UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
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2-filtered bicolimits and finite weighted bilimits commute in $\mathcal{C}at$

Nicolás Abel Canevali

Director: Eduardo J. Dubuc

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… cercare e saper riconoscere
chi e cosa,
in mezzo all'inferno,
non è inferno,
e farlo durare,
e dargli spazio.

Italo Calvino, *Le città invisibili*

2-FILTERED BICOLIMITS AND FINITE WEIGHTED BILIMITS COMMUTE IN $\mathcal{C}at$

NICOLÁS ABEL CANEVALI

Introducción

Los límites y los colímites son importantes construcciones categóricas universales que se remontan al origen mismo de la teoría de categorías. Por ejemplo, el producto cartesiano de dos conjuntos S y T es el producto categórico en la categoría $\mathcal{E}ns$ de conjuntos (pequeños): un conjunto $S \times T$ con dos funciones proyección sobre los factores $S \times T \xrightarrow{\pi_1} S$ y $S \times T \xrightarrow{\pi_2} T$, universal entre todas las ternas $(Z, \lambda_1, \lambda_2)$ con Z un conjunto y funciones $Z \xrightarrow{\lambda_1} S$ y $Z \xrightarrow{\lambda_2} T$. En otras categorías esta construcción universal da lugar a distintas nociones: en un poset la misma descripción define el ínfimo de dos elementos; en una categoría de módulos sobre un anillo base se corresponde con la suma directa. Un tipo de colímite es el coegalizador: el coegalizador de dos flechas $A \underset{g}{\overset{f}{\rightrightarrows}} B$ es un objeto C con una flecha $B \xrightarrow{h} C$ con $hf = hg$, universal entre todos los pares (Z, k) con $B \xrightarrow{k} Z$ tales que $kf = kg$.

La noción de límite o colímite en una categoría $\mathcal{C}$ se aplica a cualquier diagrama, donde por diagrama entendemos un funtor $\mathcal{B} \xrightarrow{F} \mathcal{C}$, con $\mathcal{B}$ la categoría de indexación. Los límites y colímites no necesariamente existen en general, pero cuando existen y tenemos un funtor en dos variables $\mathcal{A} \times \mathcal{P} \xrightarrow{F} \mathcal{C}$, podemos tomar límite o colímite en cualquiera de las dos variables para obtener un funtor en la otra variable. Por ejemplo, $\varprojlim_{\mathcal{A}} F$ da un funtor $P \mapsto \varprojlim_{\mathcal{A}} F(A, P)$, es decir, que a P le asigna el límite del funtor $F(-, P)$. Luego podemos tomar límite o colímite en la variable restante. En total, hay cuatro combinaciones posibles de límite o colímite en una u otra variable, y del orden en que se toman los límites o colímites. El hecho de que los límites estén dados por una propiedad de tipo *terminal* (por ejemplo, los morfismos canónicos dados por la propiedad universal del producto *llegan* al objeto universal: $Z \overset{\exists!}{\dashrightarrow} S \times T$) hace que tomar límite en una variable primero y luego en la otra dé lo mismo que hacerlo al revés, porque se definen por propiedades universales del mismo lado. Es decir,

$$\varprojlim_{\mathcal{A}} \varprojlim_{\mathcal{P}} F \cong \varprojlim_{\mathcal{P}} \varprojlim_{\mathcal{A}} F \tag{0.1}$$

y de hecho, esto coincide con tomar el límite de F como un funtor (en una variable) desde la categoría producto $\varprojlim_{\mathcal{A}\times\mathcal{P}} F$. Más precisamente, la ecuación 0.1 significa que se tienen morfismos canónicos en ambas direcciones, construidos mediante las propiedades universales y los conos universales de los límites involucrados, que determinan un isomorfismo. En breve, podemos decir que los límites conmutan con los límites.

Los colímites están dados por una propiedad universal de tipo *inicial*. Por ejemplo, en el caso del coproducto, los morfismos canónicos *salen* del objeto universal: $S \coprod T \overset{\exists!}{\dashrightarrow} Z$. Como es de esperar, al estar dados por propiedades universales del mismo lado, también tenemos que los colímites conmutan con los colímites

$$\underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{A}}\, \underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{P}}\, F \cong \underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{P}}\, \underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{A}}\, F$$

(que es lo mismo que $\underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{A}\times\mathcal{P}} F$).

Esto deja de valer en general cuando se combina un límite con un colímite: tenemos un morfismo canónico que llamamos "de comparación" sólo en una dirección

$$\underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{A}}\, \varprojlim_{\mathcal{P}} F \xrightarrow{\Diamond} \varprojlim_{\mathcal{P}}\, \underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{A}}\, F$$

que no podemos asegurar que sea un isomorfismo. Luego, resulta natural preguntarse por condiciones sobre $\mathcal{A}$, $\mathcal{P}$, $\mathcal{C}$ y F que permitan concluir que $\Diamond$ sea un isomorfismo.

Un contexto donde esto sucede es bajo las hipótesis de que $\mathcal{A}$ sea filtrante, $\mathcal{P}$ sea finita, y $\mathcal{C}$ sea la categoría de los conjuntos. Decimos entonces que los colímites filtrantes y los límites finitos conmutan en la categoría de los conjuntos:

$$\underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{A}}\, \varprojlim_{\mathcal{P}} F \cong \varprojlim_{\mathcal{P}}\, \underset{\longrightarrow}{\mathrm{colim}}_{\mathcal{A}}\, F \tag{0.2}$$

Esta es una propiedad esencial de la categoría de los conjuntos, y caracteriza a los colímites filtrantes en $\mathcal{E}ns$: una categoría $\mathcal{A}$ es filtrante precisamente si los colímites indexados por $\mathcal{A}$ conmutan con todos los límites finitos en $\mathcal{E}ns$. La ecuación 0.2 también vale en otras categorías, como las categorías de haces sobre un sitio pequeño, y tiene muchas aplicaciones.

Por ejemplo, podemos definir la noción de presentación finita en una categoría arbitraria $\mathcal{C}$: un objeto $H \in \mathcal{C}$ es de presentación finita si el funtor $\mathcal{C} \xrightarrow{[H,-]} \mathcal{E}ns$ representado por H preserva colímites filtrantes. Puede verificarse que esta noción tiene sentido en tanto caracteriza los conjuntos finitos si $\mathcal{C} = \mathcal{E}ns$ y coincide con la definición usual de anillo de presentación finita (cociente de un anillo de polinomios en finitas variables por un ideal finitamente generado) si $\mathcal{C}$ es la categoría de anillos. Un requisito importante es que la presentación finita sea preservada por colímites finitos. La demostración de este hecho sigue de la siguiente cadena de isomorfismos (donde asumimos que tenemos un diagrama finito $(H_i)_i$ con objetos de presentación finita, y un diagrama filtrante $(Y_\alpha)_\alpha$):

$$\begin{aligned}\mathcal{E}ns(\varinjlim_i H_i, \varinjlim_\alpha Y_\alpha) &\cong \varprojlim_i \mathcal{E}ns(H_i, \varinjlim_\alpha Y_\alpha)\\ &\cong \varprojlim_i \varinjlim_\alpha \mathcal{E}ns(H_i, Y_\alpha)\\ &\cong \varinjlim_\alpha \varprojlim_i \mathcal{E}ns(H_i, Y_\alpha)\\ &\cong \varinjlim_\alpha \mathcal{E}ns(\varinjlim_i H_i, Y_\alpha)\end{aligned}$$

Vemos que el hecho de que los colímites filtrantes conmutan con los límites finitos en $\mathcal{E}ns$ es esencial en esta demostración.

Para un segundo ejemplo, si tenemos un funtor

$$\mathcal{C} \xrightarrow{p} \mathcal{E}ns$$

se tiene una noción clara de lo que significa que sea exacto, es decir, que preserva límites finitos. Si la categoria $\mathcal{C}$ no admite todos los límites finitos, esta condición tiene sentido pero es algo vacua. La definición correcta es la de funtor playo: el funtor $\mathcal{C} \xrightarrow{p} \mathcal{E}ns$ es playo si su diagrama Γ_p, también llamado categoría de elementos, es una categoría filtrante (un objeto de esta categoría es un par (x, C) con $x \in pC$ y una flecha $(x, C) \to (y, D)$ está dada por un morfismo $C \xrightarrow{f} D$ en $\mathcal{C}$ tal que $(pf)(x) = y$).

Un hecho fundamental es que en el diagrama

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "h"] \arrow[dr, "p"'] & \arrow[d, phantom, "\cong"] & \mathcal{E}ns^{\mathcal{C}^{op}} \arrow[dl, "p^*"] \\ & \mathcal{E}ns & \end{tikzcd}$$

el funtor p^* es exacto precisamente cuando Γ_p es filtrante, es decir, cuando p es playo. Podemos demostrar una de estas implicaciones utilizando la conmutación de colímites filtrantes y límites finitos en $\mathcal{E}ns$: puede verse que el funtor p^* es de hecho

$$p^* = \varinjlim_{(x,C)\in\Gamma_p} ev_C(-)$$

(donde ev_C es el funtor $\mathcal{E}ns^{\mathcal{C}^{op}} \to \mathcal{E}ns$ de evaluación en un objeto $C \in \mathcal{C}$). Si Γ_p es filtrante, entonces p^* es un colímite filtrante de funtores de evaluación, que preservan todos los límites porque los límites se calculan punto a punto en las categorías de funtores. Luego p^* es un colímite filtrante de funtores exactos, y es exacto por un argumento análogo al ejemplo anterior, mediante el hecho de que los colímites filtrantes conmutan con los límites finitos en la categoría de los conjuntos.

Puede demostrarse que si $\mathcal{C}$ tiene límites finitos, entonces p es exacto si y sólo si es playo.

La demostración de la conmutación de colímites filtrantes y límites finitos en $\mathcal{E}ns$ se hace mediante la construcción del colímite filtrante, que generaliza la construcción del anillo de gérmenes de funciones continuas a valores reales en un punto de un espacio topológico.

Estamos interesados en formular y verificar la ecuación 0.2, para funtores a valores en los conjuntos, en una versión 2-categórica, para 2-funtores a valores en las categorías. Este resultado es importante en el desarrollo de la teoría de 2-funtores 2-playos y en la teoría de 2-pro-objetos y sus 2-categorías de modelos de Quillen (ver [8] y [6]). Para esto debemos realizar varias adaptaciones. Naturalmente, las categorías $\mathcal{A}$ y $\mathcal{P}$ pasan a ser 2-categorías y el funtor F pasa a ser un 2-funtor. Los límites que deben considerarse en la 2-categoría $\mathcal{C}at$ son no sólo los cónicos sino los pesados por un segundo 2-funtor $\mathcal{P} \xrightarrow{W} \mathcal{C}at$. Así, por ejemplo, podemos fijar $A \in \mathcal{A}$ y considerar el límite de $F(A,-)$ pesado por W, que denotamos $\mathrm{w}\underleftarrow{\lim}^{W} F(A,-)$. De hecho, en una 2-categoría, varias nociones distintas de propiedad universal tienen sentido y en el caso que nos concierne utilizamos la de *bilímite*. La igualdad en la fórmula 0.2, que entendemos como expresando que cierta función canónica es biyectiva, va a significar que un determinado funtor canónico es una equivalencia de categorías. De esta forma, buscamos demostrar la propiedad fundamental de conmutación

$$\mathrm{bico}\underrightarrow{\lim}_{\mathcal{A}}\ \mathrm{wb}\underleftarrow{\mathrm{ilim}}^{W}_{\mathcal{P}}\ F \simeq \mathrm{wb}\underleftarrow{\mathrm{ilim}}^{W}_{\mathcal{P}}\ \mathrm{bico}\underrightarrow{\lim}_{\mathcal{A}}\ F$$

cuando $\mathcal{A} \times \mathcal{P} \xrightarrow{F} \mathcal{C}at$ es un 2-funtor, $\mathcal{A}$ es una 2-categoría 2-filtrante y el límite pesado por W es finito, en algún sentido relevante para 2-categorías: va a ser necesario imponer condiciones de finitud no sólo sobre la 2-categoría $\mathcal{P}$, sino también sobre el peso W.

Este es un importante resultado que es inédito, y el objetivo de obtener una demostración fue el punto de partida de esta tesis.

Asumimos familiaridad con los conceptos y herramientas de la teoría básica de categorías para una lectura de este trabajo.

En la sección 2 introducimos las definiciones de 2-categorías, 2-funtores, 2-transformaciones naturales y modificaciones que aparecen en la teoría de 2-categorías, junto con ejemplos y variantes. Luego damos nociones de filtrabilidad y finitud aplicables al contexto 2-categórico en la sección 3 y varias definiciones utilizadas de 2-límites en la sección 4. En la sección 5 trabajamos algunos ejemplos de 2-límites finitos y pseudocolímites 2-filtrantes con sus construcciones en la 2-categoría de categorías, y en la sección 6 construimos 2-límites y bilímites a partir de límites más simples, siguiendo [15], [28] y [10]. La sección 7 contiene las contribuciones originales de este trabajo: demostramos la ecuación 0.2, en su versión 1-categórica, y seguidamente la conmutación de bicolímites 2-filtrantes con biproductos finitos, biegalizadores y bicotensores con una categoría finita, por separado, para concluir luego el resultado principal. Finalmente, en la sección 8, que es independiente con respecto a nuestro resultado principal, damos definiciones 2-categóricas de ends y coends, conceptos que surgieron durante el transcurso del estudio de la teoría relevante para este trabajo, y las aplicamos para la demostración de algunas propiedades básicas de los límites pesados.

Contents

1. Introduction

Limits and colimits are important categorical constructions that go back to the very origin of category theory. For example, the cartesian product of two sets S and T is the categorical product in the category $\mathcal{E}ns$ of (small) sets: a set $S \times T$ together with projection functions into the factors $S \times T \xrightarrow{\pi_1} S$ and $S \times T \xrightarrow{\pi_2} T$ universal among all triples $(Z, \lambda_1, \lambda_2)$ with Z a set and functions $Z \xrightarrow{\lambda_1} S$ and $Z \xrightarrow{\lambda_2} T$. In other categories, this universal construction gives rise to different notions: in a poset the same description defines the infimum of two elements; in a category of modules over a base ring it corresponds to the direct sum. A type of colimit is the coequalizer: the coequalizer of two arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is an object C with an arrow $B \xrightarrow{h} C$ satisfying $hf = hg$, universal among all pairs (Z, k) with $B \xrightarrow{k} Z$ such that $kf = kg$.

The notion of limit or colimit in a category $\mathcal{C}$ applies to any diagram, where by diagram we understand a functor $\mathcal{B} \xrightarrow{F} \mathcal{C}$, with $\mathcal{B}$ the indexing category. Limits and colimits don't necessarily exist in general, but when they exist and we have a functor in two variables $\mathcal{A} \times \mathcal{P} \xrightarrow{F} \mathcal{C}$, we can take the limit or the colimit in any of the two variables to obtain a functor in the other variable. For example, $\varprojlim_{\mathcal{A}} F$ is a functor $P \mapsto \varprojlim_{\mathcal{A}} F(A, P)$ sending P to the limit of the functor $F(-, P)$. Then we can take the limit or the colimit in the remaining variable. In total, there are four possible combinations of limit and colimit in one or the other variable, and in the order in which the limits or colimits are taken. The fact that limits are given by a universal property of *terminal* type (for example, the canonical morphisms given by the universal property of the product go *into* the universal object: $Z \overset{\exists!}{\dashrightarrow} S \times T$) results in that taking limit first in one variable and then in the other gives the same result as the other way around, because they are defined by universal properties on the same side. That is

$$\varprojlim_{\mathcal{A}} \varprojlim_{\mathcal{P}} F \cong \varprojlim_{\mathcal{P}} \varprojlim_{\mathcal{A}} F \tag{1.1}$$

and in fact, this coincides with taking the limit of F as a functor (in one variable) from the product category $\varprojlim_{\mathcal{A} \times \mathcal{P}} F$. More precisely, the equation 1.1 means that we have canonical morphisms in both directions, constructed from the universal properties and the universal cones of the limits involved, that determine an isomorphism. In a nutshell, we can say that limits commute with limits.

Colimits are given by a universal property of *initial* type. For example, in the case of the coproduct, canonical morphisms go *out* of the universal object: $S \coprod T \overset{\exists!}{\dashrightarrow} Z$. As expected, since they are given by universal properties on the same side, we also have that colimits commute with colimits

$$\varinjlim_{\mathcal{A}} \varinjlim_{\mathcal{P}} F \cong \varinjlim_{\mathcal{P}} \varinjlim_{\mathcal{A}} F$$

(which is the same as $\varinjlim_{\mathcal{A}\times\mathcal{P}} F$).

This is no longer true when we combine a limit with a colimit: we have a canonical morphism that we call a "comparison" morphism only in one direction

$$\varinjlim_{\mathcal{A}} \varprojlim_{\mathcal{P}} F \xrightarrow{\Diamond} \varprojlim_{\mathcal{P}} \varinjlim_{\mathcal{A}} F$$

which we can't guarantee to be an isomorphism. It is then natural to ask for conditions on $\mathcal{A}$, $\mathcal{P}$, $\mathcal{C}$ and F that allow us to conclude that $\Diamond$ is an isomorphism.

A context where this happens is under the hypotheses of $\mathcal{A}$ being filtered, $\mathcal{P}$ finite and $\mathcal{C}$ the category of sets. We then say that filtered colimits and finite limits commute in the category of sets:

$$\varinjlim_{\mathcal{A}} \varprojlim_{\mathcal{P}} F \cong \varprojlim_{\mathcal{P}} \varinjlim_{\mathcal{A}} F \tag{1.2}$$

This is an essential property of the category of sets, and characterizes filtered colimits in $\mathcal{E}ns$: a category $\mathcal{A}$ is filtered precisely when colimits indexed by $\mathcal{A}$ commute with all finite limits in $\mathcal{E}ns$. The equation 1.2 also holds in other categories, like the categories of sheaves on a small site, and has many applications.

For example, we can define the notion of finite presentation in an arbitrary category $\mathcal{C}$: an object $H \in \mathcal{C}$ is of finite presentation if the functor $\mathcal{C} \xrightarrow{[H,-]} \mathcal{E}ns$ represented by H preserves filtered colimits. It can be checked that this notion makes sense inasmuch as it characterizes finite sets if $\mathcal{C} = \mathcal{E}ns$ and coincides with the usual definition of finitely presented ring (quotient of a polynomial ring in a finite number of variables by a finitely generated ideal) if $\mathcal{C}$ is the category of rings. An important requirement is that finite presentation be preserved by finite colimits. The proof of this fact follows from the following chain of isomorphisms (where we assume that we have a finite diagram $(H_i)_i$ with objects of finite presentation, and a filtered diagram $(Y_\alpha)_\alpha$):

$$\begin{aligned}
\mathcal{E}ns(\varinjlim_i H_i, \varinjlim_\alpha Y_\alpha) &\cong \varprojlim_i \mathcal{E}ns(H_i, \varinjlim_\alpha Y_\alpha) \\
&\cong \varprojlim_i \varinjlim_\alpha \mathcal{E}ns(H_i, Y_\alpha) \\
&\cong \varinjlim_\alpha \varprojlim_i \mathcal{E}ns(H_i, Y_\alpha) \\
&\cong \varinjlim_\alpha \mathcal{E}ns(\varinjlim_i H_i, Y_\alpha)
\end{aligned}$$

We see that the fact that filtered colimits commute with finite limits in $\mathcal{E}ns$ is essential in this proof.

As a second example, if we have a functor

$$\mathcal{C} \xrightarrow{p} \mathcal{E}ns$$

there is a clear notion of what it means for it to be exact, that is, that it preserves finite limits. If the category $\mathcal{C}$ doesn't admit all finite limits, this condition makes sense but is somewhat vacuous. The correct definition is that of flat functor: the functor $\mathcal{C} \xrightarrow{p} \mathcal{E}ns$ is flat if its diagram Γ_p (also called category of elements) is a filtered category (an object of this category is a pair (x, C) with $x \in pC$ and an arrow $(x, C) \to (y, D)$ is given by a morphism $C \xrightarrow{f} D$ in $\mathcal{C}$ such that $(pf)(x) = y$).

A fundamental fact is that in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "h"] \arrow[dr, "p"'] & \arrow[d, phantom, "\cong"] & \mathcal{E}ns^{\mathcal{C}^{op}} \arrow[dl, "p^*"] \\ & \mathcal{E}ns & \end{tikzcd}$$

the functor p^* is exact precisely when Γ_p is filtered, that is, when p is flat. We can prove one of this implications using the commutation of filtered colimits and finite limits in $\mathcal{E}ns$: it can be seen that the functor p^* is in fact

$$p^* = \underset{(x,C)\in\Gamma_p}{\underrightarrow{\mathrm{colim}}} \; ev_C(-)$$

(where ev_C is the functor $\mathcal{E}ns^{\mathcal{C}^{op}} \to \mathcal{E}ns$ by evaluation at an object $C \in \mathcal{C}$). If Γ_p is filtered, then p^* is a filtered colimit of evaluation functors, which preserve all limits because in functor categories limits are computed pointwise. Then p^* is a filtered colimit of exact functors, and is exact by an argument analogous to the example above, using the fact that filtered colimits commute with finite limits in the category of sets.

It can be shown that if $\mathcal{C}$ has finite limits, then p is exact if and only if it is flat.

The proof of the commutation of filtered colimits and finite limits in $\mathcal{E}ns$ is done by means of the construction of the filtered colimit, which generalizes the construction of the ring of germs of continuous real-valued functions in a point of a topological space.

We are interested in formulating and verifying the equation 1.2, for set-valued functors, into a 2-categorical version, for category-valued 2-functors. This result is important in the development of the theory of 2-flat 2-functors and in the theory of 2-pro-objects and their Quillen model 2-categories (see [8] and [6]). In order to do this, we must perform a number of adaptations. Naturally, the categories $\mathcal{A}$ and $\mathcal{P}$ will be 2-categories, and the functor F will be a 2-functor. The limits that must be considered in the 2-category $\mathcal{C}at$ are no longer just the conical ones, but also limits weighted by a second 2-functor $\mathcal{P} \xrightarrow{W} F$. In this way, for example, we can fix $A \in \mathcal{A}$ and we can consider the limit of $F(A, -)$ weighted by W, which we denote $\underleftarrow{\mathrm{wlim}}^W F(A, -)$. In fact, in a 2-category, many different notions of universal property make sense and in the case that we consider, we utilize that of *bilimit*. The equality in formula 1.2, that we understand as expressing that a certain canonical function is bijective, will mean that a certain canonical functor is an equivalence of categories. Then, we wish to prove the fundamental commutation property

$$\mathrm{bi}\underrightarrow{\mathrm{colim}}_{\mathcal{A}}\, \mathrm{w}\underleftarrow{\mathrm{bilim}}^{W}_{\mathcal{P}}\, F \simeq \mathrm{w}\underleftarrow{\mathrm{bilim}}^{W}_{\mathcal{P}}\, \mathrm{bi}\underrightarrow{\mathrm{colim}}_{\mathcal{A}}\, F$$

when $\mathcal{A} \times \mathcal{P} \xrightarrow{F} \mathcal{C}at$ is a 2-functor, $\mathcal{A}$ is a 2-filtered 2-category and the limit weighted by W is finite, in some sense that is relevant for 2-categories: it will be necessary to impose finiteness conditions not only on the 2-category $\mathcal{P}$, but also on the weight W.

This is an important result which is original, and the objetive of obtaining a proof for this result was the starting point for this thesis.

We assume familiarity with the concepts and tools in the basic theory of categories for reading this work.

In section 2, we introduce the definitions of 2-categories, 2-functors, 2-natural transformations and modifications that appear in 2-category theory, together with examples and variants. We then give notions of filteredness and finiteness applicable to the 2-categorical context in section 3 and various utilized definitions of 2-limits in section 4. In section 5 we work through some examples of finite 2-limits and 2-filtered pseudocolimits with their constructions in the 2-category of categories, and in section 6 we construct 2-limits and bilimits from simpler limits, following [15], [28] and [10]. Section 7 contains the original contributions of this work: we prove the equation 1.2 in its 1-categorical version, and following that we prove the commutation of filtered bicolimits and finite biproducts, biequalizers and bicotensors with a finite category, separately, to later conclude the main result. Finally, in section 8, which is independent with respect to our main result, we give 2-categorical definitions of ends and coends, concepts that came up during the course of the study of the relevant theory for this work, and we apply them in proofs of certain basic properties of weighted 2-limits.

2. Basic elements of 2-categories

A 2-category is a $\mathcal{C}at$-enriched category, that is, a category in which the hom-sets are themselves categories. This definition would be technically sufficient to give the concepts of 2-categories, 2-functors, and 2-natural transformations simply as $\mathcal{C}at$-enriched versions of categories, functors, and natural transformations. However, it is useful to have a more elementary definition of these concepts, in terms of objects, morphisms and 2-cells. The definitions we give below are those of strict 2-categories and 2-functors, with on-the-nose equalities, but more lax versions are also useful. An introduction to these topics can be found in [4, chapter 7], [7], [17], or [19].

2.1. **2-categories.**

Definition 2.1. A 2-category $\mathcal{A}$ is given by

- a collection of objects or 0-cells

$$A, B, C, \ldots$$

- for each pair of objects A, B, a collection of morphisms (or arrows, or 1-cells) between them, each one indicated as follows

$$A \xrightarrow{f} B$$

- for each pair of parallel morphisms, $A \underset{g}{\overset{f}{\rightrightarrows}} B$, a collection of 2-cells (or 2-morphisms, or 2-arrows) each one indicated as follows

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

- a distinguished identity morphism for each object $id_A : A \to A$
- a distinguished identity 2-cell for each morphism $id_f : f \Rightarrow f$
- a way of (horizontally) composing two morphisms with compatible domain and codomain

$$A \xrightarrow{f} B\ ,\ B \xrightarrow{g} C \ \mapsto\ A \xrightarrow{gf} C$$

- a way of vertically composing two 2-cells with compatible domain and codomain, between the same objects

$$\begin{matrix} \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \\ \begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd} \end{matrix} \ \mapsto\ \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & B \end{tikzcd}$$

- a way of horizontally composing two 2-cells with compatible domain and codomain

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}, \begin{tikzcd} B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[r, bend left, "hf"] \arrow[r, bend right, "kg"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & C \end{tikzcd}$$

(this composite 2-cell is often denoted $\beta\alpha$ by simple juxtaposition)

subject to certain axioms:

- compositions are associative (for morphisms f, g, h, and 2-cells α, β, γ such that the compositions make sense in each case)

$$f \circ (g \circ h) = f \circ (g \circ h)$$
$$\alpha \circ (\beta \circ \gamma) = \alpha \circ (\beta \circ \gamma)$$
$$\alpha * (\beta * \gamma) = \alpha * (\beta * \gamma)$$

- identities are neutral for compositions

$$A \xrightarrow{f} B \xrightarrow{id_B} B \;=\; A \xrightarrow{f} B$$

$$A \xrightarrow{id_A} A \xrightarrow{f} B \;=\; A \xrightarrow{f} B$$

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g"' near start] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2, "\Downarrow id_g"] & B \end{tikzcd} \;=\; \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "f"' near start] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, shift left=2, "\Downarrow id_f"] \arrow[r, phantom, shift right=2, "\Downarrow\alpha"] & B \end{tikzcd} \;=\; \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "id_B"] & B \end{tikzcd} \;=\; \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "id_A"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \;=\; \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

- the interchange law:

$$(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha) \text{ in } \begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" near start] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2, "\Downarrow\beta"] & B \arrow[r, bend left=60, "f'"] \arrow[r, "g'" near start] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, shift left=2, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2, "\Downarrow\delta"] & C \end{tikzcd}$$

Observation 2.2. We draw

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "h"] & C \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

to denote the horizontal composition of 2-cells $id_h * \alpha$ or $\alpha * id_h$, respectively. This operation of composing horizontally with an identity 2-cell is called whiskering.

Example 2.3. Any category $\mathcal{C}$ can be made into a 2-category by only adding identity 2-cells for each morphism in $\mathcal{C}$.

Example 2.4. Just as $\mathcal{E}ns$ is the paradigmatic category, $\mathcal{C}at$ can be considered as a 2-category, with natural transformations as the 2-cells.
Given categories, functors and natural transformations as in

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, bend left=30, "\Downarrow\alpha"] \arrow[r, phantom, bend right=30, "\Downarrow\beta"] & \mathcal{B} \end{tikzcd}$$

the vertical composition $\beta \circ \alpha : F \Rightarrow H$ is given by components as

$$(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$$

where X is an object in $\mathcal{A}$.
Given categories, functors and natural transformations as in

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C} \end{tikzcd}$$

we can define the horizontal composition $\beta * \alpha : HF \Rightarrow KG$ in two ways: either

$$\beta_{GX} \circ H(\alpha_X)$$

or

$$K(\alpha_X) \circ \beta_{FX}$$

These compositions coincide because

$$\beta_{GX} \circ H(\alpha_X) = K(\alpha_X) \circ \beta_{FX}$$

holds by naturality of $H \overset{\beta}{\Longrightarrow} K$ with respect to the morphism $FX \xrightarrow{\alpha_X} GX$. We observe that the first definition corresponds to

$$\beta * \alpha = (\beta * id_G) \circ (id_H * \alpha)$$

and the second one to

$$\beta * \alpha = (id_K * \alpha) \circ (\beta * id_F)$$

$\mathcal{C}at$ can also be considered as an ordinary category, consisting of categories and functors. In general, any 2-category becomes an ordinary category by discarding the data for the 2-cells. To make this distinction explicit, we will sometimes say “1-category” to refer to an ordinary category (no 2-cells involved).

Example 2.5. The category $\mathcal{T}op$ of topological spaces and continuous funcions can be regarded as a sort of higher dimensional category, with homotopies between continuous functions as 2-cells, homotopies between those homotopies as 3-cells, and so on. We can discard the information above the second level to obtain a 2-category: it has topological spaces as objects, continuous functions as morphisms, and homotopies (up to homotopy, to make composition associative) as 2-cells. Furthermore, since every 2-cell is invertible, it is called a (2,1)-category (a (n,s)-category has k-cells for $k = 0, 1, \ldots, n$, where every k-cell for $k > s$ is invertible).

Example 2.6. If $\mathcal{C}$ is a 2-category, we denote $\mathcal{C}^{op}$ the 2-category obtained by reversing only the 1-cells. There are other variants that involve reversing the 2-cells.

We can organize this information in the external definition of a category enriched over $\mathcal{C}at$ (here we consider $\mathcal{C}at$ as a 1-category). This gives the following equivalent description.

Definition 2.7. A 2-category $\mathcal{A}$ is given by

- a collection of objects
- for any pair of objects A, B, a category $\mathcal{A}(A, B)$
- for any three objects A, B, C, a functor

$$\mathcal{A}(A,B) \times \mathcal{A}(B,C) \xrightarrow{c_{A,B,C}} \mathcal{A}(A,C)$$

- for each object A, a functor

$$\{\bullet\} \xrightarrow{u_A} \mathcal{A}(A,A)$$

  (here $\{\bullet\}$ is the terminal category consisting of one object and the identity morphism)

satisfying

- associativity: given any four objects A, B, C, D the following diagram commutes

(2.8)

$$\begin{tikzcd}
\mathcal{A}(A,B) \times \mathcal{A}(B,C) \times \mathcal{A}(C,D) \arrow[r, "1\times c_{B,C,D}"] \arrow[d, "c_{A,B,C}\times 1"'] \arrow[dr, phantom, "\equiv"] & \mathcal{A}(A,B) \times \mathcal{A}(B,D) \arrow[d, "c_{A,B,D}"] \\
\mathcal{A}(A,C) \times \mathcal{A}(C,D) \arrow[r, "c_{A,C,D}"'] & \mathcal{A}(A,D)
\end{tikzcd}$$

- identity: given objects A and B, the following diagrams commute

(2.9)

$$\begin{tikzcd}
\mathcal{A}(A,B) & \{\bullet\}\times\mathcal{A}(A,B) \arrow[l, "\cong"'] \arrow[d, "u_A\times 1"] \arrow[dl, phantom, "\equiv"] \\
& \mathcal{A}(A,A)\times\mathcal{A}(A,B) \arrow[ul, "c_{A,A,B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(A,B) & \mathcal{A}(A,B)\times\{\bullet\} \arrow[l, "\cong"'] \arrow[d, "1\times u_B"] \arrow[dl, phantom, "\equiv"] \\
& \mathcal{A}(A,B)\times\mathcal{A}(B,B) \arrow[ul, "c_{A,B,B}"]
\end{tikzcd}$$

These notions correspond in the following way:

- the functors $c_{A,B,C}$ give horizontal composition of morphisms and 2-cells
- the identity morphism on an object and identity 2-cell on this morphism are given by the functors u_A
- the associativity diagram translates into associativity of horizontal composition of morphisms and 2-cells
- the identity diagrams say that identity morphisms and identity 2-cells on identity morphims are neutral elements for horizontal composition of morphisms and 2-cells, respectively
- associativity and identity axioms for the vertical composition of 2-cells follow from the fact that for any two objects A and B we have $\mathcal{A}(A,B)$ a "vertical" category
- the interchange law is equivalent to functoriality of $c_{A,B,C}$

We will often write the hom-category $\mathcal{A}(A,B)$ as $[A,B]$.

In many cases it is useful to consider a weakening of the axioms of 2-categories. Namely, we could want the associativity and identity diagrams 2.8 and 2.9 to commute up to specified natural isomorphisms for each choice of objects in a coherent way, which can be made precise. This gives rise to the notion of bicategory. We refer to [4, chapter 7], [20] or [1] for an introduction to bicategories.

Example 2.10. (this is in [2, example 2.6], with more details in [25]) Given a category $\mathcal{C}$ with pullbacks, we can define a bicategory of spans with objects those of $\mathcal{C}$ and morphisms $A \to B$ given by spans

$$(f,g) = \begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B
\end{tikzcd}$$

A 2-cell $(f, g) \Rightarrow (f', g')$ in this bicategory is given by a morphism h in $\mathcal{C}$ making the two triangles commute

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\
A & & B \\
& X' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

These compose vertically in the obvious way. Now, to compose two spans

$$\begin{tikzcd}
& X_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & & X_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & B & & C
\end{tikzcd}$$

we take a pullback to obtain a new span:

$$\begin{tikzcd}
& & Y \arrow[dl, dashed, "p"'] \arrow[dr, dashed, "q"] & & \\
& X_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & X_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & & C
\end{tikzcd}$$

The choice of a pullback for each pair of objects X_1, X_2 must be given in advance so that composition of morphisms is well-defined in this bicategory. However, if we have a third span

$$\begin{tikzcd}
& X_3 \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A & & B
\end{tikzcd}$$

we can compose the three morphisms in two different ways:

$$\begin{tikzcd}
& & & Z_1 \arrow[dl, dashed, "r_1"'] \arrow[ddrr, dashed, "s_1"] & & & \\
& & Y_1 \arrow[dl, dashed, "p_1"'] \arrow[dr, dashed, "q_1"] & & & & \\
& X_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & X_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & X_3 \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A & & B & & C & & D
\end{tikzcd}$$

or

$$\begin{tikzcd}
 & & & Z_2 \arrow[ddll, dashed, "r_1"'] \arrow[dr, dashed, "s_2"] & & & \\
 & & & & Y_2 \arrow[dl, dashed, "p_2"'] \arrow[dr, dashed, "q_2"] & & \\
 & X_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & X_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & X_3 \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A & & B & & C & & D
\end{tikzcd}$$

corresponding to the compositions

$$(f_3, g_3) \circ ((f_2, g_2) \circ (f_1, g_1))$$

and

$$((f_3, g_3) \circ (f_2, g_2)) \circ (f_1, g_1)$$

respectively. Now we see that to have strict associativity we must have chosen the pullbacks in some way such these two compositions are equal: in particular, Z_1 and Z_2 must coincide. (For example, in $\mathcal{E}ns$ we have an isomorphism $A \times_X (B \times_Y C) \cong (A \times_X) \times_Y C$ but not an equality.) Instead, we can ask for associativity up to a certain natural isomorphism: the universal properties of the spans involved in these two diagrams allow us to construct an invertible 2-cell

$$(f_3, g_3) \circ ((f_2, g_2) \circ (f_1, g_1)) \Rightarrow ((f_3, g_3) \circ (f_2, g_2)) \circ (f_1, g_1)$$

This consists of an isomorphism $Z_1 \to Z_2$ in $\mathcal{C}$ that we can obtain by first using the universal property of the pullback Y_2 and the morphisms $q_1 \circ r_1$ and s_1, giving us a morphism $h : Z_1 \to Y_2$. Then from the universal property of Z_2 with the morphisms h and $p_1 \circ r_1$ we reach the desired morphism. The inverse of this morphism can be obtained arguing symmetrically.

Definition 2.11. Two objects A and B in a 2-category are isomorphic, denoted

$$A \cong B$$

if there are morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

that compose to the identity morphisms in both directions:

$$gf = id_A$$
$$fg = id_B$$

Observation 2.12. This definition also works in a 1-category since 2-cells are not involved.

When the 2-category is $\mathcal{C}at$, we obtain the following

Definition 2.13. Two categories $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, denoted

$$\mathcal{A} \cong \mathcal{B}$$

if there exist functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

that compose to the identity functors in both directions:

$$GF = Id_{\mathcal{A}}$$
$$FG = Id_{\mathcal{B}}$$

In a 2-category there is a second notion of similarity between objects:

Definition 2.14. Two objects A and B in a 2-category are equivalent, denoted

$$A \simeq B$$

if there are morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

that compose to morphisms that are isomorphic to the identity morphisms in both directions: there exist invertible 2-cells α and β such that

$$\begin{tikzcd} A \arrow[r, bend left, "gf"] \arrow[r, bend right, "id_A"'] \arrow[r, phantom, "\Downarrow\alpha\cong"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, bend right, "id_B"'] \arrow[r, phantom, "\Downarrow\beta\cong"] & B \end{tikzcd}$$

In particular, we have

Definition 2.15. Two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent, denoted

$$\mathcal{A} \simeq \mathcal{B}$$

if there exist functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

such that the compositions in both directions are naturally isomorphic to the identity functors: there exist natural isomorphisms α, β

$$GF \xRightarrow{\alpha\cong} Id_{\mathcal{A}} \qquad\qquad FG \xRightarrow{\beta\cong} Id_{\mathcal{B}}$$

In practice, it is usually the case that one of the functors is canonically defined, and the existence of the other follows from the following observation.

Observation 2.16. In [22, theorem 4.1], there is a proof of the fact that given a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, it is sufficient for the existence of G such that the pair (F, G) defines an equivalence of categories that F satisfy the following conditions

- F is full:
$$\mathcal{A}(A, B) \to \mathcal{B}(FA, FB)$$
is surjective for any $A, B \in \mathcal{A}$

- F is faithful:
$$\mathcal{A}(A,B) \to \mathcal{B}(FA,FB)$$
is injective for any $A, B \in \mathcal{A}$

- F is essentially surjective: given an object $B \in \mathcal{B}$, there exists an $A \in \mathcal{A}$ such that B is isomorphic to FA

Observation 2.17. In the light of observation 2.16 we give the following

Definition 2.18. A functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is an equivalence if there exists a functor $\mathcal{B} \xrightarrow{G} \mathcal{A}$ such that the pair (F,G) is an equivalence. The functor G is called a *quasi-inverse* for F.

We remark that this definition has a direction (it is not symmetric on F and G). The functor G is not determined by F: there can be different choices for G that result in the pair (F,G) defining an equivalence of categories, although any two such choices are naturally isomorphic.

2.2. **2-functors.**

A 2-functor between 2-categories $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an assignment of objects, morphisms and 2-cells in $\mathcal{A}$, to objects, morphisms and 2-cells in $\mathcal{B}$, respectively, that preserves all identities and compositions. In particular, it is a functor between the underlying categories.

Definition 2.19. Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ gives

- for any object A of $\mathcal{A}$, an object FA of $\mathcal{B}$
- for any morphism $A \xrightarrow{f} B$ in $\mathcal{A}$, a morphism $FA \xrightarrow{Ff} FB$ in $\mathcal{B}$
- for any 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{A}$, a 2-cell $\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \end{tikzcd}$ in $\mathcal{A}$

such that

$$F(g \circ f) = Fg \circ Ff$$
$$F(\beta \circ \alpha) = F\beta \circ F\alpha$$
$$F(\delta * \gamma) = F\delta * F\gamma$$

whenever g and f are morphisms, and α, β, γ, δ are 2-cells such that the specified compositions make sense in each case, and

$$F(id_A) = id_{FA}$$
$$F(id_f) = id_{Ff}$$

with id_A the identity morphism on an object A and id_f the identity 2-cell on a morphism f.

Observation 2.20. Since we will be working mostly with 2-functors with the 2-category $\mathcal{C}at$ as codomain, we work out precisely how such a functor $\mathcal{A} \xrightarrow{H} \mathcal{C}at$ acts.

Given a 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{A}$, we obtain a corresponding 2-cell $H\alpha$ in $\mathcal{C}at$, that is, a natural transformation between functors

$$\begin{tikzcd} HA \arrow[r, bend left, "Hf"] \arrow[r, bend right, "Hg"'] \arrow[r, phantom, "\Downarrow H\alpha"] & HB \end{tikzcd}$$

Naturality of this $Hf \overset{H\alpha}{\Longrightarrow} Hg$ means that for each $X \xrightarrow{u} Y$ in HA the following diagram commutes

$$\begin{tikzcd} (Hf)(X) \arrow[r, "(Hf)(u)"] \arrow[d, "(H\alpha)_X"'] \arrow[dr, phantom, "\equiv"] & (Hf)(Y) \arrow[d, "(H\alpha)_Y"] \\ (Hg)(X) \arrow[r, "(Hg)(u)"'] & (Hg)(Y) \end{tikzcd}$$

Observation 2.21. It can be easily checked that the composition of two 2-functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ is a 2-functor $\mathcal{A} \xrightarrow{GF} \mathcal{C}$.

The external definition of a 2-functor is as follows.

Definition 2.22. Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is an assignment of an object FA of $\mathcal{B}$ for each object A of $\mathcal{A}$, and a collection of functors

$$\mathcal{A}(A, B) \xrightarrow{F_{A,B}} \mathcal{B}(FA, FB)$$

preserving horizontal compositions:

$$\begin{tikzcd} \mathcal{A}(A, B) \times \mathcal{A}(B, C) \arrow[r, "F_{A,B} \times F_{B,C}"] \arrow[d, "c_{A,B,C}"'] \arrow[dr, phantom, "\equiv"] & \mathcal{B}(FA, FB) \times \mathcal{B}(FB, FC) \arrow[d, "c_{FA,FB,FC}"] \\ \mathcal{A}(A, C) \arrow[r, "F_{A,C}"'] & \mathcal{B}(FA, FC) \end{tikzcd}$$

for all objects A, B, C; and identities:

$$\begin{tikzcd} \{\bullet\} \arrow[r, "u_A"] \arrow[dr, "u_{FA}"'] & \mathcal{A}(A, A) \arrow[d, "F_{A,A}"] \arrow[dl, phantom, "\equiv"] \\ & \mathcal{B}(FA, FA) \end{tikzcd}$$

for any object A of $\mathcal{A}$.

Again, we can weaken this definition: if we ask that the diagrams expressing preservation of compositions and identities commute only up to a specified natural isomorphism in a coherent way we obtain the notion of a pseudofunctor.

The precise definition will not be necessary, but we make the observation that any 2-functor is a pseudofunctor, with identities as the chosen natural isomorphisms.

Example 2.23. We have representable 2-functors: for A an object in $\mathcal{A}$, the 2-functor $\mathcal{A} \xrightarrow{[A,-]} \mathcal{C}at$ is defined on objects and arrows as for ordinary representable functors: if B is an object in $\mathcal{A}$, we have $[A, B]$ the hom-category between A and B, and if $f : B \longrightarrow C$ in $\mathcal{A}$, the functor $f_* = [A, f] : [A, B] \longrightarrow [A, C]$ is defined by postcomposition

$$h \mapsto fh$$
$$\alpha : h \Rightarrow h' \mapsto id_f * \alpha$$

For $\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$ in $\mathcal{A}$, $[A, \alpha] : [A, f] \Longrightarrow [A, g]$ is a natural transformation defined on components by

$$[A, \alpha]_h = \alpha * id_h$$

Similarly, we have contravariant 2-functors $\mathcal{A}^{op} \xrightarrow{[-,A]} \mathcal{C}at$.

2.3. **2-natural transformations.**

Definition 2.24. Given parallel 2-functors between 2-categories $\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{B} \end{tikzcd}$, a 2-natural transformation $\eta : F \Longrightarrow G$ is a family of morphisms $\eta_A : FA \longrightarrow GA$ in $\mathcal{B}$ (called component of η at A) for each object A of $\mathcal{A}$, compatible with morphisms and 2-cells:

- for all $A \xrightarrow{f} B$ in $\mathcal{A}$ the following diagram commutes

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\equiv"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"'] & GB \end{tikzcd} \tag{2.25}$$

- for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{A}$ we have

$$\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \arrow[r, "\eta_B"] & GB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\eta_A"] & GA \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & GB \end{tikzcd}$$

Observation 2.26. 2-natural transformations can be composed both vertically and horizontally, with the same definitions as for natural transformations, given in example 2.4.

Example 2.27. Continuing example 2.23, if we have $A \xrightarrow{f} A'$ in $\mathcal{A}$, there is a 2-natural transformation $[f,-] : [A,-] \Rightarrow [A',-] : \mathcal{A} \longrightarrow \mathcal{C}at$ by precomposition. Its component at the object B is a functor $[f,-]_B : [A',B] \longrightarrow [A,B]$ defined as

$$g \mapsto gf$$
$$\alpha : g \Rightarrow h \mapsto \alpha * id_f$$

There is a useful relaxation of the notion of 2-natural transformation. Since the diagram 2.25 is of objects and arrows in $\mathcal{B}$, a 2-category, we could ask that it be commutative only up to specified invertible 2-cells in a coherent way.

Definition 2.28. Given parallel 2-functors between 2-categories $\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{B} \end{tikzcd}$, a pseudonatural transformation $\eta : F \Longrightarrow G$ is a family of morphisms $\eta_A : FA \longrightarrow GA$ in $\mathcal{B}$ for each object A of $\mathcal{A}$, together with invertible 2-cells η_f for each $A \xrightarrow{f} B$ in $\mathcal{A}$

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\swarrow \eta_f"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"'] & GB \end{tikzcd}$$

given in a coherent way:

- $\eta_{id_A} = id_{\eta_A}$

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, equal] \arrow[dr, phantom, "\equiv"] & GA \arrow[d, equal] \\ FA \arrow[r, "\eta_A"'] & GA \end{tikzcd}$$

- $\eta_{gf} = (\eta_g * id_{Ff}) \circ (id_{Gg} * \eta_f)$

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\swarrow\eta_f"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\swarrow\eta_g"] & GB \arrow[d, "Gg"] \\ FC \arrow[r, "\eta_C"'] & GC \end{tikzcd} \quad = \quad \begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\swarrow\eta_{gf}"] & GA \arrow[d, "G(gf)"] \\ FC \arrow[r, "\eta_C"'] & GC \end{tikzcd}$$

and compatible with 2-cells: for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{A}$ we have

$$\frac{\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, bend right, "Fg"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\eta_f"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"'] & GB \end{tikzcd} \quad = \quad \begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\swarrow\eta_g"] & GA \arrow[d, bend right, "Gg"'] \arrow[d, bend left, "Gf"] \arrow[d, phantom, "\overset{G\alpha}{\Leftarrow}"] \\ FB \arrow[r, "\eta_B"'] & GB \end{tikzcd}}{(id_{\eta_B} * F\alpha) \circ \eta_f = \eta_g \circ (G\alpha * id_{\eta_A})}$$

We observe that if the invertible 2-cells are the identities we get back the definition of 2-natural transformation.

Observation 2.29. Pseudonatural transformations can also be composed both vertically and horizontally. This requires an elaborate but straightforward verification.

2.4. **Modifications.**

We give the definition of modification between two pseudonatural transformations, which, as we have seen, subsumes the case of 2-natural transformations.

Definition 2.30. Given 2-functors and pseudonatural transformations $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta\,\Downarrow\sigma"] & \mathcal{B} \end{tikzcd}$, a modification between them $\zeta : \eta \rightsquigarrow \sigma$ is a collection of 2-cells $\eta_A \Rightarrow \sigma_A$ in $\mathcal{B}$ for each object A in $\mathcal{A}$, such that for any $A \xrightarrow{f} B$ in $\mathcal{A}$ we have

$$
\begin{tikzcd}
FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\swarrow \eta_f"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, bend left, "\eta_B"] \arrow[r, bend right, "\sigma_B"'] \arrow[r, phantom, "\Downarrow\zeta_B"] & GB
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, bend left, "\eta_A"] \arrow[r, bend right, "\sigma_A"'] \arrow[r, phantom, "\Downarrow\zeta_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\swarrow \sigma_f"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\sigma_B"'] & GB
\end{tikzcd}
$$

$$(\zeta_B * id_{Ff}) \circ \eta_f = \sigma_f \circ (id_{Gf} * \zeta_A)$$

Observation 2.31. Given any 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{A}$ we have

$$
\begin{aligned}
(\zeta_B * F\alpha) \circ \eta_f &= ((id_{\sigma_B} \circ \zeta_B) * (F\alpha \circ id_{Ff})) \circ \eta_f && \text{identities} \\
&= (id_{\sigma_B} * F\alpha) \circ (\zeta_B * iFf) \circ \eta_f && \text{interchange law} \\
&= (id_{\sigma_B} * F\alpha) \circ \sigma_f \circ (id_{Gf} * \zeta_A) && \zeta \text{ modification} \\
&= \sigma_g \circ (G\alpha * id_{\sigma_A}) \circ (id_{Gf} * \zeta_A) && \sigma \text{ pseudonatural} \\
&= \sigma_g \circ ((G\alpha \circ id_{Gf}) * (id_{\sigma_A} \circ \zeta_A)) && \text{interchange law} \\
&= \sigma_g \circ (G\alpha * \zeta_A) && \text{identities}
\end{aligned}
$$

So in fact ζ is compatible with all 2-cells α in $\mathcal{A}$, not just identity 2-cells on morphisms:

$$(\zeta_B * F\alpha) \circ \eta_f = \sigma_g \circ (G\alpha * \zeta_A)$$

Observation 2.32. Given modifications between 2-natural transformations

$$\alpha \overset{\zeta}{\rightsquigarrow} \beta \overset{\rho}{\rightsquigarrow} \gamma$$

with $\alpha, \beta, \gamma : F \Rightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$, we can compose them vertically: $\rho \circ \zeta$ is given by components as

$$(\rho \circ \zeta)_X = \rho_X \circ \zeta_X$$

if $X \in \mathcal{A}$. Horizontal composition is similarly defined: given 2-categories, 2-functors, 2-natural transformations and modifications as in

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha \overset{\zeta}{\rightsquigarrow} \Downarrow\beta"] \arrow[r, phantom, shift right=4, "\Downarrow\gamma \overset{\rho}{\rightsquigarrow} \Downarrow\delta"] & \mathcal{B}
\end{tikzcd}
$$

we can define the horizontal composite $\rho * \zeta$ by components as

$$(\rho * \zeta)_X = \rho_X * \zeta_X$$

if $X \in \mathcal{A}$.

It can be checked that these compositions, together with the obvious identities, define a 2-category $[\mathcal{A}, \mathcal{B}]$ of 2-functors, 2-natural transformations and modifications (see observation 2.26).

Example 2.33. Continuing the examples of 2-functors and 2-natural transformations $[A,-]$ and $[f,-]$ given by objects and morphisms in a 2-category $\mathcal{A}$ (examples 2.23 and 2.27), we can consider the modification $[\alpha,-]$ associated to a 2-cell $A \underset{g}{\overset{f}{\Downarrow\alpha}} B$.

For all X in $\mathcal{A}$, the component $[\alpha,-]_X : [f,-]_X \Longrightarrow [g,-]_X : [B,X] \longrightarrow [A,X]$ is a natural transformation (a 2-cell in $\mathcal{C}at$), and for $B \xrightarrow{s} X$, we have

$$[\alpha,-]_{X,s} = id_s * \alpha$$

At this point we can speak of the Yoneda embedding in the 2-categorical setting: the assignment $A \mapsto [A,-]$ is a 2-functor $h : \mathcal{A} \longrightarrow [\mathcal{A},\mathcal{C}at]^{op}$ (see [7]).

Theorem 2.34. *Given a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{C}at$, we have an isomorphism of categories*

$$\text{2Nat}([A,-],F) \cong FA$$

where the right side is the category of 2-natural transformations $[A,-] \Rightarrow F$ and modifications between them. □

This theorem can be seen as a particular case of the enriched Yoneda lemma when the enriching category is $\mathcal{C}at$, or it can be proved by elementary means. Setting $F = [A',-]$ we conclude that $h : \mathcal{A}^{op} \longrightarrow [\mathcal{A},\mathcal{C}at]$ is fully faithful as an ordinary functor and locally fully faithful. This will allow us to see $\mathcal{A}$ as a (full and locally full) subcategory of $[\mathcal{A},\mathcal{C}at]^{op}$, useful in the construction of the 2-filtered pseudocolimit (see section 5.16).

Example 2.35. We can instead consider $[\mathcal{A},\mathcal{B}]_p$: 2-functors, pseudonatural transformations and modifications also organize themselves into a 2-category. The identity on objects and the full inclusions

$$[\mathcal{A},\mathcal{B}](F,G) \xhookrightarrow{i} [\mathcal{A},\mathcal{B}]_p(F,G)$$

determine a locally full (but not full) 2-functor

$$[\mathcal{A},\mathcal{B}] \xrightarrow{i} [\mathcal{A},\mathcal{B}]_p$$

For the 2-functor $\mathcal{A} \longrightarrow [\mathcal{A},\mathcal{C}at]_p^{op}$ there is an adequate version of the Yoneda lemma (see [7, proposition 1.1.18]).

3. Notions of filteredness and finiteness

3.1. **Filtered categories.**

We first recall the definition of filteredness in an ordinary category.

Definition 3.1. A category $\mathcal{A}$ is filtered if

- it is non-empty
- for any two objects there is a third object further ahead: given objects A, B there exists an object C and arrows as in

$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd} \tag{3.2}$$

- any two parallel arrows are equalized further ahead: given f and g there exists C and an arrow h such that $hf = hg$ in

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd} \tag{3.3}$$

Observation 3.4. It is easy to prove that this is equivalent to the condition that every finite diagram in $\mathcal{A}$ admits a cone. This is done in [23, lemma 6.1].

Example 3.5. Every directed poset is filtered when seen as a category.

Example 3.6. A category with a terminal object is filtered.

Example 3.7. A category with binary products and coequalizers is filtered.

In [10] we find the following

Definition 3.8. A 2-category $\mathcal{A}$ is 2-filtered if

- it is non-empty
- (F0) for any two objects A, B there is a third object further ahead

$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd}$$

- (FF1) given $\begin{tikzcd} & A \\ E_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & \\ & B \end{tikzcd}$, $\begin{tikzcd} & A \\ E_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & \\ & B \end{tikzcd}$ there exist invertible 2-cells

$$\begin{tikzcd} & A \arrow[dr, "u"] & \\ E_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] \arrow[rr, phantom, "\Downarrow\gamma_1\cong"] & & C \\ & B \arrow[ur, "v"'] & \end{tikzcd} , \begin{tikzcd} & A \arrow[dr, "u"] & \\ E_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] \arrow[rr, phantom, "\Downarrow\gamma_2\cong"] & & C \\ & B \arrow[ur, "v"'] & \end{tikzcd}$$

- (F2) given (not necessarily invertible) 2-cells $\begin{tikzcd} & A \arrow[dr, "u_1"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\gamma_1"] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd}$,

$\begin{tikzcd} & A \arrow[dr, "u_2"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\gamma_2"] & & C_2 \\ & B \arrow[ur, "v_2"'] & \end{tikzcd}$ there exist invertible 2-cells α, β such that

$$\begin{tikzcd} & A \arrow[dr, "u_1"] & & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\gamma_1"] & & C_1 \arrow[dr, "w_1"] & \\ & B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha\cong"] & & C \\ & & C_2 \arrow[ur, "w_2"'] & \end{tikzcd} = \begin{tikzcd} & & C_1 \arrow[dr, "w_1"] & \\ & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta\cong"] & & C \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\gamma_2"] & & C_2 \arrow[ur, "w_2"'] & \\ & B \arrow[ur, "v_2"'] & & \end{tikzcd}$$

This 2-categorical analogue of filteredness can be formulated in different ways. In particular, the notion of bifiltered category appears in [18] (also see [10, definition 2.6]). In [5, section 5], these definitions are shown to be equivalent to the condition that every finite diagram admits a pseudocone.

Definition 3.8 can be weakened to give the concepts of pre-2-filteredness and pseudo-2-filteredness, which can be found in [10]. For the results that we want to prove, however, we need to use the whole strength of the definition of 2-filtered 2-category.

We observe that an ordinary 1-category is filtered if and only if it is 2-filtered when seen as a 2-category with identity 2-cells, so that 2-filteredness is a generalization of 1-filteredness.

3.2. **Finite weights.**

In the 1-dimensional case, a limit of a functor $\mathcal{P} \xrightarrow{F} \mathcal{C}$ is called finite when the category $\mathcal{P}$ is finite, and we know that finite limits can be constructed from binary products and equalizers, as we will show later on, and that these limits commute with filtered colimits. In the 2-dimensional case, we also have a weight (a 2-functor) $\mathcal{P} \xrightarrow{W} \mathcal{C}at$ and the finite condition just on $\mathcal{P}$ is not sufficient to guarantee that the 2-limit $\underleftarrow{\text{wlim}}^{W} F$ can be constructed from the corresponding three types of basic finite 2-limits, nor that such a limit will commute with

2-filtered 2-colimits: the weight functor must also satisfy finiteness conditions.

In [15] and [16], Kelly gives a definition of a finite 2-limit to be one with the indexing 2-category $\mathcal{P}$ having a finite collection of objects and finitely presentable hom-categories, and such that for all $P \in \mathcal{P}$ also WP is a finitely presentable category. This is precisely the type of 2-limit that can be constructed from binary 2-products, 2-equalizers and cotensor products with a finite category (in fact, cotensor products with the category $2 = \{0 \to 1\}$ suffice). However, we find convenient to use a simplified definition, which is enough for our purposes.

Definition 3.9. A 2-functor $\mathcal{P} \xrightarrow{W} \mathcal{C}at$ is a **finite weight** whenever the indexing 2-category $\mathcal{P}$ as well as every category WP for $P \in \mathcal{P}$ is finite (this means a finite collection of objects and morphisms in the case of a category, and also a finite collection of 2-cells in the case of a 2-category).

A weighted 2-limit $\underleftarrow{\mathrm{wlim}}^W F$ is called finite when its weight $\mathcal{P} \xrightarrow{W} \mathcal{C}at$ is a finite weight. The same terminology applies for bilimits and pseudolimits.

4. Notions of limits and colimits in a 2-category

We speak mostly about limits, as the discussion can be dualized to refer to colimits.

In an ordinary category $\mathcal{C}$, the definition of a limit can be given in several equivalent ways. The most elementary way of saying that a functor $\mathcal{B} \xrightarrow{F} \mathcal{C}$ has a limit L, with morphisms $L \xrightarrow{\pi_B} FB$ for each object B of $\mathcal{B}$, is that this data defines a cone for F, i.e. the following diagram commutes for all $B \xrightarrow{f} B'$

$$\begin{tikzcd} & L \arrow[dl, "\pi_B"'] \arrow[dr, "\pi_{B'}"] \arrow[d, phantom, "\equiv"] & \\ FB \arrow[rr, "Ff"'] & {} & FB' \end{tikzcd}$$

and every other such cone $(Z, (\mu_B)_B)$ factorizes uniquely through $(L, (\pi_B)_B)$

$$\begin{tikzcd} & Z \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "\mu_B"'] \arrow[ddr, bend left, "\mu_{B'}"] & \\ \equiv & L \arrow[dl, "\pi_B"'] \arrow[dr, "\pi_{B'}"] \arrow[d, phantom, "\equiv"] & \equiv \\ FB \arrow[rr, "Ff"'] & {} & FB' \end{tikzcd}$$

Definition 4.1. Given categories $\mathcal{B}$, $\mathcal{C}$ and an object $Z \in \mathcal{C}$, there is a functor

$$\mathcal{B} \xrightarrow{\Delta_Z} \mathcal{C}$$

sending every object $B \mapsto Z$ and every morphism $f \mapsto id_Z$. This is the constant functor at Z. In fact, we can see Δ as a functor

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^{\mathcal{B}}$$

with object function $Z \mapsto \Delta_Z$ in a straightforward way.

Using this definition, $(L, (\pi_B)_B)$ is a limit for F when $\Delta_L \xRightarrow{\pi} F$ is a natural transformation that induces by postcomposition a (natural) bijection for all Z in $\mathcal{B}$

$$\mathcal{C}(Z, L) \xrightarrow{\pi_* \cong} Nat(\Delta_Z, F)$$

The set $Nat(F,G)$ of natural transformations between two functors $F, G : \mathcal{B} \longrightarrow \mathcal{C}$ is in fact the hom-set in the functor category $\mathcal{C}^{\mathcal{B}}$, so we have a natural bijection

$$\mathcal{C}(Z, L) \xrightarrow{\pi_* \cong} \mathcal{C}^{\mathcal{B}}(\Delta_Z, F)$$

Again, we can rephrase this in terms of another ubiquitous notion in category theory: (L, π) is a limit of F when it is a representation of the functor $\mathcal{C}^{\mathcal{B}}(\Delta -, F) : \mathcal{B} \longrightarrow \mathcal{E}ns$, which maps

$$Z \mapsto \mathcal{C}^{\mathcal{B}}(\Delta_Z, F)$$

If the limit exists for all diagrams F of shape $\mathcal{B}$ in $\mathcal{C}$, we can say that $\varprojlim : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}$ is a functor, which is right adjoint to the diagonal $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{B}}$

$$\mathcal{C}(Z, \varprojlim F) \cong \mathcal{C}^{\mathcal{B}}(\Delta_Z, F)$$

In the general $\mathcal{V}$-enriched case, these definitions of limit don't make sense, because we cannot speak of morphisms $Z \to FB$, and $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{B}}$ might not exist (see [13]). We observe that we have

$$\mathcal{C}^{\mathcal{B}}(\Delta_Z, F) \cong \mathcal{E}ns^{\mathcal{B}}(\Delta_1, \mathcal{C}(Z, F-))$$

because a natural transformation $\Delta_1 \Rightarrow \mathcal{C}(Z, F-)$ amounts to a function from the singleton set $\{*\} \longrightarrow \mathcal{C}(Z, FB)$ for each B in $\mathcal{B}$ satisfying naturality conditions, and a function from the singleton set is just an element.

The functor Δ_1 might still not make sense in the enriched setting, but we can now substitute it with any other (enriched) functor W that we call the weight. This gives the definition of a weighted (or indexed) limit for categories enriched on an arbitrary monoidal category $\mathcal{V}$, which can be found in [30].

Definition 4.2. Given $\mathcal{V}$-functors $\mathcal{B} \xrightarrow{W} \mathcal{V}$ and $\mathcal{B} \xrightarrow{F} \mathcal{C}$, a W-weighted limit of F, or limit of F weighted by W is a representation of the functor

$$Z \mapsto [\mathcal{B}, \mathcal{V}](W, \mathcal{C}(Z, F-))$$

where $[\mathcal{B}, \mathcal{V}]$ is the enriched functor category, and $\mathcal{C}(Z, F-)$ is just $\mathcal{C}(Z, -) \circ F$. This representation consists of an object $\mathrm{w}\varprojlim^W F$ in $\mathcal{C}$ and the unit: a $\mathcal{V}$-natural transformation $\pi : W \Longrightarrow \mathcal{C}(\mathrm{w}\varprojlim^W F, F-)$. We also denote the limit object by $\{W, F\}$.

An introduction to weighted limits and colimits can be found in [3], [27], [26, chapter 7], or [12].

From now on, we will restrict ourselves to the case $\mathcal{V} = \mathcal{C}at$, which has many special features. In particular, we do have the diagonal 2-functor Δ. This gives the following definition, that can be found, for example, in [15]:

Definition 4.3. (weighted 2-limit) Given 2-functors $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ and $\mathcal{B} \xrightarrow{F} \mathcal{C}$, a W-weighted 2-limit of F, or 2-limit of F weighted by W is a 2-representation of the functor

$$Z \mapsto [\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(Z, F-))$$

sending an object Z in $\mathcal{C}$ to the category of 2-natural transformations and modifications between the 2-functors W and $\mathcal{C}(Z, F-)$. That is, an isomorphism of categories

$$\mathcal{C}(Z, \underset{\longleftarrow}{\mathrm{wlim}}^W F) \cong [\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(Z, F-))$$

2-natural in Z.
This representation consists of an object $\underset{\longleftarrow}{\mathrm{wlim}}^W F$ in $\mathcal{C}$ and a 2-natural transformation $\pi : W \Longrightarrow \mathcal{C}(\underset{\longleftarrow}{\mathrm{wlim}}^W F, F-)$. The object $\underset{\longleftarrow}{\mathrm{wlim}}^W F$ is also denoted $\{W, F\}$.

We call this a 2-limit, to emphasize that the universal property of the representing object is an isomorphism of categories, and not just a bijection of sets as in the ordinary 1-dimensional definition of a limit.

Observation 4.4. In [15, section 3], there is an observation that when a 2-category $\mathcal{C}$ admits a specific kind of weighted 2-limit (all cotensors with the category $2 = \{0 \to 1\}$), the 2-dimensional property of any limit follows from the 1-dimensional universal property. This observation can be proved from [14, theorem 4.85]. This means that in $\mathcal{C}at$ (a 2-category that admits cotensors with $2 = \{0 \to 1\}$), some verifications could be omitted when we want to prove that a given construction is the solution of the 2-dimensional universal property. In practice, however, an explicit description of the unique 2-cells defined by 2-dimensional universal properties may be needed. Furthermore, it is instructive to give the explicit formulation involving the 2-dimensional aspect, and we choose to do so in many cases. Cotensors are defined in 5.10.

We can dualize this definition to obtain one of weighted 2-colimits.

Definition 4.5. (weighted 2-colimit) Given 2-functors $\mathcal{B}^{op} \xrightarrow{W} \mathcal{C}at$ and $\mathcal{B} \xrightarrow{F} \mathcal{C}$, a W-weighted 2-colimit of F, or 2-colimit of F weighted by W is a 2-representation of the functor

$$Z \mapsto [\mathcal{B}^{op}, \mathcal{C}at](W, \mathcal{C}(F-, Z))$$

sending an object Z in $\mathcal{C}$ to the category of 2-natural transformations and modifications between the 2-functors W and $\mathcal{C}(F-, Z)$. That is, an isomorphism of categories

$$\mathcal{C}(\underset{\longrightarrow}{\mathrm{wcolim}}^W F, Z) \cong [\mathcal{B}^{op}, \mathcal{C}at](W, \mathcal{C}(F-, Z))$$

2-natural in Z.
This representation consists of an object $\underset{\longrightarrow}{\mathrm{wcolim}}^W F$ in $\mathcal{C}$ (also denoted $W * F$) and a 2-natural transformation $\lambda : W \Longrightarrow \mathcal{C}(F-, \underset{\longrightarrow}{\mathrm{wcolim}}^W F)$.

With the definitions we've given so far, there is a possible variation to the notion of a 2-limit or 2-colimit. In the formula

$$[\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(Z, F-))$$

we can replace the 2-functor 2-category $[\mathcal{B}, \mathcal{C}at]$ of 2-functors, 2-natural transformations and modifications by the 2-category $[\mathcal{B}, \mathcal{C}at]_p$ of 2-functors, pseudonatural transfomations, and modifications.

Definition 4.6. (weighted pseudolimit) Given 2-functors $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ and $\mathcal{B} \xrightarrow{F} \mathcal{C}$, a W-weighted pseudolimit of F, or pseudolimit of F weighted by W is a 2-representation of the functor

$$Z \mapsto [\mathcal{B}, \mathcal{C}at]_p(W, \mathcal{C}(Z, F-))$$

and is denoted $\underleftarrow{\mathrm{wpslim}}^W F$.

The weighted pseudocolimit $\underrightarrow{\mathrm{wpscolim}}^W F$ is analogously defined.

We are interested in the representation being given by an equivalence instead of an isomorphism. We have, for example, the following

Definition 4.7. (weighted bilimit) Given 2-functors $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ and $\mathcal{B} \xrightarrow{F} \mathcal{C}$, a W-weighted bilimit of F, or bilimit of F weighted by W is a birepresentation of the functor

$$Z \mapsto [\mathcal{B}, \mathcal{C}at]_p(W, \mathcal{C}(Z, F-))$$

That is,

$$\mathcal{C}(Z, \underleftarrow{\mathrm{wbilim}}^W F) \simeq [\mathcal{B}, \mathcal{C}at]_p(W, \mathcal{C}(Z, F-))$$

is an equivalence of categories.

Observation 4.8. From the definitions above it follows that any two solutions of the universal problem will be equivalent objects. Thus, when pseudolimits exist, they will be equivalent to any choice of bilimits on the same data, but not isomorphic.

Whenever the weight of a weighted 2-limit or 2-colimit is the functor Δ_1, we call such 2-limits or 2-colimits **conical**, as they can be defined in terms of universal appropriate cones. When a 2-limit is conical, we omit the letter w in the notation, as conical limits coincide with the classical notions of 2-limits and 2-colimits.

Example 4.9. We have

$$[\mathcal{B}, \mathcal{C}at]_p(\Delta_1, \mathcal{C}(Z, F-)) \cong [\mathcal{B}, \mathcal{C}]_p(\Delta_Z, F) \tag{4.10}$$

as in the unenriched, 1-categorical case. A conical pseudolimit of a functor $\mathcal{A} \xrightarrow{F} \mathcal{C}$ can be defined as a representation of the functor

$$Z \mapsto [\mathcal{B}, \mathcal{C}]_p(\Delta_Z, F)$$

It is given by a pseudocone with vertex $\underleftarrow{\mathrm{pslim}}\, F$ for the functor F: a pseudo-natural transformation

$$\Delta_{\underleftarrow{\mathrm{pslim}}\, F} \overset{\pi}{\Longrightarrow} F$$

or just written

$$\underleftarrow{\mathrm{pslim}}\, F \overset{\pi}{\Longrightarrow} F$$

universal in the sense that such that every other pseudocone with vertex Z an object of $\mathcal{C}$

$$Z \overset{\mu}{\Longrightarrow} F$$

factorizes through π in a unique way

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!"] \arrow[dr, Rightarrow, "\mu"'] \arrow[dr, phantom, shift left=3, "\equiv"] & \underleftarrow{\mathrm{pslim}}\, F \arrow[d, Rightarrow, "\pi"] \\ & F \end{tikzcd}$$

This is not sufficient to specify what a pseudolimit is, since it implies that postcomposition with the universal pseudocone π

$$\mathcal{C}(Z, \underleftarrow{\mathrm{pslim}}\, F) \xrightarrow{\pi_*} [\mathcal{B}, \mathcal{C}]_p(\Delta_Z, F)$$

is a bijection for all Z, and not an isomorphism of categories. The additional requirement is that any modification between two pseudocones with the same vertex

$$\begin{tikzcd} Z \arrow[r, Rightarrow, bend left, "\mu"] \arrow[r, Rightarrow, bend right, "\rho"'] \arrow[r, phantom, "\Downarrow\zeta"] & F \end{tikzcd}$$

must also factorize uniquely through π: if we have

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!u"] \arrow[dr, Rightarrow, "\mu"'] \arrow[dr, phantom, shift left=3, "\equiv"] & \underleftarrow{\mathrm{pslim}}\, F \arrow[d, Rightarrow, "\pi"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, dashed, "\exists!v"] \arrow[dr, Rightarrow, "\rho"'] \arrow[dr, phantom, shift left=3, "\equiv"] & \underleftarrow{\mathrm{pslim}}\, F \arrow[d, Rightarrow, "\pi"] \\ & F \end{tikzcd}$$

then for all modifications

$$\begin{tikzcd} Z \arrow[r, Rightarrow, bend left, "\mu"] \arrow[r, Rightarrow, bend right, "\rho"'] \arrow[r, phantom, "\Downarrow\zeta"] & F \end{tikzcd}$$

there exists a unique 2-cell in $\mathcal{C}$

$$\begin{tikzcd} Z \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & \underleftarrow{\mathrm{pslim}}\, F \end{tikzcd}$$

such that

$$\begin{tikzcd} Z \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & \underleftarrow{\mathrm{pslim}}\, F \arrow[r, Rightarrow, "\pi"] & F \end{tikzcd} \equiv \begin{tikzcd} Z \arrow[r, Rightarrow, bend left, "\mu"] \arrow[r, Rightarrow, bend right, "\rho"'] \arrow[r, phantom, "\Downarrow\zeta"] & F \end{tikzcd}$$

Conical 2-limits can be described in these terms making the obvious substitutions: instead of pseudonatural transformations $\Delta_Z \Longrightarrow F$ which are pseudocones, we consider 2-natural transformations $\Delta_Z \Longrightarrow F$ which we might call 2-cones. Of course, the case of colimits is identical once we revert all the arrows.

We will say **2-filtered bicolimit** to refer to a conical bicolimit where the indexing category (the shape of the diagram) is 2-filtered.

It can be instructive to spell out the details and draw the pseudocones as we usually draw cones in the 1-categorical case (see [7]). We take up this proposal for particular examples of limits and colimits in the next section.

5. Examples in $\mathcal{C}at$

5.1. **Product.**

We will consider binary products, but all the discussion in this subsection can be generalized to arbitary products in a straightforward way.

If $\mathcal{C}$ is an ordinary category (a 1-category) and A and B are two objects in $\mathcal{C}$, we say that a third object denoted $A \times B$ with projections π_1, π_2

$$\begin{tikzcd} & A \\ A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & B \end{tikzcd}$$

is a product of A and B if it is universal in the sense that every other such diagram factorizes uniquely through it: for any C, μ_1 and μ_2, there exists a unique morphism that we can call (μ_1, μ_2) such that the triangles commute

$$\begin{tikzcd} & & A \\ C \arrow[urr, bend left, "\mu_1"] \arrow[drr, bend right, "\mu_2"'] \arrow[r, dashed, "\exists!(\mu_1,\mu_2)"] \arrow[r, phantom, shift left=3, "\equiv"] \arrow[r, phantom, shift right=3, "\equiv"] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & B \end{tikzcd} \tag{5.1}$$

This is just the definition of a limit in a 1-category, as we discussed in section 4, where the functor is a diagram of shape $\{\bullet \quad \bullet\}$ with image $\{A \quad B\}$. We can say that

$$\mathcal{C}(Z, A \times B) \cong \mathcal{C}(Z, A) \times \mathcal{C}(Z, B)$$

is a natural bijection by postcomposition. We observe that

$$\mathcal{C}(Z, A) \times \mathcal{C}(Z, B) \cong (\mathcal{C} \times \mathcal{C})((Z, Z), (A, B)) \cong \mathcal{C}^{\{\bullet \quad \bullet\}}(\Delta_Z, F)$$

so that this is indeed the usual representability formula

$$\mathcal{C}(Z, A \times B) \cong \mathcal{C}^{\{\bullet \quad \bullet\}}(\Delta_Z, F) \tag{5.2}$$

This limit exists in $\mathcal{C}at$: the product of two categories is given by the usual construction with objects and morphisms of $\mathcal{A} \times \mathcal{B}$ given by pairs of objects or morphisms (the first in $\mathcal{A}$ and the second in $\mathcal{B}$) respectively.

We now consider 2-categorical analogues of this limit. This will consist in giving elementary definitions of the different variations on the notion of limit, applied to the case of a binary product.

5.2. **2-product.**

The first way of turning the product into a 2-categorical limit is by replacing the natural bijection of sets in equation 5.2 by a 2-natural isomorphism of categories

$$\mathcal{C}(Z, A \times B) \cong \mathcal{C}^{\{\bullet \quad \bullet\}}(\Delta_Z, F)$$

where now $\mathcal{C}$ is a 2-category. If this representing object $A \times B$ exists, it is the 2-product of A and B.
Analogous to the isomorphism 4.10, we have

$$[\mathcal{B}, \mathcal{C}at](\Delta_1, \mathcal{C}(Z, F-)) \cong [\mathcal{B}, \mathcal{C}](\Delta_Z, F)$$

in general. Thus $A \times B$ is defined by the formula

$$\mathcal{C}(Z, A \times B) \cong [\{\bullet \quad \bullet\}, \mathcal{C}at](\Delta_1, \mathcal{C}(Z, F-))$$

and it is then an example of a conical limit.
In addition to the 1-dimensional universal property of the usual product given by diagram 5.1, if we have two cones

$$\begin{tikzcd} & A \\ C \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \\ & B \end{tikzcd} \qquad \begin{tikzcd} & A \\ C \arrow[ur, "\rho_1"] \arrow[dr, "\rho_2"'] & \\ & B \end{tikzcd}$$

with corresponding morphisms into $A \times B$ given by the universal property

$$C \xrightarrow{(\mu_1, \mu_2)} A \times B \qquad\qquad C \xrightarrow{(\rho_1, \rho_2)} A \times B$$

then 2-cells

$$\begin{tikzcd} C \arrow[r, bend left, "(\mu_1,\mu_2)"] \arrow[r, bend right, "(\rho_1,\rho_2)"'] \arrow[r, phantom, "\Downarrow\xi"] & A \times B \end{tikzcd}$$

must correspond bijectively to modifications $\mu \rightsquigarrow \rho$ by postcomposition with π. Since $\{\bullet \quad \bullet\}$ is discrete, a modification $\mu \rightsquigarrow \rho$ amounts to a pair of 2-cells $\mu_1 \Rightarrow \rho_2$, $\mu_1 \Rightarrow \rho_2$ satisfying no compatibility equations.

The universal property of the 2-product then includes the 1-dimensional one of the ordinary product. That means that if the 2-product exists, then it is an ordinary product as well. If we have a construction of the product $A \times B$ of two objects in a category $\mathcal{C}$, the 2-product of the same objects, if it exists, must be isomorphic to the given construction.

In $\mathcal{C}at$, the 2-product exists, and is given by the same usual construction of the ordinary product.

Proposition 5.3. *Given categories $\mathcal{A}$ and $\mathcal{B}$, there exists a category $\mathcal{A} \times \mathcal{B}$ with functors*

$$\begin{tikzcd} & \mathcal{A} \\ \mathcal{A} \times \mathcal{B} \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & \mathcal{B} \end{tikzcd}$$

such that postcomposition with (π_1, π_2) gives an isomorphism of categories

$$[\mathcal{Z}, \mathcal{A} \times \mathcal{B}] \xrightarrow{\pi_* \cong} Cone(\mathcal{Z})$$

between the hom-category $[\mathcal{Z}, \mathcal{A} \times \mathcal{B}]$ and the category of cones with vertex $\mathcal{Z}$ for the diagram $\{\mathcal{A} \quad \mathcal{B}\}$ and modifications between them, for all categories $\mathcal{Z}$.

Proof. To verify this, we construct the category $\mathcal{A} \times \mathcal{B}$ and the projections π_1 and π_2 as in the 1-categorical construction of the product, and only check the 2-dimensional universal property since we already know that the 1-dimensional property holds.
Given a category $\mathcal{Z}$ and functors $F : \mathcal{Z} \longrightarrow \mathcal{A}$, $G : \mathcal{Z} \longrightarrow \mathcal{B}$, $F' : \mathcal{Z} \longrightarrow \mathcal{A}$ and $G' : \mathcal{Z} \longrightarrow \mathcal{B}$, and natural transformations $\zeta_1 : F \Longrightarrow F'$, $\zeta_2 : G \Longrightarrow G'$, we want to see that there exists a unique 2-cell (a natural transformation) $\gamma : (F, G) \Rightarrow (F', G')$ such that

$$\zeta_1 = id_{\pi_1} * \gamma$$
$$\zeta_2 = id_{\pi_2} * \gamma$$

this gives only one option for the components of γ

$$\zeta_{1,Z} = \pi_1(\gamma_Z)$$
$$\zeta_{2,Z} = \pi_2(\gamma_Z)$$

if Z is an object of $\mathcal{Z}$, so that it must be

$$\gamma_Z = (\zeta_{1,Z}, \zeta_{2,Z})$$

which gives us the uniqueness. In order to prove existence we need to check that γ defined in this way is a natural transformation: for all $Z \xrightarrow{f} Z'$ in $\mathcal{Z}$

$$\begin{tikzcd} (FZ, GZ) \arrow[r, "(\zeta_{1,Z}, \zeta_{2,Z})"] \arrow[d, "(Ff, Gf)"'] \arrow[dr, phantom, "\equiv"] & (F'Z, G'Z) \arrow[d, "(F'f, G'f)"] \\ (FZ', GZ') \arrow[r, "(\zeta_{1,Z'}, \zeta_{2,Z'})"'] & (F'Z', G'Z') \end{tikzcd}$$

but this is just, component-wise, the naturality of ζ_1 and ζ_2. $\square$

We remark that this verification of the 2-dimensional universal property is unnecessary, because of observation 4.4.

5.3. **Pseudoproduct.**

We can define a less strict version of the 2-categorical product by using pseudocones and pseudonatural transformations instead of 2-cones and 2-natural transformations:

$$\mathcal{C}(Z, A \times B) \cong [\{\bullet \quad \bullet\}, \mathcal{C}at]_p(\Delta_1, \mathcal{C}(Z, F-))$$

However, if $\mathcal{B}$ is a discrete 2-category (just a set) a pseudonatural transformation between 2-functors $\mathcal{B} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}$ is the same as a 2-natural transformation, and in fact, it is just a collection of morphisms $FB \to GB$ in $\mathcal{C}$, one for each object B, satisfying no compatibility conditions. In this case, we have

$$[\{\bullet \quad \bullet\}, \mathcal{C}at]_p(\Delta_1, \mathcal{C}(Z, F-)) \cong [\{\bullet \quad \bullet\}, \mathcal{C}at](\Delta_1, \mathcal{C}(Z, F-))$$

so the pseudoproduct is the same as the 2-product.

5.4. **Biproduct.**

Definition 5.4. (biproduct) The biproduct of two objects A and B in $\mathcal{C}$ is a pair of morphisms $Z \xrightarrow{h} A$ and $Z \xrightarrow{k} B$ inducing an equivalence of categories between $\mathcal{C}(C, Z)$ and $\mathcal{C}(C, A) \times \mathcal{C}(C, B)$, for all objects $C \in \mathcal{C}$.

Observation 5.5. This definition can be given in elementary terms, resulting in a sentence involving the 1-dimensional and the 2-dimensional universal properties. This is done by carefully unfolding the definition of equivalence (definition 2.15) or its characterization (in observation 2.16). We will be considering biproducts in the 2-category $\mathcal{C}at$, which admits pseudoproducts (see subsection 5.3), so that we can work with these because of observation 4.8.

5.5. **Equalizer.** In an ordinary category $\mathcal{C}$, an equalizer of

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is an object E with morphism $E \xrightarrow{i} A$ such that

$$E \xrightarrow{i} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

commutes, that is, $fi = gi$, universal in the sense that for every other

$$\begin{tikzcd} & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[ur, "c"] & & \end{tikzcd}$$

with $fc = gc$ there is a unique dashed morphism making the triangle commute in

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[dr, phantom, "\equiv"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dashed, "\exists! c'"] \arrow[ur, "c"'] & & \end{tikzcd}$$

This is the same as saying that if F is a diagram of shape $\{\bullet \rightrightarrows \bullet\}$ with image $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{C}$, postcomposition with i gives a bijection of sets

$$\mathcal{C}(Z, E) \cong [\{\bullet \rightrightarrows \bullet\}, \mathcal{C}](\Delta_Z, F)$$

The equalizer of two functors $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ exists in $\mathcal{C}at$: it is given by the subcategory of $\mathcal{A}$ consisting of objects and morphisms equalized by F and G.

5.6. **2-equalizer.**

We now consider

$$\mathcal{C}(Z, E) \cong [\{\bullet \rightrightarrows \bullet\}, \mathcal{C}](\Delta_Z, F)$$

a natural isomorphism of categories instead of just a bijection of sets. Since

$$[\{\bullet \rightrightarrows \bullet\}, \mathcal{C}](\Delta_Z, F) \cong [\{\bullet \rightrightarrows \bullet\}, \mathcal{C}at](\Delta_1, \mathcal{C}(Z, F-))$$

this is a conical 2-limit: the weight is Δ_1.

We spell out the 2-dimensional universal property of the 2-equalizer: if we have $fd = gd$ in

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[dr, phantom, "\equiv"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dashed, "\exists! d'"] \arrow[ur, "d"'] & & \end{tikzcd}$$

and a 2-cell

$$\begin{tikzcd} Z \arrow[r, bend left, "c"] \arrow[r, bend right, "d"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

such that

$$id_f * \alpha = id_g * \alpha$$

then there exists a unique 2-cell $\begin{tikzcd} Z \arrow[r, bend left, "c'"] \arrow[r, bend right, "d'"'] \arrow[r, phantom, "\Downarrow\varphi"] & E \end{tikzcd}$ such that

$$\begin{tikzcd} Z \arrow[r, bend left, "c'"] \arrow[r, bend right, "d'"'] \arrow[r, phantom, "\Downarrow\varphi"] & E \arrow[r, "i"] & A \end{tikzcd} = \begin{tikzcd} Z \arrow[r, bend left, "c"] \arrow[r, bend right, "d"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

that is, $id_i * \varphi = \alpha$.

Observation 5.6. Also in this case the 2-equalizer is given in $\mathcal{C}at$ by the usual construction of the equalizer. We suppose given categories and functors

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$$

and define $\mathcal{E}$ as the subcategory of $\mathcal{A}$ with morphisms and objects those equalized by F and G, with $\mathcal{E} \xrightarrow{I} \mathcal{A}$ the inclusion. If a functor $\mathcal{Z} \xrightarrow{C} \mathcal{A}$ equalizes F and G, the functor $\mathcal{Z} \xrightarrow{C'} \mathcal{E}$ induced by the universal property is given by $C'(A) = C(A)$ on objects and by $C'(f) = C(f)$ on morphisms. Given a second functor $\mathcal{Z} \xrightarrow{D} \mathcal{A}$ and a natural transformation

$$\begin{tikzcd} \mathcal{Z} \arrow[r, bend left, "C"] \arrow[r, bend right, "D"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{A} \end{tikzcd}$$

such that $id_F * \alpha = id_G * \alpha$, the corresponding natural transformation

$$\begin{tikzcd} \mathcal{Z} \arrow[r, bend left, "C'"] \arrow[r, bend right, "D'"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{E} \end{tikzcd}$$

must satisfy

$$id_I * \varphi = \alpha$$

If Z is an object in $\mathcal{Z}$, the equation $I(\varphi_z) = \alpha_z$ must hold. This gives only one possible definition $\varphi_z = \alpha_z$, and it is well-defined (i.e. it is a morphism in E) because

$$\begin{aligned} F(\alpha_Z) &= (id_F * \alpha)_Z \\ &= (id_G * \alpha)_Z \\ &= G(\alpha_Z) \end{aligned}$$

Naturality of φ is equivalent to naturality of α.

As before in proposition 5.3, this verification is unnecessary.

5.7. **Pseudoequalizer.**

We define the pseudoequalizer E as the conical pseudolimit of the same diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B$:

$$\mathcal{C}(Z,E) \cong [\{\bullet \rightrightarrows \bullet\}, \mathcal{C}at]_p(\Delta_1, \mathcal{C}(Z, F-))$$

Since the indexing category $\{\bullet \rightrightarrows \bullet\}$ has non-identity morphisms, we can't expect this new definition to be equivalent to that of the 2-equalizer, because the pseudocone condition is not vacuous in this case. In fact, a pseudoequalizer is given by a pseudocone, that is morphisms i and p and invertible 2-cells λ_1 and λ_2 as in

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ E \arrow[ur, "i"] \arrow[rr, "p"] \arrow[dr, "i"'] & \arrow[u, phantom, "\Downarrow\lambda_1\cong"] \arrow[d, phantom, "\Uparrow\lambda_2\cong"] & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

These 2-cells have no compatibility conditions since there are no non-trivial compositions in $\{\bullet \rightrightarrows \bullet\}$. The 1-dimensional universal property states that for every other such pseudocone

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ Z \arrow[ur, "h"] \arrow[rr, "q"] \arrow[dr, "h"'] & \arrow[u, phantom, "\Downarrow\varepsilon_1\cong"] \arrow[d, phantom, "\Uparrow\varepsilon_2\cong"] & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

there is a unique morphism $Z \xrightarrow{k} E$ such that the composition of k and the universal pseudocone $(i, p, \lambda_1, \lambda_2)$ gives this other pseudocone $(h, q, \varepsilon_1, \varepsilon_2)$:

$$\begin{aligned} ik &= h \\ pk &= q \\ \lambda_1 * id_k &= \varepsilon_1 \\ \lambda_2 * id_k &= \varepsilon_2 \end{aligned}$$

The 2-dimensional universal property says that given a third pseudocone

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ Z \arrow[ur, "h'"] \arrow[rr, "q'"] \arrow[dr, "h'"'] & \arrow[u, phantom, "\Downarrow\varepsilon'_1\cong"] \arrow[d, phantom, "\Uparrow\varepsilon'_2\cong"] & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

and the corresponding morphism $Z \xrightarrow{k'} E$, modifications

$$(h, q, \varepsilon_1, \varepsilon_2) \rightsquigarrow (h', q', \varepsilon'_1, \varepsilon'_2)$$

correspond bijectively to 2-cells $k \Rightarrow k'$. That is, given 2-cells $\psi_A : h \Rightarrow h'$ and $\psi_B : q \Rightarrow q'$ such that

$$\varepsilon_1' \circ (id_f * \psi_A) = \psi_B \circ \varepsilon_1$$
$$\varepsilon_2' \circ (id_g * \psi_A) = \psi_B \circ \varepsilon_2$$

there is a unique 2-cell

$$\begin{tikzcd} Z \arrow[r, bend left, "k"] \arrow[r, bend right, "k'"'] \arrow[r, phantom, "\Downarrow\varphi"] & E \end{tikzcd}$$

such that

$$id_i * \varphi = \psi_A$$
$$id_p * \varphi = \psi_B$$

Pseudoequalizers also exist in $\mathcal{C}at$, but are no longer given by the ordinary 1-categorical construction of the equalizer. We did not find a reference for the construction of pseudoequalizers in $\mathcal{C}at$.

Proposition 5.7. *Given functors and categories* $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ *, there exists a category* $\mathcal{E}$*, together with functors* $\mathcal{E} \xrightarrow{I} \mathcal{A}$ *,* $\mathcal{E} \xrightarrow{P} \mathcal{B}$ *and natural isomorphisms* $\lambda_1 : FI \Longrightarrow P$*,* $\lambda_2 : GI \Longrightarrow P$ *(that is, a pseudocone for the diagram* $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ *) that induces a natural isomorphism of categories between functors into* $\mathcal{E}$ *and pseudocones of this diagram by postcomposition*

$$I_* : \ [\mathcal{Z}, \mathcal{E}] \xrightarrow{\cong} PsCon(\mathcal{Z})$$

Proof. We propose a category $\mathcal{E}$ as a candidate for the pseudoequalizer, and prove that it satisfies the desired property.
An object of $\mathcal{E}$ is given by a tuple (A, B, γ, δ), with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\gamma\colon \ FA \xrightarrow{\cong} B$ and $\delta\colon \ GA \xrightarrow{\cong} B$ isomorphisms in $\mathcal{B}$. (In particular, we have $FA \cong GA$.)

An arrow $(A, B, \gamma, \delta) \xrightarrow{(\alpha,\beta)} (A', B', \gamma', \delta')$ is given by morphisms $\alpha\colon A \longrightarrow A'$, $\beta\colon B \longrightarrow B'$ commuting with $\gamma, \gamma', \delta, \delta'$:

$$\begin{tikzcd} FA \arrow[r, "\gamma"] \arrow[d, "F\alpha"'] \arrow[dr, phantom, "\equiv"] & B \arrow[d, "\beta"] \\ FA' \arrow[r, "\gamma'"'] & B' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} GA \arrow[r, "\delta"] \arrow[d, "G\alpha"'] \arrow[dr, phantom, "\equiv"] & B \arrow[d, "\beta"] \\ GA' \arrow[r, "\delta'"'] & B' \end{tikzcd}$$

The functor I maps $(A, B, \gamma, \delta) \mapsto A$ and $(\alpha, \beta) \mapsto \alpha$. The functor P maps $(A, B, \gamma, \delta) \mapsto B$ and $(\alpha, \beta) \mapsto \beta$. We define the natural transformations λ_1 and λ_2 by components: $\lambda_{1,(A,B,\gamma,\delta)} = \gamma$, and $\lambda_{2,(A,B,\gamma,\delta)} = \delta$.

It is straightforward to verify that these data define a category, functors and natural isomorphisms.

The functor $[\mathcal{Z},\mathcal{E}] \xrightarrow{I_*} PsCon(\mathcal{Z})$ is defined as

$$\begin{tikzcd} & & & \mathcal{A} \arrow[dr, "F"] & \\ \mathcal{Z} \arrow[r, "U"] & \mathcal{E} \arrow[r, phantom, "\mapsto"] & \mathcal{Z} \arrow[ur, "IU"] \arrow[rr, "PU"] \arrow[dr, "IU"'] \arrow[rr, phantom, shift left=4, "\Downarrow\lambda_1 * id_U \cong"] \arrow[rr, phantom, shift right=4, "\Uparrow\lambda_2 * id_U \cong"] & & \mathcal{B} \\ & & & \mathcal{A} \arrow[ur, "G"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{Z} \arrow[r, bend left, "U"] \arrow[r, bend right, "V"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathcal{E} \arrow[r, phantom, "\mapsto"] & \mathcal{Z} \arrow[r, bend left, "IU"] \arrow[r, bend right, "IV"'] \arrow[r, phantom, "\Downarrow id_I * \mu"] & \mathcal{A} \text{ and } \mathcal{Z} \arrow[r, bend left, "PU"] \arrow[r, bend right, "PV"'] \arrow[r, phantom, "\Downarrow id_P * \mu"] & \mathcal{B} \end{tikzcd}$$

To see that this functor I_* is an isomorphism of categories, we propose an inverse functor $PsCon(\mathcal{Z}) \xrightarrow{T} [\mathcal{Z},\mathcal{E}]$.

Given a pseudocone with vertex $\mathcal{Z}$ given by

$$\begin{tikzcd} & \mathcal{A} \arrow[dr, "F"] & \\ \mathcal{Z} \arrow[ur, "H"] \arrow[rr, "Q"] \arrow[dr, "H"'] \arrow[rr, phantom, shift left=4, "\Downarrow\varepsilon_1 \cong"] \arrow[rr, phantom, shift right=4, "\Uparrow\varepsilon_2 \cong"] & & \mathcal{B} \\ & \mathcal{A} \arrow[ur, "G"'] & \end{tikzcd}$$

we prescribe its image via T to be the functor $\mathcal{Z} \longrightarrow \mathcal{E}$ that sends

$$Z \mapsto (HZ, QZ, \varepsilon_{1,Z}, \varepsilon_{2,Z})$$

$$Z \xrightarrow{m} Z' \mapsto \left(HZ \xrightarrow{Hm} HZ' \ , \ QZ \xrightarrow{Qm} QZ' \right)$$

Given a second pseudocone on $\mathcal{Z}$ by the data H', Q', ε_1', ε_2' and a modification ψ between them, i.e. natural transformations $\psi_1 : H \Rightarrow H'$ and $\psi_2 : Q \Rightarrow Q'$ such that

$$\varepsilon_1' \circ (id_F * \psi_1) = \psi_2 \circ \varepsilon_1$$
$$\varepsilon_2' \circ (id_G * \psi_1) = \psi_2 \circ \varepsilon_2$$

we define $T(\psi)$ to be the natural transformation with components

$$T(\psi)_Z : T(H, Q, \varepsilon_1, \varepsilon_2) \Longrightarrow T(H', Q', \varepsilon_1', \varepsilon_2') : \mathcal{Z} \longrightarrow \mathcal{E}$$

$$T(\psi)_Z = \left(HZ \xrightarrow{\psi_{1,Z}} H'Z \ , \ QZ \xrightarrow{\psi_{2,Z}} Q'Z \right)$$

We leave to the reader the routine verifications that all the data for T is well-defined and gives an honest functor, and that T and I_* compose to the identity functors in both directions. $\square$

5.8. **Biequalizer.**

Definition 5.8. (biequalizer) The biequalizer of two parallel morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in $\mathcal{C}$ is a pseudocone with vertex E for this diagram that induces an equivalence of categories by postcomposition

$$\mathcal{C}(Z, E) \simeq PsCon(Z)$$

between the hom-category $\mathcal{C}(Z, E)$ and the category of pseudocones with vertex Z, for all $Z \in \mathcal{C}$ (see subsection 5.7 where these pseudocones are described).

Observation 5.9. The same observations in 5.5 apply in this case, because $\mathcal{C}at$ admits pseudoequalizers (proposition 5.7).

5.9. **Cotensor.**

Recall from definition 4.3 that the weighted 2-limit of a 2-functor $\mathcal{B} \xrightarrow{F} \mathcal{C}$ weighted by $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ is given by the formula

$$\mathcal{C}(Z, \underleftarrow{\text{wlim}}^W F) \cong [\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(Z, F-))$$

When $\mathcal{B}$ is the terminal 2-category $\{\bullet\}$ (with one object, the identity morphism and the identity 2-cell), we can identify the 2-functors F and W with their image:

$$F = F(\bullet) = C$$
$$W = W(\bullet) = \mathcal{A}$$

In this way, F is an object of $\mathcal{C}$ and W is a category, and the representing formula is simplified:

$$\begin{aligned}\mathcal{C}(Z, \underleftarrow{\text{wlim}}^W F) &\cong [\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(Z, F-)) \\ &\cong [\{\bullet\}, \mathcal{C}at](W, \mathcal{C}(Z, F-)) \\ &\cong \mathcal{C}at(W, \mathcal{C}(Z, F)) \\ &\cong \mathcal{C}at(\mathcal{A}, \mathcal{C}(Z, C))\end{aligned}$$

We call this type of limit a cotensor product, and denote it $\{\mathcal{A}, C\}$.

Definition 5.10. Given an object C in a 2-category $\mathcal{C}$ and a category $\mathcal{A}$, the cotensor product of $\mathcal{A}$ and C is defined as the 2-representation of the functor

$$Z \mapsto \mathcal{C}at(\mathcal{A}, \mathcal{C}(Z, C))$$

The cotensor product is also called power. We had denoted the cotensor product as $\{\mathcal{A}, C\}$. In the literature the notations $\mathcal{A} \pitchfork C$ or $C^{\mathcal{A}}$ can also be found. We then have

$$\mathcal{C}(Z, \{\mathcal{A}, C\}) \cong \mathcal{C}at(\mathcal{A}, \mathcal{C}(Z, C))$$

We can unfold the definition to obtain an explicit description of the cotensor product: the cotensor product of a category $\mathcal{A}$ and an object C of a 2-category $\mathcal{C}$ is given by morphisms and 2-cells

$$\begin{tikzcd} \{\mathcal{A}, C\} \arrow[d, bend right=50, "\xi_A"'] \arrow[d, bend left=50, "\xi_{A'}"] \arrow[d, phantom, "\overset{\xi_f}{\Longrightarrow}"] \\ C \end{tikzcd}$$

for each $A \xrightarrow{f} A'$ in $\mathcal{A}$, in a functorial way: whenever we have morphisms $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{A}$, we have

$$\begin{tikzcd} \{\mathcal{A}, C\} \arrow[d, bend right=70, "\xi_A"'] \arrow[d, "\xi'_A" description] \arrow[d, bend left=70, "\xi_{A''}"] \\ C \end{tikzcd} \quad \xi_f: \Longrightarrow,\ \xi_g: \Longrightarrow \quad = \quad \begin{tikzcd} \{\mathcal{A}, C\} \arrow[d, bend right=50, "\xi_A"'] \arrow[d, bend left=50, "\xi_{A''}"] \arrow[d, phantom, "\overset{\xi_{gf}}{\Longrightarrow}"] \\ C \end{tikzcd}$$

i.e. a functor

$$\mathcal{A} \xrightarrow{\xi} \mathcal{C}(\{\mathcal{A}, C\}, C)$$

such that

- (1-dimensional universal property) any other such functor

$$\mathcal{A} \xrightarrow{\eta} \mathcal{C}(Z, C)$$

  factorizes uniquely through ξ: there exists a unique $Z \xrightarrow{h} \{\mathcal{A}, C\}$ such that for all a in $\mathcal{A}$

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\equiv"] & \{\mathcal{A}, C\} \arrow[dl, "\xi_A"] \\ C \end{tikzcd}$$

  and that for all $A \xrightarrow{f} A'$ in $\mathcal{A}$

$$\begin{tikzcd} Z \arrow[r, bend left, "\eta_A"] \arrow[r, bend right, "\eta_{A'}"'] \arrow[r, phantom, "\Downarrow\eta_f"] & C \end{tikzcd} = \begin{tikzcd} Z \arrow[r, "h"] & \{\mathcal{A}, C\} \arrow[r, bend left, "\xi_A"] \arrow[r, bend right, "\xi_{A'}"'] \arrow[r, phantom, "\Downarrow\xi_f"] & C \end{tikzcd}$$

- (2-dimensional universal property) given two functors

$$\mathcal{A} \underset{\theta}{\overset{\eta}{\rightrightarrows}} \mathcal{C}(Z, C)$$

that factorize through ξ as

$$h^* \circ \xi = \eta$$
$$k^* \circ \xi = \theta$$

and a collection of 2-cells $\xi_A * id_h \overset{\beta_A}{\Longrightarrow} \xi_A * id_k$ indexed by the objects of $\mathcal{A}$, which is natural in A, i.e. for all $A \xrightarrow{f} A'$ in $\mathcal{A}$

$$(\xi_f * id_k) \circ \beta_A = \beta_{A'} \circ (\xi_f * id_k)$$

this β corresponds to a unique 2-cell

$$h \overset{\alpha}{\Longrightarrow} k$$

in the sense that for all objects A of $\mathcal{A}$ we have

$$id_{\xi_A} * \alpha = \beta_A$$

We observe that these diagrams no longer have the shape of a cone: there may be many morphisms from the vertex to a single object of the diagram (in this case, the many $\{\mathcal{A}, C\} \xrightarrow{\xi_A} C$), and 2-cells between these. This is unlike the limits we considered before (products and equalizers).

The definition 5.10 also works for $\mathcal{C}$ a $\mathcal{V}$-category. In that case, $\mathcal{A}$ is any object of $\mathcal{V}$ and the representation is given by a $\mathcal{V}$-natural transformation. In particular, we can consider the 1-dimensional case with $\mathcal{V} = \mathcal{E}ns$.

Example 5.11. Given a set S and an object C of a category $\mathcal{C}$ we have

$$\begin{aligned}\mathcal{C}(Z, \prod_S C) &\cong \prod_S \mathcal{C}(Z, C)\\ &\cong \prod_S \mathcal{E}ns(\{\bullet\}, \mathcal{C}(Z, C)\\ &\cong \mathcal{E}ns(\coprod_S \{\bullet\}, \mathcal{C}(Z, C))\\ &\cong \mathcal{E}ns(S, \mathcal{C}(Z, C))\end{aligned}$$

whenever the product

$$\prod_S C$$

exists. Thus, the cotensor product in this case is the iterated product (a product with all factors the same object), which justifies the notation C^S. When $\mathcal{C}$ is $\mathcal{E}ns$, for example, C^S is the set of functions $S \longrightarrow C$.

Example 5.12. The 2-category $\mathcal{C}at$ also has cotensor products: given categories $\mathcal{C}$ and $\mathcal{D}$, their cotensor product is given by the exponential

$$\{\mathcal{C}, \mathcal{D}\} = \mathcal{C}at(\mathcal{C}, \mathcal{D}) = [\mathcal{C}, \mathcal{D}] = \mathcal{D}^{\mathcal{C}}$$

This can be seen as a particular case of the general formula for 2-limits in $\mathcal{C}at$:

$$\underleftarrow{\text{wlim}}^W F = [\mathcal{B}, \mathcal{C}at](W, F)$$

which we prove in proposition 6.1. When $\mathcal{B} = \{\bullet\}$, we identify functors W and F with their images $\mathcal{C}$ and $\mathcal{D}$ respectively. We have

$$\{\mathcal{C}, \mathcal{D}\} \cong [\{\bullet\}, \mathcal{C}at](\mathcal{C}, \mathcal{D}) \cong \mathcal{C}at(\mathcal{C}, \mathcal{D}) = \mathcal{D}^{\mathcal{C}}$$

We remark that this cotensor product is not the same as $\prod\limits_{|\mathcal{D}|} \mathcal{C}$ (where $|\mathcal{D}|$ denotes the set of objects of $\mathcal{D}$), unless $\mathcal{D}$ is discrete.

Dualizing these definitions we obtain the tensor product:

Definition 5.13. Given a category $\mathcal{A}$ and an object C in a 2-category $\mathcal{C}$, the tensor product of $\mathcal{A}$ and C is defined as the 2-representation of the functor

$$Z \mapsto \mathcal{C}at(\mathcal{A}, \mathcal{C}(C, Z))$$

The tensor product is also called copower, and we denote it by $\mathcal{A} \otimes C$ or $\mathcal{A} * C$, and then

$$\mathcal{A}(\mathcal{A} \otimes C, Z) \cong \mathcal{C}at(\mathcal{A}, \mathcal{C}(C, Z))$$

Example 5.14. Making the necessary modifications, we get the definition of the tensor product of a set S and an object C of a 1-category $\mathcal{C}$. With the same reasoning as above, we have

$$\mathcal{C}(\coprod_S C, Z) \cong \prod_S \mathcal{C}(C, Z) \cong \mathcal{E}ns(S, \mathcal{C}(C, Z))$$

if the coproduct

$$\coprod_S C$$

exists. In this case, the tensor product is given by

$$S \otimes C = \coprod_S C$$

In $\mathcal{E}ns$, we also have

$$S \otimes C = \coprod_S C \cong S \times C$$

Example 5.15. It can be easily checked that the tensor product of two categories $\mathcal{C}$ and $\mathcal{D}$ (that is, the tensor product in the 2-category $\mathcal{C}at$) is given by the usual product of categories

$$\mathcal{C} \otimes \mathcal{D} = \mathcal{C} \times \mathcal{D}$$

Example 5.16. The most familiar example of tensor product (and where the name comes from) is given by rings R, S, a right R-module M, an (R, S)-bimodule N and a right S-module P: we have an isomorphism of abelian groups

$$\mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P))$$

We observe that this is the same formula that defines the tensor product in general. Choosing $R = \mathbb{Z}$, we get

$$\mathrm{Mod}_S(M \otimes_{\mathbb{Z}} N, P) \cong \mathrm{Ab}(M, \mathrm{Mod}_S(N, P))$$

Since Mod_S (the category of S-modules) is an Ab-enriched category (where Ab is the category of abelian groups), and M is a $\mathbb{Z}$-module (an abelian group), this coincides with the definition of a tensor product in a $\mathcal{V}$-category, for $\mathcal{V} = \mathrm{Ab}$.

5.10. **Pseudocotensor.**

Recall that the cotensor $\{\mathcal{A}, C\}$ is given by the formula (see subsection 5.9)

$$\mathcal{C}(Z, \{\mathcal{A}, C\}) \cong [\{\bullet\}, \mathcal{C}at](W, \mathcal{C}(Z, F-))$$

where W and F are the 2-functors sending the object of the terminal category $\{\bullet\}$ to $\mathcal{A}$ and C, respectively.

If we want to obtain a pseudolimit version of this particular weighted 2-limit, we make the usual adaptations, and define the pseudocotensor $\{\mathcal{A}, C\}_p$ by the formula

$$\mathcal{C}(Z, \{\mathcal{A}, C\}_p) \cong [\{\bullet\}, \mathcal{C}at]_p(W, \mathcal{C}(Z, F-))$$

with the same W and F. However, since $\{\bullet\}$ has no non-trivial morphisms or 2-cells, a 2-natural transformation $W \Rightarrow \mathcal{C}(Z, F-)$ consists of a functor $\mathcal{A} \to \mathcal{C}(Z, C)$ satisfying no naturality conditions, and this is the same situation in the case of a pseudonatural transformation. This gives

$$[\{\bullet\}, \mathcal{C}at](W, \mathcal{C}(Z, F-)) = [\{\bullet\}, \mathcal{C}at]_p(W, \mathcal{C}(Z, F-))$$

Observation 5.17. This means that the cotensor product and the pseudocotensor product are identical notions. This situation is analogous to that of the 2-product and the pseudoproduct, which is described in subsection 5.3.

5.11. **Bicotensor.**

Definition 5.18. Given a category $\mathcal{A}$ and an object C in a 2-category $\mathcal{C}$, the bicotensor $\{\mathcal{A}, C\}_b$ is given by a birepresentation

$$\mathcal{C}(Z, \{\mathcal{A}, C\}_b) \simeq \mathcal{C}at(\mathcal{A}, \mathcal{C}(Z, C))$$

(compare with definition 5.10).

Observation 5.19. The same observations in 5.5 apply in this case, because $\mathcal{C}at$ admits cotensors (these are the same as pseudocotensors, by observation 5.17). The fact that $\mathcal{C}at$ admits cotensor products is given in 5.12.

5.12. **Inserter and iso-inserter.**

Definition 5.20. When $\mathcal{B}$ is the category $\{\bullet \rightrightarrows \bullet\}$, the weight $\{\bullet \rightrightarrows \bullet\} \xrightarrow{W} \mathcal{C}at$ has image

$$\begin{tikzcd} 1 \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & 2 \end{tikzcd}$$

(here $1 = \{\bullet\}$, $2 = \{0 \to 1\}$, and the functors 0 and 1 are the inclusions sending $\bullet$ in 1 to the objects 0 and 1, respectively) and the functor $\{\bullet \rightrightarrows \bullet\} \xrightarrow{F} \mathcal{C}$ has image

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

the 2-limit $\underset{\longleftarrow}{\mathrm{wlim}}^W F$ is called the inserter of f and g.

Unraveling the definitions and skipping redundant information, the inserter is an object $\underset{\longleftarrow}{\mathrm{wlim}}^W F$ together with a morphism p and a 2-cell λ

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \underset{\longleftarrow}{\mathrm{wlim}}^W F \arrow[ur, "p"] \arrow[dr, "p"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

and it is the universal diagram of this form:

- (1-dimensional universal property) for any other diagram

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ Z \arrow[ur, "q"] \arrow[dr, "q"'] \arrow[rr, phantom, "\Downarrow\mu"] & & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

  there exists a unique h such that

$$\begin{tikzcd} & & A \\ Z \arrow[urr, bend left, "q"] \arrow[drr, bend right, "q"'] \arrow[r, dashed, "h"] & \underset{\longleftarrow}{\mathrm{wlim}}^W F \arrow[ur, "p"] \arrow[dr, "p"'] \arrow[u, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] & \\ & & A \end{tikzcd}$$

  and $\lambda * id_h = \mu$.

- (2-dimensional universal property) given a second diagram

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ Z \arrow[ur, "q'"] \arrow[dr, "q'"'] \arrow[rr, phantom, "\Downarrow\mu'"] & & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

factorizing through $\underset{\longleftarrow}{\mathrm{wlim}}^W F$ via $A \overset{\exists! k}{\dashrightarrow} \underset{\longleftarrow}{\mathrm{wlim}}^W F$, and a 2-cell $q \overset{\beta}{\Longrightarrow} q'$ such that

$$\mu' \circ (id_f * \beta) = (id_g * \beta) \circ \mu$$

there exists a unique 2-cell $h \overset{\alpha}{\Longrightarrow} k$ such that

$$\beta = id_p * \alpha$$

We can replace the category 2 in definition 5.20 by the category $\mathcal{I}$, which has two objects 0, 1 and an isomorphism between them, to obtain the notion of iso-inserter.

Definition 5.21. When $\mathcal{B}$ is the category $\{\bullet \rightrightarrows \bullet\}$, the weight $\{\bullet \rightrightarrows \bullet\} \xrightarrow{W} \mathcal{C}at$ has image

$$1 \underset{1}{\overset{0}{\rightrightarrows}} \mathcal{I}$$

(here $1 = \{\bullet\}$ and the functors 0 and 1 are defined as before; see definition 5.20) and the functor $\{\bullet \rightrightarrows \bullet\} \xrightarrow{F} \mathcal{C}$ has image

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

the 2-limit $\underset{\longleftarrow}{\mathrm{wlim}}^W F$ is called the iso-inserter of f and g.

The elementary description of this 2-limit is very similar to that of the inserter of f and g (see discussion following definition 5.20). The 2-cell λ is invertible, the 1-dimensional property holds for invertible 2-cells μ, and the 2-dimensional property remains as stated.

Observation 5.22. Inserters and iso-inserters exist in the 2-category $\mathcal{C}at$. The iso-inserter of functors and categories

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$$

is given by a category $\mathcal{E}$ with objects pairs (A, ρ) with $FA \xrightarrow{\rho \cong} GA$ and morphisms $(A, \rho) \to (A', \rho')$ given by morphisms $A \xrightarrow{\alpha} A'$ such that the obvious diagram commutes

$$\begin{tikzcd} FA \arrow[r, "\rho\cong"] \arrow[d, "F\alpha"'] \arrow[dr, phantom, "\equiv"] & GA \arrow[d, "G\alpha"] \\ FA' \arrow[r, "\rho'\cong"'] & GA' \end{tikzcd}$$

The universal functor $\mathcal{E} \xrightarrow{P} \mathcal{A}$ is given by projection on the first variable and the universal invertible natural transformation $FP \xRightarrow{\lambda\cong} FQ$ is given on components by projection on the second variable. The relevant universal properties are easy to verify.

The inserter of the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{B} \end{tikzcd}$$

can be constructed in the same way, with objects given by pairs (A, ρ) with $FA \xrightarrow{\rho} GA$, where ρ is not necessarily invertible.

Observation 5.23. In the description of the pseudoequalizer of functors and categories $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ given in 5.7, we might be tempted to compose isomorphisms $FA \xrightarrow{\gamma} B$ and $GA \xrightarrow{\delta} B$ to obtain an isomorphism $FA \xrightarrow{\delta^{-1}\gamma} GA$. An easy verification shows that this extends to a functor from the pseudoequalizer into the iso-inserter of F and G, which is in fact an equivalence, but not an isomorphism. (The quasi-inverse maps objects $(A, \rho) \mapsto (A, GA, \rho, id_{GA})$ and morphisms $\alpha \mapsto (\alpha, G\alpha)$.) Since limits are defined representably (see 6.3), this implies that in any 2-category, if both the iso-inserter and the pseudoequalizer of a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

exist, then they are equivalent.

For example, the pseudocoequalizer of geometric morphisms of toposes is given in [24, lemma 1.10] using the construction of the iso-inserter in definition 5.21.

5.13. **Comma-object.**

Definition 5.24. When $\mathcal{B}$ is the category $\{\bullet \to \bullet \leftarrow \bullet\}$, the weight $\{\bullet \to \bullet \leftarrow \bullet\} \xrightarrow{W} \mathcal{C}at$ has image

$$1 \xrightarrow{0} 2 \xleftarrow{1} 1$$

(here $1 = \{\bullet\}$, $2 = \{0 \to 1\}$, and the functors 0 and 1 are the inclusions sending $\bullet$ in 1 to the objects 0 and 1, respectively) and the functor $\{\bullet \to \bullet \leftarrow \bullet\} \xrightarrow{F} \mathcal{C}$ has image

$$B \xrightarrow{f} D \xleftarrow{g} C$$

the limit $\underleftarrow{\mathrm{wlim}}^W F$ is called the comma-object of f and g.

This is given by an object $\underleftarrow{\mathrm{wlim}}^W F$ with morphisms u and v and a 2-cell λ as in

$$\begin{tikzcd}
\underleftarrow{\mathrm{wlim}}^W F \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow \lambda"] & C \arrow[d, "g"] \\
B \arrow[r, "f"'] & D
\end{tikzcd}$$

such that

- (1-dimensional universal property) for any other diagram

$$\begin{tikzcd}
Z \arrow[r, "y"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow \mu"] & C \arrow[d, "g"] \\
B \arrow[r, "f"'] & D
\end{tikzcd}$$

  there exists a unique $Z \overset{\exists! h}{\dashrightarrow} \underleftarrow{\mathrm{wlim}}^W F$ such that

$$uh = x$$
$$vh = y$$
$$\lambda * id_h = \mu$$

- (2-dimensional universal property) given a second diagram

$$\begin{tikzcd}
Z \arrow[r, "y'"] \arrow[d, "x'"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow \mu'"] & C \arrow[d, "g"] \\
B \arrow[r, "f"'] & D
\end{tikzcd}$$

  factorizing through the limit via $Z \overset{\exists! k}{\dashrightarrow} \underleftarrow{\mathrm{wlim}}^W F$, and 2-cells $x \overset{\beta}{\Longrightarrow} x'$, $y \overset{\gamma}{\Longrightarrow} y'$ such that

$$\mu' \circ (id_f * \alpha) = (\beta * id_g) \circ \mu$$

there exists a unique 2-cell $h \xRightarrow{\alpha} k$ such that

$$id_u * \alpha = \beta$$
$$id_v * \alpha = \gamma$$

Example 5.25. The 2-category $\mathcal{C}at$ has comma-objects, and these are indeed the usual comma categories: if we have a span in $\mathcal{C}at$ (categories and functors)

$$\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$$

the comma category $F \downarrow G$ is defined as follows:

- an object is given by a triple (A, B, f) with A an object in $\mathcal{A}$, B an object in $\mathcal{B}$ and $FA \xrightarrow{f} GB$ a morphism in $\mathcal{C}$
- a morphism $(A, B, f) \to (C, D, g)$ is given by a pair (p, q) of morphisms $A \xrightarrow{p} C$ in $\mathcal{A}$ and $B \xrightarrow{q} D$ in $\mathcal{B}$ making the obvious diagram commute:

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "Fp"'] \arrow[dr, phantom, "\equiv"] & GB \arrow[d, "Gq"] \\ FC \arrow[r, "g"'] & GD \end{tikzcd}$$

The comma-object diagram in this case is

$$\begin{tikzcd} F \downarrow G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\Nearrow \lambda"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

where π_1 and π_2 are the projections on the first and second component, and λ is defined by $\lambda_{(A,B,f)} = f$.

A very useful particular case of the comma category is the slice category $\mathcal{C}/X$ over an object X of $\mathcal{C}$, when F is the identity functor on $\mathcal{A}$ (thus $\mathcal{A} = \mathcal{C}$), $\mathcal{B}$ is the terminal category $\{\bullet\}$ and $G(\bullet) = X$.

5.14. **Biequifier and biidentifier.**

The following definitions can be found in [28]. 2-limit versions of these can be found in [15].

Definition 5.26. (biequifier) Given 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha\,\Downarrow\beta"] & B \end{tikzcd}$$

in a 2-category $\mathcal{C}$, their biequifier is an arrow

$$E \xrightarrow{h} A$$

inducing an equivalence of categories between $\mathcal{C}(Z, E)$ and the full subcategory of $\mathcal{C}(Z, A)$ with objects those $Z \xrightarrow{z} A$ such that $\alpha * id_z = \beta * id_z$, for all $Z \in \mathcal{C}$. This is the bilimit of F weighted by W where the indexing category is

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \bullet \end{tikzcd}$$

and the images of W and F are respectively

$$\begin{tikzcd} 1 \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & 2 \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha\ \Downarrow\beta"] & B \end{tikzcd}$$

Definition 5.27. (biidentifier) The biidentifier of an endo-2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

in $\mathcal{C}$ is the biequifier of α and id_f.

5.15. **Descent object.**

The definitions of a truncated bicosimplicial diagram in a 2-category, and of its descent object can be found in [28].

Definition 5.28. A truncated bicosimplicial diagram X in a 2-category $\mathcal{C}$ is a diagram

$$\begin{tikzcd} X_0 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & X_1 \arrow[l, "\iota" description] \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=2, "\delta_2"'] & X_2 \end{tikzcd}$$

with invertible 2-cells

$$\delta_i \delta_{j-1} \xRightarrow{\sigma_{i,j} \cong} \delta_j \delta_i$$

for $i < j$, and for all i

$$id_{X_0} \xRightarrow{\mu_i \cong} \iota \delta_i$$

Definition 5.29. The descent object of a truncated bicosimplicial diagram $\mathcal{X}$ in $\mathcal{C}at$ is a category $\mathrm{Desc}(\mathcal{X})$ with objects pairs (X, θ) with X an object of $\mathcal{X}_0$ and $\delta_0(X) \xrightarrow{\theta \cong} \delta_1(X)$ an isomorphism in $\mathcal{X}_1$ such that

$$\iota(\theta) = \mu_{1,X} \mu_{0,X}^{-1} \quad \text{and} \quad \sigma_{1,2,X} \circ \delta_1(\theta) \circ \sigma_{0,1,X} = \delta_2(\theta) \circ \sigma_{0,2,X} \circ \delta_0(\theta)$$

and arrows $(X, \theta) \xrightarrow{\chi} (X', \theta')$ are morphisms $X \xrightarrow{\chi} X'$ in $\mathcal{X}_0$ such that

$$\theta' \circ \delta_0(\chi) = \delta_1(\chi) \circ \theta$$

Notice that there is a functor $\mathrm{Desc}(\mathcal{X}) \xrightarrow{P} \mathcal{X}_0$ given by projection on the first variable. The category Desc is a kind of iso-inserter or pseudoequalizer (see observations 5.22 and 5.23).

In a general 2-category $\mathcal{C}$, the descent object of a truncated bicosimplicial diagram X as before is an arrow $D \xrightarrow{h} X_0$ and an invertible 2-cell $\delta_0 * id_h \xRightarrow{\omega \cong} \delta_1 * id_h$ such that for all objects $Z \in \mathcal{C}$ these induce an equivalence of categories between $\mathcal{C}(Z, D)$ and the descent object $\mathrm{Desc}(\mathcal{C}(Z, X))$ (where $\mathcal{C}(Z, X)$ is a truncated bicosimplicial diagram in $\mathcal{C}at$): the descent object of a truncated bicosimplicial diagram is defined representably (compare with equation 6.3)

$$\mathcal{C}(Z, \mathrm{Desc}(X)) \simeq \mathrm{Desc}(\mathcal{C}(Z, X))$$

Observation 5.30. Notice that the descent object of a truncated bicosimplicial diagram is a weighted bilimit (see [28]).

5.16. **2-filtered pseudocolimit of categories.**

In this subsection, we recall the construction of a pseudocolimit in $\mathcal{C}at$ when the diagram is 2-filtered, given in [10].

Definition 5.31. Given a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{C}at$ with $\mathcal{A}$ 2-filtered, the category $\mathcal{L}(F)$ has objects and morphisms given as follows

- an object is a pair (x, A) with x an object of FA
- a premorphism $(x, A) \xrightarrow{(u,\xi,v)} (y, B)$ is given by arrows $A \xrightarrow{u} C$, $B \xrightarrow{v} C$ and $(Fu)(x) \xrightarrow{\xi} (Fv)(y)$ in FC
- two premorphisms $(x, A) \xrightarrow{(u_1,\xi_1,v_1)} (y, B)$ and $(x, A) \xrightarrow{(u_2,\xi_2,v_2)} (y, B)$ (between the same objects) are equivalent if there is an homotopy between them, i.e. invertible 2-cells α, β such that the following diagram commutes in FC

$$\begin{tikzcd}
F(w_1u_1)(x) \arrow[r, "(F\beta)_x"] \arrow[d, "(Fw_1)(\xi_1)"'] \arrow[dr, phantom, "\equiv"] & F(w_2u_w)(x) \arrow[d, "(Fw_2)(\xi_2)"] \\
F(w_1v_1)(y) \arrow[r, "(F\alpha)_y"'] & F(w_2v_2)(y)
\end{tikzcd}$$

- a morphism $(x, A) \longrightarrow (y, B)$ is an equivalence class of premorphisms $(x, A) \longrightarrow (y, B)$

We will speak of morphisms and premorphisms interchangeably, as it will be clear from context whether we are talking about an arrow in $\mathcal{L}(F)$ or a premorphism that represents it.

Using the 2-Yoneda embedding $h : \mathcal{A} \longrightarrow [\mathcal{A}, \mathcal{C}at]^{op}$ (theorem 2.34), we have

$$\begin{array}{c} x \in FA \\ \hline [A, -] \xrightarrow{x} F \text{ in } [\mathcal{A}, \mathcal{C}at] \\ \hline F \xrightarrow{x} [A, -] \text{ in } [\mathcal{A}, \mathcal{C}at]^{op} \end{array}$$

Finally we can make the abuse of notation of writing A for the representable functor $[A, -]$, and then an object of $\mathcal{L}(F)$ is just $F \xrightarrow{x} A$.

Analogously, we can write

$$\begin{tikzcd}
& A \arrow[dr, "u"] & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\Downarrow\xi"] & & C \\
& B \arrow[ur, "v"'] &
\end{tikzcd}$$

for a premorphism, and the equivalence relation between two premorphisms is given by the LL equation

$$\begin{tikzcd}
& A \arrow[dr, "u_1"] & & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\Downarrow\xi_1"] & & C_1 \arrow[dr, "w_1"] & \\
& B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha\cong"] & & C \\
& & C_2 \arrow[ur, "w_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
& & C_1 \arrow[dr, "w_1"] & \\
& A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta\cong"] & & C \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\Downarrow\xi_2"] & & C_2 \arrow[ur, "w_2"'] & \\
& B \arrow[ur, "v_2"'] & &
\end{tikzcd}$$

This category is a useful construction of the pseudocolimit of a 2-filtered diagram in $\mathcal{C}at$. The proof of the following theorem can be found in [10].

Theorem 5.32. *Given a 2-functor* $\mathcal{A} \xrightarrow{F} \mathcal{C}at$ *with* $\mathcal{A}$ *2-filtered, we have a pseudocone* $F \xrightarrow{\lambda} \mathcal{L}(F)$ *given by the formulas*

$$\lambda_A(x) = F \xrightarrow{x} A\ , \quad \lambda_A(\xi) = \begin{tikzcd}
& A \arrow[dr, "id"] & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\Downarrow\xi"] & & A \\
& A \arrow[ur, "id"'] &
\end{tikzcd}\ , \quad \lambda_u(x) = \begin{tikzcd}
& A \arrow[dr, "id"] & \\
F \arrow[ur, "ux"] \arrow[dr, "x"'] \arrow[rr, phantom, "\Downarrow id"] & & A \\
& A \arrow[ur, "u"'] &
\end{tikzcd}$$

for $A \xrightarrow{u} B$ *in* $\mathcal{A}$ *and* $x \xrightarrow{\xi} y$ *in* FA.
This pseudocone is universal in the sense that it induces an isomorphism of categories by precomposition

$$[\mathcal{L}(F), \mathcal{Z}] \xrightarrow{\lambda^* \cong} PsCon(\mathcal{Z})$$

between the category of functors and natural transformations $\mathcal{L}(F) \longrightarrow \mathcal{Z}$ *and the category of pseudocones and modifications for the diagram* F *with vertex* $\mathcal{Z}$*, for all categories* $\mathcal{Z}$. □

Observation 5.33. We remark that, like every pseudocolimit, this construction is also a bicolimit (see observation 4.8).

We state a lemma from [10] which is useful to prove the main result.

Lemma 5.34. *Given a finite family of 2-cells*
$$\begin{tikzcd} & A \arrow[dr, "u_i"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\gamma_i"] & & C_i \\ & B \arrow[ur, "v_i"'] & \end{tikzcd}$$
with $i = 1, \ldots, n$*, there exist morphisms* $A \xrightarrow{u} C$, $B \xrightarrow{v} C$, $C_i \xrightarrow{w_i} C$ *and invertible 2-cells* α_i*,* β_i*, for* $i = 1, \ldots, n$ *such that the 2-cells*

$$\begin{tikzcd} & A \arrow[dr, "u_i"] \arrow[drr, bend left, "u"] \arrow[dr, phantom, shift left=6, "\Downarrow\beta_i\cong"] & & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\gamma_i"] & & C_i \arrow[r, "w_i"] & C \\ & B \arrow[ur, "v_i"'] \arrow[urr, bend right, "v"'] \arrow[ur, phantom, shift right=6, "\Downarrow\alpha_i\cong"] & & \end{tikzcd}$$

are all equal for $i = 1, \ldots, n$*. Given a second family of 2-cells*
$$\begin{tikzcd} & A \arrow[dr, "u_i"] & \\ H \arrow[ur, "h"] \arrow[dr, "l"'] \arrow[rr, phantom, "\Downarrow\delta_i"] & & C_i \\ & B \arrow[ur, "v_i"'] & \end{tikzcd}$$
,
we can assume the same u*,* v*,* w_i*,* α_i *and* β_i *also equalize the 2-cells of the second family.* □

Observation 5.35. A 2-natural transformation $\varepsilon : F \Longrightarrow G$ induces a functor $\varepsilon : \mathcal{L}(F) \longrightarrow \mathcal{L}(G)$ by the universal property of $\mathcal{L}(F)$ applied to the pseudocone $F \overset{\varepsilon}{\Longrightarrow} G \overset{\rho_G}{\Longrightarrow} \mathcal{L}(G)$ (where ρ_G is the universal pseudocone for G). This functor is defined on objects and arrows of $\mathcal{L}(F)$ as follows

$$F \xrightarrow{a} A \;\mapsto\; G \xrightarrow{\varepsilon_A(a)} A$$

$$\begin{tikzcd}
& A \arrow[dr, "u"] & \\
F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi"] & & C \\
& A' \arrow[ur, "v"'] &
\end{tikzcd}
\;\mapsto\;
\begin{tikzcd}
& A \arrow[dr, "u"] & \\
G \arrow[ur, "\varepsilon_A(a)"] \arrow[dr, "\varepsilon_{A'}(a')"'] \arrow[rr, phantom, "\Downarrow\varepsilon_C(\xi)"] & & C \\
& A' \arrow[ur, "v"'] &
\end{tikzcd}$$

6. Constructions of all limits in terms of simpler limits

6.1. **Construction of weighted 2-limits.**

In $\mathcal{C}at$, all (small) weighted 2-limits exist:

Proposition 6.1. *Given 2-functors* $\mathcal{B} \underset{F}{\overset{W}{\rightrightarrows}} \mathcal{C}at$ *, we have*

$$\underset{\longleftarrow}{\mathrm{wlim}}^W F = [\mathcal{B}, \mathcal{C}at](W, F)$$

That is, the W-weighted 2-limit of F is the category of 2-natural transformations and modifications between W and F.

Proof. We want to prove that (see definition 4.3)

$$\mathcal{C}at(\mathcal{Z}, [\mathcal{B}, \mathcal{C}at](W, F)) \cong [\mathcal{B}, \mathcal{C}at](W, \mathcal{C}at(\mathcal{Z}, F-))$$

for $\mathcal{Z} \in \mathcal{C}at$, with unit the obvious 2-natural transformation

$$W \overset{\xi}{\Longrightarrow} \mathcal{C}at([\mathcal{B}, \mathcal{C}at](W, F), F-)$$

Given a 2-natural transformation $F \overset{\rho}{\Longrightarrow} \mathcal{C}at(\mathcal{Z}, F-)$, we have to see that there exists a unique functor $\mathcal{Z} \xrightarrow{H} [\mathcal{B}, \mathcal{C}at](W, F)$ such that $\rho = H^* \circ \xi$. This condition implies that we must define h as

$$\begin{aligned} H(Z)_B(X) &= \rho_B(X)(A) \\ H(Z)_B(f) &= (\rho_B(f))_A \\ H(g)_{B,X} &= \rho_B(X)(g) \end{aligned}$$

if $Z \xrightarrow{g} Z'$ in $\mathcal{Z}$, $B \in \mathcal{B}$ and $f : X \to X'$ is an arrow in WB. This gives uniqueness, and it's easy to verify functoriality. For the 2-dimensional universal property, we take a second $F \overset{\rho'}{\Longrightarrow} \mathcal{C}at(\mathcal{Z}, F-)$ with corresponding $Z \xrightarrow{H'} [\mathcal{B}, \mathcal{C}at](W, F)$ and a modification $\theta : \rho \rightsquigarrow \rho'$. We have to show that there exists a unique natural transformation $H \overset{\alpha}{\Longrightarrow} H'$ such that $\theta = \alpha^* \circ \xi$. We are forced to define α by components as:

$$\alpha_{Z,B,X} = \theta_{B,X,Z}$$

where $B \in \mathcal{B}$, $Z \in \mathcal{Z}$ and $X \in WB$. It's straightforward to verify that this α is well-defined and natural. □

Observation 6.2. This allows us to rewrite the definition of a weighted 2-limit in an arbitrary 2-category by the formula

$$\mathcal{C}(Z, \underset{\longleftarrow}{\mathrm{wlim}}^W F) \cong \underset{\longleftarrow}{\mathrm{wlim}}^W \mathcal{C}(Z, F-) \tag{6.3}$$

Thus, the notion of 2-limit is given by reducing to the case of 2-limits of functors into $\mathcal{C}at$. This representable definition will prove useful later on. This means that weighted limits are preserved by representable 2-functors.

To construct limits in the ordinary 1-dimensional setting, we know that it is enough to take products and equalizers: the limit of a functor $\mathcal{B} \xrightarrow{F} \mathcal{C}$ can be constructed as the equalizer of

$$\prod_A FA \underset{t}{\overset{s}{\rightrightarrows}} \prod_{f:A\to B} FB \tag{6.4}$$

where s and t are the unique morphisms defined by

$$\pi_{(f:A\to B)}s = \pi_B$$
$$\pi_{(f:A\to B)}t = (Ff)\pi_A$$

and the universal cone is given by composing with the projections of $\prod_A FA$: if $E \xrightarrow{e} \prod_A FA$ is the equalizer, the components of the cone are $\pi_A e$.

It's easy to prove that this equalizer has the same universal property as the limit: given a cone $\Delta_Z \overset{\lambda}{\Longrightarrow} F$, we can obtain $Z \xrightarrow{(\lambda_A)_A} \prod_A FA$ by the universal property of the product. Naturality of λ coincides with this morphism $(\lambda_A)_A$ equalizing s and t:

$$\pi_{(f:A\to B)}t(\lambda_A)_A = \pi_{(f:A\to B)}s(\lambda_A)_A$$
$$\iff (Ff)\lambda_A = \lambda_B$$

We then get an induced morphism $Z \overset{h}{\dashrightarrow} E$ into the equalizer, and this morphism satisfies

$$\pi_A eh = \lambda_A$$

for all $A \in \mathcal{A}$. □

In the 2-dimensional setting, one more type of 2-limit is needed: cotensor products. In fact, compositions of 2-products and 2-equalizers only give conical 2-limits. The following proposition can be found without proof in [15].

Proposition 6.5. ***(construction of weighted 2-limits)*** *Given 2-functors $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ and $\mathcal{B} \xrightarrow{F} \mathcal{C}$, if $\mathcal{C}$ has 2-equalizers, 2-products and cotensor products, then $\underleftarrow{\text{wlim}}^W F$ can be constructed as the 2-equalizer of*

$$\prod_B \{WB, FB\} \underset{t}{\overset{s}{\rightrightarrows}} \prod_{B,C} \{\mathcal{B}(B,C) \times WB, FC\}$$

where s and t are the unique morphisms arising from $\mathcal{B}(B,C) \longrightarrow \mathcal{C}(FB,FC)$ and $\mathcal{B}(B,C) \longrightarrow \mathcal{C}at(WB,WC)$.

Proof. We will first verify it for the case $\mathcal{C} = \mathcal{C}at$, and then reduce the general case to this.

We will prove that $\underleftarrow{\mathrm{wlim}}^W F \cong [\mathcal{B}, \mathcal{C}at](W, F)$ is the 2-equalizer (see proposition 6.1):

$$[\mathcal{B}, \mathcal{C}at](W, F) \xrightarrow{U} \prod_B [WB, FB] \underset{T}{\overset{S}{\rightrightarrows}} \prod_{B,C} [\mathcal{B}(B, C) \times WB, FC]$$

The functor U is the operation of taking components on 2-natural transformations and modifications. The composite functor $\pi_{B,C}S$ is given by the formulas

$$\begin{aligned}
(\pi_{B,C}S)(\alpha)(f, X) &= (\alpha_C \circ Wf)(X) \\
(\pi_{B,C}S)(\alpha)(\varepsilon, u) &= \alpha_C((W\varepsilon)_Y \circ (Wf)(u)) \\
(\pi_{B,C}S)(\eta)_{f,X} &= (\eta_C \circ Wf)(X)
\end{aligned}$$

if $\alpha = (\ WB \xrightarrow{\alpha_B} FB\ )_{B \in \mathcal{B}}$ and $\beta = (\ WB \xrightarrow{\beta_B} FB\ )_{B \in \mathcal{B}}$ are families of functors, $\eta = (\eta_B)_{B \in \mathcal{B}}$ is a family of natural transformations between them,

$$\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & C \end{tikzcd} \text{ in } \mathcal{B} \text{ and } X \xrightarrow{u} Y \text{ in } WB.$$

Likewise, the composite functor $\pi_{B,C}T$ is given by the formulas

$$\begin{aligned}
(\pi_{B,C}T)(\alpha)(f, X) &= (Ff \circ \alpha_B)(X) \\
(\pi_{B,C}T)(\alpha)(\varepsilon, u) &= (F\varepsilon)_{\alpha_B(Y)} \circ Ff(\alpha_B(u)) \\
(\pi_{B,C}T)(\eta)_{f,X} &= Ff(\eta_{B,X})
\end{aligned}$$

It's easy to see from these definitions that $\pi_{B,C}SU = \pi_{B,C}TU$, and then U equalizes S and T. Given a functor

$$\mathcal{Z} \xrightarrow{\varphi} \prod_B [WB, FB]$$

also equalizing S and T, we have to define

$$\mathcal{Z} \xrightarrow{H} [\mathcal{B}, \mathcal{C}at](F, G)$$

so that HZ is the 2-natural transformation with components those of $\varphi(Z)$, and $H(\ Z \xrightarrow{p} Z'\ )$ is the modification with components those of $\varphi(p)$, if we want $UH = \varphi$. The resulting HZ and Hp are indeed a 2-natural transformation and a modification, respectively: this follows from $S\varphi = T\varphi$. This is the 1-dimensional universal property of the 2-equalizer.

For the 2-dimensional universal property, we take a second

$$\mathcal{Z} \xrightarrow{\psi} \prod_B [WB, FB]$$

with corresponding

$$\mathcal{Z} \xrightarrow{K} [\mathcal{B}, \mathcal{C}at](F, G)$$

and a natural transformation $\varphi \xRightarrow{\sigma} \psi$ such that $id_S * \sigma = id_T * \sigma$. We want a natural transformation $H \xRightarrow{\tau} K$ such that $id_U * \tau = \sigma$. This forces us to define $\tau_{Z,B} = \pi_B(\sigma_Z)$, and this is well-defined and gives a natural transformation.

Now, if $\mathcal{C}$ is an arbitrary 2-category, we have the following chain of isomorphisms, 2-natural in A

$$\mathcal{C}(A, \mathrm{eq}(\prod_B \{WB, FB\} \rightrightarrows \prod_{B,C} \{\mathcal{B}(B,C) \times WB), FC\}))$$

$$\cong \mathrm{eq}(\mathcal{C}(A, \prod_B \{WB, FB\} \rightrightarrows \mathcal{C}(A, \prod_{B,C} \{\mathcal{B}(B,C) \times WB), FC\}))$$

$$\cong \mathrm{eq}(\prod_B \mathcal{C}(A, \{WB, FB\}) \rightrightarrows \prod_{B,C} \mathcal{C}(A, \{\mathcal{B}(B,C) \times WB), FC\})$$

$$\cong \mathrm{eq}(\prod_B \{WB, \mathcal{C}(A, FB)\} \rightrightarrows \prod_{B,C} \{\mathcal{B}(B,C) \times WB), \mathcal{C}(A, FC)\})$$

$$\cong \underleftarrow{\mathrm{wlim}}^W \mathcal{C}(A, F-) \cong \mathcal{C}(A, \underleftarrow{\mathrm{wlim}}^W F)$$

where we use the representable definition of limits in equation 6.3 to extract the equalizer, the products and the cotensor products out of the second argument of $\mathcal{C}(A, -)$, in that order, and the fact that the construction works in $\mathcal{C}at$.
By the 2-Yoneda lemma (theorem 2.34),

$$\underleftarrow{\mathrm{wlim}}^W F \cong \mathrm{eq}(\prod_B \{WB, FB\} \rightrightarrows \prod_{B,C} \{\mathcal{B}(B,C) \times WB), FC\})$$

□

Observation 6.6. When a 2-category $\mathcal{C}$ has certain classes of 2-limits, we say that a certain type of 2-limit *can be constructed* from these, instead of just saying that $\mathcal{C}$ *admits* it. Any 2-functor preserving those 2-limits also preserves the new type of 2-limit. This is suggested in [3].

6.2. **Construction of bilimits.**

In this subsection we reproduce some definitions of bilimits, results and proofs in [28] that are essential in section 7.

Observation 6.7. In [29] and [28], Street considers bilimits in a bicategory, where both the weight W and the diagram F are permitted to be pseudofunctors (also called homomorphisms of bicategories). The bilimit $\underleftarrow{\mathrm{wbilim}}^W F$ is given by a birepresentation of

$$Z \mapsto \mathrm{Hom}[\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(Z, F-))$$

where Hom$[\mathcal{B}, \mathcal{C}at]$ is the bicategory of pseudofunctors, pseudonatural transformations and modifications.
We will always assume W and F to be (strict) 2-functors, so that this notion coincides with the definition of bilimit we give in 4.7:

$$\mathrm{Hom}[\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(Z, F-)) = [\mathcal{B}, \mathcal{C}at]_p(W, \mathcal{C}(Z, F-))$$

Proposition 6.8. *The biidentifier of an auto-2-cell*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

in a 2-category $\mathcal{C}$ can be constructed using biequalizers and biproducts.

Proof. Let $\widehat{\alpha}$ be the arrow $A \to \{\mathrm{Aut}, B\}$ corresponding to the automorphism α in $\mathcal{C}(A, B)$, where Aut is the group $\mathbb{Z}$ of integers (free group on a one-element set) seen as a category. Likewise, let $\widehat{f}$ be the arrow $A \to \{\mathrm{Aut}, B\}$ corresponding to the automorphism id_f in $\mathcal{C}(A, B)$. We take $H \xrightarrow{h} A$ with $\widehat{\alpha}h \xRightarrow{\sigma \cong} \widehat{f}h$ the biequalizer of $\widehat{\alpha}$ and $\widehat{f}$, and $K \xrightarrow{k} A$ with $fh \xRightarrow{\tau \cong} fh$ the biequalizer of f and f. The unique functor $\{\bullet\} \longrightarrow \mathrm{Aut}$ induces a morphism $\{\mathrm{Aut}, B\} \xrightarrow{e} B$. We can take $H \xrightarrow{l} K$ and $kl \cong h$ with $e\sigma$ isomorphic to τl by the universal property of K, and $A \xrightarrow{d} K$ and $kd \cong id_k$ with id_f isomorphic to τ_d. The biequalizer of $d\pi_1$ and $l\pi_2$ (where π_1 and π_2 are the projections from the biproduct $A \times H$) is the biidentifier of α (see definition 5.27). $\square$

Proposition 6.9. *If*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha \Downarrow\beta"] & B \end{tikzcd}$$

are 2-cells in a 2-category $\mathcal{C}$ with β invertible, then the biequifier of α and β is the biidentifier of $\beta^{-1}\alpha$.

Proof. This is straightforward from the following equivalences:

$$\begin{aligned} \alpha * id_z &= \beta * id_z \\ \iff (\beta^{-1} * id_z) \circ (\alpha * id_z) &= (\beta^{-1} * id_z) \circ (\beta * id_z) \\ \iff \beta^{-1}\alpha * id_z &= id_f * id_z \end{aligned}$$

so that the full subcategory of $\mathcal{C}(Z, A)$ with objects those z such that $\alpha * id_z = \beta * id_z$ is the full subcategory of $\mathcal{C}(Z, A)$ with objects those z such that $\beta^{-1}\alpha * id_z = id_z$ (see definitions 5.26 and 5.27). $\square$

Proposition 6.10. *The descent object of a truncated bicosimplicial diagram (see definitions 5.28 and 5.29)*

$$\begin{tikzcd} X_0 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & X_1 \arrow[l, "\iota" description] \arrow[r, shift left=3, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=3, "\delta_2"'] & X_2 \end{tikzcd}$$

with invertible 2-cells

$$\delta_i\delta_{j-1} \overset{\sigma_{i,j}\cong}{\Longrightarrow} \delta_j\delta_i$$

for $i < j$, and for all i

$$id_{X_0} \overset{\mu_i\cong}{\Longrightarrow} \iota\delta_i$$

can be constructed using biequalizers and biidentifiers of auto-2-cells.

Proof. First, we take the biequalizer $H \xrightarrow{h} X_0$ with $id_{\delta_0} * h \overset{\theta\cong}{\Longrightarrow} id_{\delta_1} * h$. Then, we take the biequifier $K \xrightarrow{k} H$ of the two invertible 2-cells $id_\iota * \theta$ and $(\mu_1 * id_h) \circ (\mu_0^{-1} * id_h)$ (since these are invertible, their biequifier is a biidentifier of the auto-2-cell resulting from their composition). Finally, we take the biequifier $M \xrightarrow{m} H$ of the two invertible 2-cells

$$(\sigma_{1,2} * id_{hk}) \circ (id_{\delta_1} * \theta * id_k) \circ (\sigma_{0,1} * id_{hk})$$

and

$$(id_{\delta_2} * \theta * id_k) \circ (\sigma_{0,2} * id_{hk}) \circ (id_{\delta_0} * \theta * id_k)$$

The descent object is given by the object L, the arrow hkm and the invertible 2-cell $\theta * id_{km}$. ☐

Proposition 6.11. *The bilimit* $\underset{\longleftarrow}{\mathrm{wbilim}}^W F$ *of a functor* $\mathcal{B} \xrightarrow{F} \mathcal{C}$ *weighted by* $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ *can be constructed as the descent object for the truncated bicosimplicial diagram*

$$\prod_A \{WA, FA\}_b \overset{\delta_0}{\underset{\delta_1}{\overset{\longleftarrow \iota \longrightarrow}{\rightrightarrows}}} \prod_{A,B} \{\mathcal{A}(A,B) \times WA, FB\}_b$$

$$\overset{\delta_0}{\underset{\delta_2}{\overset{\xrightarrow{\delta_1}}{\rightrightarrows}}} \prod_{A,B,C} \{\mathcal{A}(B,C) \times \mathcal{A}(A,B) \times WA, FC\}_b$$

with arrows given in a similar way to those in proposition 6.5. ☐

Corollary 6.12. *An arbitrary bilimit* $\underset{\longleftarrow}{\mathrm{wbilim}}^W F$ *can be constructed by means of biproducts, biequalizers and bicotensor products. Moreover, if the weight* $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ *is finite, the bilimit can be constructed by means of finite biproducts (or binary products and the terminal object, by induction), biequalizers and bicotensor products with finite categories.*

Corollary 6.13. *Any 2-functor that preserves biproducts, biequalizers and bicotensor products also preserves all weighted bilimits. If the 2-functor preserves finite biproducts, biequalizers and bicotensor products with a finite category, then it preserves all finite weighted bilimits.*

7. Commutation of filtered bicolimits and finite weighted bilimits in $\mathcal{C}at$

7.1. The 1-dimensional case.

In ordinary category theory we have the following result.

Theorem 7.1. *Given a functor* $\mathcal{A} \times \mathcal{J} \xrightarrow{F} \mathcal{E}ns$ *, for each* $A \in \mathcal{A}$ *and* $J \in \mathcal{J}$ *we have functors*

$$\mathcal{A} \xrightarrow{F(-,J)} \mathcal{E}ns \quad \text{and} \quad \mathcal{J} \xrightarrow{F(A,-)} \mathcal{E}ns$$

by fixing one variable. Taking limits and colimits (which are computed pointwise) in the functor categories $\mathcal{E}ns^{\mathcal{A}}$ *and* $\mathcal{E}ns^{\mathcal{J}}$*, respectively, we obtain functors*

$$\mathcal{A} \xrightarrow{\varprojlim_{\mathcal{J}} F(-,J)} \mathcal{E}ns \quad \text{and} \quad \mathcal{J} \xrightarrow{\varinjlim_{\mathcal{A}} F(A,-)} \mathcal{E}ns$$

These have limits and colimits of their own, and there's a canonical function of sets

$$\varinjlim_{\mathcal{A}} \varprojlim_{\mathcal{J}} F \xrightarrow{\lozenge} \varprojlim_{\mathcal{J}} \varinjlim_{\mathcal{A}} F$$

which is a bijection provided that $\mathcal{A}$ *is filtered and* $\mathcal{J}$ *is finite (finite collection of objects and finite hom-sets).*

Proof. This can be proven either directly, or by decomposing the limit. We will follow this second method as we will also use it for the proof of the corresponding result in the 2-dimensional setting.
Using the construction of limits stated in diagram 6.4, it's enough to prove this commutation in the cases that $\mathcal{J} = \{\bullet \rightrightarrows \bullet\}$ (for the equalizer) or $\mathcal{J}$ discrete and finite (for the product). We will assume $\mathcal{J} = \{\bullet \quad \bullet\}$ in this second case, since any finite product can be constructed by iterated binary products and the case of the terminal product is trivial.

We will use the familiar constructions of limits and colimits in $\mathcal{E}ns$.

Observation 7.2. The limit of a functor $\mathcal{B} \xrightarrow{G} \mathcal{E}ns$ is the subset of the product $\displaystyle\prod_A GA$ consisting of those elements $(x_A)_A$ such that $(Gf)(x_A) = x_B$ for all $A \xrightarrow{f} B$, with the cone defined by the projections to the factors of the product (this is just the combination of the usual constructions of equalizer and product applied to the formula in diagram 6.4).

Observation 7.3. The colimit of a functor $\mathcal{B} \xrightarrow{G} \mathcal{E}ns$ with $\mathcal{B}$ filtered will be given in terms of the "germs" construction: the quotient of the disjoint union $\displaystyle\coprod_A GA$ by the equivalence relation $(x, A) \sim (y, B)$ if there exists a C

and arrows f, g as in

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
& C \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

such that $(Gf)(x) = (Gg)(y)$, with the cone defined by the inclusions into the disjoint union followed by the quotient map. We write $[x, A]$ for equivalence class represented by an element $(x, A) \in \coprod_A GA$. When there is more than one set-valued functor G of which we construct the colimit, we can make the functor G explicit by speaking of G-equivalence classes, denoted $[x, A]_G$.

(1) Case $\mathcal{J} = \{\bullet \quad \bullet\}$
We have a functor $\mathcal{A} \times \{\bullet \quad \bullet\} \longrightarrow \mathcal{E}ns$, which is the same as two functors $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{E}ns$. We define

$$\mathcal{A} \xrightarrow{F \times G} \mathcal{E}ns$$

to be the functor taking products pointwise:

$$(F \times G)(A) = FA \times GA$$
$$(F \times G)(f) = Ff \times Gf$$

if A is an object and f is a morphism in $\mathcal{A}$.
The function

$$\underrightarrow{\operatorname{colim}} F \times G \xrightarrow{\Diamond} \underrightarrow{\operatorname{colim}} F \times \underrightarrow{\operatorname{colim}} G$$

maps equivalence classes of pairs to pairs of equivalence classes

$$[(x, y), A] \mapsto ([x, A], [y, A])$$

To see that it is surjective, we pick an arbitrary

$$([x, A], [y, B]) \in \underrightarrow{\operatorname{colim}} F \times \underrightarrow{\operatorname{colim}} G$$

By axiom 3.2 there exist an object C and morphisms u, v as in

$$\begin{tikzcd}
A \arrow[dr, "u"] & \\
& C \\
B \arrow[ur, "v"'] &
\end{tikzcd}$$

Then $\Diamond([((Fu)(x), (Fv)(y)), C]) = ([(Fu)(x), C], [(Fv)(y), C])$, and in fact

$$[x, A] = [(Fu)(x), C]$$

since

$$\begin{tikzcd} A \arrow[dr, "u"] & \\ & C \\ C \arrow[ur, "id_C"'] & \end{tikzcd}$$

and

$$(F_C)(Fu)(x) = (Fu)(x)$$

Analogously,

$$[y, B] = [(Fv)(y), C]$$

Then $\Diamond([((Fu)(x), (Fv)(y)), C]) = ([x, A], [y, B])$.

To verify injectivity, we pick two elements

$$[(x_1, x_2), A], [(y_1, y_2), B] \in \underrightarrow{\operatorname{colim}}\, F \times G$$

that map to the same element in $\underrightarrow{\operatorname{colim}}\, F \times \underrightarrow{\operatorname{colim}}\, G$ via $\Diamond$:

$$([x_1, A], [x_2, A]) = ([y_1, B], [y_2, B])$$

This is the same as the two equations

$$[x_1, A] = [y_1, B]$$
$$[x_2, A] = [y_2, B]$$

which mean that there are objects and morphisms as in the diagrams

$$\begin{tikzcd} A \arrow[dr, "u_1"] & \\ & C_1 \\ B \arrow[ur, "v_1"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, "u_2"] & \\ & C_2 \\ B \arrow[ur, "v_2"'] & \end{tikzcd}$$

with

$$(Fu_1)(x_1) = (Fv_1)(y_1)$$
$$(Gu_2)(x_2) = (Gv_2)(y_2)$$

We can apply axiom 3.2 to complete a square

$$\begin{tikzcd} & C_1 \arrow[dr, dashed, "w'_1"] & \\ A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & & C'_3 \\ & C_2 \arrow[ur, dashed, "w'_2"'] & \end{tikzcd}$$

but it doesn't have to be commutative. By axiom 3.3 applied to the compositions

$$A \overset{w_1'u_1}{\underset{w_2'u_2}{\rightrightarrows}} C_3'$$

we can replace C_3' by another object C_3, and the morphisms w_1' and w_2' by another pair of morphisms w_1, w_2 such that the following diagram commutes

$$\begin{tikzcd} & C_1 \arrow[dr, "w_1"] & \\ A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\equiv"] & & C_3 \\ & C_2 \arrow[ur, "w_2"'] & \end{tikzcd}$$

We now apply axiom 3.3 to

$$A' \overset{w_1v_1}{\underset{w_2v_2}{\rightrightarrows}} C_3$$

to obtain

$$A' \overset{w_1v_1}{\underset{w_2v_2}{\rightrightarrows}} C_3 \xrightarrow{e} C_4$$

with

$$ew_1v_1 = ew_2v_2$$

Putting all this together, we have

$$\begin{tikzcd} A \arrow[dr, "ew_1u_1"] & \\ & D \\ B \arrow[ur, "ew_2v_2"'] & \end{tikzcd}$$

and

$$F(ew_1u_1)(x_1) = F(ew_1v_1)(y_1) = F(ew_2v_2)(y_1)$$
$$G(ew_1u_1)(x_2) = G(ew_2u_2)(x_2) = G(ew_2v_2)(y_2)$$

This means that $((x_1, x_2), A) \sim ((y_1, y_2), B)$ because

$$(F \times G)(ew_1u_1)(x_1, x_2) = (F \times G)(ew_2v_2)(y_1, y_2)$$

Then

$$[(x_1, x_2), A] = [(y_1, y_2), B]$$

(2) Case $\mathcal{J} = \{\bullet \rightrightarrows \bullet\}$
We have a functor $\mathcal{A} \times \{\bullet \rightrightarrows \bullet\} \longrightarrow \mathcal{E}ns$, which is the same as two

functors $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{E}ns$ with natural transformations $\varepsilon, \eta : F \Rightarrow G$. We define

$$\mathcal{A} \xrightarrow{E} \mathcal{E}ns$$

to be the functor taking equalizers pointwise (in other words, the equalizer of ε and η in the functor category $[\mathcal{A}, \mathcal{E}ns]$, which is computed pointwise):

$$EA \xrightarrow{i_A} FA \underset{\eta_A}{\overset{\varepsilon_A}{\rightrightarrows}} GA$$

(here we take EA as the subset of FA consisting of the elements equalized by ε_A and η_A), and on a morphism f

$$(Ef)(x) = (Ff)(x) \tag{7.4}$$

It can be checked that this is well-defined. We then consider the diagram

$$\underrightarrow{\operatorname{colim}} F \underset{\eta}{\overset{\varepsilon}{\rightrightarrows}} \underrightarrow{\operatorname{colim}} G$$

where ε and η are the morphisms induced in the colimit. We take the equalizer of this diagram and obtain

$$E' \xrightarrow{i} \underrightarrow{\operatorname{colim}} F \underset{\eta}{\overset{\varepsilon}{\rightrightarrows}} \underrightarrow{\operatorname{colim}} G$$

(again, we consider $E' \subseteq \underrightarrow{\operatorname{colim}} F$) There is a comparison function given by the universal property of E'

$$\underrightarrow{\operatorname{colim}} E \xrightarrow{\lozenge} E'$$

that maps an E-equivalence class to an F-equivalence class (see observation 7.3)

$$[x, A]_E \mapsto [x, A]_F$$

This function is surjective: given $[x, A]_F \in E'$, we have

$$\varepsilon([x, A]_F) = \eta([x, A]_F)$$

because $[x, A]$ belongs to E'. From the definition of ε and η, this is

$$[\varepsilon_A(x), A]_G = [\eta_A(x), A]_G$$

This means there is a C and morphisms u, v as in

$$\begin{tikzcd} A \arrow[dr, "u"] & \\ & C \\ A \arrow[ur, "v"'] & \end{tikzcd}$$

with

$$(Gu)\varepsilon_A(x) = (Gv)\eta_A(x) \tag{7.5}$$

By axiom 3.3, we obtain a D and a morphism w as in

$$A \underset{v}{\overset{u}{\rightrightarrows}} C \xrightarrow{w} D$$

with $wu = wv$. Then, by equation 7.5 and naturality of ε and η, we have

$$\varepsilon_D F(wu)(x) = \eta_D F(wv)(x) = \eta_D F(wu)(x)$$

This equation says that $F(wu)(x)$ is a well-defined element of ED because it is equalized by ε_D and η_D as in

$$ED \xrightarrow{i_D} FD \underset{\eta_D}{\overset{\varepsilon_D}{\rightrightarrows}} GD$$

We can then consider the element $[F(wu)(x), D]_E \in \varinjlim E$, and in fact, $\Diamond([F(wu)(x), D]_E) = [x, A]_F$, since we have a diagram

$$\begin{tikzcd} A \arrow[dr, "wu"] & \\ & D \\ D \arrow[ur, "id_D"'] & \end{tikzcd}$$

with

$$F(wu)(x) = id_D(F(wu)(x))$$

This shows surjectivity of $\Diamond$. To see that it is injective, we pick $[x, A]_E$, $[y, B]_E \in \varinjlim E$ with $[x, A]_F = [y, B]_F$, that is, there is a C and morphisms u and v

$$\begin{tikzcd} A \arrow[dr, "f"] & \\ & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

with

$$(Fu)(x) = (Fv)(y)$$

Since on morphisms the functor E is defined by the formula 7.4, we immediately have

$$(Eu)(x) = (Ev)(y)$$

which means $[x, A]_E = [y, B]_E$.

□

In the rest of this section we will prove that 2-filtered bicolimits and finite bilimits commute in $\mathcal{C}at$, which is a 2-categorical version of this classical result, by means of a similar general argument: decomposing a finite bilimit into certain kinds of finite limits which commute with the 2-filtered bicolimit. We assume throughout the rest of this section that $\mathcal{A}$ is 2-filtered.

7.2. **The 2-dimensional case.**

An adaptation of theorem 7.1 to the 2-categorical setting can be found in [11], which considers pseudocolimits and pseudolimits of diagrams in $\mathcal{C}at$ indexed by filtered and finite categories (these are conical pseudolimits). We extend this result to finite weighted bilimits and 2-filtered bicolimits in $\mathcal{C}at$, both indexed by 2-categories.

In the following sections, we consider particular classes of finite weighted pseudolimits and show that they commute with 2-filtered pseudocolimits. Finally, we combine these facts in the proof of the main result.

7.3. **2-product.**

Definition 7.6. Given 2-functors $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}at$, the 2-functor $\mathcal{A} \xrightarrow{F\times G} \mathcal{C}at$ is defined by the following assignments (see observation 2.20)

$$
\begin{aligned}
(F\times G)(A) &= F(A)\times G(A)\\
(F\times G)(f)(X,Y) &= ((Ff)(X),(Gf)(Y))\\
(F\times G)(f)(u,v) &= ((Ff)(u),(Gf)(v))\\
(F\times G)(\alpha)_{(P,Q)} &= ((F\alpha)_P,(G\alpha)_Q)
\end{aligned}
$$

for $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{A}$, $X \xrightarrow{u} X'$ in FA, $Y \xrightarrow{v} Y'$ in GA, and (P,Q) an object in $FA\times GA$.

It is straightforward to check that this does indeed define a 2-functor.

Observation 7.7. The 2-functor $F\times G$ is the 2-product in the 2-category $[\mathcal{A},\mathcal{C}at]$ (see [14, section 3.3].)

We consider the bicolimit of $F\times G$ using of the construction in [10] (recall that this is in fact a pseudocolimit; see definition 5.31 and theorem 5.32). Objects of the category $\mathcal{L}(F\times G)$ are given by

$$
\frac{F\times G \xrightarrow{(a,b)} A}{(a,b)\in FA\times GA}
$$

Notice that in fact $F\times G$ is a (formal) coproduct since we are in the dual category $[\mathcal{A},\mathcal{C}at]^{op}$. Premorphisms of $\mathcal{L}(F\times G)$ are given by

$$\frac{\begin{tikzcd}[column sep=small] & A \arrow[dr, "u"] & \\ F\times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] & \Downarrow(\xi,\sigma) & C \\ & A' \arrow[ur, "v"'] & \end{tikzcd}}{\begin{array}{c}(Fu)(a) \xrightarrow{\xi} (Fv)(a') \text{ in } FC \\ (Gu)(b) \xrightarrow{\sigma} (Gv)(b') \text{ in } GC\end{array}}$$

An homotopy between two premorphisms (ξ_1, σ_1) and (ξ_2, σ_2) is given by invertible 2-cells α, β satisfying the LL equation.

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "u_1"] & & \\ F\times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] & \Downarrow(\xi_1,\sigma_1) & C_1 \arrow[dr, "u"] & \\ & A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] & \Downarrow\alpha\cong & D \\ & & C_2 \arrow[ur, "v"'] & \end{tikzcd} = \begin{tikzcd}[column sep=small] & & C_1 \arrow[dr, "u"] & \\ & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] & \Downarrow\beta\cong & D \\ F\times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] & \Downarrow(\xi_2,\sigma_2) & C_2 \arrow[ur, "v"'] & \\ & A' \arrow[ur, "v_2"'] & & \end{tikzcd} \tag{7.8}$$

On the other side, each one of $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}at$ is a 2-functor from a 2-filtered category to $\mathcal{C}at$, so it makes sense to consider the categories $\mathcal{L}(F)$ and $\mathcal{L}(G)$ and take their product. We have a comparison functor $\lozenge : \mathcal{L}(F \times G) \longrightarrow \mathcal{L}(F) \times \mathcal{L}(G)$ given by the universal property of the pseudocolimit $\mathcal{L}(F \times G)$, and which is defined as follows.

$$\lozenge : \mathcal{L}(F \times G) \longrightarrow \mathcal{L}(F) \times \mathcal{L}(G)$$

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "u"] & \\ F\times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] & \Downarrow(\xi,\sigma) & C \\ & A' \arrow[ur, "v"'] & \end{tikzcd} \mapsto \left(\begin{tikzcd}[column sep=small] & A \arrow[dr, "u"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] & \Downarrow\xi & C \\ & A' \arrow[ur, "v"'] & \end{tikzcd} , \begin{tikzcd}[column sep=small] & A \arrow[dr, "u"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] & \Downarrow\sigma & C \\ & A' \arrow[ur, "v"'] & \end{tikzcd} \right)$$

We observe that equation 7.8 is equivalent to the two LL equations for the homotopies $(\alpha, \beta) : \xi_1 \Rightarrow \xi_2$ and $(\alpha, \beta) : \sigma_1 \Rightarrow \sigma_2$, reflecting the fact that the functor $\lozenge$ is well-defined.

Observation 7.9. The proof of the following theorem is similar to the proof of [10, theorem 2.4] when for $\mathcal{P}$ we take the discrete category $\{\bullet \quad \bullet\}$. For example, objects of $\mathcal{L}(F \times G)$ are given by elements $(a, b) \in FA \times GA$, while objects of $\mathcal{L}(F^{\mathcal{P}})$ are given by elements $(a, b) \in FA \times FA$.
We include the proof also because in this case (of two factors) it is much clearer.

Theorem 7.10. *Given 2-functors* $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}at$ *with* $\mathcal{A}$ *2-filtered, the comparison functor* $\Diamond : \mathcal{L}(F \times G) \longrightarrow \mathcal{L}(F) \times \mathcal{L}(G)$ *is an equivalence of categories.*

Proof. We proceed by showing that it is essentially surjective, full and faithful. This is enough to prove that $\Diamond$ is an equivalence by observation 2.16.

(1) Essential surjectivity

We pick an arbitrary object ($F \xrightarrow{a} A$, $G \xrightarrow{b} B$) in $\mathcal{L}(F) \times \mathcal{L}(G)$. Using 2-filteredness of $\mathcal{A}$, we take C, u and v as in the following diagram.

$$\begin{tikzcd} A \arrow[dr, "u"] & \\ & C \\ B \arrow[ur, "v"'] & \end{tikzcd}$$

The invertible morphism

$$\begin{tikzcd} & C \arrow[dr, "id"] & \\ F \arrow[ur, "ua"] \arrow[dr, "a"'] \arrow[rr, phantom, "\Downarrow id"] & & C \\ & A \arrow[ur, "u"'] & \end{tikzcd}$$

shows that $F \xrightarrow{a} A$ is isomorphic to $F \xrightarrow{ua} C$. Likewise, we have $G \xrightarrow{b} B \;\cong\; G \xrightarrow{vb} C$.

Then ($F \xrightarrow{a} A$, $G \xrightarrow{b} B$) $\cong \Diamond($ $F \times G \xrightarrow{(ua,vb)} C$).

(2) Fullness

Given a premorphism $\Diamond($ $F \times G \xrightarrow{(a,b)} A$) $\longrightarrow \Diamond($ $F \times G \xrightarrow{(a',b')} A'$) by the data

$$\begin{tikzcd} & A \arrow[dr, "u_1"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi"] & & C_1 \\ & A' \arrow[ur, "v_1"'] & \end{tikzcd} , \quad \begin{tikzcd} & A \arrow[dr, "u_2"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & C_2 \\ & A' \arrow[ur, "v_2"'] & \end{tikzcd}$$

we want to find premorphisms δ and ε equivalent to ξ and σ, respectively, that will allow us to pass from C_1, C_2, u_1, u_2, v_1 and

v_2 to a single C, a single u and a single v, respectively. We use 2-filteredness of $\mathcal{A}$, axiom FF1 in definition 3.8, to obtain invertible 2-cells

$$\begin{tikzcd}[row sep=small, column sep=small]
& C_1 \arrow[dr, "p"] & \\
A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\gamma^u\cong"] & & C \\
& C_2 \arrow[ur, "q"'] &
\end{tikzcd} , \quad
\begin{tikzcd}[row sep=small, column sep=small]
& C_1 \arrow[dr, "p"] & \\
A' \arrow[ur, "v_1"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\gamma^v\cong"] & & C \\
& C_2 \arrow[ur, "q"'] &
\end{tikzcd}$$

We can then define $u = pu_1$, $v = qv_2$, and δ and ε as in

$$\begin{tikzcd}[row sep=small, column sep=small]
& A \arrow[dr, "u"] & \\
F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\delta"] & & C \\
& A' \arrow[ur, "v"'] &
\end{tikzcd}
=
\begin{tikzcd}[row sep=small, column sep=small]
& A \arrow[dr, "u_1"] & & \\
F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi"] & & C_1 \arrow[dr, "p"] & \\
& A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\gamma^u"] & & C \\
& & C_2 \arrow[ur, "q"'] &
\end{tikzcd}$$

$$\begin{tikzcd}[row sep=small, column sep=small]
& A \arrow[dr, "u"] & \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & C \\
& A' \arrow[ur, "v"'] &
\end{tikzcd}
=
\begin{tikzcd}[row sep=small, column sep=small]
& & C_1 \arrow[dr, "p"] & \\
& A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\gamma^v"] & & C \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & C_2 \arrow[ur, "q"'] & \\
& A' \arrow[ur, "v_2"'] & &
\end{tikzcd}$$

Then $\xi \sim \delta$ and $\sigma \sim \varepsilon$, so $(\xi, \sigma) = \Diamond \left(\begin{tikzcd}[row sep=small, column sep=small]
& A \arrow[dr, "u"] & \\
F \times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] \arrow[rr, phantom, "\Downarrow(\delta,\varepsilon)"] & & C \\
& A' \arrow[ur, "v"'] &
\end{tikzcd} \right)$.

We observe that we essentially follow the same steps as to prove fullness of the comparison functor $\Diamond$ in theorem 7.23. This is part c) in the proof of [10, theorem 2.4]. See also the proof of [10, lemma 2.3].

(3) Faithfulness
If we have

$$\begin{tikzcd}
& A \arrow[dr, "u_1"] & \\
F\times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] \arrow[rr, phantom, "\Downarrow(\xi_1,\sigma_1)"] & & C_1 \\
& A' \arrow[ur, "v_1"'] &
\end{tikzcd}
,\quad
\begin{tikzcd}
& A \arrow[dr, "u_2"] & \\
F\times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] \arrow[rr, phantom, "\Downarrow(\xi_2,\sigma_2)"] & & C_2 \\
& A' \arrow[ur, "v_2"'] &
\end{tikzcd}$$

such that $\Diamond(\xi_1, \sigma_1) = \Diamond(\xi_2, \sigma_2)$, that is, $\xi_1 \sim \xi_2$ and $\sigma_1 \sim \sigma_2$: there are homotopies given by invertible 2-cells α_1, β_1, α_2, β_2 satisfying the LL equations

$$\begin{tikzcd}
& A \arrow[dr, "u_1"] & & \\
F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi_1"] & & C_1 \arrow[dr, "w_1"] & \\
& A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha_1\cong"] & & C \\
& & C_2 \arrow[ur, "w_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
& & C_1 \arrow[dr, "w_1"] & \\
& A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta_1\cong"] & & C \\
F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi_2"] & & C_2 \arrow[ur, "w_2"'] & \\
& A' \arrow[ur, "v_2"'] & &
\end{tikzcd}
\tag{7.11}$$

$$\begin{tikzcd}
& A \arrow[dr, "u_1"] & & \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma_1"] & & C_1 \arrow[dr, "w'_1"] & \\
& A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha_2\cong"] & & C' \\
& & C_2 \arrow[ur, "w'_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
& & C_1 \arrow[dr, "w'_1"] & \\
& A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta_2\cong"] & & C' \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma_2"] & & C_2 \arrow[ur, "w'_2"'] & \\
& A' \arrow[ur, "v_2"'] & &
\end{tikzcd}
\tag{7.12}$$

Intuitively, we would want $\xi_1 \sim \xi_2$ and $\sigma_1 \sim \sigma_2$ via the same pair (α, β) of invertible 2-cells witnessing the homotopy. We can apply lemma 5.34 to the families of 2-cells $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ to obtain (invertible!) 2-cells α, β by pasting invertible 2-cells λ_1, μ_1, λ_2, μ_2 so that

$$\begin{tikzcd} & C_1 \arrow[dr, "u"] & \\ A' \arrow[ur, "v_1"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha\cong"] & & D \\ & C_2 \arrow[ur, "v"'] & \end{tikzcd} = \begin{tikzcd} & C_1 \arrow[dr, "w_1"] \arrow[drr, bend left, "u"] \arrow[drr, phantom, "\Downarrow\lambda_1\cong"] & & \\ A' \arrow[ur, "v_1"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha_1\cong"] & & C \arrow[r, "h_1"] & D \\ & C_2 \arrow[ur, "w_2"'] \arrow[urr, bend right, "v"'] \arrow[urr, phantom, "\Downarrow\mu_1\cong"] & & \end{tikzcd}$$

$$= \begin{tikzcd} & C_1 \arrow[dr, "w_1'"] \arrow[drr, bend left, "u"] \arrow[drr, phantom, "\Downarrow\lambda_2\cong"] & & \\ A' \arrow[ur, "v_1"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha_2\cong"] & & C' \arrow[r, "h_2"] & D \\ & C_2 \arrow[ur, "w_2'"'] \arrow[urr, bend right, "v"'] \arrow[urr, phantom, "\Downarrow\mu_2\cong"] & & \end{tikzcd}$$

and

$$\begin{tikzcd} & C_1 \arrow[dr, "u"] & \\ A' \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\beta\cong"] & & D \\ & C_2 \arrow[ur, "v"'] & \end{tikzcd} = \begin{tikzcd} & C_1 \arrow[dr, "w_1"] \arrow[drr, bend left, "u"] \arrow[drr, phantom, "\Downarrow\lambda_1\cong"] & & \\ A' \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\beta_1\cong"] & & C \arrow[r, "h_1"] & D \\ & C_2 \arrow[ur, "w_2"'] \arrow[urr, bend right, "v"'] \arrow[urr, phantom, "\Downarrow\mu_1\cong"] & & \end{tikzcd}$$

$$= \begin{tikzcd} & C_1 \arrow[dr, "w_1'"] \arrow[drr, bend left, "u"] \arrow[drr, phantom, "\Downarrow\lambda_2\cong"] & & \\ A' \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\beta_2\cong"] & & C' \arrow[r, "h_2"] & D \\ & C_2 \arrow[ur, "w_2'"'] \arrow[urr, bend right, "v"'] \arrow[urr, phantom, "\Downarrow\mu_2\cong"] & & \end{tikzcd}$$

A simple calculation verifies that if we replace α_1 and β_1 by α and β in the LL equation 7.11 it remains true, by pasting appropriately λ_1 and μ_1:

$$\begin{tikzcd} & A \arrow[dr, "u_1"] & & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi_1"] & & C_1 \arrow[dr, "u"] & \\ & A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha\cong"] & & D \\ & & C_2 \arrow[ur, "v"'] & \end{tikzcd} = \begin{tikzcd} & & C_1 \arrow[dr, "u"] & \\ & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta\cong"] & & D \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi_2"] & & C_2 \arrow[ur, "v"'] & \\ & A' \arrow[ur, "v_2"'] & & \end{tikzcd} \tag{7.13}$$

We can do the same in the LL equation 7.12, replacing α and β for α_2 and β_2:

$$
\begin{tikzcd}
 & A \arrow[dr, "u_1"] & & \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma_1"] & & C_1 \arrow[dr, "u"] & \\
 & A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha\cong"] & & D \\
 & & C_2 \arrow[ur, "v"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & & C_1 \arrow[dr, "u"] & \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta\cong"] & & D \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma_2"] & & C_2 \arrow[ur, "v"'] & \\
 & A' \arrow[ur, "v_2"'] & &
\end{tikzcd}
\tag{7.14}
$$

These two resulting equations 7.13 and 7.14 are together equivalent to

$$
\begin{tikzcd}
 & A \arrow[dr, "u_1"] & & \\
F \times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] \arrow[rr, phantom, "\Downarrow(\xi_1{,}\sigma_1)"] & & C_1 \arrow[dr, "u"] & \\
 & A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha\cong"] & & D \\
 & & C_2 \arrow[ur, "v"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & & C_1 \arrow[dr, "u"] & \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta\cong"] & & D \\
F \times G \arrow[ur, "{(a,b)}"] \arrow[dr, "{(a',b')}"'] \arrow[rr, phantom, "\Downarrow(\xi_2{,}\sigma_2)"] & & C_2 \arrow[ur, "v"'] & \\
 & A' \arrow[ur, "v_2"'] & &
\end{tikzcd}
$$

which expresses that $(\xi_1, \sigma_1) \sim (\xi_2, \sigma_2)$ via (α, β), i.e. they define the same morphism in $\mathcal{L}(F \times G)$.

□

Observation 7.15. As usual, by induction, the comparison functor

$$\lozenge : \mathcal{L}\left(\prod_i F_i\right) \longrightarrow \prod_i \mathcal{L}(F_i)$$

is an equivalence of categories for any finite collection of 2-functors $\mathcal{A} \xrightarrow{F_i} \mathcal{C}at$. This is true for the empty product as well, since in this case both the domain and the codomain of $\lozenge$ reduce to the terminal category $\{\bullet\}$.

7.4. **Pseudoequalizer.**

We examine the construction of the pseudoequalizer pointwise to determine a 2-functor and then verify that it commutes with a 2-filtered pseudocolimit.

Definition 7.16. Given 2-functors and 2-natural transformations $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varepsilon \Downarrow\eta"] & \mathcal{C}at \end{tikzcd}$, the pseudoequalizer functor $E : \mathcal{A} \longrightarrow \mathcal{C}at$ is given on objects A by the construction in proposition 5.7 applied to $\begin{tikzcd} FA \arrow[r, shift left, "\varepsilon_A"] \arrow[r, shift right, "\eta_A"'] & GA \end{tikzcd}$, and on morphisms $A \xrightarrow{f} B$ and 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ as follows

$$
\begin{aligned}
Ef : EA &\to EB \\
(a, b, \gamma, \delta) &\mapsto ((Ff)(a), (Gf)(b), (Gf)(\gamma), (Gf)(\delta)) \\
(\xi, \sigma) &\mapsto ((Ff)(\xi), (Gf)(\sigma)) \\
E\alpha : Ef &\Longrightarrow Eg \\
(E\alpha)_{(a,b,\gamma,\delta)} &= ((F\alpha)_a, (G\alpha)_b))
\end{aligned}
$$

Observation 7.17. It can be checked that these assignments yield a well-defined 2-functor E, which, in fact, is the pseudoequalizer in the 2-category $[\mathcal{A}, \mathcal{C}at]$ (see [7, proposition 1.2.4]).

We now consider the pseudocolimit of E, via the construction in [10]. The category $\mathcal{L}(E)$ has as objects

$$
\frac{E \xrightarrow{(a,b,\gamma,\delta)} A}{a \in FA, b \in GA,\ \varepsilon_A(a) \xrightarrow{\gamma \cong} b\ \in GA,\ \eta_A(a) \xrightarrow{\delta \cong} b\ \in GA}
$$

(see definition 5.31) and as premorphisms

$$
\begin{tikzcd}
 & A \arrow[dr, "u"] & \\
E \arrow[ur, "{(a,b,\gamma,\delta)}"] \arrow[dr, "{(a',b',\gamma',\delta')}"'] \arrow[rr, phantom, "\Downarrow(\xi,\sigma)"] & & C \\
 & A' \arrow[ur, "v"'] &
\end{tikzcd}
$$

$$
\overline{\qquad\qquad\qquad\qquad\qquad\qquad}
$$

$$
\begin{gathered}
(Fu)(a) \xrightarrow{\xi} (Fv)(a')\ ,\ (Gu)(b) \xrightarrow{\sigma} (Gv)(b') \\
\text{such that } \sigma \circ Gu(\gamma) = Gv(\delta') \circ \varepsilon_C(\xi) \\
\text{and } \sigma \circ Gu(\gamma) = Gv(\delta') \circ \eta_C(\xi)
\end{gathered}
$$

Two premorphisms (ξ_1, σ_1) and (ξ_2, σ_2) are equivalent and define the same morphism in $\mathcal{L}(E)$ if there is an homotopy between them: α and β invertible

2-cells in $\mathcal{A}$ such that the LL equation holds:

(7.18)
$$
\begin{tikzcd}
 & A \arrow[dr, "u_1"] & & \\
E \arrow[ur, "{(a,b,\gamma,\delta)}"] \arrow[dr, "{(a',b',\gamma',\delta')}"'] \arrow[rr, phantom, "\Downarrow(\xi_1,\sigma_1)"] & & C_1 \arrow[dr, "u"] & \\
 & A' \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha\cong"] & & D \\
 & & C_2 \arrow[ur, "v"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & C_1 \arrow[dr, "u"] & \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\Downarrow\beta\cong"] & & D \\
E \arrow[ur, "{(a,b,\gamma,\delta)}"] \arrow[dr, "{(a',b',\gamma',\delta')}"'] \arrow[rr, phantom, "\Downarrow(\xi_2,\sigma_2)"] & & C_2 \arrow[ur, "v"'] & \\
 & A' \arrow[ur, "v_2"'] & &
\end{tikzcd}
$$

Following the definitions of E, this is the same as the two equations

$$(F\alpha)_{a'} \circ (Fw_1)(\xi_1) = (Fw_2)(\xi_2) \circ (F\beta)_a$$
$$(G\alpha)_{b'} \circ (Gw_1)(\sigma_1) = (Gw_2)(\sigma_2) \circ (G\beta)_b$$

which establish the homotopies $(\alpha,\beta) : \xi_1 \Rightarrow \xi_2$ and $(\alpha,\beta) : \sigma_1 \Rightarrow \sigma_2$.

Given 2-functors and 2-natural transformations $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varepsilon\,\Downarrow\eta"] & \mathcal{C}at \end{tikzcd}$ as in definition 7.16, we obtain a diagram in $\mathcal{C}at$ by observation 5.35

$$\mathcal{L}(F) \overset{\varepsilon}{\underset{\eta}{\rightrightarrows}} \mathcal{L}(G)$$

and we consider its pseudoequalizer

$$
\begin{tikzcd}
 & \mathcal{L}(F) \arrow[dr, "\varepsilon"] \arrow[d, phantom, "\Downarrow\varphi_1\cong"] & \\
\mathcal{E} \arrow[ur, "I"] \arrow[rr, "P"] \arrow[dr, "I"'] & {} \arrow[d, phantom, "\Uparrow\varphi_2\cong"] & \mathcal{L}(G) \\
 & \mathcal{L}(F) \arrow[ur, "\eta"'] &
\end{tikzcd}
$$

This category $\mathcal{E}$ has as objects (see proposition 5.7)

$$
\left(
F \xrightarrow{a} A \,,\; G \xrightarrow{b} B \,,\;
\begin{tikzcd}[ampersand replacement=\&]
 \& A \arrow[dr, "u_1"] \& \\
G \arrow[ur, "\varepsilon_A(a)"] \arrow[dr, "b"'] \arrow[rr, phantom, "\Downarrow\gamma\cong"] \& \& C_1 \\
 \& B \arrow[ur, "v_1"'] \&
\end{tikzcd}
\,,\;
\begin{tikzcd}[ampersand replacement=\&]
 \& A \arrow[dr, "u_2"] \& \\
G \arrow[ur, "\eta_A(a)"] \arrow[dr, "b"'] \arrow[rr, phantom, "\Downarrow\delta\cong"] \& \& C_2 \\
 \& B \arrow[ur, "v_2"'] \&
\end{tikzcd}
\right)
$$

Given a second object in $\mathcal{E}$

$$\left(F \xrightarrow{a'} A' \ ,\ G \xrightarrow{b'} B' \ ,\ \begin{tikzcd} & A' \arrow[dr, "u'_1"] & \\ G \arrow[ur, "\varepsilon_{A'}(a')"] \arrow[dr, "b'"'] & \Downarrow\gamma'\cong & C'_1 \\ & B' \arrow[ur, "v'_1"'] & \end{tikzcd} ,\ \begin{tikzcd} & A \arrow[dr, "u'_2"] & \\ G \arrow[ur, "\eta_{A'}(a')"] \arrow[dr, "b'"'] & \Downarrow\delta'\cong & C'_2 \\ & B' \arrow[ur, "v'_2"'] & \end{tikzcd} \right)$$

a morphism between them is represented by a pair of premorphisms in $\mathcal{L}(F)$ and $\mathcal{L}(G)$

$$\left(\begin{tikzcd} & A \arrow[dr, "r"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] & \Downarrow\xi & D_1 \\ & A' \arrow[ur, "r'"'] & \end{tikzcd} ,\ \begin{tikzcd} & A \arrow[dr, "s"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] & \Downarrow\sigma & D_2 \\ & A' \arrow[ur, "s'"'] & \end{tikzcd} \right)$$

satisfying $\sigma \circ \gamma = \gamma' \circ \varepsilon_{D_1}(\xi)$ and $\sigma \circ \delta = \delta' \circ \varepsilon_{D_2}(\xi)$ in $\mathcal{L}(G)$.

There is a comparison functor $\Diamond : \mathcal{L}(E) \longrightarrow \mathcal{E}$ given by the universal property of $\mathcal{L}(E)$. Given an object $E \xrightarrow{(a,b,\gamma,\delta)} A$, we define $\Diamond(a, b, \gamma, \delta)$ to be

$$\left(F \xrightarrow{a} A \ ,\ G \xrightarrow{b} A \ ,\ \begin{tikzcd} & A \arrow[dr, "id"] & \\ G \arrow[ur, "\varepsilon_A(a)"] \arrow[dr, "b"'] & \Downarrow\gamma\cong & A \\ & A \arrow[ur, "id"'] & \end{tikzcd} ,\ \begin{tikzcd} & A \arrow[dr, "id"] & \\ G \arrow[ur, "\eta_A(a)"] \arrow[dr, "b"'] & \Downarrow\delta\cong & A \\ & A \arrow[ur, "id"'] & \end{tikzcd} \right)$$

If we have a morphism $\begin{tikzcd} & A \arrow[dr, "u"] & \\ E \arrow[ur, "{(a,b,\gamma,\delta)}"] \arrow[dr, "{(a',b',\gamma',\delta')}"'] & \Downarrow(\xi,\sigma) & C \\ & A' \arrow[ur, "v"'] & \end{tikzcd}$ in $\mathcal{L}(E)$, its image via $\Diamond$ is

$$\Diamond(\xi, \sigma) = \left(\begin{tikzcd} & A \arrow[dr, "u"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] & \Downarrow\xi & C \\ & A' \arrow[ur, "v"'] & \end{tikzcd} ,\ \begin{tikzcd} & A \arrow[dr, "u"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] & \Downarrow\sigma & C \\ & A' \arrow[ur, "v"'] & \end{tikzcd} \right)$$

This comparison functor $\Diamond$ is in fact an equivalence of categories:

Theorem 7.19. *If $\mathcal{A}$ is a 2-filtered 2-category, given 2-functors and 2-natural transformations* $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varepsilon\,\Downarrow\eta"] & \mathcal{C}at \end{tikzcd}$ *, we call $\mathcal{E}$ is the pseudoequalizer of the induced* *functors* $\mathcal{L}(F) \overset{\varepsilon}{\underset{\eta}{\rightrightarrows}} \mathcal{L}(G)$ *and* E *the pseudoequalizer functor* $\mathcal{A} \longrightarrow \mathcal{C}at$.

Then, the comparison functor $\mathcal{L}(E) \xrightarrow{\lozenge} \mathcal{E}$ *is an equivalence of categories. That is, we have*

$$\mathcal{L}(\, E \longrightarrow F \overset{\varepsilon}{\underset{\eta}{\rightrightarrows}} G \,) \simeq \; \mathcal{E} \longrightarrow \mathcal{L}(F) \overset{\varepsilon}{\underset{\eta}{\rightrightarrows}} \mathcal{L}(G)$$

Proof. As before, we show that it is essentially surjective, full and faithful. This is enough to prove that $\lozenge$ is an equivalence by observation 2.16.

(1) Essential surjectivity

Given an object
$$\left(F \xrightarrow{a} A \;,\; G \xrightarrow{b} B \;,\; \begin{tikzcd} & A \arrow[dr, "u_1"] & \\ G \arrow[ur, "\varepsilon_A(a)"] \arrow[dr, "b"'] & \Downarrow\gamma\cong & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd} \;,\; \begin{tikzcd} & A \arrow[dr, "u_2"] & \\ G \arrow[ur, "\eta_A(a)"] \arrow[dr, "b"'] & \Downarrow\delta\cong & C_2 \\ & B \arrow[ur, "v_2"'] & \end{tikzcd} \right)$$
in $\mathcal{E}$, we can form the composition $\gamma^{-1} \circ \delta$ in $\mathcal{L}(G)$. This composition of morphisms in $\mathcal{L}(G)$ is given along an invertible 2-cell τ that exists by 2-filteredness of $\mathcal{A}$ (specifically, axiom FF1 in definition 3.8): $\gamma^{-1} \circ \delta = \gamma^{-1} \circ_\tau \delta$. We obtain

$$\gamma^{-1} \circ \delta = \begin{tikzcd} & A \arrow[dr, "u_2"] & & \\ & \Downarrow\delta\cong & C_1 \arrow[dr, dashed] & \\ G \arrow[uur, "\eta_A(a)"] \arrow[r, "b"] \arrow[ddr, "\varepsilon_A(a)'"'] & B \arrow[ur, "v_2"] \arrow[dr, "v_1"'] & \Downarrow\tau\cong & C_3 \\ & \Downarrow\gamma^{-1}\cong & C_2 \arrow[ur, dashed] & \\ & A \arrow[ur, "u_1"'] & & \end{tikzcd}$$

We refer to [10] for a full explanation of this composition.
The element (a, b, γ, δ) is isomorphic to $\lozenge(a, \varepsilon_A(a), id, \gamma^{-1} \circ \delta)$, which is

$$\left(F \xrightarrow{a} A \;,\; G \xrightarrow{\varepsilon_A(a)} A \;,\; \begin{tikzcd} & A \arrow[dr, "id"] & \\ G \arrow[ur, "\varepsilon_A(a)"] \arrow[dr, "\varepsilon_A(a)"'] & \Downarrow id\cong & A \\ & A \arrow[ur, "id"'] & \end{tikzcd} \;,\; \begin{tikzcd} & A \arrow[dr, "u_2"] & & \\ & \Downarrow\delta\cong & C_1 \arrow[dr, dashed] & \\ G \arrow[uur, "\eta_A(a)"] \arrow[r, "b"] \arrow[ddr, "\varepsilon_A(a)'"'] & B \arrow[ur, "v_2"] \arrow[dr, "v_1"'] & \Downarrow\tau\cong & C_3 \\ & \Downarrow\gamma^{-1}\cong & C_2 \arrow[ur, dashed] & \\ & A \arrow[ur, "u_1"'] & & \end{tikzcd} \right)$$

via the isomorphism

$$(id,\gamma^{-1}) = \left(\begin{tikzcd} & A \arrow[dr, "id"] & \\ F \arrow[ur, "a"] \arrow[dr, "a"'] \arrow[rr, phantom, "\Downarrow id"] & & A \\ & A \arrow[ur, "id"'] & \end{tikzcd} , \begin{tikzcd} & B \arrow[dr, "v_1"] & \\ G \arrow[ur, "b"] \arrow[dr, "\varepsilon_A(a)"'] \arrow[rr, phantom, "\Downarrow\gamma^{-1}\cong"] & & C_1 \\ & A \arrow[ur, "u_1"'] & \end{tikzcd} \right)$$

with inverse

$$(id,\gamma) = \left(\begin{tikzcd} & A \arrow[dr, "id"] & \\ F \arrow[ur, "a"] \arrow[dr, "a"'] \arrow[rr, phantom, "\Downarrow id"] & & A \\ & A \arrow[ur, "id"'] & \end{tikzcd} , \begin{tikzcd} & A \arrow[dr, "u_1"] & \\ G \arrow[ur, "\varepsilon_A(a)"] \arrow[dr, "b"'] \arrow[rr, phantom, "\Downarrow\gamma\cong"] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd} \right)$$

(2) Fullness
Suppose given objects in the image of $\Diamond$ and a morphism between them:

$$\left(\begin{tikzcd} & A \arrow[dr, "r"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi"] & & D_1 \\ & A' \arrow[ur, "r'"'] & \end{tikzcd} , \begin{tikzcd} & A \arrow[dr, "s"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & D_2 \\ & A' \arrow[ur, "s'"'] & \end{tikzcd} \right) : \Diamond(a,b,\gamma,\delta) \longrightarrow \Diamond(a',b',\gamma',\delta')$$

First, we want to obtain equivalent representatives for the morphisms ξ and σ, with $D_1 = D_2$, $r = s$ and $r' = s'$. For this goal, we observe that by application of axiom FF1 of 2-filteredness of $\mathcal{A}$ (in definition 3.8), as in the proof of fullness in the case of the product (theorem 7.10), we obtain $\begin{tikzcd} & A \arrow[dr, "u"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi'"] & & D \\ & A' \arrow[ur, "u'"'] & \end{tikzcd} , \begin{tikzcd} & A \arrow[dr, "u"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma'"] & & D \\ & A' \arrow[ur, "u'"'] & \end{tikzcd}$ with $\xi \sim \xi'$ in $\mathcal{L}(F)$ and $\sigma \sim \sigma'$ in $\mathcal{L}(G)$. The equations

$$\sigma \circ \gamma = \gamma' \circ \varepsilon_{D_1}(\xi) \tag{7.20}$$

$$\sigma \circ \delta = \delta' \circ \varepsilon_{D_2}(\xi) \tag{7.21}$$

hold in $\mathcal{L}(G)$ because (ξ,σ) is a morphism in $\mathcal{E}$. Since equivalence of premorphisms is preserved by both composition and the functors ε and η (this only means that composition and the functors are well-defined with respect to equivalence classes given by homotopies in $\mathcal{L}(F)$), we can replace ξ by ξ' and σ by σ' in the equations 7.20 and 7.21 to obtain

$$\sigma' \circ \gamma = \gamma' \circ \varepsilon_D(\xi')$$
$$\sigma' \circ \delta = \delta' \circ \varepsilon_D(\xi')$$

Then (ξ',σ') is also a morphism in $\mathcal{E}$, and it is equal to (ξ,σ). This allows us to assume that we start in the simpler case of a morphism represented by

$$\left(\begin{tikzcd}[sep=small] & A \arrow[dr, "u"] \arrow[dd, phantom, "\Downarrow\xi"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] & & D \\ & A' \arrow[ur, "u'"'] & \end{tikzcd}, \begin{tikzcd}[sep=small] & A \arrow[dr, "u"] \arrow[dd, phantom, "\Downarrow\sigma"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] & & D \\ & A' \arrow[ur, "u'"'] & \end{tikzcd}\right) : \Diamond(a,b,\gamma,\delta) \longrightarrow \Diamond(a',b',\gamma',\delta')$$

The equations $\sigma \circ \gamma = \gamma' \circ \varepsilon_D(\xi), \sigma \circ \delta = \delta' \circ \varepsilon_D(\xi)$ hold in $\mathcal{L}(G)$, and thus really are equivalences of premorphisms given, each one, by a homotopy. As in the proof of the faithfulness of the comparison functor in the case of the product (theorem 7.10), we can apply lemma 5.34 to assume that it is the same pair $\left(\begin{tikzcd}[sep=small] & D \arrow[dr, "h"] \arrow[dd, phantom, "\Downarrow\omega\cong"] & \\ A' \arrow[ur, "u'"] \arrow[dr, "u'"'] & & R \\ & D \arrow[ur, "k"'] & \end{tikzcd}, \begin{tikzcd}[sep=small] & D \arrow[dr, "h"] \arrow[dd, phantom, "\Downarrow\tau\cong"] & \\ A' \arrow[ur, "u"] \arrow[dr, "u"'] & & R \\ & D \arrow[ur, "k"'] & \end{tikzcd}\right)$ of invertible 2-cells in $\mathcal{A}$ that witnesses these homotopies. The LL equation for the first homotopy $\sigma \circ \gamma \sim \gamma' \circ \varepsilon_D(\xi)$ is

$$\begin{tikzcd}[sep=small] & & A \arrow[dr, "id"] & & & \\ & & \Downarrow\gamma\cong & A \arrow[dr, "u"] & & \\ G \arrow[uurr, "\varepsilon_A(a)"] \arrow[rr, "b"] \arrow[ddrr, "b'"'] & & A \arrow[ur, "id"] \arrow[dr, "u"'] \arrow[r, phantom, "\Downarrow id"] & & D \arrow[dr, "h"] & \\ & & \Downarrow\sigma & D \arrow[ur, "id"'] \arrow[r, phantom, "\Downarrow\omega\cong"] & & R \\ & & A' \arrow[ur, "u'"] \arrow[dr, "u'"'] & & & \\ & & & D \arrow[uurr, "k"'] & & \end{tikzcd} = \begin{tikzcd}[sep=small] & & & D \arrow[ddrr, "h"] & & \\ & & A \arrow[ur, "u"] \arrow[dr, "u"] & & \Downarrow\tau\cong & \\ & \Downarrow\varepsilon_D(\xi) & & D \arrow[dr, "id"] & & R \\ G \arrow[uurr, "\varepsilon_A(a)"] \arrow[rr, "\varepsilon_{A'}(a')"] \arrow[ddrr, "b'"'] & & A' \arrow[ur, "u'"] \arrow[dr, "id"'] \arrow[r, phantom, "\Downarrow id"] & & D \arrow[ur, "k"'] & \\ & \Downarrow\gamma'\cong & & A' \arrow[ur, "u'"'] & & \\ & & A' \arrow[ur, "id"'] & & & \end{tikzcd}$$

We define $\alpha = (Fk)(\xi)$ and $\beta = (G\omega)_{b'} \circ (Gh)(\sigma) \circ (G\tau^{-1})_b$. Precomposing with $\tau^{-1} * id_{\varepsilon_A(a)}$ on both members of this LL equation, we obtain

$$\begin{tikzcd}[sep=small] & & A \arrow[dr, "id"] & & \\ & & \Downarrow\gamma & A \arrow[dr, "ku"] & \\ G \arrow[uurr, "\varepsilon_A(a)"] \arrow[rr, "b"] \arrow[ddrr, "b'"'] & & A \arrow[ur, "id"] \arrow[dr, "ku"'] \arrow[r, phantom, "\Downarrow id"] & & R \\ & & \Downarrow\beta & R \arrow[ur, "id"'] & \\ & & A' \arrow[ur, "ku'"'] & & \end{tikzcd} = \begin{tikzcd}[sep=small] & & A \arrow[dr, "u"] & & \\ & & \Downarrow\varepsilon_D(\xi) & D \arrow[dr, "k"] & \\ G \arrow[uurr, "\varepsilon_A(a)"] \arrow[rr, "\varepsilon_{A'}(a')"] \arrow[ddrr, "b'"'] & & A' \arrow[ur, "u'"] \arrow[dr, "id"'] \arrow[r, phantom, "\Downarrow id"] & & R \\ & & \Downarrow\gamma'\cong & A' \arrow[ur, "ku'"'] & \\ & & A' \arrow[ur, "id"'] & & \end{tikzcd}$$

This says that

$$\beta \circ G(ku)(\gamma) = G(ku')(\gamma') \circ G(k)(\varepsilon_D(\xi)) \tag{7.22}$$

Since

$$\begin{aligned} G(k)(\varepsilon_D(\xi)) &= (Gk \circ \varepsilon_C)(\xi) \\ &= (\varepsilon_R \circ F(k))(\xi) \\ &= \varepsilon_R(\alpha) \end{aligned}$$

because ε is a 2-natural transformation $F \Rightarrow G$, the equation 7.22 is equivalent to

$$\beta \circ G(ku)(\gamma) = G(ku')(\gamma') \circ \varepsilon_R(\alpha)$$

In the same manner, we can obtain

$$\beta \circ G(ku)(\delta) = G(ku')(\delta') \circ \eta_R(\alpha)$$

These two equations are precisely the condition that (α, β) defines a morphism in $\mathcal{E}$:

$$\begin{tikzcd} & A \arrow[dr, "ku"] & \\ E \arrow[ur, "{(a,b,\gamma,\delta)}"] \arrow[dr, "{(a',b',\gamma',\delta')}"'] \arrow[rr, phantom, "\Downarrow(\alpha,\beta)"] & & R \\ & A' \arrow[ur, "ku'"'] & \end{tikzcd}$$

If we apply $\lozenge$ to this morphism we get

$$\lozenge(\alpha,\beta) = \left(\begin{tikzcd} & A \arrow[dr, "ku"] & \\ F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & R \\ & A' \arrow[ur, "ku'"'] & \end{tikzcd} , \begin{tikzcd} & A \arrow[dr, "ku"] & \\ G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\beta"] & & R \\ & A' \arrow[ur, "ku'"'] & \end{tikzcd} \right)$$

Finally, we observe that $\alpha \sim \xi$ and $\beta \sim \sigma$, since α and β were obtained from ξ and σ respectively by pasting of invertible 2-cells. Then in $\mathcal{E}$ we have $\lozenge(\alpha,\beta) = (\xi,\sigma)$.

(3) Faithfulness

Given morphisms $\begin{tikzcd} & A \arrow[dr, "u_1"] & \\ E \arrow[ur, "{(a,b,\gamma,\delta)}"] \arrow[dr, "{(a',b',\gamma',\delta')}"'] \arrow[rr, phantom, "\Downarrow(\xi_1,\sigma_1)"] & & C_1 \\ & A' \arrow[ur, "v_1"'] & \end{tikzcd}$ and $\begin{tikzcd} & A \arrow[dr, "u_2"] & \\ E \arrow[ur, "{(a,b,\gamma,\delta)}"] \arrow[dr, "{(a',b',\gamma',\delta')}"'] \arrow[rr, phantom, "\Downarrow(\xi_2,\sigma_2)"] & & C_2 \\ & A' \arrow[ur, "v_2"'] & \end{tikzcd}$ in $\mathcal{L}(E)$ with $\lozenge(\xi_1,\sigma_1) = \lozenge(\xi_2,\sigma_2)$, this is $\xi_1 \sim \xi_2$ in $\mathcal{L}(F)$ and $\sigma_1 \sim \sigma_2$ in $\mathcal{L}(G)$: there exist invertible 2-cells α_1, β_1, α_2, β_2 satisfying the LL equations

$$
\begin{tikzcd}
 & A \arrow[dr, "u_1"] & & \\
F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi_1"] & & C_1 \arrow[dr, "w_1"] & \\
 & A' \arrow[ur, "v_1"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha_1\cong"] & & C \\
 & & C_2 \arrow[ur, "w_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & & C_1 \arrow[dr, "w_1"] & \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\beta_1\cong"] & & C \\
F \arrow[ur, "a"] \arrow[dr, "a'"'] \arrow[rr, phantom, "\Downarrow\xi_2"] & & C_2 \arrow[ur, "w_2"'] & \\
 & A' \arrow[ur, "v_2"'] & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & A \arrow[dr, "u_1"] & & \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma_1"] & & C_1 \arrow[dr, "w'_1"] & \\
 & A' \arrow[ur, "v_1"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\Downarrow\alpha_2\cong"] & & C' \\
 & & C_2 \arrow[ur, "w'_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & & C_1 \arrow[dr, "w'_1"] & \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\Downarrow\beta_2\cong"] & & C' \\
G \arrow[ur, "b"] \arrow[dr, "b'"'] \arrow[rr, phantom, "\Downarrow\sigma_2"] & & C_2 \arrow[ur, "w'_2"'] & \\
 & A' \arrow[ur, "v_2"'] & &
\end{tikzcd}
$$

We proceed as in the proof of the fullness of $\Diamond$ for the product (theorem 7.10). Thus we can assume $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. These two equations are then precisely the equations for homotopy between (ξ_1, σ_1) and (ξ_2, σ_2) in $\mathcal{L}(E)$ given in diagram 7.18. We conclude that these premorphisms define the same morphism in $\mathcal{L}(E)$.

$\square$

7.5. **Cotensor product.** The following is proved in [10].

Theorem 7.23. *Given a 2-functor* $\mathcal{A} \xrightarrow{F} \mathcal{C}at$ *with* $\mathcal{A}$ *2-filtered and a finite category* $\mathcal{P}$*, we can consider the 2-functor* $F^{\mathcal{P}} : \mathcal{A} \longrightarrow \mathcal{C}at$ *defined by* $F^{\mathcal{P}}(A) = (FA)^{\mathcal{P}}$ *and similarly for morphisms and 2-cells. Then, the comparison functor* $\Diamond : \mathcal{L}(F^{\mathcal{P}}) \longrightarrow \mathcal{L}(F)^{\mathcal{P}}$ *(given by the universal property of* $\mathcal{L}(F^{\mathcal{P}})$ *as in theorem 5.32) is an equivalence of categories.*

7.6. **Main result.** We are now in conditions to state and prove the main result of this thesis, which is the following

Theorem 7.24. *2-filtered bicolimits and finite weighted bilimits commute in $\mathcal{C}at$. More precisely, given a 2-functor $\mathcal{A} \times \mathcal{P} \xrightarrow{F} \mathcal{C}at$ with $\mathcal{A}$ 2-filtered and $\mathcal{P} \xrightarrow{W} \mathcal{C}at$ a finite weight, the comparison functor*

$$\underrightarrow{\text{bicolim}}_{\mathcal{A}}\, \underleftarrow{\text{wbilim}}^{W}_{\mathcal{P}}\, F \xrightarrow{\lozenge} \underleftarrow{\text{wbilim}}^{W}_{\mathcal{P}}\, \underrightarrow{\text{bicolim}}_{\mathcal{A}}\, F$$

is an equivalence of categories.

Proof. We apply the construction of the bilimit given in 6.2 to decompose the weighted bilimit $\underleftarrow{\text{wbilim}}^{W}_{\mathcal{P}}$ as a composition of biequalizers, finite biproducts and bicotensors with a finite category. It this way, it suffices to prove the equivalence for these three classes of finite bilimits.

$$\underrightarrow{\text{bicolim}}_{\mathcal{A}}\, \underleftarrow{\text{wbilim}}^{V}_{\mathcal{Q}}\, G \simeq \underleftarrow{\text{wbilim}}^{V}_{\mathcal{Q}}\, \underrightarrow{\text{bicolim}}_{\mathcal{A}}\, G$$

where the finite category $\mathcal{Q}$ and the 2-functors V and G stand for the finite 2-categories and 2-functors which define biequalizers, finite biproducts and bicotensors with a finite category, in each case.

We may take these bilimits to be pseudoequalizers, finite pseudoproducts and pseudocotensors with a finite category, because these exist in $\mathcal{C}at$, and are equivalent to any choice of the corresponding bilimits (see observation 4.8). Pseudoproducts are just 2-products (see subsection 5.3), and pseudocotensors are just cotensors (see observation 5.17). It then suffices to prove that the 2-filtered pseudocolimit (of which we have a construction in subsection 5.16) commutes with each of these classes of finite pseudolimits:

$$\underrightarrow{\text{pscolim}}_{\mathcal{A}}\, \underleftarrow{\text{wpslim}}^{V}_{\mathcal{Q}}\, G \simeq \underleftarrow{\text{wpslim}}^{V}_{\mathcal{Q}}\, \underrightarrow{\text{pscolim}}_{\mathcal{A}}\, G$$

where $\mathcal{Q}$, V and G stand as before for the data that defines pseudoequalizers, finite 2-products, or cotensor products with a finite category. Since this is established in theorems 7.10 (and observation 7.15), 7.19 and 7.23, the result follows. $\square$

8. Ends

In ordinary category theory, apart from the usual conical limits of functors $\mathcal{A} \longrightarrow \mathcal{B}$, we have the notion of the end of a functor in two variables $\mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathcal{B}$. The theory of ends and coends can simplify many calculations in category theory, as shown for example in the comprehensive [21].

During the study of the 2-categorical theory for this work (in particular, [15]), we noticed that some results about weighted 2-limits could be proven quickly if we were able to apply a version of the calculus of ends and coends to the 2-categorical setting. In fact, there is a close relationship between ends and weighted limits, that extends to the 2-dimensional case.

To end this thesis, we develop basic definitions and results on a 2-categorical version of ends, which we find useful to prove propositions involving weighted 2-limits and weighted 2-colimits. Definitions and results on ends in enriched category theory can be found in [9].

Definition 8.1. Given functors $\mathcal{A}^{op} \times \mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$, a dinatural transformation between them $F \overset{\alpha}{\Longrightarrow} G$ is a collection of arrows $F(A,A) \xrightarrow{\alpha_A} G(A,A)$, one for each object $A \in \mathcal{A}$, such that for all $A \xrightarrow{f} B$ in $\mathcal{A}$ the following diagram commutes

$$\begin{tikzcd}
& F(A,A) \arrow[r, "\alpha_A"] & G(A,A) \arrow[dr, "G(id_A,f)"] & \\
F(B,A) \arrow[ur, "F(f,id_A)"] \arrow[dr, "F(id_B,f)"'] & & \equiv & G(A,B) \\
& F(B,B) \arrow[r, "\alpha_B"'] & G(B,B) \arrow[ur, "G(f,id_B)"'] &
\end{tikzcd}$$

When one of F or G is a constant functor Δ_B, we call α a wedge.

Definition 8.2. Given a functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{F} \mathcal{B}$ and an object $Z \in \mathcal{B}$, a wedge $Z \overset{\alpha}{\Longrightarrow} F$ is a collection of arrows $Z \xrightarrow{\alpha_A} F(A,A)$, one for each object $A \in \mathcal{A}$, such that for all $A \xrightarrow{f} B$ in $\mathcal{A}$ the following diagram commutes

$$\begin{tikzcd}
Z \arrow[r, "\alpha_A"] \arrow[d, "\alpha_B"'] \arrow[dr, phantom, "\equiv"] & F(A,A) \arrow[d, "F(id_A,f)"] \\
F(B,B) \arrow[r, "F(f,id_B)"'] & F(A,B)
\end{tikzcd}$$

Definition 8.3. The end of a functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{F} \mathcal{B}$ is a wedge $L \overset{\alpha}{\Longrightarrow} F$ such that every other wedge $Z \overset{\beta}{\Longrightarrow} F$ factorizes uniquely through it: if we have arrows $Z \xrightarrow{\alpha_A} F(A,A)$ for each $A \in \mathcal{A}$ such that $F(id_A, f)\beta_A = F(f, id_B)\beta_B$ for all $A \xrightarrow{f} B$ in $\mathcal{A}$, there exists a unique $Z \overset{h}{\dashrightarrow} L$ with $\alpha_A h = \beta_A$ for all $A \in \mathcal{A}$:

$$\begin{tikzcd}
 & & F(A,A) \arrow[dr, "F(id_A{,}f)"] & \\
Z \arrow[r, dashed, "h"] \arrow[urr, bend left, "\beta_A"] \arrow[drr, bend right, "\beta_B"'] \arrow[ur, phantom, "\equiv"] \arrow[dr, phantom, "\equiv"] & L \arrow[ur, "\alpha_A"] \arrow[dr, "\alpha_B"'] \arrow[rr, phantom, "\equiv"] & & F(A,B) \\
 & & F(B,B) \arrow[ur, "F(f{,}id_B"'] &
\end{tikzcd}$$

This universal wedge L is denoted $\displaystyle\int_A F(A,A)$.

We can dualize the definitions.

Definition 8.4. The coend of a functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{F} \mathcal{B}$ is a wedge $F \overset{\alpha}{\Longrightarrow} L$ such that every other wedge $F \overset{\beta}{\Longrightarrow} Z$ factorizes uniquely through it: there exists a unique $L \overset{h}{\dashrightarrow} Z$ with $h\alpha_A = \beta_A$ for all $A \in \mathcal{A}$. This universal wedge is denoted $\displaystyle\int^A F(A,A)$.

Observation 8.5. When the relevant limits exist, a collection of arrows $Z \xrightarrow{\alpha_A} F(A,A)$ is an end when the following diagram is an equalizer

$$Z \xrightarrow{\alpha} \prod_A F(A,A) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{f:A\to B} F(B,B)$$

where α, s and t are the unique morphisms such that

$$\pi_A \alpha = \alpha_A$$
$$\pi_{(f:A\to B)} s = F(id_A, f)$$
$$\pi_{(f:A\to B)} t = F(f, id_B)$$

Analogously, coends are certain coequalizers of coproducts.
This says that ends exist whenever products and equalizers exist and are preserved by representable functors.

Proposition 8.6. *Given functors* $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ *, we have a functor*

$$\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{\mathrm{Hom}(F-,G-)} \mathcal{E}ns$$

The end of this functor is the set of natural transformations $F \Rightarrow G$, i.e.

$$\int_A \mathrm{Hom}(FA, GA) = Nat(F, G)$$

Proof. We propose of course $Nat(F,G) \xrightarrow{\alpha_A} \mathrm{Hom}(FA, GA)$ as the function taking the A-component of a natural transformation: $F \overset{\lambda}{\Longrightarrow} G \quad \mapsto \quad FA \xrightarrow{\lambda_A} GA$. This gives a wedge $Nat(F,G) \overset{\alpha}{\Longrightarrow} \mathrm{Hom}(F-, G-)$ because, given a $A \xrightarrow{f} B$ in $\mathcal{A}$, we have

$$(\mathrm{Hom}(id_{FA}, Gf)\alpha_A)(\lambda) = Gf \circ \lambda_A$$

and

$$(\mathrm{Hom}(Ff, id_{GA})\alpha_B)(\lambda) = \lambda_B \circ Ff$$

and these are equal for all such f precisely when λ is a natural transformation $F \Rightarrow G$.

By the same reasoning, dinaturality of a $Z \overset{\beta}{\Longrightarrow} \mathrm{Hom}(F-, G-)$ says that $\alpha_A(z)$, with varying A, gives the components of a natural transformation, for every $z \in Z$. Then the canonical function $Z \overset{h}{\dashrightarrow} Nat(F, G)$ assigns to each $z \in Z$ the natural transformation with components $(\alpha_A(z))_A$. $\square$

We want to obtain a 2-categorical analogue of the result in proposition 8.6.

Definition 8.7. Given a 2-functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{F} \mathcal{B}$ and $Z \in \mathcal{B}$, a 2-wedge $Z \overset{\alpha}{\Longrightarrow} F$ is a family of morphisms $Z \xrightarrow{\alpha_A} F(A, A)$ for each $A \in \mathcal{A}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}(A,B) \arrow[rr, "F(A,-)"] \arrow[d, "T(-,B)"'] & & \mathcal{B}(F(A,A), F(A,B)) \arrow[d, "\mathcal{B}(\alpha_A, id_{F(A,B)})"] \\
\mathcal{B}(F(B,B), F(A,B)) \arrow[rr, "\mathcal{B}(\alpha_B, id_{F(A,B)})"'] & \equiv & \mathcal{B}(Z, F(A,B))
\end{tikzcd}$$

This is: for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & B \end{tikzcd}$ in $\mathcal{A}$ we have

$$F(A, f)\alpha_A = F(f, B)\alpha_B$$

(the condition for an ordinary wedge) and

$$T(A, \varepsilon) * id_{\alpha_A} = T(\varepsilon, B) * id_{\alpha_B}$$

Definition 8.8. Given a 2-functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{F} \mathcal{B}$ a 2-end is a 2-wedge $L \xRightarrow{\alpha} F$ such that every other 2-wedge $Z \xRightarrow{\beta} F$ factorizes uniquely through it: if we have morphisms $Z \xrightarrow{\alpha_A} F(A,A)$ for each $A \in \mathcal{A}$ such that $F(A,f)\beta_A = F(f,B)\beta_B$ and $T(A,\varepsilon) * id_{\beta_A} = T(\varepsilon,B) * id_{\beta_B}$ for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & B \end{tikzcd}$ in $\mathcal{A}$, then there exists a unique $Z \overset{h}{\dashrightarrow} L$ with $\alpha_A h = \beta_A$ for all $A \in \mathcal{A}$:

$$\begin{tikzcd}
 & & F(A,A) \arrow[dr, "F(id_A{,}f)"] & \\
Z \arrow[r, dashed, "h"] \arrow[urr, bend left, "\beta_A"] \arrow[drr, bend right, "\beta_B"'] \arrow[ur, phantom, "\equiv"] \arrow[dr, phantom, "\equiv"] & L \arrow[ur, "\alpha_A"] \arrow[dr, "\alpha_B"'] & \equiv & F(A,B) \\
 & & F(B,B) \arrow[ur, "F(f{,}id_B"'] &
\end{tikzcd}$$

This universal 2-wedge L is denoted $\displaystyle\int_A F(A,A)$.

Observation 8.9. Also in this 2-categorical setting, 2-ends are particular 2-limits, by a very similar formula:

$$\int_A F(A,A) \xrightarrow{\alpha} \prod_A F(A,A) \overset{s}{\underset{t}{\rightrightarrows}} \prod_{A,B} \{\mathcal{A}(A,B), F(B,B)\} \tag{8.10}$$

is an equalizer, and then 2-ends are preserved by representable 2-functors, just like any 2-limit.

Proposition 8.11. *Given 2-functors* $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ *, we have a 2-functor*

$$\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{\mathrm{Hom}(F-,G-)} \mathcal{C}at$$

The 2-end of this 2-functor is the category of 2-natural transformations $F \Rightarrow G$ *and modifications between them, i.e.*

$$\int_A \mathrm{Hom}(FA,GA) = [\mathcal{A},\mathcal{B}](F,G) \tag{8.12}$$

Proof. The proof is very similar to the 1-dimensional case.
We propose $[\mathcal{A},\mathcal{B}](F,G) \xrightarrow{\alpha_A} \mathrm{Hom}(FA,GA)$ as the functor taking the A-component of a 2-natural transformation or a modification: $F \xRightarrow{\lambda} G \ \mapsto \ FA \xrightarrow{\lambda_A} GA$, and $\lambda \overset{\varphi}{\rightsquigarrow} \mu \mapsto \ \lambda_A \xRightarrow{\varphi_A} \mu_A$.

Given $A \underset{g}{\overset{f}{\underset{\Downarrow\varepsilon}{\rightrightarrows}}} B$ in $\mathcal{A}$ we have

$$(\mathrm{Hom}(id_{FA}, Gf)\alpha_A)(\lambda) = Gf \circ \lambda_A$$
$$(\mathrm{Hom}(Ff, id_{GA})\alpha_B)(\lambda) = \lambda_B \circ Ff$$

and

$$(\mathrm{Hom}(id_{FA}, Gf)\alpha_A)(\varphi) = id_{Gf} \circ \varphi_A$$
$$(\mathrm{Hom}(Ff, id_{GA})\alpha_B)(\varphi) = \varphi_B \circ id_{Ff}$$

so that

$$\mathrm{Hom}(id_{FA}, Gf)(\lambda) = \mathrm{Hom}(Ff, id_{GA})(\lambda)$$
$$\mathrm{Hom}(id_{FA}, Gf)(\varphi) = \mathrm{Hom}(Ff, id_{GA})(\varphi)$$

for all such f precisely when λ is a 2-natural transformation $F \Rightarrow G$ and φ is a modification. Then

$$\mathrm{Hom}(id_{FA}, Gf)\alpha_A = \mathrm{Hom}(Ff, id_{GA})\alpha_B \tag{8.13}$$

Given a 2-natural transformation $F \overset{\lambda}{\Longrightarrow} G$, we have

$$(\mathrm{Hom}(id_{FA}, G\varepsilon) * id_{\alpha_A})_\lambda = \mathrm{Hom}(F\varepsilon, id_{GB}) * id_{\alpha_B})_\lambda$$

because this is equivalent to

$$G\varepsilon * id_{\lambda_A} = id_{\lambda_B} * F\varepsilon$$

which is true by 2-naturality of λ. Then we also have that

$$\mathrm{Hom}(id_{FA}, G\varepsilon) * id_{\alpha_A} = \mathrm{Hom}(F\varepsilon, id_{GB}) * id_{\alpha_B} \tag{8.14}$$

Equations 8.13 and 8.14 say that α defines a 2-wedge.

To see that it is universal, following the same reasoning in reverse we get that if we have a 2-wedge

$$\mathcal{Z} \overset{\beta}{\Longrightarrow} \mathrm{Hom}(F-, G-)$$

and $Z \xrightarrow{f} Z'$ in $\mathcal{Z}$, then $\beta_A(Z)$ is the A-component of a 2-natural transformation, and $\beta_A(f)$ is the A-component of a modification, for each A. Thus we define the unique functor

$$\mathcal{Z} \overset{h}{\dashrightarrow} [\mathcal{A}, \mathcal{B}](F, G)$$

as assigning to each Z the 2-natural transformation $F \Rightarrow G$ with components $(\beta_A(Z))_A$, and to each f the modification $h(Z) \rightsquigarrow h(Z')$ with components $(\beta_A(f))_A$. $\square$

Of course, we also have the following

Definition 8.15. The 2-coend of a 2-functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{F} \mathcal{B}$ is a 2-wedge $F \overset{\alpha}{\Longrightarrow} L$ such that every other 2-wedge $F \overset{\beta}{\Longrightarrow} Z$ factorizes uniquely through it: there exists a unique $L \overset{h}{\dashrightarrow} Z$ with $h\alpha_A = \beta_A$ for all $A \in \mathcal{A}$. This universal 2-wedge is denoted $\int^A F(A, A)$.

Formulas involving 2-ends and 2-coends, just like 1-categorical ends and coends, are very useful for calculations. We give an application to illustrate this.

Proposition 8.16. *Given 2-functors $\mathcal{B}^{op} \xrightarrow{F} \mathcal{C}at$ and $\mathcal{B} \xrightarrow{G} \mathcal{C}at$, we can consider both $\underrightarrow{\mathrm{wcolim}}^F G$ and $\underrightarrow{\mathrm{wcolim}}^G F$, and these are canonically isomorphic.*

Proof.

$$\begin{aligned}
[\mathcal{B}^{op}, \mathcal{C}at](F, \mathcal{C}at(G-, \mathcal{Z})) &\cong \int_B \mathcal{C}at(FB, \mathcal{C}at(GB, \mathcal{Z})) \\
&\cong \int_B \mathcal{C}at(GB, \mathcal{C}at(FB, \mathcal{Z}) \\
&\cong [\mathcal{B}^{op}, \mathcal{C}at](G, \mathcal{C}at(F-, \mathcal{Z}))
\end{aligned}$$

where we use the formula 8.12 and the 2-natural isomorphism

$$\mathcal{C}at(\mathcal{X}, \mathcal{C}at(\mathcal{Y}, \mathcal{Z})) \cong \mathcal{C}at(\mathcal{Y}, \mathcal{C}at(\mathcal{X}, \mathcal{Z}))$$

This says that

$$[\mathcal{B}^{op}, \mathcal{C}at](F, \mathcal{C}at(G-, \mathcal{Z})) \cong [\mathcal{B}^{op}, \mathcal{C}at](G, \mathcal{C}at(F-, \mathcal{Z}))$$

and thus any object representing either of these 2-functors represents the other. By the 2-Yoneda lemma (theorem 2.34),

$$\underrightarrow{\mathrm{wcolim}}^F G \cong \underrightarrow{\mathrm{wcolim}}^G F$$

□

Another useful proposition is the following co-Yoneda lemma

Proposition 8.17. *Given a 2-functor $\mathcal{B}^{op} \xrightarrow{F} \mathcal{C}at$, we have an isomorphism, 2-natural in A*

$$FA \cong \int_B \{\mathcal{B}(A, B), FB\}$$

Proof.

$$\begin{aligned}
\mathcal{C}at(\mathcal{X}, \int_B \{\mathcal{B}(A, B), FB\}) &\cong \int_B \mathcal{C}at(\mathcal{X}, \{\mathcal{B}(A, B), FB\}) \\
&\cong \int_B \mathcal{C}at(\mathcal{B}(A, B), \mathcal{C}at(\mathcal{X}, FB)) \\
&\cong [\mathcal{B}, \mathcal{C}at](\mathcal{B}(A, -), \mathcal{C}at(\mathcal{X}, F-)) \\
&\cong \mathcal{C}at(\mathcal{X}, FA)
\end{aligned}$$

where we use the defining property of cotensor products, preservation of 2-ends by representable 2-functors, and the formula 8.12. By the 2-Yoneda lemma (theorem 2.34), we obtain

$$FA \cong \int_B \{\mathcal{B}(A,B), FB\}$$

□

Finally, we remark that even though 2-ends and 2-coends are useful for computations, in categories with certain 2-limits and 2-colimits they give no new universal objects.

Observation 8.18. A limit of a 2-functor $\mathcal{B} \xrightarrow{F} \mathcal{C}$ weighted by a 2-functor $\mathcal{B} \xrightarrow{W} \mathcal{C}at$ is a 2-end, provided the 2-end and the cotensor products appearing below exist:

$$\begin{aligned}
[\mathcal{B}, \mathcal{C}at](W, \mathcal{C}(C, F-)) &\cong \int_D \mathcal{C}at(WD, \mathcal{C}(C, FD)) \\
&\cong \int_D \mathcal{C}(C, \{WD, FD\}) \\
&\cong \mathcal{C}(C, \int_D \{WD, FD\})
\end{aligned}$$

Then we have

$$\underleftarrow{\mathrm{wlim}}^W F \cong \int_D \{WD, FD\}$$

because this 2-end represents the 2-functor defining the 2-limit (see definition 4.3).

We already know that a 2-end can be expressed as a 2-limit by the formula 8.10. We now give a proof of this fact by calculus of 2-ends.

Observation 8.19. The 2-end of a 2-functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{F} \mathcal{B}$ is a 2-limit weighted by an hom 2-functor whenever the 2-limits appearing below exist:

$$\begin{aligned}
\int_A F(A,A) &\cong \int_A \int_B \{\mathcal{A}(A,B), F(A,B)\} \\
&\cong \int_{A,B} \{\mathcal{A}(A,B), F(A,B)\} \\
&\cong \underleftarrow{\mathrm{wlim}}^{\mathcal{A}(-,-)} F
\end{aligned}$$

where we use

$$F(A,A) \cong \int_B \{\mathcal{A}(A,B), F(A,B)\}$$

as a consecuence of the co-Yoneda lemma 8.17 applied to $F(A,-)$, and the fact that 2-ends commute with 2-ends (this is called the "Fubini theorem" for iterated 2-ends; see [21, remark 1.16] for a 1-categorical version of this statement, or [14, section 2.1] for the general formulation in the enriched setting).

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