheaves.

Next, we want to prove a very important property of line objects that says that every epimorphism $s : \mathbb{1} \to L$ with L a line object, is an isomorphism. This result is very useful to show that our projective space is in fact covered by affine schemes. First we will prove the following lemma:

Lemma 4.1.14. *Let L be an invertible object and $s : M \to L$ be an epimorphism. If $h, h' : A \to B$ are two morphisms such that $h \otimes s = h' \otimes s$ then $h = h'$*

Proof. Consider the epimorphism $s \otimes L^{\vee} \otimes A : M \otimes L^{\vee} \otimes A \to L \otimes L^{\vee} \otimes A \cong A$, the following elevators show that $h(s \otimes L^\vee \otimes A) = h'(s \otimes L^\vee \otimes A)$:



As $s \otimes L^\vee \otimes A$ is an epimorphism we have that $h = h'$. □

Now, Let L be a line object and $s : M \to L$ be an epimorphism and consider the two morphisms

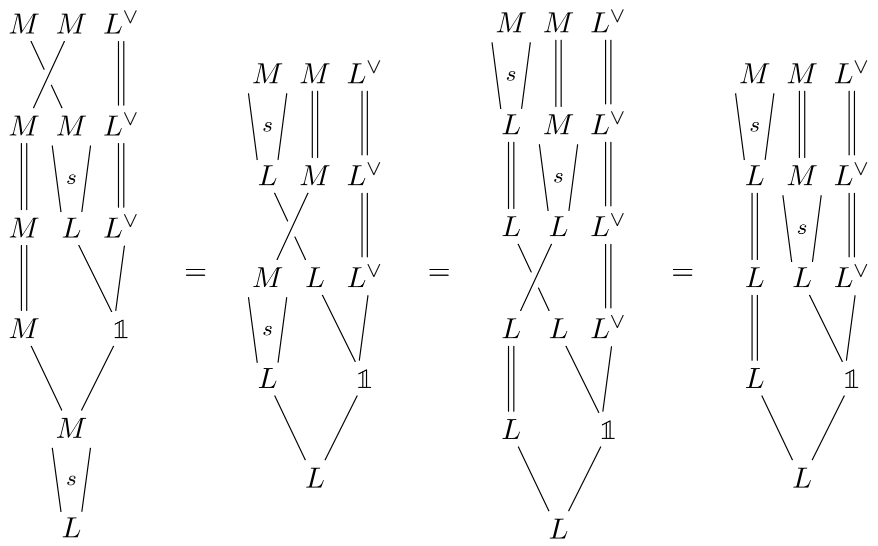
$$1 \otimes s,\ (1 \otimes s)\sigma_{M,M} : M \otimes M \to M \otimes L, \tag{4.1.2}$$

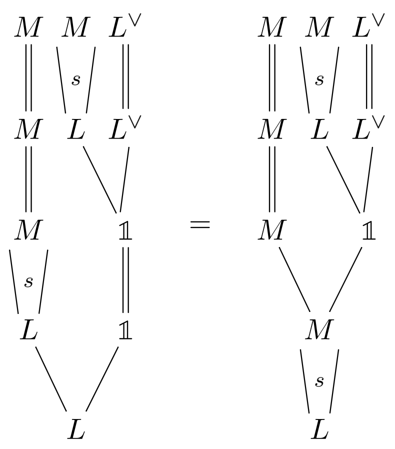
since L is invertible, these morphisms correspond to two morphisms

$$M \otimes M \otimes L^\vee \rightrightarrows M \otimes L \otimes L^\vee \cong M.$$

Lemma 4.1.15. *With notations as above, the epimorphism $s : M \to L$ is the coequalizer of these two morphisms.*

Proof. The following elevators show that $s(M \otimes S) = s(M \otimes s)\sigma_{M,M}$



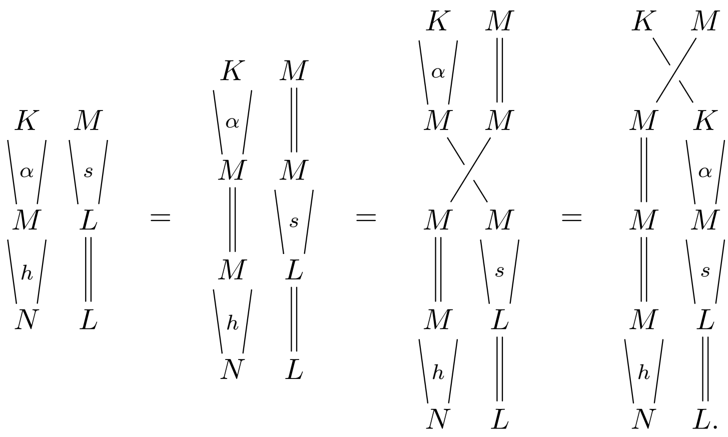


(4.1.3)

Where the first, second and last equalities are simply the naturality of the braiding and the unitor l. The third equality is because of L is symtrivial.

Now, we will show that s is universal, that is, for any $h : M \to N$ such that $h(M \otimes s) = h(M \otimes s)\sigma_{M,M}$, h factorizes through s. As s is an epimorphism, we have that s is the coequalizer of two pair of arrows, let us say $\alpha, \beta : K \to M$. The idea is to prove that $h\alpha \otimes s = h\beta \otimes s$, then by Lemma 4.1.14, $h\alpha = h\beta$, but since s is the coequalizer of α and β, there exists a unique $\tilde{h}$ such that $s\tilde{h} = h$.

The following elevator diagram shows that $h\alpha \otimes s = (h \otimes s\alpha)\sigma_{K,M}$:



Where the second equality is due to h satisfies an elevator diagram similar to (4.1.3). As $s\alpha = s\beta$, we have that $h\alpha \otimes s = (h \otimes s\beta)\sigma_{K,M}$. Going backwards with the elevators one shows that $(h \otimes s\beta)\sigma_{K,M} = h\beta \otimes s$, therefore we have the result. □

As a consequence we have the property of line objects:

Corollary 4.1.16. *With notations as above, every epimorphism $\mathbb{1} \to L$ is an isomorphism.*

Proof. Since $\sigma_{\mathbb{1},\mathbb{1}} = 1$, the two morphisms $1 \otimes s$ and $(1 \otimes s)\sigma_{\mathbb{1},\mathbb{1}}$ coincide. By Lemma 4.1.15, s coequalizes the two induced morphisms, it means that the sequence

$$\mathbb{1} \otimes \mathbb{1} \otimes L^{\vee} \xrightarrow{0} \mathbb{1} \xrightarrow{s} L \longrightarrow 0$$

is exact, then $ker(s) = 0$. Thus, s is an epimorphism and a monomorphism, so it is an isomorphism. □

Lemma 4.1.17. *Let $(SpecA_i \to SpecA)_{i \in I}$ be a finite Zariski open covering. Let the A-algebra $B = \prod_i A_{f_i}$. If for every $i \in I$, L_i is a line object in $\mathcal{M}od_{\mathcal{C}}(A_i)$ then $J = \prod_i L_i$ is a line object in $\mathcal{M}od_{\mathcal{C}}(B)$.*

Proof. We claim that J has an inverse in $\mathcal{M}od_{\mathcal{C}}(B)$ given by $J^{\vee} = \prod_i L_i^{\vee}$ with $L_i^{\vee}$ is the inverse of L_i in $\mathcal{M}od_{\mathcal{C}}(A_i)$ for all $i \in I$. If $m_i, m_i^{\vee}$ denote the actions of A_i on L_i and $L_i^{\vee}$ respectively, we will prove the following two things:

i. For every $i \in I$, $L_i \otimes_{A_i} L_i^{\vee} \cong L_i \otimes_B L_i^{\vee}$. let us consider the diagram with exact rows:

$$\begin{tikzcd}
L_i \otimes_A B \otimes_A L_i^{\vee} \arrow[r, "\bar{r}"] \arrow[d, "1\otimes p_i\otimes 1"] & L_i \otimes_A L_i^{\vee} \arrow[r, "\pi"] \arrow[d, equal] & L_i \otimes_B L_i^{\vee} \arrow[d, dashed, "\varphi"', shift right] \\
L_i \otimes_A A_i \otimes_A L_i^{\vee} \arrow[r, "r"] \arrow[u, bend left, "1\otimes\lambda_i\otimes 1"] & L_i \otimes_A L_i^{\vee} \arrow[r, "\pi'"] & L_i \otimes_{A_i} L_i^{\vee} \arrow[u, dashed, "\psi"', shift right]
\end{tikzcd} \tag{4.1.4}$$

where $\bar{r} = \bar{m_i} \otimes 1 - 1 \otimes \bar{m_i}^{\vee}$, $r = m_i \otimes 1 - 1 \otimes m_i^{\vee}$ and π, π' the cokernel maps. As $\pi' \circ \bar{r} = \pi' \circ r \circ (1 \otimes p_i \otimes 1) = 0$, there exists an arrow $\varphi : L_i \otimes_B L_i^{\vee} \to L_i \otimes_{A_i} L_i^{\vee}$ sucht that $\varphi\pi = \pi'$.

On the other hand, due to

$$(1 \otimes p_i \otimes 1)(1 \otimes \lambda_i \otimes 1) = 1,$$

then

$$\pi \circ r = \pi \circ r(1 \otimes p_i \otimes 1) \circ (1 \otimes \lambda_i \otimes 1) = \pi \circ \bar{r}(1 \otimes \lambda_i \otimes 1) = 0$$

so, there exists an arrow $\psi : L_i \otimes_{A_i} L_i^{\vee} \to L_i \otimes_B L_i^{\vee}$ satisfying $\psi \circ \pi' = \pi$. let us check they are inverse to each other.

$$\psi\varphi\pi = \psi\pi' = \pi, \qquad \varphi\psi\pi' = \varphi\pi = \pi'$$

since π, π' are epimorphisms we get $\psi\varphi = 1$ and $\varphi\psi = 1$.

ii. For every $i \neq j$, $L_i \otimes_B L_j^{\vee} = 0$: in order to prove this, we will prove that

$$r_{ij} = m_i \otimes 1 - 1 \otimes m_j^{\vee} : L_i \otimes_A B \otimes_A L_j^{\vee} \to L_i \otimes_A L_j^{\vee}$$

is an epimorphism, thence its cokernel $L_i \otimes_B L_j^{\vee}$ would be the zero object. For this, consider for every $i \in I$, the morphism $\lambda^{(i)} : A \to B$ given by

$(0, \cdots, \eta_i, 0 \cdots,)$ with $\eta_i : A \to A_i$ the unit of A_i as an A-algebra, in the i-th position

$$r_{ij}(1 \otimes \lambda_i \otimes 1) : L_i \otimes_A A \otimes_A L_j^{\vee} \to L_i \otimes_A B \otimes_A L_j^{\vee}$$

is the identity arrow for $i \neq j$, this means that r_{ij} is an epimorphism for $i \neq j$.

combining i. and ii. we have that

$$J \otimes_B J^{\vee} \cong \prod_{i,j} coKer\left(L_i \otimes_A B \otimes_A L_j^{\vee} \to L_i \otimes_A L_j^{\vee} \to L_i \otimes_B L_j^{\vee}\right) \cong$$
$$\prod_{i} coKer\left(L_i \otimes_A B \otimes_A L_i^{\vee} \to L_i \otimes_A L_i^{\vee} \to L_i \otimes_B L_i^{\vee}\right) \cong \prod_i A_i = B$$

Now we show that J is a symtrivial object in $\mathcal{M}od_{\mathcal{C}}(B)$ provided that each L_i is symtrivial in $\mathcal{M}od_{\mathcal{C}}(A_i)$ for all i. let us denote $\sigma, \sigma^i, \sigma^B$ the symmetries in $\mathcal{M}od_{\mathcal{C}}(A)$,$\mathcal{M}od_{\mathcal{C}}(A_i)$, $\mathcal{M}od_{\mathcal{C}}(B)$ respectively and consider the following diagram, where unadorned tensor means $\otimes_A$

$$\begin{tikzcd}
L_i \otimes A_i \otimes L_i \arrow[r] \arrow[d, "1\otimes\lambda_i\otimes 1"'] & L_i \otimes L_i \arrow[r] \arrow[d, equal] & L_i \otimes_{A_i} L_i \arrow[d, "\psi"] \arrow[ddd, bend left, "\sigma^i_{L_i,L_i}"] \\
L_i \otimes B \otimes L_i \arrow[r] \arrow[d, "(\sigma_{B,L_i}\otimes 1)(1\otimes\sigma_{L_i,L_i})(\sigma_{L_i,B}\otimes 1)"'] & L_i \otimes L_i \arrow[r] \arrow[d, "\sigma_{L_i,L_i}"] & L_i \otimes_B L_i \arrow[d, "\sigma^B_{L_i,L_i}"] \\
L_i \otimes B \otimes L_i \arrow[r] \arrow[d, "1\otimes p_i\otimes 1"'] & L_i \otimes L_i \arrow[r] \arrow[d, equal] & L_i \otimes_B L_i \arrow[d, "\varphi"] \\
L_i \otimes A_i \otimes L_i \arrow[r] & L_i \otimes L_i \arrow[r] & L_i \otimes_{A_i} L_i
\end{tikzcd}$$

with φ, ψ defined as in 4.1.4 with $L_i^{\vee} = L_i$. By naturality of σ and the identity

$$(1 \otimes p_i \otimes 1)(1 \otimes \lambda_i \otimes 1) = 1$$

we have that

$$(1\otimes p_i\otimes 1)(\sigma_{B,L_i}\otimes 1)(1\otimes\sigma_{L_i,L_i})(\sigma_{L_i,B}\otimes 1)(1\otimes\lambda_i\otimes 1) = (\sigma_{A_i,L_i}\otimes 1)(1\otimes\sigma_{L_i,L_i})(\sigma_{L_i,A_i}\otimes 1)$$

consequently $\varphi\,\sigma^B_{L_i,L_i}\varphi^{-1} = \sigma^i_{L_i,L_i} = 1$ which implies $\sigma^B_{L_i,L_i} = 1$.

So far, we have proved that each L_i is a symtrivial object in $\mathcal{M}od_{\mathcal{C}}(B)$. To finally get the result, we use the fact that $L_i \otimes_B L_j = 0$ for every $i \neq j$ and proposition 4.1.13 (2), so $\bigoplus_i L_i$ is symtrivial in $\mathcal{M}od_{\mathcal{C}}(B)$ □

4.2 The scheme $\mathbb{P}^n_{\mathcal{C}}$

As a motivation for the definition of the projective space, we first recall a characterisation of the functor of points of the scheme $\mathbb{P}^n_{\mathbb{Z}}$. Let us denote $Mor(X,Y)$ the set of morphisms in the category of schemes Sch, then we have that:

Theorem 4.2.1 (See [11]). *For any ring A,*

$$\mathit{Mor}(SpecA, \mathbb{P}^n_{\mathbb{Z}}) = \{L \subset A^{n+1} \ : L \text{ is a locally rank 1 direct summand of } A^{n+1}\}$$

$$\cong \{\text{invertible } A-\text{modules } P \text{ with an epimorphism } A^{n+1} \to P\}/\{\text{isomorphisms}\}$$

where by invertible module we mean a finitely generated, locally free A-module of rank 1 and an isomorphism from $\varphi : A^{n+1} \to P$ to $\varphi' : A^{n+1} \to P$ is an automorphism $\alpha : P \to P$ such that $\alpha\varphi = \varphi'$.
Moreover, for any scheme X, one has the natural bijection

$$Mor(X, \mathbb{P}^n_{\mathbb{Z}}) = \{\text{Invertible sheaves } P \text{ in } \mathcal{Q}coh(X) \text{ with an epimorphism } \mathcal{O}^{n+1}_X \to P\}/\{iso\}$$

Having this characterisation in mind and by example 3 and remark 4.1.9 item iii), we define the projective space relative to the category $\mathcal{C}$ as the functor $\mathbb{P}^n_{\mathcal{C}} : Aff^{op}_{\mathcal{C}} \to \mathcal{E}ns$, as follows:

Definition 4.2.2. [Relative Projective Scheme] Let $n \geq 1$ a fixed integer. For every affine scheme $Spec(A)$ we define $\mathbb{P}^n_{\mathcal{C}}(A)$ to be the set of submodules L of A^{n+1} satisfying

- L is a line object in $\mathcal{M}od_{\mathcal{C}}(A)$
- For the monomorphism $\mathrm{x} : L \to A^{n+1}$, there exists a retraction $A^{n+1} \to L$, this is, L is a direct summand of A^{n+1}.

For every morphism $Spec(B) \to Spec(A)$ in $Aff_{\mathcal{C}}$, the function $\mathbb{P}^n_{\mathcal{C}}(A) \to \mathbb{P}^n_{\mathcal{C}}(B)$ assigns to $L \in \mathbb{P}^n_{\mathcal{C}}(A)$ the corresponding direct summand $B \otimes_A L \hookrightarrow B^{n+1}$.

Note that $B \otimes_A L$ is a line object in $\mathcal{M}od_{\mathcal{C}}(B)$ since line objects are preserved by strong monoidal functors.

Remark 4.2.3. Note that for every $A \in Comm(\mathcal{C})$ a pair (L, x) in $\mathbb{P}^n_{\mathcal{C}}(A)$ is a subobject, that is, a class of monomorphisms of A^{n+1}, where $(L_1 \mathrm{x}_1)$, (L_2, x_2) represent the same element subobject, if there exists an isomorphism $\lambda : L_1 \to L_2$ such that the diagram commutes

$$\begin{tikzcd} L_1 \arrow[r, hook, "\mathrm{x}_1"] \arrow[dr, "\lambda"'] & A^{n+1} \\ & L_2 \arrow[u, hook, "\mathrm{x}_2"'] \end{tikzcd}$$

Since L is an invertible object we have that $Aut(L) \cong Aut(A)$, therefore the equivalence relation is given by scalar multiplication by invertible elements in A. So if we think of the pair (L, x) as a vector in A^{n+1}, its class in $\mathbb{P}^n_{\mathcal{C}}(A)$ represents the "line" in A^{n+1}. This is kind of the intuition one has of the classical projective space.

Theorem 4.2.4. *Let $\mathcal{C}$ be an abelian strong relative context. Then the presheaf $\mathbb{P}^n_{\mathcal{C}}$ is a $\mathcal{C}$-scheme.*

Proof. Let us check the sheaf condition in the Zariski topology: Let $\{SpecA_i \to SpecA\}_i$ be a Zariski covering, we have to prove the exactness of the sequence

$$\mathbb{P}^n_{\mathcal{C}}(A) \longrightarrow \prod_i \mathbb{P}^n_{\mathcal{C}}(A_i) \rightrightarrows \prod_{i,j} \mathbb{P}^n_{\mathcal{C}}(A_{ij}). \tag{4.2.1}$$

Let $L \in \mathbb{P}^n_{\mathcal{C}}(A)$, by the equivalence given in (3.1.1) and Proposition 3.2.5 the following sequence is exact

$$L \longrightarrow \prod_i L_i \rightrightarrows \prod_{i,j} L_{ij}$$

then L is determined by $L_i \in \mathbb{P}^n_{\mathcal{C}}(A_i)$ therefore $\mathbb{P}^n_{\mathcal{C}}(A)$ is a cone of the diagram. Now we have to check that $\mathbb{P}^n_{\mathcal{C}}(A)$ is universal. To see this, consider the compatible family $(L_i)_i \in \prod_i \mathbb{P}^n_{\mathcal{C}}(A_i)$. The compatibility says that we have a family of isomorphisms

$$\begin{tikzcd} L_i \otimes_A A_j \arrow[r, "\cong", "\theta_{ij}"'] \arrow[dr, hook] & L_j \otimes_A A_i \arrow[d, hook] \\ & A^{n+1}_{ij}, \end{tikzcd}$$

let us prove that $(L_i, \theta_{i,j})_{i,j}$ is a descent data, that is, θ_{ij} satisfies the cocycle condition $\theta_{j,k} \circ \theta_{i,j} = \theta_{i,k}$ in $\mathcal{M}od(A_{i,j,k})$. In fact, the following diagram of subobjects of $A^{n+1}_{i,j,k}$

$$\begin{tikzcd} L_i \otimes_A A_j \otimes_A A_k \arrow[r, "\theta_{i,j}\otimes A_k"] \arrow[rr, bend left, "\theta_{i,k}\otimes A_j"] & L_j \otimes_A A_i \otimes_A A_k \arrow[r, "\theta_{j,k}\otimes A_i"] & L_k \otimes_A A_i \otimes_A A_j \end{tikzcd}$$

says that the two arrows coincide since between two subobjects there is at most one arrow. Thus by the equivalence given in (3.1.1) and Proposition 3.2.5, we have that the descent data $(L_i, \theta_{i,j})_{i,j}$ defines an A-module L as the limit of the diagram

$$\begin{tikzcd} & L_i \arrow[r] & L_i \otimes A_j \arrow[d, "\theta_{i,j}"] \\ L \arrow[ur] \arrow[dr] & & \\ & L_j \arrow[r] & L_j \otimes A_i. \end{tikzcd}$$

To prove that $L \in \mathbb{P}^n_{\mathcal{C}}(A)$, consider the product algebra $B = \prod_i A_i$, note that B is a faithfully flat A-algebra as $\mathcal{M}od_{\mathcal{C}}(B) \cong \prod_i \mathcal{M}od_{\mathcal{C}}(A_i)$ and the functor $- \otimes_A B$ is naturally isomorphic to $\bigtimes_i(- \otimes_A A_i)$. Now take the B-module $L \otimes B$, then we have that

$$L \otimes B \cong \prod_i L \otimes A_i \cong \prod_i L_i.$$

By Lemma 4.1.17, $L \otimes B$ is a line object in $\mathcal{M}od(B)$ therefore by proposition 4.1.13 we have that L is a line object in $\mathcal{M}od(A)$. Finally by Lemma 4.1.7, L is a direct summand of A^{n+1}.

$\mathbb{P}^n_{\mathcal{C}}$ is covered by the affine open sub-functors U_i for $i = 1, \cdots n+1$

$$U_i(A) = \{L \in \mathcal{M}od_{\mathcal{C}}(A) : \ L \overset{\mathbf{x}}{\hookrightarrow} A^{n+1} \xrightarrow{\pi_i} A \quad \pi_i \circ \mathbf{x} \text{ is an isomorphism}\}. \tag{4.2.2}$$

Representability of the subfunctors U_i: let us fix the index i. Given any element $(L, \mathbf{x}) \in U_i(A)$, we identify L with A as submodules of A^{n+1} via the isomorphism $\pi_i \mathbf{x} : L \cong A$, then we obtain $\tilde{\mathbf{x}} = \mathbf{x}(\pi_i \mathbf{x})^{-1} : A \to A^{n+1}$, this means that $(L, \mathbf{x}) = (A, \tilde{\mathbf{x}})$ as subobjects of A^{n+1}. Since $\pi_i \tilde{\mathbf{x}} = 1$, $\tilde{\mathbf{x}}$ is completely determined by specifying the morphisms $\pi_j \tilde{\mathbf{x}} : A \to A$ for $j = 1, \cdots n+1$ and $j \neq i$, i.e., the functor U_i is isomorphic to the functor

$$A \mapsto \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \mathrm{Hom}_A(A, A) \cong \prod_{j=1}^{n} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, A)$$

$$\cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}^n, A) \cong \mathrm{Hom}_{Comm(\mathcal{C})}(\mathbb{1}[x_1, \cdots x_n], A) = \mathbb{A}^n_{\mathcal{C}}(A),$$

therefore U_i is representable by an affine scheme.

The sub functors U_i are Zariski open immersions: let us see that for affine scheme h_A and any morphism $h_A \to \mathbb{P}^n_{\mathcal{C}}$, the pullback $h_A \times_{\mathbb{P}^n_{\mathcal{C}}} U_i$ is a Zariski immersion of h_A.

By Yoneda's Lemma the morphism $h_A \to \mathbb{P}^n_{\mathcal{C}}$ corresponds to $(L, \mathbf{x})$ in $\mathbb{P}^n_{\mathcal{C}}(A)$. Consider the pullback

$$\begin{tikzcd} V_i = h_A \times_{\mathbb{P}^n_{\mathcal{C}}} U_i \arrow[r] \arrow[d] & U_i \arrow[d, hook] \\ h_A \arrow[r] & \mathbb{P}^n_{\mathcal{C}}. \end{tikzcd}$$

Now, an element in $V_i(B)$ is the same as a morphism $f : A \to B \in Comm(\mathcal{C})$ such that $B \otimes_A L \twoheadrightarrow B^{n+1} \twoheadrightarrow B$ is an isomorphism. If I_i denotes the ideal in A defined by the image of $\pi_i \circ \mathbf{x}$, then tensoring the factorization of this arrow with B, we get a diagram

$$
\begin{tikzcd}
L \arrow[rr, "\pi_i \circ \mathbf{x}"] \arrow[dr, twoheadrightarrow, "p"'] & & A \\
& I_i \arrow[ur, hook, "j"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes_A L \arrow[rr, twoheadrightarrow, "B \otimes (\pi_i \circ \mathbf{x})"] \arrow[dr, twoheadrightarrow, "B \otimes p"'] & & B \otimes_A A \cong B. \\
& B \otimes_A I_i \arrow[ur, "m_B(B \otimes f \circ j)"'] &
\end{tikzcd}
$$

We have that all the arrows in the triangle on the right are isomorphisms. On the other hand, consider the ideal BI_j, which by definition is the image of $m_B(B \otimes f \circ j)$ then we have $B \otimes_A I_i \cong BI_i$. This means that V_i is contained in the complementary open subscheme associated to the ideal I_i. Let us see that the complementary open U_{I_i} defined by the ideal I_i is contained in V_i . Let $f \in U_{I_i}(B)$, i.e., $f : A \to B$ satisfies that the induced ideal BI_i is isomorphic with B. Then we have that, by the triangle in the right, $B \otimes_A L \to B$ is an epimorphism and $B \otimes_A L$ is a line object in $\mathcal{M}od(B)$, so tensoring this epimorphism with the inverse of $B \otimes_A L$, we have again an epi $B \twoheadrightarrow B \otimes_A L^\vee$ which by Corollary 4.1.16 is an isomorphism in $\mathcal{M}od(B)$, then tensoring again with the inverse we get $B \otimes_A L \xrightarrow{\cong} B$. This means that $f \in V_i(B)$. Finally by Lemma 4.1.3, $V_i \subset SpecA$ is a Zariski open, so is $U_i \subset \mathbb{P}^n_{\mathcal{C}}$.

The family $(U_i)_i$ **is an affine Zariski open covering**: We have to prove that

$$\coprod_i U_i \to \mathbb{P}^n_{\mathcal{C}}$$

is an epimorphism of sheaves. By lemma 4.1.4 is enough to prove that $\coprod_i U_i(\mathbb{K}) \to \mathbb{P}^n_{\mathcal{C}}(\mathbb{K})$ is surjective for every field $\mathbb{K} \in Comm(\mathcal{C})$.

Let $L \in \mathbb{P}^n_{\mathcal{C}}(\mathbb{K})$, i.e., $\mathbf{x} : L \hookrightarrow \mathbb{K}^{n+1}$, then there exists an index j such that the arrow $\pi_j \circ \mathbf{x} : L \to \mathbb{K}$ is non zero but then the image ideal I_j in $\mathbb{K}$ must be exactly $\mathbb{K}$ thus we have an epi $L \twoheadrightarrow \mathbb{K}$. Since L is a line object we have $L \otimes_{\mathbb{K}} L^{\vee} \cong \mathbb{K}$ therefore $\mathbb{K} \twoheadrightarrow L^{\vee}$, thus by Corollary 4.1.16, $\mathbb{K} \cong L^{\vee}$ so $\mathbb{K} \cong L$. $\square$

Definition 4.2.5. $\mathscr{M}$ is a symmetric monoidal category, $A \in Comm(\mathscr{M})$ and $\mathcal{C} = \mathcal{M}od_{\mathscr{M}}(A)$ then we define $\mathbb{P}^n_A := \mathbb{P}^n_{\mathcal{C}}$.

Definition 4.2.6. We define the octonionic projective space $\mathbb{P}^n_{\mathbb{O}}$ to be the functor $\mathbb{P}^n_{\mathcal{C}}$ relative to the category $\mathcal{C} = \mathcal{M}od_{\mathcal{U}}(\mathbb{O})$.

Theorem 4.2.7. *$\mathbb{P}^n_{\mathbb{O}}$ is a relative scheme.*

Proof. In Section 2.4 we have proved that $\mathcal{M}od_{\mathcal{U}}(\mathbb{O})$ is an abelian strong relative context, then by Theorem 4.2.4 we get the result. $\square$

Proposition 4.2.8. *The fiber product $U_{ij} = U_i \times_{\mathbb{P}^n_{\mathcal{C}}} U_j$ is representable by an affine scheme.*

Proof. For any A in $Comm(\mathcal{C})$, an element in $U_{ij}(A)$ is an isomorphism class of pairs $(L, \mathbf{x})$ where $L \overset{\mathbf{x}}{\hookrightarrow} A^{n+1}$ satisfies that $\pi_i \mathbf{x}$, $\pi_j \mathbf{x} : L \to A$ are isomorphisms. We denote these isomorphisms by x_i, x_j respectively. Using these isomorphisms, we identify the pair $(L, \mathbf{x})$ with a family of arrows

$$\left(\frac{x_k}{x_i} : A \to A\right) \quad \text{for} \quad k = 0, \ldots \hat{\imath}, \ldots n,$$

with the property that $\frac{x_j}{x_i}$ is an isomorphism (thence invertible). By the universal property of the localization and the polynomial algebra, we have that

$$U_{ij}(A) \cong Hom_{Comm(\mathcal{C})}(\mathbb{1}[\tfrac{x_0}{x_i}, \ldots, \tfrac{x_n}{x_i}][\tfrac{x_j}{x_i}]^{-1}, A)$$

$\square$

Now we give another definition of the relative projective space in terms of quotients instead of submodules. This definition is somehow dual to the one given in definition 4.2.2 and in fact we show the equivalence between these two definitions.

Definition 4.2.9. Let $n \geq 1$ a fixed integer. For every affine scheme $Spec(A)$ we define $\overline{\mathbb{P}^n_{\mathcal{C}}}(A)$ to be the set of quotients L of A^{n+1} with L a line object in $\mathcal{M}od_{\mathcal{C}}(A)$. For every morphism $Spec(B) \to Spec(A)$, the function $\overline{\mathbb{P}^n_{\mathcal{C}}}(A) \to \overline{\mathbb{P}^n_{\mathcal{C}}}(B)$ assigns to $L \in \overline{\mathbb{P}^n_{\mathcal{C}}}(A)$ the corresponding epimorphism $B^{n+1} \twoheadrightarrow B \otimes_A L$.

As before, $B \otimes_A L$ is a line object in $\mathcal{M}od_{\mathcal{C}}(B)$ since line objects are preserved by strong monoidal functors.

Theorem 4.2.10. *If $\mathcal{C}$ is an abelian strong relative context then $\overline{\mathbb{P}^n_{\mathcal{C}}}$ is a $\mathcal{C}$-scheme.*

The proof of this theorem is quite similar to its analogous result 4.2.4, however by the very definition we will not need Lemmas 4.1.6 and 4.1.7.

Proof. The sheaf condition is proven similarly as we did for $\mathbb{P}^n_{\mathcal{C}}$. Let $(A \to A_i)_{i \in I}$ be a Zariski covering for $SpecA$, we have to check the exactness of the diagram

$$\overline{\mathbb{P}^n_{\mathcal{C}}}(A) \longrightarrow \prod_i \overline{\mathbb{P}^n_{\mathcal{C}}}(A_i) \rightrightarrows \prod_{i,j} \overline{\mathbb{P}^n_{\mathcal{C}}}(A_{ij}).$$

We proceed as before to show that $\overline{\mathbb{P}^n_{\mathcal{C}}}(A)$ is a cone for the diagram. To show that is universal consider the compatible family $(L_i)_i \in \prod_i \overline{\mathbb{P}^n_{\mathcal{C}}}(A_i)$. The compatibility says that we have a family of isomorphisms θ_{ij} making the diagram commute

$$\begin{tikzcd} & A_{ij}^{n+1} \arrow[dl] \arrow[d] & \\ L_i \otimes_A A_j \arrow[r, "\cong", "\theta_{ij}"'] & L_j \otimes_A A_i, & \end{tikzcd}$$

$(L_i, \theta_{i,j})_{i,j}$ is a descent data, that is, θ_{ij} satisfies the cocycle condition $\theta_{j,k} \circ \theta_{i,j} = \theta_{i,k}$ in $\mathcal{M}od(A_{i,j,k})$. Thus by the equivalence given in (3.1.1), we have that the descent data $(L_i, \theta_{i,j})_{i,j}$ defines an A-module L as the limit of the diagram

$$\begin{tikzcd} & L_i \arrow[r] & L_i \otimes A_j \arrow[d, "\theta_{i,j}"] \\ L \arrow[ur] \arrow[dr] & & \\ & L_j \arrow[r] & L_j \otimes A_i. \end{tikzcd}$$

This L is a line object by propositions 4.1.13, 4.1.17. Finally to see that $A^{n+1} \to L$ is an epimorphism we use the fact that for every $i \in I$, we have the family of epimorphisms $A_i^{n+1} \twoheadrightarrow L_i \cong A_i \otimes_A L$, since the family of functors $A_i \otimes_A -$ is jointly conservative we get the result.

We now prove that $\overline{\mathbb{P}^n_{\mathcal{C}}}$ has an affine Zariski open covering. For this, we define for $i = 0, \dots n$

$$\overline{U_i}(A) = \{(L, \mathbf{x}),\ \text{such that the composition } A \xrightarrow{\lambda_i} A^{n+1} \overset{\mathbf{x}}{\twoheadrightarrow} L \text{ is an isomorphism}\}$$

the isomorphism $\mathbf{x}\lambda_i$ occurs in $\mathcal{M}od_{\mathcal{C}}(A)$. We will check the representability of these functors by showing that $U_i \cong \overline{U_i}$ for $i = 0, \dots n$.

In fact, for every affine scheme $SpecA$, we will define a bijection $U_i(A) \longleftrightarrow \overline{U_i}(A)$. First let us make a simplification: if $(L, \mathbf{x})$ belongs to $U_i(A)$ we can make the identification $L \cong A$ as subobjects of A^{n+1}, we will denote the pair $(A, \mathbf{x})$ in $U_i(A)$. The same goes for a pair $(L, \mathbf{y})$ in $\overline{U_i}(A)$. let us fix the index i:

$$U_i(A) \xrightarrow{\varphi} \overline{U_i}(A)$$

$$(A, \mathbf{x}) \longmapsto (A, \mathbf{y})$$

with $\mathbf{y}$ defined by the following: for every $j = 0, \dots n$, the diagram commutes

$$\begin{tikzcd} A \arrow[d, "\lambda_j"] \arrow[dr, "\pi_j \mathbf{x}"] & \\ A^{n+1} \arrow[r, dashed, "\exists ! \mathbf{y}"'] & A \end{tikzcd}$$

since $\mathbf{y}\lambda_i = \pi_i \mathbf{x}$ is an isomorphism we have that $\mathbf{y}$ is an epimorphism, even more that $(A, \mathbf{y})$ is in $\overline{U_i}(A)$.

For the arrow in the other direction:

$$\begin{aligned} \overline{U_i}(A) &\xrightarrow{\psi} U_i(A) \\ (A, \mathbf{y}) &\longmapsto (A, \mathbf{x}) \end{aligned} \qquad (4.2.3)$$

with x defined analogously by the following diagram for every $j = 0, \dots n$:

$$\begin{tikzcd} & A \\ A \arrow[ur, "\mathbf{x}\lambda_j"] \arrow[r, dashed, "\mathbf{x}"'] & A^{n+1}, \arrow[u, "\pi_j"'] \end{tikzcd}$$

as $\pi_i \mathbf{x} = \mathbf{y}\lambda_i$ is an isomorphisms it says that $(A, \mathbf{x})$ is in $U_i(A)$.

We will check that $\psi\varphi = 1$ the other one is similar.

$$\psi\varphi(A, \mathbf{x}) = \psi(A, \mathbf{y}) = (A, \tilde{\mathbf{x}})$$

with $\pi_j \tilde{\mathbf{x}} = \mathbf{y}\lambda_j = \pi_j \mathbf{x}$ for all $j = 0, \dots n$, then $\tilde{\mathbf{x}} = \mathbf{x}$.

The next step is to prove that every $\overline{U_i}$ is a Zariski open immersion of $\overline{\mathbb{P}^n_{\mathcal{C}}}$. Again, we will show that for any affine scheme h_A and morphism $h_A \to \overline{\mathbb{P}^n_{\mathcal{C}}}$, the pullback

$$\begin{tikzcd} \overline{V_i} \arrow[r] \arrow[d] & \overline{U_i} \arrow[d] \\ h_A \arrow[r] & \overline{\mathbb{P}^n_{\mathcal{C}}} \end{tikzcd}$$

is a Zariski open in h_A. We proceed as we did before, that is, we show that the subfunctor V_i is equivalent to the complementary open subscheme of h_A associated to an ideal I of A. For $B \in Comm(\mathcal{C})$, $\overline{V_i}(B)$ consists of morphisms $f : A \to B$ in $Comm(\mathcal{C})$ such that the if $(L, \mathbf{x})$ is in $\overline{U_i}(A)$, the induced morphism

$$B \otimes_A A \to B \otimes_A A^{n+1} \to B \otimes_A L$$

is an isomorphism. Take the dual morphism (as they are dualizable objects in $\mathcal{M}od_{\mathcal{C}}(A)$) of the composition $A \xrightarrow{\lambda_i} A^{n+1} \xrightarrow{\mathbf{x}} L$ and take its image ideal I, as we see in the factorization diagram

$$\begin{tikzcd} L \arrow[r, "\mathbf{x}^\vee"] \arrow[dr, two heads] & A^{n+1} \arrow[r, "\lambda_i^\vee"] & A \\ & I \arrow[ur, tail, "j"'] & \end{tikzcd}$$

applying the functor $B \otimes_A -$ we obtain the diagram

$$\begin{tikzcd} B \otimes_A L \arrow[r, "1\otimes\mathbf{x}^\vee"] \arrow[dr, two heads] & B \otimes_A A^{n+1} \arrow[r, "1\otimes\lambda_i^\vee"] & B \otimes_A A \\ & B \otimes_A I \arrow[ur, "1\otimes j"'] & \end{tikzcd}$$

the morphism in the top of the triangle is an isomorphism as it is the dual of an isomorphism, then the epimorphism $B \otimes_A L \twoheadrightarrow B \otimes_A I$ is a monomorphism, therefore all arrows in the triangle are isomorphisms, this means that $\overline{V_i}(B) \subset U_I$ where U_I denotes the complementary open subscheme associated to the ideal I. The other inclusion is obtained similarly.

Finally to show that the family $(\overline{U_i})_{i=0,\ldots n}$ is a covering we will prove that for every field $\mathbb{K} \in Comm(\mathcal{C})$ we have a surjection $\coprod_i \overline{U_i}(\mathbb{K}) \to \overline{\mathbb{P}^n_{\mathcal{C}}}(\mathbb{K})$. In fact, let $(L, \mathbf{x})$ in $\overline{\mathbb{P}^n_{\mathcal{C}}}(\mathbb{K})$, then there exists an index j such that $\mathbf{x}\lambda_j : \mathbb{K} \to L$ is the non zero arrow, then taking its dual morphism

$$L^\vee \xrightarrow{(\mathbf{x}\lambda_i)^\vee} \mathbb{K}$$

we have that this morphism must be an epimorphism since its image is an ideal in $\mathbb{K}$ and $\mathbb{K}$ is simple. After we tensor this epi with L we get an epimorphism $\mathbb{K} \twoheadrightarrow L$ which by Corollary 4.1.16 is an isomorphism. $\square$

Theorem 4.2.11. *$\mathbb{P}^n_{\mathcal{C}}$ and $\overline{\mathbb{P}^n_{\mathcal{C}}}$ are isomorphic as $\mathcal{C}$-schemes.*

Proof. Since the category of $\mathcal{C}$- schemes is a full subcategory of $Sh(Aff_{\mathcal{C}})$, we will prove the isomorphism as sheaves. Let us define for every $A \in Comm(\mathcal{C})$ a function

$$\overline{\mathbb{P}^n_{\mathcal{C}}}(A) \xrightarrow{\quad\Psi\quad} \mathbb{P}^n_{\mathcal{C}}(A)$$

$$A^{n+1} \xrightarrow{\mathbf{x}} L \quad \longmapsto \quad L^\vee \xrightarrow{\mathbf{x}^\vee} A^{n+1}.$$

Since $\mathbf{x}$ is an epimorphism, $\mathbf{x}^\vee$ is a monomorphism. As L is invertible, hence projective, there exists a section s for $\mathbf{x}$, then $r = s^\vee$ is a retraction for $\mathbf{x}^\vee$ thence $(L^\vee, \mathbf{x}^\vee)$ is an element in $\mathbb{P}^n_{\mathcal{C}}(A)$. let us see that Ψ is injective. Take $\mathbf{x}_i : A^{n+1} \twoheadrightarrow L_i$, $i = 1, 2$ two elements in $\overline{\mathbb{P}^n_{\mathcal{C}}}(A)$ such that their images coincide, then we have that $L_1^\vee$ and $L_2^\vee$ are isomorphic as subobjects of A^{n+1} as it's seen in the diagram

$$\begin{tikzcd} L_1^\vee \arrow[r, "\mathbf{x}_1^\vee"] \arrow[d, "\cong"'] & A^{n+1} \\ L_2^\vee \arrow[ur, "\mathbf{x}_2^\vee"'] & \end{tikzcd}$$

by dualizing we obtain that L_1 and L_2 are isomorphic as quotients of A^{n+1}, therefore they represent the same element in $\overline{\mathbb{P}^n_{\mathcal{C}}}(A)$.

To see that Ψ is an epimorphism, we check that for every i, the following diagram commutes:

$$\begin{tikzcd} \overline{\mathbb{P}^n_{\mathcal{C}}} \arrow[r, "\Psi"] & \mathbb{P}^n_{\mathcal{C}} \\ \overline{U_i} \arrow[u] \arrow[r, "\psi"] & U_i \arrow[u] \end{tikzcd}$$

with ψ defined in (4.2.3). As before, for every $A \in Comm(\mathcal{C})$, we identify the object $(L, \tilde{\mathbf{x}})$ in $\overline{U_i}(A)$ with $(A, \mathbf{x})$. Take $(A, \mathbf{x})$ in $\overline{U_i}(A)$, then $\Psi(A, \mathbf{x}) = (A, \mathbf{x}^\vee)$, since $\mathbf{x}\lambda_i$ is an isomorphism and for all $j = 0, \ldots n$, $\lambda_j^\vee = \pi_j$, then $\lambda_i^\vee \mathbf{x}^\vee = \pi_i \mathbf{x}^\vee$ is an isomorphism. This says that the pair $(A, \mathbf{x}^\vee)$ is in $U_i(A)$. On the other hand, $\psi(A, \mathbf{x}) = (A, \mathbf{y})$ with $\mathbf{y} : A \to A^{n+1}$ satisfying that $\pi_j \mathbf{y} = \mathbf{x}\lambda_j$. To prove the commutativity of the diagram, that is, the compatibility between Ψ and ψ, it is enough to show that both pairs $(A, \mathbf{x}^\vee)$ and $(A, \mathbf{y})$ are the same subobject in A^{n+1}. The result follows from the fact that the dual of the morphism $\mathbf{x}\lambda_j : A \to A$ is itself in $\mathcal{M}od_{\mathcal{C}}(A)$, therefore:

$$\pi_j \mathbf{x}^\vee = \lambda_j^\vee \mathbf{x}^\vee = (\mathbf{x}\lambda_j)^\vee = \mathbf{x}\lambda_j = \pi_j \mathbf{y}$$

for every $j = 0, \ldots n$, then $\mathbf{x}^\vee = \mathbf{y}$.

To finish the proof, consider the commutative diagram in $Sh(Aff_{\mathcal{C}})$

$$\begin{tikzcd} \overline{\mathbb{P}^n_{\mathcal{C}}} \arrow[r, "\Psi"] & \mathbb{P}^n_{\mathcal{C}} \\ \coprod_i \overline{U_i} \arrow[u, two heads] \arrow[r, "\psi", "\cong"'] & \coprod_i U_i \arrow[u, two heads] \end{tikzcd}$$

therefore Ψ is a sheaf epimorphism. □

4.3 Quasi-Coherent Sheaves

For $X \in Sch(\mathcal{C})$, $Zar(X)$ denotes the category of Zariski open immersions of X, i.e., the full sub category of $Sh(Aff_{\mathcal{C}})/X$ whose objects are $u : Y \to X$ with Y a $\mathcal{C}$-scheme and u a Zariski open immersion. A morphism between two objects $u_1 : Y_1 \to X$, $u_2 : Y_2 \to X$ in $Zar(X)$ is a Zariski open immersion $f : Y_1 \to Y_2$ such that $u_1 = u_2 \circ f$.

$Zar(X)$ has a topology induced by the topology in $Sh(Aff_{\mathcal{C}})$, then a family of morphisms $(Y_i \to Y)_i$ is a covering family in $Zar(X)$ if the morphism $\coprod_i Y_i \to Y$ is a sheaf epimorphism. The full subcategory of $Zar(X)$ consisting of $Y \to X$ with Y affine is denoted by $ZarAff(X)$. Again by restriction of the topology in $Sh(Aff_{\mathcal{C}})$, $ZarAff(X)$ is also a site.

Given an object $Spec(A) = Y \to X$, we will denote it simply by Y or $Spec(A)$. Let us consider the functor

$$\mathcal{O}_X : ZarAff(X)^{op} \to Comm(\mathcal{C}), \quad Spec(A) \mapsto \mathcal{O}_X(SpecA) = A.$$

Given two affine open $U = Spec(A)$, $V = Spec(B)$ in $ZarAff(X)$ and a morphism $\varphi : V \to U$, the induced morphism is denoted $\varphi^{\#} : A \to B$, it is a flat epimorphism of finite presentation in $Comm(\mathcal{C})$. $\mathcal{O}_X$ is in fact a sheaf, see [26, Corollaire 2.11. 1]. The sheaf $\mathcal{O}_X$ will be called the structure sheaf.

Definition 4.3.1 ($\mathcal{O}_X$-modules)**.** A sheaf $\mathcal{F} : ZarAff(X)^{op} \to \mathcal{C}$ is said to be a

$\mathcal{O}_X$-module if for every affine open U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and for every morphism $\varphi : V \to U$ in $ZarAff(X)$ an $\mathcal{O}_X(U)$-morphism $\mathcal{F}(U) \xrightarrow{\varphi^{\#}} \mathcal{F}(V)$.

Definition 4.3.2 (Quasi-coherent $\mathcal{O}_X$-module.)**.** An $\mathcal{O}_X$-module is said to be quasi-coherent if for every morphism $V \to U$ in $ZarAff(X)$, the morphism

$$\mathcal{O}_X(V) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{F}(U) \to \mathcal{F}(V)$$

is an isomorphism in $\mathcal{M}od_{\mathcal{C}}(\mathcal{O}_X(V))$.

The categories of $\mathcal{O}_X$-modules and quasi-coherent modules are denoted by $\mathcal{M}od(\mathcal{O}_X)$ and $Qcoh(X)$ respectively.

Gluing Quasi-Coherent Sheaves

Let $X \in Sch(\mathcal{C})$ and let $\mathcal{U} = (U_i \to X)_{i \in I}$ be a finite covering. We denote by$Desc(\mathcal{U}/X, Qcoh())$ to the category whose objects are pairs $(\mathscr{F}_i, \phi_{ij})_{ij \in I \times I}$ with $\mathscr{F}_i \in Qcoh(U_i)$ and a family of isomorphims $\phi_{ij} : \mathscr{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{F}_j|_{U_{ij}}$ in $Qcoh(U_{ij})$ such that the cocycle condition $\phi_{jk}\phi_{ij} = \phi_{ik}$ holds in $Qcoh(U_{ijk})$.
The following result says that every quasi-coherent sheaf is obtained by gluing quasi-coherent sheaves defined in the affine open immersion of the covering.

Proposition 4.3.3. *There is an equivalence of categories* $Qcoh(X) \cong Desc(\mathcal{U}/X, Qcoh(-))$.

Proof. Given $\mathscr{F} \in Qcoh(X)$ and $(U_i \to X)$ a finite covering, we define $\mathscr{F}_i := \mathscr{F}|_{U_i}$ and ϕ_{ij} to be the identity, this defines an object in $Desc(\mathcal{U}/X, Qcoh(-))$.
On the other hand, given $(\mathscr{F}_i, \phi_{ij})_{ij} \in Desc(\mathcal{U}/X, Qcoh(-))$, we define for any object $V \hookrightarrow X \in ZarAff(X)$, $\mathscr{F}(V)$ as the limit of the diagram

$$\prod_k \mathscr{F}_k(V \times_X U_k) \overset{s}{\underset{t}{\rightrightarrows}} \prod_{k,l} \mathscr{F}_{kl}(V \times_X U_{kl}).$$

We have to check that this defines an object in $Qcoh(X)$ and both assignments give rise to an equivalence.

The immersion $U_k \to U$ induces a morphism $A = \mathscr{O}_X(U) \to \mathscr{O}_X|_{U_k}(V_k) = A_k$ in $Comm(\mathcal{C})$, then each $\mathscr{F}_k(V \cap U_k)$ is an A-module, then $\prod_k \mathscr{F}_k(V \cap U_k)$ is also an A-module. Similarly, $\prod_{k,l} \mathscr{F}_{kl}(V \cap U_{kl})$ is in $\mathcal{M}od_{\mathcal{C}}(A)$. Even more, the arrows t, s above, are in fact A-module morphisms.

Let $V \to W \in ZarAff(X)$, then $V \to W$ is a Zariski open immersion, i.e., $\mathscr{O}_X(W) \to \mathscr{O}_X(V)$ is a flat epimorphism of finite presentation in $Comm(\mathcal{C})$, so tensoring $\mathscr{F}(W)$ with $\mathscr{O}_X(V)$ as $\mathscr{O}_X(W)$-modules , we obtain a diagram with exact rows

$$\begin{tikzcd}
\mathscr{F}(W) \underset{\mathscr{O}_X(W)}{\otimes} \mathscr{O}_X(V) \arrow[r] & \prod_k \mathscr{F}_k(W_k) \underset{\mathscr{O}_X(W)}{\otimes} \mathscr{O}_X(V) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\cong"] & \prod_{k,l} \mathscr{F}_{kl}(W_{kl}) \underset{\mathscr{O}_X(W)}{\otimes} \mathscr{O}_X(V) \arrow[d, "\cong"] \\
\mathscr{F}(V) \arrow[r] & \prod_k \mathscr{F}_k(V_k) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \prod_{k,l} \mathscr{F}_{kl}(V_{kl}),
\end{tikzcd}$$

where $V_k = V \cap U_k$, $W_k = W \cap U_k$. The second and third columns are isomorphisms as we have the following isomorphism for each k:

$$\mathscr{F}_k(W_k) \underset{\mathscr{O}_X(W_k)}{\otimes} \left(\mathscr{O}_X(W_k) \underset{\mathscr{O}_X(W)}{\otimes} \mathscr{O}_X(V) \right) \cong \mathscr{F}_k(W_k) \underset{\mathscr{O}_X(W_k)}{\otimes} \mathscr{O}_X(V_k) \cong \mathscr{F}_k(V_k),$$

therefore we have an isomorphism in the first column, so $\mathscr{F}$ is quasi-coherent. Finally, it is clear that both constructions give rise to an equivalence. $\square$

4.4 The construction of twisted sheaves or the Operation $\mathscr{F}(m)$

In this section, we construct the twisting sheaves in the projective space $\mathbb{P}^n_{\mathcal{C}}$. The motivation of this construction is to relate the global sections of the twisted sheaf $\mathscr{O}_{\mathbb{P}^n_{\mathcal{C}}}(m)$ with the m-th symmetric power $S^m(\mathbb{1}^{n+1})$, the ultimate goal is to give a characterization of the quasi-coherent sheaves on $\mathbb{P}^n_{\mathcal{C}}$ and graded modules over the graded algebra $S(\mathbb{1}^{n+1})$. So far, this relation isn't yet completed, it is work in progress though.

Let us introduce some notations that will be useful to make the analogy with the construction in the classical projective space.

Let $\mathscr{O}_X$ be the structure sheaf on $X = \mathbb{P}^n_{\mathcal{C}}$. We denote $\mathscr{O}_X|_{U_i}$ the restriction of the sheaf to the affine open U_i. For any integer m we define a family of isomorphisms

$$\theta_{ij}(m) : \mathscr{O}_X|_{U_{ij}}(V) \to \mathscr{O}_X|_{U_{ij}}(V), \qquad a \mapsto (\mathbf{x}_j\mathbf{x}_i^{-1})^m a$$

for every affine open immersion $V = Spec(A) \hookrightarrow U_{ij}$, where $\mathbf{x}_i$ is, for every i, as in Proposition 4.2.8, the composition $L \xrightarrow{\mathbf{x}} A^{n+1} \xrightarrow{\pi_i} A$. Therefore we have the morphism $\mathbf{x}_j\mathbf{x}_i^{-1} : A \to A$ which in turn, by adjunction is considered as an element in A. So the morphism $\theta_{ij}(m)(a)$ is defined as a multiplication by the section $(\mathbf{x}_j\mathbf{x}_i^{-1})^m$.

The inverse is given by the formula $\theta_{ij}(-m)$ and for every point in U_{ijk} we have the cocycle condition $\theta_{jk}(m)\theta_{ij}(m) = \theta_{ik}(m)$.

More general, for every $\mathscr{F} \in Qcoh(X)$, we have for every integer, the isomorphisms $\theta_{ij}(m) : \mathscr{F}|_{U_{ij}} \to \mathscr{F}|_{U_{ij}}$ given by the multiplication by the section $(\mathbf{x}_j\mathbf{x}_i^{-1})^m$ in the $\mathscr{O}_X|_{U_{ij}}(V)$-module $\mathscr{F}|_{U_{ij}}(V)$.

The data $(\mathscr{F}|_{U_i}, \theta_{ij}(m))_{ij}$ defines by gluing, a quasi-coherent sheaf which we denote by $\mathscr{F}(m)$. By construction we have canonical isomorphisms

$$\mathscr{F} \cong \mathscr{F}(0), \qquad \mathscr{F}(m)(l) \cong \mathscr{F}(m+l).$$

Proposition 4.4.1. *For every quasi coherent sheaf $\mathscr{F}$ we have a canonical isomorphism*

$$\mathscr{F}(m) \cong \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(m)$$

Proof. For every integer m, the sheaf $\mathscr{O}_X(m)$ is obtained by gluing the isomorphisms $\theta_{ij}(m)$ and $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(m)$ by gluing with $1 \otimes \theta_{ij}(m)$. The isomorphism comes from the identification of $\mathscr{F}|_{U_i} \otimes_{\mathscr{O}_X} \mathscr{O}_X|_{U_i}$ with $\mathscr{F}|_{U_i}$. $\square$

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