UNIVERSIDAD DE BUENOS AIRES
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Tesis de Licenciatura

On the renormalization group of quantum field theory (after R. Bocherds)

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Introduction

The aim of this thesis is to analyze part of the results presented by R.E. Borcherds in his article on renormalization and quantum field theories [23]. To facilitate this, the first chapter presents the basic facts on category theory, sheaf theory and the theory of distributions on a manifold. In the particular case of distributions on manifolds we differentiate between the H-distribution (introduced by Hörmander [15]) and the D-distributions (described in the book of Dieudonné [14]) and extend the product of distributions defined by Hörmander only for H-distibution, to an action from H-distributions over D-distributions. Each one of these facts will be used along the thesis in order to exhibit more detailed descriptions and explanations for the results.

In Chapter 2 the concepts of spacetime, fields, Lagrangians, propagators and Feynman measure are defined. The notion of Feynman measure is introduced with a more detailed explanation of the Gaussian condition and its relation with the cut propagator.

In the last chapter we define the renormalization group and immediately characterize it using the results from Chapter 1. The algebraic structure of this group is described in detail. We present a descending filtration of the renormalization group by normal subgroups, compute the corresponding quotients and commutators, and show that any element of the renormalization group can be written as an infinite product of elements each one living in one of the aforementioned subgroups. Lastly we show that an action from the renormalization group over the set of Feynman measures is well defined and prove that this action is transitive over the Feynman measures associated with a given cut local propagator.

Contents

Chapter 1

Preliminary results

Along this thesis we will need certain level of knowledge in category theory, sheaf theory and the theory of distributions on manifolds. Since we shall just use some specific result in each of these areas, we shall present them. The reader familiarized with these concepts can omit this chapter and eventually come back next to regard a specific result if needed.

1.1 Results on category theory

1.1.1 Basic notions

For the basic definitions on category theory, we refer the reader to [1], Ch.9, or [7], Ch.XI.1. Moreover, the definitions of monoidal (or tensor) category, monoidal (or tensor) functor and monoidal natural transformation can be found in [7], Ch.XI.2. and XI.4.

We shall use the letters $\mathcal{C}$, $\mathcal{D}$,... to denote a monoidal category. If $\mathcal{C}$ is a monoidal category we will denote by $\otimes_{\mathcal{C}} : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ the *tensor product* bifunctor and by $\mathbf{1}_{\mathcal{C}}$ the unit object. The left and right unit morphisms will be denoted by $l_M : \mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{C}} M \to M$ and $r_M : M \otimes_{\mathcal{C}} \mathbf{1}_{\mathcal{C}} \to M$ for any $M \in Obj(\mathcal{C})$; whereas the associativity morphism will be denoted by $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for any A, B and C objects of $\mathcal{C}$.

A strict monoidal category is a category where the associativity and unit morphism are the identities of the category, i.e. for all X, Y and Z objects of $\mathcal{C}$ we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $\mathbf{1}_{\mathcal{C}} \otimes X = X = X \otimes \mathbf{1}_{\mathcal{C}}$.

We present a well-known result about monoidal categories, whose proof can be found in [7] Theorem XI.5.3

Theorem 1.1.1. *Any monoidal category is monoidal (or tensor) equivalent to a strict one.*

This theorem implies Mac Lane´s coherence theorem which states that in a monoidal category all diagrams built with the unit, associativity or identity morphisms commute.

Hence from now on we will suppose for simplicity in the exposition that all our monoidal categories are strict.

Definition 1.1.2. *Given a monoidal category $\mathcal{C}$, a triplet (A, m, u) with $A \in Obj(\mathcal{C})$, $m \in Hom_{\mathcal{C}}(A \otimes A, A)$ and $u \in Hom_{\mathcal{C}}(\mathbf{1}, A)$ is called an* unitary algebra in $\mathcal{C}$ *if we have:*

$$m \circ (m \otimes Id_A) = m \circ (Id_A \otimes m),$$
$$m \circ (u \otimes Id_A) = Id_A,$$
$$m \circ (Id_A \otimes u) = Id_A.$$

Remark 1.1.3. $\mathbf{1}_{\mathcal{C}}$ with the product $l_{\mathbf{1}_{\mathcal{C}}} = r_{\mathbf{1}_{\mathcal{C}}}$ (see [7], Lemma XI.2.3) and unit $Id_{\mathbf{1}_{\mathcal{C}}}$ is a unitary algebra.

Notice that the natural domain of $m \otimes Id_A$ is $(A \otimes A) \otimes A$ and for $Id_A \otimes m$ it is $A \otimes (A \otimes A)$, so the only reason why the first equation of the definition makes sense is because in a strict monoidal category the two domains are equal. A similar observation applies to the second and third equations: if we were not in a strict monoidal category the first of the equations above should take the form $m \circ (m \otimes Id_A) = m \circ (Id_A \otimes m) \circ a_{A,A,A}$, and the same applies to the rest.

The morphism between two unitary algebras (A, m_A, u_A) and (B, m_B, u_B) in $\mathcal{C}$ is a morphism $f \in Hom_{\mathcal{C}}(A, B)$ such that

$$u_B \circ f = u_A \qquad \text{and} \qquad m_B \circ (f \otimes f) = f \otimes m_A.$$

The class of unitary algebras in $\mathcal{C}$ together with the previous morphisms form a category which we note by $Alg_{\mathcal{C}}$. Since we will be interested in unitary algebras, we will drop the adjective unitary and refer to them as algebras.

Definition 1.1.4. *Let $\mathcal{C}$ be a monoidal category and (A, m, u) a unitary algebra in $\mathcal{C}$. A* right module over *A is a pair (M, ρ) such that $M \in Obj(\mathcal{C})$ and $\rho \in Hom_{\mathcal{C}}(M \otimes A, M)$ satisfying the following equations*

$$\rho \circ (Id_M \otimes m) = m \circ (\rho \otimes Id_A),$$
$$\rho \circ (Id_M \otimes u) = Id_M.$$

One defines left modules over an algebra A analogously. A *bimodule over* A is a triple (M, λ, ρ) such that (M, λ) is a left module and (M, ρ) is a right module over A and $\rho \circ (\lambda \otimes Id_A) = \lambda \circ (Id_A \otimes \rho)$. Note that (A, μ, μ) is a bimodule, called *regular*. A morphism of bimodules is a morphism of left and right modules. They clearly form a category.

Definition 1.1.5. *Let $\mathcal{C}$ be a monoidal category and (A, m, u) a unitary algebra in $\mathcal{C}$. An* ideal *I of A is a subobject of A in the category of bimodules over (A, m, u).*

Let A be a commutative algebra and ${}_AMod$ be the category of A-modules. It is a symmetric monoidal category for the canonical tensor product over A and the usual braiding denote by τ. A *good category $\mathcal{C}$* will be a symmetric monoidal category together with a fully faithfull monoidal functor from $\mathcal{C}$ to the symmetric monoidal category ${}_AMod$, for some A.

For the following definition we suppose our category satisfies Grothendieck's axiom $Ab3^*$ (see [5], Appendix for the definition).

Definition 1.1.6. *Given an algebra A in a monoidal category $\mathcal{C}$, and a subobject X of A, the* ideal generated *by X is the intersection of the family of ideals I of A such that X is a subobject of I (see* [3] *Ch.I, section 9).*

Definition 1.1.7. *Given a monoidal category $\mathcal{C}$, a* counitary coalgebra *in $\mathcal{C}$ is a triplet $(C, \varepsilon_C, \Delta_C)$ where $C \in Obj(\mathcal{C})$, $\varepsilon_C \in Hom(C, \mathbf{1})$, $\Delta_C \in Hom(C, C \otimes C)$ such that*

$$(\Delta_C \otimes Id_C) \circ \Delta_C = (Id_C \otimes \Delta_C) \circ \Delta_C,$$
$$Id_C = (\varepsilon_C \otimes Id_C) \circ \Delta_C,$$
$$Id_C = (Id_C \otimes \varepsilon_C) \circ \Delta_C.$$

The morphism ε_C is called the *counit* of C and Δ_C the *coproduct*.

Remark 1.1.8. The object $\mathbf{1}_\mathcal{C}$ together with coproduct $l^{-1}_{\mathbf{1}_\mathcal{C}} = r^{-1}_{\mathbf{1}_\mathcal{C}}$ (see previous remark) and counit $Id_\mathcal{C}$ is a counitary coalgebra.

Since we are mainly interested in counitary coalgebras, we will only refer to them as coalgebras.

Similarly to the case of algebras the class of coalgebras in $\mathcal{C}$ constitute a category if we take as morphism from $(C, \varepsilon_C, \Delta_C)$ to $(D, \varepsilon_D, \Delta_D)$ the maps $g \in Hom_\mathcal{C}(C, D)$, satisfying

$$f \circ \varepsilon_C = \varepsilon_D \qquad \text{and} \qquad (f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

We denote this category by $Coalg_\mathcal{C}$. Similarly one defines coalgebras without counit and their morphisms. We denote this category ${}_nCoalg_\mathcal{C}$

We refer the reader to [7], Ch XIII for the definition of braided category. For each pair of objects V and W of the category $\mathcal{C}$ denote by $c_{V,W} \in Hom(V \otimes W, W \otimes V)$ the *braiding* isomorphism. A symmetric monoidal category is a braided monoidal category such that $c_{V,W} \circ c_{W,V} = Id_{W \otimes V}$ for all objects V and W in the category.

Definition 1.1.9. *Given a symmetric monoidal category $\mathcal{C}$, a unitary algebra (A, m, u) in $\mathcal{C}$ is called a* commutative *if* $m = m \circ c_{A,A}$.

If (A, m_A, u_A) and (B, m_B, u_B) are two commutative unitary algebras, a morphism between them will be a map $f \in Hom_{Alg_{\mathcal{C}}}(A, B)$. The class of commutative unitary algebras together with these morphisms forms a category which we note by $Alg^c_{\mathcal{C}}$.

Definition 1.1.10. *Given a symmetric monoidal category $\mathcal{C}$, a cocommutative counitary coalgebra in $\mathcal{C}$ is a coalgebra $(C, \Delta_C, \varepsilon_C)$ such that $\Delta_C = c_{C,C} \circ \Delta_C$.*

If (C, Δ_C, u_C) and (D, Δ_D, Δ_D) are two cocommutative counitary coalgebras, a morphism between them will be a map $f \in Hom_{Coalg_{\mathcal{C}}}(C, D)$. The class of cocommutative counitary coalgebras together with these morphisms forms a category which we note by $Coalg^c_{\mathcal{C}}$.

The basic example of a braided (non-strict) monoidal category is the category of vector spaces over a field. Note that the category of algebras in a braided monoidal category is also braided, as recalled in [27], Ch.1.

Definition 1.1.11. *Given a monoidal category $\mathcal{C}$ and a coalgebra (C, Δ, ε) in $\mathcal{C}$ a* right comodule over C *is pair (M, δ) where $M \in Obj(\mathcal{C})$ and $\delta \in Hom(M, M \otimes C)$ satisfy*

$$(Id_M \otimes \Delta) \circ \delta = (\delta \otimes Id_C) \circ \delta, \quad (Id_M \otimes \varepsilon) \circ \delta = Id_M. \tag{1.1}$$

The morphism δ is called the *coaction*. We denote by Com_C the category whose objects are right comodules over C and whose morphisms are given as follows. If (M, δ_M) and (N, δ_N) are two right comodules over C, a morphism f from M to N is a map $f \in Hom_{\mathcal{C}}(M, N)$ such that

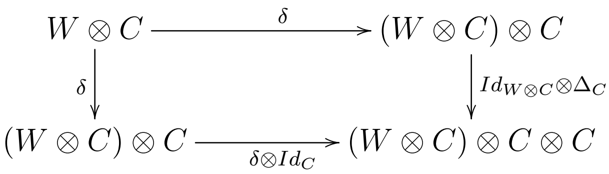
$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \delta_N} \\ M \otimes C & \xrightarrow{f \otimes Id_C} & N \otimes C \end{array} \tag{1.2}$$

is commutative.

Since we shall work only with right comodules, we shall refer to them just as comodules.

Proposition 1.1.12. *Let $\mathcal{C}$ be a monoidal category, $W \in Obj(\mathcal{C})$ and $(C, \Delta_C, \varepsilon_C)$ a coalgebra in $\mathcal{C}$. Then $W \otimes C$ has structure of right comodule over C, with coaction $\delta := Id_W \otimes \Delta_C : W \otimes C \to (W \otimes C) \otimes C$.*

Proof. To see that this is a right comodule over C, one must check equation (1.1). First we will show that the following diagram



commutes.

Indeed, we have the following chain of identities

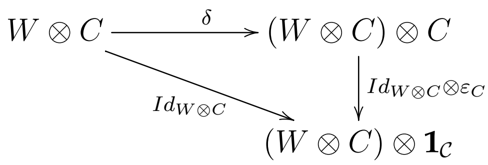
$$\begin{aligned}&(\delta \otimes Id_C) \circ \delta = ((Id_W \otimes \Delta_C) \otimes Id_C) \circ (Id_W \otimes \Delta_C)\\ &= (Id_W \otimes (\Delta_C \otimes Id_C)) \circ (Id_W \otimes \Delta_C) = (Id_W \circ Id_W) \otimes ((\Delta_C \otimes Id_C) \circ \Delta_C),\end{aligned}$$

where we have used the strictness of the category in the second equality and the property $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$. Moreover,

$$\begin{aligned}&(\delta \otimes Id_C) \circ \delta = (Id_W \circ Id_W) \otimes ((Id_C \otimes \Delta_C) \circ \Delta_C)\\ &= (Id_W \otimes (Id_C \otimes \Delta_C)) \circ (Id_W \otimes \Delta_C) = ((Id_W \otimes Id_C) \otimes \Delta_C) \circ (Id_W \otimes \Delta_C)\\ &= (Id_{W\otimes C} \otimes \Delta_C) \circ (Id_W \otimes \Delta_C) = (Id_{W\otimes C} \otimes \Delta_C) \circ \delta,\end{aligned}$$

where in the last row we have used that $Id_{W\otimes C} = Id_W \otimes Id_C$.

Finally we must show the commutation of the following diagram

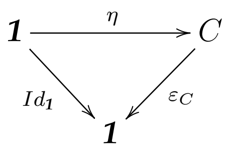


Indeed, we have the following chain of identities

$$\begin{aligned}&(Id_{W\otimes C} \otimes \varepsilon_C) \circ \delta = (Id_{W\otimes C} \otimes \varepsilon_C) \circ (Id_W \otimes \Delta_C)\\ &= ((Id_W \otimes Id_C) \otimes \varepsilon_C) \circ (Id_W \otimes \Delta_C) = (Id_W \otimes (Id_C \otimes \varepsilon_C)) \circ (Id_W \otimes \Delta_C)\\ &= (Id_W \circ Id_W) \otimes ((Id_C \otimes \varepsilon_C) \circ \Delta_C) = Id_W \otimes Id_C = Id_{W\otimes C},\end{aligned}$$

where we have used the strictness of the category in the third equality. □

Definition 1.1.13. *Given a (respectively symmetric) monoidal category $\mathcal{C}$, a* (respectively cocommutative) coaugmented coalgebra in $\mathcal{C}$ *is a cocommutative coalgebra (C, Δ, ε) in $\mathcal{C}$ provided with a morphism of (respectively cocommutative) coalgebras $\eta \in Hom(\mathbf{1}_{\mathcal{C}}, C)$ such that*



(1.3)

Given two cocommutative coaugmented coalgebras $(C, \Delta_C, \varepsilon_C, \eta_C)$ and $(D, \Delta_D, \varepsilon_D, \eta_D)$, a morphism of cocommutative coaugmented coalgebras from C to D is an element $f \in Hom_{Coalg_\mathcal{C}}(C, D)$ such that $f \circ \eta_C = \eta_D$.

The class of all cocommutative coaugmented coalgebras in $\mathcal{C}$ with morphisms between them form a category for the usual composition and identity morphisms, that we denote by ${}_cCoalg_\mathcal{C}$.

Example 1.1.14. If V is a K-vector space, let $TV = \oplus_{n \geqslant 0} V^{\otimes n}$ its tensor construction. We define a counit morphism as the map $\varepsilon : TV \to K$ given by the projection on the zero degree component and a coproduct as follows. If $v = v_1 \otimes \cdots \otimes v_n$ is an element of TV then

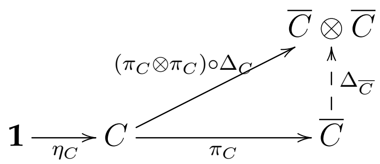
$$\Delta(v) = \sum_{i=1}^{n-1} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n + 1_K \otimes v + v \otimes 1_K.$$

We also define $\Delta(1_K) = 1 \otimes 1$. The coproduct is a linear extension of the previous definitions. The coaugmentation morphism is the map $\eta : K \to TV$ given by $\eta(1_K) = 1 \in T^0V$. It is easy to verify that $(TV, \Delta, \varepsilon, \eta)$ is a coaugmented coalgebra in the symmetric monoidal category of vector spaces over the field K.

1.1.2 Relation between coaugmented and noncounitary coalgebras in an abelian category

All along this section $\mathcal{C}$ will be an abelian symmetric monoidal category. We shall define two functors $F : {}_cCoalg_\mathcal{C} \to {}_nCoalg_\mathcal{C}$ and $G : {}_nCoalg_\mathcal{C} \to {}_cCoalg_\mathcal{C}$ which are quasi-inverses of each other.

If $(C, \Delta_C, \varepsilon_C, \eta_C)$ is a coaugmented coalgebra in $\mathcal{C}$, we will denote $(\overline{C}, \pi_C) = Coker(\eta_C)$. The universal property of the cokernel gives us a map $\Delta_{\overline{C}}$ such that the following diagram



(1.4)

commutes. The morphism $\Delta_{\overline{C}}$ is the only one such that the diagram commutes, and it exists because $(\pi_C \otimes \pi_C) \circ \Delta_C \circ \eta_C = 0$. Indeed by the definition of cocommutative coaugmented coalgebra we have that $(\pi_C \otimes \pi_C) \circ \Delta_C \circ \eta_C = (\pi_C \otimes \pi_C) \circ (\eta_C \otimes \eta_C) \circ \iota = ((\pi_C \circ \eta_C) \otimes (\pi_C \circ \eta_C)) \circ \iota$, which is zero by definition of the cokernel of η_C.

If $(C, \Delta_C, \varepsilon_C, \eta_C) \in obj({}_cCoalg_\mathcal{C})$ we define $F(C, \Delta_C, \varepsilon_C, \eta_C) = (\overline{C}, \Delta_{\overline{C}})$. If $f : (C, \Delta_C, \varepsilon_C, \eta_C) \to (D, \Delta_D, \varepsilon_D, \eta_D)$ is a morphism in the category ${}_cCoalg_\mathcal{C}$, we see that there exists a unique map in the category ${}_{n}Coalg_{\mathcal{C}}$, which we call $\overline{f}$, satisfying that

$$\begin{array}{ccccc} & & & & \overline{D} \\ & & & \overset{\pi_D \circ f}{\nearrow} & \uparrow{\scriptstyle \overline{f}} \\ \mathbf{1} & \xrightarrow[\eta_C]{} & C & \xrightarrow[\pi_C]{} & \overline{C} \end{array} \tag{1.5}$$

In order to guarantee the existence of a mapping $\overline{f}$ in $\mathcal{C}$ making (1.5) commute, we must only see that $(\pi_D \circ f) \circ \eta_C = 0$. By definition of $Hom_{{}_cCoalg_{\mathcal{C}}}(C, D)$ we have that $(\pi_D \circ f) \circ \eta_C = \pi_D \circ (f \circ \eta_C) = \pi_D \circ \eta_D$ which is zero because $(\overline{D}, \pi_D)$ is the cokernel of η_D. In order to see that $\overline{f} \in Hom_{{}_nCoalg_{\mathcal{C}}}(\overline{C}, \overline{D})$, it is enough to prove the diagram

$$\begin{array}{ccc} \overline{C} & \xrightarrow{\overline{f}} & \overline{D} \\ {\scriptstyle \Delta_{\overline{C}}}\downarrow & & \downarrow{\scriptstyle \Delta_{\overline{D}}} \\ \overline{C} \otimes \overline{C} & \xrightarrow[\overline{f} \otimes \overline{f}]{} & \overline{D} \otimes \overline{D} \end{array} \tag{1.6}$$

commutes.

We will prove the equality $\Delta_{\overline{D}} \circ \overline{f} = (\overline{f} \otimes \overline{f}) \circ \Delta_{\overline{C}}$ by using that $\pi_C : C \to \overline{C}$ is an epimorphism, because it is a cokernel. So it is enough proving $\pi_{\overline{D}} \circ \overline{f} \circ \pi_C = (\overline{f} \otimes \overline{f}) \circ \Delta_{\overline{C}} \circ \pi_C$. Indeed,

$$\begin{aligned} \Delta_{\overline{D}} \circ \overline{f} \circ \pi_C &= \Delta_{\overline{D}} \circ \pi_D \circ f = (\pi_D \otimes \pi_D) \circ \Delta_D \circ f \\ &= (\pi_D \otimes \pi_D) \circ (f \otimes f) \circ \Delta_C = (\overline{f} \otimes \overline{f}) \circ (\pi_C \otimes \pi_C) \circ \Delta_C \\ &= (\overline{f} \otimes \overline{f}) \circ \Delta_{\overline{C}} \circ \pi_C \end{aligned}$$

where we have used that $\overline{f} \circ \pi_C = \pi_D \circ f$ in the first equality and the fact that the morphisms of coalgebras satisfy $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ in the third one together with common properties of the morphisms of monoidal categories.

If $f \in Hom_{{}_cCoalg_{\mathcal{C}}}(C, D)$ we define $F(f) = \overline{f} \in Hom_{{}_nCoalg_{\mathcal{C}}}(\overline{C}, \overline{D})$. To prove that this assignment is a functor we must see that

- $F(g \circ f) = F(g) \circ F(f)$,
- $F(Id_C) = Id_{\overline{C}}$.

In the following we suppose that $f \in Hom_{{}_cCoalg_{\mathcal{C}}}(C, D)$ and $g \in Hom_{{}_cCoalg_{\mathcal{C}}}(D, E)$.

We know that $F(g \circ f)$ is the only morphism such that $F(g \circ f) \circ \pi_C = \pi_E \circ (g \circ f)$. Since

$$F(g) \circ F(f) \circ \pi_C = F(g) \circ \overline{f} \circ \pi_C = \overline{g} \circ \pi_D \circ f = \pi_E \circ g \circ f,$$

then, by uniqueness we conclude that $F(g \circ f) = F(g) \circ F(f)$. Diagram (1.5) implies that $F(Id_C) = Id_{\overline{C}}$. Then $F : {}_cCoalg_{\mathcal{C}} \to {}_nCoalg_{\mathcal{C}}$ is a functor. Moreover; we will prove that it is a categorical equivalence, and in order to see that we will construct its quasi-inverse functor $G : {}_nCoalg_{\mathcal{C}} \to {}_cCoalg_{\mathcal{C}}$.

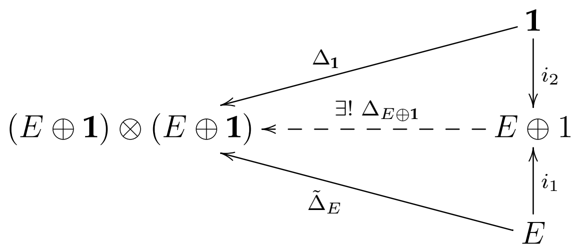
Given an object (E, Δ_E) in the category ${}_nCoalg_{\mathcal{C}}$, we define its image under G by $(E \oplus \mathbf{1}_{\mathcal{C}}, \Delta_{E\oplus\mathbf{1}_{\mathcal{C}}}, \pi_2, i_2)$, where $\pi_2 : E \oplus \mathbf{1}_{\mathcal{C}} \to \mathbf{1}_{\mathcal{C}}$ is the canonical projection of the direct sum and $i_2 : \mathbf{1}_{\mathcal{C}} \to E \oplus \mathbf{1}_{\mathcal{C}}$ is the canonical inclusion map of the direct sum. The definition of $\Delta_{E\oplus\mathbf{1}_{\mathcal{C}}}$ is a little more involved. We write **1** instead of $\mathbf{1}_{\mathcal{C}}$ to denote the unit object. We first define the morphisms

$$\tilde{\Delta}_E = (i_1 \otimes i_1) \circ \Delta_E + (i_1 \otimes i_2) \circ r_E^{-1} + (i_2 \otimes i_1) \circ l_E^{-1},$$
$$\Delta_{\mathbf{1}} = (i_2 \otimes i_2) \circ \iota^{-1}$$

Hence, there is a unique morphism

$$\Delta_{E\oplus\mathbf{1}} \in Hom_{\mathcal{C}}(E \oplus \mathbf{1}, (E \oplus \mathbf{1}) \otimes (E \oplus \mathbf{1}))$$

such that $\Delta_{E\oplus\mathbf{1}} \otimes i_1 = \tilde{\Delta}_E$ and $\Delta_{E\oplus\mathbf{1}} \otimes i_2 = \Delta_{\mathbf{1}}$, i.e. making the following diagram



(1.7)

commute.

From now on, for simplicity we shall write $E_+ = E \oplus \mathbf{1}$. The next step is to define G on morphisms. If f is in $Hom_{{}_nCoalg_{\mathcal{C}}}(C, D)$, then we define $G(f) = i_1^D \circ f \circ \pi_1^C + i_2^D \circ \pi_2^C$, where $i_1^D : D \to D_+$, and $\pi_{D_+} : D_+ \to \overline{(D_+)}$ is the cokernel of $i_2^D : \mathbf{1} \to D_+$ (i.e. the coaugmentation of D_+). We will denote $G(f)$ by f_+.

Example 1.1.15. If we are in the category of vector spaces over a field K, the previous constructions are of the form:

- $G(E) = E \oplus \mathbf{1}$ will actually be $E \oplus K \cdot 1_{E_+}$,
- The coaugmentation $\eta(1_K) = 1_{E_+}$,
- $\Delta_{E\oplus K}(e) = \Delta_E(e) + 1_{E_+} \otimes e + e \otimes 1_{E_+}$,
- If $f \in Hom_{{}_nCoalg_{\mathcal{C}}}(E, H)$, then $G(f)(e + \lambda \cdot 1_{E_+}) = f(e) + \lambda \cdot 1_{H_+}$.

Theorem 1.1.16. *Let $\mathcal{C}$ be a monoidal abelian category. Then, the functor*

$$F : {}_c Coalg_{\mathcal{C}} \to {}_n Coalg_{\mathcal{C}}$$

is a categorical equivalence whose quasi-inverse functor is

$$G : {}_n Coalg_{\mathcal{C}} \to {}_c Coalg_{\mathcal{C}}.$$

Proof. If $X \in Obj({}_n Coalg_{\mathcal{C}})$ we construct the natural map $N_X : X \to \overline{(X_+)}$ given by the composition $N_X = \pi_{X_+} \circ i_1^X$. Given two objects $X, Y \in Obj({}_n Coalg_{\mathcal{C}})$ and $f \in Hom_{{}_n Coalg_{\mathcal{C}}}(X, Y)$, we will show that

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle N_X}\downarrow & & \downarrow{\scriptstyle N_Y} \\ \overline{(X_+)} & \xrightarrow{\quad \overline{(f_+)} \quad} & \overline{(Y_+)} \end{array} \tag{1.8}$$

is commutative. In order to do that, note that

$$\begin{aligned} f_+ \circ i_1^X &= (i_1^Y \circ f \circ \pi_1^X + i_2^Y \circ \pi_2^X) \circ i_1^X \\ &= i_1^Y \circ f \circ \pi_1^X \circ i_1^X + i_2^Y \circ \pi_2^X \circ i_1^X \\ &= i_1^Y \circ f \circ Id_X + i_2^Y \circ 0 = i_1^Y \circ f, \end{aligned}$$

where we have used the bilinearity of the composition in an abelian category in the second equality and the facts that $\pi_1^X \circ i_1^X = Id_X$ and $\pi_2^X \circ i_1^X = 0$ in the third one. Hence,

$$N_Y \circ f = \pi_{Y_+} \circ i_1^Y \circ f = \pi_{Y_+} \circ f_+ \circ i_1^X = \overline{(f_+)} \circ \pi_{X_+} \circ i_1^X = \overline{(f_+)} \circ N_X$$

where in the third equality we use the commutation of the diagram (1.5) and in the last one the definition of the natural transformation. Then the functor $F \circ G$ is naturally isomorphic to $Id_{{}_n Coalg_{\mathcal{C}}}$.

Also it is possible to construct a natural map between $Id_{\mathcal{C}_{{}_c Coalg_{\mathcal{C}}}}$ and $G \circ F$. Suppose $(X, \Delta_X, \varepsilon_X, \eta_X) \in Obj({}_c Coalg_{\mathcal{C}})$, we define $L_X : X \to (\overline{X})_+$ by $L_X = i_1^{\overline{X}} \circ \pi_X + i_2^X \circ \varepsilon_X$, where $i_1^{\overline{X}} : \overline{X} \to (\overline{X})_+$ and $i_2^X : \mathbf{1} \to (\overline{X})_+$ are the inclusion maps, $\pi_X : X \to \overline{X}$ is the cokernel of η_X and $\pi_1^{\overline{X}} : (\overline{X})_+ \to \overline{X}$ and $\pi_1^{\overline{X}} : (\overline{X})_+ \to \mathbf{1}$ are the projections.

Note that

$$\begin{aligned} (\overline{f})_+ \circ L_X &= (i_1^{\overline{Y}} \circ \overline{f} \circ \pi_1^{\overline{X}} + i_2^Y \circ \pi_2^{\overline{X}}) \circ (i_1^{\overline{X}} \circ \pi_X + i_2^X \circ \varepsilon_X) \\ &= i_1^{\overline{Y}} \circ \overline{f} \circ \pi_1^{\overline{X}} \circ i_1^{\overline{X}} \circ \pi_X + i_1^{\overline{Y}} \circ \overline{f} \circ \pi_1^{\overline{X}} \circ i_2^X \circ \varepsilon_X \\ &\quad + i_2^Y \circ \pi_2^{\overline{X}} \circ i_1^{\overline{X}} \circ \pi_X + i_2^Y \circ \pi_2^{\overline{X}} \circ i_2^X \circ \varepsilon_X, \end{aligned}$$

where we have used the bilinearity of the composition, the definition of the functor G and the natural map N_X defined above. Using that $\pi_1^{\overline{X}} \circ i_1^{\overline{X}} = Id_{\overline{X}}$, $\pi_1^{\overline{X}} \circ i_2^{X} = 0$, $\pi_2^{\overline{X}} \circ i_1^{\overline{X}} = 0$ and $\pi_2^{\overline{X}} \circ i_2^{\overline{X}} = Id_{\mathbf{1}}$, we get that

$$(\overline{f})_+ \circ L_X = i_1^{\overline{Y}} \circ \overline{f} \circ \pi_X + i_2^{Y} \circ \varepsilon_X.$$

On the other hand, the computation of $N_Y \circ f$ gives us

$$N_Y \circ f = (i_1^{\overline{Y}} \circ \pi_Y + i_2^{Y} \circ \varepsilon_Y) \circ f = i_1^{\overline{Y}} \circ \pi_Y \circ f + i_2^{Y} \circ \varepsilon_Y \circ f$$
$$= i_1^{\overline{Y}} \circ \overline{f} \circ \pi_X + i_2^{Y} \circ \varepsilon_X,$$

where we have used the definition of $\overline{f}$ and the fact that f is a morphism of (counitary) coalgebras in the last equality. Hence $N_Y \circ f = (\overline{f})_+ \circ L_X$ which together with the previous result implies that F and G are quasi-inverse functors. □

Let us denote $\Delta^{(1)} = Id_{\overline{C}}$, $\Delta^{(2)}_{\overline{C}} = \Delta_{\overline{C}}$ and $\Delta^{(n)}_{\overline{C}} = (\Delta_{\overline{C}} \otimes Id^{\otimes(n-2)}) \circ \Delta^{(n-1)}_{\overline{C}}$ if $n \geqslant 3$. Notice that there is a canonical morphism $Ker\Delta^{(n)}_{\overline{C}} \hookrightarrow Ker\Delta^{(n+1)}_{\overline{C}}$ for all $n \geqslant 2$.

Definition 1.1.17. *A (respectively cocommutative) coaugmented coalgebra* $(C, \Delta, \varepsilon, \eta)$ *is* cocomplete (or conilpotent) *if* $\overline{C} = colim_{n\to\infty} Ker\Delta^{(n)}_{\overline{C}}$.

1.1.3 Symmetric coalgebras

Let us begin this section with an example.

Example 1.1.18. Let $\mathcal{C}$ be a good symmetric monoidal category. Given $V \in Obj(\mathcal{C})$, let us consider the *tensor algebra*

$$T(V) = \bigoplus_{n\in\mathbb{N}_0} V^{\otimes n}, \tag{1.9}$$

where $V^{\otimes 0} = A$ and the product is given by concatenation. The algebra $S(V)$ is defined by $S(V) = T(V)/I$, where I is the ideal in $T(V)$ generated by the image of $Id_V^{\otimes 2} - \tau_{V,V}$ as a subobject of $V^{\otimes 2}$ (where τ is the braiding given by the usual twist). We write

$$S(V) = \bigoplus_{n\in\mathbb{N}_0} S^n V. \tag{1.10}$$

We will define a structure of cocommutative coaugmented coalgebra on it, denoted by $S^c V$. The coproduct is defined as follows. We set $\Delta(1_{S^cV}) = 1_{S^cV} \otimes 1_{S^cV}$, and for an element of the form $v_1 \cdots v_n$ with $v_i \in V$ we define

$$\Delta(v_1 \cdots v_n) = \sum_{I,J/I\sqcup J=\{1\cdots n\}} v_I \otimes_A v_J, \tag{1.11}$$

where $v_I = v_{i_1} \cdots v_{i_k}$ if $I = \{i_1 < \cdots < i_k\} \neq \emptyset$ and $v_\emptyset = 1_{S^cV}$ The counit $\varepsilon : S^c(V) \to A$ is given by the canonical projection onto the zeroeth component, and the coaugmentation $\eta : A \to S^c(V)$ is the canonical inclusion.

From now on, we denote this coalgebra only by SV, unless we say the opposite. Note that $\overline{SV}$ defined in 1.1.2. is just $\overline{SV} = \oplus_{n\in\mathbb{N}} S^nV$ and $\Delta_{\overline{SV}}$ is

$$\Delta_{\overline{SV}}(v_1 \cdots v_n) = \sum_{I,J\neq\emptyset/I\sqcup J=\{1\cdots n\}} v_I \otimes v_J \tag{1.12}$$

for $n \geqslant 2$ and is zero over V.

Remark 1.1.19. Notice that SV is cocomplete. To prove that, it is sufficient note that $\Delta^{(n)}_{\overline{SV}}$ vanishes on elements of $\oplus_{k=1}^n S^kV$.

Remark 1.1.20. Note that the construction of the coalgebra SV can be done for any object V in a symmetric monoidal category $\mathcal{C}$ with arbitrary direct sums such that $\otimes_{\mathcal{C}}$ commutes with $\oplus$. The Theorem 1 in Chapter XI.1 from [25] give us a canonical action from $\mathcal{S}_n$ to the tensor construction of degree n, TV. This theorem allow us to construct the symmetric coalgebra as equivalence classes of this action. Let us denote by $\pi_I : S^nV \to S^{|I|}V$ the projection map in the symmetric product with component in $I \subset \{1, \cdots, n\}$. Then we can consider the maps $\Delta_n : S^nV \to SV \otimes SV$ given by

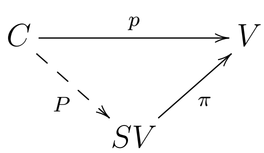
$$\Delta_n = \bigoplus_{I,J|I\sqcup J=\{1,\cdots,n\}} \pi_I \otimes \pi_J,$$

and define the coproduct $\Delta : SV \to SV \otimes SV$ for SV as $\Delta = \bigoplus_{n\in\mathbb{N}_0} \Delta_n$.

However in [25] there is another result in Chapter XI, establishing that there is a strong monoidal functor between the tensor algebra construction and a free algebra in the category algebras in a monoidal category. This result is equally valid for the case of coalgebras.

If C is a coaugmented coalgebra in $\mathcal{C}$, we shall denote by 1_C the image $\eta(1_K)$. Note it is a distinguished *group-like* element of C.

Proposition 1.1.21. *Given a symmetric monoidal category $\mathcal{C}$ with arbitrary direct sums such that $\otimes_{\mathcal{C}}$ commutes with $\oplus$, for any $(C, \Delta_C, \varepsilon_C, \eta_C) \in {}_cCoalg^c_{\mathcal{C}}$, any $V \in Obj(\mathcal{C})$ and $p : C \to V$ morphism in $Hom_{\mathcal{C}}(C, V)$ such that $p \circ \eta_C = 0$, there exists a unique morphism $P \in Hom_{{}_cCoalg^c_{\mathcal{C}}}(C, SV)$ such that the following diagram*



(1.13)

commutes, where π is the canonical projection to the elements of degree one.

Proof. The reader may suppose that $\mathcal{C}$ is a good category. If he or she is willing to, but the proof is exactly the same.

If C is as in the statement, by Theorem 1.1.16 we can write $C \simeq \overline{C} \oplus A$. So, it suffices to define P by its restrictions to each direct summand. We first set P such that $P \circ \eta_C = \eta_{SV}$. In the sequel we shall denote $p^{\odot\alpha} : \overline{C}^{\otimes\alpha} \to S^\alpha V$ to denote the symmetric product of the map $p : \overline{C} \to V$, i.e. $p^{\otimes l}$ composed with the quotient projection $T^n V \to S^n V$. Moreover, we define $\overline{P} = P|_{\overline{C}}$ by

$$\overline{P} = \sum_{n\in\mathbb{N}} \frac{1}{n!}(p^{\odot n} \circ \Delta_{\overline{C}}^{(n)}), \tag{1.14}$$

where we are committing the abuse of calling p the morphism $\overline{C} \to V$ induced by $p : C \to V$. This induced map exists because of the universal property of the cokernel since $p \circ \eta_C = 0$, as depicted below

$$\begin{array}{ccccc} & & & & V \\ & & & \overset{p}{\nearrow} & \uparrow{\scriptstyle \exists! p} \\ A & \xrightarrow{\eta_C} & C & \xrightarrow{\pi_C} & \overline{C} \end{array} \tag{1.15}$$

In order to see that $\overline{P}$ is a coalgebra morphism we must prove the commutation of

$$\begin{array}{ccc} \overline{C} & \xrightarrow{\overline{P}} & \overline{SV} \\ {\scriptstyle \Delta_{\overline{C}}}\downarrow & & \downarrow{\scriptstyle \Delta_{\overline{SV}}} \\ \overline{C} \otimes \overline{C} & \xrightarrow[\overline{P}\otimes\overline{P}]{} & \overline{SV} \otimes \overline{SV} \end{array} \tag{1.16}$$

Changing some parentheses in the firsts equalities and using the definition of $\overline{P}$ and the equation (1.12) in the fourth equality, we have

$$\Delta_{\overline{SV}} \circ \overline{P} = \Delta_{\overline{SV}} \circ (\sum_{n\in\mathbb{N}} \frac{1}{n!}(p^{\odot n} \circ \Delta_{\overline{C}}^{(n)})) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \Delta_{\overline{SV}} \circ (p^{\odot n} \circ \Delta_{\overline{C}}^{(n)})$$
$$= \sum_{n\in\mathbb{N}} \frac{1}{n!} (\Delta_{\overline{SV}} \circ p^{\odot n}) \circ \Delta_{\overline{C}}^{(n)} = \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{I,J\neq\emptyset / I\sqcup J=\{1\cdots n\}} (p^{\odot|I|} \otimes p^{\odot|J|}) \circ \Delta_{\overline{C}}^{(n)}.$$

By rewriting $\Delta_{\overline{C}}^{(n)}$ and using that in a monoidal category the morphisms satisfy $(f \otimes g) \circ (h \otimes r) = (f \circ h) \otimes (g \circ r)$, we see that the latter sum coincides with

$$\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{I,J\neq\emptyset/I\sqcup J=\{1\cdots n\}}(p^{\odot|I|}\otimes p^{\odot|J|})\circ(\Delta_{\overline{C}}\otimes Id)^{|I|-1}\circ(Id\otimes\Delta_{\overline{C}})^{|J|-1}\circ\Delta_{\overline{C}}$$
$$=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{I,J\neq\emptyset/I\sqcup J=\{1\cdots n\}}(p^{\odot|I|}\circ\Delta_{\overline{C}}^{(|I|)}\otimes p^{\odot|J|})\circ(Id\otimes\Delta_{\overline{C}})^{|J|-1}\circ\Delta_{\overline{C}}$$
$$=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{I,J\neq\emptyset/I\sqcup J=\{1\cdots n\}}(p^{\odot|I|}\circ\Delta_{\overline{C}}^{(|I|)}\otimes p^{\odot|J|}\circ\Delta_{\overline{C}}^{(|J|)})\circ\Delta_{\overline{C}}$$
$$=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{I\subseteq\{1\cdots n\}/1<|I|<n}(p^{\odot|I|}\circ\Delta_{\overline{C}}^{(|I|)}\otimes p^{\odot n-|I|}\circ\Delta_{\overline{C}}^{(n-|I|)})\circ\Delta_{\overline{C}}.$$

Since $\odot$ is a symmetric product, any term who has $p^{\odot t}\circ\Delta_{\overline{C}}^{(t)}\otimes p^{\odot n-t}\circ\Delta_{\overline{C}}^{(n-t)}$ is equal to any other who has $p^{\odot r}\circ\Delta_{\overline{C}}^{(r)}\otimes p^{\odot n-r}\circ\Delta_{\overline{C}}^{(n-r)}$ if and only if $t=r$. Therefore for a fixed $r\in\{1,...,n-1\}$ there are $\binom{n}{r}$ terms equal to $p^{\odot r}\circ\Delta_{\overline{C}}^{(r)}\otimes p^{\odot n-r}\circ\Delta_{\overline{C}}^{(n-r)}$. Hence,

$$=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{1<|I|<n}\binom{n}{|I|}(p^{\odot|I|}\circ\Delta_{\overline{C}}^{(|I|)}\otimes p^{\odot n-|I|}\circ\Delta_{\overline{C}}^{(n-|I|)})\circ\Delta_{\overline{C}}$$
$$=\sum_{n\in\mathbb{N}}\sum_{1<|I|<n}\frac{1}{|I|!}\frac{1}{(n-|I|)!}(p^{\odot|I|}\circ\Delta_{\overline{C}}^{(|I|)}\otimes p^{\odot n-|I|}\circ\Delta_{\overline{C}}^{(n-|I|)})\circ\Delta_{\overline{C}}$$
$$=[\sum_{n\in\mathbb{N}}\sum_{1<k<n}\frac{1}{k!}\frac{1}{(n-k)!}(p^{\odot k}\circ\Delta_{\overline{C}}^{(k)}\otimes p^{\odot n-k}\circ\Delta_{\overline{C}}^{(n-k)})]\circ\Delta_{\overline{C}}$$
$$=(\sum_{r\in\mathbb{N}}\frac{1}{r!}(p^{\odot r}\circ\Delta_{\overline{C}}^{(r)})\otimes\sum_{s\in\mathbb{N}}\frac{1}{s!}(p^{\odot s}\circ\Delta_{\overline{C}}^{(s)}))\circ\Delta_{\overline{C}}$$
$$=(\overline{P}\otimes\overline{P})\circ\Delta_{\overline{C}},$$

which is exactly the commutation we want. □

Remark 1.1.22. The previous proposition implies that there is a bijection

$$Hom_{{}_cCoalg^c_{\mathcal{C}}}(C,SV)\leftrightsquigarrow\{f\in Hom_{{}_nCoalg_{\mathcal{C}}}(C,V)/f\circ\eta_C=0\}\tag{1.17}$$

for all cocommutative coaugmented coalgebra $(C,\Delta_C,\varepsilon_C,\eta_C)\in{}_cCoalg^c_{\mathcal{C}}$ in any symmetric monoidal category $\mathcal{C}$.

Since $\overline{P}$ is the restriction of P, we shall omit the bar in order to simplify the notation.

Given $V\in Obj(\mathcal{C})$, we specialize Proposition 1.1.21 in the case $C=SV$. To every $P\in Hom_{{}_cCoalg^c_{\mathcal{C}}}(SV,SV)$ it corresponds a unique $p=\pi\circ P|_{\overline{SV}}\in$

$Hom_{\mathcal{C}}(\overline{SV}, V)$ that will be represented by a denumerable family of maps $\pi \circ P|_{S^nV}$, which we denote $\{p_n\}_{n\in\mathbb{N}}$ and call it the *sequential representation* of P. If we apply the formula $P|_{\overline{SV}} = \sum_{n\in\mathbb{N}} \frac{1}{n!}(p^{\odot n} \circ \Delta^{(n)}_{SV})$ in this case, we get

$$P|_{S^nV}(v_1 \cdots v_n) = \sum_{m=1}^{n} \sum_{I_1\cdots I_m\neq\emptyset; I_1\sqcup\cdots\sqcup I_m=\{1\cdots n\}} \frac{1}{m!} p_{|I_1|}(v_1) \odot \ldots \odot p_{|I_m|}(v_{I_m}) \quad (1.18)$$

where $|X|$ is the cardinality of the set X. Note that the sequential representation of $\pi \circ Id_{SV}$ is $\{Id_V, 0, 0, \cdots\}$.

Proposition 1.1.23. *Let $\mathcal{C}$ be a symmetric monoidal category with arbitrary direct sums such that $\otimes_{\mathcal{C}}$ commutes with $\oplus$, the morphisms $P \in Hom_{{}_cCoalg^c_{\mathcal{C}}}(SV, SV)$ are represented by maps $p \in Hom_{\mathcal{C}}(\overline{SV}, V)$. Moreover P is an isomorphism if and only if p_1 is an isomorphism, where we use the notation of sequential representation introduced previously.*

Proof. As before, the reader may assume that $\mathcal{C}$ is a good category if he or she is willing to, but the proof applies analogously to the general case. It is sufficient to prove the second statement, for the first is a consequence of Remark 1.1.22. We will prove that $P : SV \to SV$ is an isomorphism if and only if $\pi \circ P|_{S^1V} = p_1$ is an isomorphism. Consider $P, Q \in Hom_{{}_cCoalg^c_{\mathcal{C}}}(SV, SV)$. By formula (1.18), we obtain that $\pi \circ P \circ Q|_{\overline{SV}} = \sum_{n\in\mathbb{N}} h_n$ where

$$h_n(v_1 \cdot v_2 \cdots v_n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{I_1\cdots I_m\neq\emptyset; I_1\sqcup\cdots\sqcup I_m=\{1\cdots n\}} p_m(q_{|I_1|}(v_{I_1}) \cdots q_{|I_m|}(v_{I_m})). \quad (1.19)$$

Suppose that P is bijective and take $Q = P^{-1}$. Then (1.19) for $n = 1$ implies that q_1 is the inverse of p_1.

On the other hand assume that p_1 is bijective. We will prove that P is bijective. In order to do so, we will exhibit a left and right inverses of P.

Define recursively the family $\{q_n\}_{n\in\mathbb{N}}$ where $q_n \in Hom_{\mathcal{C}}(S^nV, V)$ is given as follows. Set $q_1 = p_1^{-1}$ and if we have defined all the q_m's with $m < n$ then

$$q_n(w_1 \cdot w_2 \cdots w_n) = -\sum_{m=1}^{n-1} \sum_{I_1\cdots I_m\neq\emptyset; I_1\sqcup\cdots\sqcup I_m=\{1\cdots n\}} q_m(p_{|I_1|}(\tilde{w}_{I_1}) \cdots p_{|I_m|}(\tilde{w}_{I_m})) \quad (1.20)$$

for $n \geqslant 2$,where $\tilde{w}_{I_k} = q_1^{\otimes|I_k|}(w_{I_k})$. Let $Q : SV \to SV$ be the morphism of coalgebras whose sequential representation is $\{q_n\}_{n\in\mathbb{N}}$. Then by means of (1.19) one can see that the sequential representation of $\pi \circ (Q \circ P)$ is $\{Id_V, 0, \cdots\}$ and thus by uniqueness of the correspondence in Proposition 1.1.21 we get that $Q \circ P = Id_{\overline{S^cV}}$ and consequently P is injective.

Let us now prove the surjectivity of P. Define a family of morphisms $\{\tilde{q}_n\}_{n\in\mathbb{N}}$, where $\tilde{q}_n \in Hom_{\mathcal{C}}(S^nV, V)$ is defined as follows. Set $\tilde{q}_1 = p_1^{-1}$ and if $\tilde{q}_1, \cdots \tilde{q}_{n-1}$ for $n \in \mathbb{N}_{\geqslant 2}$ are defined, we fix

$$\tilde{q}_n(w_1 \cdot w_2 \cdots w_n) = -\sum_{m=2}^{n} \frac{1}{m!} \sum_{I_1 \cdots I_m \neq \emptyset; I_1 \sqcup \cdots \sqcup I_m = \{1 \cdots n\}} \tilde{q}_1(p_m(\tilde{q}_{|I_1|}(w_{I_1}) \cdots \tilde{q}_{|I_m|}(w_{I_m}))) \tag{1.21}$$

for $n \geqslant 2$.

This sequential representation defines a unique $\tilde{Q} \in Hom_{{}_c Coalg^c_{\mathcal{C}}}(SV, SV)$ for which $P \circ \tilde{Q} = Id_{SV}$. Then, P is surjective. As a consequence, P is bijective if and only if p_1 is bijective. □

Proposition 1.1.24. *Let $\mathcal{C}$ be a monoidal abelian category, $(D, \Delta_D, \varepsilon_D)$ a coalgebra in $\mathcal{C}$ and M a D-comodule in $\mathcal{C}$. If $W \in Obj(\mathcal{C})$, then there is a one to one correspondence*

$$\boxed{Hom_{Com_D}(M, W \otimes D) \longleftrightarrow Hom_{\mathcal{C}}(M, W)} \tag{1.22}$$

given by $f \mapsto (Id_W \otimes \varepsilon_D) \circ f$, where $W \otimes D$ is the right D comodule described in Proposition 1.1.12. The inverse is given by sending g to $\tilde{g} = (g \otimes Id_D) \circ \rho_M$, where ρ_M is the coaction of M.

Proof. Let us show that if $h \in Hom_{\mathcal{C}}(M, W)$ then the morphism $\tilde{h} : M \to W \otimes D$ is an isomorphism of D-comodules, i.e. satisfies that $(\tilde{h} \otimes Id_D) \circ \rho_M = \rho_{W\otimes D} \circ \tilde{h}$, where $\rho_{W\otimes D}$ is the coaction of $W \otimes D$. Indeed, we have the equations

$$\begin{aligned}(\tilde{h} \otimes Id_D) \circ \rho_M &= (((h \otimes Id_D) \circ \rho_M) \otimes Id_D) \circ \rho_M \\ &= ((h \otimes Id_D) \otimes Id_D) \circ (\rho_M \otimes Id_D) \circ \rho_M \\ &= (h \otimes (Id_D \otimes Id_D)) \circ (\rho_M \otimes Id_D) \circ \rho_M,\end{aligned} \tag{1.23}$$

where we have used the definition of $\tilde{h}$ in this first equality. Moreover, since $Id_D \otimes Id_D = Id_{D\otimes D}$, the latter member of (1.23) coincides with

$$\begin{aligned}&(h \otimes Id_{D\otimes D}) \circ (Id_M \otimes \Delta_D) \circ \rho_M = (h \circ Id_M \otimes Id_{D\otimes D} \circ \Delta_D) \circ \rho_M \\ &= (h \otimes \Delta_D) \circ \rho_M = (Id_W \otimes \Delta_D) \circ (h \otimes Id_D) \circ \rho_M \\ &= \rho_{W\otimes D} \circ (h \otimes Id_D) \circ \rho_M = \rho_{W\otimes D} \circ \tilde{h},\end{aligned}$$

where we have used equation (1.1) in the first equality. This proves that the map $g \mapsto \tilde{g}$ is well defined.

To see that the correspondence is biunivocal take $f \in Hom_{Com_D}(M, W \otimes D)$. If we apply (1.22) to f we get $(Id_W \otimes \varepsilon_D) \circ f$, and the allowed inverse gives us

$$\begin{aligned}&(((Id_W \otimes \varepsilon_D) \circ f) \otimes Id_D) \circ \rho_M = ((Id_W \otimes \varepsilon_D) \otimes Id_D) \circ (f \otimes Id_D) \circ \rho_M \\ &= ((Id_W \otimes \varepsilon_D) \otimes Id_D) \circ \rho_{W\otimes D} \circ f = (Id_W \otimes (\varepsilon_D \otimes Id_D)) \circ (Id_W \otimes \Delta_D) \circ f \\ &= (Id_W \circ Id_W \otimes (\varepsilon_D \otimes Id_D) \circ \Delta_D) \circ f = (Id_W \otimes Id_D) \circ f = Id_{W\otimes D} \circ f = f,\end{aligned}$$

where we have used the property $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$ in the first step, the fact that f commutes with the coactions in the second equality, the definition of the coaction of $W \otimes D$ and the strictness of the category in the third equality and lastly the definition of coalgebra by mean of equation $(\varepsilon_D \otimes Id_D) \circ \Delta_D = Id_D$ and the property $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$ again. □

1.2 The category of sheaves and the topology of its sections

For the basic definitions and general references about sheaf theory we refer the reader to [4] chapters 1 to 4.

Given a topological space $\mathcal{M}$ and a ringed space $(\mathcal{M}, \mathcal{O})$, we shall denote by ${}_{\mathcal{O}}Mod$ the category of sheaves of $\mathcal{O}$-modules. If $F, G \in {}_{\mathcal{O}}Mod$, $F \otimes_{\mathcal{O}} G$ stands for the tensor product in the category ${}_{\mathcal{O}}Mod$. If $F \in {}_{\mathcal{O}}Mod$, we denote by $\tau_{\mathcal{M}}$ the topology of $\mathcal{M}$ and by $r_{V \subseteq U} : F(U) \to F(V)$ the corresponding restriction map. $\Gamma(F)$ will denote the sections of the sheaf and, $\Gamma_c(F)$ will be sections of compact support.

Given two sheaves of $\mathcal{O}$-modules F and G and a morphism f from F to G, we shall denote by $f_U : F(U) \to G(U)$ the family of maps indexed by the open sets U in $\mathcal{M}$ that form the morphism f.

If $\mathcal{M}$ is a smooth manifold, and $\tau_{\mathcal{M}}$ its topology, then the assignment $\mathcal{C}^{\mathbb{R}} : \tau_{\mathcal{M}} \to {}_{\mathbb{R}}Alg$ such that $\mathcal{C}^{\mathbb{R}}(U) = \mathcal{C}^{\infty}(U) = \mathcal{C}^{\infty}(U, \mathbb{R})$ is an example of sheaf of $\mathbb{R}$-algebras, where the restrictions maps are the usual restrictions of functions. Moreover the pair $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$ satisfy the definition of ringed space of $\mathbb{R}$-algebras.

We refer the reader to [20] to general references about vector bundles, adapted coordinated system and related concepts.

Let $p : E \to \mathcal{M}$ be a complex finite dimensional vector bundle over a smooth manifold $\mathcal{M}$. Let us denote by Φ the assignment sending each $U \subseteq \mathcal{M}$ open to $\Phi(U) = \Gamma(U, E) = \{\sigma \in \mathcal{C}^{\infty}(U, E) / p \circ \sigma = Id_U\}$. It is clear that $\Phi(U)$ is a module over $\mathcal{C}^{\infty}(U)$. Let be $U, V \in \tau_{\mathcal{M}}$ such that $V \subseteq U$. We define restriction maps $\Phi(U) \to \Phi(V)$ by $r_{V \subseteq U}(\sigma) = \sigma|_V$ where $\sigma \in \Phi(U)$. It is clear that they satisfies the axiom of presheaf.

Moreover the restriction map $r_{V \subseteq U} : \Phi(U) \to \Phi(V)$ is a $\mathcal{C}^{\infty}(U)$-linear morphism, because it is linear and if $f \in \mathcal{C}^{\infty}(U)$ and $\sigma \in \Phi(U)$, then $r_{V \subseteq U}(f\sigma) = (f\sigma)|_V = f|_V \sigma|_V = r_{V \subseteq U}(f) r_{V \subseteq U}(\sigma)$, where $r_{V \subseteq U}(f)$ are the restriction maps of the aforementioned sheaf $\mathcal{C}^{\mathbb{R}}$. This proves that Φ is a presheaf of modules over the ringed space $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$.

Furthermore, the following is also true, whose proof is immediate.

Lemma 1.2.1. *Let $\mathcal{M}$ be a smooth manifold and $p : E \to \mathcal{M}$ a finite dimensional complex vector bundle over $\mathcal{M}$. Then the above mentioned assignment*

$\Phi : \tau_{\mathcal{M}} \to {}_{\mathbb{R}}Vect$ *is a sheaf of* $\mathcal{C}^{\mathbb{R}}$*-modules.*

1.2.1 Sheaf of jets and sheaf of densities

Definition 1.2.2. *Consider a finite dimensional complex vector bundle* $(E, p, \mathcal{M})$ *and let* $q \in \mathcal{M}$*. Two local sections* $\phi, \psi \in \Gamma_q(p)$ *around* q *are said to be* k-equivalent at q *if* $\phi(q) = \psi(q)$ *and if in some adapted coordinate system* (x^i, u^α) *around* $\phi(p)$*, we have that*

$$\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}|_q = \frac{\partial^{|I|}\psi^\alpha}{\partial x^I}|_q \tag{1.24}$$

for each $1 \leqslant |I| \leqslant k$ *and* $1 \leqslant \alpha \leqslant Rank(E)$*. This is an equivalence relation on the local sections around* q*, and the equivalent class containing* ϕ *is called the* k-jet of ϕ at q *and is denoted by* $j^k_q\phi$

Given $q \in \mathcal{M}$, define the set of *k-th jets of* p at q

$$J^k p = \{ j^k_q \phi \mid q \in \mathcal{M} \text{ and } \phi \text{ is a local section of } p \text{ arround } q \}$$

The *k-th jet manifold of* p is the disjoint union $\sqcup_{q\in\mathcal{M}} J^k_q p$. Lots of properties of this set can be found in [8]. The most important to us are the following two.

Lemma 1.2.3 (See [8], Proposition 4.1.7)**.** $J^k p$ *is a smooth finite dimensional manifold for any* $k \in \mathbb{N}_0$*.*

For the manifold $J^k p$ we define the projection $\pi : J^k p \to \mathcal{M}$ such that $\pi(j^k_q\phi) = q$.

Lemma 1.2.4 (See [8], Proposition 6.2.13)**.** *Let* $(E, p, \mathcal{M})$ *be a finite dimensional vector bundle. Then* $(J^k p, \pi_k, \mathcal{M})$ *is a finite dimensional vector bundle for* $k \in \mathbb{N}_0$*.*

For a fixed $k \in \mathbb{N}$, the last lemma enables us to consider sections of the finite dimensional vector bundle $(J^k p, \pi_k, \mathcal{M})$ of jets of order k which by Lemma 1.2.1 is a sheaf of modules over the ringed space $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$. We shall denote this sheaf by $J^k\Phi$. Analogously one define the jets of infinite order over a fibre bundle $p : E \to \mathcal{M}$ (see [8], Chapter 7) which we denote by $J^\infty\Phi$ and also conform a vector bundle over $\mathcal{M}$, whose projection is called π, the only difference is that the fibres are infinite dimensional.

All along this thesis when we write $J\Phi$ we will refer to J^k for $k \in \mathbb{N} \cup \{\infty\}$.

The category of sheaves over a sheaf of commutative algebras is a symmetric monoidal category which satisfies $Ab3^*$ (even $Ab4^*$) axioms and $\otimes_{\mathcal{O}}$ commutes with $\oplus$ (see [4], Chapter 4). Therefore, given any sheaf $F \in {}_{\mathcal{O}}Mod$, we define SF the symmetric sheaf of F as in Remark 1.1.20.

We shall also use the notion of *sheaf of densities* of a smooth manifold. The reader is refereed to [22], Chapter 7, or [29], Chapter 1, for the definition and basic properties. Given a smooth manifold $\mathcal{M}$, we shall denote by ω the sheaf of 1-densities over $\mathcal{M}$. We recall it is the sheaf of sections of a complex vector bundle over $\mathcal{M}$, so it has structure of $\mathcal{C}^{\mathbb{R}}$-module in the natural way.

Given a vector bundle $p : E \to \mathcal{M}$ and we have just defined the sheaf $J\Phi$ of jets and its symmetric sheaf $SJ\Phi$, also we have the sheaf of densities ω and both have structure of $\mathcal{C}^{\mathbb{R}}$-modules. Then we consider its tensor product $\omega \otimes_{\mathcal{C}^{\mathbb{R}}} SJ\Phi$ in the category of $\mathcal{C}^{\mathbb{R}}$-modules and denote it only by $\omega SJ\Phi$.

1.2.2 Topology on the spaces of sections

We refer the reader to [30] or [13] for general references about locally convex spaces (LCS).

We will now describe the topology of several spaces of sections of sheaves. We begin with $\Gamma_c \omega SJ\Phi$, following the steps of [14] Chapter 17, Section 2.

Let Ω be an open set of $\mathbb{R}^n$ and consider a fundamental sequence of compact sets $\{K_m\}_{m\in\mathbb{N}}$, i.e. $K_m \subseteq K^\circ_{m+1}$ for all $m \in \mathbb{N}$ and $\bigcup_{m\in\mathbb{N}} K_m = \Omega$. Denote by $\mathcal{C}^\infty_m(\Omega)$ the subspace of $\mathcal{C}^\infty(\Omega)$ formed by functions whose support is contained in K_m. One can define a family of *seminorms* on $\mathcal{C}^\infty(\Omega)$ by

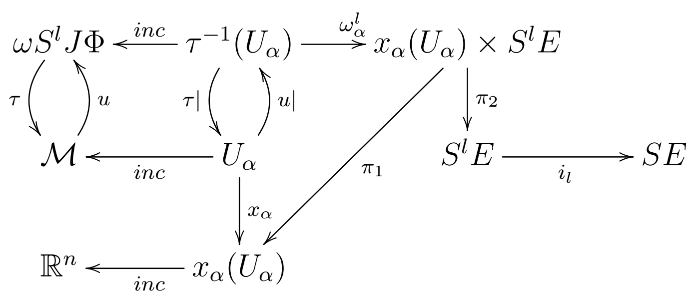
$$p_m(f) = \max\{|D^\alpha f(x)|/x \in K_m, |\alpha| \leqslant m\}$$

that turns $\mathcal{C}^\infty(\Omega)$ into a *Fréchet* space (see [30], 1.46). It is easy to prove that $\mathcal{C}^\infty_c(\Omega)$ is a closed subspace of $\mathcal{C}^\infty(\Omega)$.

We regard $\mathcal{C}^\infty_m(\Omega)$ as a topological subspace of $\mathcal{C}^\infty(\Omega)$ with its Fréchet topology. Finally we endow $\mathcal{C}^\infty_c(\Omega)$ with the final LCS topology from the family of inclusions $\{i_m : \mathcal{C}^\infty_m(\Omega) \to \mathcal{C}^\infty_c(\Omega)\}$. One can see that with this topology on $\mathcal{C}^\infty_c(\Omega)$ is a complete LCS.

The space $\mathcal{C}^\infty_c(\Omega)$ is an example of *LF-space*, and we shall topologize the space of compact supported sections in a very similar way as a LF-space.

We will now recall the topology on $\Gamma_c \omega SJ\Phi$. We will use the following diagram



(1.25)

where τ is the projection of the vector bundle $\omega S^l J\Phi$ and ω^l_α is a local trivialization of such bundle over the domain U_α for $l \in \mathbb{N}$. By reducing the domains if necessary one can always think that the domains of the charts are trivializants for the vector bundle.

Topology for the fibres of the vector bundle $J\Phi$

We began for define a topology on the fibres of the vector bundle $J\Phi$. This fibres are typically the space $\prod_{k\in\mathbb{N}_0} L^k_{sym}(\mathbb{C}^n,\mathbb{C}^m)$ where L^k_{sym} denotes the space of symmetric k-multilinear maps from $\mathbb{C}^n$ to $\mathbb{C}^m$. Let us denote this fibre by E, as we did in the diagram (1.25). In [2], Lewis introduced a family of seminorms $\{\lambda_r\}_{r\in\mathbb{N}}$ on the space $\prod_{k\in\mathbb{N}_0} L^k_{sym}(\mathbb{C}^n,\mathbb{C}^m)$ which turns it into a Frechét space. More precisely if $A\in\prod_{k\in\mathbb{N}_0} L^k_{sym}(\mathbb{C}^n,\mathbb{C}^m)$ and we write $A=\prod_{k\in\mathbb{N}_0} A_k$ then λ_r is given by

$$\lambda_r(A)=\max\{||A_0||'_0,||A_1||'_1,\cdots,||A_r||'_r\},$$

where $||-||'_j$ are norms on $L^j_{sym}(\mathbb{C}^n,\mathbb{C}^m)$ and we recall that, if $M\in L^j_{sym}(\mathbb{C}^n,\mathbb{C}^m)$, then

$$||M||'_j=\sup\{||M(v,\cdots,v)||/v\in\mathbb{C}^n,||v||=1\}.$$

Possibles topologies for the tensor product of LCS

We shall use the seminorms λ_r to define the *projective topology* over T^2E following which was done at [13] Chapter 3, Section 6. Given two LCS F and E, the projective topology for $F\otimes E$ is the finest LCS topology such that the universal mapping $\chi: F\times E\to F\otimes E$ is continuous. We recall that in a LCS there is a correspondence between the Minkowski functionals and the zero open neighbourhoods. Indeed, given U an open neighbourhood, set $p_U(x)=\inf\{t\in\mathbb{R}_{>0}/x\in tU\}$, called the *associated Minkowski functional*. Conversely, given a Minkowski functional p, then $p^{-1}([0,1))$ is an open neighbourhood of zero. Given two LCS E and F, the projective topology of $E\otimes F$ can be explicitly defined by its Minkowski functionals. If $\chi: E\times F\to E\otimes F$ is the universal mapping, then a base of zero neighbourhoods for $E\otimes F$ is the balanced convex hull of

$$\{\chi(U\times V)/U \text{ is a open neighbourhood of zero in } E \text{ and V in } F\}$$

which is denoted by $CH(U\otimes V)$. Moreover if p_U is the Minkowski functional associated to $U\subseteq E$ and p_V of $V\subseteq F$, then

$$p_{CH(U\otimes V)}(u)=\inf\{\sum_i p_U(x_i)p_V(y_i)/u=\sum_i x_i\otimes y_i\} \tag{1.26}$$

is a seminorms on $E\otimes F$ which coincide with the Minkowski functional associated to $CH(U\otimes V)$ (see result 6.3. from [13], chapter 3, section 6.). We denote it by $p_U\otimes p_V$.

We denote by $E\otimes_\pi F$ the algebraic tensor product with the projective topology just described and call it the projective tensor product between E and F. If E and F are LCS then also is $E\otimes_\pi F$. But with this topology if E and F are Fréchet spaces then $E\otimes_\pi F$ is not necessarily a Fréchet space and the same holds for LF (inductive strict limit of Fréchet, whose principal example are the compactly supported sections over a manifold).

Denoting by $E\hat{\otimes}_\pi F$ the completion of $E \otimes_\pi F$, we obtain a categorical definition of tensor product in the category of Fréchet spaces or LF.

In both cases, if E is a LCS by recursive applications of the LCS structure on the tensor products give us a family of countable seminorms on $T^l E$ for $l \in \mathbb{N}$ (that may be is a Fréchet topology, depending if E is Fréchet and if we take $\otimes_\pi$ or $\hat{\otimes}_\pi$). We note them by $|| - ||_{T^l E,\beta}$ with $\beta \in \mathbb{N}_0$.

Topologies for the symmetric product $S^l E$

Let us consider the linear inclusion of $S^l E$ inside $T^l E$ given by the symmetrization map $x_1 \cdots x_l \mapsto \sum_{\sigma \in \mathbb{S}_l} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(l)}$. It is easy to show that it is a closed subspace of $T^l E$. Then, we may consider the induced LCS topology on $S^l E$ by means of this inclusion. We shall denote it by $(T^l E)^{\mathbb{S}_l}$. If E is a Fréchet space, note that $T^l E$ is also a Fréchet space and the same holds for the closed subspace $(T^l E)^{\mathbb{S}_l}$, provided we are using the completion of the projective topology for the algebraic tensor product.

On the other hand, since $S^l E$ is a quotient of $T^l E$, the LCS topology on the latter induces a LCS topology on the former (see [17], section 3). If E is a Fréchet space, then $S^l E$ is also.

Note that the composition $(T^l E)^{\mathbb{S}_l} \hookrightarrow T^l E \twoheadrightarrow S^l E$ is clearly linear and continuous, because the inclusion is linear and continuous by definition of subspace topology at $(T^l E)^{\mathbb{S}_l}$ and the quotient projection is clearly linear and continuous. Also, one can see that the composition is bijective. If E is Fréchet the open map theorem for Fréchet spaces (see [30] chapter 1), implies that the inverse map of this composition is also continuous and as a conclusion $S^l E$ and $(T^l E)^{\mathbb{S}}$ are homeomorphic. Note that the open map theorem can be used only if the LCS involved are Fréchet spaces and it is true only if we use the tensor product $\hat{\otimes}_\pi$.

Adding the fibre of the density bundle

Continuing the construction of the topology for the section space given the vector bundle $\tau : \omega S^l J\Phi \to \mathcal{M}$ we denote by $\omega_\alpha^l : \tau^{-1}(U_\alpha) \to x_\alpha(U_\alpha) \times S^l E$ as its local trivialization. Notice that this vector bundle has the same local fibre as $S^l J\Phi$, because the tensor product on the fibres are over $\mathbb{R}$ and ω is a line bundle, the typical fibre of $S^l J\Phi$ is $S^l E$ and for $\omega S^l J\Phi$ is $\mathbb{R} \otimes_{\mathbb{R}} S^l E$.

Topology on $\Gamma\omega SJ\Phi$ and $\Gamma_c\omega SJ\Phi$

For each chart (U_α, x_α) of the aforementioned atlas of $\mathcal{M}$ we consider a fundamental sequence of compacts $\{K_m^\alpha\}_{m\in\mathbb{N}}$ and denote by $\Gamma_m \omega S^l J\Phi$ the sections with compact support contained in K_m.

Given $n \in \mathbb{N}$, define the following family of seminorms on $\Gamma_m \omega S^l J\Phi$ (see [14], chapter XV, section 2) defined as

$$p_{s,\beta,\alpha,m}(u) = \sup_{x \in K_m \subseteq U_\alpha} \sum_{|j| \leqslant s, j \in \mathbb{N}_o^n} ||\partial^j(\pi_2 \circ \omega_\alpha^l \circ u \circ x_\alpha^{-1})(x)||_{S^l E, \beta},$$

where s indicates the maximum order of derivation, α is the index of the chart and β the index of the seminorm of $S^l E$.

Note that the (partial) derivatives appearing in the previous definition are the classical ones (of any function of the form $f : W \subseteq \mathbb{R}^n \to L$, where L is a Fréchet space).

We finally consider the final LCS topology on $\Gamma_c \omega S^l J\Phi$ given by the family of inclusions $i_{m,l} : \Gamma_m \omega S^l J\Phi \hookrightarrow \Gamma_c \omega S^l J\Phi$, and on $\Gamma_c \omega S J\Phi$ with the LCS topology of the direct sum, i.e. the final LCS topology given by the inclusions $i_l : \Gamma_c \omega S^l J\Phi \to \Gamma_c \omega S J\Phi$. With this final topology $\Gamma_c \omega S^l J\Phi$ is a LF-space.

1.3 Distributions on manifolds

1.3.1 H-distributions & D-distributions

We refer the reader to [15], [14], [29] and [18] for general references about the theory of distributions on manifolds. During this section X and Y will be open sets of $\mathbb{R}^n$, and $\mathcal{M}$ will be a manifold of dimension n. We denote by $\mathcal{D}'(X)$ the space of distributions on $X \subseteq \mathbb{R}^n$ (see [15] for a definition). Finally we denote by $H_d(\mathcal{M})$ to the distributions with density character 0 on $\mathcal{M}$ and by $D_d(\mathcal{M})$ to the distributions with *density character* 1 over $\mathcal{M}$ (see [18] for a reference). We will call these two types of distributions H-distributions and D-distributions respectively and shall present a formal definition later in this subsection.

We recall that a subset V of $\mathbb{R}^n$ is said to be *conic* if for each $\xi \in V$ and $t > 0$, $t\xi$ belongs to V. If $\mathcal{E}'(\mathbb{R}^n)$ are the distributions of compact support then for any $u \in \mathcal{E}'(\mathbb{R}^n)$ we define the set $\Sigma(u) \subseteq \mathbb{R}^n \setminus \{\vec{0}\}$ as the points having no conic neighbourhood $V \subseteq \mathbb{R}^n \setminus \{\vec{0}\}$ and positive constants C_N such that

$$|\hat{u}(\xi)| \leqslant C_N(1 + |\xi|)^{-N}$$

holds for all $\xi \in V$ and all $N \in \mathbb{N}$, where $\hat{u}$ is the Fourier transform of u. With this definition $\Sigma(u)$ is clearly a closed cone in $\mathbb{R}^n \setminus \{\vec{0}\}$.

For any $u \in \mathcal{D}'(X)$ and $\psi \in \mathcal{C}_c(X)$, the distribution ψu belongs to $\mathcal{E}'(X)$. Given $u \in \mathcal{D}'(X)$, for each $x \in X$ we set

$$\Sigma_x(u) = \bigcap \{\Sigma(\phi u) / \phi \in \mathcal{C}_0^\infty(X), \phi(x) \neq 0\}$$

and call it the *cone of u at x*. For a nice exposition of these concepts, and together with a definition of the wavefront set of a distribution on $\mathbb{R}^n$ and also the definition of wavefront set for an H-distribution on a manifold, see [29], Chapter 2.

The objective of this section is to generalise the product of H-distributions defined by L.Hörmander in [15] (see [29], Chapter 2.3) to an action of the space of H-distributions over the space of D-distributions. It will be very useful to describe $H_d(\mathcal{M})$ and $D_d(\mathcal{M})$ as families of distributions in open sets of $\mathbb{R}^n$ satisfying some sort of covariance, which we briefly recall.

If $f : X \to Y$ is a diffeomorphism, the pullback of f, noted by $f^* : \mathcal{D}'(Y) \to \mathcal{D}'(X)$, is defined as

$$f^*(w)(\phi) = w(|\det(f^{-1})'|\phi \circ f^{-1}), \tag{1.27}$$

where $w \in \mathcal{D}'(Y)$ and ϕ is a test function in X, i.e. $\phi \in \mathcal{C}_c^\infty(X)$.

Take a differentiable structure on $\mathcal{M}$ given by charts $(U_\alpha, \varphi_\alpha)_{\alpha\in\mathbb{N}}$. Then $\{\varphi_\alpha(U_\alpha)\}_{\alpha\in\mathbb{N}}$ is a family of open sets of $\mathbb{R}^n$, so it makes sense to consider $\mathcal{D}'(\varphi_\alpha(U_\alpha))$.

Definition 1.3.1. *The space $H_d(\mathcal{M})$ is the set whose elements are families of the form $u = \{u_\alpha\}_{\alpha\in\mathbb{N}}$, where $u_\alpha \in \mathcal{D}'(\varphi_\alpha(U_\alpha))$, obeying the equality*

$$u_\beta = (\varphi_\alpha \circ \varphi_\beta^{-1})^* u_\alpha \tag{1.28}$$

of distributions on $\varphi_\beta(U_\alpha \cap U_\beta)$.

Analogously, if f is a diffeomorphism, the *semi-pullback* is given by

$$f^\bullet(w)(\phi) = w(\phi \circ f^{-1}), \tag{1.29}$$

where $w \in \mathcal{D}'(Y)$ and ϕ is a test function in X, i.e. $\phi \in \mathcal{C}_c^\infty(X)$. Notice that

$$f^\bullet(w) = |\det(f')| f^*(w), \tag{1.30}$$

for all $w \in \mathcal{D}'(Y)$

Definition 1.3.2 (see [14])**.** *The space $D_d(\mathcal{M})$ is the set whose elements are the families $u = \{u_\alpha\}_{\alpha\in\mathbb{N}}$, where $u_\alpha \in \mathcal{D}'(\varphi(U_\alpha))$, satisfying that*

$$u_\beta = (\varphi_\alpha \circ \varphi_\beta^{-1})^\bullet u_\alpha \tag{1.31}$$

as distributions on $\varphi_\beta(U_\alpha \cap U_\beta)$.

Note that (1.30) tells us that equation (1.31) can be rewritten as

$$u_\beta = |\det(\varphi_\alpha \circ \varphi_\beta^{-1})'|(\varphi_\alpha \circ \varphi_\beta^{-1})^* u_\alpha. \tag{1.32}$$

Definitions 1.3.1 and 1.3.2 are particular examples of what is called *distribution with density character q* (see [18] Chapter 3, Section 1, where they are also described as continuous dual spaces of sections of certain density bundles).

We will extend the definition of wavefront set of H-distributions to D-distributions. We remark that the definition of wavefront set of H-distribution was introduced by L. Hörmander in [15] (see [29] for a short exposition).

Lemma 1.3.3. *Let $f \in \mathcal{C}^\infty(W)$ positive, $W \subseteq R^n$ open and $\mu \in \mathcal{D}'(W)$. Hence*

$$\Sigma_x(\mu) = \Sigma_x(f\mu)$$

for all $x \in W$.

Proof. We take $\psi \in \mathcal{C}_c^\infty(W)$ such that ψ takes the value 1 in a closed neighbourhood of x and 0 in the complement of an open set containing that closed set. We also consider $\varphi \in \mathcal{C}_c^\infty(W)$ such that $\varphi(x) \neq 0$. Then, $\varphi\mu \in \mathcal{D}'(W)$ has compact support. If we apply [29], Lemma 2.5, to the compact support distribution $\varphi\mu$ and the smooth compact support function $f\psi$, we see that $\Sigma((f\psi)(\varphi\mu)) \subseteq \Sigma(\varphi\mu)$. Since $\Sigma((f\psi)(\varphi\mu)) = \Sigma((\varphi\psi)(f\mu))$, intersecting the above contention for all $\varphi \in \mathcal{C}_c^\infty(W)$ such that $\varphi(x) \neq 0$, we get that $\bigcap_\varphi \Sigma((\varphi\psi)(f\mu)) \subseteq \Sigma_x(\mu)$. On the other hand, as $\varphi = \varphi\psi$ on a neighbourhood of x and Σ_x is a local property, we also conclude that $\Sigma_x(f\mu) \subseteq \Sigma_x(\mu)$.

We apply the argument of the previous paragraph to the positive function $\phi = \frac{1}{f}$ and the distribution $u = f\mu$, to deduce that

$$\begin{aligned} &\Sigma_x(\phi u) \subseteq \Sigma_x(u), \\ &\Sigma_x(\frac{1}{f} f\mu) \subseteq \Sigma_x(f\mu), \\ &\Sigma_x(\mu) \subseteq \Sigma_x(f\mu). \end{aligned}$$

As a consequence $\Sigma_x(\mu) = \Sigma_x(f\mu)$, which is what we want. □

Equation (2.133) of [29], tells us that

$$\Sigma_{x(p)}(u_x) = \Sigma_{x(p)}(|\det(y \circ x^{-1})'|(y \circ x^{-1})^*(u_y)) = \Sigma_{x(p)}((y \circ x^{-1})^*(u_y)), \quad (1.33)$$

where (x, U_x) is an atlas for $\mathcal{M}$, $p \in \mathcal{M}$ and we have used equation (1.32) and Lemma 1.3.3.

As we can see in [29], equation (2.111), each diffeomorphism $f : X \to Y$ satisfies

$$\Sigma_x(f^*u) = f'(x)^T \Sigma_{f(x)}(u) \quad (1.34)$$

for any $x \in X$, where $f'(x)$ is the differential of f. Now if $u = \{u_z\}_{(z,U_z)}$ is a D-distribution on $\mathcal{M}$ and $p \in \mathcal{M}$ is such that the charts (x, U_x) and (y, U_y) satisfy that $p \in U_x$ and $p \in U_y$, then using the Einstein's summation convention, we get that

$$\{\xi_k dx_p^k / \xi \in \Sigma_{x(p)}(u_x)\} = \{\xi_k dx_p^k / \xi \in \Sigma_{x(p)}((y \circ x^{-1})^* u_y)\}$$
$$= \{((y \circ x^{-1})'(x(p))^T \eta)_k dx_p^k / \eta \in \Sigma_{y(p)}(u_y)\} = \{((\frac{\partial y^j}{\partial x^k})_p \eta_j dx_p^k / \eta \in \Sigma_{y(p)}(u_y)\}$$
$$= \{\eta_j dy_p^j / \eta \in \Sigma_{y(p)}(u_y)\},$$

where we have used equations (1.33) and (1.34) in the first equality and in the second one respectively. This allow us to introduce the following notion.

Definition 1.3.4. *Let $u \in D_d(\mathcal{M})$. The wave front set of u is the set*

$$WF(u) = \{\{p\} \times \Sigma_p(u) / p \in \mathcal{M}\} \subseteq T^*\mathcal{M},$$

where $\Sigma_p(u) = \{\xi_k dx_p^k / \xi \in \Sigma_{x(p)}(u_x)\}$ for any chart (x, U_x) such that $p \in U_x$, and $T^*\mathcal{M}$ is the cotangent bundle of the manifold.

The pullback of distributions defined on open sets of $\mathbb{R}^n$ is also defined for maps which are not diffeomorphisms.

Theorem 1.3.5 ([29], Theorem 2.61)**.** *Let X and Y be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and let $f : X \to Y$ be an smooth map. Define*

$$N_f = \{(f(x), \nu) \in Y \times \mathbb{R}^n / x \in X, f'(x)^T \nu = \vec{0}\}.$$

Then there is a unique way of defining the pullback

$$f^* : \{u \in \mathcal{D}'(Y) / N_f \cap WF(u) = \emptyset\} \to \mathcal{D}'(X),$$

such that $f^* u = u \circ f$ for all $u \in \mathcal{C}^\infty(Y)$.

We recall that given $u \in \mathcal{D}'(X)$ and $v \in \mathcal{D}'(Y)$, the exterior tensor product $u \otimes v$ is a distribution over $X \times Y$ (see for instance [29], Definition 1.48). The hypotheses of Theorem 1.3.5 are satisfied for $\Delta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ and a distribution of the form $\mu \otimes \nu$ if there is no $(x, \xi) \in WF(\mu)$ such that $(x, -\xi) \in WF(\nu)$ (as explained in [29], Theorem 2.167) where μ and ν are distributions over $\mathbb{R}^n$.

The family $\{\Delta_\alpha^*(u_\alpha \otimes v_\alpha)\}_{\alpha \in \mathbb{N}}$ will be of interest to us: these are distributions $\Delta_\alpha^*(u_\alpha \otimes v_\alpha) \subseteq \mathcal{D}'(\varphi_\alpha(U_\alpha))$ where $\Delta_\alpha : \varphi_\alpha(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^{2n}$ are the diagonal maps.

The following will be useful.

Lemma 1.3.6. *Given $f : Z \to X$ and $g : W \to Y$ diffeomorphisms between open sets of $\mathbb{R}^n$, let $f \times g : Z \times W \to X \times Y$ be defined by $(f \times g)(z, w) = (f(z), g(w))$. Then*

$$(f \times g)^*(u \otimes v)(\varphi \otimes \psi) = (f^*(u) \otimes g^*(v))(\varphi \otimes \psi), \tag{1.35}$$

for all $u \in \mathcal{D}'(X)$, $v \in \mathcal{D}'(Y)$, $\varphi \in \mathcal{C}_c^\infty(Z)$ and $\psi \in \mathcal{C}_c^\infty(W)$. Since $\mathcal{C}_c^\infty(Z) \otimes \mathcal{C}_c^\infty(W) \subseteq \mathcal{C}_c^\infty(Z \times W)$ is dense and the pullback is sequentially continuous (see [29]*), we can conclude that*

$$(f \times g)^*(u \otimes v) = f^*(u) \otimes g^*(v) \tag{1.36}$$

as elements of $\mathcal{D}'(Z \times W)$.

Proof. First of all we take sequences $\{u_j\}_{j\in\mathbb{N}}$ and $\{v_m\}_{m\in\mathbb{N}}$ of continuous functions over X and Y, respectively, such that $u_n \to u$ and $v_n \to v$. We recall that $\mathcal{D}'(X)$ has the weak*-topology induced by $\mathcal{C}_c^\infty(X)$, and the same holds for Y. Then,

$$((f \times g)^*(u \otimes v))(\varphi \otimes \psi) = \lim_{l\to\infty}((f \times g)^*(u_l \otimes v_l))(\varphi \otimes \psi),$$

because the pullback is sequentially continuous. Hence,

$$\lim_{l\to\infty}((f\times g)^*(u_l\otimes v_l))(\varphi\otimes\psi) = \lim_{l\to\infty}\int_{Z\times W}((u_l\otimes v_l)\circ(f\times g))(z,w)\cdot(\varphi\otimes\psi)(z,w)dzdw,$$

for the pullback map is an extension of the precompositions from functions to distributions. Moreover, by definition we see that

$$\lim_{l\to\infty}\int_{Z\times W}(u_l \circ f(z) \cdot \varphi(z)) \cdot ((v_l \circ g)(w) \cdot \psi(w))dzdw,$$

which by Fubinni's Theorem gives us

$$\lim_{l\to\infty}[\int_Z (u_l \circ f(z) \cdot \varphi(z))dz \cdot \int_W ((v_l \circ g)(w) \cdot \psi(w))dw]$$

$$= \lim_{l\to\infty}\int_Z (u_l \circ f(z) \cdot \varphi(z))dz \cdot \lim_{l\to\infty}\int_W ((v_l \circ g)(w) \cdot \psi(w))dw$$

$$= \lim_{l\to\infty} f^*u_l(\varphi) \cdot \lim_{l\to\infty} g^*v_l(\psi) = f^*u(\varphi) \cdot g^*v(\psi) = (f^*u \otimes g^*v)(\varphi \otimes \psi).$$

The lemma is thus proved. □

Consider now the pullback of $\Delta_\alpha^*(u_\alpha \otimes v_\alpha) \in \mathcal{D}'(\varphi_\alpha(U_\alpha \cap U_\beta))$ by $\varphi_\alpha \circ \varphi_\beta^{-1}$. Note that the following diagram

$$\begin{array}{ccc}
\varphi_\beta(U_\alpha \cap U_\beta) & \xrightarrow{\Delta_\beta} & \varphi_\beta(U_\alpha \cap U_\beta) \times \varphi_\beta(U_\alpha \cap U_\beta) \\
\Big\downarrow{\scriptstyle \varphi_\alpha\circ\varphi_\beta^{-1}} & & \Big\downarrow{\scriptstyle (\varphi_\alpha\circ\varphi_\beta^{-1})\times(\varphi_\alpha\circ\varphi_\beta^{-1})} \\
\varphi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\Delta_\alpha} & \varphi_\alpha(U_\alpha \cap U_\beta) \times \varphi_\alpha(U_\alpha \cap U_\beta)
\end{array}$$

is commutative. As a consequence,

$$(\varphi_\alpha \circ \varphi_\beta^{-1})^*(\Delta_\alpha^*(u_\alpha \otimes v_\alpha)) = (\varphi_\alpha \circ \varphi_\beta^{-1})^* \circ \Delta_\alpha^*(u_\alpha \otimes v_\alpha) = (\Delta_\alpha \circ \varphi_\alpha \circ \varphi_\beta^{-1})^*(u_\alpha \otimes v_\alpha)$$

$$= [((\varphi_\alpha \circ \varphi_\beta^{-1}) \times (\varphi_\alpha \circ \varphi_\beta^{-1})) \circ \Delta_\beta]^*(u_\alpha \otimes v_\alpha) = \Delta_\beta^* \circ ((\varphi_\alpha \circ \varphi_\beta^{-1}) \times (\varphi_\alpha \circ \varphi_\beta^{-1}))^*(u_\alpha \otimes v_\alpha), \tag{1.37}$$

where we have used Theorem 2.126 of [29] in the second equality, the commutation of the previous diagram and Lemma 2.130 of [29].

By Lemma 1.3.6, Definitions 14 and 15, and equation (1.32), the last member of (1.37) coincides with

$$\Delta_\beta^* \circ ((\varphi_\alpha \circ \varphi_\beta^{-1})^*(u_\alpha) \otimes (\varphi_\alpha \circ \varphi_\beta^{-1})^*(v_\alpha)) = \Delta_\beta^*(u_\beta \otimes \frac{1}{|\det(\varphi_\alpha \circ \varphi_\beta^{-1})'|} v_\beta)$$

$$= \frac{1}{|\det(\varphi_\alpha \circ \varphi_\beta^{-1})'|} \Delta_\beta^*(u_\beta \otimes v_\beta).$$

Hence , by multiplying this equality by $|\det(\varphi_\alpha \circ \varphi_\beta^{-1})'|$ we obtain

$$\Delta_\beta^*(u_\beta \otimes v_\beta) = |\det(\varphi_\alpha \circ \varphi_\beta^{-1})'|(\varphi_\alpha \circ \varphi_\beta^{-1})^*(\Delta_\alpha^*(u_\alpha \otimes v_\alpha)).$$

By equation (1.32) we get that

$$\Delta_\beta^*(u_\beta \otimes v_\beta) = (\varphi_\alpha \circ \varphi_\beta^{-1})^\bullet(\Delta_\alpha^*(u_\alpha \otimes v_\alpha)). \tag{1.38}$$

Hence, the family $\{\Delta_\alpha^*(u_\alpha \otimes v_\alpha)\}_{\alpha \in \mathbb{N}}$ defines an element of $D_d(\mathcal{M})$ which we will denote $u \cdot v$. Indeed we have just proved the following proposition.

Proposition 1.3.7. *There exist an action of $H_d(\mathcal{M})$ over $D_d(\mathcal{M})$ which we note*

$$H_d(\mathcal{M}) \times D_d(\mathcal{M}) \longrightarrow D_d(\mathcal{M}) \tag{1.39}$$

$$(u, v) \longrightarrow u \cdot v$$

provided that for each $\alpha \in \mathbb{N}$, there are not points (x, ξ) in $WF(u_\alpha)$ such that $(x, -\xi)$ is in $WF(v_\alpha)$.

Remark 1.3.8. Lastly we want mention that there is an alternative definition of H-distributions on a manifold which was introduced by [15] or [29], as elements of the continuous dual of $\Gamma_c(\omega)$ (the compact support sections of the density bundle of $\mathcal{M}$ (the manifold)) respect to a topology given by seminorms (this topology is explicitly defined in both text and was described here in the previous section) which makes of $\Gamma_c(\omega)$ a Fréchet space.

On the other hand if we follow the text [14] the distributions were introduced as elements of the continuous dual of $\mathcal{C}_c^\infty(\mathcal{M})$ (in [15] these were called distribution densities), we will call these D-distributions.

The relation between this and the above definitions is clearly established in both texts, but we say that for any family $\{u_\alpha\}_{\alpha\in\mathbb{N}}$ which satisfies the Definition 1.3.1 there exists only one element $\tilde{u}$ in the continuous dual of $\Gamma_c(\omega)$ such that locally works in the following way,

$$\tilde{u}(\delta) = \int_{\varphi_\alpha(U_\alpha)} u_\alpha(\varphi_\alpha^{-1})^*\delta$$

where δ is an element of $\Gamma_c(\omega)$ whose support is in U_α. Using the Definition 1.3.1 one sees that $\tilde{u}$ is well defined independently of U_α. Similar considerations are valid for the D-distributions. We invite the reader to deepen on these topics in the aforementioned books.

Chapter 2

Basic facts on Feynman measures

2.1 Lagrangian formulation

We begin by expressing in mathematical terminology certain ideas that are common in physics. The manifolds will allways be T_2 and satisfy the second axiom of countability in this thesis and consequently they are paracompact and a have locally finite partition of the unity.

A relation $\mathcal{R}$ in a topological space X is said to be *closed* if the set $\{(x, y) \in X^2/x\mathcal{R}y\} \subseteq X^2$ is closed for the product topology of X^2.

Definition 2.1.1. *A* spacetime *will be a smooth finite-dimensional manifold $\mathcal{M}$, together with a closed, reflexive and transitive relation $\preceq$. Two points in the spacetime will be called* spacelike separated *if $x \npreceq y$ and $y \npreceq x$.*

Example 2.1.2. A first example of spacetime is $\mathbb{R}$ with the usual manifold structure and the classical order $\leqslant$ relation, which is obviously reflexive and transitive. The relation $\leqslant$ is also closed because the set $\{(x, y) \in \mathbb{R}^2/x \leqslant y\}$ has as complement the positivity set of the continuous function $f(x, y) = x - y$. In this example there are no spacelike separated points, for $\leqslant$ is a total order. $\square$

Example 2.1.3. We shall now consider the space $\mathbb{R}^{1,3}$, also denoted by $\mathbb{M}^4$, and called the *Minkowski spacetime*. Its underlying manifold is $\mathbb{R}^4$ and the relation is given as follows. Define the bilinear form $\langle -; - \rangle : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ by $\langle z^1, z^2 \rangle = z_0^1 z_0^2 - z_1^1 z_1^2 - z_2^1 z_2^2 - z_3^1 z_3^2$ where $z^i = (z_0^i, z_1^i, z_2^i, z_3^i)$ for $i = 1, 2$. Then the relation is the following, $z_1 \preceq z_2$ iff $\langle z_2 - z_1; z_2 - z_1 \rangle \geqslant 0$ and $z_0^2 \geqslant z_0^1$.

The relation $\preceq$ is clearly reflexive. It is transitive, because if $z^3 = (z_0^3, z_1^3, z_2^3, z_3^3)$

and $z_1 \preceq z_2$ and $z_2 \preceq z_3$, then:

$$\begin{aligned}
&(z^3 - z^1; z^3 - z^1) = (z_0^3 - z_0^1)^2 - (z_1^3 - z_1^1)^2 - (z_2^3 - z_2^1)^2 - (z_3^3 - z_3^1)^2 \\
&= (z_0^3 - z_0^2 + z_0^2 - z_0^1)^2 - (z_1^3 - z_1^2 + z_1^2 - z_1^1)^2 \\
&- (z_2^3 - z_2^2 + z_2^2 - z_2^1)^2 - (z_3^3 - z_3^2 + z_3^2 - z_3^1)^2 \\
&= (z_0^3 - z_0^2)^2 + (z_0^2 - z_0^1)^2 + 2(z_0^3 - z_0^2)(z_0^2 - z_0^1) \\
&- ||((z_1^3, z_2^3, z_3^3) - (z_1^2, z_2^2, z_3^2)) + ((z_1^2, z_2^2, z_3^2) - (z_1^1, z_2^1, z_3^1))||^2.
\end{aligned}$$

Applying the triangle inequality we have that

$$\begin{aligned}
(z^3 - z^1; z^3 - z^1) &\geqslant (z_0^3 - z_0^2)^2 + (z_0^2 - z_0^1)^2 + 2(z_0^3 - z_0^2)(z_0^2 - z_0^1) \\
&-||(z_1^3, z_2^3, z_3^3) - (z_1^2, z_2^2, z_3^2)||^2 - ||(z_1^2, z_2^2, z_3^2) - (z_1^1, z_2^1, z_3^1)||^2 \\
&= [(z_0^3 - z_0^2)^2 - ||(z_1^3, z_2^3, z_3^3) - (z_1^2, z_2^2, z_3^2)||^2] \\
&+[(z_0^2 - z_0^1)^2 - ||(z_1^2, z_2^2, z_3^2) - (z_1^1, z_2^1, z_3^1)||^2] + 2(z_0^3 - z_0^2)(z_0^2 - z_0^1) \\
&= (z^3 - z^2; z^3 - z^2) + (z^2 - z^1; z^2 - z^1) + 2(z_0^3 - z_0^2)(z_0^2 - z_0^1).
\end{aligned}$$

The three last summands are $\geqslant 0$ because of the hypotheses $z_1 \preceq z_2$ and $z_2 \preceq z_3$. The transitivity of $\preceq$, follows from the fact that $z_0^3 \geqslant z_0^1$.

This relation is also closed, because the complement of $\{(z^1, z^2) \in \mathbb{R}^8 / z^1 \preceq z^2\}$ is the preimage of the open set $\{(x, y) \in \mathbb{R}^2 / x > 0$ or $y > 0\}$ by the continuous function $f : \mathbb{R}^8 \to \mathbb{R}^2$ defined by $f(z^1, z^2) = (||(z_1^1, z_2^1, z_3^1) - (z_1^2, z_2^2, z_3^2)||^2 - (z_0^1 - z_0^2)^2, z_0^1 - z_0^2)$. □

As we mentioned in Section 1.2, if $\mathcal{M}$ is a spacetime then it has an structure $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$ of ringed space over $\mathbb{R}$. Let $(E, p, \mathcal{M})$ be a finite dimensional complex vector bundle over $\mathcal{M}$ and Φ the sheaf of modules over the ringed space $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$ associated to it (see Lemma 1.2.1). This vector bundle and the corresponding sheaf are fixed from now on. It will be called the *sheaf of classical fields* over $\mathcal{M}$ and its local sections, i.e. elements of $\Phi(U) = \Gamma(U, E)$ where U is open in $\mathcal{M}$, are called *classical fields*.

Example 2.1.4. (Classical mechanics) Consider the spacetime with the underlying manifold $\mathbb{R}^3$ and the identity relation, and the vector bundle $\pi : T\mathbb{R}^3 \to \mathbb{R}^3$ given by the tangent bundle. A classical field in classical mechanics is a section of the previous vector bundle. □

Example 2.1.5. (Classical field theory) In a classical field theory, one usually considers the trivial line vector bundle over the Minkowski spacetime $\pi : \mathbb{M}^4 \times \mathbb{R} \to \mathbb{M}^4$. A classical field in this situation is just a section of π. So a classical field Φ is given by a function $\phi : \mathbb{M}^4 \to \mathbb{R}$.

We recall that the d'Alembertian operator $\Box$ on a function $g : \mathbb{M}^4 \to \mathbb{R}$ is given by $\Box g = \frac{\partial^2 g}{\partial t^2} - \frac{\partial^2 g}{\partial x_1^2} - \frac{\partial^2 g}{\partial x_2^2} - \frac{\partial^2 g}{\partial x_3^2}$. A *massive scalar field of mass* m is a classical field $\Phi(z) = (z, \phi(z))$ such that ϕ satisfies the *Klein-Gordon equation* $(-\Box + m^2)\phi = 0$. □

Example 2.1.6. (Electromagnetism) Let $\mathbb{M}^4$ be the Minkowski spacetime and $\Omega^2\mathbb{M}^4$ be the vector bundle of antisymmetric two forms over $\mathbb{M}^4$. If $U \subseteq \mathbb{M}^4$ is an open set, a local section of $\Omega^2\mathbb{M}^4$ over U is of the form

$$F(t,\vec{x}) = \frac{1}{2}F_{\mu\nu}(t,\vec{x})dx^\mu \wedge dx^\nu$$

and is a classical field (where the Einstein's summation convention and the notation $x^0 = t$, $x^1 = x_1$,$x^2 = x_2$ and $x^3 = x_3$ were used).

We define the electric $E : U \to \mathbb{R}^3$ and the magnetic $B : U \to \mathbb{R}^3$ fields by

$$\begin{aligned} E_x(t,\vec{x}) = F_{01}(t,\vec{x}), \quad E_y(t,\vec{x}) = F_{02}(t,\vec{x}), \quad E_z(t,\vec{x}) = F_{02}(t,\vec{x}), \\ B_z(t,\vec{x}) = -F_{12}(t,\vec{x}), \quad B_y(t,\vec{x}) = F_{13}(t,\vec{x}), \quad B_x(t,\vec{x}) = -F_{23}(t,\vec{x}). \end{aligned}$$

The classical Maxwell's equations in electromagnetism are written in the form

$$\partial_\eta F_{\mu\nu} + \partial_\mu F_{\nu\eta} + \partial_\nu F_{\eta\mu} = 0 \qquad \text{and} \qquad \partial_\nu F^{\mu\nu} = j^\mu,$$

where $j^o(t,\vec{x}) = \rho(t,\vec{x})$ is the charge density, j^i the density current in the direction of x_i for $1 \leqslant i \leqslant 3$, and $F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$ with g the metric tensor $g = diag(-1,1,1,1)$. □

Definition 2.1.7. *Given the vector bundle $(E, p, \mathcal{M})$ over the spacetime $\mathcal{M}$ and Φ the associated sheaf, the sheaf of jets $J\Phi$ constructed after Lemma 1.2.4 is called the sheaf of* derivatives of classical fields.

Definition 2.1.8. *The sheaf $S_{\mathcal{C}^\mathbb{R}}J\Phi$ given by the symmetric construction of the sheaf of modules $J\Phi$ over the ringed space $(\mathcal{M}, \mathcal{C}^\mathbb{R})$ is called the* sheaf of Lagrangians *(or* composite fields*).*

If ϕ is a massive scalar field of mass m as in Example 2.1.5, then

$$l_1 = \frac{1}{2}\partial_\mu\phi \odot \partial^\mu\phi - \frac{1}{2}m^2\phi^{\odot 2}$$

is an example of Lagrangian, where we have used Einstein's convention for the sum. We will usually omit the symmetric product to lighten the notation and just write $l_1 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$.

An example of Lagrangian for Example 2.1.6 is

$$l_2 = F_{\mu,\nu}F^{\mu,\nu}.$$

We will give a few more examples. If φ and ψ are classical fields over an spacetime $\mathcal{M}$ (see Example 2.1.5) and f, g are smooth functions over $\mathcal{M}$; then

$$l_3 = \overline{\psi}\psi,$$
$$l_4 = f\varphi\partial_1\psi + g\varphi^2\partial_1\partial_3\psi,$$
$$l_5 = (f\varphi)\psi = \varphi(f\psi) = f(\varphi\psi),$$

are examples of Lagrangians.

Any combination of this Lagrangians with coefficients in the ring of smooth functions over the spacetime is also a Lagrangian.

We recall that we denote by ω the sheaf of densities of the spacetime.

Definition 2.1.9. *The* sheaf of Lagrangian densities *is $\omega SJ\Phi$. A Lagrangian density is any of its sections.*

For example, if $\mathcal{M}$ is orientable of dimension n and φ is a classical field with support contained in an open set U of $\mathcal{M}$, we can regard $d^n x$ (where (U, x) is a chart) as a local section of the density bundle ω which together with any Lagrangian over U can produce a Lagrangian density $\mathcal{L}$ as for example

$$\mathcal{L} = (\varphi + m\varphi^2\partial\varphi)d^n x,$$

where m is a constant.

Definition 2.1.10. *A* non-local action *is an element of the symmetric algebra $S\Gamma\omega SJ\Phi$ of the $\mathbb{R}$-vector space of global sections of the sheaf $\omega SJ\Phi$.*

Remark 2.1.11. In order to topologize the symmetric algebra $S\Gamma\omega SJ\Phi$ of the $\mathbb{R}$-vector space of global sections of the sheaf $\omega SJ\Phi$, we must first consider the algebra $T\Gamma\omega SJ\Phi$. And as we saw in Section 1.2.2 there are two ways of topologize $T^k\Gamma\omega SJ\Phi$ (for $k \in \mathbb{N}$) one considering the projective $\otimes_\pi$ topology over the tensor product and the second one considering its completation $\hat{\otimes}_\pi$.

We can choose the topology we want for $T^k\Gamma\omega SJ\Phi$, but if we take the topology given by $\hat{\otimes}_\pi$ then the two ways of topologize $S^k\Gamma\omega SJ\Phi$ will be homeomorphic as we saw in Section 1.2.2.

The proofs for the next facts can be found in [16], Theorem 7.5.5 and Corollary 7.5.6.

Theorem 2.1.12. *Let E and F be two finite dimensional complex vector bundles $\mathcal{M}$. The $\mathcal{C}^\infty(\mathcal{M})$-linear map $\alpha : \Gamma(E)\otimes_{\mathcal{C}^\infty(\mathcal{M})}\Gamma(F) \to \Gamma(E\otimes F)$, defined by $\alpha(s\otimes t)(x) = s(x) \otimes t(x) \in E_x \otimes F_x = (E \otimes F)_x$, where $s \in \Gamma(E)$, $t \in \Gamma(F)$ and $x \in \mathcal{M}$, is a canonical isomorphism of $\mathcal{C}^\infty(\mathcal{M})$-modules.*

As a corollary of this theorem we have

Corollary 2.1.13. *Given a finite dimensional complex vector bundle E over $\mathcal{M}$, there exists a canonical isomorphism of $\mathcal{C}^\infty(\mathcal{M})$-modules between $\Gamma(S^k E)$ and $S^k_{\mathcal{C}^\infty(\mathcal{M})}\Gamma(E)$ for all $k \in \mathbb{N}_0$.*

In Section 1.2.1 we told that $J\Phi$ denote the vector bundle $J^k\Phi$ for some $k \in \mathbb{N} \cup \{\infty\}$ and unless $k = \infty$, $J^k\Phi$ is a finite dimensional vector bundle over $\mathcal{M}$ and we can apply the Theorem 2.1.12 and Corollary 2.1.13 to it.

We want to mention a result similar to Theorem 2.1.12, whose proof involves the Serre-Swan Theorem for non compact manifolds, which establishes that the $\mathcal{C}^\infty(\mathcal{M})$-module of global sections of a smooth vector bundle over a manifold $\mathcal{M}$ is projective over $\mathcal{C}^\infty(\mathcal{M})$ (see [12]).

Proposition 2.1.14. *Given E and F two finite dimensional complex vector bundles over a manifold $\mathcal{M}$, the restriction of the morphism α of Theorem 2.1.12 establishes an isomorphism $\alpha| : \Gamma_c(E) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(F) \to \Gamma_c(E \otimes F)$.*

Proof. Given a finite dimensional vector bundle H over a manifold $\mathcal{M}$, we consider the map $\chi : \mathcal{C}^\infty_c(\mathcal{M}) \times \Gamma(H) \to \Gamma_c(H)$ defined by $\chi(f, \varphi) = f\varphi$ where $f \in \mathcal{C}^\infty_c(\mathcal{M})$, $\varphi \in \Gamma(H)$ and if $p \in \mathcal{M}$, $f\varphi(p) = f(p)\varphi(p)$, then $f\varphi$ is a compact support section of H. The map χ is clearly $\mathcal{C}^\infty(\mathcal{M})$-balanced, hence it factors throughout the tensor product as $\overline{\chi} : \mathcal{C}^\infty_c(\mathcal{M}) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(H) \to \Gamma_c(H)$.

The map $\overline{\chi}$ is an isomorphism. It is surjective because if $\psi \in \Gamma_c(H)$ then suppose U is an open set containing the support of ψ and V is a compact set such that $supp(\psi) \subseteq U \subseteq V$, then if we take $f \in \mathcal{C}^\infty_c(\mathcal{M})$ such that f takes the value 1 on U and the value zero outside V and $\overline{\chi}(f \otimes \psi) = f\psi = \psi$.

On the other hand the map $\overline{\chi}$ is injective because if $\overline{\chi}(f \otimes \varphi) = 0 \in \Gamma_c(H)$, then there is no point $p \in \mathcal{M}$ such that both $f(p) \neq 0$ and $\varphi(p) \neq 0$. Let us consider an elemental sequence of compact sets $C_n \nearrow supp(\phi)$ and using the Urysohn's lemma a sequence of functions $g_n \in \mathcal{C}^\infty_c(\mathcal{M})$ such that $g \equiv 1$ over C_n and $g \equiv 0$ over $supp(f)$. Then we have $f \otimes_{\mathcal{C}^\infty(\mathcal{M})} \varphi = \lim_{n\to\infty} f \otimes_{\mathcal{C}^\infty(\mathcal{M})} g_n\varphi = \lim_{n\to\infty} fg_n \otimes_{\mathcal{C}^\infty(\mathcal{M})} \varphi = 0 \in \mathcal{C}^\infty_c(\mathcal{M}) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(H)$, as we want.

As the module $\Gamma(F)$ is $\mathcal{C}^\infty_c(\mathcal{M})$-projective then we can apply the functor $(-) \otimes_{\mathcal{C}^\infty_c(\mathcal{M})} \Gamma(F)$ to the precedent isomorphism

$$\overline{\chi} : \mathcal{C}^\infty_c(\mathcal{M}) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(E) \to \Gamma_c(E)$$

taking $H = E$ and still have an isomorphism of $\mathcal{C}^\infty_c(\mathcal{M})$-modules,

$$\mathcal{C}^\infty_c(\mathcal{M}) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(E) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(F) \to \Gamma_c(E) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(F).$$

Moreover by applying Theorem 2.1.12 we have an isomorphism,

$$\mathcal{C}^\infty_c(\mathcal{M}) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(E \otimes F) \to \Gamma_c(E) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(F)$$

and by applying the above result again using $H = E \otimes F$ then we have the isomorphism

$$\Gamma_c(E \otimes F) \to \Gamma_c(E) \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma(F)$$

which is what we established.

□

Remark 2.1.15. In this remark we will apply the above results to the case of sections of the vector bundle $J\Phi$ (always under the hypothesis of finite order jets).

Given $k \in \mathbb{N}$, the sections of the vector bundle $S^k J\Phi$, that is $\Gamma S^k J\Phi$ are isomorphic by Corollary 2.1.13 to $S^k_{\mathcal{C}^\infty(\mathcal{M})}\Gamma J\Phi$. Moreover by Theorem 2.1.12 together with the recent comment we have an isomorphism between $\Gamma\omega S^k J\Phi$ and $\Gamma\omega \otimes_{\mathcal{C}^\infty(\mathcal{M})} S^k_{\mathcal{C}^\infty(\mathcal{M})}\Gamma J\Phi$.

Lastly by Proposition 2.1.14 we have an isomorphism similar to the last one, but very important if we want integrate a local action over the spacetime, $\Gamma_c\omega S^k J\Phi \simeq \Gamma_c\omega \otimes_{\mathcal{C}^\infty(\mathcal{M})} S^k_{\mathcal{C}^\infty(\mathcal{M})}\Gamma J\Phi$.

As a consequence a *monomial element* of the space of non-local actions $A \in S^n\Gamma\omega S^k J\Phi$ with $n \in \mathbb{N}$ can be written $A = \prod_{i=1}^n \alpha_i \otimes (\odot_{j=1}^k l^i_j)$, $\alpha_i \in \Gamma(\omega)$ and $l^i_j \in \Gamma(J\Phi)$, and $\prod$ denotes the product of SV for $V = \Gamma\omega SJ\Phi$. In view of the above remarks the densities $\alpha_i \in \Gamma(\omega)$ can be taken of compact support, i.e. $\alpha_i \in \Gamma_c(\omega)$, if A has compact support.

2.2 Propagators

2.2.1 Types of propagators

Given a complex finite dimensional vector bundle $\pi : E \to \mathcal{M}$ over the spacetime and $\varphi \in \Gamma(J\Phi)$ we denote by φ^* its complex conjugate section, i.e. the section such that $\varphi^*(p) = \overline{\varphi(p)}$ for all $p \in \mathcal{M}$ (the bar indicates the complex conjugate). And if $f = \delta\varphi \in \Gamma_c\omega J\Phi$, we denote by f^* the section $\delta\varphi^*$ where $*$ does not work over the real densities, i.e. $f^*(p) = \delta(p) \otimes \overline{\varphi(p)}$ if δ is a real density an $p \in \mathcal{M}$.

Definition 2.2.1. *A* propagator *associated to the vector bundle $\pi : E \to \mathcal{M}$ (or its associated sheaf Φ) is a continuous and $\mathbb{R}$-bilinear function*

$$\Delta : \Gamma_c\omega J\Phi \times \Gamma_c\omega J\Phi \to \mathbb{C}.$$

We denote the space of propagators associated to $\pi : E \to \mathcal{M}$ by $Prop(E)$. We say that the propagator is

- local *if $\Delta(f, g) = \Delta(g, f)$ for each f and g whose supports are spacelike separated (see Definition 2.1.1) .*
- Feynman *if it is symmetric.*
- Hermitian *if $\Delta^* = \Delta$, where $\Delta^*(f^*, g^*) := \overline{\Delta(g, f)}$ and f^* is the section of $\Gamma_c\omega J\Phi$ given by a $*$-operation on it.*

- positive *if* $\Delta(f^*, f) \geqslant 0$.

Given two finite dimensional vector bundles $\pi_E : E \to M$ and $\pi_F : F \to N$, we consider the manifold $N \times M$ and pull-back these bundles by the projections $pr_1 : N \times M \to N$ and $pr_2 : N \times M \to M$ to construct the bundles, $pr_1^*(\pi_F) : pr_1^*(F) \to N \times M$ and $pr_2^*(\pi_E) : pr_2^*(E) \to N \times M$, so we have two bundles over $N \times M$ and we denote by $F \boxtimes E$ the bundle $pr_1^*(F) \otimes pr_2^*(E)$ over $N \otimes M$.

Proposition 2.2.2. *Given two finite dimensional K (real or complex) vector bundles E and F, we have an isomorphism $\Gamma(E \boxtimes F) \simeq \Gamma(E)\hat{\otimes}_K\Gamma(F)$ where the completion is with respect to the just described Fréchet topologies.*

The next proposition give us a link between propagators and distributions over $\mathcal{M} \times \mathcal{M}$.

Proposition 2.2.3. *There is a linear map $\iota : Prop(E) \to \Gamma(\omega_{\mathcal{M}\times\mathcal{M}} \otimes (J\Phi \boxtimes J\Phi))'$ given by $\iota(\Delta)((\delta_1 \otimes \delta_2) \otimes (f \otimes g)) = \Delta(\delta_1 \otimes f, \delta_2 \otimes g)$, for all $\delta_1, \delta_2 \in \Gamma(\omega)$ and $f, g \in \Gamma(J\Phi)$.*

Proof. Since $\Gamma(\omega_{\mathcal{M}\times\mathcal{M}} \otimes (J\Phi \boxtimes J\Phi)) = \Gamma((\omega_{\mathcal{M}} \boxtimes \omega_{\mathcal{M}}) \otimes (J\Phi \boxtimes J\Phi))$ for $\omega_{\mathcal{M}\times\mathcal{M}} = \omega_{\mathcal{M}} \boxtimes \omega_{\mathcal{M}}$, the previous conditions determine precisely one unique $(J\Phi \boxtimes J\Phi)^*$-valued distribution on $\mathcal{M} \times \mathcal{M}$

□

Definition 2.2.4. *The space of bilinear maps with respect to $\mathcal{C}^\infty(M)^{\otimes 2}$ associated to a bundle $\pi : E \to \mathcal{M}$*

$$\Gamma_c J\Phi \times \Gamma_c J\Phi \to \{\textit{H-distibutions of compact support on } \mathcal{M} \times \mathcal{M}\} = \Gamma(\omega_{\mathcal{M}\times\mathcal{M}})'$$

will be denoted by $Prop'(E)$

Remark 2.2.5. There is a linear map $\xi : Prop(E) \to Prop'(E)$ satisfying that $\xi(\Delta)(A \otimes B)(f \otimes g) = \Delta(fA, gB)$ for all $A, B \in \Gamma_c J\Phi$ and $f, g \in \Gamma(\omega_{\mathcal{M}})$. This map is clearly injective, because if $\xi(\Delta) = \xi(\tilde{\Delta})$ then by using the Proposition 2.1.14 and the definition of ξ we conclude that $\Delta(A, B) = \tilde{\Delta}(A, B)$ for all $A, B \in \Gamma_c \omega J\Phi$, i.e. Δ is equal to $\tilde{\Delta}$.

Reciprocally each map in $Prop'(E)$ induces a morphism of $\mathcal{C}^\infty(M)^{\otimes 2}$-modules between $\Gamma_c J\Phi\hat{\otimes}_K\Gamma_c J\Phi$ and $\Gamma(\omega_{\mathcal{M}\times\mathcal{M}})'$, then ξ is bijective.

There is an important family of propagators that we want to use.

Definition 2.2.6. *Given a vector space V, we say that a set $C \subset V$ is a cone if $\xi \in C$ implies that $t\xi \in C$ for all $t > 0$.*

If we are working in a t.v.s. we can talk about closed cones.

Definition 2.2.7. *A cone which is properly contained in a semispace is called proper.*

Note that a semispace is not a proper cone.

Given a closed proper convex cone C of a t.v.s. Z, we can define a partial order on Z of the form $x \preccurlyeq y$ if $y - x \in C$.

Definition 2.2.8. *A propagator $\Delta : \Gamma_c \omega J\Phi \times \Gamma_c \omega J\Phi \to \mathbb{C}$ is called* cut, *if for each $\varphi, \psi \in \Gamma_c J\Phi$ and each $z \in \mathcal{M}$ the H-distribution over $\mathcal{M} \times \mathcal{M}$: $\xi(\Delta)(\varphi, \psi)$ possesses a partial order in the cotangent space defined by a proper closed convex cone C_z, such that if (p, q) is in the wavefront set of $\xi(\Delta)(\varphi, \psi)$ at some point $(x, y) \in \mathcal{M} \times \mathcal{M}$ then $p \preccurlyeq 0$ and $0 \preccurlyeq q$. Moreover if $x = y$ then $p + q = 0$.*

Example 2.2.9. In Quantum Field Theory one often encounters the so-called advanced propagator $\Delta_{(+)}$ such that its associated element in $Prop'(E)$. If it is evaluated in the scalar field, that is $\xi(\Delta_{(+)})(\phi, \phi)$, (see [11] chapter 6) in local coordinates is given by the function

$$\Delta_+(x) = \frac{i}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{-ip_\mu x^\mu}}{2p_0} dp_1 dp_2 dp_3,$$

where $p_0 = \sqrt{|\vec{p}|^2 + m^2}$. This means that the H-distribution $\xi(\Delta_{(+)})(\phi, \phi)$ evaluated in two compact support densities $f(x)d^4x$ and $g(y)d^4y$ is given by

$$\xi(\Delta_{(+)})(\phi, \phi)(f(x)d^4x, g(y)d^4y) = -i \int_{supp(f) \times supp(g)} \Delta_+(x - y) f(x) g(y) d^4x d^4y.$$

The wavefront set of this distributions was calculated in [19], Theorem IX.48, and it gives

$$WF(\Delta_+) = \{(0, -|\vec{p}|, \vec{p})/0 \in \mathbb{R}^4, \vec{p} \in \mathbb{R}^3 \backslash \{0\}\} \cup \{(\pm|\vec{x}|, \vec{x}, -\lambda|\vec{x}|, \mp\lambda\vec{x})/\vec{x} \in \mathbb{R}^3, \lambda > 0\} \tag{2.1}$$

By pulling back this distribution with the map $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$, $(x, y) \mapsto x - y$ we obtain the wavefront set of $\xi(\Delta_{(+)})(\varphi, \varphi)$ over $\mathbb{M}^4 \times \mathbb{M}^4$, which is

$$WF(\xi(\Delta_{(+)})(\varphi, \varphi)) = \{(x, p, y, -p) \in \mathbb{R}^{16}/(x - y, p) \in WF(\Delta_+)\}.$$

Example 2.2.10. Consider the distribution θ_{x^0} on $\mathbb{M}^4$ defined by $\theta_{x^0}(f) = \int_{x^0 \geqslant 0} f(x) d^3\vec{x}$. In this case

$$WF(\theta_{x^0}) = \{(0, \vec{x}, \lambda, \vec{0})/\vec{x} \in \mathbb{R}^3, \lambda \in \mathbb{R} \setminus \{0\}\} \tag{2.2}$$

(see [29]). Since $WF(\theta_{x^0})$ and $WF(\Delta_+)$ satisfy that there is no point $(x, p) \in WF(\theta_{x^0})$ such that $(x, -p) \in WF(\Delta_+)$, because the only x which is in both singular supports is 0 but in this case the cones has not intersections (see formulas (2.1) and (2.2)), we can define their product $\theta_{x^0}\Delta_+$ (see [19]). The Feynman propagator is given by

$$\Delta_F(x) = \theta_{x^0} \cdot \Delta_+(x) + \theta_{-x^0} \cdot \Delta_+(-x).$$

The wavefront set of this distribution was computed in [6], Proposition 26. If we pull back the Feynman propagator by the diagonal map we obtain an H-distribution on $\mathbb{M}^4 \times \mathbb{M}^4$.

Definition 2.2.11. *A propagator* $\Delta : \Gamma_c \omega J\Phi \times \Gamma_c \omega J\Phi \to \mathbb{C}$ *is called* polynomially smooth *if the expression in local coordinates of* $\xi(\Delta)(A, B)$ *is a sum of products of smooth functions and powers of logarithms of polynomials, for any* $A, B \in \Gamma_c J\Phi$.

Example 2.2.12. We recall that the Feynman propagator of the massive scalar field described in Example 2.2.10 is

$$\Delta_F(t, \vec{x}) = i \int \frac{e^{-i\omega_p|t| + i\vec{p}\cdot\vec{x}}}{2\omega_p} \frac{d^3\vec{p}}{(2\pi)^3},$$

where $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$. The integral can be explicitly computed (see [11] Chapter 6, Section 5) and it gives

$$\Delta_F(t, \vec{x}) = \frac{im}{4\pi^2\sqrt{|\vec{x}|^2 - t^2}} K_1\left(m\sqrt{|\vec{x}|^2 - t^2}\right) \qquad on \qquad \mathbb{R}^4 \setminus \{(t, \vec{x})/|t| = |\vec{x}|\},$$

where K_1 is the modified Bessel function of order 1. Then Δ_F is polynomially smooth. For more examples of smooth polynomial propagators see the Dirac and the Proca propagators in Chapter 6, Section 5 of [11].

2.2.2 Generalized propagators

We now proceed to extend the propagator to a bigger domain.

We recall that a propagator Δ gives a bilinear map $\xi(\Delta) : \Gamma_c J\Phi \times \Gamma_c J\Phi \to \Gamma(\omega_{\mathcal{M}\times\mathcal{M}})'$ (see Remark 2.2.5).

We first define $\tilde{\Delta} : \Gamma_c SJ\Phi \times \Gamma_c SJ\Phi \to \Gamma(\omega_{\mathcal{M}\times\mathcal{M}})'$ by means of the formula

$$\tilde{\Delta}(a_1 \odot \dots \odot a_n, b_1 \odot \dots \odot b_n) = \sum_{\sigma\in S_n} \xi(\Delta)(a_1, b_{\sigma 1}) \cdot \xi(\Delta)(a_2, b_{\sigma_2}) \cdot \dots \cdot \xi(\Delta)(a_n, b_{\sigma_n}), \tag{2.3}$$

where the $\cdot$ is a product of H-distributions (recall the product of H-distribution is described in [29], Chapter 2) and we define $\tilde{\Delta}$ to be zero if it is evaluated at different degree elements.

If $V = \Gamma SJ\Phi$, then SV admits two structures of coalgebra. The first one is the cocommutative cofree coaugmented symmetric coalgebra of V, with this structure the elements of V are primitives. Let us call Δ_C to the coproduct of this coalgebra.

For the other hand if we make the construction described in 1.1.20 to the object $J\Phi$, we obtain the symmetric coalgebra $SJ\Phi$, in which the elements of

$J\Phi$ are primitive elements. This structure has a unique extension to the algebra SV, such that SV is a Hopf algebra (see [28]). Let us denote by Δ_T the coproduct of SV with this structure.

If $\varphi, \psi \in J\Phi$, then $\varphi\psi \in S^2 J\Phi \subseteq S^1\Gamma S^2 J\Phi \subseteq SV$ (we will denote with an underline the elements of $J\Phi$ viewed as elements of $S^1 J\Phi$) and,

$$\Delta_C(\underline{\varphi\psi}) = 1 \otimes \underline{\varphi\psi} + \underline{\varphi\psi} \otimes 1$$

but

$$\begin{aligned}\Delta_T(\underline{\varphi\psi}) &= \Delta_T(\underline{\varphi}) \cdot \Delta_T(\underline{\psi}) \\ &= (1 \otimes \underline{\varphi} + \underline{\varphi} \otimes 1) \cdot (1 \otimes \underline{\psi} + \underline{\psi} \otimes 1) \\ &= 1 \otimes \underline{\varphi\psi} + \underline{\psi} \otimes \underline{\varphi} + \underline{\varphi} \otimes \underline{\psi} + \underline{\varphi\psi} \otimes 1.\end{aligned}$$

In the following when we refer to the coproduct of an element in SV, it will be in the sense of Δ_T.

Finally, we extend $\tilde{\Delta}$ to a map $\hat{\Delta}$ from $S^m\Gamma_c SJ\Phi \times S^n\Gamma_c SJ\Phi$ to H-distributions of compact support on $\mathcal{M}^m \times \mathcal{M}^n$ by the recursive expression

$$\hat{\Delta}(AB, C) = \sum \hat{\Delta}(A, C') \otimes \hat{\Delta}(B, C''), \tag{2.4}$$

$$\hat{\Delta}(A, 1) = \varepsilon(A),$$

where $\sum C' \otimes C''$ is the coproduct of C in $S^n\Gamma_c SJ\Phi$, and $\hat{\Delta}(A, C') \otimes \hat{\Delta}(B, C'')$; an H-distribution on $\mathcal{M}^m \times \mathcal{M}^n$. Note that (2.4) is precisely the definition of a Laplace pairing (see [27], Definition 10.2).

As an example suppose A, B and C are Lagrangian fields of compact support, i.e. $A, B, C \in \Gamma_c SJ\Phi$, then $AB \in S^2\Gamma_c SJ\Phi$ and $C \in S^1\Gamma_c SJ\Phi$. Then the distribution $\hat{\Delta}(AB, C)$ must be evaluated in densities on $\mathcal{M}^2 \times \mathcal{M}$. Assume $\alpha, \beta, \gamma \in \Gamma\omega$, then $(\alpha \otimes \beta) \otimes \gamma$ is a density over $\mathcal{M}^2 \times \mathcal{M}$ and the definition of $\hat{\Delta}$ indicates the following,

$$\hat{\Delta}(AB, C)((\alpha \otimes \beta) \otimes \gamma) = \sum \tilde{\Delta}(A, C')(\alpha \otimes \gamma) \cdot \tilde{\Delta}(B, C'')(\beta \otimes \gamma) \tag{2.5}$$

where we put $\tilde{\Delta}$ in the right hand side, instead of $\hat{\Delta}$, because when we restrict to elements of degree zero or one it takes the same values, and $A, B, C', C'' \in S^{\leqslant 1}\Gamma_c SJ\Phi$. The product $\cdot$ in (2.5), is a product of complex numbers.

With this successive extensions of a given propagator Δ, we define $\hat{\Delta}$ whose restrictions $\hat{\Delta}|_{S^m\Gamma_c SJ\Phi \times S^n\Gamma_c SJ\Phi}$ satisfy the following.

Definition 2.2.13. *A* generalized propagator *is a family* $\hat{\Delta} = \{\Delta_{m,n}\}_{m,n\in\mathbb{N}}$ *of continuous and* $\mathbb{R}$*-bilinear functions*

$$\Delta_{m,n} : S^m\Gamma_c SJ\Phi \times S^n\Gamma_c SJ\Phi \longrightarrow H_d(\mathcal{M}^m \times \mathcal{M}^n)$$

where the topology for the codomain is the weak-$*$ topology (see [30], Chapter 3, Section 11 and Chapter 6 for a general description of the weak-$*$ topology and Section 16 for the specific case $\mathcal{M}^m \times \mathcal{M}^n = \mathbb{R}^k$).

Definition 2.2.14. *A generalized propagator $\Delta = \{\Delta_{m,n}\}_{m,n\in\mathbb{N}}$ is called* cut *if for all $x \in \mathcal{M}$ there is a partial order in the cotangent space of $\mathcal{M}$ at x defined by a proper closed convex cone C_x, such that for all $(m, n) \in \mathbb{N}^2$ and $(p_1, \cdots, p_m, q_1, \cdots, q_n)$ in the wavefront set of $\Delta_{m,n}$ at the point $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ all the $p_i \preccurlyeq 0$ and all the $0 \preccurlyeq q_j$. And if $(p_1, \cdots, p_m, q_1, \cdots, q_n)$ is in the wavefront set of $\Delta_{m,n}$ on the diagonal of $\mathcal{M}^m \times \mathcal{M}^n$ then $p_1 + \cdots + p_m + q_1 + \cdots + q_n = 0$.*

2.3 Feynman measure

We are interested in integrating expressions of the form $e^{iL_F}\mathcal{L}$, where $\mathcal{L}$ is a lagrangian density.

Let us define a collection of maps $\chi_n : (S\Gamma_c\omega SJ\Phi)' \times S^n\Gamma_c\omega SJ\Phi \to \mathcal{C}^\infty(\mathcal{M}^n)'$ for $n \in \mathbb{N}$ by $\chi_n(\delta, A_1 \odot \cdots \odot A_n)(f_1 \otimes \cdots \otimes f_n) = \delta(f_1A_1 \odot \cdots \odot f_nA_n)$, where $f_i \in \mathcal{C}^\infty(\mathcal{M})$ and $A_i \in \Gamma_c\omega SJ\Phi$, $\forall i = 1, \cdots, n$. Since the subspace of $\mathcal{C}^\infty(\mathcal{M})$ formed by sums of functions of the form $f_1 \otimes \cdots \otimes f_n$ is dense, the maps χ_n are well-defined.

Set

$$\chi : (S\Gamma_c\omega SJ\Phi)' \times S\Gamma_c\omega SJ\Phi \to \oplus_{n\in\mathbb{N}}\mathcal{C}^\infty(\mathcal{M}^n)'$$

the direct sum of those maps.

Definition 2.3.1. *Let $\delta : S\Gamma_c\omega SJ\Phi \to \mathbb{C}$ be a continuous and linear map. We say that it is* smooth on the diagonal *if $(p_1, ..., p_n) \in \Sigma_{(q,...,q)}\chi_n(\delta, A)$ implies that $p_1 + \cdots + p_n = 0$, for all $A \in S^n\Gamma_c\omega SJ\Phi$.*

We are now ready to introduce the notion of Feynman measure.

Definition 2.3.2. *A Feynman measure is a continuous linear map from $S\Gamma_c\omega SJ\Phi$ to $\mathbb{C}$. Let $\delta : S\Gamma_c\omega SJ\Phi \to \mathbb{C}$ be a continuous linear map and let $\Delta : \Gamma_c\omega J\Phi \times \Gamma_c\omega J\Phi \to \mathbb{C}$ be a propagator. Then, δ is said to be associated to the propagator Δ if it satisfies the following conditions:*

1. *δ is smooth in the diagonal;*
2. *(non-degeneracy) There is a smooth nowhere vanishing function g so that $\delta(v) = \int_{\mathcal{M}} gv$ for all $v \in S^1\Gamma_c\omega S^0J\Phi = \Gamma_c\omega$;*
3. *(Gaussian condition or weak translational invariance) Let $A \in S^m\Gamma_c\omega SJ\Phi$ and $B \in S^n\Gamma_c\omega SJ\Phi$ such that there is no point in $supp(A)$ that is $\leqslant$ to some point in the $supp(B)$. This means that there is not point at $supp(A)$ which is $\leqslant$ to some point at $supp(B)$, we actually should say that there is no point in*

$supp(A) \subseteq \mathcal{M}^m$ such that any of its coordinate be $\leqslant$ to some coordinate of some point in $supp(B) \subseteq \mathcal{M}^n$. Then the following equality holds

$$\chi(\delta, A \cdot B) = \sum (\chi(\delta, A') \otimes \chi(\delta, B')) \cdot \hat{\Delta}(A'', B'') \tag{2.6}$$

where $\sum A' \otimes A'' \in S\Gamma_c \omega SJ\Phi \otimes S\Gamma_c SJ\Phi$ is the image of A given by the coaction $S^m\Gamma_c\omega SJ\Phi \to S^m\Gamma_c\omega SJ\Phi \otimes S^m\Gamma_c SJ\Phi$.

Remark 2.3.3. The product in (2.6) will be regarded in the following manner. Given $A'' \in S^m\Gamma_c SJ\Phi$ and $B'' \in S^n\Gamma_c SJ\Phi$, $\Delta(A'', B'')$ defines an element of $H_d(\mathcal{M}^m \times \mathcal{M}^n)$ that can be multiplied by $\chi(\delta, A') \otimes \chi(\delta, B') \in D_d(\mathcal{M}^m \times \mathcal{M}^n)$ using the product in 1.39.

Remark 2.3.4. Let us explain the coaction involved in the Gaussian condition in more detail.

By using the Theorem 2.1.12 and its Corollary 2.1.13, $\Gamma_c\omega SJ\Phi$ can be written in the form $\Gamma_c\omega \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma SJ\Phi$. Then by using Proposition 1.1.12 we conclude that $\Gamma_c\omega SJ\Phi$ is a comodule over $\Gamma SJ\Phi$ (call ρ the coaction), and consequently $S\Gamma_c\omega SJ\Phi$ is a comodule over $S\Gamma SJ\Phi$ and then it has sense to talk about a coaction.

In Sweedler notation if $\mathcal{L} \in \Gamma\omega SJ\Phi$ then $\rho(\mathcal{L}) = \sum_{()} \mathcal{L}_{(0)} \otimes \mathcal{L}_{(1)}$ where $\mathcal{L}_{(0)} \in \Gamma\omega SJ\Phi$ and $\mathcal{L}_{(1)} \in \Gamma SJ\Phi$, then we can define a coaction $\rho^n : S^n\Gamma\omega SJ\Phi \to S^n\Gamma\omega SJ\Phi \otimes S^n\Gamma SJ\Phi$ that over sections works as

$$\rho^n(\mathcal{L}_1 \odot \cdots \odot \mathcal{L}_n) = \sum_{()} (\mathcal{L}_{1(0)} \odot \cdots \odot \mathcal{L}_{n(0)}) \otimes (\mathcal{L}_{1(1)} \odot \cdots \odot \mathcal{L}_{n(1)})$$

where $\mathcal{L}_{j(0)} \in \Gamma\omega SJ\Phi$ for all $1 \leqslant j \leqslant n$ and $\mathcal{L}_{k(1)} \in \Gamma SJ\Phi$ for all $1 \leqslant k \leqslant n$, note that $\mathcal{L}_{1(1)} \odot \cdots \odot \mathcal{L}_{n(1)}$ belongs to $\Gamma SJ\Phi$. We denote by σ the coaction induced by $\oplus_{n\in\mathbb{N}_0}\rho^n$ which makes of $S\Gamma\omega SJ\Phi$ a $\Gamma SJ\Phi$-comodule.

Remark 2.3.5. Provided our propagator Δ is cut, the smoothness on the diagonal allows us to compute the product at the right hand of (2.6). Indeed, suppose we have $(p_1, \cdots, p_m, q_1, \cdots q_n)$ in the wave front set of $\Delta(A'', B'')$ at a point $(x, \cdots, x, y, \cdots, y)$, then $p_i \preccurlyeq 0$ and $0 \preccurlyeq q_j$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

If $(-p_1, \cdots, -p_m, -q_1, \cdots - q_n)$ is in the wave front set of $\chi(\delta, A') \otimes \chi(\delta, B')$ Lemma 2.175 in [29] implies that $q_j = 0$ for all $1 \leqslant j \leqslant n$ or $(-q_1, \cdots, -q_n)$ is in the wave front set of $\chi(\delta, B')$ at $(x, \cdots, x)$. Consequently $-q_1 - q_2 \cdots - q_n = 0$ by the smoothness on the diagonal condition of the Feynman measure. In other words we get that $q_1 + \cdots + q_n = 0$. This condition together with the previous one stating that $0 \preccurlyeq q_i$ (i.e. q_i is in the proper convex closed cone C_x) implies that all the q_j must be zero. A similar argument can be used to see that all the p_i are zero. This is absurd, for $(p_1, \cdots, p_m, q_1, \cdots q_n)$ is in the wave front set of $\Delta(A'', B'')$. The contradiction came from the assumption that $(-p_1, \cdots, -p_m, -q_1, \cdots - q_n)$ was in the wave front set of $\chi_m(\delta, A') \otimes \chi_n(\delta, B')$.

As a consequence, we conclude that the product between these distributions is well defined.

The idea is to think of the Feynman measure in A as the integral of A over all the classical fields, that is $\delta(A) = \int_\Phi A(\varphi)\mathcal{D}\varphi$.

Remark 2.3.6. Given Δ a Feynman measure and $A \in S^n\Gamma_c\omega SJ\Phi$, then $\delta(A)$ is a complex number and $\chi_n(\delta, A)$ is an element of $D_d(\mathcal{M}^n)$. The latter can be thought of a density over $\mathcal{M}^n$ in the sense that the densities take functions and assign numbers (by integration over $\mathcal{M}$), i.e. $\chi_n(\delta, A)(f) = \delta(fA)$ for $f \in \mathcal{C}^\infty(\mathcal{M}^n)$. By using the notation for the complex number $\delta(A) = \chi_n(\delta, A)(1)$, where 1 is the function identically 1 over $\mathcal{M}^n$, and as a consequence of this, if we want to see that $\delta(A) = \delta(B)$ for two elements $A, B \in S^n\Gamma_c\omega SJ\Phi$ then we need only to examine $\chi_n(\delta, A)$ and $\chi_n(\delta, B)$ as elements of $D_d(\mathcal{M}^n)$, and lastly conclude the equality by evaluating in the function 1, $\chi_n(\delta, A)(1) = \chi_n(\delta, B)(1)$ or $\delta(A) = \delta(B)$ as complex numbers.

As a conclusion, if we want to prove that $\delta(A) = \delta(B)$ for two elements $A, B \in S\Gamma_c\omega SJ\Phi$ then it is sufficient to prove it for $\chi(\delta, A)$ and $\chi(\delta, B)$.

Remark 2.3.7. A last comment concerning the physics picture of the Feynman measure is that the smoothness on the diagonal is exactly the conservation of the momentum, because the momentum coordinates are the coordinates of the Fourier transform of a field.

Chapter 3

Renormalization

3.1 Characterization of the renormalization group

As we saw in the precedent chapter the sheaf $S\omega SJ\Phi$ has a structure of comodule over $SJ\Phi$.

Definition 3.1.1. *A* renormalization *is an automorphism of the sheaf of coalgebras $S\omega SJ\Phi$ which preserves the coaction of $SJ\Phi$. The set of renormalizations is a group under composition and is called the renormalization group.*

The next theorem characterizes the renormalization group in a nice manner.

Theorem 3.1.2. *The elements of the renormalization group are in correspondence with the elements in $Hom(S\omega SJ\Phi, \omega)$ that are zero over $S^0\omega SJ\Phi$ and are isomorphisms when they are restricted to $S^1\omega S^0 J\Phi = \omega$.*

Proof. By Proposition 1.1.23 there is a correspondence between the coalgebra automorphisms $R \in Hom_{Coalg}(S\omega SJ\Phi, S\omega SJ\Phi)$ and the morphisms $r \in Hom_{\mathcal{C}^{\mathbb{R}}-Mod}(S\omega SJ\Phi, \omega SJ\Phi)$ whose sequential representation satisfies $r_0 = 0$ and r_1 is an isomorphism.

At the beginning of this chapter we mentioned that $S\omega SJ\Phi$ is a comodule over $SJ\Phi$. We use Proposition 1.1.24 in the monoidal abelian category of sheafs over the ringed space $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$, taking D as the sheaf $SJ\Phi$, M as the sheaf $S\omega SJ\Phi$ of $SJ\Phi$-comodules and W as the sheaf ω. It gives a correspondence between the comodule morphisms $r \in Hom_{Com_{SJ\Phi}}(S\omega SJ\Phi, \omega SJ\Phi)$ and the morphisms $\eta \in Hom_{\mathcal{C}^{\mathbb{R}}-Mod}(S\omega SJ\Phi, \omega)$.

If we begin this process with an $R \in Hom_{Coalg}(S\omega SJ\Phi, S\omega SJ\Phi)$ which is a renormalization, then it is a coalgebra morphism and also a comodule morphism so we successively apply both correspondences to obtain the desired total correspondence between the renormalizations and the subset of $\eta \in Hom_{\mathcal{C}^{\mathbb{R}}-Mod}(S\omega SJ\Phi, \omega)$ such that $\eta_0 = 0$, $\eta_1 = (Id_\omega \otimes \varepsilon_{SJ\Phi}) \circ r_1|_{\omega S^0 J\Phi}$ is an isomorphism (because r_1 is). □

3.2 The structure of the renormalization group

Notice that the renormalization group (which we will denote by G) preserves the increasing filtration $S^{\leqslant n}\omega SJ\Phi \subseteq S^{\leqslant n+1}\omega SJ\Phi$, because if we apply $P \in G$ to an element $v \in S^{\leqslant n}\omega SJ\Phi$, from equation (1.18) is clear that $P(v) \in S^{\leqslant n}\omega SJ\Phi$.

If $\mathcal{C}$ is a monoidal abelian subcategory of ${}_AMod$ or $\mathcal{C}^{\mathbb{R}}$-Modules, we denote by G' the group of automorphisms of the cocommutative coaugmented coalgebra S^cV (with $V \in Obj(\mathcal{C})$) and $G'_{>a} = \{g \in G'/g(\alpha) = \alpha, \forall \alpha \in S^{\leqslant a}V\}$, where $a \in \mathbb{N}_0$. Notice that $G'_{>0}$ is exactly G'.

Proposition 3.2.1. *Given $a \in \mathbb{N}$ and $P \in G'$. Then $P \in G'_{>a}$ if and only if the sequential representation of P satisfies $p_1 = Id_V$ and $p_n = 0, \forall 2 \leqslant n \leqslant a$.*

Proof. The proof follows directly from equation (1.18). If we apply the formula for $n = 1$ and $P \in G'_{>a}$, then we obtain:

$$v = P(v) = \sum_{m=1}^{1} \sum_{I_1 \neq \emptyset; I_1 = \{1\}} \frac{1}{1!} p_1(v_{I_1}) = p_1(v),$$

that is p_1 is the identity of V.

Let us do the same for $n = 2$ (supposing that $a \geqslant 2$), in this case the equation (1.18) gives us,

$$v_1 \cdot v_2 = \frac{1}{2} p_1(v_1) \cdot p_1(v_2) + \frac{1}{2} p_1(v_2) \cdot p_1(v_1) + p_2(v_1 \cdot v_2),$$

where we use the previous result, $p_1(v_1) \cdot p_1(v_2) = v_1 \cdot v_2$, and $P(v_1 \cdot v_2) = v_1 \cdot v_2$ and so $p_2(v_1 \cdot v_2) = 0$.

Let us proceed by induction. Given $j \in \mathbb{N}$ such that $2 \leqslant j \leqslant a$ and supposing that $p_k = 0$ for all $k \in \mathbb{N}$ satisfying $2 \leqslant k < j$ and $p_1 = Id_V$, we write the equation (1.18)(taking $n = j$),

$$\begin{aligned} v_1 \cdots v_j = P(v_1 \cdots v_j) &= \sum_{\text{all permutations}} \frac{1}{j!} p_1(v_1) \cdots p_1(v_j) \\ &+ \sum_{\text{terms with } p_k \text{ with } 2 \leqslant k < j} \cdots + p_j(v_1 \cdots v_j) \\ &= j! \frac{1}{j!} v_1 \cdots v_j + 0 + \cdots + 0 + p_j(v_1 \cdots v_j), \end{aligned}$$

hence $p_j(v_1 \cdots v_j) = 0$ for all $v_1 \cdots v_j \in S^jV$.

Reciprocally if the sequential representation of P satisfies $p_1 = Id_V$ and $p_n = 0$ for all $2 \leqslant n \leqslant a$ then if we apply P to $v_1 \cdots v_n \in S^nV$ by means of equation (1.18), only the terms such that $|I_j| = 1$ survive, that is when $m = n$. Hence,

$$P(v_1 \cdot v_2 \cdots v_n) = \sum_{m=1}^{n} \sum_{I_1 \cdots I_m \neq \emptyset; I_1 \sqcup \cdots \sqcup I_m = \{1 \cdots n\}} \frac{1}{m!} p_{|I_1|}(v_{I_1}) \cdots p_{|I_m|}(v_{I_m})$$

$$= \sum_{I_1 \cdots I_n \neq \emptyset; I_1 \sqcup \cdots \sqcup I_n = \{1 \cdots n\}} \frac{1}{n!} p_1(v_{I_1}) \cdots p_1(v_{I_n})$$

$$= \sum_{I_1 \cdots I_n \neq \emptyset; I_1 \sqcup \cdots \sqcup I_n = \{1 \cdots n\}} \frac{1}{n!} Id_V(v_{I_1}) \cdots Id_V(v_{I_n})$$

$$= \sum_{I_1 \cdots I_n \neq \emptyset; I_1 \sqcup \cdots \sqcup I_n = \{1 \cdots n\}} \frac{1}{n!} v_1 \cdots v_n = \frac{n!}{n!} v_1 \cdots v_n = v_1 \cdots v_n,$$

where we use the fact that there are $n!$ forms of take intervals I_i such that $|I_i| = 1$ and $I_1 \sqcup \cdots \sqcup I_n = \{1, \cdots, n\}$. □

If $P \in G'$, we call p its sequential representation and usually say that $p \in G'$.

Proposition 3.2.2. *$G'_{>a}$ is a subgroup of G' for all $a \in \mathbb{N}$*

Proof. By composing with $\pi : S^cV \to V$ it is clear that $1_{G'}$ has the sequential representation $\{Id_V, 0, 0, \cdots\}$, then by the precedent proposition $1_{G'} \in G_{>a}$ for all $a \in \mathbb{N}$.

If the sequential representation of P and $Q \in G'$ are $\{p_n\}_{\mathbb{N}}$ and $\{q_n\}_{\mathbb{N}}$ respectively, then the product representation of $P \circ Q$ has sequential representation $\{p_n\}_{\mathbb{N}} \cdot \{q_n\}_{\mathbb{N}} = \{(p \cdot q)_n\}_{\mathbb{N}}$ given by the following formula (see equation (1.19)),

$$(p \cdot q)_n(v_1 \cdots v_n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{I_1 \cdots I_m \neq \emptyset; I_1 \sqcup \cdots \sqcup I_m = \{1 \cdots n\}} p_m(q_{|I_1|}(v_{I_1}) \cdots q_{|I_m|}(v_{I_m})).$$

With this formula it is possible to show that $p \cdot q \in G'_{>a}$ provided that p and q are there. For example if $2 \leqslant n \leqslant a$, having in account that $q_k = 0$ for all $2 \leqslant k \leqslant a$, we see that in the sum only the terms with all q_1 survive, i.e. $m = n$. Also as q_1 is the identity: $q_1(v_{I_1}) \cdots q_1(v_{I_n}) = v_{I_1} \cdots v_{I_n}$, but $v_{I_i} = v_{j_i}$ because $|I_i| = 1$ when $m = n$, then $q_1(v_{I_1}) \cdots q_1(v_{I_n}) = v_{j_1} \cdots v_{j_n}$. But as $\cdot$ is a symmetric product then all the terms $v_{j_1} \cdots v_{j_n}$ are equal, so there are $n!$ summands which are equal and consequently the last equality can be written as

$$(p \cdot q)_n(v_1 \cdots v_n) = n! \frac{1}{n!} p_n(v_1 \cdots v_m).$$

This is zero because $p_n = 0$. From the same equation (1.19), but this time taking $n = 1$, we can conclude $(p \cdot q)_1 = Id_V$.

Finally if $p \in G'_{>a}$, we take $q = p^{-1}$ in G'. Suppose $q \notin G'_{>a}$, and let n_0 be the first natural $\geqslant 2$ such that $q_{n_0} \neq 0$. Equation (1.19) show us that $q_1 = p_1^{-1}$ (taking $n = 1$). But $q \notin G'_{>a}$, then $n_0 \leqslant a$. Then by using again the equation (1.19) taking $n = n_0$ and using the fact that $\{(p \cdot q)_n\}_{\mathbb{N}} = \{Id_V, 0, 0 \cdots\}$ we obtain $0 = p_1(q_{n_0}(v_1 \cdots v_{n_0}))$. But as $p_1 = Id_V$ the equation turns to: $0 = q_{n_0}(v_1 \cdots v_{n_0})$ i.e. $q_{n_0} = 0$, which is absurd. Hence $q \in G'_{>a}$. □

Notice that the precedent proposition is trivially valid for the case $a = 0$.

For $a \geqslant 2$ we define the sets

$$G'_a = \{P \in G'/p = \{Id_V, 0 \cdots, 0, p_a, 0 \cdots\}\},$$

and

$$G'_1 = \{P \in G'/p = \{p_1, 0 \cdots\}\}$$

were we use the just described notation of sequential representation introduced in Chapter 1. An element in $P \in G'_a$ will be an automorphism $P \in G'$ such that when we restrict $\pi \circ P$ to $S^n V$ it is zero ($\pi : S^n V \to V$ is the canonical projection), and less $n = 1$ or $n = a$.

The next proposition is proved by induction on a.

Proposition 3.2.3. *Let $a \in \mathbb{N}$ and $P \in G'$. There are unique $P_1, P_2, \cdots P_a, P' \in G'$ such that $P_i \in G'_i$, $P' \in G'_{>a}$ and $P = P_1 \circ P_2 \cdots P_a \circ P'$*

The proof is deduced from the next lemma:

Lemma 3.2.4. *Given $a \in \mathbb{N}$ and $P \in G'_{>a}$ there exist unique P_{a+1} and $P' \in G'$ such that $P_{a+1} \in G'_{a+1}$, $P' \in G'_{>a+1}$ and $P = P_{a+1} \circ P'$.*

Proof. We give the proof for the case $a \geqslant 2$, the case $a = 1$ is similar. If the representation of P is given by $\{Id_V, 0 \cdots 0, p_{a+1}, p_{a+2} \cdots\}$, we take P_{a+1} as the only one whose sequential representation is $\tilde{p} = \{Id_V, 0 \cdots p_{a+1}, 0 \cdots\}$ and P' whose representation is $\{(\tilde{p}^{-1} \cdot p)_n\}_{n \in \mathbb{N}}$. By using the equation (1.19) it is not difficult to see that $(\tilde{p}^{-1} \cdot p)_1 = Id_V$, and as $\tilde{p}, p \in G'_{>a}$ (which is a group as we saw in Proposition 3.2.2) then $(\tilde{p}^{-1} \cdot p)_n = 0$ for all $2 \leqslant n \leqslant a$. By means of (1.19) and computing $\tilde{p}^{-1}$ up to degree $a + 1$ with (1.20), we can see that $\tilde{p}^{-1} = \{Id_V, 0 \cdots 0, -p_{a+1}, \cdots\}$ and $(\tilde{p}^{-1} \cdot p)_{a+1} = 0$, so $\tilde{p}^{-1} \cdot p \in G'_{>a+1}$. Let us see in detail that, if $\tilde{p} = \{Id_V, 0 \cdots p_{a+1}, 0 \cdots\}$ then its inverse satisfies $(\tilde{p}^{-1})_1 = (\tilde{p}_1)^{-1} = Id_V^{-1} = Id_V$ and then by using the equation (1.21) for $2 \leqslant k \leqslant a$ we have

$$\begin{aligned}
&(\tilde{p}^{-1})_k(v_1 \cdot v_2 \cdots v_k) \\
&= -\sum_{m=2}^{k} \frac{1}{m!} \sum_{I_1 \cdots I_m \neq \emptyset; I_1 \sqcup \cdots \sqcup I_m = \{1 \cdots k\}} (\tilde{p}^{-1})_1(\tilde{p}_m((\tilde{p}^{-1})_{|I_1|}(v_{I_1}) \cdots (\tilde{p}^{-1})_{|I_m|}(v_{I_m}))),
\end{aligned}$$

but as we say $(\tilde{p}^{-1})_1$ is the identity and then

$$(\tilde{p}^{-1})_k(v_1 \cdot v_2 \cdots v_k) \\ = -\sum_{m=2}^{k} \frac{1}{m!} \sum_{I_1 \cdots I_m \neq \emptyset; I_1 \sqcup \cdots \sqcup I_m = \{1 \cdots k\}} \tilde{p}_m((\tilde{p}^{-1})_{|I_1|}(v_{I_1}) \cdots (\tilde{p}^{-1})_{|I_m|}(v_{I_m})),$$

which is zero because all the $\tilde{p}_m$ has $m \leqslant k \leqslant a$, and then $\tilde{p}_m = 0$.

Lastly if we take $n = a+1$ in (1.21), we obtain:

$$(\tilde{p}^{-1})_{a+1}(v_1 \cdot v_2 \cdots v_{a+1}) \\ = -\sum_{m=2}^{a+1} \frac{1}{m!} \sum_{I_1 \cdots I_m \neq \emptyset; I_1 \sqcup \cdots \sqcup I_m = \{1, \cdots, a+1\}} \tilde{p}_m((\tilde{p}^{-1})_{|I_1|}(v_{I_1}) \cdots (\tilde{p}^{-1})_{|I_m|}(v_{I_m})).$$

By the same aforementioned considerations all the summands are zero except the only ones in which appears $\tilde{p}_{a+1}$, so

$$(\tilde{p}^{-1})_{a+1}(v_1 \cdot v_2 \cdots v_{a+1}) \\ = -\frac{1}{(a+1)!} \sum_{I_1 \cdots I_{a+1} \neq \emptyset; I_1 \sqcup \cdots \sqcup I_{a+1} = \{1, \cdots, a+1\}} \tilde{p}_{a+1}((\tilde{p}^{-1})_{|I_1|}(v_{I_1}) \cdots (\tilde{p}^{-1})_{|I_{a+1}|}(v_{I_{a+1}})).$$

The only way such that $a+1$ non-empty sets have disjoint union equal to $\{1, \cdots, a+1\}$ is that each one of them has cardinal 1 and then for all $1 \leqslant j \leqslant a+1$ we have $(\tilde{p}^{-1})_{|I_j|}(v_{I_j}) = (\tilde{p}^{-1})_1(v_{i_j}) = I_V(v_{i_j}) = v_{i_j}$. Consequently,

$$(\tilde{p}^{-1})_{a+1}(v_1 \cdot v_2 \cdots v_{a+1}) \\ = -\frac{1}{(a+1)!} \sum_{I_1 \cdots I_{a+1} \neq \emptyset; I_1 \sqcup \cdots \sqcup I_{a+1} = \{1, \cdots, a+1\}} \tilde{p}_{a+1}(v_{i_1} \cdots v_{i_{a+1}}).$$

Since the $\cdot$ is a symmetric product we can write,

$$(\tilde{p}^{-1})_{a+1}(v_1 \cdot v_2 \cdots v_{a+1}) = -\tilde{p}_{a+1}(v_1 \cdots v_{a+1}).$$

That is $(\tilde{p}^{-1})_{a+1} = -\tilde{p}_{a+1}$. Now we know that $\tilde{p}^{-1} = \{Id_V, 0 \cdots 0, -p_{a+1}, \cdots\}$, it is no so difficult to see that $(\tilde{p}^{-1} \cdot p)_{a+1} = 0$ by using the equation (1.19) for $n = a+1$.

The uniqueness is deduced from a similar argument, if $P = P_{a+1} \cdot P' = Q_{a+1} \cdot Q'$ with $P_{a+1}, Q_{a+1} \in G'_{a+1}$ and $P', Q' \in G'_{>a+1}$ then $P_{a+1}^{-1} \cdot Q_{a+1} = P' \cdot Q'^{-1}$.

If the sequential representation of P_{a+1} is $\{Id_V, 0 \cdots, \tilde{p}_{a+1}, 0, \cdots\}$ the element $a+1$ of the sequential representation of P_{a+1}^{-1} is $-\tilde{p}_{a+1}$ and if Q_{a+1} has representation $\{Id_V, 0 \cdots, \tilde{q}_{a+1}, 0, \cdots\}$ so the $a+1$-element of the representation of $P_{a+1}^{-1} \cdot Q_{a+1}$ is $-\tilde{p}_{a+1} + \tilde{q}_{a+1}$, but $P_{a+1}^{-1} \cdot Q_{a+1} \in G'_{>a+1}$ so $-\tilde{p}_{a+1} + \tilde{q}_{a+1} = 0$ and $P_{a+1} = Q_{a+1}$, which says us that $P' = Q'$. □

The proof of Proposition 3.2.3 is deduced by repeated application of the precedent lemma.

For Proposition 3.2.3 and Lemma 3.2.4 there exist similar statements in which the order of the compositions is the opposite. We rewrite Proposition 3.2.3 in these terms:

Proposition 3.2.5. *Let $a \in \mathbb{N}$ and $P \in G'$. There are unique $P_1, P_2, \cdots P_a, P' \in G'$ such that $P_i \in G'_i$, $P' \in G'_{>a}$ and $P = P' \circ P_a \circ P_{a-1} \cdots P_1$*

Proposition 3.2.6. *$G'_{>a}$ is a normal subgroup of G' for all $a \in \mathbb{N}$.*

Proof. Thanks to Propositions 3.2.2 and 3.2.3 it is sufficient to prove that $P^{-1} \cdot G'_{>a} \cdot P \subseteq G'_{>a}$, for all $P \in G'_i$ with $1 \leqslant i \leqslant a$.

If $i = 1$ and $Q \in G'_{>a}$ then abusing the notation $Q = \{Id_V, 0 \cdots, 0, q_{a+1}, \cdots\}$, and with (1.19) is deduced $P^{-1} \cdot Q \cdot P = \{Id_V, 0 \cdots, 0, p_1 \circ q_{a+1} \circ p_1^{\odot(a+1)}, \cdots\}$ which is in $G'_{>a}$ (where $\odot$ denotes the symmetric product).

If $2 \leqslant i \leqslant a$; $Q = \{Id_V, 0, \cdots, 0, q_{a+1}, \cdots\}$ and $P = \{Id_V, 0, \cdots, 0, p_i, 0 \cdots\}$ by using (1.19) obtain $Q \cdot P = \{Id_V, 0, \cdots, 0, p_i, 0, \cdots, 0, q'_{a+1}, \cdots\}$.

By using (1.20) $P^{-1} = \{Id_V, 0 \cdots, 0, -p_i, r_{i+1}, \cdots, r_a, \cdots\}$.But when we multiply $P^{-1} \cdot Q \cdot P$ the first a elements are calculated in the same manner which the first a elements of $P^{-1} \cdot P$, because the elements with order bigger than a do not appear. So $(\pi \circ P^{-1} \cdot Q \cdot P)|_{S^nV} = (\pi \circ P^{-1} \cdot P)|_{S^nV}$ for all $1 \leqslant n \leqslant a$, which implicates that $P^{-1} \cdot Q \cdot P \in G'_{>a}$. □

The preceding proposition is valid in the case $a = 0$ too but in this case the proof is immediate. The next corollary is a consequence of the above results.

Corollary 3.2.7. *For all $a \in \mathbb{N}$ we have $G'/G'_{>a} \simeq G'_{\leqslant a}$.*

Proposition 3.2.8. *For all $a, b \in \mathbb{N}$; $[G'_{>a}, G'_{>b}] \subseteq G'_{>a+b}$*

Proof. Without loss of generality we can suppose $a \leqslant b$. Let $P \in G'_{>a}$ and $Q \in G'_{>b}$. Then the sequential representation of P is $\{p_n\}_{n\in\mathbb{N}} = \{Id_V, 0, \cdots, 0, p_{a+1}, p_{a+2}, \cdots\}$ and similarly for Q, $\{q_n\}_{n\in\mathbb{N}} = \{Id_V, 0, \cdots, 0, q_{b+1}, q_{b+2}, \cdots\}$. By the composition formula we have

$$(p \cdot q)_n(v_1 \cdots v_n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{I_i \neq \emptyset; \sqcup_{i=1}^m I_i = \{1, \cdots, n\}} p_m(q_{|I_1|}(v_{I_1}) \cdots q_{|I_m|}(v_{I_m})) \tag{3.1}$$

If $n = 1$ in (3.1) we have $(p \cdot q)_1 = Id_V$. For $n \in \{2, \cdots, a\}$, in the sum over m only the terms with $m = 1$ survive, because $p_j = 0$ for $2 \leqslant j \leqslant n \leqslant a$, then $(p \cdot q)_n(v_1 \cdots v_n) = q_n(v_1 \cdots v_n)$ which is equal to zero because of $2 \leqslant n \leqslant a \leqslant b$.

For $a+1 \leqslant n \leqslant b$, in (3.1) only the terms with $m = 1, a+1, a+2, \cdots n$ survive, so

$$(p \cdot q)_n(v_1 \cdots v_n) = q_n(v_1 \cdots v_n) + \sum_{m=a+1}^{n} \frac{1}{m!} \sum_{I_i \neq \emptyset; \sqcup_{i=1}^{m} I_i = \{1, \cdots, n\}} p_m(q_{|I_1|}(v_{I_1}) \cdots q_{|I_m|}(v_{I_m})) \tag{3.2}$$

but $q_n = 0$. Moreover, if $m \in \{a+1, \cdots, n-1\}$ then there exists at least one I_{i_0} such that $|I_{i_0}| \geqslant 2$ (and $|I_{i_0}| \leqslant b$) and consequently $q_{|I_{i_0}|} = 0$ and (3.2) becomes

$$(p \cdot q)_n(v_1 \cdots v_n) = p_n(v_1 \cdots v_n).$$

Lastly if $b+1 \leqslant n \leqslant a+b$, as before in the sum over m only the terms with $m = 1, a+1, a+2, \cdots n$ persist. Then with similar arguments as in the previous cases we have,

$$(p \cdot q)_n(v_1 \cdots v_n) = q_n(v_1 \cdots v_n) + p_n(v_1 \cdots v_n).$$

So we have

$$\{(p \cdot q)\}_{n \in \mathbb{N}} = \{Id_v, 0, \cdots, 0, p_{a+1}, \cdots, p_b, p_{b+1} + q_{b+1}, \cdots, p_{b+a} + q_{b+a}, \cdots\}. \tag{3.3}$$

The computation of the sequential representation of $Q \circ P$ is a bit easier and gives us the same result up to the $a+b$ term. For this composition we have the formula

$$(q \cdot p)_n(v_1 \cdots v_n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{I_i \neq \emptyset; \sqcup_{i=1}^{m} I_i = \{1, \cdots, n\}} q_m(p_{|I_1|}(v_{I_1}) \cdots p_{|I_m|}(v_{I_m})). \tag{3.4}$$

It is easy to check the $(q \cdot p)_1 = Id_V$, and that if $n \in \{2, \cdots, a\}$ then $(q \cdot p)_n = 0$, because (3.4) becomes

$$(q \cdot p)_n(v_1 \cdots v_n) = q_1(p_{|I_1|}(v_{I_1})) = p_n(v_1 \cdots v_n) = 0, \tag{3.5}$$

for $I_1 = \{1, \cdots, n\}$.

For $n \in \{a+1, \cdots, b\}$ we have

$$(q \cdot p)_n(v_1 \cdots v_n) = q_1(p_{|I_1|}(v_{I_1})) = p_n(v_1 \cdots v_n), \tag{3.6}$$

but this time $p_n \neq 0$, because $n \geqslant a+1$.

Finally for $n \in \{b+1, \cdots, b+a\}$ in (3.4) only the terms with $m = 1, b+1, b+2, \cdots, n$ survive so,

$$(q \cdot p)_n(v_1 \cdots v_n) = q_1(p_n(v_1 \cdots v_n)) + \sum_{m=b+1}^{n} \frac{1}{m!} \sum_{I_i \neq \emptyset; \sqcup_{i=1}^{m} I_i = \{1, \cdots, n\}} q_m(p_{|I_1|}(v_{I_1}) \cdots p_{|I_m|}(v_{I_m})),$$

and for $m \in \{b+1, \cdots, n-1\}$ there exists at least one I_{i_0} with more than one element and less then $a+1$, hence $p_{|I_{i_0}|} = 0$. That is, $(q \cdot p)_n(v_1 \cdots v_n) = p_n(v_1 \cdots v_n) + q_n(v_1 \cdots v_n)$.

Then the sequential representation of $Q \circ P$ is the same that the right hand side of (3.3) up to the $a+b$-term. Whence $P \circ Q = Q \circ P$ in $G'_{\leqslant a+b}$, which proves what we want. □

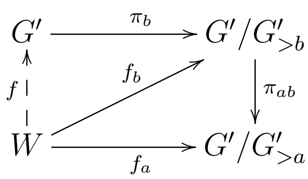
We have just seen that it has sense to talk about $G'/G'_{>a}$ for all $a \in \mathbb{N}$ it is possible to see that G' is the inverse limit of $G'/G'_{>n}$.

Lemma 3.2.9. *G' is the direct limit of the projective system $\{G'/G'_{>n}\}_{n \in \mathbb{N}}$.*

Proof. If $a < b$ are natural numbers then we can define $\pi_{ab} : G'/G'_{>b} \to G'/G'_{>a}$ as $\pi_{ab}(P_1 P_2 \cdots P_b) = P_1 P_2 \cdots P_a$ where we have used that each element that belongs to $G'/G'_{>b}$ can be written uniquely as a product by Lemma 3.2.7. This function clearly satisfies $\pi_{ab} \circ \pi_{bc} = \pi_{ac}$ and so we have a projective system.

Moreover the morphisms $\pi_a : G' \to G'/G'_{>a}$ such that $\pi_a(P) = [P]_{G'/G'_{>a}} = P_1 \cdots P_a$ commutes with π_{ab} $(a < b)$ because $\pi_{ab} \circ \pi_b(P) = \pi_{ab}(P_1 \cdots P_b) = P_1 \cdots P_a = \pi_a(P)$.

Suppose that W is another group with a family of arrows $f_a : W \to G'/G'_{>a}$ satisfying $\pi_{ab} \circ f_b = f_a$ too, for $a < b$. We can define a unique morphism $f : W \to G'$ such that the following diagram



commutes. □

We will define a topology on G' (its projective topology, the finest such that π_n are all continuous), where a subbase of the open neighbourhoods of zero is given by the $G'_{>n}$. Then in the above diagram we can define as $f(w) = \lim_{n\to\infty} f_n(w) \in G'/G'_{>n} = G'_1 \cdots G'_n \subseteq G'$ which exists because f_n commutes with π_{ab}. With this definition we have $\pi_b(f(w)) = \pi_b(\lim_{n\to\infty} f_n(w)) = \lim_{n\to\infty} \pi_b f_n(w)$ where used the continuity of π_b with respect to this topology. Note that for all $n > b$, $f_n(w)$ has more than b "elements", and so $\pi_b(f_n(w)) = f_b(w)$ and consequently $\pi_b \circ f = f_b$.

As a major consequence of the last results we can stablish the next corollary, its proof is immediate.

Corollary 3.2.10. *G' is equal to $G'_1 G'_2 G'_3 \cdots$ in the sense that any element $P \in G'$ can be written as an infinite product $P = P_1 \circ P_2 \circ P_3 \cdots$ of elements $P_i \in G'_i$ and conversely each of this infinite products in such conditions belongs to G'.*

Proof. Given $P \in G'$, call P_i the morphisms belonging to G'_i proportioned by Proposition 3.2.3. The limit $P_1 \circ P_2 \circ P_3 \cdots$ (i.e. $\lim_{n\to\infty} P_1 \cdots P_n$) where $P_i \in G'_i$ converges in G' because a subbase of open sets for the projective topology in G' are $G'_{>k}$ for $k \in \mathbb{N}$ and then the difference between P and $P_1 \cdots P_n$ belongs to any $G'_{>a}$, provided n is sufficiently large. □

This corollary has also a version with the products in the opposite order.

Corollary 3.2.11. *G' is equal to $\cdots G'_3 G'_2 G'_1$in the sense that any element $P \in G'$ can be written as an infinite product $P = \cdots P_3 \circ P_2 \circ P_1$ of elements $P_i \in G'_i$ and conversely each of this infinite products in such conditions belongs to G'.*

It is of central interest for us the case $V = \omega SJ\Phi$. In this case G' is not the renormalization group, because an element belonging to G' may not preserve the coaction of $SJ\Phi$.

The natural map $S\Gamma\omega SJ\Phi \to \Gamma S\omega SJ\Phi$ is not an isomorphism because as we mentioned, there is a difference between the symmetric product in the category of sheafs of modules over the ringed space $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$ and the symmetric product in the category of $\mathbb{R}$-vector spaces. For example if we denote by $\odot$ the symmetric product in the category of sheafs of modules over the ringed space $(\mathcal{M}, \mathcal{C}^{\mathbb{R}})$ and by $\cdot$ in the category of $\mathbb{R}$-vector spaces, the natural map sends $\psi \cdot \eta \to \psi \odot \eta$ where ψ and η are lagrangian densities.

The difference between these product is that if f is a smooth function over $\mathcal{M}$ then $f\psi \cdot \eta \neq \psi \cdot f\eta$ but $f\psi \odot \eta = \psi \odot f\eta$, hence the natural map is not injective.

Lemma 3.2.12. *The induced action of a renormalization over $\Gamma S\omega SJ\Phi$ can be lifted to an action over $S\Gamma\omega SJ\Phi$ preserving the coproduct, the coaction of $\Gamma SJ\Phi$ and the product of elements with disjoint support.*

Proof. By Theorem 2.1.12 $\Gamma\omega SJ\Phi \simeq \Gamma\omega \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma SJ\Phi$, then applying Proposition 1.1.12, $\Gamma\omega \otimes_{\mathcal{C}^\infty(\mathcal{M})} \Gamma SJ\Phi$ can be seen as a $\Gamma SJ\Phi$-comodule over $\mathcal{C}^\infty(\mathcal{M})$.

Call N the natural map $S\Gamma\omega SJ\Phi \to \Gamma S\omega SJ\Phi$, and φ the isomorphism existent between $\Gamma S\omega SJ\Phi$ and $S_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega SJ\Phi$ (see Corollary 2.1.13). In this context, given a renormalization P, we still call P the induced map on the sections of $S\omega SJ\Phi$ and then we consider the composition $p := \pi_1 \circ \varphi \circ P \circ N : S\Gamma\omega SJ\Phi \to \Gamma\omega SJ\Phi$.

By using Proposition 1.1.23, $p : S\Gamma\omega SJ\Phi \to \Gamma\omega SJ\Phi$ is in correspondence with a morphism which we call $P' : S\Gamma\omega SJ\Phi \to S\Gamma\omega SJ\Phi$ and is written (as in Proposition 1.1.21)

$$P'(c) = \sum_{n\in\mathbb{N}} \frac{1}{n!} p^{\odot n} \circ \Delta_C^{(n)}(c), \tag{3.7}$$

where $C = S\Gamma\omega SJ\Phi$ and $\Delta_C^{(n)}$ was defined in Section 1.1.2.

Now we prove that P' preserves the product of elements with disjoint support, so let $\varphi = \prod_{i=1}^{k} f_i$ and $\psi = \prod_{l=1}^{r} g_l$ be elements of $S\Gamma\omega SJ\Phi$ with disjoint support, $\varphi \cdot \psi \in S\Gamma\omega SJ\Phi$, call $\chi_i = f_i$ for all $1 \leqslant i \leqslant k$ and $\chi_{k+l} = g_l$ for all $1 \leqslant l \leqslant r$. From the statement over the supports we can suppose that $supp(f_i) \subseteq U$ and $supp(g_l) \subseteq V$ for some U and V disjoint open sets in $\mathcal{M}$.

Consider

$$\Delta^{(n)}_{S\Gamma\omega SJ\Phi}(\varphi \cdot \psi) = \sum_{I_1,\cdots,I_n \neq \emptyset; I_1 \sqcup \cdots \sqcup I_n = \{1,\cdots,k+r\}} \chi_{I_1} \otimes \cdots \otimes \chi_{I_n},$$

so by applying $p^{\odot n}$ we obtain

$$p^{\odot n} \circ \Delta^{(n)}_{S\Gamma\omega SJ\Phi}(\varphi \cdot \psi) = \sum_{I_1,\cdots,I_n \neq \emptyset; I_1 \sqcup \cdots \sqcup I_n = \{1,\cdots,k+r\}} p(\chi_{I_1}) \odot \cdots \odot p(\chi_{I_n}), \tag{3.8}$$

Let I_k be a subset of $\{1, \cdots, k+r\}$ such that has at least one index $i_0 \in \{1, \cdots, k\}$ and one index $j_0 \in \{k+1, \cdots, k+r\}$, and $h \in \mathcal{C}^{\infty}_0(\mathcal{M})$ such that $h|_{Sup(\varphi)} \equiv 1$ and $h|_{Sup(\psi)} \equiv 0$, this function exists because the spacetime is paracompact. Then,

$$\begin{aligned} N(\chi_{I_k}) &= N(\chi_\alpha \cdots \chi_{i_0} \cdots \chi_{j_0} \cdots \chi_{|I_k|}) \\ &= \chi_\alpha \odot \cdots \odot \chi_{i_0} \odot \cdots \odot \chi_{j_0} \odot \cdots \odot \chi_{|I_k|} \\ &= \chi_\alpha \odot \cdots \odot h\chi_{i_0} \odot \cdots \odot \chi_{j_0} \odot \cdots \odot \chi_{|I_k|} \\ &= \chi_\alpha \odot \cdots \odot \chi_{i_0} \odot \cdots \odot h\chi_{j_0} \odot \cdots \odot \chi_{|I_k|} \\ &= \chi_\alpha \odot \cdots \odot \chi_{i_0} \odot \cdots \odot 0 \odot \cdots \odot \chi_{|I_k|} = 0 \end{aligned}$$

where α is the the first element of I_k.

Then, in equation (3.8) only terms such that all the I_j are totally included in $\{1, \cdots, k\}$ or in $\{k+1, \cdots, k+r\}$ survive. We will call I to the first and J to the seconds.

In the equation (3.8), we have terms with only one I and $n-1$ subsets J, or terms with two I and $n-2$ J's, three I's, four I's ... $n-1$. In the following sums all the I's are contained in $\{1, \cdots k\}$ and the J's in $\{k+1, \cdots k+r\}$, all of them are $\neq \emptyset$ and its disjoint union is $\{1, \cdots, k+r\}$.

$$\begin{aligned} &p^{\odot n} \circ \Delta^{(n)}_{S\Gamma\omega SJ\Phi}(\varphi \cdot \psi) \\ &= \sum_{J_1 \cdots J_{n-1}} n[p(\chi_{I_1}) \odot p(\chi_{J_1}) \odot \cdots \odot p(\chi_{J_{n-1}})] \\ &+ \sum_{I_1, I_2; J_1 \cdots J_{n-3} = \{k+1,\cdots,k+r\}} \binom{n}{2} [p(\chi_{I_1}) \odot p(\chi_{I_2}) \odot p(\chi_{J_1}) \odot \cdots \odot p(\chi_{J_{n-2}})] \\ &+ \sum_{I_1 \cdots I_3; J_1 \cdots J_{n-3}} \binom{n}{3} [p(\chi_{I_1}) \odot \cdots \odot p(\chi_{I_3}) \odot p(\chi_{J_1}) \odot \cdots \odot p(\chi_{J_{n-3}})] + \cdots , \end{aligned}$$

where the binomial coefficients appear because of the possible reorderings of the expression. We will make the computations for $n = 1, 2, 3, 4...$ For the general case one must proceed with a combinatorial argument as in the proof of Proposition 1.1.21. For the case **n=1** the result of the above computation is zero because there are no subset J.

Case **n=2**, $p^2 \circ \Delta_{S\Gamma\omega SJ\Phi}(\varphi \cdot \psi) = 2p(\chi_{I_1}) \odot p(\chi_{J_1}) = 2p(\varphi) \odot p(\psi)$.

Case **n=3**:

$$p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\varphi \cdot \psi) = 3p(\varphi) \odot [\sum_{J_1,J_2} p(\chi_{J_1}) \odot p(\chi_{J_2})] + 3[\sum_{J_1,J_2} p(\chi_{I_1}) \odot p(\chi_{I_2})] \odot p(\psi)$$

$$= 3p(\varphi) \odot p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\psi) + 3p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p(\psi)$$

For **n=4**,

$$p^{\odot 4} \circ \Delta^{(4)}_{S\Gamma\omega SJ\Phi}(\varphi \cdot \psi) = 4p(\varphi) \odot p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\psi)$$
$$+ \binom{4}{2} p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\psi) + 4p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p(\psi)$$

Multiplying the last equations by $\frac{1}{n!}$ and adding them as in equation (3.7), we can see how are the first terms of $P'(\varphi \cdot \psi)$, using easy facts as $\frac{1}{4!}\binom{4}{2} = \frac{1}{2}\frac{1}{2}$. Then we conclude,

$$P'(\varphi \cdot \psi) = p(\varphi) \odot p(\psi) + \frac{1}{2} p(\varphi) \odot p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\psi) + \frac{1}{2} p^{\odot 2} \circ \Delta^{2}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p(\psi)$$
$$+ \frac{1}{3!} p(\varphi) \odot p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\psi) + \frac{1}{2}\frac{1}{2} p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\psi)$$
$$+ \frac{1}{3!} p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p(\psi) + \cdots \tag{3.9}$$

On the other hand if we compute $P'(\varphi) \odot P'(\psi)$, we have

$$\left(\sum_{k \in \mathbb{N}} \frac{1}{k!} p^{\odot k} \circ \Delta^{(k)}_{S\Gamma\omega SJ\Phi}(\varphi) \right) \odot \left(\sum_{k \in \mathbb{N}} \frac{1}{s!} p^{\odot s} \circ \Delta^{(s)}_{S\Gamma\omega SJ\Phi}(\psi) \right)$$
$$= \left(p \circ Id_{S\Gamma\omega SJ\Phi}(\varphi) + \frac{1}{2} p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\varphi) + \frac{1}{3!} p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\varphi) + \cdots \right)$$
$$\odot \left(p \circ Id_{S\Gamma\omega SJ\Phi}(\psi) + \frac{1}{2} p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\psi) + \frac{1}{3!} p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\psi) + \cdots \right)$$

$$= p(\varphi) \odot p(\psi) + \frac{1}{2} p(\varphi) \odot p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\psi) + \frac{1}{2} p^{\odot 2} \circ \Delta^{2}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p(\psi)$$
$$+ \frac{1}{3!} p(\varphi) \odot p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\psi) + \frac{1}{2} p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\varphi) \odot \frac{1}{2} p^{\odot 2} \circ \Delta^{(2)}_{S\Gamma\omega SJ\Phi}(\psi)$$
$$+ \frac{1}{3!} p^{\odot 3} \circ \Delta^{(3)}_{S\Gamma\omega SJ\Phi}(\varphi) \odot p(\psi) + \cdots,$$

which is equal to $P'(\varphi \cdot \psi)$ by equation (3.9). Notice that we used the fact that $\Delta_{S\Gamma\omega SJ\Phi} = Id_{S\Gamma SJ\Phi}$ (see the definition of $\Delta^{(n)}$ in Section 1.1.2). $\square$

As an observation we can say that renormalizations do not necessarily preserve the product of elements of $S\Gamma_c\omega SJ\Phi$ if they have not disjoint support.

3.3 Renormalizations $\curvearrowright$ Feynman measures

Thanks to the Lemma 3.2.12, given a renormalization P we can think it as an automorphism of $S\Gamma\omega SJ\Phi$ which preserves the coaction of $\Gamma SJ\Phi$. A Feynman measure (see Definition 2.3.2) is a linear map from $S\Gamma_c\omega SJ\Phi$ to $\mathbb{C}$; then we can define the action from the renormalization group over the Feynman measures as follows. If P is a renormalization and δ is a Feynman measure, we can lift P by using Lemma 3.2.12 that is still called P. Then a new Feynman measure $P(\delta)$ is define by $P(\delta)(A) := \delta(P^{-1}(A))$ if $A \in S\Gamma_c\omega SJ\Phi$, .

It will be interesting to prove that $P(\delta)$ is still a Feynman measure associated with the propagator Δ if δ is associated with Δ too. First of all we will prove the smoothness on the diagonal.

Suppose δ is a Feynman measure and let P the lift of some renormalization, and take $A \in S^n\Gamma_c\omega SJ\Phi$. P^{-1} is an element of the renormalization group that preserves the filtration, so $P^{-1}(A)$ is a polynomial in monomials of degree at least n , say $B_1 + \cdots + B_n$. Given $(p_1, \cdots, p_k) \in \sum_{(q,\cdots,q)} \chi_k(\delta, B_k)$ one can affirm that $p_1 + \cdots + p_k = 0$ because δ is smooth in the diagonal, as a conclusion $P\delta$ **is smooth in the diagonal**.

To see that $P\delta$ is non-degenerate we can use Theorem 3.1.2 to establish that if P is a renormalization $P|_{\Gamma_c\omega}$ is an isomorphism. But the isomorphisms of sheafs (induced by vector bundles) are in correspondence whith the isomorphisms of vector bundles supported by the identity over $\mathcal{M}$. Given two vector bundles, an isomorphism between them must be linear on each fiber, but in our case the bundle has range 1, because it is the density bundle. Hence the isomorphism is given (on the fibers) by multiplication by a constant $\lambda \neq 0$, so the bundle morphism is given globally by the multiplication by a nowhere-vanishing function $\lambda \in \mathcal{C}^\infty(\mathcal{M})$.

Applying this to P^{-1}, given $v \in \Gamma_c\omega$, we obtain,

$$P\delta(v) = \delta(P^{-1}(v)) = \int_{\mathcal{M}} gP^{-1}(v),$$

where g is a nowhere-vanishing function that exists because δ is non-degenerate. Moreover,

$$P\delta(v) = \int_{\mathcal{M}} (g\lambda)(v),$$

where λ is the nowhere-vanishing function described above. Then $P\delta$ **is non-degenerate**.

Let δ be a Feynman measure, Q an element of the renormalization group (whose inverse is P) and let $A \in S^m\Gamma_c\omega SJ\Phi$ and $B \in S^n\Gamma_c\omega SJ\Phi$ such that there is no coordinate of any element in $Sup(A)$ which is $\leqslant$ to some coordinate of any element in $Sup(B)$, this particularly implies that $Sup(A) \cap Sup(B) = \emptyset$, and then by Lemma 3.2.12, P preserves the product. Then,

$$Q\delta(A \cdot B) = \delta(P(A \cdot B)) = \delta(P(A) \cdot P(B))$$

and as Δ is cut the products involved in the Gaussian condition are well defined (see Observation 2.3.5), so

$$Q\delta(A \cdot B) = \sum(\chi(\delta, (P(A))') \otimes \chi(\delta, (P(B))')) \cdot \hat{\Delta}((P(A))'', (P(B))''). \tag{3.10}$$

We will now compute $(P(A))'$, $(P(A))''$, $(P(B))''$ and $(P(B))'$. Consider

$$\begin{array}{ccc} S\Gamma\omega SJ\Phi & \xrightarrow{\sigma} & S\Gamma\omega SJ\Phi \otimes_{\mathcal{C}^\infty(\mathcal{M})} S\Gamma SJ\Phi \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P\otimes Id} \\ S\Gamma\omega SJ\Phi & \xrightarrow{\sigma} & S\Gamma\omega SJ\Phi \otimes_{\mathcal{C}^\infty(\mathcal{M})} S\Gamma SJ\Phi \end{array}$$

which is commutative.

Applying this maps to the element A, and remembering that $\sigma(A) = \sum A' \otimes A''$ we have

$$A'' = (P(A))''$$

and the same for B. Substituting the results in equation (3.10), we obtain

$$Q\delta(A \cdot B) = \sum(\chi(\delta, (P(A))') \otimes \chi(\delta, (P(B))')) \cdot \hat{\Delta}(A'', B'').$$

The commutative diagram tells us that $(P(A))' = P(A')$, using this we have

$$\begin{aligned} Q\delta(A \cdot B) &= \sum(\chi(\delta, P(A')) \otimes \chi(\delta, P(B'))) \cdot \hat{\Delta}(A'', B'') \\ &= \sum(\chi(Q\delta, A') \otimes \chi(Q\delta, B')) \cdot \hat{\Delta}(A'', B''), \end{aligned}$$

which is exactly the Gaussian condition for $Q\delta$. As a conclusion we can say that if P is a renormalization and δ a Feynman measure then $P\delta$ is a Feynman measure too. So we have an action of the renormalization group over the set of Feynman measures associated to a fixed cut propagator.

We will need the following result.

Lemma 3.3.1. *Given $k \in \mathbb{N}$, if $\lambda : S^k\Gamma_c\omega SJ\Phi \to \mathbb{C}$ is a continuous function satisfying $Sup(\lambda) \subseteq Diag(\mathcal{M}^k)$ then λ is $\mathcal{C}^\infty(\mathcal{M})$-balanced.*

Proof. Let f be a smooth function over $\mathcal{M}$. To prove that $\lambda(A_1 \cdots fA_{i_0} \cdots A_{j_0} \cdots A_k) = \lambda(A_1 \cdots A_{i_0} \cdots fA_{j_0} \cdots A_k)$ (A_i's are Lagrangian densities of compact support) as complex numbers, we use Remark 2.3.6 and reduce it to prove the equality in $D_d(\mathcal{M}^k)$.

Let $g_1 \otimes \cdots \otimes g_k$ be an element of $\mathcal{C}^\infty(\mathcal{M})^{\otimes k}$ (recall these element are dense in $\mathcal{C}^\infty(\mathcal{M}^k)$, and λ is continuous) whose support is included in $Diag(\mathcal{M}^k)$, then we must compare $\chi_k(\lambda, A_1 \cdots fA_{i_0} \cdots A_{j_0} \cdots A_k)(g_1 \otimes \cdots \otimes g_k)$ and $\chi_k(\lambda, A_1 \cdots A_{i_0} \cdots fA_{j_0} \cdots A_k)(g_1 \otimes \cdots \otimes g_k)$, so if they are equal then $\lambda(A_1 \cdots fA_{i_0} \cdots A_{j_0} \cdots A_k)$ and $\lambda(A_1 \cdots A_{i_0} \cdots fA_{j_0} \cdots A_k)$ will also be equal. But,

$$\chi_k(\lambda, A_1 \cdots fA_{i_0} \cdots A_{j_0} \cdots A_k)(g_1 \otimes \cdots \otimes g_k)$$

$$= \lambda(g_1A_1 \cdots g_{i_0}fA_{i_0} \cdots g_{j_0}A_{j_0} \cdots g_kA_k)$$

$$= \chi_k(\lambda, A_1 \cdots A_{i_0} \cdots A_{j_0} \cdots A_k)(g_1 \otimes \cdots \otimes fg_{i_0} \otimes \cdots \otimes g_k) \tag{3.11}$$

using that $g_1 \otimes \cdots \otimes fg_{i_0} \otimes \cdots \otimes g_k = g_1 \otimes \cdots \otimes fg_{j_0} \otimes \cdots \otimes g_k$ (that is because evaluating the left hand side in a point $(x_1, \cdots, x_k) \in \mathcal{M}^k$ we obtain $g_1(x_1) \cdots f(x_{i_0})g_{i_0}(x_{i_0}) \cdots g_k(x_k)$, which is zero if there are two points $x_i \neq x_j$. But in the relevant case $(x_1, \cdots, x_k) \in$ Diag$(\mathcal{M}^k)$, that is $(x, \cdots, x)$ we obtain $g_1(x) \cdots f(x)g_{i_0}(x) \cdots g_k(x) = g_1(x) \cdots g_{i_0}(x) \cdots f(x)g_{j_0}(x) \cdots g_k(x)$ where we can move $f(x)$ because it is a product on $\mathbb{C}$) we conclude that the last term in Equation (3.11) is equal to

$$\chi_k(\lambda, A_1 \cdots A_{i_0} \cdots A_{j_0} \cdots A_k)(g_1 \otimes \cdots \otimes fg_{j_0} \otimes \cdots \otimes g_k)$$

$$\chi_k(\lambda, A_1 \cdots fA_{j_0} \cdots A_{j_0} \cdots A_k)(g_1 \otimes \cdots \otimes g_k).$$

Then from Remark 2.3.6 we conclude that $\chi_k(\lambda, A_1 \cdots fA_{i_0} \cdots A_{j_0} \cdots A_k) = \chi_k(\lambda, A_1 \cdots A_{i_0} \cdots fA_{j_0} \cdots A_k)$ as element of $D_d(\mathcal{M}^k)$ for all $k \in \mathbb{N}$, which implies that $\lambda(A_1 \cdots fA_{i_0} \cdots A_{j_0} \cdots A_k) = \lambda(A_1 \cdots A_{i_0} \cdots fA_{j_0} \cdots A_k)$ as complex numbers. □

To prove the next theorem we will use a well-known result,

Lemma 3.3.2 (See [24]). *If $E \to \mathcal{M}$ is a finite rank vector bundle over a manifold, then $\Gamma_c(E)$ is a projective $\mathcal{C}^\infty(\mathcal{M})$-module.*

Consider the vector bundle $\omega S^k J\Phi$, which will be of finite rank if and only we take finite order jets $J\Phi$. Then from now on we must understand $J\Phi$ as the vector bundle of finite jets up to some fixed order, i.e. $J\Phi = J^k\Phi$ for some $k \in \mathbb{N}$.

Another result we will use is,

Lemma 3.3.3. *If A and B are projective modules over a commutative ring R, then the symmetric tensor product $(A \otimes_R B)/I$ is also a projective R-module (I is the ideal of the vectors which give symmetry).*

Proof. Can be found in many books of elementary algebra as [5], and it is deduced from the fact that the tensor product commutes with the direct sum. □

With all these results we are ready to establish and prove the following theorem which gives us an important property of this action.

Theorem 3.3.4. *The group of renormalizations acts transitively on the set of Feynman measures associated with a given cut local propagator.*

Proof. We already proved that the action is well defined in the sense that the result is also a Feynman measure associated to the same given cut local propagator.

To finish the proof we must prove the transitivity. Given δ and δ' Feynman measures associated with the same cut local propagator, we want to prove the existence of a $g \in G$ ($g : S\Gamma\omega SJ\Phi \to S\Gamma\omega SJ\Phi$) such that $g\delta = \delta'$, expressing g as $\cdots g_3 \cdot g_2 \cdot g_1$ (see Corollary 3.2.11). We proceed by induction, on the degree of the elements of $S\Gamma\omega SJ\Phi$ i.e. $S^n\Gamma\omega S^kJ\Phi$ for $(n, k) \in \mathbb{N} \times \mathbb{N}_0$ in the lexicographic order.

We begin proving the existence of $g_1 : S\Gamma\omega SJ\Phi \to S\Gamma\omega SJ\Phi$ whose sequential representation is $\{g_1\}_{n\in\mathbb{N}} = \{f_1, 0, \cdots\}$ where $f_1 : \Gamma\omega SJ\Phi \to \Gamma\omega SJ\Phi \simeq \Gamma_c\omega \otimes S_{\mathcal{C}^\infty(\mathcal{M})}\Gamma J\Phi$ (where we use Theorems 2.1.12 and 2.1.13). Hence by Proposition 1.1.24, there exits a map $h_1 : \Gamma\omega SJ\Phi \to \Gamma_c\omega$ such that $f_1 = (h_1 \otimes Id_{\Gamma SJ\Phi}) \circ (Id_{\Gamma_c\omega} \otimes \Delta_{\Gamma SJ\Phi})$.

Taking $n = 1$ and $k = 0$ in $S^n\Gamma\omega S^kJ\Phi$, we want to define f_1 over the elements $\mathcal{L} = \alpha \otimes l$ where $\alpha \in \Gamma_c\omega$ and $l = 1 \in S^0\Gamma J\Phi$. We do that by means of h_1,

$$
\begin{aligned}
&g_1(\mathcal{L}) = f_1(\mathcal{L}) \\
&= f_1(\alpha \otimes l) = (h_1 \otimes Id_{\Gamma SJ\Phi}) \circ (Id_{\Gamma_c\omega} \otimes \Delta_{\Gamma SJ\Phi})(\alpha \otimes 1) \\
&= (h_1 \otimes Id_{\Gamma\omega SJ\Phi}) \circ (\alpha \otimes (1 \otimes 1)) \\
&= (h_1 \otimes Id_{\Gamma\omega SJ\Phi}) \circ ((\alpha \otimes 1) \otimes 1) \\
&= h_1(\alpha \otimes 1) \otimes Id_{\Gamma SJ\Phi}(1) = h_1(\mathcal{L}) \otimes 1 = h_1(\mathcal{L}).
\end{aligned}
$$

Applying δ' we have

$$\delta'(g_1(\mathcal{L})) = \delta'(f_1(\mathcal{L})) = \delta'(h_1(\mathcal{L})).$$

We want h_1 to satisfy $\delta' \circ h_1 = \delta$ over the elements with degree $n = 1$ and $k = 0$. In the case where $\Gamma\omega S^0J\Phi$ is a free $\mathcal{C}^\infty(\mathcal{M})$-module so it is easy to define $h_1|_{\Gamma\omega S^0J\Phi}$ over a basis such that $h_1|_{\Gamma\omega S^0J\Phi}$ satisfies $\delta' \circ h_1|_{\Gamma\omega S^0J\Phi} = \delta$ because δ'

is epimorphism.
In the general case were $\Gamma\omega S^0 J\Phi$ (is not necessarily a free $\mathcal{C}^\infty(\mathcal{M})$-module) is a projective $\mathcal{C}^\infty(\mathcal{M})$-module, then it is a direct summand of a free $\mathcal{C}^\infty(\mathcal{M})$-module (call it F), so we apply the previous argument to define $\tilde{h}_1$ over F and then by restricting $\tilde{h}_1$ to the direct summand we are interested to have $h_1|_{\Gamma\omega S^0 J\Phi}$ satisfying $\delta' \circ h_1|_{\Gamma\omega S^0 J\Phi} = \delta$. Hence the diagram,

$$\begin{array}{ccc} & \overset{h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^0 J\Phi}}{\dashrightarrow} & \Gamma_c\omega \\ & & \downarrow \delta' \\ S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^0 J\Phi & \xrightarrow[\delta]{} & \mathbb{C} \end{array} \tag{3.12}$$

can be completed with a map $h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^0 J\Phi}$ or which is the same $h_1|_{\Gamma\omega S^0 J\Phi}$.

But $g_1\delta(\mathcal{L}) = \delta'(\mathcal{L})$ if and only if $\delta(\mathcal{L}) = \delta'(g_1(\mathcal{L}))$, and this is exactly what we have with the aforementioned election of $h_1|_{\omega S^0 J\Phi}$.

Continuing this process by defining h_1 over elements of $S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^1 J\Phi$ (i.e. $n = 1$ and $k = 1$), we consider the Lagrangian density $\mathcal{L} = \alpha \otimes l$ where $\alpha \in \Gamma_c\omega$ and $l = \varphi \in S^1\Gamma J\Phi$ and compute

$$\begin{aligned} & g_1(\mathcal{L}) = f_1(\mathcal{L}) \\ & = f_1(\alpha \otimes l) = (h_1 \otimes Id_{\Gamma SJ\Phi}) \circ (Id_{\Gamma_c\omega} \otimes \Delta_{\Gamma SJ\Phi})(\alpha \otimes \varphi) \\ & = (h_1 \otimes Id_{\Gamma SJ\Phi}) \circ (\alpha \otimes [1 \otimes \varphi + \varphi \otimes 1]) \\ & = (h_1 \otimes Id_{\Gamma SJ\Phi})((\alpha \otimes 1) \otimes \varphi + (\alpha \otimes \varphi) \otimes 1) \\ & = h_1(\alpha \otimes 1) \otimes Id_{\Gamma SJ\Phi}(\varphi) + h_1(\alpha \otimes \varphi) \otimes Id_{\Gamma\omega SJ\Phi}(1) \\ & = h_1(\alpha \otimes 1) \otimes \varphi + h_1(\mathcal{L}). \end{aligned}$$

Applying δ' to both sides of this equation we have,

$$\delta'(g_1(\mathcal{L}) = \delta'(h_1(\alpha \otimes 1) \otimes \varphi) + \delta' \circ h_1(\mathcal{L}).$$

But as we want $\delta' \circ g_1 = \delta$ then we are looking for a $h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^1 J\Phi}$ such that the equation

$$\begin{aligned} & \delta'(h_1(\alpha) \otimes \varphi) + \delta' \circ h_1(\mathcal{L}) = \delta(\mathcal{L}), \\ & \text{or } \delta' \circ h_1(\mathcal{L}) = (\delta - \delta' \circ (h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^0 J\Phi} \otimes Id_{\Gamma SJ\Phi}))(\mathcal{L}) \end{aligned}$$

holds. Take a $h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^1 J\Phi}$ such that the next diagram

$$\begin{array}{ccc} & \overset{h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^0 J\Phi}}{\dashrightarrow} & \Gamma_c\omega \\ & & \downarrow \delta' \\ S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^1 J\Phi & \xrightarrow[\delta - \delta' \circ (h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^1 J\Phi} \otimes Id_{\Gamma SJ\Phi})]{} & \mathbb{C} \end{array} \tag{3.13}$$

commutes. The existence of this $h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^1 J\Phi}$ follows from the same arguments given for $h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^0 J\Phi}$.

Continuing this process we can recursively define $h_1|_{S^1_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega S^k J\Phi}$ for all $k \in \mathbb{N}_0$ and so we have just defined $h_1 : S^1\Gamma\omega SJ\Phi \to \Gamma_c\omega$. Proceeding by induction we can define $h_n : S^n\Gamma\omega SJ\Phi \to \Gamma_c\omega$ for each $n \in \mathbb{N}$ (using the Lemma 3.3.2 and Lemma 3.3.3 to affirm that $\Gamma\omega SJ\Phi$ is a projective module and so is $S\Gamma\omega SJ\Phi$. And consequently a is direct addend of a free $\mathcal{C}^\infty(\mathcal{M})$-module), each one of is in correspondence with a $g_n : S\Gamma\omega SJ\Phi \to S\Gamma\omega SJ\Phi$ whose sequential representation is $\{0, \cdots, 0, f_n, 0, \cdots\}$.

The maps g_n define by Corollary 3.2.11 a renormalization $g = ...g_3 g_2 g_1$, such that $\delta(\mathcal{L}) = \delta'(g(\mathcal{L}))$ for all $\mathcal{L} \in S_{\mathcal{C}^\infty(\mathcal{M})}\Gamma\omega SJ\Phi$. That is $g(\delta) = \delta'$. □

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