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Soluciones Especiales de Sistemas A-hipergeométricos

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La mentira tiene patas cortas..., pero a veces usa zancos.

Unas palabras...

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Abstract

The A-hypergeometric systems of differential equations introduced by Gelfand, Kapranov and Zelevinsky are a generalization of a broad class of differential equations in the complex domain, incorporating analytical, algebro-geometrical and combinatorial tools. In this work, we study two different types of special (holomorphic multivalued) A-hypergeometric functions, that is, two types of special solutions of A-hypergeometric systems. On one hand, we introduce a proper notion of Nilsson solutions for the space of formal solutions of irregular A-hypergeometric systems, we explore the dimension of this space and convergence issues. The second problem addressed in the thesis is the characterization of algebraic A-hypergeometric functions admitting a Laurent series expansion, for regular configurations that are Cayley configurations of two planar configurations, in terms of appropriate multidimensional residues.

Keywords: A-hypergeometric, irregular D-module, Nilsson series, multidimensional residue, algebraic function.

Resumen

Los sistemas de ecuaciones diferenciales A-hipergeométricos introducidos por Gelfand, Kapranov y Zelevinsky constituyen una generalización de una amplia clase de ecuaciones diferenciales en el campo complejo, incorporando herramientas analíticas, algebro-geométricas y combinatorias. En este trabajo se estudian dos tipos distintos de funciones (holomorfas multivaluadas) A-hipergeométricas especiales, es decir dos tipos de soluciones especiales de sistemas A-hipergeométricos. Por un lado, se introduce una noción apropiada de soluciones de Nilsson para el espacio de soluciones formales de sistemas A-hipergeométricos irregulares y se estudia la dimensión de este espacio así como la convergencia. El segundo problema abordado en la tesis ha sido la caracterización de funciones A-hipergeométricas algebraicas que admitan un desarrollo como series de Laurent, para configuraciones regulares A, que sean configuraciones de Cayley de dos configuraciones planas, en términos de apropiados residuos multidimensionales.

Palabras clave: A-hipergeométrico, D-módulo, series de Nilsson, residuo multidimensional, función algebraica.

Introduction

The solutions of the *Gauss hypergeometric equation* are described by means of the *Gauss hypergeometric series*. Their formal study begun, precisely, with Gauss and it is still an active area of research, involving many areas of mathematics such as complex analysis, number theory, combinatorics, mathematical physics, etc.

In the early 90's, Gel'fand, Kapranov and Zelevinsky studied several differential equations of hypergeometric type (Gauss, Horn, Lauricella, etc.) and introduced a common framework for all of them (see [GKZ89], [GKZ88],[GKZ90]), namely, the *A-hypergeometric systems* (or *GKZ-hypergeometric systems*), whose solutions are called *A-hypergeometric functions*. Their work involves D-modules, toric varieties, combinatorics and other tools. The information of the system is codified in a integer matrix A (that can also be thought as a configuration of integer points), and a complex vector β. The A-hypergeometric system with parameter β is denoted by $H_A(\beta)$.

In this thesis we study two special kinds of solutions of A-hypergeometric systems: Nilsson solutions of irregular A-hypergeometric systems, in chapters 3 and 4, and algebraic Laurent solutions of Cayley configurations, in chapters 5 and 6.

In 2000, Saito, Sturmfels and Takayama gave a *Gröbner Basis* reformulation of the GKZ theory. Our way to deal with A-hypergeometric systems is based on their work. We overview some of their ideas and results in chapter 1. In chapter 2 we explain two combinatorial tools that we strongly use in the rest of the work: Gale dual and coherent mixed subdivisions. The other fundamental tool from combinatorics is that of coherent triangulations that is treated in Section 1.4.2.

In the first part of our work, we study solutions of irregular A-hypergeometric systems. For an integer matrix A and a complex parameter β, the system $H_A(\beta)$ is a holonomic D-ideal [Ado94, GKZ89]. It is also known that $H_A(\beta)$ is regular holonomic if and only if the $\mathbb{Q}$-rowspan of the matrix A contains the vector $(1, \dots, 1)$. The if direction was proved by Hotta in his work on equivariant D-modules [Hot91]; Saito, Sturmfels and Takayama gave a partial converse in [SST00, Theorem 2.4.11], assuming that the parameter β is generic.

The Frobenius method is a symbolic procedure for solving a linear ordinary differential equation in a neighborhood of a regular singular point. The solutions are represented as convergent logarithmic Puiseux series that belong to the Nilsson class. In the multivariate case, the Saito, Sturmfels and Takayama method, called the *canonical series algorithm*, applied to a regular holonomic left D-ideal, yields a basis of the solution space [SST00, Chapter 2]. The basis elements belong to an explicitly described Nilsson ring, and are therefore called *Nilsson series*, or *Nilsson solutions*. Each Nilsson ring is constructed using a weight vector; the choice of weight vector is a way of determining the common domain of convergence of the corresponding solutions. The canonical series procedure requires a regular holonomic input; although one can run this algorithm on holonomic left D-ideals that have irregular singularities, there is no guarantee that the output series converge, or even that the correct number of basis elements will be produced.

In Section 3.1, we extend the notion of Nilsson solution to general A-hypergeometric systems. For arbitrary A and β, we denote by $\mathscr{N}_w(H_A(\beta))$ the space of Nilsson solutions of the system $H_A(\beta)$. In order to obtain the elements of $\mathscr{N}_w(H_A(\beta))$, we introduce, in section 3.2, an application ρ of *homogenization* that goes from $\mathscr{N}_w(H_A(\beta))$ to an associated regular system.

For generic parameters, we calculate in Section 3.3 the dimension of $\mathscr{N}_w(H_A(\beta))$ in combinatorial terms, and construct an explicit basis. Ohara and Takayama [OT09] showed that the method of canonical series for a weight vector which is a perturbation of $(1,\dots,1)$ produces a basis for the solution space of $H_A(\beta)$ consisting of (convergent) Nilsson series that contain no logarithms. We extend their results and give in section 4.2 a criteria to decide which elements of the constructed basis of $\mathscr{N}_w(H_A(\beta))$ converge, for arbitrary weight vectors w.

In order to produce a basis of solutions for $H_A(\beta)$ when β is not generic, logarithmic series cannot be avoided, even in the regular case. We study them in section 3.4. Dealing with logarithmic solutions of $H_A(\beta)$ poses technical challenges that we resolve here, allowing us to lift the genericity hypotheses from the results of Ohara and Takayama: running the canonical series algorithm on $H_A(\beta)$ with weight vector (a perturbation of) $(1,\dots,1)$ always produces a basis of (convergent) Nilsson solutions of $H_A(\beta)$, if the cone spanned by the columns of A is *strongly convex* (i.e., the cone contains no lines). This is done in section 4.1. On the other hand, formal solutions of irregular hypergeometric systems that are not Nilsson series need to be considered, even in one variable (see, for instance, [Cop34].)

Finally, in section 3.5, we extend the proof of Saito, Sturmfels and Takayama of the converse of Hotta's regularity theorem mentioned above, assuming that the cone over the columns of A is strongly convex. This gives an alternative proof to a result given by Schulze and Walther.

In fact, a different strategy to show that a D-ideal is not regular holonomic is to prove that it has *slopes*. The analytic slopes of a D-module were introduced in the work of Mebkhout [Meb89], while an algebraic version was given by Laurent [Lau87]. These authors have shown that the analytic and algebraic slopes of a D-module along a hypersurface agree [LM99]. From a computational perspective, Assi, Castro–Jiménez and Granger gave a Gröbner basis algorithm to find algebraic slopes [ACG96]. There has been an effort to compute the (algebraic) slopes of $H_A(\beta)$ along a coordinate hypersurface. In the cases $d = 1$ and $n - d = 1$, these slopes were determined by Castro–Jiménez and Takayama [CT03], and Hartillo–Hermoso [Har03, Har05]. More generally, Schulze and Walther [SW08] have calculated the slopes of $H_A(\beta)$ under the *strongly convex* assumption. The fact that slopes of $H_A(\beta)$ always exist when the vector $(1,\dots,1)$ does not belong to the rowspan of A, implies that $H_A(\beta)$ has irregular singularities. Thus, [SW08, Corollary 3.16] gives a converse for Hotta's regularity theorem in the strongly convex case. Our proof is done by extending ideas of Saito, Sturmfels and Takayama, where the main technical obstacle to overcome is the potential existence of logarithmic hypergeometric series.

Further insight into the solutions of hypergeometric system comes from the analytic approach taken up by Castro–Jiménez and Fernández–Fernández [CF11, Fer10], who studied the Gevrey filtration on the irregularity complex of an A-hypergeometric system. Since formal series solutions of irregular systems need not converge, a study of the Gevrey filtration provides information on how far such series are from convergence.

In the second part of our work, we study in chapters 5 and 6, Laurent solutions of a regular A-hypergeometric system where the configuration A is a Cayley configuration of k configuration in dimension k. A configuration A is Cayley if given $A_1, \ldots, A_s$ configurations in $\mathbb{Z}^r$

$$A = \{e_1\} \times A_1 \cup \cdots \cup \{e_s\} \times A_s.$$

If we consider polynomials with these supports

$$f_i(t) = \sum_{j=1}^{|A_i|} x_j t^{a_j}, \quad i = 1, \ldots, r,$$

generically in the coefficients x_j, the common zeros of f_i $i = 1, \ldots, r$ are finite and simple. Then the local Grothendieck residue

$$Res_{f_1,\ldots,f_r,\xi}(t^m) = \frac{\xi^m}{J_f^T(\xi)}$$

is well-defined. If V is the set of common zeros of the polynomials f_i, then the global residue

$$Res_f^m = \sum_{\xi \in V} Res_{f_1,\ldots,f_r,\xi}(t^m)$$

is an algebraic A-hypergeometric function with homogeneity $(-1, \ldots, -1, -m)$.

The study of univariate algebraic hypergeometric functions is a classical subject. Beukers and Heckman [BH89] gave an explicit classification of all algebraic univariate hypergeometric series (see also [Rod]). There exists a small number of general results on algebraicity of A-hypergeometric functions in the multivariate case (that is, for configurations of codimension greater than one). A recent work of Beukers ([Beu10]), characterizes those configurations A for which exists generic parameters such that *all* solutions are algebraic. Our study is based on the determination and explicitation of algebraic A-hypergeometric functions for certain *resonant* values of the parameters (for which there also exists logarithmic solutions and the monodromy is not finite) in terms of multidimensional residues.

The existence of non-trivial rational A-hypergeometric functions imposes severe combinatorial constraints on the configuration A. In [CDS01] it was conjectured that A needs to carry an essential Cayley structure. This was elucidated in several articles (see, eg., [CDS01], [CDD99],[CDR11]), in connection with the structure of the full A-determinant [GKZ94], which defines the singular locus of $H_A(\beta)$ for any β.

It follows from those papers that in the case of the Cayley configurations of k configurations in dimension k that we consider, there are no non-trivial rational solutions. Our results show that the algebraicity of A-hypergeometric Laurent series is also imposed by purely combinatorial conditions. We next explain more in detail our statements.

As a generalization of [CDD99], we work with the case of two polynomials f_1, f_2 in the variables t_1, t_2. We study the possible minimal regions, that is, subsets of $\{1, \ldots, |A_1| + |A_2|\}$ indexing the variables that appear in the denominators of the Laurent solutions associated to the configuration A and to homogeneity parameters $(-1, -1, -m)$ lying in the Euler-Jacobi cone of A (see Definition 5.15.) We establish in Section 5.2 a precise relation between infinite minimal regions and interior points of the Minkowski sum of the convex hulls of A_1 y A_2.

The common roots of f_1, f_2 can be obtained from mixed cells of the coherent mixed subdivision of the configuration, following work of Huber and Sturmfels [HS95]). To each mixed cell σ in a coherent mixed subdivision of the Minkowski sum of the given configurations, one can associate as many roots of f_1, f_2 as the normalized volume of σ. We define in Section 6.1 the *residue Res_f^σ relative to σ* adding the local residues over the roots corresponding to σ.

Theorem 6.2.2, our main result in Section 6.2, gives an explicit way of writing each residue Res_f^σ as a linear combination of the canonical solutions associated to minimal regions corresponding to interior points of the Minkowski sum of the convex hulls of A_1 y A_2 that are vertices of σ. The coefficients of this linear combination are combinatorially defined. In Theorem 6.3.1 of Section 6.3, we give a complete description of algebraic Laurent A-hypergeometric functions in terms of residues, in case $|A_1| + |A_2| = 6$ and no A_i has an interior point. In the last section, we highlight the complications that arise in more general situations and we state general conjectures which would extend our results.

Contents

Chapter 1

Regular A-hypergeometric systems

A-hypergeometric systems are a generalization of classical hypergeometric equations and lead us to consider partial differential equations in $\mathbb{C}^n$ instead of ordinary differential equations in $\mathbb{C}$. The theory of D-modules gives the appropriated tools to deal with them, because the coefficients of these partial differential equations are polynomials in $\mathbb{C}^n$. Saito, Sturmfels and Takayama presented in [SST00] an algorithm to solve regular systems using Gröbner bases in the Weyl algebra. Moreover, they showed that in the case of an hypergeometric D-module, due to its combinatorial nature, their methods are more accurate. Our objective in Chapter 3 is to solve irregular hypergeometric systems, but inspired on the techniques in [SST00], so we introduce in this chapter the basics on D-modules, the method of Saito, Sturmfels and Takayama and the combinatorial aspects of the A-hypergeometric systems.

1.1 From the classical equation to A-hypergeometric systems

In this informal section, we present an overview of the "evolution" of hypergeometric functions. After introducing the *Gauss hypergeometric series* we present the A-hypergeometric series related to it and the A-hypergeometric system associated as well as some examples.

The *Gauss hypergeometric equation*

$$[x(x-1)\frac{d^2}{dx^2} + (c - x(a+b+1))\frac{d}{dx} - ab] \bullet f = 0 \tag{1.1}$$

where a, b, c are complex parameters has been widely studied since Gauss. Note that the singular points of this equations are $x = 0$, $x = 1$ and $x = \infty$. Its importance in mathematics as well as in physics relies in the fact that any equation with three regular singular points can be written in this form. We will explain in Section 1.3 the notion of regularity in one and more variables with more detail, but for the purpose of this introductory section we will describe the manner to obtain the solutions of the equation (1.1) around its singular points.

1.1.1 Solving equation 1.1 via the Frobenius method

We will operate in a pure algebraic way to obtain formal solutions around the singular point $x = 0$. The general procedure to obtain the solutions in this case will be discussed when we deal with regularity in Chapter 1.

Suppose we are looking for a (multivalued) solution of the shape:

$$f = x^s \cdot \sum_{k=0}^{\infty} c_k x^k \tag{1.2}$$

that is a power series around $x = 0$ times x^s with $s \in \mathbb{C}$ (choose an appropriated branch of the logarithm).

Regard the differential operator in (1.1) as and element P in the Weyl algebra $D = \mathbf{k}\langle x, \partial\rangle$. Thus P can be written as

$$P = [x(x-1)\partial^2 - (a+b+1)x\partial + c\partial - ab]. \tag{1.3}$$

Applying it to both sides of (1.2) we have

$$xP \bullet f = \sum_{k=0}^{\infty} c_k(s+k)(s+k+c-1)x^{s+k} - \sum_{k=0}^{\infty} c_k(s+k+a)(s+k+b)x^{s+k+1} = 0$$

and we obtain the following recurrence relations for the coefficients c_k:

$$\begin{aligned} s(s+c-1) &= 0, \\ (s+k+1)(s+k+c)c_{k+1} - (s+k+a)(s+k+b)c_k &= 0, \qquad k = 0, 1, 2, \ldots \end{aligned}$$

Assume $c_0 = 1$. Once that the exponent s is chosen by means of the first equation, we can obtain the coefficients c_k through the second one. If $s = 0$ then

$$c_k = \frac{(a)_k (b_k)}{(1)_k (c)_k}$$

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

is the *Pochhammer symbol*. If, on the other hand, $s = 1 - c$ then

$$c_k = \frac{(a+1-c)_k (b+1-c)_k}{(1)_k (2-c)_k}.$$

Note that in both cases we need $c \notin \mathbb{Z}$ to obtain c_k for all $k = 0, 1, 2, \ldots$. We have obtained two formal solutions to (1.1) around $x = 0$:

$$F(a, b, c; x) := \sum_{k=0}^{\infty} \frac{(a)_k (b_k)}{(1)_k (c)_k} x^k \tag{1.4}$$

and

$$x^{1-c}.F(a+1-c, b+a-c, 2-c; x). \tag{1.5}$$

The function $F(a, b, c; x)$ is also denoted by ${}_2F_1(a, b, c; x)$ and it is called *Gauss Hypergeometric Function* with parameters (a, b, c).

1.1.2 Introducing homogeneities in the classical hypergeometric equation

The following proposition, which is straightforward (see [SST00, Proposition 1.3.7]), summarizes the prominence of the change of the point of view about hypergeometric functions introduced by Gel'fand, Kapranov and Zelevinsky.

Proposition 1.1.1. *The function*

$$x_1^{c-1}x_2^{-a}x_3^{-b}F\left(a,b,c;\frac{x_1x_4}{x_2x_3}\right)$$

is annihilated by the following operators in the Weyl algebra $\mathbb{C}\langle x_1,x_2,x_3,x_4,\partial_1,\partial_2,\partial_3,\partial_4\rangle$*:*

$$\partial_2\partial_3-\partial_1\partial_4,\ x_1\partial_1-x_4\partial_4+1-c,\ x_2\partial_2+x_4\partial_4+a,\ x_3\partial_3+x_4\partial_4+b. \tag{1.6}$$

This proposition states if one adds extra homogeneities to the Gauss hypergeometric series, one gets that the new function satisfies a PDE system which consists of three equations expressing (an infinitesimal version of) homogeneities, and a equation of superior order.

The notable fact about this is its intimate relation with toric varieties in algebraic geometry. Indeed, the equations (1.6) can be introduced in the following way:

Let

$$A=\begin{pmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&1\end{pmatrix}\in\mathbb{Z}^{3\times4}\qquad\text{and}\qquad\beta=\begin{pmatrix}c-1\\-a\\-b\end{pmatrix}\in\mathbb{C}^3.$$

Consider the following D-ideal

$$H_A(\beta)=I_A+\langle x_1\partial_1-x_4\partial_4+1-c,x_2\partial_1-x_4\partial_4+a,x_3\partial_3-x_4\partial_4+b\rangle \tag{1.7}$$

where $I_A=\langle\partial_2\partial_3-\partial_1\partial_4\rangle$ is the toric ideal associated to the matrix A. Note that the homogeneity equations correspond with the rows of A and β and that the series

$$x_1^{c-1}x_2^{-a}x_3^{-b}\sum_{k=0}^{\infty}\frac{(a)_k(b_k)}{(1)_k(c)_k}\left(\frac{x_1x_4}{x_2x_3}\right)^k, \tag{1.8}$$

which is a solution to this system, can also we written in terms of the "exponent" $v=(c-1,-a,-b,0)$ and a weight vector $w=(1,0,0,1)$. Consider the set

$$C_w=\{u\in\ker_{\mathbb{Z}}(A)/u\cdot w\geq 0\}$$

and now write the series in equation (1.8) in this way

$$\sum_{u\in C_w}c_ux^{v+u},$$

where

$$c_u=c_{(k,-k,-k,k)}=\frac{(a)_k(b_k)}{(1)_k(c)_k}.$$

As we will see, the combinatorics of the configuration A, the parameter β and a weight vector $w\in\mathbb{R}^n$ determine the features of the equations of hypergeometric type, including the number of solutions and its domain of convergence, regularity of the system, and so on.

1.1.3 The GKZ hypergeometric systems

We introduce the Weyl algebra. As usual, ∂_i stands for the partial derivative with respect to x_i.

Definition 1.1.2. *The Weyl algebra of dimension n is the free associative $\mathbb{C}$-algebra*

$$D_n = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$$

modulo the commutation rules

$$x_i x_j = x_j x_i,\ \partial_i \partial_j = \partial_j \partial_i,\ \partial_i x_j = \partial_i x_j \textit{ for } i \neq j, \textit{ and } \partial_i x_i = x_i \partial_i + 1.$$

When no confusion arises, we simply write D for D_n.

The following definition is due to Gelfand, Kapranov and Zelevinsky.

Definition 1.1.3. *Let $A = [a_{ij}] \in \mathbb{Z}^{d\times n}$ whose rows $\mathbb{Z}$-span $\mathbb{Z}^d$, and let $\beta \in \mathbb{C}^d$. The **A-hypergeometric (or GKZ-hypergeometric) system with parameter** β is the left D-ideal*

$$H_A(\beta) = I_A + \langle E_1 - \beta_1, \dots, E_d - \beta_d \rangle \subset D,$$

*where $E_i = \sum_{j=1}^n a_{ij} x_j \partial_j$, $1 \leq i \leq d$, and I_A denotes the **toric ideal***

$$I_A = \langle \partial^u - \partial^v \mid A \cdot u = A \cdot v \rangle \subseteq \mathbb{C}[\partial].$$

The second "easiest" example after the Gauss system is the following.

Example 1.1.4. *The matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.9}$$

and a parameter $\beta \in \mathbb{C}^2$ define the A-hypergeometric system

$$H_A(\beta) = D \cdot \left\{ \partial_0\partial_2 - \partial_1^2, x_0\partial_0 + x_1\partial_1 + x_2\partial_2 - \beta_1, x_1\partial_1 + 2x_2\partial_2 - \beta_2 \right\}.$$

in the Weyl algebra $D = \mathbb{C}\langle x_0, x_1, x_2, \partial_0, \partial_1, \partial_2 \rangle$. If $\beta = (0, -1)$ we have the solutions

$$\frac{-x_1 \pm (x_1^2 - 4x_0x_2)^{1/2}}{2x_2}. \tag{1.10}$$

These are the two roots of a quadratic polynomial $f(t) = x_2t^2 + x_1t + x_0$ in terms of the coefficients x_0, x_1, x_2. Moreover, the roots of a polynomial of any degree are A-hypergeometric functions of the coefficients of that polynomial for suitable A (see [May37], [Stu00].)

1.2 Holonomic systems

A system of linear differential equations with polynomial coefficients can be identified with a left ideal in D (or a left D-ideal), considering the natural action of the Weyl algebra D as follows:

$$\partial_i \bullet f = \frac{\partial f}{\partial x_i}, \quad x_i \bullet f = x_i f, \tag{1.11}$$

where f may belong to many different D-modules, such as *formal power series* $\mathbb{C}[[x_1, \dots, x_n]]$, *holomorphic functions* $\mathcal{O}(U)$ on an open subset U of $\mathbb{C}^n$, etc. Thus, we say that the element f is a solution of the (left) ideal I of D if $p \bullet f = 0$ for all $p \in I$.

Any element p of D has a unique expression of the form

$$p = \sum_{u,v \in \mathbb{N}^n} c_{uv} x^u \partial^v$$

where $c_{uv} = 0$ for all but finitely many pairs (u, v). Associated with D we consider the commutative polynomial ring $R = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$. Given any non-zero $p \in D$ let $\nu(p) := \max\{|v| : c_{uv} \neq 0 \text{ for some } u \in \mathbb{N}^n\}$ be the *order* of p and set

$$\sigma(p) := \sum_{u \in \mathbb{N}^n, |v| = \nu(p)} c_{uv} x^u \xi^v \in R.$$

The polynomial $\sigma(p)$ is called the (principal) symbol of the differential operator p.

Definition 1.2.1. *Given a left ideal $I \subset D$, its* characteristic variety *is the affine variety in $\mathbb{C}^{2n}$ defined by the* characteristic ideal *ch*$(I) := \langle \sigma(p) : p \in I \rangle \subset R$.

Definition 1.2.2. *A left ideal $I \subset D$ is called* holonomic *if and only if its characteristic ideal has (Krull) dimension n. The* holonomic rank *of I is the dimension of the following vector space over the field of rational functions $\mathbb{C}(x)$:*

$$\operatorname{rank}(I) := \dim_{\mathbb{C}(x)} \left(\frac{\mathbb{C}(x)[\xi]}{(\mathbb{C}(x)[\xi] \cdot ch(I))} \right).$$

Holonomic systems have the following nice property ([SST00, Proposition 1.4.9]).

Proposition 1.2.3. *If I is a holonomic D-ideal, then* $\operatorname{rank}(I)$ *is finite.*

Definition 1.2.4. *Let $I \subset D$ be a left ideal and $\mathbb{V}(ch(I)) \subset \mathbb{C}^{2n}$ its characteristic variety. The singular locus Sing(I) is defined as the Zariski closure of the projection on $\mathbb{C}^n_x$ of*

$$\mathbb{V}(ch(I)) - \{\xi_1 = \dots = \xi_n = 0\}.$$

The following theorem ([SST00, Theorem 1.4.19]) is a result of Kashiwara that relates the holonomic rank of a D-ideal I to its solution space.

Theorem 1.2.5. *Let I be a holonomic ideal and U a simply connected domain in $\mathbb{C}^n - Sing(I)$. Consider the system of differential equations $I \bullet f = 0$. Then the dimension of the complex vector space of holomorphic solutions is equal to* $\operatorname{rank}(I)$.

Example 1.2.6. *The prototypical example of a holonomic D-ideal is the principal D-ideal defined by a linear ordinary (this is $n = 1$) differential equation of order m:*

$$I = D \cdot \{a_m(x)\partial^m + a_{m-1}(x)\partial^{m-1} + \dots + a_0(x)\},$$

where the a_i's are polynomials in x and $a_m \neq 0$. Here $ch(I) = \langle a_m(x)\xi^m \rangle$, hence I is holonomic with $\operatorname{rank}(I) = m$. *The singular locus is the zero set $\{x \in \mathbb{C} : a_m(x) = 0\}$ of the polynomial a_m. Theorem 1.2.5 reduces in this case to the classical theorem of existence of solutions around non-singular points.*

1.3 Canonical solutions

In [SST00], Saito, Sturmfels and Takayama presented a method to obtain series solutions for regular holonomic D-ideals. In this section we briefly describe this method.

The key idea is to obtain the "first part" of the solutions solving the "first part" of the equations and then produce an actual solution, in an analogous way to the Frobenius method in one variable. For this, we define the initial of a D-ideal.

Definition 1.3.1. *For an element $p = \sum_{u,v} c_{uv}x^u\partial^v$ in the Weyl algebra D, and a vector $w \in \mathbb{R}^n$, we define $\mathrm{in}_{(-w,w)}(p)$ to be the sum of the terms of p in which the inner product $(u,v)\cdot(-w,w)$ achieves its maximum. If I is a left D-ideal, we define*

$$\mathrm{in}_{(-w,w)}(I) = \langle \mathrm{in}_{(-w,w)}(p) \mid p \in I\rangle \subset D.$$

For a holonomic ideal, we have the following result which is Theorem 2.2.1 in [SST00].

Theorem 1.3.2. *Let I be a holonomic D-ideal and $w \in \mathbb{R}^n$. The initial D-ideal $\mathrm{in}_{(-w,w)}(I)$ is also holonomic and*

$$\mathrm{rank}(\mathrm{in}_{(-w,w)}(I)) \leq \mathrm{rank}(I). \tag{1.12}$$

Remark 1.3.3. *The vector $w \in \mathbb{R}^n$ is called* weight vector *in [SST00]. We will ask stronger conditions to w in Chapters 3 and 4 when the ideal is not regular, so we will reserve the term* **weight vector** *for that case. See Definition 3.1.1.*

The following step is to look at the shape of the solutions so that the expression "first part" makes sense. This will be possible if we assume that the system is *regular*.

We first introduce the notion of regularity for an ordinary differential equation. Consider the equation

$$a_m(x)\frac{\partial^m}{\partial x^m}f(x) + a_{m-1}(x)\frac{\partial^{m-1}}{\partial x^{m-1}}f(x) + \cdots + a_1(x)\frac{\partial}{\partial}xf(x) + a_0(x) = 0 \tag{1.13}$$

where the functions $a_i(x)$ are holomorphic in an open set $\mathcal{U} \subset \mathbb{C}$ and let $x_0 \in \mathcal{U}$ such that $a_m(x_0) = 0$. We say that x_0 is a *regular singular point* of the equation (1.13) if the functions

$$b_i(x) := \frac{a_i(x)}{a_m(x)}$$

have at worst a pole of order $m - i$ at x_0.

The following theorem is classical and its proof can be found in [CL55].

Theorem 1.3.4. *Let x_0 be a regular singular point of* (1.13)*. Then the following statements hold:*

1. *The vector space of multivalued holomorphic functions in a sufficiently small punctured disk $\{0 < |x-x_0| < \varepsilon\}$, which are solutions of* (1.13)*, has dimension m and is generated by functions of the form*

   $$(x - x_0)^\lambda(\ln(x - x_0)^j)f(x),$$

   where $\lambda \in \mathbb{C}$, $j \in \mathbb{Z}$, $0 \leq j \leq m - 1$, $f(x)$ is holomorphic in the disk $\{|x - x_0| < \varepsilon\}$ and $f(x_0) \neq 0$.

2. *If the series*

$$g(x) = (x - x_0)^\lambda \sum_{j=0}^{k} \left(\sum_{l=0}^{\infty} c_l x^l \right) \ln(x - x_0)^j,$$

where $\lambda, c_l \in \mathbb{C}$ *and* $k \in \mathbb{N}$, *formally satisfies the equation* (1.13) *then there exists a punctured disk* $\{0 < |x - x_0| < \varepsilon\}$ *such that* $g(x)$ *defines a multivalued holomorphic function in there.*

We now present a definition of regularity of an arbitrary D-module as given in [SST00]. The definition involves complicated tools which we not discuss in this thesis, and we include it for the sake of completeness, being the properties of regular D-modules what is important for us. Denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators on $X = \mathbb{C}^n$.

Definition 1.3.5. *Let* I *be a holonomic ideal in the Weyl algebra* D. *Let* C *be a smooth curve in* $X = \mathbb{C}^n$ *and* $j : C \to \mathbb{C}^n$ *an embedding. A holonomic* $\mathcal{D}_X$*-module* $\mathcal{D}_X/\mathcal{D}_X I$ *is called* regular holonomic *when* $L^k j^*(\mathcal{D}_X/\mathcal{D}_X I)$ *is regular holonomic on a smooth compactification* $\overline{C}$ *for any such curve* C *and for all* $k = 0, -1, \ldots, -n+1$. *When* $\mathcal{D}_X/\mathcal{D}_X I$ *is regular holonomic, we call* I *regular holonomic. Here* $L^k j^*$ *is the* k*-th derived functor of* j^*.

The following result is known and appears in [SST00] as Theorem 2.4.12 and Corollary 2.4.14.

Theorem 1.3.6. *Let* I *be a regular holonomic* D*-ideal. Assume that the singular locus of* I *is contained in an algebraic hypersurface that is a normal crossing divisor locally at the origin. Then there exist vectors* $\alpha_1, \ldots, \alpha_m$ *in* $\mathbb{C}^n$ *such that any multivalued holomorphic solution of* I *on* $\{\prod_{i=1}^n x_i \neq 0\}$ *near the origin has a series expression in the ring*

$$\mathbb{C}[[x_1, \ldots, x_n]][x^{\alpha_1}, \ldots, x^{\alpha_m}, \log(x_1), \ldots, \log(x_n)].$$

These series converge around the origin, and they are polynomials in $\log(x_i)$ *of degree at most* $\operatorname{rank}(I) - 1$.

The hypotheses of Theorem 1.3.6 mean that if $s(x)$ is the polynomial defining $\operatorname{Sing}(I)$, then it can be written as

$$s(x) = x^a \cdot \left(1 + \sum_{u \in B} c_u x^u\right) \tag{1.14}$$

where $c_u \in \mathbb{C}^*$, $B \subset \mathbb{N}^n$ and $a \in \mathbb{Z}^n$. In a general case, we can always find a change of variables that leads to a expression like (1.14) and it turns out that the shape of the solutions provided by Theorem 1.3.6 is governed by the convex geometry of the singular locus.

In fact, assume that I is a regular holonomic D-ideal and suppose that

$$s(x) = \sum_{k=1}^{l} c_{m_k} x^{m_k}$$

with $m_k \in \mathbb{N}^n$, $k = 1, \ldots, l$ is the defining polynomial of a hypersurface that contains $\operatorname{Sing}(I)$. Applying a multiplicative change of coordinates:

$$x_j = y_1^{v_{1j}} y_2^{v_{2j}} \ldots y_n^{v_{nj}} \text{ for } j = 1, \ldots, n \tag{1.15}$$

we obtain

$$s(x_1, \dots, x_n) = y^{V \cdot m_1} \left(1 + \sum_{k=2}^{l} c_{m_k} y^{V \cdot (m_k - m_1)} \right) \tag{1.16}$$

where V is the matrix whose rows are the vectors $(v_{i1}, \dots, v_{in})$. How do we choose V so that the condition (1.14) is satisfied, that is $V \cdot (m_k - m_1) \geq 0$ for all $k = 2, \dots, l$?

Suppose that the monomial x^q occurs in s. Consider the *Newton polytope* $\mathrm{New}(s)$ of s. Then q is a vertex of $\mathrm{New}(s)$. Take a cone C which has its vertex at q and contains $\mathrm{New}(s)$ with generators $\gamma^1, \dots, \gamma^n \in \mathbb{Z}^n$. We can assume that the determinant of $(\gamma^1, \dots, \gamma^n) = \pm 1$. The $n \times n$ matrix $U = (\gamma^1, \dots, \gamma^n)$ is invertible over $\mathbb{Z}$ and let $V = (v_{ij})$ be its inverse matrix. This implies that the row vectors of V span the polar cone C^*, that is, the cone of all vector $w \in \mathbb{R}^n$ such that $\langle w, \gamma^i \rangle \geq 0$ for $i = 1, \dots, n$.

Then $m_1 = q$ and $m_k = q + \sum_{i=1}^{n} \lambda_{ik} \gamma^i$ with λ_{ik} non-negative integers and the conditions required are satisfied, because

$$V \cdot (m_k - m_1) = \sum_{i=1}^{n} \lambda_{ik} V \cdot \gamma^i = (u^1, \dots, u^n)$$

is a non-negative integer vector, for $k = 2, \dots, l$.

Then we have the following result, [SST00, Corollary 2.4.16].

Theorem 1.3.7. *The regular holonomic D-ideal I has a fundamental set of solutions on $0 < |x^{u^i}| \ll 1$ each of which is represented by a series in*

$$N = \mathbb{C}[[x^{u^1}, \dots, x^{u^n}]][x^{\alpha_1}, \dots, x^{\alpha_m}, \log(x_1), \dots, \log(x_n)]. \tag{1.17}$$

where $\alpha_1, \dots, \alpha_m$ are suitable vectors in $\mathbb{C}^n$

The ring N gives the appropriated context to work with initial of series solutions. In fact, a vector $w \in \mathbb{R}^n$ defines a partial term order $\leq$ on N as follows:

$$x^a \log(x)^b \leq x^c \log(x)^d \Leftrightarrow \mathrm{Re}(a \cdot w) \leq \mathrm{Re}(c \cdot w). \tag{1.18}$$

If $g = \sum_{a,b} c_{ab} x^a \log(x)^b$ is a non zero element of N, then the set of real parts $\{\mathrm{Re}(a \cdot w) \mid c_{ab} \neq 0 \text{ for some } b\}$ achieves a (finite) minimum, denoted by $\mu(g)$. Moreover, the subseries of g consisting of terms $c_{ab} x^a \log(x)^b$ such $c_{ab} \neq 0$ and $\mathrm{Re}(a \cdot w) = \mu(g)$ is finite by [SST00, Proposition 2.5.2]. We call this finite initial sum the *initial series of g with respect to w* and we denote it by $\mathrm{in}_w(g)$.

Remark 1.3.8. *The ring N is not closed under differentiation in general. These and other technical reasons are dealt carefully in [SST00, Section 2.5]. Here we just explain the basics to reach to a general understanding of the Saito, Sturmfels and Takayama method.*

By means of the following theorem ([SST00, Theorem 2.5.5]) it is possible to identify what we called the "first part" of a series solution.

Theorem 1.3.9. *If $f \in N$ is a solution to I then $\mathrm{in}_w(f)$ is a solution to $\mathrm{in}_{(-w,w)}(I)$.*

The term order (1.18) can be refined by the lexicographic term order. We denote this refinement by $\prec_w$. Every element g of N has a unique *initial monomial* $\mathrm{in}_{\prec_w}(g)$ with respect to $\prec_w$. The following lemma is Lemma 2.5.6 and Proposition 2.5.7 in [SST00].

Lemma 1.3.10. *Let $g_1, \dots, g_k \in N$.*

1. *If the initial monomials $\mathrm{in}_{\prec_w}(g_1), \dots, \mathrm{in}_{\prec_w}(g_k)$ are distinct, then the initial series defined by $\mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_k)$ are $\mathbb{C}$-linearly independent.*

2. *If the initial series $\mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_k)$ are $\mathbb{C}$-linearly independent, then $g_1, \dots, g_k$ are $\mathbb{C}$-linearly independent.*

3. *If $g_1, \dots, g_k$ are $\mathbb{C}$-linearly independent, there exists a $k \times k$ complex matrix (λ_{ij}) such that the initial series of $\psi_i = \sum_{j=1}^k \lambda_{ij} g_j$ for $i = 1, \dots, k$ are $\mathbb{C}$-linearly independent.*

The following theorem (Theorem 2.5.1 in [SST00]) states a notable fact about regular holonomic systems.

Theorem 1.3.11. *Let I be a regular holonomic D-ideal and $w \in \mathbb{R}^n$. Then*

$$\mathrm{rank}(I) = \mathrm{rank}(\mathrm{in}_{(-w,w)}(I)). \tag{1.19}$$

Remark 1.3.12. *The proof of Theorem 1.3.11 is not difficult. First, note that by Theorem 1.3.2 we just need to prove that $\mathrm{rank}(I) \leq \mathrm{rank}(\mathrm{in}_{(-w,w)}(I))$. The crucial fact is that a fundamental set of solutions to I (with $\mathrm{rank}(I)$ many elements) on an open ball $\mathcal{U} \subset \mathbb{C}^n$ can be represented by series in N, by Theorem 1.3.7. Then apply Theorem 1.3.9 and Lemma 1.3.10 to obtain the desired result. We want to emphasize that this argument cannot be replicated in the non-regular case, because we do not have that nice description of the solutions.*

The shape of the solutions of a regular holonomic ideal can be more accurate than Theorem 1.3.7. We first introduce the important notion of *exponent*.

Definition 1.3.13. *A vector $v \in \mathbb{C}$ is an **exponent** of a D-ideal I with respect to $w \in \mathbb{R}^n$ if x^v is a solution of $\mathrm{in}_{(-w,w)}(I)$.*

Remark 1.3.14. *For a holonomic ideal, the set of exponents is finite because of Proposition 1.2.3 and Theorem 1.3.2.*

The vectors u^i in Theorem 1.3.7 can also be better understood by means of the convex geometry of the ideal I.

Definition 1.3.15. *Let I be a regular holonomic D-ideal and $w \in \mathbb{R}^n$ generic. The* Gröbner cone *of I containing the vector w is defined by*

$$C_w(I) = \left\{ w' \in \mathbb{R}^n : \ \mathrm{in}_{(-w,w)}(I) = \mathrm{in}_{(-w',w')}(I) \right\} \tag{1.20}$$

$C_w(I)$ is a union of open convex polyhedral cones in $\mathbb{R}^n$, since I is a regular holonomic D-ideal and w is generic. The polar cone of $C_w(I)$ is, by definition, the set $C_w(I)^*$ of vectors $u \in \mathbb{R}^n$ such that $\mathrm{in}_{(-w,w)}(I) = \mathrm{in}_{(-w',w')}(I)$ implies $u \cdot w' \geq 0$. Since $C_w(I)$ is n-dimensional, $C_w(I)^*$ is *strongly convex*, i.e., $C_w(I)^*$ contains no non-zero linear subpaces.

Consider the monoid consisting of all integer vectors

$$C_w(I)^*_{\mathbb{Z}} := C_w(I)^* \cap \mathbb{Z}^n$$

Write $\mathbb{C}[[C_w(I)^*_{\mathbb{Z}}]]$ for the ring of formal power series $f = \sum_u c_u x^u$ where $u \in C_w(I)^*_{\mathbb{Z}}$ and $c_u \in \mathbb{C}^*$. Since every non-constant term $c_u x^u$ appearing in f satisfies $w \cdot u > 0$, $\mathrm{in}_w(f)$ is well-defined for all $f \in \mathbb{C}[[C_w(I)^*_{\mathbb{Z}}]]$.

Finally, we get to a full description of the solutions to a regular holonomic ideal (see Theorems 2.5.12, 2.5.14 and 2.5.16 of [SST00]).

Theorem 1.3.16. *Let I be a regular holonomic D-ideal and $w \in \mathbb{R}^n$ generic. Put $r := \mathrm{rank}(I)$. Then there exist a set $\mathcal{C} = \{f_1, \ldots, f_r\}$ such that for $i = 1, \ldots, r$:*

1. *f_i is a solution to I.*

2. $\mathrm{in}_{\prec_w}(f_i) = x^{a_i} \log(x)^{b_i}$ *and* $\mathrm{in}_{\prec_w}(f_i) \neq \mathrm{in}_{\prec_w}(f_j)$ *for $i \neq j$.*

3. *f_i have the form $x^{a_i} \cdot p_i$ where a_i is and exponent and p_i is an element of the ring*
$$\mathbb{C}[[C_w(I)^*_{\mathbb{Z}}]][\log(x_1), \ldots, \log(x_n)].$$

4. *The degree of the logarithmic series p_i of the previous item with respect to $\log(x_k)$ is at most $\mathrm{rank}(I) - 1$ for every $k = 1, \ldots, n$.*

5. *There exists a point $x_0 \in \mathbb{C}_w(I)$ such that f_i converges for $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ satisfying*
$$(-\log|x_1|, \ldots, -\log|x_n|) \in x_0 + \mathbb{C}_w(I).$$

The series f_i, $i = 1, \ldots, r$, of Theorem 1.3.16 are called the *canonical (series) solutions* for I with respect to $\prec_w$. In Section 3.4 we will deal with logarithm free canonical series solutions and in Section 5.1 with the particular case of canonical series which are Laurent (i.e., with integer exponents).

Let $\{a_1, \ldots, a_r\} \subset \mathbb{C}^n$ is the set of exponents of I. The ring

$$N_w(I) := \mathbb{C}[[C_w(I)^*_{\mathbb{Z}}]][x^{a_1}, \ldots, x^{a_r}, \log(x_1), \ldots, \log(x_n)] \tag{1.21}$$

is called the *Nilsson ring* with respect to w.

Of course, we can consider an open set $U \subset \mathbb{C}^n$ and a (multivalued) holomorphic function in U that is a solution of the system $H_A(\beta)$ without regarding at any weight vector w, or consider the weight vector afterwards the holomorphic solution is defined. The following definition clarifies this.

Definition 1.3.17. *We say that a series*

$$\phi = \sum_{u \in \mathrm{supp}(\phi)} x^u p_u(log(x)) \tag{1.22}$$

that defines a (multivalued) holomorphic solution of the system $H_A(\beta)$ is a series solution of $H_A(\beta)$ in the direction of w *if the following conditions holds:*

1. *The p_u are polynomials.*

2. $\mathrm{supp}(\phi)$ *is contained in a strongly convex cone.*

3. *For every $u \in \mathrm{supp}(\phi)$, $\langle w, u \rangle \geq 0$.*

1.4 Hypergeometric regular ideals

In this section we show how the techniques explained in Section 1.3 can be applied to the system $H_A(\beta)$ in Definition 1.1.3. We analyze the combinatorial properties of the system $H_A(\beta)$ that allow us to obtain the canonical series described in Theorem 1.3.16.

We first observe that A-hypergeometric systems are holonomic, which was proved first by Gel'fand, Kapranov and Zelevinsky under hypothesis a) and b) below ([GKZ89]) and then by Adolphson (see [Ado94]) without assumptions on the configuration A.

Theorem 1.4.1. *The D-ideal $H_A(\beta)$ is holonomic.*

Hotta proved (see [Hot91]) that the A-hypergeometric systems are regular, provided that the associated toric variety is projective.

Theorem 1.4.2. *If the vector $(1, 1, \ldots, 1)$ is in the $\mathbb{Q}$-span of the row vectors of A, then $H_A(\beta)$ is regular holonomic for all $\beta \in \mathbb{C}$.*

We think of A, not just as a matrix, but also as the point configuration $\{a_1, \ldots, a_n\} \subset \mathbb{Z}^d$. We will assume the following conditions:

a) The vector $(1, 1, \ldots, 1)$ is in the $\mathbb{Q}$-span of A.

b) The elements of A span the lattice $\mathbb{Z}^d$ over $\mathbb{Z}$.

Important combinatorial features were developed in [GKZ94] from A-hypergeometric systems, such as Principal Determinants, Secondary Polytopes, Regular (coherent) triangulations, Secondary Fan, and more. Moreover, in [SST00] these structures are reviewed from a Commutative Algebra and Gröbner Theory point of view. We discuss this and show that more accurate results can be obtained if we assume that the parameter β is generic.

1.4.1 Invariants of the configuration A

The configuration A determines important properties of the system $H_A(\beta)$. Let $\mathrm{conv}(A)$ be the convex hull of the columns $a_1, \ldots, a_n$ of A. We consider a particular notion of volume (see [GKZ89]).

Definition 1.4.3. *The* normalized volume $\mathrm{vol}(A)$ *is the Euclidean volume of* $\mathrm{conv}(A)$ *normalized so that the unit simplex in the lattice $\mathbb{Z}A$ has volume one. As we have assumed that $\mathbb{Z}A = \mathbb{Z}^d$, the normalization is achieved by multiplying the Euclidean volume of* $\mathrm{conv}(A)$ *by* $d!$.

The singularities of $H_A(\beta)$ are also made explicit by the structure of A.

Definition 1.4.4. *Consider a generic polynomial with exponents in A:*

$$f(t_1, \ldots, t_d) = \sum_{i=1}^{n} x_i t^{a_i}.$$

The principal A-determinant *is defined as the resultant*

$$E_A(f) = R_A \left(t_1 \frac{\partial f}{\partial t_1}, \ldots, t_d \frac{\partial f}{\partial t_d}, f \right). \tag{1.23}$$

The following theorem summarizes the known results about the singularities and the calculation of the holonomic rank of $H_A(\beta)$. The two first statements are due to Gel'fand, Kapranov and Zelevinsky [GKZ89] under assumptions a) and b) and to Adolphson [Ado94] in the general case. The third one is Theorem 3.5.1 in [SST00]. The if part of the last statement is due to Gel'fand, Kapranov and Zelevinsky [GKZ89] and to Adolphson [Ado94]. The only if part was proved by Matusevich, Miller and Walther [MMW05].

Theorem 1.4.5. *Let $H_A(\beta)$ be an A-hypergeometric system.*

1. *The singular locus of the system is the zero set of the principal A-determinant:*
$$\mathit{Sing}(A) = \{E_A = 0\}.$$
2. *For arbitrary A and generic β,* $\operatorname{rank}(H_A(\beta)) = \operatorname{vol}(A)$.
3. *For arbitrary A and β,* $\operatorname{rank}(H_A(\beta)) \geq \operatorname{vol}(A)$.
4. *Given A,* $\operatorname{rank}(H_A(\beta)) = \operatorname{vol}(A)$ *for all $\beta \in \mathbb{C}^d$ if and only if the ideal I_A is Cohen-Macaulay.*

1.4.2 The fake initial ideal of $H_A(\beta)$

In order to describe the series solutions with respect to $w \in \mathbb{R}^n$ of a regular holonomic D-ideal I, we must first calculate the exponents, that is, the vectors $v \in \mathbb{C}^n$ such that x^v is a solution of $\mathrm{in}_{(-w,w)}(I)$. In the case $I = H_A(\beta)$, an easy manipulable description of $\mathrm{in}_{(-w,w)}(H_A(\beta))$ is known when the parameters β are generic (see Theorem 3.1.3 in [SST00]).

Lemma 1.4.6. *For β generic,*
$$\mathrm{in}_{(-w,w)}(H_A(\beta)) = \mathrm{in}_w(I_A) + \langle E - \beta \rangle.$$

Definition 1.4.7. *The ideal*
$$\mathrm{in}_w(I_A) + \langle E - \beta \rangle \tag{1.24}$$
is called the fake initial *ideal of $H_A(\beta)$ and if x^v is a solution to it, the vector $v \in \mathbb{C}^n$ is called a* fake exponent *of $H_A(\beta)$ with respect to w.*

Corollary 1.4.8. *For the ideal $H_A(\beta)$, the set of exponents is included in the set of fake exponents.*

Proof. This is clear because I_A and $E - \beta$ are both contained in $H_A(\beta)$. □

Fake exponents are easier to manipulate than exponents (in the generic case they are the same) and in general, they are useful to obtain the solutions of the system, as we will show.

Note that for $v \in \mathbb{C}^n$ to be an fake exponent it has to satisfy that x^v is a solution of the monomial ideal $\mathrm{in}_w(I_A) \subset \mathbb{C}[\partial_1, \ldots, \partial_n]$ and that for each row $(a_{i1}, \ldots, a_{in})$ of A it holds:
$$\sum_{j=1}^{n} a_{ij} x_j \partial_j (x^v) = \beta_i x^v$$

which turns into the following linear equations for v:

$$\sum_{j=1}^{n} a_{ij} v_j = \beta_i \text{ for } i = 1, \dots, d.$$

That is $A \cdot v = \beta$. So now we pay attention to the solutions of the ideal $\mathrm{in}_w(I_A) \subset \mathbb{C}[\partial_1, \dots, \partial_n]$. Note that if w is generic then $\mathrm{in}_w(I_A)$ is a monomial ideal.

A useful tool to find the solutions of a monomial ideal in $\mathbb{C}[\partial] := \mathbb{C}[\partial_1, \dots, \partial_n]$ is that of *standard pairs*.

Definition 1.4.9. *The set $\mathcal{S}(M)$ of* standard pairs *of the monomial ideal $M \subset \mathbb{C}[\partial]$ is the set of pairs (∂^a, σ) where $a \in \mathbb{N}^n$ and σ is a subset of the index set $\{1, \dots, n\}$ such that*

1. *$a_i = 0$ if $i \in \sigma$.*
2. *$\{x^v : v \in a + \mathbb{N}^\sigma\} \cap M = \emptyset$, where $\mathbb{N}^\sigma := \{\gamma \in \mathbb{N}^n : \gamma_l = 0, l \notin \sigma\}$.*
3. *For each $l \notin \sigma$, $\{x^v : v \in a + \mathbb{N}^{\sigma \cup \{l\}}\} \cap M \neq \emptyset$.*

If the dimension of M in $\mathbb{C}[\partial]$ is d we call top-dimensional *to an element $(\partial^a, \sigma) \in \mathscr{S}(M)$ such that $|\sigma| = d$. We denote $\mathscr{T}(M)$ the set of top-dimensional standard pairs of M.*

Top-dimensional standard pairs are useful to describe the set $\mathrm{top}(M)$ of top-dimensional components of a monomial ideal M in $\mathbb{C}[\partial]$, as Lemma 3.2.4 of [SST00] shows:

Lemma 1.4.10. *Let M be a monomial ideal in $\mathbb{C}[\partial]$.*

$$\mathrm{top}(M) = \bigcap_{(\partial^a, \sigma) \in \mathscr{T}(M)} \langle \partial_i^{a_i + 1} : i \notin \sigma \rangle. \tag{1.25}$$

As mentioned, we can describe the solutions of a monomial ideal in $\mathbb{C}[\partial]$ by means of its standard pairs.

Proposition 1.4.11. *For a monomial ideal $M \subset \mathbb{C}[\partial]$, the solutions of the system of partial differential equations*

$$\partial^u F = 0; \ \partial^u \in M$$

are functions of the form:

$$F(x) = \sum_{(\partial^a, \sigma) \in \mathcal{S}(M)} x^a \cdot F_\sigma(x)$$

where the function $F_\sigma(x)$ depends only on the variables x_i, $i \in \sigma$.

For a better understanding of the fake initial ideal of $H_A(\beta)$, we introduce the notion of *simplicial complex* and subsequently, the notion of *coherent triangulation* of a convex polytope.

Definition 1.4.12. *A family Δ of subsets of $\{1, \dots, n\}$ is called a* simplicial complex *when the following two conditions are satisfied:*

1. *if $\sigma \in \Delta$, then any subset of σ belongs to Δ.*

2. *if* $\sigma, \tau \in \Delta$, *then* $\sigma \cap \tau \in \Delta$.

Each subset in Δ *is called a* face *of the simplicial complex* Δ.

There is a connection between simplicial complexes and monomial ideals.

Definition 1.4.13. *The* Stanley-Reisner ideal *of* Δ *is the ideal in* $\mathbb{C}[\partial_1, \ldots, \partial_n]$ *generated by monomials* $\prod_{i\in\sigma} \partial_i$ *where* σ *runs over all the subsets of* $\{1, \ldots, n\}$ *such that* $\sigma \notin \Delta$.

Example 1.4.14.

$$\Delta = \{\{1,2,3\}, \{2,3,4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$$

is a simplicial complex on $\{1, 2, 3, 4\}$ *and its Stanley-Riesner ideal is* $\langle \partial_1\partial_4 \rangle$. *Note that only listing top-dimensional faces (facets) is enough to determine a simplicial complex. For instance, in the above example we can write* $\Delta = \{\{1, 2, 3\}, \{2, 3, 4\}\}$.

Definition 1.4.15. *A* triangulation *of a configurations of points* $A = \{a_1, \ldots, a_n\}$ *in* $\mathbb{R}^m$ *is a decomposition of its convex hull* $\mathrm{conv}(A)$ *into simplices with vertices in* A *which are the (top) faces of a simplicial complex* Δ.

A vector $w \in \mathbb{R}^n_{>0}$ *induces a subdivision* Δ_w *of the configuration* A, *by projecting the lower hull of* $\mathrm{conv}(\{(w_i, a_i) : i = 1, \ldots, n\})$ *onto* $\mathrm{conv}(A)$ *(see [Stu96, Chapter 8] for details). If* w *is generic,* Δ_w *is a triangulation of* A. *Such subdivisions are usually called regular, but we use the alternative term* coherent.

We always write triangulations of A as simplicial complexes on $\{1, \ldots, n\}$, but think of them geometrically: a simplex σ in such a triangulation corresponds to the geometric simplex $\mathrm{conv}(\{a_i : i \in \sigma\})$. Example 1.4.14 gives a possible triangulation of the convex hull of four affinely independent points in $\mathbb{R}^3$.

There is a deep connection between initials of toric ideals, coherent triangulations and standard pairs, explained in the following theorem due to Sturmfels (see [Stu96, Chapter 8]), that will help to understand the solutions of the A-hypergeometric systems.

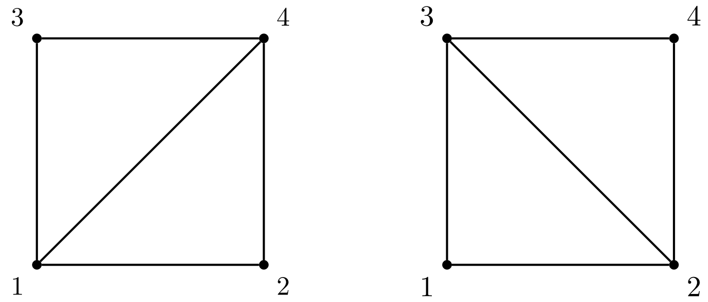
Theorem 1.4.16. *Let* A *be a homogeneous integer matrix and* $w \in \mathbb{R}^n$ *generic. Let* $M = \mathrm{in}_w(I_A)$. *The radical ideal* $\sqrt{M}$ *is the Stanley-Reisner ideal of the regular triangulation* Δ_w *of* A *defined by* w *and it can be described using the top-dimensional standard pairs of* M *in the following way:*

$$\Delta_w = \{\sigma : (\partial^a, \sigma) \in \mathscr{T}(M) \text{ for some } a\}. \tag{1.26}$$

Example 1.4.17. *Let* I_A *be the toric ideal associated to the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

For $w \in \mathbb{R}^4$ *generic there are two possible initial ideals of* I_A. *Take, for instance,* $w_1 = (0, 1, 1, 0)$ *and* $w_2 = (1, 0, 0, 1)$. *Then* $\mathrm{in}_{w_1} = \langle \partial_2\partial_3 \rangle$ *and* $\mathrm{in}_{w_2} = \langle \partial_1\partial_4 \rangle$. *The standard pairs of* $\langle \partial_1\partial_4 \rangle$ *are* $(1, \{1, 2, 4\})$ *and* $(1, \{1, 3, 4\})$ *and the standard pairs of* $\langle \partial_2\partial_3 \rangle$ *are* $(1, \{1, 2, 3\})$ *and* $(1, \{2, 3, 4\})$. *In Figure 1.1 we depicted the corresponding associated coherent triangulations. The configuration* A *lives in* $\mathbb{R}^2$, *but by the assumption of regularity the points lie in a hyperplane off the origin (in this case, the hyperplane* $x_1 + x_2 + x_3 = 1$*) and then we can actually consider the points in the plane).*



$\Delta_{w_1} = \{\{1,2,4\},\{1,3,4\}\} \qquad \Delta_{w_2} = \{\{1,2,3\},\{2,3,4\}\}$

Figure 1.1: The coherent triangulations of A for Example 1.4.17.

Example 1.4.18. *Consider the configuration A defined by the matrix* (1.9) *in Example 1.1.4. The convex hull of A is the segment $\{1\} \times [0,2] \backsimeq [0,2]$ and the toric ideal is given by $I_A = \langle \partial_0\partial_2 - \partial_1^2 \rangle$. The initial ideal $\langle \partial_0\partial_2 \rangle$ gives the standard pairs $(1,\{0,1\})$ and $(1,\{1,2\})$ that is, the triangulation consisting in the simplices $[0,1]$ and $[1,2]$. On the other hand, the initial ideal $\langle \partial_1^2 \rangle$ gives the standard pairs $(1,\{0,2\})$ and $(\partial_1,\{0,2\})$ and the corresponding triangulation is the whole segment $[0,2]$.*

There is a better description of the fake initial ideal for β generic ([SST00, Theorem 3.2.11]):

Theorem 1.4.19. *Let w a generic vector in $\mathbb{R}^n$. For generic parameters β, we have*

$$\mathrm{in}_{(-w,w)}(H_A(\beta)) = \mathrm{top}(\mathrm{in}_w(I_A)) + \langle E - \beta \rangle. \tag{1.27}$$

Suppose that v is a fake exponent of $H_A(\beta)$ with respect to a generic vector $w \in \mathbb{R}^n$, then Theorem 1.4.19 together with Lemma 1.4.10 and Proposition 1.4.11 allow us to easily calculate it: pick $(\partial^a, \sigma) \in \mathscr{T}(\mathrm{in}_w(I_A))$ and set $v_i = a_i$ for $i \notin \sigma$. Then solve the system $A \cdot v$ which is now a $d \times d$ invertible system, because σ is actually a simplex in the triangulation Δ_w. We denote $v = \beta^{(\partial^\alpha,\sigma)}$ the fake exponent thus obtained. In the generic case, we can obtain $\mathrm{vol}(A)$ (fake) exponents by this procedure and consequently a basis of the space of solutions with respect to w, as we explain in the following section. In the non-generic case the picture is a bit more complicated, and we show in Section 3.4 how to obtain the exponents and the series solutions.

1.4.3 Hypergeometric canonical series

The following step for a better understanding of the solutions of the system is to describe its canonical solutions.

Lemma 1.4.20. *Let $w \in \mathbb{R}^n$ generic. The canonical solutions of the D-ideal $H_A(\beta)$ have the shape*

$$\phi = x^v \sum_{u \in C} x^u p_u(log(x)) \tag{1.28}$$

where

1. $v \in \mathbb{C}^n$ *is an exponent of* $H_A(\beta)$,
2. *the cone* C *is contained in* $\ker_{\mathbb{Z}}(A) := \ker(A) \cap \mathbb{Z}^n$, *with the property that*
3. $\langle w, u\rangle \geq 0$ *for every* $u \in \mathbb{C}$ *and*
4. *the polynomials* p_u *belong to the symmetric algebra of* $\ker_{\mathbb{Z}}(A)$.

Proof. See [Sai02]. □

The regions of convergence of the canonical solutions of A-hypergeometric solutions with respect to a generic weight w can be described in a more manageable manner than Theorem 1.3.16.5.

Let w be a weight vector for $H_A(\beta)$ and let $\{\gamma_1, \dots, \gamma_{n-d}\} \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-basis for $\ker_{\mathbb{Z}}(A)$ such that $\gamma_i \cdot w > 0$ for $i = 1, \dots, n-d$. For any $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{n-d}) \in \mathbb{R}^{n-d}_{>0}$, we define the (non empty) open set

$$\mathscr{U}_{w,\boldsymbol{\varepsilon}} = \{x \in \mathbb{C}^n \mid |x^{\gamma_i}| < \varepsilon_i \text{ for } i = 1, \dots, n-d\}. \tag{1.29}$$

If we take ε such that $\varepsilon_i \ll 1$ for $i = 1, \dots, n-d$, then the canonical A-hypergeometric series clearly converge in the open set $\mathscr{U}_{w,\boldsymbol{\varepsilon}}$.

1.4.4 Gamma series for generic parameters

When the parameters β are generic, it is easier to get a full picture of the solutions.

The following result is restatement of Proposition 3.4.1, Theorem 3.4.2 and Lemma 3.4.6 in [SST00], which do not need homogeneity for I_A.

Proposition 1.4.21. *For any* $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ *such that* $A \cdot v = \beta$, *the formal series*

$$\phi_v = \sum_{u \in \ker_{\mathbb{Z}}(A)} \frac{[v]_{u_-}}{[u+v]_{u_+}} x^{u+v} \tag{1.30}$$

where

$$[v]_{u_-} = \prod_{u_i<0} \prod_{j=1}^{-u_i} (v_i - j + 1); \quad [u+v]_{u_+} = \prod_{u_i>0} \prod_{j=1}^{u_i} (v_i + j) \tag{1.31}$$

is well defined and is annihilated by the hypergeometric D*-ideal* $H_A(\beta)$.

If moreover β is a generic parameter vector, the support of ϕ_v is the set $\mathrm{supp}(\phi_v) = \{u \in \ker_{\mathbb{Z}}(A) \mid u_i + v_i \geq 0 \quad \forall i \notin \sigma\}$. Thus,

$$\phi_v = \sum_{\{u \in \ker_{\mathbb{Z}}(A) \mid u_i+v_i \geq 0 \ \ \forall i \notin \sigma\}} \frac{[v]_{u_-}}{[u+v]_{u_+}} x^{u+v}. \tag{1.32}$$

Theorem 1.4.22. *For a generic vector $w \in \mathbb{R}^n$ and a generic parameter β, the set*

$$\{\phi_{\beta(\partial^\alpha,\sigma)} \mid (\partial^\alpha, \sigma) \in \mathscr{T}(\mathrm{in}_w(I_A))\} \tag{1.33}$$

is a basis for the (multivalued) holomorphic solutions of $H_A(\beta)$ in an open set of the form (1.29) *for some $\varepsilon \in \mathbb{R}^{n-d}_{>0}$.*

Corollary 1.4.23. *For a generic vector $w \in \mathbb{R}^n$ and a generic parameter β it holds*

$$\mathrm{rank}(H_A(\beta)) = \mathrm{rank}(\mathrm{in}_w(H_A(\beta))) = |\mathscr{T}(\mathrm{in}_w(H_A(\beta)))| = \mathrm{vol}(A) \tag{1.34}$$

The following example summarizes how to obtain the solutions of an A-hypergeometric system with respect to $w \in \mathbb{R}^n$ for generic parameters.

Example 1.4.24. *Take A as in Example 1.4.17 and $\beta \in \mathbb{C}^3$ a generic parameter. Consider the A-hypergeometric system associated to this data:*

$$H_A(\beta) = I_A + \langle x_1\partial_1 - x_4\partial_4 + 1 - \beta_1, x_2\partial_1 - x_4\partial_4 - \beta_2, x_3\partial_3 - x_4\partial_4 - \beta_3\rangle \tag{1.35}$$

and $w = (1,0,0,1)$. The fake exponents of $H_A(\beta)$ with respect to w are $v = (\beta_1, \beta_2, \beta_3, 0)$ and $v' = (0, \beta_2 + \beta_1, \beta_3 + \beta_1, -\beta_1)$; they correspond to the initial ideal $\langle \partial_1\partial_4 \rangle$. The genericity on β consists in that none of the coordinates of v nor v' is a negative integer. Since $\ker_{\mathbb{Z}}(A) = \{(n,-n,-n,n), n \in \mathbb{Z}\}$, it follows that for $u = (n,-n,-n,n)$ with $n \geq 1$, the expression $[v]_{u_-}$ vanishes since $v_4 = 0$. Hence

$$\begin{aligned}
\phi_v &= x^v \cdot \sum_{n=0}^{\infty} \frac{[v]_{(0,n,n,0)}}{[v + (n,-n,-n,n)]_{(n,0,0,n)}} \left(\frac{x_1x_4}{x_2x_3}\right)^n \\
&= x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3} \cdot \sum_{n=0}^{\infty} \frac{(-1^n)(-\beta_2)_n(-1^n)(-\beta_3)_n}{(\beta_1+1)_n n!} \left(\frac{x_1x_4}{x_2x_3}\right)^n \\
&= x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3} \cdot F\left(-\beta_2, -\beta_3, \beta_1 + 1; \frac{x_1x_4}{x_2x_3}\right),
\end{aligned}$$

i.e., we recover the series hypergeometric series (1.4)*, with $c = \beta_1 + 1$, $b = -\beta_2$ and $a = -\beta_3$ multiplied by a appropriated monomial to adjust the homogeneity. Similarly, if we consider v', we recover the hypergeometric series* (1.5).

Chapter 2

Tools from combinatorics

This chapter summarizes the tools from combinatorics that we need in the rest of the thesis. Coherent triangulations were treated in section 1.4.2 in the context of initial of toric ideals, but they are intimately related with the topics seen here. The Secondary Fan is the object that describes all coherent triangulations for a given configuration. Coherent mixed subdivision will be used to define algebraic Laurent solutions of particular A-hypergeometric systems in Chapter 6.

2.1 Secondary Fan

We now present the object that parametrizes all coherent triangulations of the configuration A and, consequently, it also describes the regions of convergence of A-hypergeometric canonical series. This object is called the *secondary fan* of A, and was introduced by Gelfand, Kapranov and Zelevinsky (see [GKZ94, Chapter 7]). We summarize the construction of the secondary fan and give some examples related with the ones previously given.

To construct the secondary fan of A, we need the following notion

Definition 2.1.1. *A Gale dual for A is a matrix $B \in \mathbb{Z}^{n\times(n-d)}$ whose columns form a $\mathbb{Z}$-basis for* $\ker_{\mathbb{Z}}(A)$.

Denote by $b_1, \dots, b_n$ the rows of B, a Gale dual for A, and let Δ be a coherent triangulation of A. For each maximal simplex $\sigma \in \Delta$, we define a cone

$$\mathscr{K}_\sigma = \left\{ \sum_{i\notin\sigma} \lambda_i b_i \;\middle|\; \lambda_i \geq 0 \right\}.$$

The set $\{b_i \mid i \notin \sigma\}$ is linearly independent [GKZ94, Lemma 7.1.16], and therefore $\mathscr{K}_\sigma$ is full-dimensional.

Define $\mathscr{K}_\Delta = \cap_{\sigma\in\Delta}\mathscr{K}_\sigma$. Then $w \in \mathbb{R}^n$ is such that $\Delta_w = \Delta$ if and only if $w \cdot B$ belongs to the interior of $\mathscr{K}_\Delta$. The cones $\mathscr{K}_\Delta$ for all coherent triangulations Δ of A are the maximal cones in a polyhedral fan, called the *secondary fan of A* (see also [GKZ94, Theorem 7.1.17]). It is in this sense in which we say that the secondary fan *parametrizes* the coherent triangulations of A.

Example 2.1.2. *Consider A as in Example 1.1.4. We describe its coherent triangulations in Example 1.4.18. A Gale dual for the configuration A is given by*

$$B = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Consider the coherent triangulation $\Delta_1 = \{\{0,1\},\{1,2\}\}$. The one-dimensional cone $\mathscr{K}_{\Delta_1}$ is the intersection of the cones $\mathscr{K}_{\{0,1\}}$ and $\mathscr{K}_{\{1,2\}}$ both equal to $\{\lambda \in \mathbb{R}_{\geq 0}\} = \mathbb{R}_{\geq 0}$, that is $\mathscr{K}_{\Delta_1} = \mathbb{R}_{\geq 0}$. The vectors $w \in \mathbb{R}^3$ that induce this triangulation are the ones such that $w_1 - 2w_2 + w_3 > 0$.

On the other hand, the triangulation $\Delta_2 = \{\{0,2\}\}$ corresponds to the cone $\mathscr{K}_{\Delta_2} = \mathscr{K}_{\{0,2\}} = \{-2 \cdot \lambda \in \mathbb{R}_{\geq 0}\} = \mathbb{R}_{\leq 0}$. The vectors $w \in \mathbb{R}^3$ that induce this triangulation are the ones such that $w_1 - 2w_2 + w_3 < 0$.

Next we consider an example where the secondary fan is two-dimensional.

Example 2.1.3. *Let*

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 3 \\ 0 & 4 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 1 & -3 \\ 1 & 0 \\ -2 & -1 \\ 1 & 1 \end{pmatrix}.$$

Being the configuration A a subset of $\mathbb{R}^4$ we cannot present the triangulations graphically. To obtain the secondary fan, we simply draw the vectors b_i, $i = 1, \dots, 6$ in a plane and consider the positive cones generated pairwise by the row vectors b_i. These are the intersections of the cones $\mathscr{K}_\sigma$ as σ runs over a coherent triangulation. Then the coherent triangulations of $\operatorname{conv}(A)$ are:

$$\begin{aligned}
\Delta_1 =& \{\{1,2,3,5\},\{1,3,5,6\},\{1,2,4,5\},\{2,3,5,6\},\{1,4,5,6\}\} \\
\Delta_2 =& \{\{1,3,4,5\},\{2,3,4,5\},\{1,3,5,6\},\{1,4,5,6\},\{2,3,5,6\},\{1,2,3,4\}\} \\
\Delta_3 =& \{\{3,4,5,6\},\{2,3,4,5\},\{1,3,4,6\},\{2,3,5,6\},\{1,2,3,4\}\} \\
\Delta_4 =& \{\{2,3,4,6\},\{2,4,5,6\},\{1,3,4,6\},\{1,2,3,4\}\} \\
\Delta_5 =& \{\{1,2,4,6\},\{2,4,5,6\},\{1,2,3,6\}\} \\
\Delta_6 =& \{\{1,2,5,6\},\{1,2,4,5\},\{1,2,3,6\},\{1,4,5,6\}\}
\end{aligned}$$

For instance, the triangulation Δ_1 is associated to any vector $w \in \mathbb{R}^6$ such that the planar vector $b_w = \sum_{i=1}^{6} w_i b_i$ lies in the positive cone $\mathbb{R}_{>0} b_4 + \mathbb{R}_{>0} b_6$. This implies that the complementary indices $\{1,2,3,5\}$ form a maximal cell of Δ. Indeed, maximal cells correspond precisely to the complementary indices of those pairs of vectors b_i, b_j such that $b_w \in \mathbb{R}_{>0} b_i + \mathbb{R}_{>0} b_j$.

In Figure 2.1 we depict the secondary fan and indicate, in the interior of each cone, the corresponding coherent triangulation.

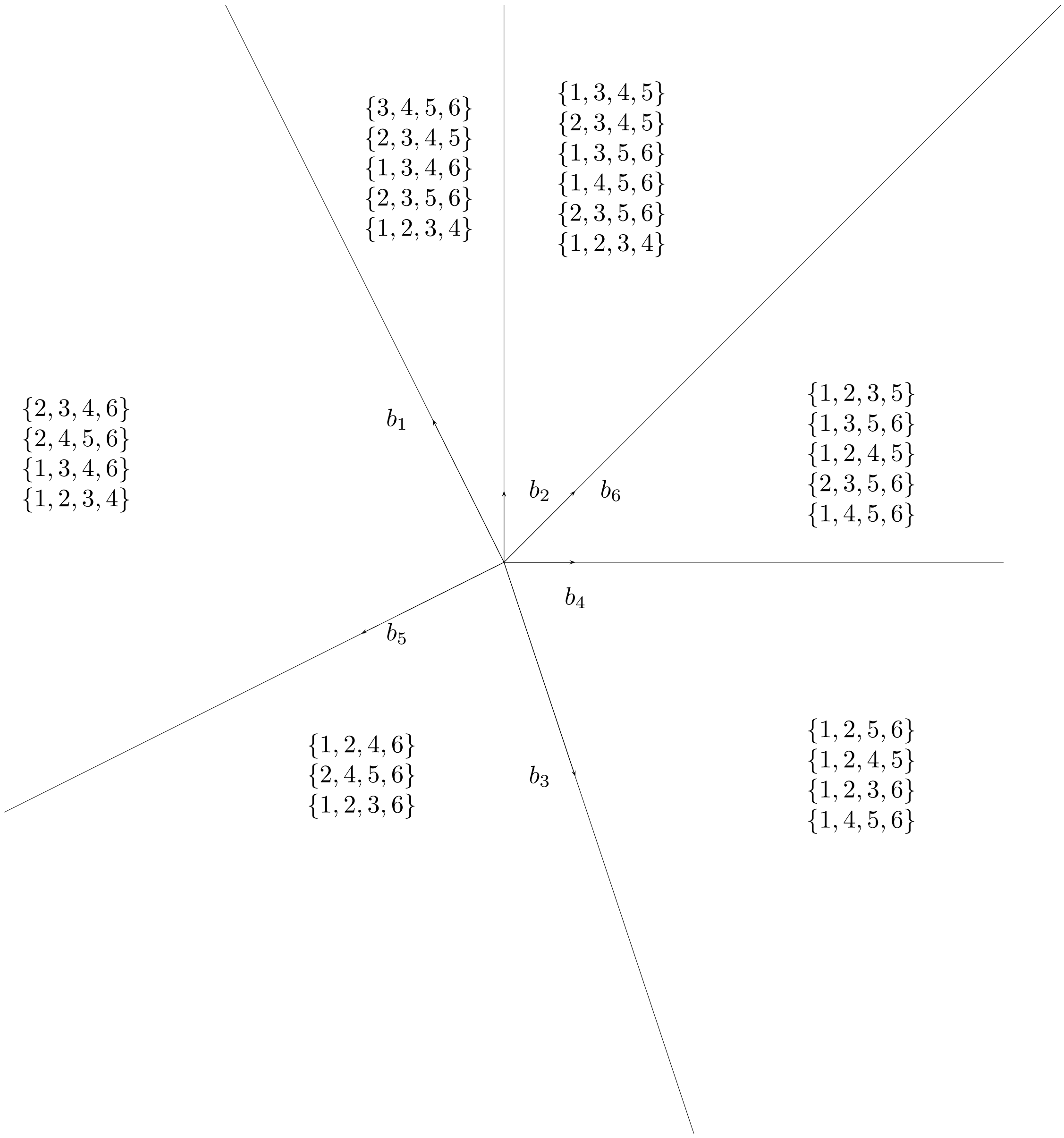


Figure 2.1: The secondary fan parametrizes the coherent triangulations

2.2 Coherent mixed subdivisions

In this section we describe the combinatorial objects that play the role of the triangulations in the description of the common roots of the polynomials (5.24), by the work of Huber and Sturmfels [HS95]. These are the *coherent mixed subdivisions* of the Minkowski sum of the corresponding supports. Their relation with polynomial systems will be explained in section 6.1 in order to define algebraic solutions of Cayley configurations. Here we define them and give some properties.

We begin with the definition of Cayley configurations.

Definition 2.2.1. *Let $A_1, \ldots, A_k \subset \mathbb{Z}^k$ be k lattice configurations in k-th dimensional space. Set $n = |A_1| + \cdots + |A_k|$ and $d = 2k$. We call Cayley configuration associated with $A_1, \ldots, A_k$, denoted by $A = Cayley(A_1, \ldots, A_k)$, the configuration in $\mathbb{Z}^d$ defined by*

$$A \;=\; \{e_1\} \times A_1 \,\cup\, \cdots \,\cup\, \{e_k\} \times A_k. \tag{2.1}$$

Our goal in Chapter 6 will be to describe the Laurent algebraic solutions for the A-hypergeometric systems associated to Cayley configurations (see Definition 2.2.1) (and integer homogeneities). Based on Example 5.3.4, we will show that the common roots of the polynomials (5.24) play a special role in the description of such solutions.

In the univariate case, there is an intimate relation between the combinatorics of the configuration A and the regions where the roots $\rho_1(x), \ldots, \rho_d(x)$ define holomorphic functions. Examples 5.1.6 and 1.4.18 show this relation: for $w \in \mathbb{R}^3$ such that the coherent triangulation of the configuration A is $\{\{0,1\},\{1,2\}\}$, the roots can be written as Laurent series (see (5.8)) converging in the open set

$$\left|\frac{x_0 x_2}{x_1^2}\right| << 1.$$

On the other hand, for $w \in \mathbb{R}^3$ such that the coherent triangulation of the configuration A is $\{\{0,2\}\}$, there are no Laurent series in the direction of w, but the roots are still solutions of $H_A(\beta)$ in the open set

$$\left|\frac{x_1^2}{x_0 x_2}\right| << 1$$

and therefore, they can be written as Puiseaux series there with exponents in $\frac{1}{2} \cdot \mathbb{Z}$. The roots of any degree univariate polynomial behave in a similar way.

Although the definitions can be given for an arbitrary $k \in \mathbb{N}$ we assume $k = 2$ for simplicity of the exposition.

Suppose that $A_1 = \{\alpha_1, \ldots, \alpha_r\}$ and $A_2 = \{\beta_1, \ldots, \beta_s\}$ are planar lattice configurations. Set $n = r + s$.

Definition 2.2.2. *Given $w \in \mathbb{R}^n$ a generic weight we denote by Δ_w the corresponding coherent triangulation of $A = Cayley(A_1, A_2)$. The corresponding mixed coherent subdivision Π_w of $A_1 + A_2$ is defined as follows. The top dimensional cells of Π_w are in bijection with the top dimensional cells in Δ_w. Given any top dimensional cell τ in Δ_w, must contain an index in each of A_1, A_2. Note that $\dim(\tau) = 3$ and so $|\tau| = 4$. The vertices of the corresponding cell σ_τ in Π_w are the points $\alpha_i + \beta_j$, for each pair $i, r + j \in \tau$ with $1 \leq i \leq r$ and $1 \leq j \leq s$.*

We indicate the cell σ_τ in Π_w by the cell $\tau \subset \{1, \ldots, n\}$ of Δ_w. We say that σ_τ is mixed if $|\tau \cap A_i| = 2, i = 1, 2$, and unmixed otherwise.

Remark 2.2.3. *Mixed subdivisions can be defined in terms of liftings of $A_1 + A_2$ as in [HS95]; see [DRS10, Section 9.2] for the equivalence of both definitions.*

Let $w \in \mathbb{R}^n$ generic. The following proposition is an easy consequence of the definiton of coherent mixed subdivision, and Theorem 1.4.16.

Proposition 2.2.4. *Let w be a generic weight in $\mathbb{R}^n$ and call Π_w the associated mixed coherent subdivision of $A_1 + A_2$. Let $\sigma \in \Pi_w$. The following statements are equivalent:*

1. *σ is a cell in Π_w.*

2. *For all $\sigma' \subseteq \sigma$ call $\Gamma_{\sigma'} = \{u \in \ker_A : \ u_i < 0 \Leftrightarrow i \in \sigma'\}$. Then $\langle w, u\rangle > 0$ for all $u \in \Gamma_{\sigma'}$, $u \neq 0$.*

3. *No initial monomial in $\mathrm{in}_w(I_A)$ has support (contained) in σ.*

Remark 2.2.5. *Note that $\Gamma_{\sigma'}$ could be empty. For instance, if $\alpha_i + \beta_j$ is a vertex of $A_1 + A_2$ and $\sigma' = \{i, r + j\}$, then $\Gamma_{\sigma'} = \emptyset$. This implies that all vertices automatically satisfy 2. and then, they belong to any Π_w, which is obvious.*

Before giving examples of coherent mixed subdivisions, we prove a technical lemma that will be used in section 6.2.

Lemma 2.2.6. *Let $w \in \mathbb{R}^n$ generic and $\sigma \subset \Pi_w$ a mixed cell. Let $I \subsetneq \sigma$ and $r_I \neq \emptyset$. Then there exists $w' \in \mathbb{R}^n$ such that $\sigma \subset \Pi_{w'}$ and $I \nsubseteq \Pi_{w'}$.*

Proof. Suppose $I = \{i_1, \dots, i_k\}$. Given that $r_I \neq \emptyset$, there exists a binomial of the form

$$b_I = \partial_{i_1}^{u_1} \dots \partial_{i_k}^{u_k} - \partial^{v(I)} \in I_A,$$

where $u_i \in \mathbb{N}$, $i = 1, \dots, k$ and $v(I)_j = 0$ for all $j \in I$. As $I \subsetneq \sigma$ there exists $1 \leq j \leq k$ such that $i_j \in I - \sigma$. Set $w' = w + \lambda e_{i_j}$ with λ sufficiently large so that $\mathrm{in}_{w'}(b_I) = \partial_{i_1}^{u_1} \dots \partial_{i_k}^{u_k}$. Then $I \nsubseteq \Pi_{w'}$.

To see that $\sigma \subset \Pi_{w'}$, note that by hypothesis and Proposition 2.2.4, no initial monomial in $\mathrm{in}_w(I_A)$ has support contained in σ. Given that $i_j \notin \sigma$, it is clear that no initial monomial with respect to w' will have support contained in σ. □

Example 2.2.7. *Consider $A_1 = \{\alpha_1 = (1,0), \alpha_2 = (0,4), \alpha_3 = (1,1)\}$ and $A_2 = \{\beta_1 = (1,1), \beta_2 = (2,1), \beta_3 = (3,0)\}$. In Example 2.1.3 we considered the coherent triangulations of $A = Cayley(A_1, A_2)$ and showed how the secondary fan of A can be used to parametrize them. In order to understand the mixed subdivisions of $A_1 + A_2$, we depict the Minkowski sum in Figure 2.2.*

In Figure 2.2.7 we depict the coherent mixed subdivisions of the configuration A, also parametrized by the secondary fan of A, as in Figure 2.1. Recall that a weight $w \in \mathbb{R}^n$ induces the subdivision of $A_1 + A_2$ (equivalently, triangulation of A) corresponding to a cone in a Gale dual of A, if and only if $w \cdot B$ belongs to the interior of such cone. In this sense, the cone generated by b_3 and b_5 corresponds to the subdivision $\{\{1,2,4,6\}, \{2,4,5,6\}, \{1,2,3,6\}\}$. Here, the index set $\{1,2,4,6\}$ indicates the appearance in the subdivision of the cell with vertices $\alpha_1 + \beta_1$, $\alpha_1 + \beta_3$, $\alpha_2 + \beta_1$, $\alpha_2 + \beta_3$.

Example 2.2.8. *Consider $A_1 = \{\alpha_1 = (0,1), \alpha_2 = (2,0), \alpha_3 = (1,2)\}$ and $A_2 = \{\beta_1 = (1,0), \beta_2 = (0,1), \beta_3 = (0,0)\}$. Note that the inner normal directions corresponding to A_1, A_2 do not alternate. Their Minkowski sum is Figure 2.4 and the coherent mixed subdivisions in Figure 2.5.*

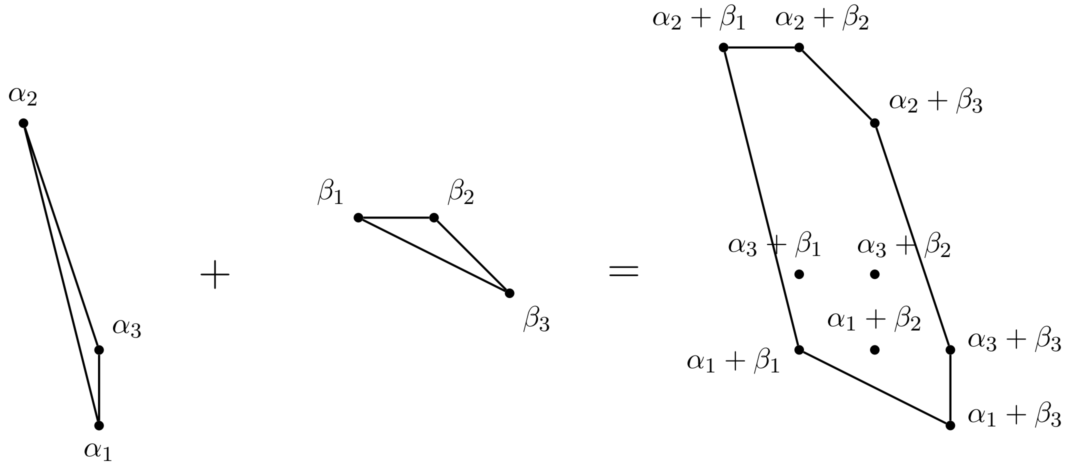


Figure 2.2: The Minkowski sum of $A_1 + A_2$ for Example 2.2.7

Example 2.2.9. *For $A_1 = \{\alpha_1 = (1,0), \alpha_2 = (0,1), \alpha_3 = (0,0)\}$ and $A_2 = \{\beta_1 = (2,0), \beta_2 = (0,2), \beta_3 = (0,0)\}$ we depict the Minkowski sum in Figure 2.6. In this case the convex hulls of A_1 and A_2 have the same inner normals, and so this is an example of two non-developed polytopes (cf. Section 5.4.1). Notice that the Minkowski sum does not have any interior points of the form $\alpha_i + \beta_j$. Also notice that the six possible coherent mixed subdivisions, depicted in Figure 2.7, consist of one mixed cell and two unmixed cells.*

The following two examples concern the case $|A_1| = 2$ and $|A_2| = 4$, to be discussed in Chapter 6.

Example 2.2.10. *Let $A_1 = \{\alpha_1 = (0,0), \alpha_2 = (1,1)\}$ and $A_2 = \{\beta_1 = (0,0), \beta_2 = (1,0), \beta_3 = (0,1), \beta_4 = (1,1)\}$. The Minkowski sum of the segment and the square is depicted in Figure 2.8. In this case the two interior point coincide, that is $\alpha_1 + \beta_4 = \alpha_2 + \beta_1$. The coherent mixed subdivisions are depicted in Figure 2.9.*

Example 2.2.11. *Let $A_1 = \{\alpha_1 = (0,0), \alpha_2 = (1,1)\}$ and $A_2 = \{\beta_1 = (0,0), \beta_2 = (3,0), \beta_3 = (0,3), \beta_4 = (1,1)\}$, which has an interior point. Their Minkowski sum is depicted in Figure 2.10. In this case, there are two interior points: $(1,1) = \alpha_1 + \beta_4 = \alpha_2 + \beta_1$ and $(2,2) = \alpha_2 + \beta_4$. The coherent mixed subdivisions are depicted in Figure 2.11.*

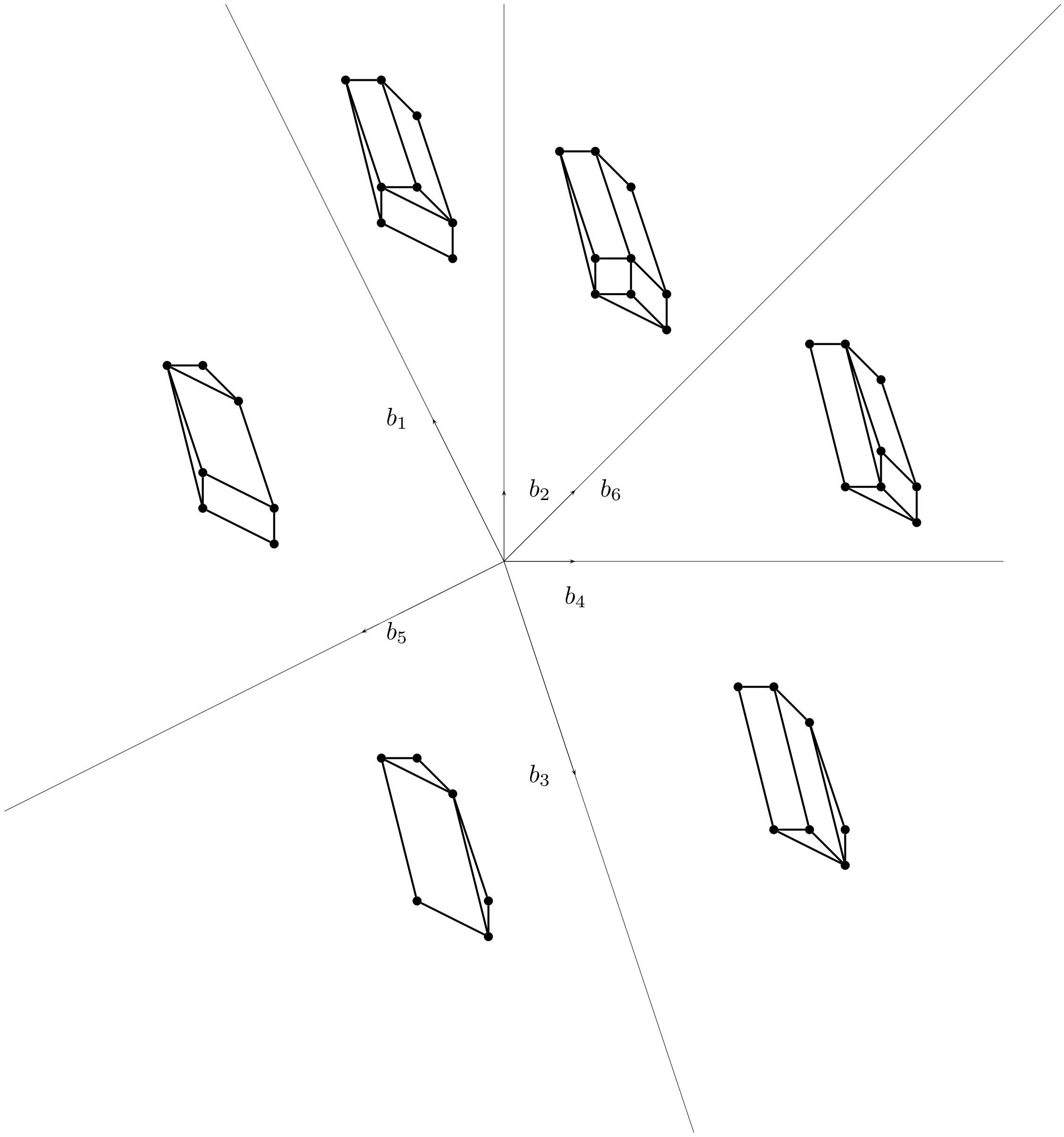


Figure 2.3: The mixed subdivisions of A for Example 2.2.7.

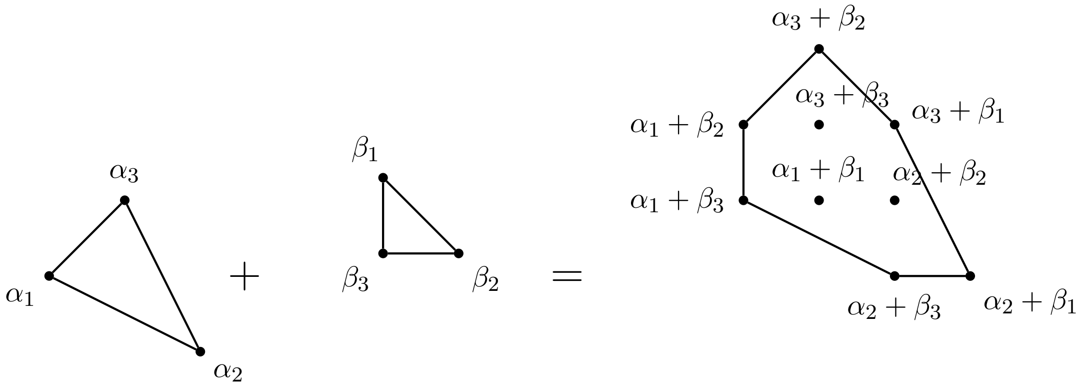


Figure 2.4: The Minkowski sum of $A_1 + A_2$ for example 2.2.8

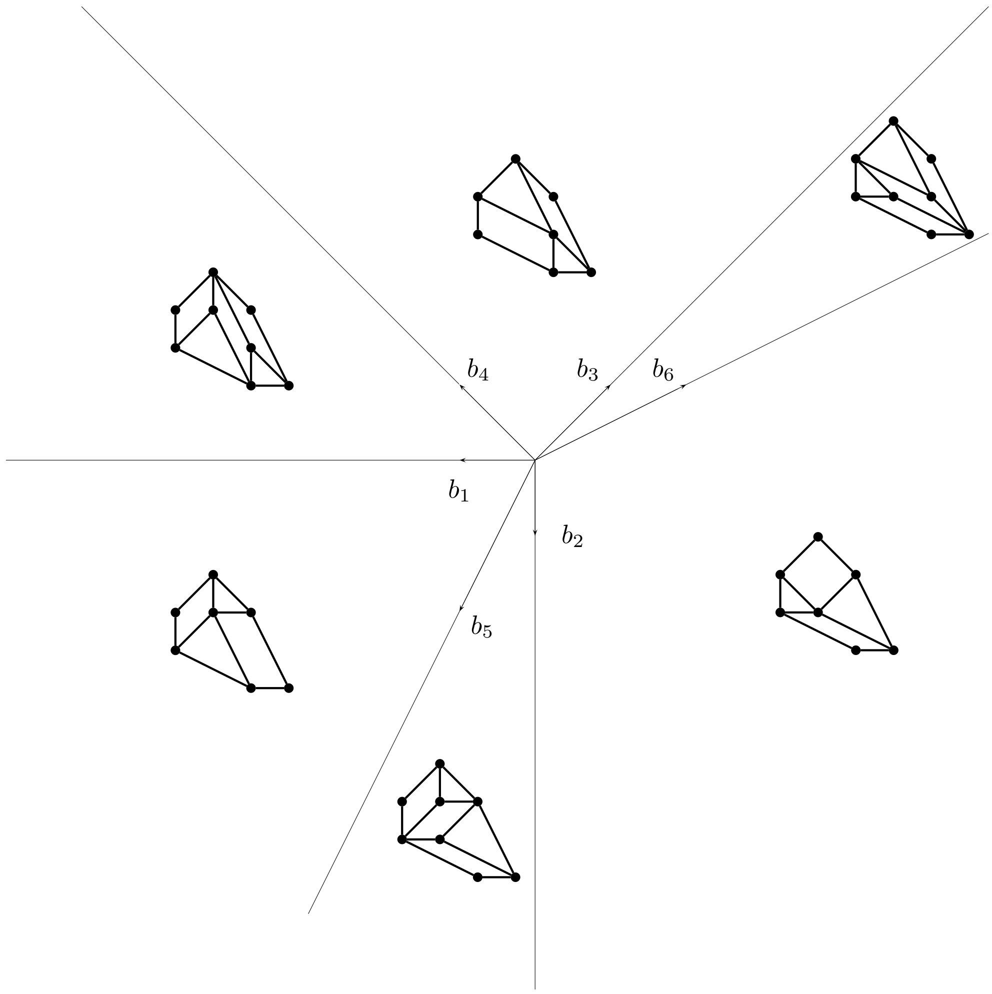


Figure 2.5: The mixed subdivisions of A for Example 6.3.3

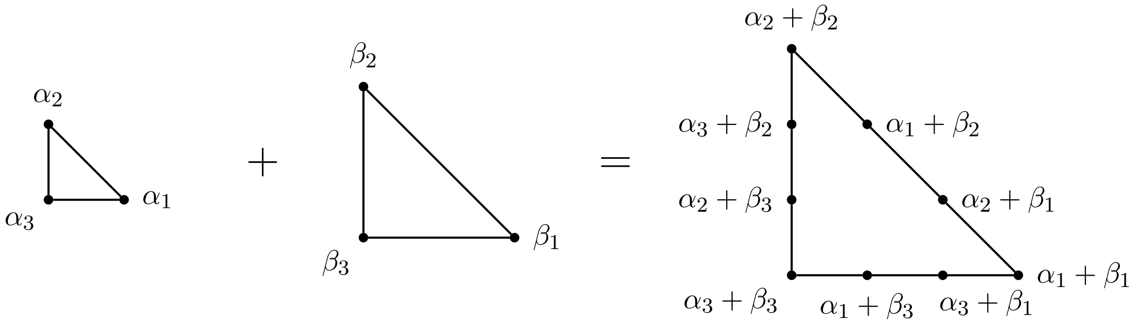


Figure 2.6: The non-developed case

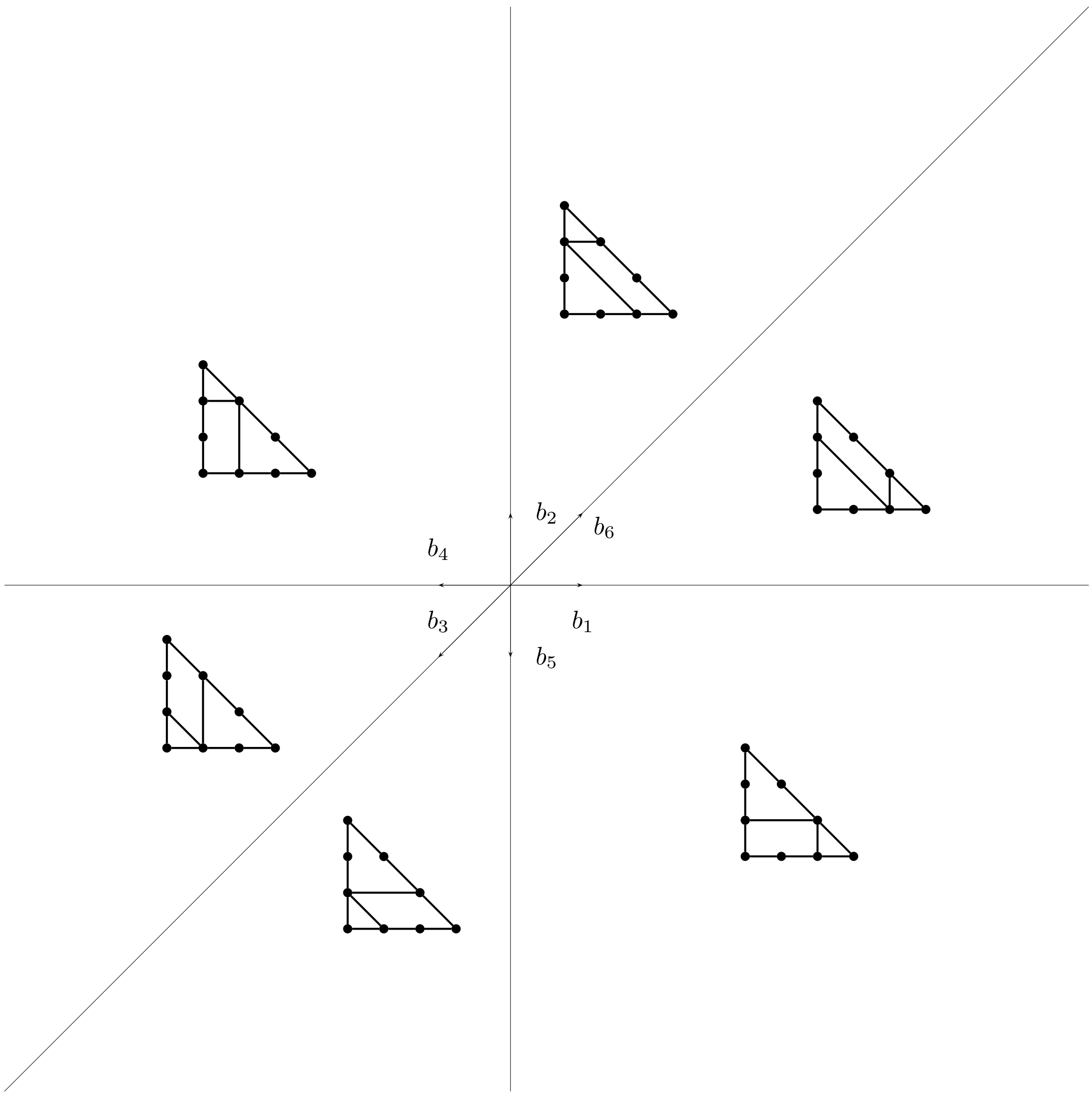


Figure 2.7: The mixed subdivisions of A for Example 2.2.9

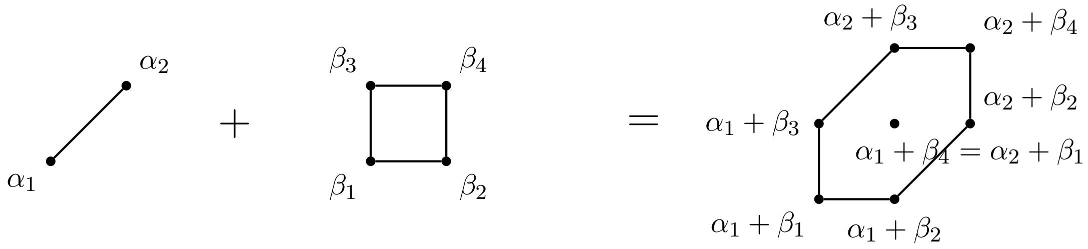


Figure 2.8: The Minkowski sum of $A_1 + A_2$ for Example 2.2.10

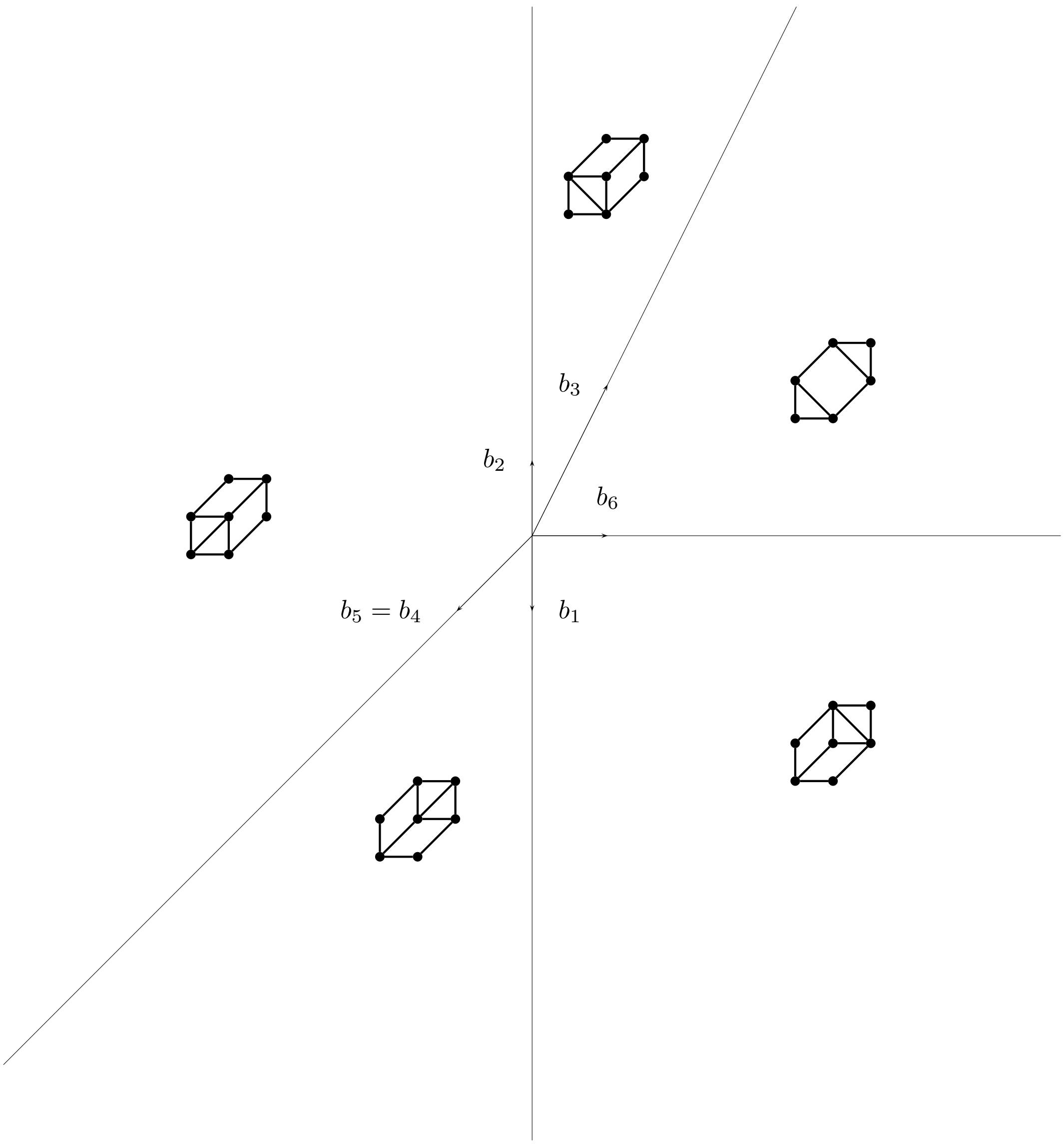


Figure 2.9: The mixed subdivisions of A for Example 2.2.10

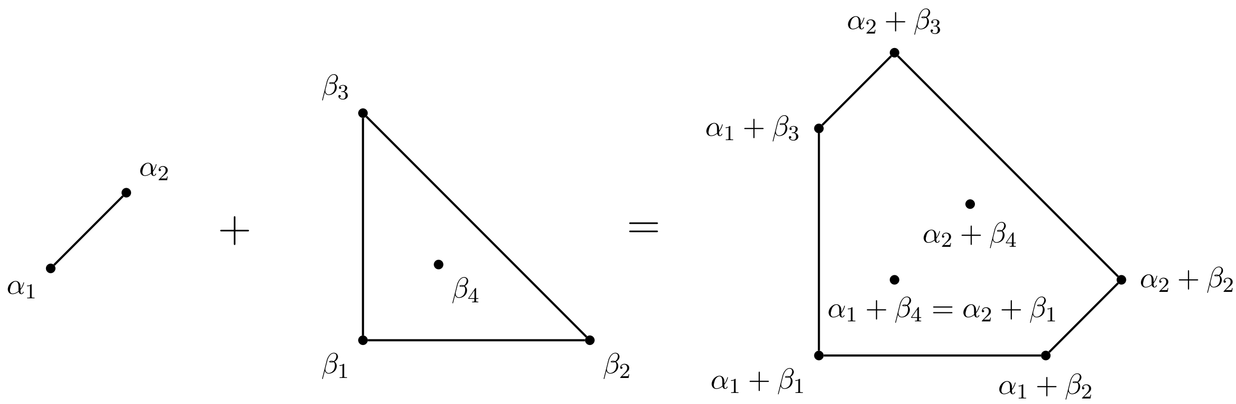


Figure 2.10: The Minkowski sum of $A_1 + A_2$ for Example 2.2.11

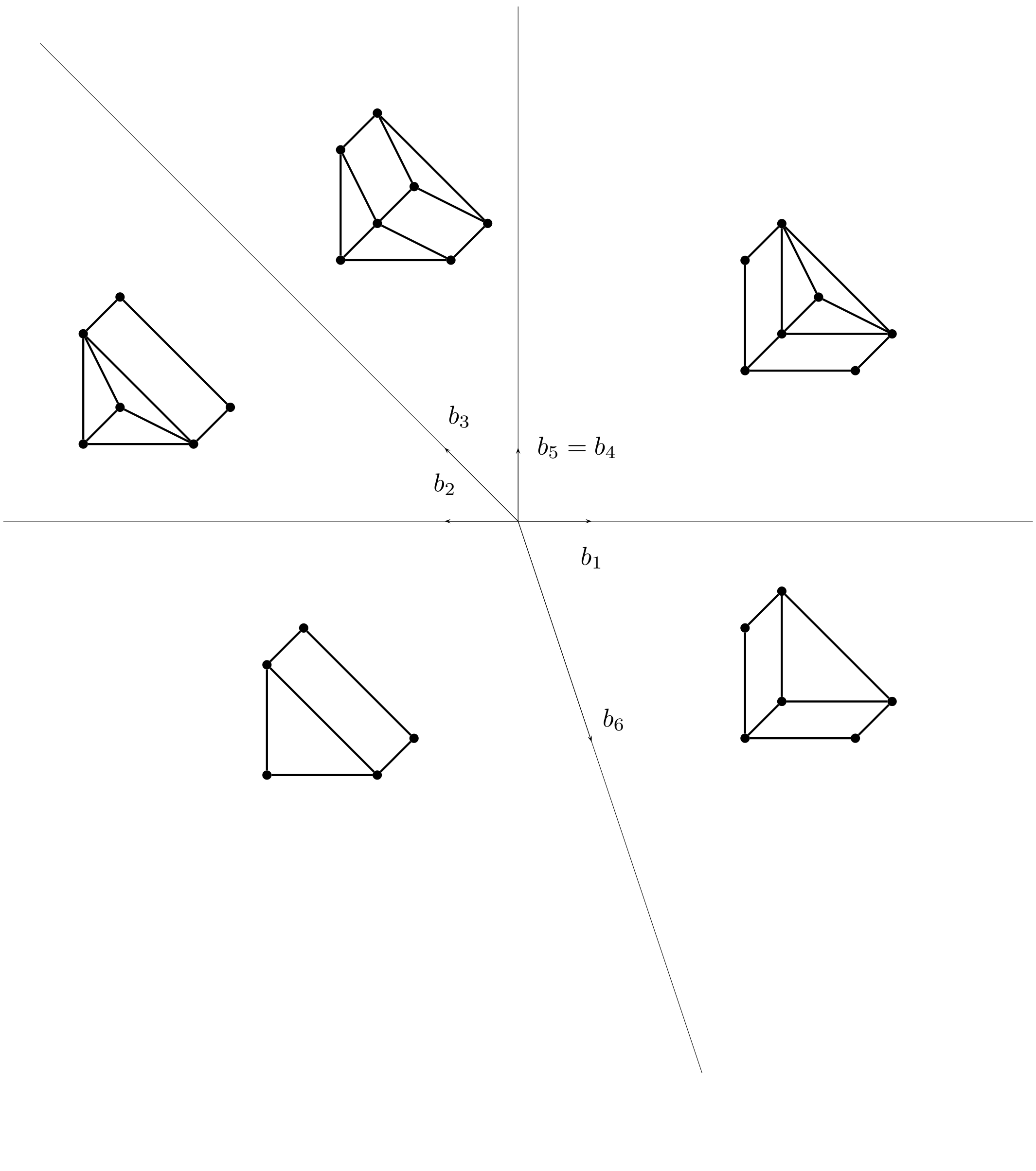


Figure 2.11: The mixed subdivisions of A for Example 2.2.11

Chapter 3

Formal Nilsson solutions of irregular A-hypergeometric systems

One of the main properties of regular systems, based on important results by Malgrange in the unidimensional case and Mebkhout in the general case, resides in that one does not need to prove that a formal solution converges; as we saw in Chapter 1, this happens “automatically”. We recalled in Theorem 1.4.2 that if the row span of A contains the vector $(1, \ldots, 1)$, then the system $H_A(\beta)$ is regular. If this is not the case, Schulze and Walther [SW08] proved that the system is not regular. In this chapter we will not assume that the row span of A contains the vector $(1, \ldots, 1)$ and we will reprove in Theorem 3.5.6 their result. Our proof will deal with formal solutions of general A-hypergeometric systems, so that, in Section 3.1 we introduce the notion of *formal Nilsson solutions*, which is an extension of the regular case. To that end, we introduce a suitable notion of *weight vector* $w \in \mathbb{R}^n_{\geq 0}$. The space of formal Nilsson solutions is then established and we prove its relation with an associated regular system in Section 3.2, using the operation of *homogenization*. When the parameters are generic, the action of the homogenization becomes more clear and we will be able, in Section 3.3, to calculate in combinatorial terms the dimension of the space of formal Nilsson solutions as well as to give an explicit basis of it. For general parameters, we study homogenization of logarithm-free formal Nilsson solutions in Section 3.4. Finally, in Section 3.5 we use the developed tools to give the mentioned alternative proof of the result of Schulze and Walther.

3.1 Initial ideals and formal Nilsson series

The following definition characterizes the weight vectors we consider in this and in the following chapter. Recall that a cone is *strongly convex* if it contains no lines.

Definition 3.1.1. *A vector $w \in \mathbb{R}^n_{>0}$ is a **weight vector** for $H_A(\beta)$ if there exists a strongly convex open rational polyhedral cone $\mathscr{C}$, $\mathscr{C} \setminus \{0\} \subset \mathbb{R}^n_{>0}$, with $w \in \mathscr{C}$, such that, for all $w' \in \mathscr{C}$, we have*

$$\operatorname{in}_w(I_A) = \operatorname{in}_{w'}(I_A) \quad \textit{and} \quad \operatorname{in}_{(-w,w)}(H_A(\beta)) = \operatorname{in}_{(-w',w')}(H_A(\beta)).$$

In particular, $\operatorname{in}_w(I_A)$ *is a monomial ideal.*

It follows from the existence of the Gröbner fan (see [MR88] for the commutative version, and [ACG00] for the situation in the Weyl algebra) that weight vectors form an open dense subset of $\mathbb{R}^n_{>0}$. For an introduction to the theory of Gröbner bases in the Weyl algebra, we refer to Chapters 1 and 2 of [SST00].

Remark 3.1.2. *In the case when the cone spanned by the columns of A is strongly convex, the assumption that weight vectors have positive coordinates is not necessary. To see this, select $h \in \mathbb{R}^d$ such that the vector $h \cdot A$ has all positive entries. (This is the definition of strong convexity for the cone over the columns of A.) If $w \in \mathbb{R}^n$, choose a positive number λ such that $w' = w + \lambda h \cdot A$ is coordinatewise positive. We claim that w and w' define the same initial ideal for $H_A(\beta)$ (and in particular, for I_A).*

The reason our claim is true is that the ideal $H_A(\beta)$ is homogeneous with respect to the $\mathbb{Z}^d$-grading in the Weyl algebra defined by $\deg(x^u\partial^v) = A \cdot (v-u)$.

We need to verify that, if $f \in D$ is A-homogeneous, then $\mathrm{in}_{(-w,w)}(f) = \mathrm{in}_{(-w',w')}(f)$. *Write $f = \sum c_{uv} x^u \partial^v$. Being A-homogeneous means that the vectors $A \cdot (v-u)$ for $c_{uv} \neq 0$ are all the same. But*

$$-w' \cdot u + w' \cdot v = -w \cdot u + w \cdot v + \lambda\,[\,h \cdot A \cdot (-u+v)\,].$$

Since f is A-homogeneous, using w' instead of w simply adds a constant to the weights of the terms in f, from which it follows that the initial forms with respect to w and w' coincide.

A special case of this phenomenon occurs when $(1,\dots,1)$ belongs to the rowspan of A, which happens if and only if the toric ideal I_A is homogeneous with respect to the usual $\mathbb{Z}$-grading of the polynomial ring $\mathbb{C}[\partial]$.

Our aim is to define a notion of formal solutions to the system $H_A(\beta)$ which includes the space of canonical solutions described in Theorem 1.3.16. We have set in Definition 3.1.1 stronger conditions for the weight vector w with respect to which the solutions are considered. Now we define the space of formal Nilsson series solutions of $H_A(\beta)$ associated to a weight vector w, which we denote $\mathscr{N}_w(H_A(\beta))$.

Recall that if w is a weight vector for $H_A(\beta)$, and $\mathscr{C}$ is an open cone as in Definition 3.1.1, the *polar cone* $\mathscr{C}^*$ consisting of elements $u \in \mathbb{R}^n$ such that $u \cdot w' \geq 0$ for all $w' \in \mathscr{C}$ is strongly convex. Moreover, for any nonzero $u \in \mathscr{C}^*$ and any $w' \in \mathscr{C}$, we have $u \cdot w' > 0$.

Definition 3.1.3. *Let w be a weight vector for $H_A(\beta)$. A formal solution ϕ of $H_A(\beta)$ that has the form*

$$\phi = \sum_{u \in C} x^{v+u} p_u(\log(x)), \quad \textit{where } v \in \mathbb{C}, \tag{3.1}$$

and satisfies

1. *C is contained in $\mathscr{C}^* \cap \ker_{\mathbb{Z}}(A)$, where $\mathscr{C}$ is an open cone as in Definition 3.1.1,*
2. *The p_u are polynomials, and there exists $K \in \mathbb{Z}$ such that $\deg(p_u) \leq K$ for all $u \in C$,*
3. *$p_0 \neq 0$,*

*is called a **basic Nilsson solution of** $H_A(\beta)$ **with respect to** w. The set*

$$\mathrm{supp}(\phi) = \{u \in C \mid p_u \neq 0\} \subset \ker_{\mathbb{Z}}(A)$$

*is called the **support** of ϕ.*

*The $\mathbb{C}$-span of the basic Nilsson solutions of $H_A(\beta)$ with respect to w is called the **space of formal Nilsson series solutions of** $H_A(\beta)$ **with respect to** w and is denoted by $\mathscr{N}_w(H_A(\beta))$.*

Remark 3.1.4. *Given a weight vector w, we may replace the first requirement in Definition 3.1.3 by either of the following equivalent conditions:*

i) $C \subset \ker_{\mathbb{Z}}(A)$ and there exists an open neighborhood U of w such that, for all $w' \in U$ and all $u \in C \smallsetminus \{0\}$, we have $w' \cdot u > 0$.

ii) There exist $\mathbb{R}$-linearly independent $\gamma_1, \ldots, \gamma_n \in \mathbb{Q}^n$ with $w \cdot \gamma_i > 0$ for all $i = 1, \ldots, n$, such that $C \subset \big(\mathbb{R}_{\geq 0}\gamma_1 + \cdots + \mathbb{R}_{\geq 0}\gamma_n\big) \cap \ker_{\mathbb{Z}}(A)$.

Remark 3.1.5. *If w is a weight vector for $H_A(\beta)$ and $\mathscr{C}$ is a strongly convex open cone as in Definition 3.1.1, then for any $w' \in \mathscr{C}$, the exponents of $H_A(\beta)$ with respect to w and w' coincide. Moreover, the basic Nilsson solutions of $H_A(\beta)$ with respect to w and w' are the same, and therefore $\mathscr{N}_w(H_A(\beta)) = \mathscr{N}_{w'}(H_A(\beta))$.*

Lemma 3.1.6. *If ϕ is a basic Nilsson solution of $H_A(\beta)$ with respect to w as in* (3.1)*, then v is an exponent of $H_A(\beta)$ with respect to w.*

Proof. Since w is a weight vector, $w \cdot u > 0$ for all $u \in \mathscr{C}^*$. This implies that $\mathrm{in}_w(\phi) = x^v p_0(\log(x))$. Thus, $x^v p_0(\log(x))$ is a solution of $\mathrm{in}_{(-w,w)}(H_A(\beta))$ by Theorem 1.3.9. But then x^v is a solution of the initial ideal $\mathrm{in}_{(-w,w)}(H_A(\beta))$ by [SST00, Theorems 2.3.3(2) and 2.3.11]. □

Compare the following definition with Definition 1.3.17.

Definition 3.1.7. *Solutions of $H_A(\beta)$ of the form* (3.1) *that satisfy the first two conditions in Definition 3.1.3 are called series solution of $H_A(\beta)$ **in the direction of** w. The $\mathbb{C}$-span of all such series is called the **space of series solutions of** $H_A(\beta)$ **in the direction of** w.*

Most of the considerations about the ring N_w can be made also on $\mathscr{N}_w(H_A(\beta))$. We can consider the term order (1.18) and manipulate the series in the same way. This is a version of Lemma 1.3.10 that clearly holds for $\mathscr{N}_w(H_A(\beta))$:

Lemma 3.1.8. *Let $\phi_1, \ldots, \phi_k \in \mathscr{N}_w(H_A(\beta))$.*

1. *If the initial series $\mathrm{in}_w(\phi_1), \ldots, \mathrm{in}_w(\phi_k)$ are $\mathbb{C}$-linearly independent, then $\phi_1, \ldots, \phi_k$ are $\mathbb{C}$-linearly independent.*

2. *If $\phi_1, \ldots, \phi_k$ are $\mathbb{C}$-linearly independent, there exists a $k \times k$ complex matrix (λ_{ij}) such that the initial series of $\psi_i = \sum_{j=1}^{k} \lambda_{ij}\phi_j$ for $i = 1, \ldots, k$ are $\mathbb{C}$-linearly independent.*

We can compare the dimension of the space of formal Nilsson solutions of $H_A(\beta)$ with respect to w with the holonomic rank of the associated initial ideal.

Proposition 3.1.9. *Let w be a weight vector for $H_A(\beta)$. Then*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) \leq \mathrm{rank}(\mathrm{in}_{(-w,w)}(H_A(\beta))). \tag{3.2}$$

Proof. Choose $\psi_1, \ldots, \psi_k$ linearly independent elements of $\mathscr{N}_w(H_A(\beta))$. The second part of Lemma 3.1.8 allows us to assume that the series $\mathrm{in}_w(\psi_1), \ldots, \mathrm{in}_w(\psi_k)$ are linearly independent solutions of the initial system $\mathrm{in}_{(-w,w)}(H_A(\beta))$. These initial series have a non empty common open domain of convergence since they have a finite number of terms. Therefore $\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta)))$ cannot exceed the holonomic rank of $\mathrm{in}_{(-w,w)}(H_A(\beta))$. □

We show in Corollary 3.3.12 that this inequality is, in fact, an equality for generic β. If I_A is a homogeneous ideal, $\dim_{\mathbb{C}} \mathscr{N}_w(H_A(\beta))$, $\mathrm{rank}(\mathrm{in}_{(-w,w)}(H_A(\beta)))$, and $\mathrm{rank}(H_A(\beta))$ are the same (see Proposition 3.1.10 below). However, if I_A is not homogeneous, we will prove in Corollary 3.3.13 that $\mathrm{rank}(\mathrm{in}_{(-w,w)}(H_A(\beta)))$ does not always equal $\mathrm{rank}(H_A(\beta))$.

Finally, we show that this notion of formal solution coincide with the known space of solutions for the regular case.

Proposition 3.1.10. *If the row span of A contains the vector $(1, \ldots, 1)$ and $w \in \mathbb{R}^n$ is a weight vector then the space generated by the canonical series coincides with $\mathscr{N}_w(H_A(\beta))$. In particular*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \mathrm{rank}(H_A(\beta)). \tag{3.3}$$

Proof. By Theorem 1.4.2 we know that $H_A(\beta)$ is regular holonomic. Then, by Proposition 3.1.9 and Theorem 1.3.11 we have that

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) \leq \mathrm{rank}(H_A(\beta)). \tag{3.4}$$

On the other hand, any canonical solution of $H_A(\beta)$ is, by definition, a basic solution of $H_A(\beta)$. This means that the space spanned by the canonical series solutions of $H_A(\beta)$, whose dimension is $\mathrm{rank}(H_A(\beta))$, is less or equal that $\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta)))$. This concludes the proof. □

3.2 Homogenization of formal Nilsson solutions of $H_A(\beta)$

The goal of this section is to obtain the solutions of the system $H_A(\beta)$ by solving a related hypergeometric system that is regular holonomic. For generic parameters, this idea was used in other works, such as [OT09]; here, we require no genericity hypotheses on β. The key concept is that of homogenization.

Notation 3.2.1. *Throughout this and the following chapter, the letter ρ is used to indicate the homogenization of various objects: polynomials, ideals, and later on, Nilsson series.*

If $f \in \mathbb{C}[\partial_1, \ldots, \partial_n]$ is a polynomial, we denote by $\rho(f) \in \mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$ its homogenization, that is,

$$f = \sum_{u \in \mathbb{N}^n} c_u \partial^u \Longrightarrow \rho(f) = \sum_{u \in \mathbb{N}^n} c_u \partial_0^{\deg(f)-|u|} \partial^u, \quad |u| = u_1 + \cdots + u_n.$$

If $I \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]$ is an ideal, then

$$\rho(I) = \langle \rho(f) \mid f \in I \rangle \subseteq \mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n].$$

If $A = [a_{ij}]$ is a $d \times n$ integer matrix, then $\rho(A) \in \mathbb{Z}^{(d+1)\times(n+1)}$ is obtained by attaching a column of zeros to the left of A, and then attaching a row of ones to the resulting matrix, namely

$$\rho(A) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{d1} & \dots & a_{dn} \end{pmatrix}.$$

Note that

$$\rho(I_A) = I_{\rho(A)}.$$

Let w be a weight vector for $H_A(\beta)$ and fix $\beta_0 \in \mathbb{C}$. Let $H_{\rho(A)}(\beta_0, \beta)$ be the (regular holonomic) hypergeometric system associated to the matrix $\rho(A)$ from Notation 3.2.1 and the vector $(\beta_0, \beta) \in \mathbb{C}^{n+1}$. Since the set of weight vectors for $H_{\rho(A)}(\beta_0, \beta)$ is an open dense subset of $\mathbb{R}^{n+1}_{>0}$, there exists $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}$ and $\varepsilon_0 > 0$ such that $(0, w) + \varepsilon\alpha$ is a weight vector for $H_{\rho(A)}(\beta_0, \beta)$ for all $0 < \varepsilon < \varepsilon_0$. But then $(0, w) + \varepsilon\alpha - \varepsilon\alpha_0(1, \dots, 1)$ is also a weight vector for $H_{\rho(A)}(\beta_0, \beta)$, because $I_{\rho(A)}$ is homogeneous (see Remark 3.1.2). If ε is sufficiently small, then $w' = w + \varepsilon((\alpha_1, \dots, \alpha_n) - \alpha_0(1, \dots, 1))$ belongs to the open cone $\mathscr{C}$ from Definition 3.1.1. This means that we can use w' instead of w as weight vector for $H_A(\beta)$, with the same open cone, initial ideals, and basic Nilsson solutions as w, and guarantee that $(0, w')$ is a weight vector for $H_{\rho(A)}(\beta_0, \beta)$.

This justifies assuming, as we do from now on, that any time we choose a weight vector w for $H_A(\beta)$, the vector $(0, w)$ is a weight vector for $H_{\rho(A)}(\beta_0, \beta)$.

We have chosen a weight vector w, and we wish to use the auxiliary system $H_{\rho(A)}(\beta_0, \beta)$ to study the solutions of $H_A(\beta)$. The matrix $\rho(A)$ is fixed, but we have freedom in the choice of the parameter $\beta_0 \in \mathbb{C}$, and it is convenient to assume that β_0 is generic. The correct notion of genericity for β_0 can be found in Definition 3.2.4. Under that hypothesis, our objective is to construct an injective linear map

$$\rho : \mathscr{N}_w(H_A(\beta)) \longrightarrow \mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0, \beta)), \tag{3.5}$$

whose image is described in Theorem 3.2.12. For some weight vectors, ρ is guaranteed to be surjective (Proposition 3.5.1). However, if the cone over the columns of A is strongly convex and I_A is not homogeneous, there always exist weights for which surjectivity fails (Proposition 3.5.4).

Let $\phi = \sum_{u \in C} x^{v+u} p_u(\log(x))$ be a basic Nilsson solution of $H_A(\beta)$ with respect to w as in (3.1). Since ϕ is annihilated by the Euler operators $E_1 - \beta_1, \dots, E_d - \beta_d$, the polynomials p_u appearing in ϕ belong to the symmetric algebra of the lattice $\ker_{\mathbb{Z}}(A)$ by [Sai02, Proposition 5.2]. More explicitly, let $\{\gamma_1, \dots, \gamma_{n-d}\} \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-basis of $\ker_{\mathbb{Z}}(A)$. Then we can write

$$p_u(t_1, \dots, t_n) = \sum_{\alpha \in \mathbb{N}^{n-d}} c_\alpha \prod_{j=1}^{n-d} (\gamma_j \cdot (t_1, \dots, t_n))^{\alpha_j}.$$

For such a p_u, define

$$\widehat{p_u}(t_0, t_1, \dots, t_n) = \sum_{\alpha \in \mathbb{N}^{n-d}} c_\alpha \prod_{j=1}^{n-d} ((-|\gamma_j|)\, t_0 + \gamma_j \cdot (t_1, \dots, t_n))^{\alpha_j}, \tag{3.6}$$

where, for any vector $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$, $|v| = \sum_{i=1}^n v_i$. Note that $\widehat{p_u}$ specializes to p_u when $x_0 = 1$, or equivalently, when $\log(x_0) = 0$.

The formal definition of the homogenization of the series $\phi = \sum_{u\in C} x^{v+u} p_u(\log(x))$ is

$$\rho(\phi) = \sum_{u\in C} \partial_0^{|u|} x_0^{\beta_0 - |v|} x^{v+u} \widehat{p_u}(\log(x_0), \ldots, \log(x_n)). \tag{3.7}$$

If $|u| \geq 0$ for all $u \in C$, the above formula makes sense, and it easily checked that $\rho(\phi)$ is a basic Nilsson solution of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$. The bulk of the work in this section concerns the definition and properties of the operator ∂_0^k when $k \in \mathbb{Z}_{<0}$.

We point out that there is one case when the elements of the supports of all basic Nilsson solutions of $H_A(\beta)$ with respect to w are guaranteed to have non negative coordinate sum, namely when the weight vector w is close to $(1, \ldots, 1)$. We make this notion precise in the following definition.

Definition 3.2.2. *Let w be weight vector for $H_A(\beta)$. We say that w is a* ***perturbation*** *of $w_0 \in \mathbb{R}^n_{>0}$ if there exists an open cone $\mathscr{C}$ as in Definition 3.1.1 with $w \in \mathscr{C}$, such that w_0 lies in the closure of $\mathscr{C}$.*

Suppose that $\phi = x^v \sum_{u\in C} x^u p_u(\log(x))$ is a basic Nilsson solution of $H_A(\beta)$ with respect to a weight vector w which is a perturbation of $(1, \ldots, 1)$. Since $u \in C$ implies $u \in \mathscr{C}^* = (\overline{\mathscr{C}})^*$, we have $u \cdot w \geq 0$ for all $u \in C$. But then, as $C \subset \mathbb{Z}^n$ and w is a perturbation of $(1, \ldots, 1)$, it follows that $|u| = u \cdot (1, \ldots, 1) \geq 0$ for all $u \in C$. Thus, the operator $\partial_0^{|u|}$ is defined, and so is (3.7).

As we mentioned before, in order to work with other weight vectors, we need to define the operator ∂_0^k when k is negative.

Recall that $\mathrm{in}_{(0,w)}(I_{\rho(A)})$ is a monomial ideal, as $(0, w)$ is a weight vector for $H_{\rho(A)}(\beta_0, \beta)$. A standard pair of $\mathrm{in}_{(0,w)}(I_{\rho(A)})$ is said to **pass through zero** if $0 \in \sigma$.

Definition 3.2.3. *We say that a basic Nilsson solution of $H_A(\beta)$ as in* (3.1) *is* ***associated*** *to the standard pair $(\partial^\alpha, \sigma)$ if v is the (fake) exponent corresponding to this standard pair.*

The following definition gives the correct notion of genericity for $\beta_0 \in \mathbb{C}$ so that we can study the solutions of $H_A(\beta)$ using those of $H_{\rho(A)}(\beta_0, \beta)$.

Definition 3.2.4. *Let w be a weight vector for $H_A(\beta)$. We say that β_0 is a* ***homogenizing value*** *for A, β, and w if $\beta_0 \notin \mathbb{Z}$ and for any fake exponent v of $H_A(\beta)$ with respect to w, we have $v_0 = \beta_0 - \sum_{j=1}^n v_j \notin \mathbb{Z}$.*

Given a weight vector w for $H_A(\beta)$, we fix a homogenizing value β_0 for A, β, and w. Let ϕ as in (3.1) be a basic Nilsson solution of $H_A(\beta)$ with respect to w. We want to construct a basic Nilsson solution $\rho(\phi)$ of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$.

The following lemma tells us how to differentiate logarithmic terms.

Lemma 3.2.5. *[Sai02, Lemma 5.3] Let h be a polynomial in r variables, $\nu \in \mathbb{N}^r$ and $s \in \mathbb{C}^r$. Then*

$$\partial^\nu x^s h(\log(x)) = x^{s-\nu} \left(\sum_{0 \leq \nu' \leq \nu} \lambda_{\nu'} \left[\partial^{\nu - \nu'} h \right] (\log(x)) \right),$$

where the sum is over nonnegative integer vectors ν' that are coordinatewise smaller than ν, and the $\lambda_{\nu'}$ are complex numbers. □

Lemma 3.2.6. *Let $\hat{p}$ be a polynomial in $n+1$ variables, and $s \in \mathbb{C}^n$. If $s_0 \in \mathbb{C}$ and $s_0 \neq -1$, there exists a unique polynomial $\hat{q}$ with $\deg(\hat{q}) = \deg(\hat{p})$ such that*

$$\partial_0 \big[x_0^{s_0+1} x^s \hat{q}(\log(x_0), \dots, \log(x_n))\big] = x_0^{s_0} x^s \hat{p}(\log(x_0), \dots, \log(x_n)). \tag{3.8}$$

Proof. Writing

$$\hat{p}(\log(x)) = \sum_{i=0}^{k} p_i(\log(x_1), \dots, \log(x_n)) \log(x_0)^i$$

and

$$\hat{q}(\log(x)) = \sum_{i=0}^{k} q_i(\log(x_1), \dots, \log(x_n)) \log(x_0)^i,$$

we can equate coefficients in (3.8) to obtain

$$p_k = (s_0+1) q_k \ ; \quad p_i = (s_0+1) q_i + (i+1) q_{i+1} \quad 0 \leq i \leq k-1.$$

Therefore

$$q_i = \sum_{\ell=0}^{k-i} (-1)^\ell \frac{\prod_{j=1}^{\ell}(i+j)}{(s_0+1)^\ell} p_{i+\ell} \quad 0 \leq \ell \leq k,$$

where the empty product is defined to be 1. □

Definition 3.2.7. *With the notation of Lemma 3.2.6, define*

$$\partial_0^{-1} \big[x_0^{s_0} x^s \hat{p}(\log(x_0), \dots, \log(x_n))\big] = x_0^{s_0+1} x^s \hat{q}(\log(x_0), \dots, \log(x_n)).$$

Note that if $s_0 \neq -2, \dots, -k$, the construction of ∂_0^{-1} can be iterated $(k-1)$ times. We denote by $\partial_0^{-k}\big[x_0^{s_0} x^s \hat{p}(\log(x_0), \dots, \log(x_n))\big]$ the outcome of this procedure.

Lemma 3.2.8. *Use the same notation and hypotheses as in Lemma 3.2.6, and assume furthermore that $x_0^{s_0} x^s \hat{p}(\log(x))$ is a solution of $\langle E_0 - \beta_0, E - \beta\rangle$. Then $\partial_0^{-1}[x_0^{s_0} x^s \hat{p}(\log(x))]$ is a solution of $\langle E_0 - (\beta_0+1), E-\beta\rangle$. If $s_0 \neq -2, \dots, -k$, then $\partial_0^{-k}[x_0^{s_0} x^s \hat{p}(\log(x))]$ is a solution of the system $\langle E_0 - (\beta_0+k), E-\beta\rangle$.*

Proof. If $i > 0$, $\partial_0(E_i - \beta_i) = (E_i - \beta_i)\partial_0$, so that

$$\begin{aligned} \partial_0(E_i - \beta_i)(\partial_0^{-1} x_0^{s_0} x^s \hat{p}(\log(x))) &= (E_i - \beta_i)\partial_0(\partial_0^{-1} x_0^{s_0} x^s \hat{p}(\log(x))) \\ &= (E_i - \beta_i) x_0^{s_0} x^s \hat{p}(\log(x)) \\ &= 0. \end{aligned}$$

This means that $(E_i - \beta_i)(\partial_0^{-1} x_0^{s_0} x^s \hat{p}(\log(x)))$ is constant with respect to x_0. On the other hand, $s_0 + 1 \neq 0$ and $(E_i - \beta_i)(\partial_0^{-1} x_0^{s_0} x^s \hat{p}(\log(x)))$ is a multiple of x^{s_0+1}. Thus, in order to be constant with respect to x_0, $(E_i - \beta_i)(\partial_0^{-1} x_0^{s_0} x^s \hat{p}(\log(x)))$ must vanish. For $i=0$, the argument is similar since $\partial_0(E_0 - (\beta_0+1)) = (E_0 - \beta_0)\partial_0$. The last assertion follows by induction on k. □

Lemma 3.2.9. *Let $\nu \in \mathbb{N}^n$, $k \in \mathbb{Z}$, and assume that $s_0 \notin \mathbb{Z}$. Then*

$$\partial^\nu \left[\partial_0^k \left[x_0^{s_0} x^s p(\log(x_0), \ldots, \log(x_n))\right]\right] = \partial_0^k \left[\partial^\nu \left[x_0^{s_0} x^s p(\log(x_0), \ldots, \log(x_n))\right]\right].$$

Proof. This is clear if $k \geq 0$. For $k < 0$, the result follows by induction from the uniqueness part of Lemma 3.2.6. □

We are now ready to define the homogenization of a basic Nilsson solution of $H_A(\beta)$.

Definition 3.2.10. *Let $\phi = x^v \sum_{u \in C} x^u p_u(\log(x))$ be a basic Nilsson solution of $H_A(\beta)$ with respect to a weight vector w, and let $\beta_0 \in \mathbb{C}$ a homogenizing value for A, β and w, so that $v_0 = \beta_0 - \sum_{i=1}^n v_i$ is not an integer. Thus, we may define*

$$\rho(\phi) = \sum_{u \in C} \partial_0^{|u|} \left[x_0^{v_0} x^{v+u} \widehat{p_u}(\log(x_0), \log(x_1), \ldots, \log(x_n))\right],$$

where $\widehat{p_u}$ is obtained from p_u as in (3.6).

Proposition 3.2.11. *Let w be a weight vector with respect to $H_A(\beta)$ (which, by assumption implies that $(0, w)$ is a weight vector for $H_{\rho(A)}(\beta_0, \beta)$, where β_0 is our fixed homogenizing value for A, β and w). For any basic Nilsson solution $\phi = x^v \sum x^u p_u(\log(x))$ of $H_A(\beta)$ with respect to w, the (formal) series $\rho(\phi)$ from Definition 3.2.10 is a basic Nilsson solution of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$. We extend ρ linearly to obtain a map*

$$\rho : \mathscr{N}_w(H_A(\beta)) \to \mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0, \beta)).$$

Proof. We first show that the series $\rho(\phi)$ has the shape required in Definition 3.1.3. Conditions 2 and 3 are clearly satisfied by the construction of the polynomials $\hat{p}_u$ and Lemma 3.2.5. Thus, we need to verify that $\rho(\phi)$ satisfies condition (i) from Remark 3.1.4. The support of ϕ is in bijection with the support of $\rho(\phi)$ via $u \mapsto (-|u|, u)$, which sends $\ker_{\mathbb{Z}}(A)$ into $\ker_{\mathbb{Z}}(\rho(A))$. We can assume that C is the intersection of $\ker_{\mathbb{Z}}(A)$ with the dual $\mathscr{C}^*$ of an open cone $\mathscr{C}$ such that its closure $\overline{\mathscr{C}}$ is a strongly convex rational polyhedral cone of maximal dimension. Let $\{\gamma_1, \ldots, \gamma_m\}$ be a Hilbert basis of $\mathscr{C}^* \cap \ker_{\mathbb{Z}}(A) = (\overline{\mathscr{C}})^* \cap \ker_{\mathbb{Z}}(A)$. Then $w' \cdot \gamma_i > 0$ for all $w' \in \mathscr{C}$ and all $i = 1, \ldots, m$. Let $\delta > 0$ such that, for all $\varepsilon \in \mathbb{R}^{n+1}_{>0}$ whose Euclidean distance to the origin is $||\varepsilon|| < \delta$, and all $i = 1, \ldots, m$, we have $\left[(0, w) + \varepsilon\right] \cdot (-|\gamma_i|, \gamma_i) > 0$. It follows that for any non zero $u \in C$ and any $\tilde{w}$ in the ball centered at $(0, w)$ with radius δ, we have $\tilde{w} \cdot (-|u|, u) > 0$, which proves our claim.

Now we prove that $\rho(\phi)$ is a formal solution of $H_{\rho(A)}(\beta_0, \beta)$. Since $\widehat{p_u}$ belongs to the symmetric algebra of $\ker_{\mathbb{Z}}(\rho(A))$, and

$$\rho(A) \cdot (v_0, v + u) = (\beta_0 - |v| + |v| + |u|, A \cdot (v + u)) = (\beta_0 + |u|, \beta),$$

the term $x_0^{v_0} x^{v+u} \widehat{p_u}(\log(x_0), \log(x_1), \ldots, \log(x_n))$ is a solution of the system of Euler operators $\langle E_0 - (\beta_0 + |u|), E_1 - \beta_1, \ldots, E_d - \beta_d \rangle$. By Lemma 3.2.8 with $(s_0, s) = (v_0, v+u)$ and $k = -|u|$, each term of $\rho(\phi)$ is therefore a solution of $\langle E - (\beta_0 + |u| - |u|), E_1 - \beta_1, \ldots, E_d - \beta_d \rangle$.

To verify that the elements of $I_{\rho(A)}$ annihilate $\rho(\phi)$, first note that Lemma 3.2.5 implies that for any $\mu \in \ker_{\mathbb{Z}}(A)$

$$\partial^{\mu_+} x^{v+u} p_u(\log(x)) = \partial^{\mu_-} x^{v+u-\mu} p_{u-\mu}(\log(x)),$$

because ϕ is a solution of $H_A(\beta)$.

We claim that

$$\partial^{\mu_+} x_0^{v_0} x^{v+u} \widehat{p_u}(\log(x)) = \partial^{\mu_-} x_0^{v_0} x^{v+u-\mu} \widehat{p_{u-\mu}}(\log(x)).$$

To see this, use Lemma 3.2.5 and the fact that, if $i > 0$ and p is an element of the symmetric algebra of $\ker_{\mathbb{Z}}(A)$, then $\partial_i p$ is also in the symmetric algebra of $\ker_{\mathbb{Z}}(A)$, and $\partial_i \widehat{p} = \widehat{\partial_i p}$.

Now, using Lemma 3.2.9 with $(s_0, s) = (v_0, v + u)$ and $k = |u|$, and the fact that $v_0 = \beta_0 - |v| \notin \mathbb{Z}$, we obtain from

$$\partial_0^{|u|} \partial^{\mu_+} x_0^{v_0} x^{v+u} \widehat{p_u}(\log(x)) = \partial_0^{|\mu|} \partial_0^{|u|-|\mu|} \partial^{\mu_-} x_0^{v_0} x^{v+u-\mu} \widehat{p_{u-\mu}}(\log(x)),$$

that

$$\partial^{\mu_+} \left[\partial_0^{|u|} x_0^{v_0} x^{v+u} \widehat{p_u}(\log(x))\right] = \partial_0^{|\mu|} \partial^{\mu_-} \left[\partial_0^{|u|-|\mu|} x_0^{v_0} x^{v+u-\mu} \widehat{p_{u-\mu}}(\log(x))\right].$$

Assuming $|\mu| > 0$, we conclude

$$\partial^{(-|\mu|,\mu)_+} \left[\partial_0^{|u|} x_0^{v_0} x^{v+u} \widehat{p_u}(\log(x))\right] = \partial^{(-|\mu|,\mu)_-} \left[\partial_0^{|u|-|\mu|} x_0^{v_0} x^{v+u-\mu} \widehat{p_{u-\mu}}(\log(x))\right].$$

□

We linearly extend ρ to $\mathscr{N}_w(H_A(\beta))$, and show that this map identifies $\mathscr{N}_w(H_A(\beta))$ with a subspace of $\mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0, \beta))$. Therefore, the inverse map ρ^{-1} allows us to obtain Nilsson solutions of $H_A(\beta)$ from Nilsson solutions of the regular holonomic system $H_{\rho(A)}(\beta_0, \beta)$.

Theorem 3.2.12. *Let w be a weight vector for $H_A(\beta)$ and β_0 a homogenizing value for A, β and w. The linear map*

$$\rho : \mathscr{N}_w(H_A(\beta)) \to \mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0, \beta))$$

is injective and its image is spanned by basic Nilsson solutions of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$ associated to standard pairs of $\mathrm{in}_{(0,w)}(I_{\rho(A)})$ that pass through zero.

Proof. If $\phi = \sum_{u \in C} x^{v+u} p_u(\log(x))$ is a basic Nilsson solution of $H_A(\beta)$ with respect to w, then we have $\mathrm{in}_{(0,w)}(\rho(\phi)) = x_0^{v_0} x^v \widehat{p_0}(\log(x_0), \ldots, \log(x_n))$. Choose a basis of $\mathscr{N}_w(H_A(\beta))$ consisting of basic Nilsson series whose initial terms are linearly independent (use the second part of Lemma 3.1.8). Then the initial series of their images are also linearly independent, as $\hat{p}(1, \log(x)) = p(\log(x))$. Now apply the first part of Lemma 3.1.8 to complete the proof that ρ is injective.

Observe that, by construction, $\rho(\mathscr{N}_w(H_A(\beta)))$ is contained in the span of the basic Nilsson solutions of $H_{\rho(A)}(\beta_0, \beta)$ corresponding to standard pairs that pass through zero, because the powers of x_0 appearing in $\rho(\phi)$ are non integer for any basic Nilsson solution ϕ of $H_A(\beta)$.

To show the other inclusion, let ψ be a basic Nilsson solution of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$ corresponding to a standard pair that passes through zero, with starting exponent $(\beta_0 - |v|, v)$. We wish to prove that ψ can be dehomogenized. We can write

$$\psi = x_0^{\beta_0 - |v|} x^v \sum_{(-|u|,u) \in \ker_{\mathbb{Z}}(\rho(A))} x_0^{-|u|} x^u h_u(\log(x)),$$

where $\beta_0 - |v|$ is not an integer because β_0 is a homogenizing value for A, β and w. Then we can perform

$$\partial_0^{|u|}\left(x_0^{\beta_0-|v|-|u|}x^{v+u}h_u(\log(x))\right) = x_0^{\beta_0-|v|}x^{v+u}\widehat{p_u}(\log(x))$$

and use this to define $\phi = \sum x^{v+u}p_u(\log(x))$ (with the same relation between p and $\widehat{p}$ as in (3.6)). We claim that ϕ is a basic Nilsson solution of $H_A(\beta)$ and $\psi = \rho(\phi)$. The proof of this claim is a reversal of the arguments in Proposition 3.2.11. □

Definition 3.2.13. *For a series $\psi \in \mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0, \beta))$ with $\phi \in \mathscr{N}_w(H_A(\beta))$ such that $\rho(\phi) = \psi$, we call ϕ a **dehomogenized Nilsson series**, or say that ϕ is the **dehomogenization** of ψ.*

3.3 Hypergeometric Nilsson series for generic parameters

When the parameter vector β is sufficiently generic (see Convention 3.3.2), the Nilsson solutions of $H_A(\beta)$ are completely determined by the combinatorics of the initial ideals of I_A. The goal of this section is to study this case in detail.

In order to precisely describe the genericity condition used in this section, we restate the description of $\mathrm{in}_{(-w,w)}(H_A(\beta))$ made in Section 1.4.2 for generic parameter vectors.

Lemma 3.3.1. *For β generic,*

$$\mathrm{in}_{(-w,w)}(H_A(\beta)) = \mathrm{in}_w(I_A) + \langle E - \beta\rangle.$$

Therefore, all the fake exponents of $H_A(\beta)$ with respect to w are actual exponents.

Moreover, under suitable genericity conditions for β, a better description of the initial ideal $\mathrm{in}_{(-w,w)}(H_A(\beta))$ is available, namely

$$\mathrm{in}_{(-w,w)}(H_A(\beta)) = \mathrm{top}(\mathrm{in}_w(I_A)) + \langle E - \beta\rangle.$$

Proof. This is a version of [SST00, Theorem 3.1.3 and Theorem 3.2.11] for non homogeneous toric ideals, which holds with the same proof, since I_A is always A-graded. □

Convention 3.3.2. *In this section, we assume that β is generic enough that the second displayed formula in Lemma 3.3.1 is satisfied, so that all exponents of $H_A(\beta)$ with respect to w come from top-dimensional standard pairs of $\mathrm{in}_w(I_A)$.*

Moreover, we require that the only integer coordinates of these exponents are the ones imposed by the corresponding standard pairs. In particular, the exponents of $H_A(\beta)$ with respect to w have no negative integer coordinates.

Finally, we ask that no two exponents differ by an integer vector. Note that these integrality conditions force us to avoid an infinite (but locally finite) collection of affine spaces.

The series ϕ_v from Proposition 1.4.21 (which is proved without assumptions on A) is our “model” of solution in the generic case.

Proposition 3.3.3. *If v is a fake exponent with respect to a weight vector w of $H_A(\beta)$, then ϕ_v is a basic Nilsson solution of $H_A(\beta)$.*

Proof. Note that $u \in \operatorname{supp}(\phi_v)$ implies $u \cdot w \geq 0$ as is shown in the proof of [SST00, Theorem 3.4.2]. Since v is a fake exponent of $H_A(\beta)$ with respect to any w' in an open neighborhood of w, we have in fact that $u \cdot w > 0$ for any nonzero u in the support of ϕ_v. This argument shows that ϕ_v satisfies the first requirement from Definition 3.1.3. The remaining conditions are readily verified. □

In this section, we describe the space $\mathscr{N}_w(H_A(\beta))$ using the homogenization map ρ defined in Section 3.2. An explicit basis for $\mathscr{N}_w(H_A(\beta))$ is constructed using the exponents of the ideal $H_A(\beta)$. Our first step is to relate the exponents of $H_A(\beta)$ with respect to w to the exponents of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$.

The following is a well known result, whose proof we include for the sake of completeness.

Lemma 3.3.4. *Let $I \subset \mathbb{C}[\partial_1, \ldots, \partial_n]$ be an ideal and let $\rho(I) \subset \mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$ be its homogenization. Let $w \in \mathbb{R}^n_{\geq 0}$ sufficiently generic so that $\operatorname{in}_w(I)$ and $\operatorname{in}_{(0,w)}(\rho(I))$ are both monomial ideals. Suppose that*

$$\operatorname{in}_{(0,w)}(\rho(I)) = \bigcap Q_i$$

is a primary decomposition of the monomial ideal $\operatorname{in}_{(0,w)}(\rho(I))$. Then

$$\operatorname{in}_w(I) = \bigcap_{\partial_0 \notin \sqrt{Q_i}} \langle f(1, \partial_1, \ldots, \partial_n) \mid f \in Q_i \rangle$$

is a primary decomposition of the monomial ideal $\operatorname{in}_w(I)$.

Proof. Let $f \in I$ such that $\operatorname{in}_w(f)$ is a monomial. Then $\operatorname{in}_{(0,w)}(\rho(f)) = \partial_0^h \operatorname{in}_w(f)$ for some $h \in \mathbb{N}$, with $h = 0$ if f is homogeneous. Therefore $\operatorname{in}_w(I)$ is obtained by setting $\partial_0 \mapsto 1$ in the generators of $\operatorname{in}_{(0,w)}(\rho(I))$. Now the result follows by observing that if Q is primary monomial ideal one of whose generators is divisible by ∂_0, then Q must contain a power of ∂_0 as a minimal generator. □

Notation 3.3.5. *Given $\beta_0 \in \mathbb{C}$ and $v \in \mathbb{C}^n$, we set*

$$\rho_{\beta_0}(v) = (\beta_0 - \sum_{i=1}^{n} v_i, v) = (\beta_0 - |v|, v).$$

In particular, $\rho_0 = \rho$ maps $\ker_{\mathbb{Z}}(A)$ to $\ker_{\mathbb{Z}}(\rho(A))$.

Lemma 3.3.6. *Let β_0, β generic and w a weight vector for $H_A(\beta)$. Then the map $v \mapsto \rho_{\beta_0}(v)$ is a bijection between the exponents of $H_A(\beta)$ with respect to w and the exponents of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$ associated to standard pairs that pass through zero.*

Proof. By Lemma 3.3.4, v is the exponent of $H_A(\beta)$ with respect to w corresponding to a standard pair $(\partial^\alpha, \sigma)$, if and only if $\rho_{\beta_0}(v)$ is the exponent of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$ corresponding to the standard pair $(\partial^\alpha, \{0\} \cup \sigma)$. □

The following result is immediate.

Lemma 3.3.7. *Let β generic, w a weight vector for $H_A(\beta)$ and β_0 a homogenizing value for A, β and w. Let v be an exponent of $H_A(\beta)$ with respect to w and consider the map ρ from Proposition 3.2.11. Then*

$$\rho(\phi_v) = \phi_{(\rho_{\beta_0}(v),v)},$$

where $\phi_v, \phi_{(\rho_{\beta_0}(v),v)}$ are as in Proposition 1.4.21. □

We now come to the main result in this section.

Theorem 3.3.8. *Let β generic and w a weight vector for $H_A(\beta)$. Then*

$$\{\phi_v \mid v \text{ is an exponent of } H_A(\beta) \text{ with respect to } w\}$$

is a basis for $\mathscr{N}_w(H_A(\beta))$.

Proof. Fix a homogenizing value β_0 for A, β and w, and let $\psi \in \mathscr{N}_w(H_A(\beta))$. Then, by Theorem 3.2.12, Then $\rho(\psi) \in \mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0, \beta))$. Since $I_{\rho(A)}$ is homogeneous, $H_{\rho(A)}(\beta_0, \beta)$ is regular holonomic, and we can use Lemma 3.3.7 and the results from the previous subsection to write

$$\rho(\psi) = \sum c_v \phi_{(\rho_{\beta_0}(v),v)} = \sum c_v \rho(\phi_v) \tag{3.9}$$

where the sum is over the exponents of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$ corresponding to standard pairs that pass through zero, and the c_v are complex numbers. By Lemma 3.3.6, the sum is over the exponents of $H_A(\beta)$ with respect to w. But ρ is injective, so (3.9) implies

$$\psi = \sum c_v \phi_v.$$

Thus $\mathscr{N}_w(H_A(\beta))$ is contained in the $\mathbb{C}$-span of the series ϕ_v associated to the exponents of $H_A(\beta)$ with respect to w. Since the series ϕ_v are basic Nilsson solutions of $H_A(\beta)$ with respect to w (Proposition 1.4.21), the reverse inclusion follows. Linear independence is proved using Lemma 3.1.8. □

Recall that we have assumed that, if w is a weight vector for $H_A(\beta)$, then $(0, w)$ is a weight vector for $H_{\rho(A)}(\beta_0, \beta)$. In particular, this implies that the subdivision of $\rho(A)$ induced by $(0, w)$ is in fact a triangulation [Stu96, Chapter 8].

Corollary 3.3.9. *Assume $\beta \in \mathbb{C}^d$ is generic and let w be a weight vector for $H_A(\beta)$. The dimension of the space of Nilsson solutions of $H_A(\beta)$ with respect to w is*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \sum_{\substack{\sigma \text{ facet of } \Delta_{(0,w)} \\ \text{such that } 0 \in \sigma}} \operatorname{vol}(\sigma).$$

Proof. By Theorem 3.3.8, $\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta)))$ is the number of exponents of $H_A(\beta)$ with respect to w, which is the number of top dimensional standard pairs of $\operatorname{in}_w(I_A)$, because β is generic. Using the bijection from Lemma 3.3.6, we conclude that $\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta)))$ is the number of top dimensional standard pairs of $\operatorname{in}_{(0,w)}(I_{\rho(A)})$ that pass through zero. Given $\sigma \subset \{0, 1, \ldots, n\}$ of cardinality $d+1$ such that $0 \in \sigma$, the number of top dimensional standard pairs of $\operatorname{in}_{(0,w)}(I_{\rho(A)})$ of the form $(\partial^\alpha, \sigma)$ is the multiplicity of $\langle \partial_i \mid i \notin \sigma \rangle$ as an associated prime of $\operatorname{in}_{(0,w)}(I_{\rho(A)})$ by [STV95, Lemma 3.3]. This number equals the normalized volume of the simplex $\{0\} \cup \sigma$ by [Stu96, Theorem 8.8], and the result follows. □

Proposition 3.3.10. *Suppose that β is generic, and let w be a weight vector for $H_A(\beta)$. Then*

$$\operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta)) = \deg(\operatorname{in}_w(I_A)).$$

Proof. By Lemma 3.3.1, $\operatorname{in}_{(-w,w)}(H_A(\beta)) = \operatorname{top}(\operatorname{in}_w(I_A)) + \langle E-\beta\rangle$. Since $\operatorname{top}(\operatorname{in}_w(I_A))$ is a monomial ideal, $\operatorname{in}_{(-w,w)}(H_A(\beta))$ and $\langle x^u\partial^u \mid \partial^u \in \operatorname{top}(\operatorname{in}_w(I_A))\rangle + \langle E-\beta\rangle$ have the same holomorphic solutions.

We denote $x_i\partial_i = \theta_i$, and observe that $\mathbb{C}[\theta_1,\dots,\theta_n]$ is a commutative polynomial subring of D. Also recall that the Euler operators $E_i-\beta_i$ belong to $\mathbb{C}[\theta]$. Since $x^u\partial^u = \prod_{i=1}^n\prod_{j=0}^{u_i-1}(\theta_i - j)$, [SST00, Proposition 2.3.6] can be applied to conclude that

$$\operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta))) = \dim_{\mathbb{C}}\left(\frac{\mathbb{C}[\theta]}{\langle\prod_{i=1}^n\prod_{j=0}^{u_i-1}(\theta_i-j)\mid \partial^u\in\operatorname{top}(\operatorname{in}_w(I_A))\rangle + \langle E-\beta\rangle}\right). \tag{3.10}$$

Considered as a system of polynomial equations in n variables, the zero set of $\langle\prod_{i=1}^n\prod_{j=0}^{u_i-1}(\theta_i - j)\mid \partial^u\in\operatorname{top}(\operatorname{in}_w(I_A))\rangle$ is a subvariety of $\mathbb{C}^n$ consisting of $\deg(\operatorname{in}_w(I_A))$ irreducible components, each of which is a translate of a d-dimensional coordinate subspace of $\mathbb{C}^n$. By [SST00, Corollary 3.2.9], each of these components meets the zero set of $\langle E-\beta\rangle$ in exactly one point. Therefore the dimension in the right hand side of (3.10) equals $\deg(\operatorname{in}_w(I_A))$, and the proof is complete. □

Corollary 3.3.11. *Let β generic and w a weight vector for $H_A(\beta)$. Then*

$$\operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta))) = \sum_{\substack{\sigma \text{ facet of } \Delta_{(0,w)}\\ \text{such that } 0\in\sigma}} \operatorname{vol}(\sigma), \tag{3.11}$$

Proof. We need to show that the sum on the right hand side of (3.11) equals $\deg(\operatorname{in}_w(I_A))$. This degree equals the number of top dimensional standard pairs of $\operatorname{in}_w(I_A)$ by [STV95, Lemma 3.3], which equals the number of top dimensional standard pairs of $\operatorname{in}_{(0,w)}(I_{\rho(A)})$ passing through zero by Lemma 3.3.6. As in the proof of Corollary 3.3.9, the number of such standard pairs is the desired sum. □

Corollary 3.3.12. *Suppose that β is generic and w is a weight vector for $H_A(\beta)$. Then*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta))).$$

Proof. Immediate from Corollary 3.3.9 and Corollary 3.3.11. □

The following corollary states that, for certain weight vectors, the dimension of $\mathscr{N}_w(H_A(\beta))$ equals $\operatorname{rank}(H_A(\beta))$. However, this fails in general, as Example 3.3.14 shows. This means that, as expected, formal Nilsson series are not enough to understand the solutions of irregular hypergeometric systems.

Corollary 3.3.13. *Suppose that β is generic, and w is a weight vector for $H_A(\beta)$. The equality*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \operatorname{vol}(A) = \operatorname{rank}(H_A(\beta))$$

holds if and only if 0 belongs to every maximal simplex in the triangulation $\Delta_{(0,w)}$ of $\rho(A)$.

Proof. Note that $\mathrm{vol}(A) = \mathrm{vol}(\rho(A))$, which is the sum of the volumes of all the maximal simplices in $\Delta_{(0,w)}$. Therefore $\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \mathrm{vol}(A)$ if and only if all maximal simplices in $\Delta_{(0,w)}$ pass through zero. Now use a result of Adolphson [Ado94] that, for generic β, $\mathrm{rank}(H_A(\beta)) = \mathrm{vol}(A)$. □

Example 3.3.14. *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ so that } I_A = \langle \partial_3 - \partial_1\partial_2 \rangle .$$

Then $\mathrm{vol}(\rho(A)) = \mathrm{vol}(A) = 2$ *is the generic rank of both* $H_A(\beta)$ *and* $H_{\rho(A)}(\beta_0, \beta)$*.*

If w *is a perturbation of* $(1,1,1)$*, we have* $\mathrm{in}_w(I_A) = \langle \partial_1\partial_2 \rangle$ *and the corresponding triangulation is* $\Delta_{(0,w)} = \{\{0,1,3\},\{0,2,3\}\}$*. In this case*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \mathrm{vol}(\{0,1,3\}) + \mathrm{vol}(\{0,2,3\}) = 2 = \mathrm{rank}(H_A(\beta)).$$

On the other hand, if w *is a perturbation of* $(1,1,3)$*, then we have* $\mathrm{in}_w(I_A) = \langle \partial_3 \rangle$ *and the corresponding triangulation is* $\Delta_{(0,w)} = \{\{0,1,2\},\{1,2,3\}\}$*. In this case*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \mathrm{vol}(\{0,1,2\}) = 1 < \mathrm{rank}(H_A(\beta)).$$

3.4 Logarithm-free Nilsson series

If we assume that β is generic, then all of the Nilsson solutions of $H_A(\beta)$ are automatically logarithm-free. We now turn our attention to the logarithm-free Nilsson solutions of $H_A(\beta)$ without any assumptions on the parameter β.

Definition 3.4.1. *For a vector* $v \in \mathbb{C}^n$*, its* ***negative support*** *is the set of indices*

$$\mathrm{nsupp}(v) = \big\{ i \in \{1,\dots,n\} \mid v_i \in \mathbb{Z}_{<0} \big\}.$$

A vector $v \in \mathbb{C}^n$ *has* ***minimal negative support*** *if* $\mathrm{nsupp}(v)$ *does not properly contain the set* $\mathrm{nsupp}(v+u)$ *for any nonzero* $u \in \ker_{\mathbb{Z}}(A)$*. We denote*

$$N_v = \{ u \in \ker_{\mathbb{Z}}(A) \mid \mathrm{nsupp}(u+v) = \mathrm{nsupp}(v) \}. \tag{3.12}$$

When β is arbitrary, the fake exponents of $H_A(\beta)$ with respect to a weight vector w can have negative integer coordinates. For such a v, we wish to construct an associated basic Nilsson solution of $H_A(\beta)$, in the same way as we did in Proposition 1.4.21.

Proposition 3.4.2. *Let* w *be a weight vector for* $H_A(\beta)$ *and let* $v \in \mathbb{C}^n$ *be a fake exponent of* $H_A(\beta)$ *with respect to* w*. The series*

$$\phi_v = \sum_{u \in N_v} \frac{[v]_{u_-}}{[u+v]_{u_+}} x^{u+v} , \tag{3.13}$$

where $[v]_{u_-}$ *and* $[u+v]_{u_+}$ *are as in (1.30), is well defined. This series is a formal solution of* $H_A(\beta)$ *if and only if* v *has minimal negative support, and in that case,* ϕ_v *is a basic Nilsson solution of* $H_A(\beta)$ *with respect to* w*. Consequently,* v *is an exponent of* $H_A(\beta)$ *with respect to* w*.*

Proof. The series is well defined because, as the summation is over N_v, there cannot be any zeros in the denominators of the summands. The second assertion holds with the same proof as Proposition 3.4.13 of [SST00]. To see that ϕ_v is a basic Nilsson solution of $H_A(\beta)$, we can argue in the same way as in the proof of Proposition 1.4.21. $\square$

Lemma 3.4.12 of [SST00] shows that if the negative support of v is empty, then equations (3.13) and (1.32) coincide.

We now consider Nilsson series in the direction of a weight vector (see Definition 3.1.7). The $\mathbb{C}$-vector space of *logarithm-free* formal A-hypergeometric series with parameter β in the direction of w is denoted by $\mathscr{S}_w(H_A(\beta))$.

Theorem 3.4.3. *Let w be a weight vector for $H_A(\beta)$. The set*

$$\{\phi_v \mid v \textit{ is an exponent of } H_A(\beta) \textit{ with minimal negative support}\} \tag{3.14}$$

is a basis for $\mathscr{S}_w(H_A(\beta))$.

The previous result was stated in [Sai02, Display (7)], in the special case when I_A is homogeneous. Its proof for the case when $\beta \in \mathbb{Z}^d$ appeared in [CDR11, Proposition 4.2]; we generalize that argument here.

Proof. Linear independence of the proposed basis elements follows from Lemma 3.1.8, so we need only show that these series span $\mathscr{S}_w(H_A(\beta))$.

Let $G(x) \in \mathscr{S}_w(H_A(\beta))$, and suppose that x^ν appears in G with nonzero coefficient $\lambda_\nu \in \mathbb{C}$. We claim that ν has minimal negative support. By contradiction, let $u \in \ker_{\mathbb{Z}}(A)$ such that $\operatorname{nsupp}(\nu+u)$ is strictly contained in $\operatorname{nsupp}(\nu)$. This means that there is $1 \leq i \leq n$ such that $\nu_i \in \mathbb{Z}_{<0}$ and $\nu_i + u_i \in \mathbb{N}$. In particular, $u_i > 0$.

Since G is a solution of $H_A(\beta)$, the operator $\partial^{u_+} - \partial^{u_-} \in I_A$ annihilates G. Note that $\operatorname{nsupp}(\nu+u) \subset \operatorname{nsupp}(\nu)$ implies that $\partial^{u_-}x^\nu \neq 0$. Then some term from $\partial^{u_+}G$ needs to equal $\lambda_v\partial^{u_-}x^\nu$, which is a nonzero multiple of $x^{\nu-u_-}$. But any function f such that $\partial^{u_+}f = x^{\nu-u_-}$ must involve $\log(x_i)$. This produces the desired contradiction.

Fix ν such that x^ν appears with nonzero coefficient λ_ν in G, and let ψ be the subseries of G consisting of terms of the form $\lambda_{\nu+u}x^{\nu+u}$ with $u \in \ker_{\mathbb{Z}}(A)$ and $\lambda_{\nu+u} \in \mathbb{C}$, such that $\operatorname{nsupp}(\nu+u) = \operatorname{nsupp}(\nu)$. Our goal is to show that ψ is a multiple of one of the series from (3.14). This will conclude the proof.

We claim that ψ is a solution of $H_A(\beta)$. That the Euler operators $\langle E-\beta\rangle$ annihilate ψ follows since they annihilate every term of G. To deal with the toric operators, recall that $\partial^{u_+}G = \partial^{u_-}G$ for all $u \in \ker_{\mathbb{Z}}(A)$. But terms in $\partial^{u_+}G$ that come from ψ can only be matched by terms in $\partial^{u_-}G$ that also come from ψ, so $\partial^{u_+} - \partial^{u_-}$ must annihilate ψ, for all $u \in \ker_{\mathbb{Z}}(A)$.

Since G is a solution of $H_A(\beta)$ in the direction of w, so is ψ. This implies that $\operatorname{in}_w(\psi)$ is a logarithm-free solution of $\operatorname{in}_{(-w,w)}(H_A(\beta))$, and therefore, by [SST00, Theorems 2.3.9 and 2.3.11], $\operatorname{in}_w(\psi)$ is a linear combination of (finitely many) monomial functions arising from exponents of $H_A(\beta)$ with respect to w. By construction of ψ, these exponents differ by elements of $\ker_{\mathbb{Z}}(A)$. Arguing as in the proof of [SST00, Theorem 3.4.14], we see that $\operatorname{in}_w(\psi)$ can only have one term, that is, $\operatorname{in}_w(\psi) = \lambda_v x^v$ where v is an exponent of $H_A(\beta)$ with respect to w and $\lambda_v \neq 0$. Since $\lambda_v x^v$ is a term in G, v has minimal negative support. Thus, v is an exponent of $H_A(\beta)$ with respect to w that has minimal negative support.

To finish the proof, we show that $\psi = \lambda_v \phi_v$. Suppose that $u \in N_v$, which means that $u \in \ker_{\mathbb{Z}}(A)$ and $\operatorname{nsupp}(v+u) = \operatorname{nsupp}(v)$. The equality of negative supports implies that $\partial^{u_-} x^v = [v]_{u_-} x^{v-u_-}$ is nonzero. Since $\partial^{u_-}\psi = \partial^{u_+}\psi$, $\partial^{u_+}\psi$ must contain the term $\lambda_v [v]_{u_-} x^{v-u_-}$, which can only come from $\partial^{u_+}\lambda_{v+u} x^{v+u}$. Thus

$$\lambda_v [v]_{u_-} x^{v-u_-} = \partial^{u_+}\lambda_{v+u} x^{v+u} = \lambda_{v+u}[v+u]_{u_+} x^{v+u-u_+}.$$

Consequently $\lambda_{v+u} = \lambda_v \frac{[v]_{u_-}}{[v+u]_{u_+}}$, that is, the coefficient of x^{v+u} in ψ equals λ_v times the coefficient of x^{v+u} in ϕ_v. Therefore $\psi = \lambda_v \phi_v$, as we wanted. □

The next theorem gives a bijective map between the space of logarithm-free series solutions of $H_A(\beta)$ in the direction of w and a subspace of the logarithm-free solutions of $H_{\rho(A)}(\beta_0, \beta)$ in the direction of $(0, w)$.

Theorem 3.4.4. *Let w be a weight vector for $H_A(\beta)$ and let β_0 be a homogenizing value for A, β, and w. Then $\rho(\mathscr{S}_w(H_A(\beta)))$ equals the $\mathbb{C}$-linear span of the logarithm-free basic Nilsson solutions of $H_{\rho(A)}(\beta_0, \beta)$ which are associated to standard pairs with respect to $(0, w)$ that pass through zero.*

Proof. Since ρ is linear, we only need to consider the image of the elements of a basis for the space $\mathscr{S}_w(H_A(\beta))$, such as the one given in Theorem 3.4.3.

We claim that if v is a fake exponent of $H_A(\beta)$ with respect to w that has minimal negative support, then $\rho_{\beta_0}(v) = (\beta_0 - |v|, v)$ is a fake exponent of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$ corresponding to a standard pair that passes though zero, and moreover, $\rho_{\beta_0}(v)$ has minimal negative support. The first part is proved using Lemma 3.3.4. To see that $\rho_{\beta_0}(v)$ has minimal negative support, first recall that $\beta_0 - |v| \notin \mathbb{Z}$ because β_0 is a homogenizing value for A, β and w. This implies that 0 is not in the negative support of $\rho_{\beta_0}(v) + \mu$, for any $\mu \in \ker_{\mathbb{Z}}(\rho(A))$. Now use the bijection $u \mapsto (-|u|, u)$ between $\ker_{\mathbb{Z}}(A)$ and $\ker_{\mathbb{Z}}(\rho(A))$ and the fact that v has minimal negative support, to conclude that $\rho_{\beta_0}(v)$ has minimal negative support.

To complete the proof, we show that $\rho(\phi_v) = \phi_{\rho_{\beta_0}(v)}$. Lemma 3.3.7 is this statement in the case when $\operatorname{nsupp}(v) = \emptyset$, but now we have to pay attention to the supports of these series.

The same argument we used to check that $\rho_{\beta_0}(v)$ has minimal negative support yields $N_v = \pi(N_{\rho_{\beta_0}(v)})$, where π is the projection onto the last n coordinates, and therefore $\rho(\phi_v)$ and $\phi_{\rho_{\beta_0}(v)}$ have the same support. The verification that the corresponding coefficients are the same is straightforward. □

3.5 The irregularity of $H_A(\beta)$ via its Nilsson solutions

In this section, we give an alternative proof of [SW08, Corollary 3.16] using our study of Nilsson solutions of $H_A(\beta)$. We assume as in [SW08] that the columns of A span a strongly convex cone.

In Theorem 3.2.12 we computed the image of ρ. There is one case in which this map is guaranteed to be onto.

Proposition 3.5.1. *Suppose that w is a perturbation of $(1, \dots, 1)$. Then the map ρ is surjective.*

Proof. First note that none of the minimal generators of the monomial ideal $\mathrm{in}_{(0,w)}(I_{\rho(A)})$ have ∂_0 as a factor. This implies that, if $u \in \mathbb{N}^{n+1}$ and $\partial^u \notin \mathrm{in}_{(0,w)}(I_{\rho(A)})$, then $\partial_0^k \partial^u \notin \mathrm{in}_{(0,w)}(I_{\rho(A)})$ for all $k \in \mathbb{N}$. We conclude that all the standard pairs of $\mathrm{in}_{(0,w)}(I_{\rho(A)})$ pass through zero. Theorem 3.2.12 completes the proof. □

We wish to find weight a vector w for $H_A(\beta)$ for which ρ is not surjective. We require the following statement.

Lemma 3.5.2. *Let $A \in \mathbb{Z}^{d\times n}$ of full rank d whose columns span a strongly convex cone. If the row span of A does not contain the vector $(1,\dots,1)$, there exists $w \in \mathbb{R}^n_{>0}$ such that the coherent triangulation $\Delta_{(0,w)}$ of $\rho(A)$ has a maximal simplex that does not pass through zero. Given $\beta \in \mathbb{C}^d$, the vector w can be chosen to be a weight vector for $H_A(\beta)$.*

Proof. We use the description of the secondary fan of $\rho(A)$ from Section.

Let B be a Gale dual matrix of $\rho(A)$ with rows $b_0,\dots,b_n$. Since $(1,\dots,1)$ is not in the rowspan of A, the zeroth row of B is nonzero. Because B has full rank $n-d$, we can choose $\sigma \subset \{1,\dots n\}$ of cardinality $d+1$ such that $\{b_i \mid i \notin \sigma\}$ is linearly independent.

The assumption that the columns $a_1,\dots,a_n$ of A span a strongly convex cone means that there exists a vector $h \in \mathbb{R}^d$ such that $h\cdot A$ is coordinatewise positive. As $\rho(A)\cdot B = 0$, $\sum_{i=1}^n (h\cdot a_i) b_i = 0$.

Choose $w \in \mathbb{R}^n_{>0}$ and positive real λ_i for $i \notin \sigma$, such that

$$w_i + \lambda_0 = h\cdot a_i \ \text{ for } i \in \sigma\,, \quad \text{and } w_i + \lambda_0 - \lambda_i = h\cdot a_i \ \text{ for } i \notin \sigma\cup\{0\}.$$

There is enough freedom in the choice of w that we may assume that $(0,w)$ induces a triangulation $\Delta_{(0,w)}$ of $\rho(A)$ and not merely a subdivision. This also implies that w can be chosen a weight vector for $H_A(\beta)$, if $\beta \in \mathbb{C}^d$ is given.

We claim that $(0,w)\cdot B \in \mathscr{K}_\sigma$. This implies that σ is a maximal simplex in $\Delta_{(0,w)}$ which does not pass through zero.

To prove the claim, note that

$$\sum_{i\in\sigma}(w_i+\lambda_0)b_i + \sum_{i\notin\sigma\cup\{0\}}(w_i+\lambda_0-\lambda_i)b_i = \sum_{i=1}^n (h\cdot a_i)b_i = 0.$$

Then

$$\sum_{i=1}^n w_i b_i = \sum_{i\notin\sigma\cup\{0\}}(\lambda_i-\lambda_0)b_i - \lambda_0\sum_{i\in\sigma} b_i = \sum_{i\notin\sigma\cup\{0\}}\lambda_i b_i - \lambda_0\sum_{i=1}^n b_i = \sum_{i\notin\sigma}\lambda_i b_i,$$

where the last equality follows from $-b_0 = \sum_{i=1}^n b_i$. But then, $\lambda_i > 0$ for $i \notin \sigma$ implies that $(0,w)\cdot B \in \mathscr{K}_\sigma$, which is what we wanted. □

The hypothesis that the columns of A span a strongly convex cone cannot be removed from Lemma 3.5.2, as the following example shows.

Example 3.5.3. *Let $A = (-1\ 1)$. Then*

$$\rho(A) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \text{ and choose } B = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

There are only two coherent triangulations of $\rho(A)$, namely

$$\Delta_1 = \{\{-1,1\}\} \text{ and } \Delta_2 = \{\{-1,0\},\{0,1\}\}.$$

Their corresponding cones in the secondary fan are $\mathscr{K}_{\Delta_1} = \mathbb{R}_{\leq 0}$ and $\mathscr{K}_{\Delta_2} = \mathbb{R}_{\geq 0}$. For any vector $w = (w_{-1}, w_1) \in \mathbb{R}^2_{>0}$, the number $0 \cdot b_0 + w_{-1} \cdot b_{-1} + w_1 \cdot b_1 = w_{-1} + w_1$ belongs to the cone $\mathscr{K}_{\Delta_2} = \mathbb{R}_{\geq 0}$ and consequently w always induces a triangulation of $\rho(A)$ all of whose maximal simplices pass through zero.

Proposition 3.5.4. *Assume that the columns of A span a strongly convex cone. If the row span of A does not contain the vector $(1,\ldots,1)$, there exists a weight vector w such that, the linear map*

$$\rho : \mathscr{N}_w(H_A(\beta)) \to \mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0,\beta)),$$

where β_0 is a homogenizing value for A, β and w, is not surjective.

Proof. Use Lemma 3.5.2 to pick w so that the triangulation $\Delta_{(0,w)}$ of $\rho(A)$ has a maximal simplex that does not pass through zero. Then $\mathrm{in}_{(0,w)}(I_{\rho(A)})$ has top dimensional standard pairs that do not pass through zero. Choose a homogenizing value β_0 for A, β and w, and let (v_0, v) be a fake exponent of $H_{\rho(A)}(\beta_0,\beta)$ corresponding to such a standard pair (fake exponents associated to top-dimensional standard pairs always exist). In particular, $v_0 \in \mathbb{N}$. If (v_0, v) has minimal negative support, it is an exponent of $H_{\rho(A)}(\beta_0,\beta)$ corresponding to a standard pair that does not pass through zero, and the associated logarithm-free solution $\phi_{(v_0,v)}$ of $H_{\rho(A)}(\beta_0,\beta)$ cannot belong to the image of ρ by Theorem 3.2.12.

If (v_0, v) does not have minimal negative support, the argument given in [SST00, Proposition 3.4.16] produces an element $(v_0', v') \in ((v_0,v) + \ker_{\mathbb{Z}}(A)) \cap \mathrm{Minex}_{\rho(A),(\beta_0,\beta),(0,w)}$ whose negative support is strictly contained in that of (v_0, v), so that v_0' is still a non negative integer. Thus, the standard pair corresponding to (v_0', v_0) cannot pass through zero, and $\phi_{(v_0',v')}$ is not in the image of ρ. $\square$

The following result is due to Berkesch [Ber10, Theorem 7.3].

Theorem 3.5.5. *If the cone over the columns of A is strongly convex and β_0 is generic,*

$$\mathrm{rank}(H_A(\beta)) = \mathrm{rank}(H_{\rho(A)}(\beta_0,\beta)).$$

We are finally ready to show that, in the case when the columns of A span a strongly convex cone, non homogeneous A-hypergeometric systems are irregular for all β, thus generalizing the argument in [SST00, Theorem 2.4.11], and providing an alternative proof of [SW08, Corollary 3.16].

Theorem 3.5.6. *Assume that the columns of A span a strongly convex cone and I_A is not homogeneous. Then $H_A(\beta)$ is not regular holonomic for any $\beta \in \mathbb{C}^d$.*

Proof. Choose w a weight vector for $H_A(\beta)$ as in Proposition 3.5.4 and β_0 a homogenizing value for A, β and w. Assume that $H_A(\beta)$ is regular holonomic. Then, by Proposition 3.1.10 it holds that

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \operatorname{rank}(H_A(\beta)).$$

We have

$$\begin{aligned}\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) &< \dim_{\mathbb{C}}(\mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0,\beta)))\\ &= \operatorname{rank}(H_{\rho(A)}(\beta_0,\beta))\\ &= \operatorname{rank}(H_A(\beta)).\end{aligned}$$

The equality in the second line follows again from Proposition 3.1.10 because the system $H_{\rho(A)}(\beta_0,\beta)$ is regular holonomic, as $I_{\rho(A)}$ is homogeneous. The last equality is by Theorem 3.5.5 (we may have to make β_0 more generic for this result to hold, but this does not affect Proposition 3.5.4). We obtain $\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) < \operatorname{rank}(H_A(\beta))$, a contradiction. Then $H_A(\beta)$ is not regular holonomic. $\square$

Chapter 4

Convergence of hypergeometric Nilsson series in the irregular case

Until now, we have made no convergence considerations in our study of $\mathscr{N}_w(H_A(\beta))$. The purpose of this chapter is to investigate convergence issues in detail. In particular, Theorem 4.1.2 states that, if w is a perturbation of $(1,\ldots,1)$, the elements of $\mathscr{N}_w(H_A(\beta))$ have a common domain of convergence. Moreover, assuming that the cone spanned by the columns of A is strongly convex, results from Section 3.5 imply that $\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \operatorname{rank}(H_A(\beta))$. This provides an explicit construction for the space of (multivalued) holomorphic solutions of $H_A(\beta)$ in a particular open subset of $\mathbb{C}^n$.

When the parameter β is generic and w is a perturbation of $(1,\ldots,1)$, the convergence of the elements of $\mathscr{N}_w(H_A(\beta))$ was shown in [OT09]. In Subsection 4.2 we complete this study by considering other weight vectors.

Notation 4.0.7. *We have already used the notation $|\cdot|$ to mean the coordinate sum of a vector. When applied to a monomial, such as x^u, $|\cdot|$ means complex absolute value. Let w be a weight vector for $H_A(\beta)$ and let $\{\gamma_1,\ldots,\gamma_{n-d}\} \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-basis for $\ker_{\mathbb{Z}}(A)$ such that $\gamma_i \cdot w > 0$ for $i=1,\ldots,n-d$. For any $\boldsymbol{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_{n-d}) \in \mathbb{R}^{n-d}_{>0}$, we have defined in Subsection 1.4.3 the (non empty) open set*

$$\mathscr{U}_{w,\boldsymbol{\varepsilon}} = \{x \in \mathbb{C}^n \mid |x^{\gamma_i}| < \varepsilon_i \text{ for } i=1,\ldots,n-d\}. \tag{4.1}$$

4.1 General parameters

The following result is the main technical tool in this section.

Theorem 4.1.1. *Let w be a weight vector for $H_A(\beta)$ and let $\{\gamma_1,\ldots,\gamma_{n-d}\}$ be a basis for $\ker_{\mathbb{Z}}(A)$ such that $w\cdot\gamma_i > 0$ for $i=1,\ldots,n-d$. Let $\phi = \sum x^{v+u} p_u(\log(x))$ be a basic Nilsson solution of $H_A(\beta)$ as in* (3.1)*, such that $|u| \geq 0$ for almost all $u \in \operatorname{supp}(\phi)$, meaning that the set $\{u \in \operatorname{supp}(\phi) \mid |u| < 0\}$ is finite. Then there exists $\boldsymbol{\varepsilon} \in \mathbb{R}^{n-d}_{>0}$ such that ϕ converges in the open set $\mathscr{U}_{w,\boldsymbol{\varepsilon}}$.*

Proof. We may assume without loss of generality that $|u| \geq 0$ for all $u \in \operatorname{supp}(\phi)$. Choose β_0 a homogenizing value for A, β and w, and recall from Definition 3.2.10 that the homogenization of ϕ is

$$\rho(\phi) = \sum_{u \in \operatorname{supp}(\phi)} \partial_0^{|u|} \left[x_0^{\beta_0 - |v|} x^{v+u} \widehat{p_u}(\log(x_0), \log(x_1), \ldots, \log(x_n)) \right],$$

where p_u and $\widehat{p_u}$ are related by (3.6). By Theorem 3.2.11, $\rho(\phi)$ is a basic Nilsson solution of $H_{\rho(A)}(\beta_0, \beta)$ with respect to $(0, w)$. Since $H_{\rho(A)}(\beta_0, \beta)$ has regular singularities, [SST00, Theorem 2.5.16] implies that there exists $\varepsilon \in \mathbb{R}^{n-d}_{>0}$ such that $\rho(\phi)$ converges (absolutely) in the open set

$$\mathscr{U}_{(0,w),\varepsilon} = \{(x_0, x) \in \mathbb{C}^{n+1} \mid |x_0^{-|\gamma_i|} x^{\gamma_i}| < \varepsilon_i, i = 1, \ldots, n-d\}.$$

We make use of the convergence of $\rho(\phi)$ to prove convergence for ϕ.

As $\rho(\phi)$ converges absolutely in $\mathscr{U}_{(0,w),\varepsilon}$, convergence is preserved when we reorder terms. Use the fact that $|u| \geq 0$ for all $u \in \operatorname{supp}(\phi)$ to rewrite

$$\rho(\phi) = \sum_{m=0}^{\infty} \partial_0^m \left[x_0^{\beta_0 - |v|} f_m \right]$$

where

$$f_m(x_0, \ldots, x_n) = \sum_{|u|=m} x^{v+u} \widehat{p_u}(\log(x_0), \log(x_1), \ldots, \log(x_n))$$

is a polynomial in $\log(x_0)$ whose coefficients are (multivalued) holomorphic functions of the n variables $x_1, \ldots, x_n$. Recall that, by Definition 3.1.3, there exists a positive integer K such that the degree of f_m in $\log(x_0)$ is less than or equal to K for all $m \in \mathbb{N}$. A key observation is that

$$\sum_{m=0}^{\infty} \left(x_0^{\beta_0 - |v| - m} f_m(x_0, x_1, \ldots, x_n) \right) \Big|_{x_0 = 1} = \phi(x_1, \ldots, x_n). \tag{4.2}$$

Since $\{1\} \times \mathscr{U}_{w,\varepsilon} \subset \mathscr{U}_{(0,w),\varepsilon}$, if we show that $\sum_{m=0}^{\infty} x_0^{\beta_0 - |v| - m} f_m$ converges absolutely on $\mathscr{U}_{(0,w),\varepsilon}$, the convergence of ϕ on $\mathscr{U}_{w,\varepsilon}$ will follow.

For $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, we denote the m-th descending factorial by

$$[\lambda]_m = \lambda(\lambda - 1) \ldots (\lambda - m + 1).$$

Set $\lambda = \beta_0 - |v|$. Since β_0 is a homogenizing value for A, β and w, we have $\lambda \notin \mathbb{Z}$. We claim that the domain of convergence of $\sum_{m=0}^{\infty} [\lambda]_m x_0^{\lambda - m} f_m$ contains $\mathscr{U}_{(0,w),\varepsilon}$. But if this is true, the convergence of $\sum_{m=0}^{\infty} x_0^{\lambda - m} f_m$ on $\mathscr{U}_{(0,w),\varepsilon}$ follows by comparison, since the absolute value of $[\lambda]_m$ grows like $(m-1)!$ as m goes to ∞. Thus, all we need to show in order to finish our proof is that $\sum_{m=0}^{\infty} [\lambda]_m x_0^{\lambda - m} f_m$ converges absolutely on $\mathscr{U}_{(0,w),\varepsilon}$.

Consider f_m as a polynomial in $\log(x_0)$. By construction, the coefficients of f_m are constant with respect to x_0. Denote by $f_m^{(r)}$ the r-th derivative of f_m with respect to $\log(x_0)$. Then $f_m^{(K+1)} = 0$ since the degree of f_m in $\log(x_0)$ is at most K. We compute $\partial_0^m (x_0^\lambda f_m)$ for $m \geq K$, using the fact that $f_m^{(r)} = 0$ if $r > K$.

$$\begin{aligned}
\partial_0^m (x_0^\lambda f_m) &= \partial_0^{m-1} \partial_0 (x_0^\lambda f_m) = \partial_0^{m-1} (x_0^{\lambda - 1} (\alpha f_m + f_m')) \\
&= \partial_0^{m-2} (x_0^{\lambda - 2} (\lambda(\lambda - 1) f_m + (\lambda + (\lambda - 1)) f_m' + f_m'')) \\
&= \cdots \\
&= x_0^{\lambda - m} (c_0(\lambda, m) f_m + c_1(\lambda, m) f_m' + \ldots + c_K(\lambda, m) f_m^{(K)}),
\end{aligned}$$

where

$$c_j(\lambda, m) = \sum_{1 \leq i_1 < \cdots < i_{m-j} \leq m} \prod_{k=1}^{j} (\lambda - i_k + 1).$$

Note that, as m goes to ∞, the dominant term in absolute value in $c_j(\lambda, m)$ is $\prod_{r=j}^{m-1}(\lambda - r + 1)$, which grows like $\prod_{r=j}^{m-1} r = \frac{(m-1)!}{(j-1)!}$. But then, if $j > 0$, $[\lambda - j]_m$ grows faster than $c_j(\alpha, m)$ as m goes to ∞, because $[\lambda - j]_m$ grows like $\frac{(m+j-1)!}{(j-1)!}$. In other words,

$$\lim_{m\to\infty} \frac{c_j(\lambda, m)}{[\lambda - j]_m} = 0 \quad \text{for } j \geq 1. \tag{4.3}$$

Since $\rho(\phi)$ converges absolutely on the open set $\mathscr{U}_{(0,w),\varepsilon}$, $\partial_0\rho(\phi)$ is also absolutely convergent on $\mathscr{U}_{(0,w),\varepsilon}$, and

$$\begin{aligned}
\partial_0\rho(\phi) &= \partial_0 \sum_{m=0}^{\infty} \partial_0^m \left[x_0^\lambda f_m\right] = \sum_{m=0}^{\infty} \partial_0^m \left[\partial_0 x_0^\lambda f_m\right] \\
&= \sum_{m=0}^{\infty} \partial_0^m \left[\lambda x_0^{\lambda-1} f_m + x_0^\lambda f_m'\right] x_0^{-1} \\
&= \lambda \sum_{m=0}^{\infty} \partial_0^m \left[x_0^{\lambda-1} f_m\right] + \sum_{m=0}^{\infty} \partial_0^m \left[x_0^{\lambda-1} f_m'\right].
\end{aligned}$$

The series $\sum_{m=0}^{\infty} \partial_0^m \left[x_0^{\lambda-1} f_m\right]$ converges in $\mathscr{U}_{(0,w),\varepsilon}$ because it is a basic Nilsson solution of the regular hypergeometric system $H_{\rho(A)}(\beta_0 - 1, \beta)$ with respect to $(0, w)$. (We may need to decrease ε coordinatewise for the previous assertion to hold.) This, and the convergence of $\partial_0(\rho(\phi))$, imply that $\sum_{m=0}^{\infty} \partial_0^m \left[x_0^{\lambda-1} f_m'\right]$ converges absolutely on $\mathscr{U}_{(0,w),\varepsilon}$. Proceeding by induction, we conclude that

$$\sum_{m=0}^{\infty} \partial_0^m x_0^{\lambda-j} f_m^{(j)} \quad \text{converges absolutely on } \mathscr{U}_{(0,w),\varepsilon} \text{ for } j = 1, \ldots, K. \tag{4.4}$$

Now we induct on $K - \ell$ to show that

$$\sum_{m=0}^{\infty} x_0^{\lambda-m-\ell}[\lambda - \ell]_m f_m^{(\ell)} \quad \text{converges absolutely on } \mathscr{U}_{(0,w),\varepsilon} \text{ for } \ell = 0, 1, \ldots, K. \tag{4.5}$$

The $\ell = 0$ case of this assertion is exactly what we needed to verify in order to finish the proof.

If $K - \ell = 0$, then $f_m^{(\ell)} = f_m^{(K)}$ is a (maybe zero) constant with respect to x_0, and therefore (4.5) is the $j = K$ case of (4.4). For the inductive step, compute the m-th derivative inside the series:

$$\begin{aligned}
\sum_{m=0}^{\infty} \partial_0^m x_0^{\lambda-\ell} f_m^{(\ell)} &= \sum_{m=0}^{\infty} x_0^{\lambda-\ell-m}([\lambda - \ell]_m f_m^{(\ell)} + c_1(\lambda - \ell, m) f_m^{(\ell+1)} + \cdots + c_{K-\ell}(\lambda - \ell, m) f_m^{(K)}) \\
&= \sum_{m=0}^{\infty} x_0^{\lambda-\ell-m}[\lambda - \ell]_m f_m^{(\ell)} \\
&\quad + \sum_{m=0}^{\infty} x_0^{\lambda-\ell-m} c_1(\lambda - \ell, m) f_m^{(\ell+1)} + \cdots + \sum_{m=0}^{\infty} x_0^{\lambda-\ell-m} c_{K-\ell}(\lambda - \ell, m) f_m^{(K)}.
\end{aligned}$$

We want to show that $\sum_{m=0}^{\infty} x_0^{\lambda-\ell-m}[\lambda-\ell]_m f_m^{(\ell)}$ converges absolutely on $\mathscr{U}_{(0,w),\varepsilon}$. We know that $\sum_{m=0}^{\infty} \partial_0^m x_0^{\lambda-\ell} f_m^{(\ell)}$ converges absolutely on $\mathscr{U}_{(0,w),\varepsilon}$ by (4.4), so we need to control the other summands. But the inductive hypothesis tells us that (4.5) is true for $\ell+1,\dots,K$. By comparison using (4.3), and harmlessly multiplying by x_0^j, we conclude that the series $\sum_{m=0}^{\infty} x_0^{\lambda-\ell-m} c_j(\lambda-\ell,m) f_m^{(\ell+j)}$ converges absolutely on $\mathscr{U}_{(0,w),\varepsilon}$ for $1 \leq j < K-\ell$. □

The following result gives the construction of a basis of series solutions of $H_A(\beta)$ that have a common domain of convergence, without any assumptions on β. While such constructions are well known in the regular case, when I_A is inhomogeneous, important theoretical tools become unavailable. A way of bypassing this difficulty is to assume that the parameters are generic and w is a perturbation of $(1,\dots,1)$ as in [OT09].

Theorem 4.1.2. *Assume that the cone over the columns of A is strongly convex, and let w be a weight vector for $H_A(\beta)$ that is a perturbation of $(1,\dots,1)$. Then*

$$\dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta))) = \operatorname{rank}(H_A(\beta))$$

and there exists $\varepsilon \in \mathbb{R}_{>0}^{n-d}$ such that every element of $\mathscr{N}_w(H_A(\beta))$ converges in the open set $\mathscr{U}_{w,\varepsilon}$.

Proof. Given β_0 a homogenizing value for A, β and w, Proposition 3.5.1 states that the spaces $\mathscr{N}_w(H_A(\beta)))$ and $\mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0,\beta)))$ are isomorphic. We may further assume that β_0 is sufficiently generic that Theorem 3.5.5 holds.

As $I_\rho(A)$ is homogeneous, $H_{\rho(A)}(\beta_0,\beta)$ is regular holonomic, and [SST00, Corollary 2.4.16] implies that $\dim_{\mathbb{C}}(\mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0,\beta))) = \operatorname{rank}(H_{\rho(A)}(\beta_0,\beta))$. Then

$$\begin{aligned} \dim_{\mathbb{C}}(\mathscr{N}_w(H_A(\beta)))) &= \dim_{\mathbb{C}}(\mathscr{N}_{(0,w)}(H_{\rho(A)}(\beta_0,\beta)))) \\ &= \operatorname{rank}(H_{\rho(A)}(\beta_0,\beta)) = \operatorname{rank}(H_A(\beta)), \end{aligned}$$

where the last equality is by Theorem 3.5.5.

Since w is a perturbation of $(1,\dots,1)$, if ϕ is a basic Nilsson solution of $H_A(\beta)$ with respect to w, then $|u| \geq 0$ for all $u \in \operatorname{supp}(\phi)$. Therefore we can use Theorem 4.1.1 to find $\varepsilon \in \mathbb{R}_{>0}^{n-d}$ such that all basic Nilsson solutions of $H_A(\beta)$ with respect to w converge on $\mathscr{U}_{w,\varepsilon}$. □

4.2 Generic parameters

In this subsection, we assume that β is generic as in Convention 3.3.2. In this case, by Theorem 3.3.8, the set

$$\mathscr{B}_w = \{\phi_v \mid v \text{ is an exponent of } H_A(\beta) \text{ with respect to } w\} \tag{4.6}$$

is a basis for $\mathscr{N}_w(H_A(\beta))$, and we can write the series ϕ_v is as in (1.32), by the genericity of the parameters.

We wish to determine which elements of $\mathscr{B}_w$ converge. This depends on the choice of the weight vector w.

Theorem 4.2.1. *Let β generic, w a weight vector for $H_A(\beta)$, and $\phi_v \in \mathscr{B}_w$. There exists $\varepsilon \in \mathbb{R}^{n-d}_{>0}$ such that ϕ_v converges in $\mathscr{U}_{w,\varepsilon}$ if and only if $|u| \geq 0$ for almost all $u \in \operatorname{supp}(\phi_v)$.*

Proof. If $|u| \geq 0$ for almost all $u \in \operatorname{supp}(\phi_v)$, then ϕ_v converges on an open set $\mathscr{U}_{w,\varepsilon}$ by Theorem 4.1.1.

Now assume that there exist an infinite number of elements $u \in \operatorname{supp}(\phi_v)$ such that $|u| < 0$. Using the description of $\operatorname{supp}(\phi_v)$ from (1.32), which applies when β is generic, we can find $\nu \in \operatorname{supp}(\phi_v)$ such that $|\nu| < 0$ and $\{m\nu \mid m \geq m_0 \in \mathbb{N}\} \subset \operatorname{supp}(\phi_v)$ for some $m_0 \in \mathbb{N}$.

Let ψ be the subseries of ϕ_v whose terms are indexed by the set $\{m\nu \mid m \geq m_0 \in \mathbb{N}\}$. The coefficient of $x^{v+m\nu}$ in ψ is

$$\frac{\prod_{\nu_i<0}\prod_{j=1}^{-m\nu_i}(v_i - j + 1)}{\prod_{\nu_i\geq 0}\prod_{j=1}^{m\nu_i}(v_i + j)},$$

which grows like $\lambda_m = \prod_{\nu_i<0}(-\nu_i m)!/\prod_{\nu_i>0}(\nu_i m)!$ as m goes to ∞. Since $|\nu| < 0$, $\lim_{m\to\infty}\lambda_m = \infty$. Therefore ψ cannot absolutely converge unless $x_i = 0$ for some i such that $\nu_i > 0$, and consequently ϕ_v does not have an open domain of convergence. □

Remark 4.2.2. *In this section we study convergence of Nilsson solutions of $H_A(\beta)$ with respect to a weight vector w. We can change the point of view and fix a basis $\{\gamma_1, \dots, \gamma_{n-d}\} \subset \mathbb{Z}^n$ of $\ker_{\mathbb{Z}}(A)$; then our results apply to any weight vector w such that $\gamma_i \cdot w > 0$ for $i = 1, \dots, n-d$.*

Since β is generic, all the information necessary to compute the Nilsson solutions of $H_A(\beta)$ associated to a weight vector w can be extracted from the top-dimensional standard pairs of $\operatorname{in}_{(0,w)}(I_{\rho(A)})$; the simplices appearing in these standard pairs are the maximal simplices of the coherent triangulation $\Delta_{(0,w)}$ of $\rho(A)$. These triangulations also control the possible regions of convergence of basic Nilsson solutions of $H_A(\beta)$ with respect to w, as a change in triangulation changes $\mathscr{U}_{w,\varepsilon}$. Recall from Section 2.1 that the Gale dual of a configuration parametrizes its coherent triangulations.

Given β generic, fix w a weight vector for $H_A(\beta)$. Then the supports of the basic Nilsson solutions of $H_A(\beta)$ with respect to w can be described by means of the cones associated to maximal simplices of the triangulation $\Delta_{(0,w)}$ of $\rho(A)$. Indeed, if (v_0, v) is the exponent associated to a standard pair $(\partial^\alpha, \sigma)$ of the monomial ideal $\operatorname{in}_{(0,w)}(I_{\rho(A)})$, so that $\sigma \in \Delta_{(0,w)}$ and $0 \in \sigma$, we know that the support of the dehomogenized series ϕ_v is

$$\operatorname{supp}(\phi_v) = \{u \in \ker_{\mathbb{Z}}(A) \mid u_i + v_i \geq 0 \;\; \forall i \notin \sigma\}.$$

Note that, as $0 \in \sigma$, the zeroth row of the Gale dual B of $\rho(A)$ is not present in this description. Since the columns of B span $\ker_{\mathbb{Z}}(\rho(A))$, the support of ϕ_v is naturally identified with

$$\operatorname{supp}(\phi_v) = \left\{(b_1 \cdot \nu, \dots, b_n \cdot \nu) \mid \nu \in \mathbb{Z}^{n-d} \text{ and } \nu \cdot b_i \geq -v_i,\; i \notin \sigma\right\}. \tag{4.7}$$

The following statement gives a necessary and sufficient condition for a series ϕ_v associated to a cone $\mathscr{K}_\sigma$ (that is, to a standard pair $(\partial^\alpha, \sigma)$) to have an open domain of convergence. Note that several series may be associated with a single cone.

Theorem 4.2.3. *For β generic, let w be a weight vector for $H_A(\beta)$. Let*

$$\{(-|\gamma_1|, \gamma_1), \dots, (-|\gamma_{n-d}|, \gamma_{n-d})\}$$

be a $\mathbb{Z}$*-basis of* $\ker_{\mathbb{Z}}(\rho(A))$ *such that for any* $i = 1, \ldots, n-d$*, we have* $\gamma_i \cdot w > 0$*. The vectors in this basis are the columns of a Gale dual matrix of* $\rho(A)$*, whose rows we denote by* $b_0, \ldots, b_n$*. Let* (v_0, v) *be an exponent of* $\rho(A)$ *corresponding to a standard pair* $(\partial^\alpha, \sigma)$ *of* $\mathrm{in}_{(0,w)}(I_{\rho(A)})$ *that passes through zero, so that* σ *is a maximal simplex in the triangulation* $\Delta_{(0,w)}$*, and* $0 \in \sigma$*. The dehomogenized basic Nilsson series* ϕ_v *has an open domain of convergence if and only if* $-b_0 \in \mathscr{K}_\sigma$*. In this case, there exists* $\varepsilon \in \mathbb{R}^{n-d}_{>0}$ *such that* ϕ_v *converges in* $\mathscr{U}_{w,\varepsilon}$*.*

Proof. Suppose that $-b_0 \in \mathscr{K}_\sigma$, so that $-b_0 = \sum_{i \notin \sigma} \lambda_i b_i$ with $\lambda_i \geq 0$. Let $u \in \mathrm{supp}(\phi_v)$, and choose ν using (4.7). Then

$$|u| = \sum_{i=1}^{n} b_i \cdot \nu = -b_0 \cdot \nu = \sum_{i \notin \sigma} \lambda_i b_i \cdot \nu \geq -\sum_{i \neq \sigma} \lambda_i v_i.$$

This means that the set $\{|u| \mid u \in \mathrm{supp}(v)\}$ is bounded below, and therefore $|u|$ must be non negative for almost all $u \in \mathrm{supp}(\phi_v)$. Now apply Theorem 4.1.2 to conclude that ϕ_v has an open domain of convergence.

Let us now prove the converse. As $\{b_i \mid i \notin \sigma\}$ is a basis of $\mathbb{R}^{n-d}$, we can write $-b_0 = \sum_{i \notin \sigma} \lambda_i b_i$. Suppose that $\lambda_{i_0} < 0$ for some $i_0 \notin \sigma$, and consider the infinite set

$$\{\nu \in \mathbb{Z}^{n-d} \mid \nu \cdot b_i = 0 \text{ for } i \notin \sigma \cup \{i_0\} \text{ and } \nu \cdot b_{i_0} > 0\}.$$

For each element ν of this set, $(\nu \cdot b_1, \ldots, \nu \cdot b_n)$ is an element of $\mathrm{supp}(\phi_v)$, and the sum of its coordinates is

$$\sum_{i=1}^{n} \nu \cdot b_i = \nu \cdot \big(\sum_{i=1}^{n} b_i\big) = \nu \cdot (-b_0) = \nu \cdot (\sum_{i \notin \sigma} \lambda_i b_i) = \lambda_{i_0}(\nu \cdot b_{i_0}).$$

Thus, $\mathrm{supp}(\phi_v)$ has an infinite subset consisting of vectors whose coordinate sum is negative, and by Theorem 4.2.1, ϕ_v does not have an open domain of convergence. □

We now give a combinatorial formula for the dimension of the space of convergent Nilsson solutions of $H_A(\beta)$ with respect to w, for generic parameters β.

Corollary 4.2.4. *Let* β *be generic, and* w *a weight vector for* $H_A(\beta)$*. Then*

$$\dim_{\mathbb{C}}(\{\phi \in \mathscr{N}_w(H_A(\beta)) \textit{ convergent }\}) = \sum_{\sigma \in T} \mathrm{vol}(\sigma)\,,$$

where $T = \{\sigma \textit{ facet of } \Delta_{(0,w)} \mid 0 \in \sigma \textit{ and } -b_0 \in \mathscr{K}_\sigma\}$*.* □

The following is essentially a restatement of Theorem 4.2.3.

Corollary 4.2.5. *Let* β *be generic, and* w *a weight vector for* $H_A(\beta)$*. An element of the set* $\mathscr{B}_w$ *from* (4.6) *has an open domain of convergence if and only if its associated maximal simplex in the triangulation* $\Delta_{(0,w)}$ *of* $\rho(A)$ *also belongs to a coherent triangulation of* $\rho(A)$ *defined by a perturbation of* $(0, 1, \ldots, 1)$*.*

Proof. Using Theorem 4.2.3 and its notation, $-b_0 = (0, 1, \ldots, 1) \cdot B \in \mathscr{K}_\sigma$ if and only if σ is a maximal simplex in the coherent triangulation of $\rho(A)$ defined by a perturbation of $(0, 1, \ldots, 1)$. □

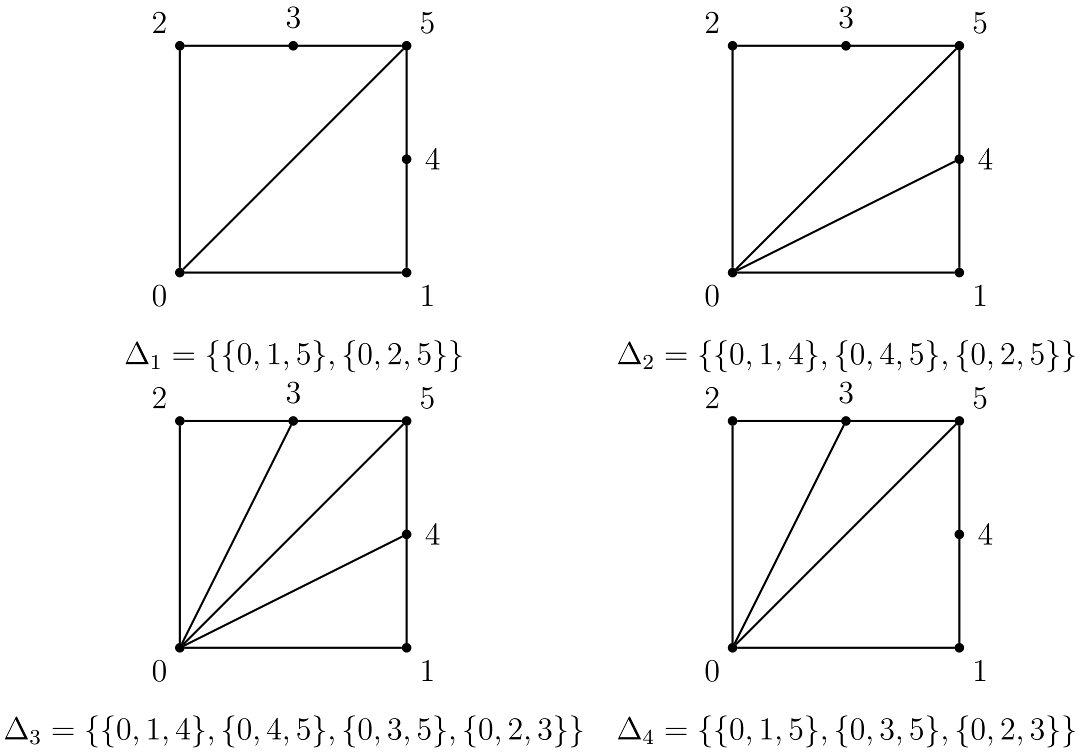


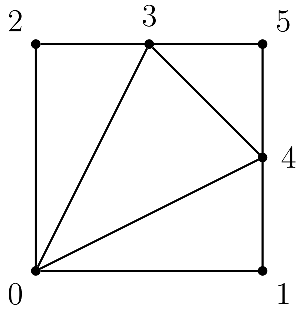
Figure 4.1: The coherent triangulations of $\rho(A)$ corresponding to perturbations of the vector $(0,1\ldots,1)$ for Example 4.2.6.

Example 4.2.6. *Corollary 4.2.5 allows us to decide whether $H_A(\beta)$ has Nilsson solutions with respect to a weight vector that do not converge by inspecting the triangulations of $\rho(A)$. Take for instance*

$$A = \begin{bmatrix} 2 & 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 1 & 2 \end{bmatrix}$$

and consider the coherent triangulations of $\rho(A)$, or, equivalently, the coherent triangulations of $A \cup \{0\}$. Note that the triangulations $\Delta_i,\ i = 1,\ldots,4$ appearing in Figure 4.1 are all the triangulations induced by perturbations of $(0,1,\ldots,1)$, in particular, if β is generic, all the Nilsson solutions of $H_A(\beta)$ associated to maximal simplices in these triangulations have open domains of convergence.

Now consider the triangulation Δ_5 drawn in Figure 4.2. The simplex $\{0,3,4\}$ belongs to Δ_5 and passes through zero, but does not appear in any triangulation of $\rho(A)$ induced by a perturbation of $(0,1,\ldots,1)$; therefore, Corollary 4.2.5 ensures that the corresponding Nilsson solutions of $H_A(\beta)$ (for generic β) do not have open domains of convergence.



$$\Delta_5 = \{\{0,1,4\},\{0,3,4\},\{0,2,3\},\{3,4,5\}\}$$

Figure 4.2: A coherent triangulation of $\rho(A)$ from Example 4.2.6.

Chapter 5

Laurent A-hypergeometric solutions and residues

In this chapter we study Laurent A-hypergeometric series when A is a Cayley configuration $A = Cayley(A_1, A_2)$ with A_1 and A_2 subsets of $\mathbb{Z}^2$ (see Definition 2.2.1). This configuration is associated to two polynomials f_1 and f_2 in two variables. This generalizes Example 5.3.4, in which we had only one configuration in $\{1\} \times \mathbb{Z}$. We also give several definitions and results which will lead us to the definition and properties of special Laurent A-hypergeometric series in terms of combinatorially defined residues. These residues will be the main characters in Chapter 6.

Section 5.1 summarizes the basics of A-hypergeometric Laurent series. These are a special case of logarithm-free hypergeometric series, treated in Section 3.4, but assuming that the parameters β are integer. We study in Section 5.2 the minimal regions that may appear in the case $A = Cayley(A_1, A_2)$ and a parameter $\gamma = (-1, -1, -m)$ with $m \in \mathbb{Z}^2$. In Section 5.3 we introduce local residues associated to Cayley configurations and show that they are special solutions for these configurations. The Gel'fond-Khovanskii's method for the computation of global residues on the torus is presented in Section 5.4. This method will be useful to prove Theorem 6.2.2. Theorem 5.5.2 in Section 5.5 imposes a necessary condition for the algebraicity of A-hypergeometric Laurent series.

5.1 A-hypergeometric Laurent series

Let $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^d$ be a regular configuration. As before, we assume, without loss of generality, that the points of A are all distinct and that they span $\mathbb{Z}^d$.

We consider the $\mathbb{C}$-vector space:

$$S \;=\; \left\{ \sum_{v \in \mathbb{Z}^n} c_v x^v \;;\; c_v \in \mathbb{C} \right\}$$

of formal Laurent series in the variables $x_1, \ldots, x_n$. The matrix A defines a $\mathbb{Z}^d$-valued grading in S by

$$\deg(x^v) \;:=\; A \cdot v \quad ; \quad v \in \mathbb{Z}^n \,. \tag{5.1}$$

We will say that $\Phi \in S$ is an A-hypergeometric Laurent series of degree β (or Laurent solution of $H_A(\beta)$) if it is annihilated by $H_A(\beta)$.

Remark 5.1.1. *By [PST05, Proposition 5], if a hypergeometric Laurent series has a non trivial domain of convergence, then its exponents must lie in a strictly convex cone.*

Given $\beta \in \mathbb{Z}^d$, consider the fiber

$$M_\beta \;:=\; \{v \in \mathbb{Z}^n : A \cdot v = \beta\}.$$

Recall that for any vector $v \in \mathbb{Z}^n$ we define its *negative support* as:

$$\operatorname{nsupp}(v) \;:=\; \{i \in \{1,\dots,n\} : v_i < 0\}, \tag{5.2}$$

and given $I \subset \{1,\dots,n\}$, we let $r_{(I,\beta)} = \{v \in M_\beta : \operatorname{nsupp}(v) = I\}$. We call $r_{(I,\beta)}$ a *region* in M_β.

Definition 5.1.2. *We say that $r_{(I,\beta)}$ is a* minimal region *if $r_{(I,\beta)} \neq \emptyset$ and $r_{(J,\beta)} = \emptyset$ for $J \subsetneq I$. We write r_I instead of $r_{(I,\beta)}$ is it is clear by the context.*

Given a minimal region $r_I = r_{(I,\beta)}$ we let

$$\Phi^\beta_{r_I}(x) := \sum_{u \in r_{(I,\beta)}} (-1)^{\sum_{i\in I} u_i} \, \frac{\prod_{i \in I}(-u_i - 1)!}{\prod_{j \notin I}(u_j)!} \, x^u \,. \tag{5.3}$$

Given a weight $w \in \mathbb{R}^n$, $\varepsilon > 0$ and $\nu_1, \dots, \nu_{n-d}$ a $\mathbb{Z}$-basis of the lattice $M = \{v \in \mathbb{Z}^n : A \cdot v = 0\}$ satisfying $\langle w, \nu_i \rangle > 0$ for all $i = 1, \dots, n-d$, we recall from Subsection 1.4.3, the definition of the open set:

$$\mathscr{U}_{w,\boldsymbol{\varepsilon}} = \{x \in \mathbb{C}^n \mid |x^{\nu_i}| < \varepsilon_i \;\text{ for }\; i = 1, \dots, n-d\}\,. \tag{5.4}$$

Definition 5.1.3. *We will denote by r_w the collection of minimal regions $r_{(I,\beta)}$ for which the inner product with w is bounded below and the minimum is attained at a unique point.*

The following theorem shows the importance of the Laurent series defined in (5.3).

Theorem 5.1.4. *Let $A \subset \mathbb{Z}^d$ and $\beta \in \mathbb{Z}^d$ as before and $w \in \mathbb{R}^n$ such that r_w is non-empty. Then:*

(i) For ε sufficiently small, the open set $\mathscr{U}_{w,\boldsymbol{\varepsilon}}$ of the form (5.4) *is a common domain of convergence of all $\Phi^\beta_{r_I}$ with $r_I = r_{(I,\beta)} \in r_w$ and*

(ii) These $\Phi^\beta_{r_I}$ are a basis of the vector space of A-hypergeometric Laurent series of degree β convergent in $\mathscr{U}_{w,\boldsymbol{\varepsilon}}$.

Proof. First note that the series $\Phi^\beta_{r_I}$ is a constant multiple of the Gamma series ϕ_v from (3.13). Here v is the element of r_I where the inner product $\langle w, \cdot \rangle$ reaches its minimum. In fact,

$$\Phi^\beta_{r_I} = (-1)^{\sum_{i\in I} v_i} \, \frac{\prod_{i \in I}(-v_i - 1)!}{\prod_{j \notin I}(v_j)!} \cdot \phi_v. \tag{5.5}$$

Considering that in the regular case formal solutions coincide with holomorphic solutions, the proof of the theorem follows from Proposition 3.4.2 and Theorem 3.4.3. Theorem 1.3.16 and Subsection 1.4.3 clarify convergence issues. $\square$

We can deduce from Theorem 5.1.4 and Remark 5.1.1 an equivalent characterization of r_w: it is the collection of those regions $r_{(I,\beta)}$ which are contained in the intersection of a half-space $\{v \in \mathbb{R}^n : \langle w, v\rangle \geq \lambda,\ \lambda \in \mathbb{R}\}$ with a strictly convex cone.

Definition 5.1.5. *We call the series $\Phi^{\beta}_{r_I}$, for $r_I \in r_w$ minimal,* canonical Laurent *A-hypergeometric series in the direction of w (see Section 3.4 and Definition 1.3.17).*

Example 5.1.6. *Recall from Example 1.1.4 that the roots of the generic polynomial $x_2t^2 + x_1t + x_0$ are solutions of an A-hypergeometric system with integer parameters. We can express these roots as Laurent series in the variables x_0, x_1, x_2. In fact, developing the square root we obtain:*

$$\rho_1 = \frac{-x_1 + (x_1^2 - 4x_0x_2)^{1/2}}{2x_2} = -\frac{x_1}{2x_2} + \left(\frac{x_1}{2x_2} - \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}\frac{x_2^k x_0^{k+1}}{x_1^{2k+1}}\right), \tag{5.6}$$

$$\rho_2 = \frac{-x_1 - (x_1^2 - 4x_0x_2)^{1/2}}{2x_2} = -\frac{x_1}{2x_2} - \left(\frac{x_1}{2x_2} - \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}\frac{x_2^k x_0^{k+1}}{x_1^{2k+1}}\right). \tag{5.7}$$

For A and β as in Example 1.1.4, the only subsets of $\{0,1,2\}$ such that the corresponding regions are minimal are $\{1\}$ and $\{2\}$ with $r_{\{1\}} = \{(1+k, -2(k+1), k), k \in \mathbb{N}_0\}$ and $r_{\{2\}} = \{(0,1,-1)\}$. Note that

$$\rho_1 = -\Phi^{\beta}_{r_{\{1\}}} \text{ and } \rho_2 = -\Phi^{\beta}_{r_{\{2\}}} - \Phi^{\beta}_{r_{\{1\}}}. \tag{5.8}$$

and that both $r_{\{1\}}$ and $r_{\{2\}}$ belong to r_w for any $w = (w_0, w_1, w_2) \in \mathbb{R}^3$ such that $w_0 + w_2 > 2w_1$. Moreover, there are no Laurent series in the direction of w for w not satisfying this condition.

Remark 5.1.7. *Let Φ a Laurent solution of $H_A(\beta)$ in the direction of w. This implies, by Theorem 5.1.4 that*

$$\Phi = \sum_{r_I \in r_w} k_I \Phi^{\beta}_{r_I} \tag{5.9}$$

with $k_I \in \mathbb{C}$ and the series Φ converges in $\mathscr{U}_{w,\varepsilon}$. In particular $\langle w, u\rangle \geq 0$ for all $u \in r_I \in r_w$.

If we consider another weight vector w' such that $\langle w', u\rangle \geq 0$ for all $u \in r_I \in r_{w'}$, according to 5.1.4(ii), we obtain another expansion of Φ converging in an open set $\mathscr{U}_{w',\varepsilon'}$. These two expansions have to coincide. In fact, the minimal regions appearing in both have to belong to r_w and $r_{w'}$ simultaneously, otherwise there would be elements in the support of Φ with negative inner product either with w or w', contradicting the assumptions. Therefore we obtain a unique expansion converging in $\mathscr{U}_{w,\varepsilon} \cap \mathscr{U}_{w',\varepsilon'}$.

The following Proposition is an easy calculation.

Proposition 5.1.8. *If $\alpha \in \mathbb{N}^n$ then the derivative ∂^{α} of a canonical Laurent A-hypergeometric series with homogeneity β is a canonical Laurent A-hypergeometric series with homogeneity $\beta - A\cdot\alpha$.*

Since an A-hypergeometric series of degree β satisfies d independent homogeneity relations it may be viewed as a function of $n-d$ variables. To make this more precise we use a Gale dual $\{b_1, \ldots, b_n\} \subset \mathbb{Z}^{n-d}$ of the configuration A (see Definition 2.1.1).

Remark 5.1.9. *(i) Our definition of Gale dual depends on the choice of a basis of* $M_0 = \ker_{\mathbb{Z}}(A)$*; this amounts to an action of* $\mathrm{GL}(n-d,\mathbb{Z})$ *on the configuration* B*.*

(ii) B *is primitive, i.e., if* $\delta \in \mathbb{Z}^{n-d}$ *has relatively prime entries then so does* $B\delta$*. This follows from the fact that if* $rv \in M_0$ *for* $r \in \mathbb{Z}$ *and* $v \in \mathbb{Z}^n$ *then* $v \in M_0$*. Equivalently, the rows of* B *span* $\mathbb{Z}^{n-d}$*.*

(iii) The regularity condition on A *is equivalent to the requirement that*

$$\sum_{j=1}^{n} b_j \;=\; 0. \tag{5.10}$$

(iv) A *is not a pyramid if and only if none of the vectors* b_j *vanishes.*

As we saw in previous chapters, given $v \in M_\beta$, and the choice of a Gale dual B we may identify $M_\beta \;\cong\; \mathbb{Z}^{n-d}$ by $u \in M_\beta \mapsto \mu \in \mathbb{Z}^{n-d}$ with

$$u = v + \mu_1\nu_1 + \cdots + \mu_{n-d}\nu_{n-d}.$$

In particular, $u_i < 0$ if and only if $\ell_i(\mu) < 0$, where

$$\ell_i(\mu) := \langle b_i, \mu\rangle + v_i. \tag{5.11}$$

The linear forms in (5.11) define a hyperplane arrangement oriented by the normals b_i and each minimal region $r_{(I,\beta)}$ corresponds to the closure of a certain connected components c_I in the complement of this arrangement.

Let $\Phi^{\beta}_{r_I}(x)$ as in (5.3). We can also write for $v \in M_\beta$

$$\Phi^{\beta}_{r_I}(x) = x^v \sum_{\mu\in c_I\cap\mathbb{Z}^{n-d}} \frac{\prod_{i\in I}(-1)^{\ell_i(\mu)}(-\ell_i(\mu)-1)!}{\prod_{j\notin I}\ell_j(\mu)!}\, x^{B\mu}\,. \tag{5.12}$$

Setting

$$y_j = x^{\nu_j}, \quad j = 1,\dots,n-d, \tag{5.13}$$

we can now rewrite, the series (5.3) in the coordinates y as $\Phi^{\beta}_{r_I}(x) = x^v\varphi_{c_I}(y)$, where

$$\varphi_{c_I}(y) := \sum_{\mu\in c_I\cap\mathbb{Z}^{n-d}} \frac{\prod_{\ell_i(\mu)<0}(-1)^{\ell_i(\mu)}(-\ell_i(\mu)-1)!}{\prod_{\ell_j(\mu)>0}\ell_j(\mu)!}\, y^m\,. \tag{5.14}$$

Moreover, since changing $v \in M_\beta$ only changes (5.3) by a constant, we can assume that in order to write (5.14) we have chosen $v \in r_{(I,\beta)}$ and this guarantees that $-v_i - 1 > 0$ for $i \in I$ and $v_j \geq 0$ for $j \notin I$.

If $F(x)$ is an A-hypergeometric function of degree β, then $\partial_j(F) = \partial F/\partial x_j$ is A-hypergeometric of degree $\beta - a_j$. In terms of the hyperplane arrangement in $\mathbb{R}^{n-d}$ this has the effect of changing the hyperplane $\{\langle b_j,\cdot\rangle + v_j\}$ to the hyperplane $\{\langle b_j,\cdot\rangle + v_j - 1\}$.

The cone of parameters

$$\mathcal{E} = \mathcal{E}_A \;:=\; \left\{\sum_{i=1}^{d+2}\lambda_i a_i \;:\; \lambda_i\in\mathbb{R},\ \lambda_i < 0\right\} \tag{5.15}$$

is called the *Euler-Jacobi cone of* A. We note that if $\beta \in \mathcal{E}$ then $\beta - a_j \in \mathcal{E}$ for all $j = 1,\dots,n$.

We also recall the following result of Saito, Sturmfels and Takayama [SST00, Corollary 4.5.13], which we will use in the following sections:

Theorem 5.1.10. *If F is an A-hypergeometric function of degree $\beta \in \mathcal{E}$ then, for any $j = 1, \dots, n$, $\partial_j(F) = 0$ if and only if $F = 0$.*

In particular, all non-zero A-hypergeometric functions F whose degree lies in the Euler-Jacobi cone are stable; that is, no partial derivative $\partial^\alpha F$ vanishes.

5.2 Minimal regions and Minkowski sum

We consider the Cayley configuration $A = Cayley(A_1, A_2)$ given by the lattice points

$$A_1 = \{\alpha_1, \dots, \alpha_r\} \subset \mathbb{Z}^2$$

$$A_2 = \{\beta_1, \dots, \beta_s\} \subset \mathbb{Z}^2$$

with $r, s > 1$, that is, corresponding to the bivariate polynomials

$$f_1(t_1, t_2) = x_1 t^{\alpha_1} + \dots + x_r t^{\alpha_r} \tag{5.16}$$

and

$$f_2(t_1, t_2) = x_{r+1} t^{\beta_1} + \dots + x_{r+s} t^{\beta_s}. \tag{5.17}$$

We set $n = r + s$. The corresponding integer matrix is

$$A = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \\ \alpha_1 & \dots & \alpha_r & \beta_1 & \dots & \beta_s \end{pmatrix} \in \mathbb{Z}^{4 \times n}. \tag{5.18}$$

As always, we assume that $\mathbb{Z}A = \mathbb{Z}^4$, or equivalently that $\mathbb{Z}A_1 + \mathbb{Z}A_2 = \mathbb{Z}^2$. Denote by P_1 the convex hull of A_1, P_2 the convex hull of A_2 and by $P_1 + P_2$ the Minkowski sum of both, which has dimension two.

In this section we study the Laurent solutions of the A-hypergeometric system $H_A(\gamma)$ with parameter

$$\gamma = (-1, -1, -m), \quad m \in \mathbb{Z}^2. \tag{5.19}$$

To that end, in this section we inspect which are the minimal regions $r_I = r_{(I,\gamma)}$ (cf. Definition 5.1.2). Let A be as in (5.18) and γ as in (5.19). Our first observation is the following lemma.

Lemma 5.2.1. *There are no minimal regions r_I with $I \subset \{1, \dots, r\}$ or $I \subset \{r+1, \dots, r+s\}$.*

Proof. This is clear since for any $u \in \mathbb{R}^n$ such that $A \cdot u = \gamma$ we have that

$$\sum_{i=1}^{r} u_i = \sum_{j=1}^{s} u_{r+j} = -1$$

□

Hence the first case to study possible minimal regions r_I is when $I = \{i_0, j_0\}$ with $i_0 \in \{1, \dots, r\}$ and $j_0 \in \{r+1, \dots, r+s\}$. These regions are related with the points $\alpha_{i_0} + \beta_{j_0} \in P_1 + P_2$ and their occurrence will depend on $m \in \mathbb{Z}^2$. We need the following definition:

Definition 5.2.2. *We say that a point $v \in P_1 + P_2$ may be seen from $m \in \mathbb{Z}^2$ if there exists non-negative integers λ_{ij} such that*

$$-m + v = \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{ij}(\alpha_i + \beta_j - v)$$

In Figure 5.1 we look at the geometric interpretation of Definition 5.2.2: the vertex v of $P_1 + P_2$ may be seen from any $m \in \mathbb{Z}^2$ in the cone with apex at v and edge directions $v - v'$ and $v - v''$.

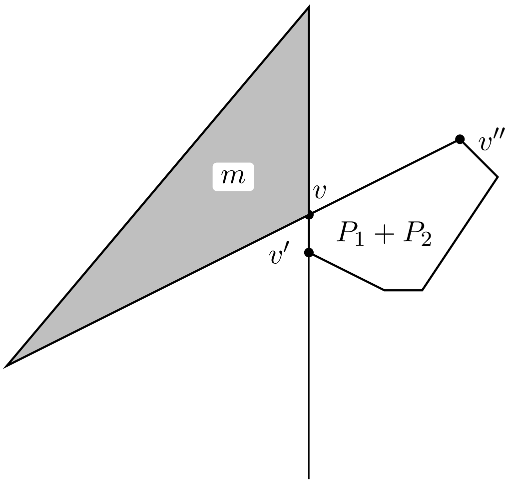


Figure 5.1: v may be seen from m

Remark 5.2.3. *If we take $v = \alpha_{i_0} + \beta_{j_0}$, Definition 5.2.2 is equivalent to saying that there exists non-negative integers $\lambda_i,\ \mu_j$ such that*

$$-m + v = \sum_{i \neq i_0}^{r} \lambda_i(\alpha_i - \alpha_{i_0}) + \sum_{j \neq j_0}^{s} \mu_j(\beta_j - \beta_{j_0})$$

In fact,

$$-m + \alpha_{i_0} + \beta_{j_0} = \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{ij}(\alpha_i + \beta_j - (\alpha_{i_0} + \beta_{j_0})) =$$
$$= \sum_{i \neq i_0}^{r} \left(\sum_{k=1}^{r} \lambda_{ik} \right) (\alpha_i - \alpha_{i_0}) + \sum_{j \neq j_0}^{s} \left(\sum_{k=1}^{r} \lambda_{kj} \right) (\beta_j - \beta_{j_0}),$$

and $\sum_{k=1}^{r} \lambda_{ik}$ and $\sum_{k=1}^{r} \lambda_{kj}$ also are non-negative integers. For the converse, just note that

$$\sum_{i \neq i_0}^{r} \lambda_i(\alpha_i - \alpha_{i_0}) + \sum_{j \neq j_0}^{s} \mu_j(\beta_j - \beta_{j_0}) =$$
$$= \sum_{i \neq i_0}^{r} \lambda_i(\alpha_i + \beta_{j_0} - (\alpha_{i_0} + \beta_{j_0})) + \sum_{j \neq j_0}^{s} \mu_j(\alpha_{i_0} + \beta_j - (\alpha_{i_0} + \beta_{j_0})).$$

Remark 5.2.4. *From Definition 5.2.2 and Figure 5.1 it is clear that no vertex of $P_1 + P_2$ is seen from a different vertex of $P_1 + P_2$.*

Now we can express how this geometric-arithmetic property of m seeing a point determines the occurrence of the regions $r_{\{i_0,j_0\}}$.

Lemma 5.2.5. *Let $m \in \mathbb{Z}^2$. Then the region $r_{\{i_0,j_0\}}$ is non-empty if and only if the point $\alpha_{i_0} + \beta_{j_0} \in P_1 + P_2$ may be seen from m.*

Proof. For simplicity we assume $i_0 = j_0 = 1$. Suppose that there exists a point $u \in \mathbb{Z}^{r+s}$ such that $A \cdot u = \gamma$ and $u_1 < 0$, $u_{r+1} < 0$ and $u_j \geq 0$ for other j. Then we have

$$u_1 + \ldots + u_r = -1,$$

$$u_{r+1} + \ldots + r_{r+s} = -1 \text{ and}$$

$$-m = \sum_{i=1}^{r} u_i \alpha_i + \sum_{j=1}^{s} u_{r+j} \beta_j.$$

Combining these three equations we obtain

$$-m + \alpha_1 + \beta_1 = \sum_{i=1}^{r} u_i(\alpha_i - \alpha_1) + \sum_{j=1}^{s} u_{r+j}(\beta_j - \beta_1) \tag{5.20}$$

with $u_2, \ldots, u_r, u_{r+2}, \ldots, u_{r+s} \geq 0$, then by Remark 5.2.3 we obtain that $\alpha_1 + \beta_1 \in P_1 + P_2$ may be seen from m. For the converse just step backwards. □

According to Lemma 5.2.5, we can consider different cases of $m \in \mathbb{Z}^2$ in order to establish which minimal regions occur for A and γ. Note that if $v = \alpha_{i_0} + \beta_{j_0} \in (P_1 + P_2)^\circ$, then it is geometrically clear that any $m \in \mathbb{Z}^2$ sees the point v and only arithmetic issues may imply that v be not seen from m, but this never happen, see Corollary 5.2.9.

To study regions corresponding to vertices, we state the following lemma.

Lemma 5.2.6. *Let $m \in \mathbb{Z}^2$ and $\alpha_{i_0} + \beta_{j_0} \in P_1 + P_2$ such that $\alpha_{i_0} + \beta_{j_0}$ is seen from m. If $\alpha_{i_0} + \beta_{j_0}$ is a vertex of $P_1 + P_2$ then the region $r_{\{i_0,r+j_0\}}$ is finite.*

Proof. Assume again $i_0 = 1, j_0 = 1$. Note that, being $\alpha_1 + \beta_1$ a vertex, there exists a linear functional ξ such that

$$\xi(\alpha_1 + \beta_1) < \xi(\alpha_i + \beta_j) \text{ for } i \neq 1 \text{ or } j \neq 1.$$

Hence, if

$$\begin{aligned} -m + \alpha_1 + \beta_1 &= \sum_{i=2}^{r} u_i(\alpha_i - \alpha_1) + \sum_{j=2}^{s} u_{r+j}(\beta_j - \beta_1) = \\ &= \sum_{i=2}^{r} u_i[(\alpha_i + \beta_1) - (\alpha_1 + \beta_1)] + \sum_{j=2}^{s} u_{r+j}[(\alpha_1 + \beta_j) - (\alpha_1 + \beta_1)], \end{aligned}$$

where $u_i, u_{r+j} \geq 0$ for all i, j, we have

$$\sum_{i=2}^{r} u_i h_i + \sum_{j=2}^{s} u_{r+j} k_j = \xi(-m + \alpha_1 + \beta_1)$$

where h_i and k_i are positive real numbers. Since u_i, u_{r+j} are non-negative integers, their possible values are bounded and the region with support $\{1, r+1\}$ is a bounded minimal region. □

Remark 5.2.7. *It can be easily seen that*

$$m \in (P_1 + P_2)^\circ \text{ if and only if } \gamma = (-1, -1, -m) \in \mathcal{E},$$

where $\mathcal{E}$ is the Euler-Jacobi cone defined in (5.15)*.*

The following results relates supports of vectors in the kernel of the Cayley configuration A with interior lattice points in the Minkowski sum $A_1 + A_2$.

Proposition 5.2.8. *Fix $i \in \{1, \dots, r\}$, $j \in \{1, \dots, s\}$. The following conditions are equivalent:*

1. *There exists $\ell = n - 4$ linearly independent vectors $u^{(1)}, \dots, u^{(\ell)} \in \ker_{\mathbb{Z}}(A)$ with* $\mathrm{nsupp}(u^{(i)}) = \{i, r+j\}$.
2. *$\alpha_i + \beta_j \in (P_1 + P_2)^\circ$.*
3. *The minimal region $r_{(\{i,r+j\},(-1,-1,-(\alpha_i+\beta_j))}$ has a recession cone of maximal dimension.*
4. *There exists $m \in \mathbb{Z}^2$ such that $r_{(\{i,r+j\},\gamma)}$ with γ as in* (5.19) *has a recession cone of maximal dimension.*

Proof. Assume that 1. holds and let $m = \alpha_i + \beta_j$. Then, $\gamma = A.(-e_i - e_{r+j})$ and so the region $r_{(\{i,r+j\},(-1,-1,-(\alpha_i+\beta_j))}$ is non empty. Since $(-e_i - e_{r+j}) + u^{(k)} \in r_{(\{i,r+j\},\gamma)}$ for any $k = 1, \dots, \ell$, it follows that the recession cone is of maximal dimension, proving 3. Item 3. clearly implies 4., which in turns implies 1., as the difference of any two vectors in $r_{(\{i,r+j\},\gamma)}$ lies in the kernel of A.

We now see the equivalence of items 2 and 3. Let $\alpha_i + \beta_j \in (P_1 + P_2)$. If the point lies in $(P_1 + P_2)^\circ$, then $\gamma = (-1, -1, (\alpha_i + \beta_j))$ lies in the Euler-Jacobi cone and so any minimal region has maximal dimension recession cone If instead $\alpha_i + \beta_j$ lies in the boundary of $P_1 + P_2$, let η be an inner normal vector such that $\langle \eta, \alpha_i + \beta_j \rangle = c_\eta$ is minimal and $\langle \eta, v \rangle \geq c_\eta$ for all $v \in P_1 + P_2$. We denote by F the corresponding face and F_1, F_2 the faces of P_1, P_2 such that $F = F_1 + F_2$, Any vector in r_I, is a $\mathbb{Z}$-linear combination of the vectors $(1, 0, \alpha_k), k = 1, \dots, r$ and $(0, 1, \beta_\ell)$ with coefficients $(u_1, \dots, u_{r+s})$ which non negative coefficients except from $U_i, u_{r+j} \in \mathbb{Z}_{<0}$. Moreover, the sum of the first r and last s coordinates equal to -1. We then have

$$\alpha_i + \beta_j = \sum_{k \neq 1} u_k((\alpha_k + \beta_j) - (\alpha_i + \beta_j) + \sum_{\ell \neq j} u_{\ell+r}((\alpha_i + \beta_\ell) - (\alpha_i + \beta_j)).$$

As, $\sum_{k \neq i} u_i = 0, \sum_{\ell \neq j} u_{\ell+r} = 0$, we can argue as in Lemma 5.2.6. We take the inner product on both sides with η and using the positivity, we deduce that $u_k = 0$ if $\alpha_k \notin F_1$, $u_\ell = 0$ if $\beta_e ll \notin F_2$. Therefore, r_I cannot be full dimensional. □

Corollary 5.2.9. *If $\alpha_{i_0} + \beta_{j_0}$ is an interior point of $P_1 + P_2$ then it may be seen from any $m \in \mathbb{Z}^2$.*

Proof. Given $m \in \mathbb{Z}^2$, our assumption that $\mathbb{Z}A = \mathbb{Z}^4$ ensures that we can always take $v \in \mathbb{Z}^n$ such that $A \cdot v = (-1, -1, -m)$. By the first item in Proposition 5.2.8, there exists $u \in \ker_{\mathbb{Z}}(A)$ with $\mathrm{nsupp}(u^{(i)}) = \{i, r + j\}$ and all coordinates u_i are non zero. Then, for any sufficiently big $\lambda \in \mathbb{N}$ it holds that $A \cdot (\lambda u + v) = (-1, -1, -m)$ and $\mathrm{nsupp}(\lambda u + v) = \{i_0, r + j_0\}$. We deduce from Lemma 5.2.5 that v may be seen from m. □

5.3 Residues and Cayley configurations

In this section we introduce local and global residues and show that they satisfy a particular A-hypergeometric system. The local Grothendieck residue associated with a family of k-variate Laurent polynomials $f_1, \ldots, f_k$ with an isolated zero at a point ξ, is defined by

$$Res_{f,\xi}(h) = \frac{1}{(2\pi i)^k} \int_{\Gamma_\xi(\varepsilon)} \frac{h(t)}{f_1(t) \ldots f_k(t)} \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_k}{t_k} \tag{5.21}$$

where $\Gamma_\xi(\varepsilon)$ is the real k-cycle $\Gamma_\xi(\varepsilon) = \{|f_i(t)| = \varepsilon_i\}$ oriented by the k-form $d(arg(f_1)) \wedge \ldots \wedge d(arg(f_k))$. For almost every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ in a neighborhood of the origin, $\Gamma_\xi(\varepsilon)$ is smooth and by Stokes' Theorem the integral (5.21) is independent of ε. Note that this definition makes sense as long as h is holomorphic in a neighborhood of ξ. If ξ is a simple zero then the toric Jacobian

$$J_f^T(\xi) = \det\left(\frac{t_j \partial f_i}{\partial t_j}(\xi)\right)$$

is non-zero and

$$\mathrm{Res}_{f,\xi}(h) = \frac{h(\xi)}{J_f^T(\xi)}. \tag{5.22}$$

This identity follows from the change of coordinates $y_i = f_i(t)$ and iterated integration.

Assuming that the set V of common zeros of $f_1, \ldots, f_k$ in the torus $T = (\mathbb{C}^*)^k$ is finite, we can define the global residue as the sum of local residues:

$$Res_{f_1,\ldots,f_k}(h) = \sum_{\xi \in V} Res_{f,\xi}(h). \tag{5.23}$$

where h is holomorphic around each $\xi \in V$ (for instance $h = \frac{h_1}{h_2}$ with h_1, h_2 polynomials with $h_2(\xi) \neq 0$ for each $\xi \in V$).

Residues are solutions of certain A-hypergeometric systems. Fix k configurations of integer points, $A_i = \{\alpha_1^i, \ldots, \alpha_{|A_i|}^i\} \subset \mathbb{Z}^k$ for $i = 1, \ldots, k$ and denote

$$x = (x_{\alpha_1^1}, \ldots, x_{\alpha_{|A_1|}^1}, \ldots, x_{\alpha_1^k}, \ldots, x_{\alpha_{|A_k|}^k}), \quad t = (t_1, \ldots, t_k).$$

Let $f_1(x;t), \ldots, f_k(x;t)$ be k-variate polynomials with these exponents and $|A_1| + \cdots + |A_k|$ indeterminate coefficients

$$f_i(x;t) = \sum_{l=1}^{|A_i|} x_{\alpha_l^i} t^{\alpha_l^i};\ i = 1, \ldots, k. \tag{5.24}$$

Then, for generic values of the coefficients x, $f_1, \ldots, f_k$ will have a finite set V of common zeros in the torus.

Let $n = |A_1| + \cdots + |A_k|$ and $d = 2k$. Recall that the Cayley configuration $A = Cayley(A_1, \ldots, A_k)$ associated with $A_1, \ldots, A_k$ is the configuration in $\mathbb{Z}^d$ defined by

$$A \ = \ \{e_1\} \times A_1 \ \cup \ \cdots \ \cup \ \{e_k\} \times A_k. \tag{5.25}$$

We have the following proposition, whose proof can be found in [Dic04, Proposition 3.4].

Proposition 5.3.1. *For any $m \in \mathbb{Z}^k$, the local residue*

$$Res_{f,\xi}^m := Res_{f,\xi}(t^m) \tag{5.26}$$

where $t^m = t_1^{m_1} \ldots t_k^{m_k}$ and $\xi \in V$ is an A-hypergeometric algebraic function of the coefficients x of $f_1, \ldots, f_k$ with homogeneity

$$\gamma = (-1, \ldots, -1, -m_1, \ldots, -m_k) \in \mathbb{Z}^d.$$

Consequently, the global residue (5.23) *is a rational A-hypergeometric function with the same homogeneity.*

For $q \in \mathbb{N}^n$ and $i = 1, \ldots, k$ denote by $q^i = (0, \ldots, 0, q_{|A_{i-1}|+1}, \ldots, q_{|A_{i-1}|+|A_i|}, 0, \ldots, 0)$. The following proposition follows easily by differentiating under the integral sign.

Proposition 5.3.2. *Let $q \in \mathbb{N}^n$ then*

$$\partial^q \left(Res_{f,\xi}(t^m)\right) = (-1)^{|q|} \prod_{i=1}^{n} (|q^i| - 1)! Res_{\bar{f},\xi}(t^{\bar{m}}) \tag{5.27}$$

where $\bar{f}$ denotes the collection of polynomials $f^{|q^1|}, \ldots, f^{|q^n|}$ and

$$\bar{m} = m + \sum_{i=1}^{n} \sum_{j=1}^{|A_i|} q_{|A_{i-1}|+j} \alpha_j^i,$$

where $|A_0|$ is understood as 0.

Let $A_1, \ldots, A_k$ be k configurations in $\mathbb{Z}^k$. As usual, we denote by $A_1 + \cdots + A_k$ their *Minkowski sum* $\{\alpha_{j1}^i + \cdots + \alpha_{jk}^i,\ \alpha_{jl}^i \in A_i\}$.

The following result about vanishing of global residues, due to A. Khovanskii [Kho78], is the sparse version of the classical Euler-Jacobi Theorem.

Theorem 5.3.3. *Let $f_1, \ldots, f_k$ be generic polynomials with coefficients in $A_1, \ldots, A_k$. For any Laurent polynomial h with support contained in the interior of the convex hull of $A_1 + \cdots + A_k$, the global residue $Res_{f_1,\ldots,f_k}(h)$ is equal to* 0.

Example 5.3.4. *Consider the 1-dimensional Cayley configuration*

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & k_1 & \dots & k_m & \ell \end{pmatrix}$$

and let

$$f(x;t) := x_0 + x_{k_1} t^{k_1} + \dots + x_{k_m} t^{k_m} + x_\ell t^\ell$$

denote the polynomial with exponents $0, k_1, \dots, k_m, \ell$ and indeterminate coefficients. The powers $\rho^s(x)$, $s \in \mathbb{Z}$ of the roots of $f(x;t)$, viewed as functions of the coefficients, are algebraic solutions of the A-hypergeometric system with exponent $(0, -s)$ and the total sum

$$p_s(x) := \rho_1^s(x) + \dots + \rho_d^s(x)$$

is then a rational solution with the same exponent. Similarly, one can show that the local residues

$$Res_{\rho(x)} \left(\frac{t^b}{f^a(x;t)} \frac{dt}{t} \right), \quad a, b \in \mathbb{Z}; a \geq 1$$

give algebraic solutions with exponent $(-a, -b)$ and, again, the total sum of residues is a rational solution. In [CDD99] a family of algebraic A-hypergeometric functions is defined in terms of the roots of the polynomial $f(x;t)$. These functions are used to construct a basis of the space of solutions for any integer parameter.

5.4 A-hypergeometric solutions associated to a vertex

In this section we show that canonical A-hypergeometric series with homogeneity γ as in (5.19) corresponding to minimal regions associated with a vertex $\alpha_{i_0} + \beta_{j_0}$ of $P_1 + P_2$, are residues associated to f_1 and f_2 as defined in (5.16),(5.17).

5.4.1 The Gelfond-Khovanskii method for calculating residues

We say that two lattice polygones Q_1 and Q_2 are *developed* if no edge of Q_1 is allowed to be parallel to an edge of Q_2. This implies that each edge E of $Q_1 + Q_2$ is either a translate of an edge of Q_1 ("a 1–edge") or a translate of an edge of Q_2 ("a 2–edge"). Along this section we suppose that Q_1 and Q_2 are developed.

Given $h \in \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}]$, let $Res_{g_1,g_2}(h)$ as in (5.23), for $k = 2$. In [GK96], the following method to calculate $Res_{g_1,g_2}(h)$ is introduced, assuming that $Q_i := \mathrm{New}(g_i)$, $i = 1, 2$, are developed.

To each vertex $v = (v_1, v_2)$ of $Q_1 + Q_2$ we assign a *combinatorial coefficient* $k_v \in \{-1, 0, +1\}$ according to the following rule:

Let E_1, E_2 be the two edges of $Q_1 + Q_2$ adjacent to v in counterclockwise order. We set

$$k_v := \begin{cases} -1 & \text{if } E_1 \text{ is a } 1 \text{ -edge and } E_2 \text{ is a } Q_2\text{-edge,} \\ +1 & \text{if } E_1 \text{ is a 2-edge and } E_2 \text{ is a } Q_1 \text{ -edge,} \\ 0 & \text{if } E_1 \text{ and } E_2 \text{ are both } Q_1 \text{ -edges or both } Q_2\text{-edges.} \end{cases} \tag{5.28}$$

We call the coefficient k_v, *combinatorial Khovanskii coefficient* of v.

Let $a_v \cdot t_1^{v_1} t_2^{v_2}$ be the extreme term of $g_1 \cdot g_2$ corresponding to v. We define $L(v)$ to be the formal Laurent series

$$L(v) = \frac{1}{a_v \cdot t_1^{v_1} t_2^{v_2}} \cdot \sum_{n=0}^{\infty} \left(1 - \frac{g_1(t_1, t_2) \cdot g_2(t_1, t_2)}{a_v \cdot t_1^{v_1} t_2^{v_2}} \right)^n . \tag{5.29}$$

Denote by $R_{v,g_1,g_2}(h)$ the constant term of $h \cdot L(v)$. Then, the theorem of Gel'fond and Khovanskii states:

Theorem 5.4.1. *With the previous notation,*

$$Res_{g_1,g_2}(h) \quad = \sum_{v \text{ a vertex of } Q_1+Q_2} k_v \cdot R_{v,g_1,g_2}(h). \tag{5.30}$$

5.4.2 Laurent polynomial solutions

Now, we can combine the Gelfond-Khovanskii method with the notion of m "seeing" v from Definition 5.2.2. Take f_1, f_2, P_1 and P_2 as in the beginning of Section 5.2, with P_1 and P_2 developed.

Proposition 5.4.2. *Let $m \in \mathbb{Z}^2$ and v a vertex of $P_1 + P_2$, then $R_{v,f_1,f_2}(t^m) = 0$ if v is not seen from m.*

Proof. It is clear from Definition 5.2.2 and Equation (5.29) that if v is not seen from m, no constant term appears in the product $t^m \cdot L(v)$. □

Theorem 5.4.3. *Let $m \in \mathbb{Z}^2$ and $v = \alpha_{i_0} + \beta_{j_0} \in P_1 + P_2$ a vertex such that v is seen from m. Then $Res_{v,f_1,f_2}(t^m)$ agrees with the canonical hypergeometric Laurent polynomial corresponding to the minimal region $r_{\{i_0,j_0\}}$.*

Proof. Suppose that $v = \alpha_1 + \beta_1$ so that $a_v t^v = x_1 x_{r+1} t^{\alpha_1+\beta_1}$. Let x^u be a monomial appearing in $R_v(t^m)$ with coefficient C_u. Then, we have that all exponents u_k are non-negative except u_1, u_{r+1} which are strictly negative and

$$u_1 = -1 - u_2 - \cdots - u_r \quad \text{and} \quad u_{r+1} = -1 - u_{r+2} - \cdots - u_{r+s}.$$

As usual, let us denote by u_+, the non-negative entries of u and consider the derivative

$$\partial^{u_+} \cdot R_v(t^m)$$

We claim that

$$\partial^{u_+} \cdot R_v(t^m) \;=\; C_u \, (\prod_{u_j > 0} u_j!) \, x_1^{u_1} \cdot x_{r+1}^{u_{r+1}}.$$

Indeed, suppose $\tilde{u} \in \mathbb{Z}^{r+s}$ is such that $A \cdot \tilde{u} = \gamma$ and $\mathrm{nsupp}(\tilde{u}) = \{1, r+1\}$. Then $\partial^{u_+} x^{\tilde{u}} = 0$ unless $\tilde{u}_+ \geq u_+$, in the sense that the entries of $\tilde{u}_+$ are greater than or equal than those of u_+. Suppose then that there exists such $\tilde{u}$ with $\tilde{u}_+ \geq u_+$. It follows that

$$\tilde{u}_1 \leq u_1 \; ; \quad \tilde{u}_{r+1} \leq u_{r+1}.$$

But then the vector $\tilde{u} - u \in \ker_{\mathbb{Z}} A$ has negative support contained in $\{1, r+1\}$. It follows that, since the region is bounded by Lemma 5.2.6, we must have $\tilde{u} = u$ and there is only one term left.

On the other hand, we have that $\partial^{u_+} \cdot R_v(t^m)$ is the zero order coefficient in the expansion of

$$\partial^{u_+} \left(\frac{t^m}{x_1 x_{r+1} t^{\alpha_1} t^{\beta_1} (f_1/x_1 t^{\alpha_1})(f_2/x_{r+1} t^{\beta_1})} \right) = \frac{(-1)^{a+b}\, a!\, b!\, t^{\tilde{m}}}{x_1 x_{r+1} t^{\alpha_1} t^{\beta_1} (f_1/x_1 t^{\alpha_1})^a (f_2/x_{r+1} t^{\beta_1})^b},$$

where $a = u_2 + \cdots + u_r = -u_1 - 1$, $b = u_{r+2} + \cdots + u_{r+s} = -u_{r+1} - 1$ and

$$\tilde{m} = m + \sum_{i=2}^{r} u_i\, \alpha_i + \sum_{j=2}^{s} u_{r+j}\, \beta_j.$$

Now, note that since x^u was a term in $R_v(t^m)$ we have

$$-m + \alpha_1 + \beta_1 = \sum_{i=2}^{r} u_i\, (\alpha_i - \alpha_1) + \sum_{j=2}^{s} u_{r+j}\, (\beta_j - \beta_1)$$

and therefore

$$-\tilde{m} + (a+1)\alpha_1 + (b+1)\beta_1 = 0.$$

Hence, the zero order coefficient in the expansion is simply:

$$\frac{(-1)^{a+b}\, a!\, b!}{x_1^{a+1} x_{a+1}^{b+1}}.$$

Comparing the two expressions we get:

$$C_u = \frac{(-1)^{a+b}\, a!\, b!}{\prod_{u_j > 0} u_j!}.$$

This implies that $R_v(t^m)$ is the canonical hypergeometric Laurent polynomial (cf. (5.3)) with homogeneity γ corresponding to the bounded minimal region $r_{\{1,r+1\}}$. $\square$

5.5 A necessary condition for the algebraicity of A-hypergeometric Laurent series

In this section we give a necessary condition for the algebraicity of the canonical Laurent A-hypergeometric series $\Phi^{\gamma}_{r_I}$ (cf. (5.3)). We will need the following result proved in [CDR11].

Theorem 5.5.1. *The hypergeometric series*

$$u(z) := \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (p_i n + k_i)!}{\prod_{j+i}^{s} (q_j n)!} z^n\ , \quad k_i \in \mathbb{N} \tag{5.31}$$

is algebraic if and only if $s-r=1$ and the factorial ratios

$$A_n := \frac{\prod_{i=1}^{r}(p_i n)!}{\prod_{j+i}^{s}(q_j n)!}$$

are integral for every $n \in \mathbb{N}$.

Let $A \in \mathbb{Z}^{n\times d}$ and $\beta \in \mathbb{Z}^d$ and $B \in \mathbb{Z}^{n\times n-d}$ a Gale dual of A with rows $b_1,\ldots,b_n$. We say that a canonical series $\phi_{(I,\beta)}$ with unbounded $r_{(I,\beta)}$ has *regular support* if there exists an infinite extremal ray of $r_{(I,\beta)}$ defined by the vanishing of the linear forms associated to indices K in the complement of I, such that for any $J \in K$ with cardinality $n-d-1$ such that $\{b_i, i\in J\}$ are linearly independent, and any $i \in I$, b_i is not in the linear span of those vectors. Any such ray will be called *regular*.

We then have:

Theorem 5.5.2. *Given $r_{(I,\beta)} \in r_w$, if the canonical Laurent A-hypergeometric series $\Phi^{\beta}_{r_I}$ is algebraic and has regular support, then $|I| \leq \frac{d}{2}$.*

Proof. By the discussion in Section 5.1 (cf. (5.13) and (5.14) for the notations), it is enough to show that if the series

$$\varphi_{c_I}(y) := \sum_{\mu\in c_I\cap\mathbb{Z}^{n-d}} \frac{\prod_{\ell_i(\mu)<0}(-1)^{\ell_i(\mu)}(-\ell_i(\mu)-1)!}{\prod_{\ell_j(\mu)>0}\ell_j(\mu)!}\, y^{\mu}\ .$$

is algebraic, then $|I| \leq \frac{d}{2}$. Note that the number of factorials in the numerator is $|I|$ and the number of factorials in the denominator is $n-|I|$. Suppose then that $\varphi_{c_I}(y)$ is algebraic. Consider the restriction of $\varphi_{c_I}(y)$ to an (infinite) regular ray of c_I given by $\ell_{j_1}(\mu)=0,\ldots,\ell_{j_\ell}(\mu)=0$, $\ell \geq n-d-1$. Then this restriction must also be algebraic. But this is a univariate hypergeometric series of the form (5.31) with integral ratios and $|I|$ factorials in the numerator (which are all non trivial by the hypothesis of regular support) and $n-|I|-\ell$ factorials in the denominator. Then, by Theorem 5.5.1, we have that $n-|I|-\ell = |I|+1$ and we conclude that $|I| \leq \frac{d}{2}$.

□

The following example explains the inclusion of the hypothesis of regular support.

Remark 5.5.3. *Let $A_1 = \{\alpha_1=(0,0), \alpha_2=(1,1)\}$ and $A_2=\{\beta_1=(0,0),\beta_2=(1,0),\beta_3=(0,1),\beta_4=(0,2)\}$. Note that β_3 is an interior point of the edge with vertices β_1,β_4. The dimension of a Gale dual configuration to A is two. Take any integer homogeneity in $\mathcal{E}$. There is an unbounded minimal region with negative support* $\{1,4,5\}$. *Note that* b_1 *and* b_4 *are linearly dependent with b_2, and one of the infinite rays of its recession cones is defined by $\ell_2=0$. The restriction of the associated canonical series Φ to this ray is an algebraic hypergeometric function. However, the other infinite ray allows to preclude the algebraicity of* Φ.

In the case of Cayley configurations, we obtain the following corollary.

Corollary 5.5.4. *Let A be the Cayley configuration in $\mathbb{Z}^{2k}$ associated to k configurations in $\mathbb{Z}^k$. Let $\gamma=(u,-m)$, with $u\in\mathbb{Z}^k_{<0}, m\in\mathbb{Z}^k$ a homogeneity in the Euler-Jacobi cone $\mathcal{E}$. Let Φ_I be an algebraic canonical A-hypergeometric Laurent series of homogeneity γ with negative support I. If ϕ_I has regular support then $|I|=k$.*

Proof. As $\gamma \in \mathcal{E}$, no minimal region is bounded. We deduce from Theorem 5.5.2 that $|I| \leq \frac{2k}{2} = k$. On the other side, as the k first entries of γ are negative I needs to "intersect" the index set of each of the k configurations and so also $|I| \geq k$. $\square$

Chapter 6

Algebraic A-hypergeometric solutions and residues

We now turn our attention to algebraic solutions of regular A-hypergeometric systems $H_A(\beta)$, with β an integer (resonant) parameter. Rational solutions (which are, in particular, algebraic solutions) of A-hypergeometric systems have been studied in, e.g., [CDS01, CDR11, CDD99, PST05]. A characterization of A-hypergeometric systems having a full set of algebraic solutions is given in [Beu10], using tools from number theory.

In this final chapter we study the algebraic A-hypergeometric Laurent solutions of the Cayley configuration $A = Cayley(A_1, A_2)$ with A_1 and A_2 subsets of $\mathbb{Z}^2$. This configuration is associated to two polynomials f_1 and f_2 in two variables. In the case of one univariate polynomial with fix monomial support, the algebraic solutions of the associated configuration were described by means of residues in Example 5.3.4. Those residues are defined in terms of the roots of the polynomial, which can in turn be explicitly described in terms of the coherent triangulations of the convex hull of the configuration. Here, the coherent mixed subdivisions of the Minkowski sum of A_1 and A_2, studied in Chapter 2, will help us to describe the roots of f_1 and f_2 over which we will add the local residues, to produce algebraic A-hypergeometric Laurent series convergent in explicit open sets.

In section 6.1 we introduce this notion of residues associated to a mixed cell of a coherent mixed subdivision of the Minkowski sum of A_1 and A_2. In section 6.2, we show how these residues can be written in terms of canonical Laurent A-hypergeometric series. This is the content of Theorem 6.2.2, a main result in this chapter. We prove in Theorem 6.3.1 of section 6.3 that all algebraic solutions for these configurations can be described in terms of residues in case the Cayley configuration associated to A_1, A_2 has codimension two. Finally, we present in the last section general conjectures for further study.

6.1 Coherent mixed subdivisions and residues

In [HS95], an algorithm based on coherent mixed subdivision to find the solutions of a polynomial system is introduced. We explain it briefly:

Let $w \in \mathbb{R}^n$ be a generic weight and Π_w the coherent mixed subdivision associated to it.

Let $\sigma = \{i_0, i_1, j_0, j_1\} \in \Pi_w$ a mixed cell and

$$f_1^\sigma = x_{i_0} t^{\alpha_{i_0}} + x_{i_1} t^{\alpha_{i_1}}, \tag{6.1}$$

$$f_2^\sigma = x_{r+j_0} t^{\beta_{j_0}} + x_{r+j_1} t^{\beta_{j_1}}. \tag{6.2}$$

Denote $A_1^\sigma = \{i_0, i_1\}$, $A_2^\sigma = \{j_0, j_1\}$, so that $\sigma = A_1^\sigma + A_2^\sigma$. Let

$$\xi_1^\sigma, \ldots, \xi_{\mathrm{vol}(\sigma)}^\sigma$$

be the roots of f_1^σ, f_2^σ. Then we can "extend" them to $\xi_1, \ldots, \xi_{\mathrm{vol}(\sigma)}$ roots of f_1, f_2 such that their coordinates are Puiseaux series in $x = (x_1, \ldots, x_n)$. More explicitly, for each i, $\xi_i = \xi_i(x)$ is an algebraic function of x such that ξ_i converges in $\mathscr{U}_{w,\varepsilon}$ for ε sufficiently small. Moreover

Lemma 6.1.1. *There exists $\tilde{w} \in w + \mathrm{rowspan}(A)$ such that $\tilde{w}_t = 0$ for $t = i, j, k$ or l and $\tilde{w}_t > 0$ otherwise. Consequently*

$$\mathrm{in}_{\tilde{w}}\left(\xi_{ih}(x)\right) = \xi_{ih}^\sigma(x) \quad h = 1, 2, \tag{6.3}$$

i.e. $\xi_{ih}(x) = \xi_{ih}^\sigma(x) + h.o.t.(x)$.

Now we use this information to define algebraic solutions to our A-hypergeometric system $H_A(\gamma)$. Since the functions defining the roots are clearly algebraic, it follows that the local residue of t^m with respect of f_1, f_2 is also an algebraic function, because generically in x we can write

$$Res_{f_1,f_2,\xi_i(x)}^m := Res_{f_1,f_2,\xi_i(x)}(t^m) = \frac{\xi_{i1}^{m_1} \xi_{i2}^{m_2}}{J_f^T(\xi_i)}, \tag{6.4}$$

and the right-hand side of this equality is clearly algebraic. Thus

$$Res_\sigma^m(x) := \sum_{i=1}^{\mathrm{vol}(\sigma)} Res_{f_1,f_2,\xi_i(x)}^m \tag{6.5}$$

is an algebraic Laurent A-hypergeometric function in $\mathscr{U}_{w,\varepsilon}$ with homogeneity $(-1,-1,-m)$. We call it the *local residue relative to the cell* σ.

It follows from Theorem 5.3.3 that for any $m \in (P_1 + P_2)^\circ$ and any generic weight $w \in \mathbb{R}^n$, the residues $\{Res_\sigma^m, \sigma$ maximal cell in $\Pi_w\}$ satisfy the linear relation

$$\sum_\sigma Res_\sigma^m = 0. \tag{6.6}$$

Given $w \in \mathbb{R}^n$ generic and $\gamma \in \mathbb{Z}^4$, we denote by $Alg_{\gamma,w}$ the subspace of A-hypergeometric Laurent algebraic series with homogeneity γ that converge in the direction of w.

We then obtain the following Corollary:

Corollary 6.1.2. *For generic $w \in \mathbb{R}^n$, $m \in (P_1 + P_2)^\circ$ and $\gamma = (-1,-1,-m)$ it holds*

$$R_{\gamma,w} := \langle Res_\sigma^m,\ \sigma \textit{ a mixed cell in } \Pi_w \rangle \subseteq Alg_{\gamma,w}. \tag{6.7}$$

Moreover

$$\dim(R_{\gamma,w}) \leq \#(\textit{mixed cell in } \Pi_w) - 1. \tag{6.8}$$

6.2 Algebraic solutions as residues

In this section we present one of our fundamental result in this chapter: Theorem 6.2.2. where we find the expression of the residues associated to cells in a mixed subdivision in terms of canonical series solutions.

Notation 6.2.1. *Let $w \in \mathbb{R}^n$ be a generic weight and consider a cell $\sigma \in \Pi_w$. We will denote by $k_{\sigma,v} \in \{1,-1,0\}$ the combinatorial Khovanskii coefficient of the vertex $v \in \sigma$ (cf.* (5.28)*) associated to f_1^σ and f_2^σ (cf.* (6.1) *and* (6.2)*). Here $Q_1 = A_1^\sigma$ and $Q_2 = A_2^\sigma$ which are developed since* $\dim(\sigma) = 2$.

Being a Laurent A-hypergeometric series converging in $\mathscr{U}_{w,\varepsilon}$, Theorem 5.1.4 says that the local residue relative to the cell σ can be written in terms of the canonical Laurent series $\Phi^\gamma_{r_I}$ in the direction of w. The next theorem shows that, for $m \in (P_1+P_2)^\circ \cap \mathbb{Z}^2$, in the corresponding expansion of Res^m_σ the only $\Phi^\gamma_{r_I}$ that appear are those such that $I = \{i,j\}$ and r_I is a infinite region corresponding to a point $\alpha_i + \beta_j \in \sigma \cap (P_1+P_2)^\circ$. Moreover, the coefficients are determined.

Theorem 6.2.2. *Let $w \in \mathbb{R}^n$, $m \in (P_1+P_2)^\circ \cap \mathbb{Z}^2$ and σ a mixed cell in a subdivision Π_w of P_1+P_2. Assume that all non empty $r_{(I,(-1,-1,-m)}$ with $I\{1,\dots,n\} \subset \sigma$ with $|I| = 3$ are regular supports with regular ray extremal for some weight. Then*

$$Res^m_\sigma(x) = \sum k_{\sigma,v} \Phi^\gamma_{r_{\{i,j\}}}(x) \tag{6.9}$$

where v runs over the points $\alpha_i + \beta_j \in \sigma \cap (P_1+P_2)^\circ$.

In order to prove Theorem 6.2.2 we will need the following Lemmas.

Lemma 6.2.3. *Let $w \in \mathbb{R}^n$ generic such that σ is mixed cell with $\sigma \subset \Pi_w$ and $m \in \mathbb{Z}^2$, $\gamma = (-1,-1,-m)$. In the expansion:*

$$Res^m_\sigma = \sum_{r_I \in r_w} k_I \Phi^\gamma_{r_I}, \tag{6.10}$$

the coefficients k_I are independent of the choice of m.

Proof. Take $m, m' \in \mathbb{Z}^2$. Then there exists $q, q' \in \mathbb{N}^n$ such that

$$\partial^q(Res^m_\sigma) = \partial^{q'}(Res^{m'}_\sigma).$$

In fact, by Proposition 5.3.2 it is enough to take q, q' such that $q_1 + \cdots + q_r = q'_1 + \cdots + q'_r$, $q_{r+1} + \cdots + q_{r+s} = q'_{r+1} + \cdots + q'_{r+s}$ and $m + \sum_{i=1}^r q_i\alpha_i + \sum_{j=1}^s q_{r+j}\beta_j = m' + \sum_{i=1}^r q'_i\alpha_i + \sum_{j=1}^s q'_{r+j}\beta_j$. This can always be done, because of our hypothesis $\mathbb{Z}A = \mathbb{Z}^4$.

Taking into account the respective expansions of both functions (cf. Theorem 5.1.4(ii)) and interchanging infinite sum with derivative (which is allowed by uniform convergence) we obtain:

$$\sum_{r_I \in r_w} k_I(m)\partial^q\left(\Phi^\gamma_{r_I}\right) = \sum_{r_I \in r_w} k_I(m')\partial^{q'}\left(\Phi^{\gamma'}_{r_I}\right)$$

where $\gamma' = (-1, -1, -m)$. By Proposition 5.1.8 we obtain the identity:

$$\sum_{r_I \in r_w} k_I(m) \left(\Phi_{r_I}^{\gamma - Aq}\right) = \sum_{r_I \in r_w} k_I(m') \left(\Phi_{r_I}^{\gamma' - Aq'}\right). \tag{6.11}$$

Let $\delta = m - Aq = m' - Aq'$. Since the series $(\Phi_{r_I}^{\delta})_I$ are a basis of the space of Laurent A-hypergeometric functions with homogeneity δ, and that, by Theorem 5.1.10 and Remark 5.2.7, all the coefficients $k_I(m)$ and $k_I(m')$ appearing in the expansions of Res_σ^m and $Res_\sigma^{m'}$ respectively also appear in (6.11), we conclude that the coefficients $k_I(m)$ coincide with $k_I(m')$. □

Lemma 6.2.4. *Let $w \in \mathbb{R}^n$ generic such that σ is mixed cell with $\sigma \subset \Pi_w$. Suppose that $m = \alpha_{i_0} + \beta_{j_0} \in \sigma \cap (P_1 + P_2)^\circ$. Then there exists $\tilde{w}$ such that $\Pi_w = \Pi_{\tilde{w}}$ and*

$$\text{in}_{\tilde{w}}(Res_\sigma^m(x)) = Res_{f_1^\sigma, f_2^\sigma}^m = k_{\sigma,m} x_{i_0}^{-1} x_{r+j_0}^{-1}. \tag{6.12}$$

Proof. Let $\tilde{w}$ as in 6.1.1. Putting f_1^σ and f_1^σ instead of f_1 and f_2 as in (6.1) and (6.2) respectively, and A_1^σ and A_2^σ instead of P_1 and P_2 in Theorem 5.4.3 we have that $Res_{f_1^\sigma, f_2^\sigma}^m \neq 0$. Then

$$\text{in}_{\tilde{w}}(Res_\sigma^m(x)) = \sum_{i=1}^{\text{vol}(\sigma)} \text{in}_{\tilde{w}}\left(Res_{f_1, f_2, \xi_i(x)}^m\right) = \sum_{i=1}^{\text{vol}(\sigma)} \text{in}_{\tilde{w}}\left(\frac{\xi_i^m}{J_f^T(\xi_i)}\right) = \sum_{i=1}^{\text{vol}(\sigma)} \frac{\text{in}_{\tilde{w}}(\xi_i^m)}{\text{in}_{\tilde{w}}(J_f^T(\xi_i))} =$$

$$= \sum_{i=1}^{\text{vol}(\sigma)} \frac{(\xi_i^\sigma)^m}{J_{f^\sigma}^T(\xi_i^\sigma)} = Res_{f_1^\sigma, f_2^\sigma}^m \tag{6.13}$$

To see that the last equality in (6.12) holds, we use Theorem 5.4.1 for f_1^σ, f_2^σ . Name $v_{hl} = \alpha_{i_h} + \beta_{j_l}$, $h, l = 0, 1$ (note that $m = v_{00}$). Then we obtain:

$$Res_{f_1^\sigma, f_2^\sigma}(t^m) = \sum_{h,l=0}^{1} k_{\sigma.v_{hl}}(v_{hl}) \cdot R_{v_{kl}}(t^m). \tag{6.14}$$

From Remark 5.2.4 and Proposition 5.4.2 we obtain that $R_{v_{kl}}(t^m) = 0$ if $v_{hl} \neq v_{00} = m$. On the other hand, $R_m(t^m) = k_{\sigma,m} x_{i_0}^{-1} x_{r+j_0}^{-1}$ is an easy calculation. □

Now we are ready to prove Theorem 6.2.2.

Proof (of Theorem 6.2.2). By Lemma 6.2.3 we write

$$Res_\sigma^m = \sum_{r_I \in r_w} k_I \Phi_{r_I}. \tag{6.15}$$

We will prove that all the coefficients k_I in (6.10) are zero except for those corresponding to $I = \{i, j\}$ with $v = \alpha_i + \beta_j \in (P_1 + P_2)^\circ \cap \sigma$, which will be equal to the Khovanskii combinatorial coefficients $k_{\sigma,m}$ (see Notation 6.2.1).

First consider a minimal region $r_I \neq \emptyset$ such that $I \subsetneq \sigma$. By Lemma 2.2.6 there exists $w' \in \mathbb{R}^n$ weight vector such that $\sigma \in \Pi_{w'}$ and $I \nsubseteq \Pi_{w'}$. Then Φ_{r_I} does not appear in the expansion in the direction of w', that is, the corresponding coefficient is zero. As Res_σ^m is also a Laurent solution with respect to w, both expansions coincide (see Remark 5.1.7). So

$$Res_\sigma^m = \sum_{\substack{I \subset \sigma \\ r_I \neq \emptyset}} k_I \Phi_{r_I}. \tag{6.16}$$

Now we are left with minimal regions $r_I \neq \emptyset$ such that $I \subseteq \sigma$. It is easy to see that if $k_I \neq 0$, then I cannot be the whole of σ, since in this case I is not contained in any other mixed cell and the correspoding summand cannot be canceled as the Euler Jacobi theorem 5.3.3 asserts. We discard all regions r_I with $|I| = 3$ using Corollary 5.5.4 by our hypothesis on these supports.

So we see that the only coefficients that may be distinct to zero are those corresponding to regions $r_I \neq \emptyset$ with $I = \{i, j\}$, with $\alpha_i + \beta_j \in (P_1 + P_2)^\circ \cap \sigma$. As $\gamma \in \mathcal{E}$, we deduce from Proposition 5.2.8 that $\alpha_i + \beta_j$ lies in the interior $(P_1 + P_2)^\circ$ of the Minkowski sum.

Finally we compute the coefficient corresponding to $I = \{i, j\}$ with $\alpha_i + \beta_j \in (P_1 + P_2)^\circ \cap \sigma$. Assume, by Lemma 6.2.3, that $m = \alpha_i + \beta_j$. Then Lemma 6.2.4 shows that $k_{\sigma,m} x_i^{-1} x_{r+j}^{-1}$ appear in the expansion of Res_σ^m. But this is a term in the canonical Laurent series $\Phi^\gamma_{r_{\{i,j\}}}$ and then the whole series must appear multiplied by the coefficient $k_{\sigma,m}$. □

We will show in the next Section that the hypotheses of regularity and extremality of negative supports with cardinality three hold in codimension two if all points α_i, β_j are vertices.

We obtain the following variant of Theorem 5.3.3.

Corollary 6.2.5. *If σ has no vertices in $(P_1 + P_2)^\circ$ and $m \in (P_1 + P_2)^\circ$ then $Res_\sigma^m = 0$.*

Example 6.2.6. *Assume that we have two configurations of dimension one, for instance let $A_1 = \{\alpha_{i+1} = (i, 0), 0 \leq i \leq r - 1\}$ and $A_2 = \{\beta_{j+1} = (0, j), 0 \leq j \leq s - 1\}$, $r, s \geq 2$. Then, a possible coherent mixed subdivision of $A_1 + A_2$ consists of the following $s - 1$ cells σ_j, $j = 1, \ldots, s - 1$, which are all mixed and rectangular, with the four vertices in the boundary of $A_1 + A_2$:*

$$\sigma_j = \{1, r, j, j + 1\}.$$

Then, by Corollary 6.2.5, the sum of the residues $R^m_{\sigma_j}$ over each one of these cells is 0, which gives new linear relations among them, different from the Euler-Jacobi vanishing condition. As before, denote by $f_1 = f_1(x)$, $f_2 = f_2(y)$ polynomials with variable coefficients associated to A_1, A_2. These zero residues $R^m_{\sigma_j}$ correspond to the sum of the residues over all the points (a, y_j), $j = 1, \ldots, s - 1$, here a is a fixed zero of f_1 and y_j ranges over all the zeros of f_2.

6.3 The case $n = 6$

In this section we will make a detailed description of algebraic Laurent series solutions of the system $H_A(\gamma)$ in terms of residues, in case $n = r + s = 6$, $r, s \geq 2$, which corresponds to non pyramidal codimension two Cayley configurations. As before $\gamma = (-1, -1, -m)$ with $m \in (P_1 + P_2)^\circ$. In view of Example 6.2.6, we will deal with the following cases: A_1, A_2 are two trinomials with $\dim(A_1) = \dim(A_2) = 2$ or A_1 is a binomial and A_2 quatrinomial with $\dim(A_2) = 2$ and all α_i, β_j are vertices.

Under these hypotheses, we have the following precise version of Corollary 6.1.2. Recall that given $w \in \mathbb{R}^n$ generic and $\gamma \in \mathbb{Z}^4$, we denote by $Alg_{\gamma,w}$ the subspace of A-hypergeometric Laurent algebraic series with homogeneity γ that converge in the direction of w.

Theorem 6.3.1. *Let w be a generic weight in $\mathbb{R}^n$, $n = 6$ and A_1, A_2 satisfying the hypotheses at the beginning of section 6.3. Then, for any $\gamma = (-1, -1, -m)$ with $m \in (A_1 + A_2)^\circ \cap \mathbb{Z}^2$,*

$$Alg_{\gamma,w} = R_{\gamma,w} \tag{6.17}$$

and

$$\dim(R_{\gamma,w}) = \#(\textit{mixed cell in } \Pi_w) - 1 = \#\textit{interior vertices of } \Pi_w. \tag{6.18}$$

In particular, the only non-trivial linear relation (up to multiplicative constant) among the residues R^m_σ, where σ runs over the mixed cells of Π_w, is the Euler-Jacobi vanishing condition in Theorem 5.3.3.

Proof. We list the following facts that follow by inspecting all possible cases, which we do in the following subsections 6.3.1 and 6.3.2.

(a) Let $I = \{i, j\}$. By Proposition 2.2.4, the minimal region $r_{I,\gamma}$ lies in the direction of w if and only if $\alpha_i + \beta_j$ is an interior vertex of Π_w. In the $n = 6$ case, all these vertices lie in a mixed cell σ.

(b) The number of mixed cells in Π_w equals the number of interior vertices of Π_w plus one.

To prove equality in (6.17), we need to show that all canonical series $\Phi_{ij} := \Phi^\gamma_{r_{\{i,j\}}}$ corresponding to interior vertices of Π_w can be written in terms of residues. Theorem 6.2.2 gives the explicit expressions:

$$Res^m_\sigma(x) = \sum k_{\sigma,v} \Phi^\gamma_{r_{\{i,j\}}}(x),$$

that we need to invert. If $\alpha_i + \beta_j \in (P_1 + P_2)^\circ$ is the only interior vertex of a mixed cell $\sigma \in \Pi_w$, then the canonical series coincides with the residue associated to this cell up to sign. Otherwise, it is not so immediate but it is still straightforward in all cases we are considering. For instance, let $r = s = 3$ and A_1, A_2 as in example 2.2.7. Take $w \in \mathbb{R}^6$ with $\sum_{i=1}^6 w_i b_i$ in the positive cone generated by b_2 and b_6. The corresponding subdivision has four mixed cells $\sigma_1 = \{2, 3, 5, 6\}$, $\sigma_2 = \{2, 3, 4, 5\}$, $\sigma_3 = \{1, 3, 4, 5\}$, and $\sigma_4 = \{1, 3, 5, 6\}$, and three interior vertices corresponding to the indices $\{3, 5\}, \{3, 4\}$ and $\{1, 5\}$. Then,

$$\Phi_{3,5} = Res^m_{\sigma_1}, \Phi_{3,4} = Res^m_{\sigma_1} + Res^m_{\sigma_2}, \Phi_{1,5} = Res^m_{\sigma_1} + Res^m_{\sigma_4}.$$

Then, the residues $Res^m_{\sigma_1}, Res^m_{\sigma_2}, Res^m_{\sigma_4}$ are linearly independent and there cannot be any linear relations other than the Euler-Jacobi relation. □

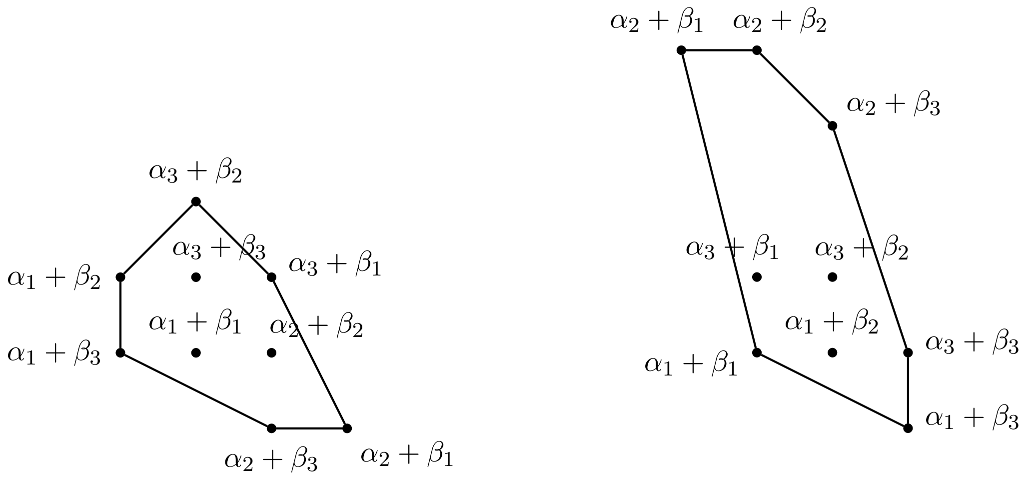
6.3.1 The case $r = s = 3$

We study in detail the developed case. In the non-developed case, we only present the "extreme" Example 6.3.8.

In this case we notice the following situation peculiar to the Minkowski sum $P_1 + P_2$, assuming that P_1 and P_2 are developed: it is known that the facets of the Minkowski sum of two polygons are parallel to a facet (edge) of one of them. Note that there are two possible configurations of the edges of $P_1 + P_2$ with respect to their alternation, that is, the way they concatenate. In the *alternate* case, either no facet parallel to a facet of P_1 is adjacent to any other facet of P_1 (and consequently, the same phenomenon occurs with P_2). The other possibility, which we call *non-alternate*, corresponds to the occurrence of two consecutive facets parallel to facets of P_1 (and then also two consecutive facets parallel to facets of P_2). Call, by simplicity, Case 1 to the alternate case and Case 2 to the non-alternate case.

Before giving precise statements and proofs. we show the two different patters of coherent mixed subdivisions and minimal regions corresponding to homogeneities $(-1, -1, -m)$ with m in the interior of the Minkowski sum $P_1 + P_2$. These examples feature the general patterns.

Example 6.3.2 (Case 1: alternate polygons)**.** *In Figure 2.4 we drew the Minkowski sum of the polytopes from Example 2.2.8. In this case, the Minkowski sum is alternate (Case 1). In Figure 6.1(a) we depicted again $P_1 + P_2$. Consider the corresponding Cayley configuration A*



(a) Case 1 (b) Case 2

Figure 6.1: The two possible cases for $P_1 + P_2$

and a choice B of Gale dual:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ -1 & 1 \\ -1 & -2 \\ 2 & 1 \end{pmatrix}.$$

The recession cones of the minimal regions in the hyperplane arrangement defined by B (cf. (5.11)*) are depicted in Figure 6.2. There are only three minimal regions, with negative support of cardinality two, corresponding to the three interior points $\alpha_i + \beta_j$ of $P_1 + P_2$.*

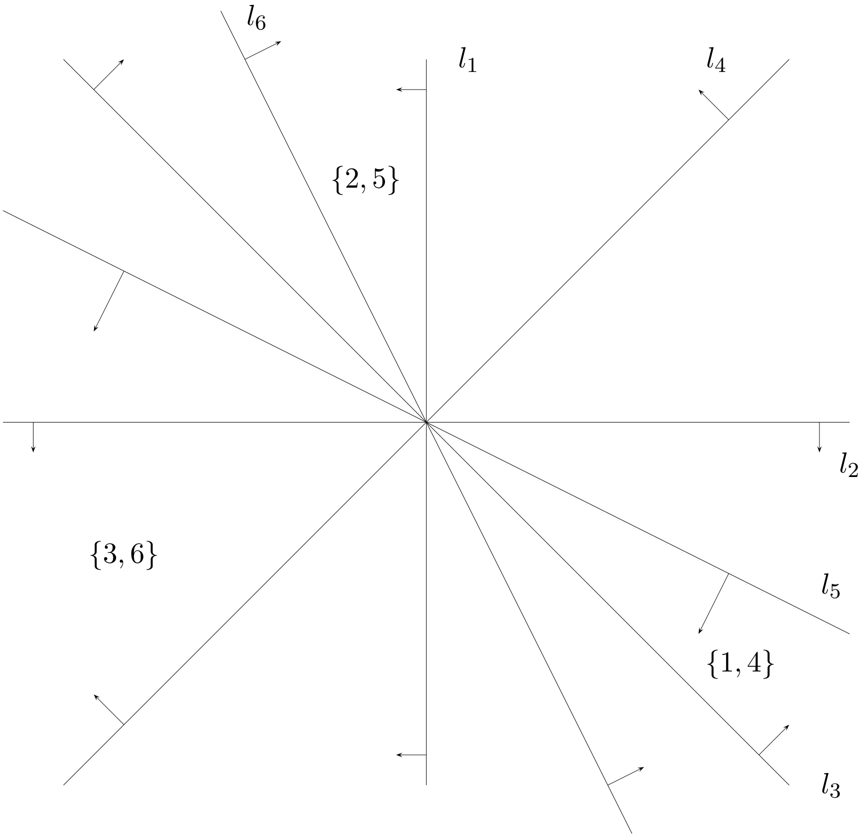


Figure 6.2: Minimal regions for an alternate case

Example 6.3.3 (Case 2: non-alternate polygons)**.** *In Figure 2.2 we drew the Minkowski sum of the polytopes from Example 2.2.7. In this case, the Minkowski sum is non-alternate (Case 2). In Figure 6.1(b) we depicted again $P_1 + P_2$. Consider the corresponding Cayley configuration A and a choice B of Gale dual:*

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 3 \\ 0 & 4 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 1 & -3 \\ 1 & 0 \\ -2 & -1 \\ 1 & 1 \end{pmatrix}.$$

The corresponding Minkowski sum is depicted in Figure 6.1(b). In Figure 6.3 we illustrate the recession cones of the minimal regions that appear. Note that the consecutive edges coming from each polygons give rise to minimal regions with negative support of cardinality three, besides the three minimal regions coming from the interior points $\alpha_i + \beta_j$.

The following lemma is a reformulation of Lemma 5.2.1.

Lemma 6.3.4. *The possible negative supports for $u \in \mathbb{Z}^6$ and $A \cdot u = \gamma = (-1, -1, -m)$ are of the shape $\{i_1, i_2, j_1, j_2\}$ (possibly $i_1 = i_2$ or $j_1 = j_2$).*

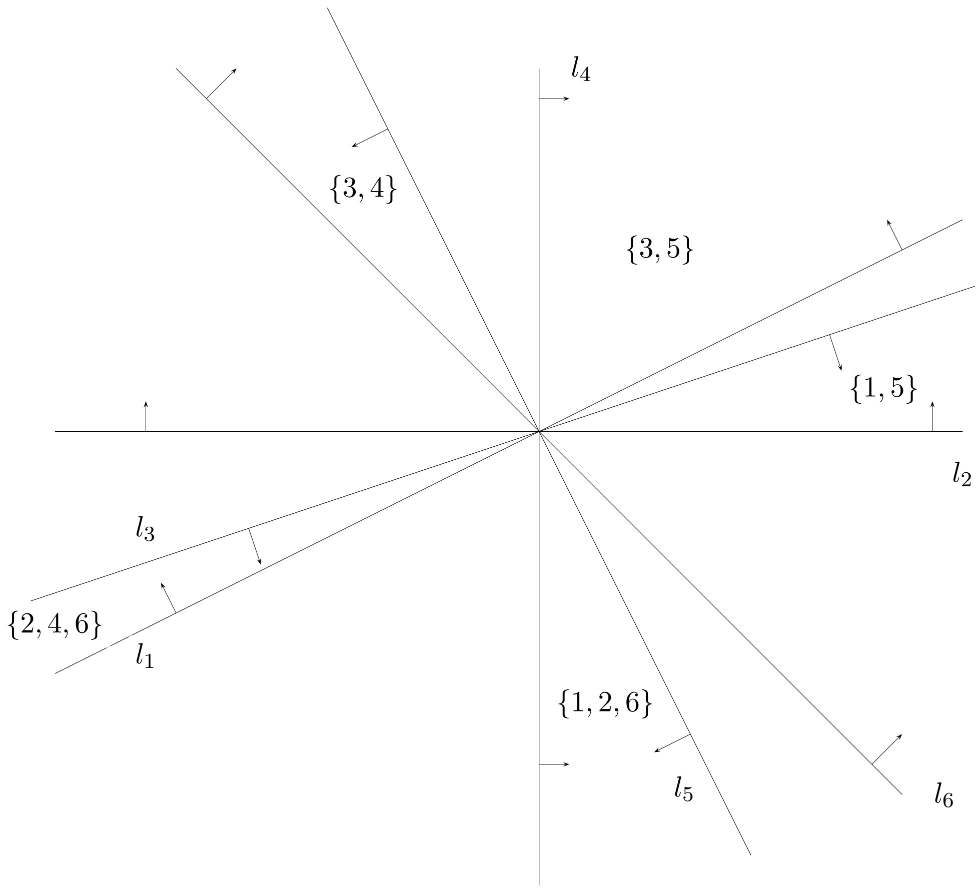


Figure 6.3: Minimal regions for a non-alternate case

Proof. If $u \in \mathbb{Z}^6$ and $A \cdot u = (-1, -1, -m)$ then $u_1 + u_2 + u_3 = -1$ and $u_4 + u_5 + u_6 = -1$. □

Lemma 6.3.5. *In Case 2 the regions $r_{\{2,4,6\}}$ and $r_{\{1,2,6\}}$ are non empty and infinite.*

Proof. To show that the region $r_{\{2,4,6\}}$ is non empty and infinite (the other case is similar) we will find $u \in \ker_{\mathbb{Z}}(A)$ with $u_2, u_4, u_6 < 0$ and all other coordinates strictly positive. The lemma follows because by our assumptions on A we can always find $\nu \in \mathbb{Z}^6$ with $A \cdot \nu = \gamma$ and then $\lambda u + \nu \in r_{\{2,4,6\}}$ for infinite $\lambda \in \mathbb{N}$.

It is clear from Figure6.1(b) that the segment between $\alpha_2 + \beta_1$ and $\alpha_2 + \beta_3$ and the segment between $\alpha_2 + \beta_2$ and $\alpha_3 + \beta_3$ intersect at a point that we can write in two different ways, i.e., there exists with $q_1, q_2, p_1, p_2 \in \mathbb{N}$ with $q_1 + q_2 = Q$ and $p_1 + p_2 = P$

$$Pq_1(\alpha_2 + \beta_1) + Pq_2(\alpha_2 + \beta_3) = Qp_1(\alpha_2 + \beta_2) + Qp_2(\alpha_3 + \beta_3)$$

from which

$$0 = 0 \cdot \alpha_1 + Q(p_1 - P)\alpha_2 + Qp_2\alpha_3 + (-Pq_1)\beta_1 + Qp_1\beta_2 + (Qp_2 - Pq_2)\beta_3$$

Then the element $v = (0, Q(p_1 - P), Qp_2, -Pq_1, Qp_1, Qp_2 - Pq_2)$ belongs to $\ker_{\mathbb{Z}}(A)$. Note that $v_2, v_4 < 0$, $v_3, v_5 > 0$ and $v_1 = 0$. Similarly, but using $\alpha_1 + \beta_1$ instead of $\alpha_3 + \beta_3$ we obtain an element $\hat{v} \in \ker_{\mathbb{Z}}(A)$ with $\hat{v}_2, \hat{v}_6 < 0$, $\hat{v}_1, \hat{v}_5 > 0$ and $\hat{v}_3 = 0$. If $v_6 \leq 0$ or $\hat{v}_4 \leq 0$ it is possible to find a suitable linear combination to obtain $u \in \ker_{\mathbb{Z}}(A)$ with $u_2, u_4, u_6 < 0$ and all other coordinates strictly positive and we are done. Otherwise, that is, if $v_6 > 0$ and $\hat{v}_4 > 0$ we have a Gale dual B for A with the following configuration of signs:

$$B = \begin{pmatrix} =0 & >0 \\ <0 & <0 \\ >0 & =0 \\ <0 & >0 \\ >0 & >0 \\ >0 & <0 \end{pmatrix}. \tag{6.19}$$

Denote b_i, $i = 1, \ldots, 6$ the rows of B. Then, we have that

$$\frac{\det(A_{\{1,2,3,4\}})}{\det\begin{pmatrix} b_5 \\ b_6 \end{pmatrix}} = -\frac{\det(A_{\{1,2,3,5\}})}{\det\begin{pmatrix} b_4 \\ b_6 \end{pmatrix}}$$

where A_I is the square matrix obtained from A after subtracting the columns not in I. Here

$$\det(A_{\{1,2,3,5\}}) = \det(A_{\{1,2,3,4\}}) = \det(\alpha_2 - \alpha_1, \alpha_3 - \alpha_2).$$

Thus, being $\det\begin{pmatrix} b_5 \\ b_6 \end{pmatrix} < 0$ it must hold that $\det\begin{pmatrix} b_4 \\ b_6 \end{pmatrix} > 0$, which implies,together with the sign configuration in 6.19, the existence of $\lambda, \mu \in \mathbb{Z}_{>0}$ such that

$$\begin{pmatrix} b_4 \\ b_6 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} <0 \\ <0 \end{pmatrix}.$$

Consequently $u = \lambda v + \mu \hat{v} \in \ker_{\mathbb{Z}}(A)$ with $u_2, u_4, u_6 < 0$ and all other coordinates strictly positive and we are done. $\square$

Proposition 6.3.6. *The following regions for $\gamma = (-1, -1, -m) \in \mathcal{E}$ are non empty and minimal:*

i) $r_{\{1,4\}}$, $r_{\{3,6\}}$ *and* $r_{\{2,5\}}$ *in Case 1, and*

ii) $r_{\{3,4\}}$, $r_{\{3,5\}}$ *and* $r_{\{1,5\}}$ *in Case 2.*

Proof. By Corollary 5.2.9 and Lemma 5.2.5 regions $\{i, 3+j\}$ with $\alpha_i + \beta_j$ an interior point of $P_1 + P_2$ are non empty. Moreover, they are infinite, by Proposition 5.2.8. By Lemma 6.3.4 they are minimal. $\square$

Theorem 6.3.7. *The only minimal regions for $\gamma = (-1, -1, -m) \in \mathcal{E}$ are*

i) $r_{\{1,4\}}$, $r_{\{3,6\}}$ *and* $r_{\{2,5\}}$ *in Case 1, and*

ii) $r_{\{3,4\}}$, $r_{\{3,5\}}$ *and* $r_{\{1,5\}}$, $r_{\{1,2,6\}}$ *and* $r_{\{2,4,6\}}$ *in Case 2.*

Proof. The proof is similar for both Case 1 and Case 2. We show that all others possible regions either are empty or cannot be minimal. For instance, in Case 1, all possible negative supports with four distinct elements contain either $\{1,4\}$, $\{3,6\}$ or $\{2,5\}$ (respectively $\{3,4\}$, $\{3,5\}$ or $\{1,5\}$ in Case 2), that is, they do not give minimal regions. Regions of the shape $r_{\{i,3+j\}}$ with $\alpha_i + \beta_j$ a vertex of $P_1 + P_2$ are finite by Lemma 5.2.6, and finite regions cannot occur if the homogeneity belongs to Euler-Jacobi cone of A, as in this case (cf. Remark 5.2.7). In fact, finite regions are clearly annihilated by some partial derivative (cf. Theorem 5.1.10).

By inspecting the Case 1 in Figure 6.1 we see that if $\{i, 3+j, 3+j'\}$ or $\{i, i', 3+j\}$ do not contain neither $\{1,4\}$, $\{3,6\}$ nor $\{2,5\}$ (which means that the corresponding region could be minimal) then $\alpha_i + \beta_j$ and $\alpha_i + \beta_{j'}$ are necessarily vertices of an edge of $P_1 + P_2$. With arguments similar to those in the proof of Lemma 5.2.6, we can show that if the corresponding minimal region is non empty, it has to be finite, which is again impossible. The same occurs in Case 2, except for the regions $r_{\{1,2,6\}}$ and $r_{\{2,4,6\}}$, which thus turn out to be minimal (and infinite). □

In the non-developed case the study is similar, but we only present the following example of two polygons with the same inner normal directions.

Example 6.3.8. *Take the following matrix A*

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \end{pmatrix}$$

with Gale dual

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ -2 & -2 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

We consider the homogeneity $\gamma = (-1, -1, -m)$, with $(1,1) = \alpha_1 - \alpha_2 + \alpha_3 + \beta_2$ an interior point of $P_1 + P_2$. Let $v = (-1, 1, -1, 0, -1, 0)$. We draw in Figure 6.4 the hyperplane arrangement and the corresponding minimal regions. Note that all support indices have cardinality three. So, by Corollary 5.5.4, there are no (nonzero) algebraic Laurent A-hypergeometric functions. This is in concordance with the description of all possible coherent mixed subdivisions in Figure 2.7 of Example 2.2.9. As there is always a single mixed cell, by the Euler-Jacobi condition we have that the associated residue is 0. Then, both items in Theorem 6.3.1 are verified.

6.3.2 The case $r = 2, s = 4$

The remaining case to study for two polynomials in two variables is the one of a binomial and a quatrinomial, that is P_1 is the convex hull of the points $\alpha_1, \alpha_2 \in \mathbb{Z}^2$ and P_2 is the (two dimensional) convex hull of four points $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z}^2$. There are two possibilities: either P_2 has four facets or it is a triangle with β_4 as an interior point (see Figure 6.5).

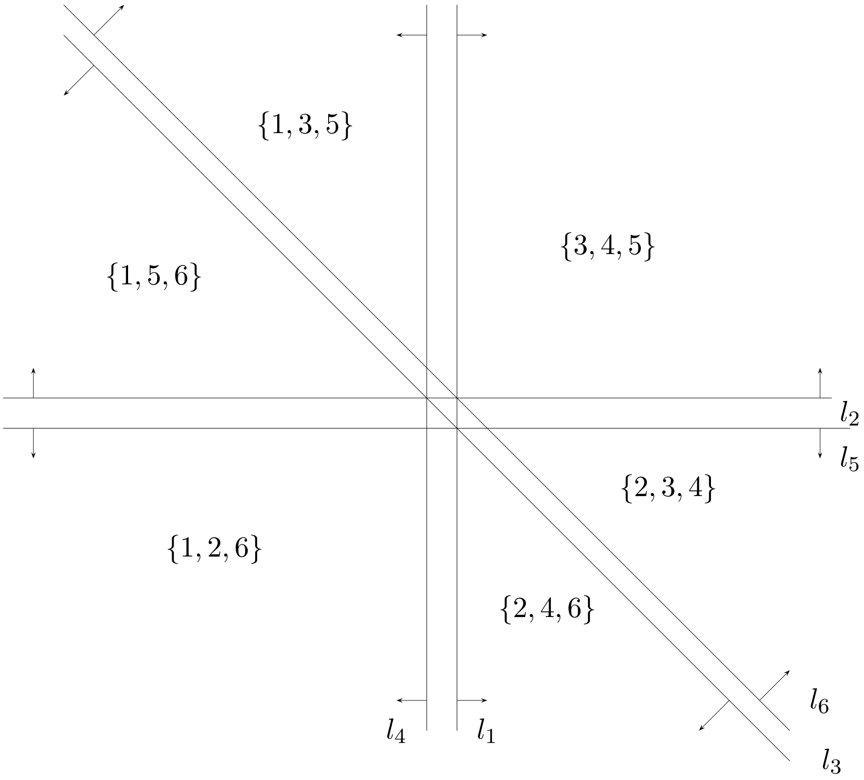
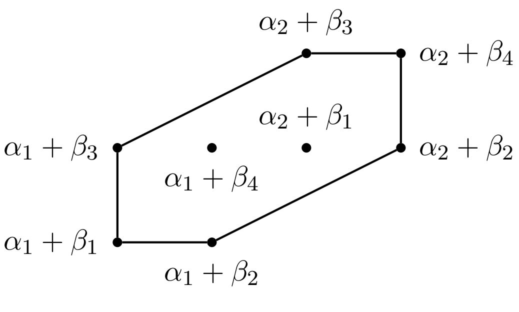
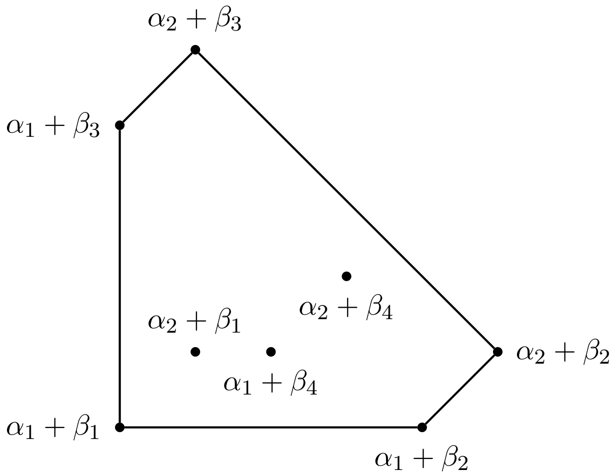


Figure 6.4: Minimal regions for a non developed case



(a) Case 1



(b) Case 2

Figure 6.5: The two possible cases for $P_1 + P_2$ for $r = 2, s = 4$

We will describe the minimal regions that appear in these two cases, in the developed case, that is when the segment P_1 is not parallel to any of the edges of P_2. The remaining cases are also straightforward. We proceed in a similar fashion than in the the case $r = s = 3$. Note that Lemma 6.3.4 holds in the case as well, since it only depends on the fact that $\gamma_1 = \gamma_2 = -1$ are negative.

Lemma 6.3.9. *The regions $r_{\{1,4,5\}}$ and $r_{\{2,4,5\}}$ in Case 1 and the region $r_{\{1,4,5\}}$ in Case 2 are non empty and infinite.*

Proof. For the region $r_{\{1,4,5\}}$ in Case 1, take $q_1, q_2, p_1, p_2 \in \mathbb{N}$ with $q_1+q_2 = Q$ and $p_1+p_2 = P$ to write

$$Pq_1(\alpha_1 + \beta_2) + Pq_2(\alpha_1 + \beta_3) = Qp_1(\alpha_1 + \beta_1) + Qp_2(\alpha_2 + \beta_4).$$

Then, the vector $(Q(p_1 - P), Qp_2, Qp_1, -Pq_1, -Pq_2, Qp_2) \in \ker_{\mathbb{Z}}(A)$. For the region $r_{\{2,4,5\}}$ in Case 1 proceed similarly.

For the region $r_{\{1,4,5\}}$ in Case 2 we can proceed in a similar fashion as in Lemma 6.3.5. □

Finally, we can argue as in the case $r = s = 3$ to give a complete picture of the minimal regions in this case. We have the following analog results and similar conclusions than in the previous case.

Proposition 6.3.10. *Let $m \in (P_1 + P_2)^\circ$ then the only minimal regions for $\gamma = (-1, -1, -m)$ are*

i) $r_{\{1,6\}}$, $r_{\{2,3\}}$, $r_{\{1,4,5\}}$ *and* $r_{\{2,4,5\}}$ *in the Case 1 and*

ii) $r_{\{1,6\}}$, $r_{\{2,6\}}$, $r_{\{2,3\}}$ *and* $r_{\{1,4,5\}}$ *in the Case 2.*

In Case 1, the six different mixed coherent subdivisions Π_w, depicted in Figure 2.9, have two possible different shapes. Either there is one interior vertex ($\alpha_1 + \beta_4 = \alpha_2 + \beta_1$) and two mixed cells σ_1, σ_2 which contain it, or there is a single mixed cell and the only minimal regions in the direction of w have negative support $r_{\{1,4,5\}}$ and $r_{\{2,4,5\}}$, which cannot give rise to Laurent algebraic solutions. Thus, we check the validity of the facts in the proof of Theorem 6.3.1. In Case 2, however, the situation is not covered exactly by this result and we detail it in the next section.

6.4 General conjectures

We end this chapter with natural general conjectures for future work. We expect that under very general conditions, Conjecture 6.4.3 below holds true.

Note that for configurations with interior points, there might be vertices of coherent mixed subdivisions which need not be vertices of a mixed cell. Our next example shows this behaviour and some new associated features.

Example 6.4.1. *Let A be as in Example 2.2.11, corresponding to a configuration termed as Case 2 at the end of the previous section. Note that the two interior points $(1,1) = \alpha_1 + \beta_4 = \alpha_2 + \beta_1$ and $(2,2) = \alpha_2 + \beta_4$ of $A_1 + A_2$ show the following "new" behaviour:*

- *Both occur as vertices of a coherent mixed subdivision, but not as vertices of a mixed cell of this subdivision.*

- *The point $(2,2)$ never occurs as a vertex of a mixed cell without $(1,1)$ being another such vertex. They "share" the same mixed cells.*

Let $\gamma = (-1,-1,-1,-1)$. The corresponding central arrangement is depicted in Figure 6.6, using the Gale dual configuration given by $\{(-1,1),(0,1),(0,1),(1,-3),(1,0),(-1,0)\}$.

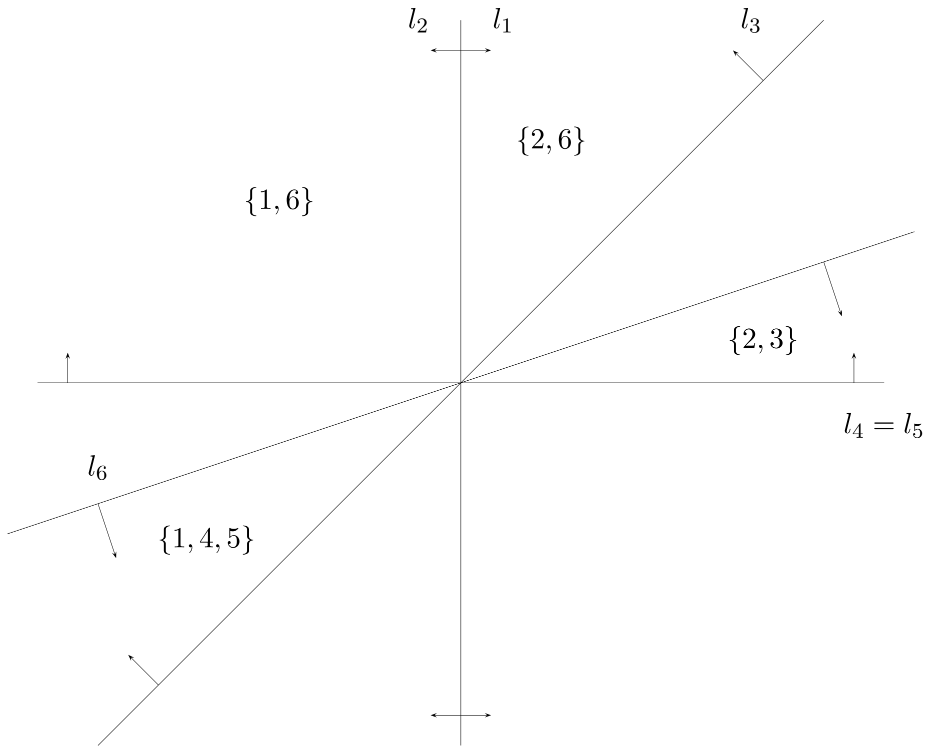


Figure 6.6: Minimal regions for Example 6.4.1

There are three minimal regions with negative support of cardinality three (one corresponding to $(2,2)$ and the two others corresponding to the two expressions for $(1,1)$.) Let w with $b_w := \sum_{i=1}^{6} w_i b_i$ lying in the positive cone spanned by b_1 and b_4, b_5. We see that the point $(2,2)$ is a vertex of Π_w which is not in a mixed cell. The corresponding region (with minimal support $I = \{2,6\}$) gives rise to a non-algebraic *canonical Laurent series Φ_I. This can be seen by restricting Φ_I to the vertical boundary ray.*

The canonical series Φ_J, $J = \{2,3\}$ corresponding to the decomposition $(1,1) = \alpha_2 + \beta_2$ is algebraic since this point is a vertex of a mixed cell in Π_w, for any w with b_w in the positive cone spanned by b_6 and b_4, b_5. On the other side, the canonical series Φ_K, $K = \{1,6\}$, corresponding to the decomposition $(1,1) = \alpha_1 + \beta_4$ is again not algebraic (seen by restriction to its vertical boundary ray). In this case the point does not occur as a vertex of a mixed cell (but it is a vertex of Pi_w for any b_w in the positive cone spanned by b_2 and b_3.).

Indeed, let any w with b_w in the positive cone spanned by b_3 and b_4, b_5. The corresponding coherent mixed decomposition has two mixed cells σ_1, σ_2 and the dimension of the space of *residues is one. However, there are two interior vertices* $(2,2)$ *and* $(1,1)=\alpha_1+\beta_4$*; choosing* m *equal to any of them, Theorem 6.2.2 gives us the decomposition*

$$Res_{\sigma_i}^m = \Phi_K - \Phi_J.$$

Thus, the linear combination $\Phi_K - \Phi_J$ *is algebraic and the dimension of the space of algebraic* A*-hypergeometric functions in the direction of* w *is one, that is, it is equal to the number of mixed cells minus one, but it does not equal the number of interior vertices of these cells, which is two.*

We now state our first conjecture:

Conjecture 6.4.2. *Let* $w \in \mathbb{R}^n$ *generic and* $v = \alpha_i + \beta_j \in \Pi_w$ *an interior point of* $P_1 + P_2$ *which does not belong to any mixed cell. Then the canonical Laurent series with negative support* $\{i, j\}$ *is not algebraic. Here it is assumed that if* v *can also be written as* $\alpha_\ell + \alpha_k$*, with* $i \neq \ell, j \neq k$ *then* $w_i + w_j < w_\ell + w_k$*.*

We now state our main conjecture:

Conjecture 6.4.3. *Let* w *be a generic weight in* $\mathbb{R}^n$*, and* A_1, A_2 *two general planar configurations. Then, for any* $\gamma = (-1,-1,-m)$ *with* $m \in (A_1 + A_2)^\circ \cap \mathbb{Z}^2$*,*

$$Alg_{\gamma,w} = R_{\gamma,w} \tag{6.20}$$

It would be interesting to find combinatorial conditions that ensure that all canonical series $\Phi_{ij} := \Phi^\gamma_{r_{\{i,j\}}}$ corresponding to interior vertices of Π_w which lie in mixed cells, can be written in terms of residues. We would need to invert the explicit expressions given in Theorem 6.2.2:

$$Res_\sigma^m(x) = \sum k_{\sigma,v} \Phi^\gamma_{r_{\{i,j\}}}(x).$$

This seems to be easy in the case all the points of the configurations are vertices but rather difficult in general. The understanding of this question would lead to an explicit combinatorial formula for the dimension of $Alg_{\gamma,w}$.

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