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Tesis de Licenciatura

Fibraciones como Representaciones: Fibraciones Esféricas. Implementación en Teoría Homotópica de Tipos

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Introducción

El objetivo de esta tesis es entender por qué las fibraciones sobre un espacio pueden ser interpretadas como representaciones del tipo homotópico de este espacio. El trabajo está dividido en dos capítulos.

El primer capítulo repasa la teoría de fibraciones esféricas y apunta a estudiar algunos conceptos que podrían acercar la K-teoría a la Teoría Homotópica de Tipos (HoTT). La parte principal de este capítulo es la demostración de un conocido teorema que relaciona los grupos de homotopía del espacio clasificante de fibraciones esféricas con los grupos de homotopía estables de las esferas.

En el segundo capítulo intentamos traducir las nociones básicas de la teoría de fibraciones esféricas a HoTT. Esto nos lleva a conjeturar la validez de resultados conocidos en teoría de homotopía y teoría de categorías, como la conmutatividad de limites homotópicos finitos y colímites homotópicos filtrantes. El estudio de espacios clasificantes nos lleva naturalmente a considerar el concepto de *diagramas secuenciales* y *tipos compactos*. Si bien estos son temas relativamente nuevos en este contexto, es claro que son buenas abstracciones y una teoría sobre estas sería útil. Varias ideas en esta sección se deben a Egbert Rijke. Cuando consideramos los grupos de homotopía del espacio clasificante de fibraciones esféricas en HoTT debemos también estudiar el grado de funciones $\mathbb{S}^n \to \mathbb{S}^n$ y la acción del grupo fundamental de un espacio en sus grupos de homotopía.

En el segundo capítulo el lenguaje de HoTT formaliza la analogía entre representaciones de un tipo homotópico y fibraciones sobre el mismo, y teniendo esto en cuenta concluimos definiendo las nociones básicas de una teoría de ∞-grupos y sus representaciones en HoTT. El segundo capítulo es exploratorio, pocos argumentos están formalizados y verificados en una computadora y algunos argumentos son bastante informales. Está pensada como una colección de ideas bajo estudio y que serán objeto de trabajo futuro.

Agradecimientos

A mi familia: mis padres, mis hermanos, mis tíos y tías, mis primos y primas, mis sobrinos y sobrinas.

A Carina.

A todos mis mejores amigos.

A Nacho por los dibujos.

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Contents

Introduction

Scope

The objective of this thesis is to understand why fibrations over a space can be regarded as representations of its homotopy type. The work is divided in two chapters.

The first chapter reviews the theory of spherical fibrations and aims to study some concepts that could bring K-theory closer to Homotopy Type Theory (HoTT). The main content of the chapter is the statement and proof of a well known theorem that relates the homotopy groups of the classifying space of spherical fibrations to the stable homotopy groups of the spheres.

In the second chapter we attempt to translate the basics of the theory of spherical fibrations into HoTT. This leads us to conjecture the validity of well known results of homotopy theory and category theory in the HoTT setting, such as the commutativity of finite (homotopy) limits and filtered (homotopy) colimits. When studying classifying spaces we are naturally lead to the concept of *sequential diagram* and the concept of *compact type*. Although these are relatively new topics in this context it is already clear that they are good abstraction worthy of the development of a theory about it. Many ideas in this section are due to Egbert Rijke. When considering the homotopy groups of the classifying space of spherical fibrations in HoTT we are also lead to the study of the degree of maps $\mathbb{S}^n \to \mathbb{S}^n$ and the action of the fundamental group of a space on its homotopy groups.

In the second chapter the language of HoTT formalizes the analogy between representations of a homotopy type and fibrations over the type, and having this in mind we conclude by making an attempt to define the very basics of the theory of ∞-groups and their representations in the HoTT setting. The second chapter is exploratory, few of the arguments given there are formalized and some of them are quite informal. It is intended as a collection of ideas that are currently under study and that will be the subject of future work.

Prerequisites

For both chapters it is convenient to have some background in category theory since this language simplifies many explanations. The reader is referred to the classical text [ML98]. A good understanding of the basic notions in topology and of some basic notions of algebraic topology and homotopy theory will also be useful. Standard references for algebraic topology are [Spa94], [Hat02], [Swi02]. References for topological K-theory are [Ati94], [Kar08], [JM74] and [Hus94].

For the second chapter the reader is assumed to have some knowledge of the basics of Homotopy Type Theory. The canonical reference for this is the HoTT book [Uni13]. For a shorter but complete introduction to HoTT we also recommend [Rij12].

Notation and conventions

Generalities

0.0.0.1 Notation (Categories)**.** Standard categories that we will be using are:

Set	for sets.
Cat	for categories.
Space	for "nice" topological spaces. We will use CW-complexes.
hSpace	for "nice" topological spaces up to homotopy: The objects are CW-complexes and the arrows, maps up to homotopy.
cSpace	for "nice" compact topological spaces.
hcSpace	for "nice" compact topological spaces up to homotopy.
Grp	for groups.
Ring	for unital rings.
Ab	for abelian groups.
csGrp	for commutative semigroups.
$\mathsf{vec}_{\mathbb{K}}$	for finite dimensional vector spaces over a field $\mathbb{K}$.

Notice that instead of CW-complexes up to homotopy we can also use topological spaces up to weak equivalence. See [Qui67] for the theory that lets us identify these two categories.

0.0.0.2 Notation ($\hom$ spaces)**.** If $\mathcal{C}$ is a category and A and B are two objects of $\mathcal{C}$ we will use $\mathcal{C}(A, B)$ to denote the arrows between A and B. When the category $\mathcal{C}$ is implicit we may also use the notation $[A, B]$. When doing so for spaces A and B (see Notation 0.0.0.7) notation $[A, B]$ will *always* mean $\mathsf{hSpace}(A, B)$, the set of maps between A and B *up to homotopy*.

0.0.0.3 Notation (Pointed things)**.** When using pointed "things" such as pointed sets, pointed functions between pointed sets, etc, we will use the symbol "$\bullet$". For example the set of pointed functions between two pointed sets A and B will be denoted by $\mathsf{Set}_\bullet(A, B)$. And the set of pointed maps up to homotopy between two spaces will be denoted by $[A, B]_\bullet$.

0.0.0.4 Notation (Equivalences and analogies)**.** Many times we will state that there is an equivalence between two classes of objects. In general we could be referring to a bijection of sets, an equivalence of categories, an equivalence of types, etc. We will use a *double line* for this. For example, we can state (part) of the Yoneda lemma as:

$$\frac{[c, -] \to [d, -]}{d \to c}$$

We can read this as "A natural transformation between the representable functors $[c, -]$ and $[d, -]$ is the same as an arrow $c \to d$."

When we only have a construction that assigns to each object of a certain kind an object of another kind we will denote so with a *single line*. For example given a group G and a subgroup $H \leq G$ we can construct an action on the cosets $G \curvearrowright G/H$, but there is no way to assign to each action $G \curvearrowright X$ a subgroup $H \leq G$ in a natural way. We write this as:

$$\frac{H \leq G}{G \curvearrowright X}$$

0.0.0.5 Notation (Conjectures)**.** Results marked with a dagger "†" depend on conjectures. These will appear in the second chapter.

0.0.0.6 Notation (Spheres)**.** In both chapters the n-dimensional sphere will by denoted by $\mathbb{S}^n$.

Conventions for the first chapter

0.0.0.7 Notation (Spaces and maps)**.** We reserve the word *map* for continuous functions between topological spaces. The word *function* will usually mean function between sets. The word *space* will mean CW-complex.

0.0.0.8 Notation (Paths)**.** Some times we will write paths in some space A as a map $p : I \to A$, where I is the interval space. But other times it will be useful to explicitly state what the endpoints of the path are. For a path that starts at a and ends at b we will also use the notation $p : a \rightsquigarrow b$.

0.0.0.9 Definition (H-spaces)**.** An H-space is a space X together with an associative operation $\mu : X \times X \to X$ and a two-sided unit $e \in X$. *The associativity and unit law must hold on the nose, not just up to homotopy.*

Conventions for the second chapter

0.0.0.10 Notation (Universe)**.** For simplicity we will use just one universe which we denote by $\mathcal{U}$ although strictly speaking some constructions will actually live in a higher universe.

0.0.0.11 Notation (Spaces and maps)**.** Here *space* will mean type, and *map* will mean computable map between types.

0.0.0.12 Notation (Paths)**.** Path spaces in HoTT will be denoted by the usual $a =_A b$ where $a, b : A$. When the type that the elements inhabit is implicit we will omit the subscript and we will write just $a = b$.The composition of two composable paths p and q will be denoted by $p \cdot q$.

0.0.0.13 Notation (Definitionaly equality)**.** Definitional equality will be denoted by $\equiv$ as usual in HoTT. When defining an inhabitant of a type we will often use the notation $a :\equiv b : A$, which means that a is just a renaming of $b : A$. Of course in this case we have $a \equiv b$.

0.0.0.14 Notation (Dependent products and dependent sums)**.** If $B : A \to \mathcal{U}$ is a type family indexed by $A : \mathcal{U}$ we will denote the induced dependent product by $(a : A) \to B(a)$. If $B(a)$ is again a function type we will usually omit parenthesis, using the standard convention that $C \to D \to E$ means $C \to (D \to E)$.

In the same setting, the induced dependent sum will be denoted by $(a : A) \times B(a)$. Inhabitants of a dependent sum will be denoted by tuples. Again, if $B(a)$ is a dependent sum we will omit parenthesis.

0.0.0.15 Definition (H-spaces)**.** In the second chapter the notion of H-space will be the one given in [Uni13, Definition 8.5.4].

0.0.0.16 Notation (Mere things)**.** A *mere proposition* is a -1-truncated type as defined in [Uni13, Definition 3.3.1]. Sometimes we will say "*mere* inhabitant $a : A$" meaning an inhabitant $m : \|A\|_{-1}$. Will mostly do so when proving a *mere* proposition and thus we will immediately assume that m is of the form $|a|_{-1}$ justifying the abuse of terminology.

Standard algebraic topology definitions and results

The following are standard definitions and well known propositions that can be found in the cited bibliography about algebraic topology. The reader might wish to skip this part since this propositions will be referenced when needed.

0.0.0.17 Notation (Automorphisms space)**.** For X any space we write $\mathrm{hAut}(X)$ for the space of self homotopy equivalences of X with the compact open topology. Notice that this space has a natural H-space structure given by composition of self homotopy equivalences (composition is associative on the nose).

0.0.0.18 Definition (Suspension and loop space)**.** Given a space X we can construct its *suspension* as a quotient $\Sigma X := X \times I/\sim$, where the equivalence relation is given by $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. For a map $f : X \to Y$ we can consider the product map $f \times I : X \times I \to Y \times I$ that is the identity in the second component. It is straightforward to check that this induces a map between the quotients $\Sigma f : \Sigma Y \to \Sigma Y$. This makes Σ a functor $\Sigma : \mathsf{Space} \to \mathsf{Space}$.

When working with pointed spaces one usually uses the *reduced suspension* functor. The construction of the reduced suspension of a space is the same as the suspension of the space, but one also collapses the "line" that goes through the distinguished point of the space: If (X, x) is the pointed space then its reduced suspension collapses $\{(x, t) \in \Sigma X \mid t \in I\}$ to a point. It is easy to check that this construction is also functorial.

For a pointed space (X, x) there is another related construction, the *loop space* ΩX. This is the subspace of X^I given by paths starting and ending in x. This construction is also functorial.

0.0.0.19 Proposition (Exponential laws)**.** *For spaces X, Y, Z there is a natural isomorphism:*

$$\mathsf{Space}(X \times Y, Z) \simeq \mathsf{Space}(X, Z^Y).$$

If X, Y and Z are pointed spaces we have the natural isomorphism:

$$\mathsf{Space}_\bullet(X \wedge Y, Z) \simeq \mathsf{Space}_\bullet(X, Z^Y),$$

where Z^Y is the pointed exponential.

0.0.0.20 Definition (Stable homotopy groups of spheres)**.** Define for each $k \in \mathbb{N}$ the group $\pi_k^{\mathbb{S}} = \operatorname{colim}_n \pi_{n+k}(\mathbb{S}^n)$, where the colimit is taken to the sequential diagram of the suspension morphisms $\pi_{n+k}(\mathbb{S}^n) \to \pi_{n+k+1}(\mathbb{S}^{n+1})$ given by the functoriality of the reduced suspension.

0.0.0.21 Lemma. *If the fundamental group of a space X acts trivially on the n-th homotopy group there is a bijection $[\mathbb{S}^n, X]_\bullet \simeq [\mathbb{S}^n, X]$.*

0.0.0.22 Definition (Degree of an endomap of a sphere)**.** For a map $\mathbb{S}^n \to \mathbb{S}^n$ one can consider the induced map $\mathbb{Z} \to \mathbb{Z}$ in the n-th homology group. For this one has to choose the same orientation on both copies of $\mathbb{S}^n$. Moreover, the degree is multiplicative, which means that composition of endomaps corresponds to multiplication of integers. One can check that this is the same morphism induced by regarding the map as a representative of the n-th homotopy group of the sphere $\mathbb{S}^n$. To do this one must use Lemma 0.0.0.21 to identify pointed maps with non-pointed ones using the fact that the fundamental group of the n-dimensional sphere is trivial for $n > 1$, and for example Hurewicz's theorem to relate homotopy and homology. For our purposes it suffices to take the characterization using homotopy groups as the definition of degree.

0.0.0.23 Proposition (Degree determines the homotopy class)**.** *From the fact that the n-th homotopy group of the n-dimensional sphere is $\mathbb{Z}$ (and the remark made in the definition above) one deduces that the homotopy equivalence class of pointed maps $\mathbb{S}^n \to_\bullet \mathbb{S}^n$ is completely determined by its degree.*

0.0.0.24 Proposition (Action of fundamental group of an H-space)**.** *For any H-space A the action of $\pi_1(A)$ on $\pi_n(A)$ is trivial.*

1

Classical spherical fibrations

The idea of this chapter is to motivate the study of spherical bundles and spherical fibrations. We will see that spherical fibrations are classified in a suitable way by a space, and that the homotopy groups of this space are the stable homotopy groups of spheres. On the other hand we will introduce real K-theory and using this classifying space we will construct a morphism from the real K-theory of the spheres to the stable homotopy groups of spheres. This is called the J-homomorphism.

The reason why this morphism is so important is that it is far from trivial and yields a lot of elements of the stable homotopy groups of spheres. These elements are easy to compute since the real K-theory of spheres is completely described by the Bott periodicity theorem. For many applications of orthogonal spherical bundles and the J-homomorphism the reader can take a look at the series of four papers by Adams that starts with [Ada63], at Atiyah's article [Ati61] and at Hatcher's [Hat09, Chapter 4].

We will start defining fibrations and fiber bundles, then we will define real K-theory and spherical bundles. The main focus of this chapter are Sections 1.5 and 1.7, and in particular the proof of Theorem 1.6.0.4 that essentially states that the homotopy groups of the classifying space of spherical bundles are isomorphic to the stable homotopy groups of the spheres. This seems to be a well known theorem but it is not easy to find a complete proof in the literature.

1.1 Fibrations

Let us motivate the definition of *fibration* by asking some questions. Consider a function between sets $f : E \to B$. This function induces the function that takes an element $b \in B$ to its fiber $f^{-1}(b) \in \mathsf{Set}$.

$$\begin{aligned} F : B &\to \mathsf{Set} \\ b &\mapsto f^{-1}(b) \end{aligned}$$

One can read this as "a function between sets induces a family of sets parametrized by the codomain of the map".

The following remark takes this idea further.

1.1.0.1 Remark (Indexed families)**.** Given a set B there is a bijection between families of sets indexed by B and functions between sets with B as its codomain. We write this as:

$$\frac{E \to B}{\overline{B \to \mathsf{Set}}}$$

In fact this bijection can be lifted to an equivalence of categories. To be precise consider the category that has as objects families of sets indexed by B. We can denote an object of this category as $\{E_b\}_{b\in B}$. An arrow between two families $\{E_b\}_{b\in B}$ and $\{E'_b\}_{b\in B}$ is given by a family of functions $\{f_b : E_b \to E'_b\}_{b\in B}$. The statement is that this category is equivalent to the slice category $\mathsf{Set}_{/B}$. The proof of this last fact is a simple exercise in category theory: To construct a function $E \to B$ out of a B-indexed family of sets assume given a family $\{E_b\}_{b\in B}$ and define E to be the disjoint union $\coprod_{b\in B} E_b = \{(b,x) \mid b \in B, x \in E_b\}$. Given an arrow between two B-indexed families just "glue" the family $\{f_b\}_{b\in B}$ to form a function f such that $f(b,x) = f_b(x)$. It is easy to show that this construction constitutes an inverse for the functor that takes a function $E \to B$ to the family of its fibers.

Now translate everything we just said to the world of maps and spaces: E and B are now spaces, f is now a (continuous) map and F takes values in Space since the fibers $f^{-1}(b)$ have a natural topology, the subspace topology. This might lead us to ask to ourselves: Does the fact that f is a *continuous* map imply that F is in some sense continuous? This is an ill-posed question because it does not say in which sense F should be continuous, but let us try to make sense out of it. From a homotopy theoretic point of view we could say that F is continuous if paths $p : b \rightsquigarrow a$ in B induce continuous maps $\tilde{p} : F(b) \to F(a)$. One way to formalize this is to consider the fundamental groupoid of B, usually denoted as $\pi_1(B)$. The fundamental groupoid of B is a category where the objects are the points $b \in B$ and the arrows are paths $p : b \rightsquigarrow a$ up to homotopy (notice that this category is indeed a groupoid since paths are reversible). Let us refine our first question as: Is there a functor $\mathcal{F} : \pi_1(B) \to \mathsf{Space}$ that behaves like the function F on objects? Since we are using paths up to homotopy it might be more reasonable to ask for a functor $\mathcal{F} : \pi_1(B) \to \mathsf{hSpace}$. How should this functor act on arrows? Remember that arrows in this category are paths in B up to homotopy. The functor should then assign to each path $p : a \rightsquigarrow b$ a homotopy equivalence between $f^{-1}(a)$ and $f^{-1}(b)$. Observe that this potential construction would yield a representation of the fundamental groupoid of B: The image of this construction is again a groupoid that remembers some of the structure of the fundamental groupoid of B, and maybe collapses part of the structure also. Notice that if B were to be path connected then the fiber would be essentially unique: $f^{-1}(a)$ would be homotopy equivalent to $f^{-1}(b)$ for any $a, b \in B$.

If the map f is a fibration then this construction can indeed be carried on. Although we won't do it, it is even possible to lift the representation of the fundamental group of the base space B of a fibration to a representation of the homotopy type of B using some standard higher category theory. Nonetheless the reader should keep in mind the following analogy.

1.1.0.2 Remark (Representations of homotopy types)**.** Fibrations over a connected space B are the representations (in hSpace) of the homotopy type of B in the same way that G-sets are representations (in Set) of a group G. Under this analogy the (essentially unique) fiber F of a fibration over B corresponds to the G-set X of a representation of a group G.

The following remark will be of use in the second chapter to exemplify in which way fibrations over a homotopy type and representations of this homotopy type and can be identified in HoTT:

1.1.0.3 Remark. Although in the category of sets we have the equivalence

$$\frac{E \to B}{\overline{B \to \mathsf{Set}.}}$$

When looking at (topological) spaces this is a priory not true. We only stated that a fibration $E \to B$ induces a functor $\pi_1(B) \to \mathsf{hSpace}$:

$$\frac{E \to B}{\pi_1(B) \to \mathsf{hSpace}.}$$

Before giving the definition of fibration let us give two examples of maps where the domain is parametrized by the codomain in a "continuous" way.

1.1.0.4 Example (Double coverings of the circle)**.** Consider the map:

$$\begin{aligned} \mathbb{S}^1 &\to \mathbb{S}^1 \\ z &\mapsto z^2 \end{aligned}$$

We can picture it as:

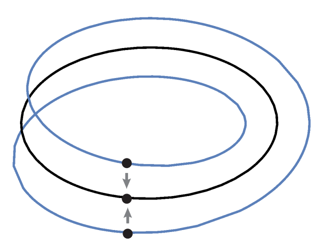


Figure 1.1: Möbius covering of the circle.

Notice how the fiber of each element is $\mathbb{S}^0$. This map is sometimes called the *Möbius covering* of the circle. A similar situation is the one of the map $\mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$ that maps each copy of $\mathbb{S}^1$ homeomorphically to $\mathbb{S}^1$ (here $\sqcup$ denotes the disjoint union):

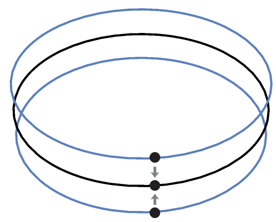


Figure 1.2: Trivial two-sheeted covering of the circle.

Once again the fiber of each point is $\mathbb{S}^0$. If we think of the domain of a map as being parametrized by its codomain we see in this example that there are at least two distinct ways in which we can put a topology in the disjoint union of the fibers $E = \coprod_{x\in\mathbb{S}^1} \mathbb{S}^0$ to get a (continuous) map $E \to \mathbb{S}^1$. Informally, there are at least two distinct ways in which we can glue $\mathbb{S}^1$-many $\mathbb{S}^0$'s.

1.1.0.5 Definition (Homotopy lifting property with respect to cubes)**.** A map $p : E \to B$ has the homotopy lifting property with respect to cubes if every time we have a cube $c : I^n \to B$ in B and a lift of one of its faces $f : I^{n-1} \to E$ there exists a lift of the entire cube $\overline{c} : I^n \to E$. Diagrammatically:

$$\begin{array}{ccc} I^{n-1} & \xrightarrow{\;f\;} & E \\ {\scriptstyle i}\big\downarrow & \overset{\overline{c}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ I^n & \xrightarrow[\;c\;]{} & B \end{array}$$

Here i is the inclusion of the $(n-1)$-dimensional cube I^{n-1} in one of the faces of the n-dimensional cube I^n.

1.1.0.6 Definition ((Weak) fibrations)**.** A map $p : E \to B$ that has the homotopy lifting property with respect to cubes is called a *weak fibration* or a Serre fibration. We will call them simply *fibrations*[1]. We call B the *base space* and E the *total space*.

[1]Since we are only considering CW-complexes there is really no difference between weak fibrations and standard (Hurewicz) fibrations. Concretely, the lifting property with respect to cubes implies the lifting property with respect to any CW-complex, as one can show by induction on the dimension of the cells.

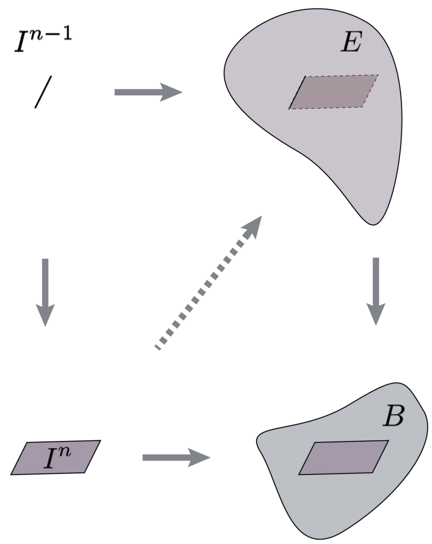


Figure 1.3: Homotopy lifting property with respect to cubes.

1.1.0.7 Notation (The fiber point of view)**.** Since we are interested in the way the fiber of a map varies when we move along its codomain we will frequently use the following notation. If $f : E \to B$ is a fibration and $b \in B$ we will write E_b for the fiber of b through f.

We will also make many constructions that will have a very concrete characterization when we look at each fiber. When applying the same construction to each fiber we will say that the construction is made *fiberwise*. We will refer to the idea of regarding the total space as a space parametrized by the base space as the *fiber point of view*.

It is not obvious that the definition of fibration we just gave implies that there exists a functor $\pi_1(B) \to \mathsf{hSpace}$ that maps every point in B to its fiber. To prove this one must use some basic theory about fibrations (see for example [Spa94, Chapter 2, Section 8, Theorem 12]).

A very important consequence of this fact is that fibrations have an essentially unique fiber in the following sense.

1.1.0.8 Remark (Uniqueness of the fiber)**.** Let B be a connected space and let $E \to B$ be a fibration. Then the fibers of any pair of points are homotopy equivalent. This follows from the result stated above since a path between the two points induces a homotopy equivalence between the fibers.

For this reason it makes sense to introduce the following notation.

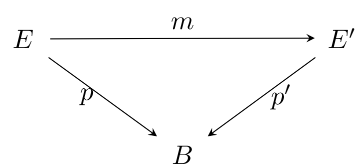
1.1.0.9 Notation (Standard notation for fibrations)**.** Let B be a pointed connected space an let $E \to B$ be a fibration. One usually denotes this fibration as $F \hookrightarrow E \to B$ where the first map is the inclusion of the fiber of the base point of B in the total space.

By abuse of notation if we know that the fiber in each component is homotopy equivalent to F we will use the notation $F \hookrightarrow E \to B$ even if B is not connected.

Another very important property of fibrations is that, up to homotopy, every map is equivalent to a fibration. Concretely this means that given a map $f : A \to B$ there exists a homotopy equivalence $e : A \to E$ and a fibration $p : E \to B$ such that $f = p \circ e$. The canonincal reference for this subject –that leads to an abstract notion of fibration and eventually to the concept of model category– is [Qui67].

Regarding fibrations over a fixed space B as the objects of study we now define an appropriate notion of morphism between these objects.

1.1.0.10 Definition (Fibration maps)**.** A morphism m between two fibrations over the same base $E \to B$ and $E' \to B$ is given by a commutative triangle of the form:



1.1.0.11 Remark. From the fiber point of view this is the same as a family of maps:

$$\{m_b : E_b \to E'_b\}_{b \in B}$$

such that the whole glued map m is continuous. We say that m *respects* or *preserves* the fibers.

A concept of particular interest for us will be the one of F-fibrations over a space B. As a matter of fact we will be interested in classifying this objects.

1.1.0.12 Definition (F-fibrations)**.** Fix two spaces F and B. Fibrations of the form $F \hookrightarrow E \to B$ will be called *F-fibrations over B*, or simply *F-fibrations* if the base space is implicit.

There is a notion of homotopy equivalence between F-fibrations called *fiber homotopy equivalence*. Essentially a fiber homotopy equivalence between fibrations is a homotopy equivalence of the total spaces that respects the fibers.

1.1.0.13 Definition (Fiber homotopy equivalence)**.** A map m between two fibrations $E_1 \to B$ and $E_2 \to B$ is called a fiber homotopy equivalence if there exists a map m going from $E_2 \to B$ to $E_1 \to B$ and fiber preserving homotopies $h : E_1 \times I \to E_1$ and $h' : E_2 \times I \to E_2$ between $m' \circ m$ and the identity of E_1, and $m \circ m'$ and the identity of E_2 respectively. When such an m exists we say that E_1 and E_2 have the same *fiber homotopy type*.

Finally let us state a very useful property of fibrations that will be needed later, namely the long exact sequence of homotopy groups induced by a fibration. For a proof of this fact see any book with an introduction to algebraic topology, for example the construction in [Hat02, Proposition 4.66].

1.1.0.14 Proposition (Long exact sequence induced by a fibration)**.** *Given three pointed spaces F, E and B and a point preserving fibration $F \hookrightarrow E \xrightarrow{p} B$ there exists a long exact sequence:*

$$\begin{aligned} \cdots \to \pi_{n+1}(F) \to \pi_{n+1}(E) \to \pi_{n+1}(B) \to \pi_n(F) \to \cdots \\ \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B). \end{aligned}$$

Where exactness means that the image of each morphism is equal to the kernel of the following morphism. Morphisms for $n > 1$ are abelian group morphisms, for $n = 1$ they are group morphisms and for $n = 0$ they are pointed set morphisms.

1.1.0.15 Remark (Morphisms of the long exact sequence and homotopy pullbacks)**.** To get the long exact sequence one can make the following constructions: The morphisms $\pi_k(F) \to \pi_k(E)$ are induced by the inclusion of the fiber $F \hookrightarrow E$ and the functoriality of the loop space construction (recall Definition 0.0.0.18 and the fact that the homotopy groups are defined as the connected components of an iterated loop space). Similarly the map $p : E \to B$ induces morphisms $\pi_k(E) \to \pi_k(B)$. What we need now is a so-called *connecting* map $\Omega B \to F$. This can be done with the *homotopy fiber* construction as showed in [Hat02, Proposition 4.66]. More generally, the homotopy fiber of a pointed map can be defined as the homotopy pullback:

$$\begin{array}{ccc} \mathsf{hfib}(p) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ \{*\} & \longrightarrow & B \end{array}$$

where $\{*\} \to B$ is the inclusion of the base point of B.

The homotopy pullback of a cospan $A \xrightarrow{f} C \xleftarrow{g} B$ is defined as the space of triples $\{(a, b, p) \in A \times B \times C^I \mid p(0) = f(a), p(1) = g(b)\}$ with the subspace topology[2]. See how with this definition the following is a homotopy pullback:

$$\begin{array}{ccc} \Omega B & \longrightarrow & \{*\} \\ \big\downarrow & & \big\downarrow \\ \{*\} & \longrightarrow & B \end{array}$$

By a simple application of the lifting property of fibrations one sees that if $F \hookrightarrow E \xrightarrow{p} B$ is a fibration the natural inclusion $F \to \mathsf{hfib}(p)$ is a homotopy equivalence. On the other hand a pullback-pasting lemma for homotopy pullbacks gives us a map $\Omega B \to \mathsf{hfib}(p)$:

$$\begin{array}{ccc} \Omega B & \longrightarrow & \{*\} \\ \big\downarrow & & \big\downarrow \\ \mathsf{hfib}(p) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ \{*\} & \longrightarrow & B \end{array}$$

Together with a choice of homotopy inverse for the equivalence $F \xrightarrow{\sim} \mathsf{hfib}(p)$ we obtain the connecting map $\Omega B \to F$. If we do not make this choice we get a map defined up to homotopy. Notice that iterating this construction we get in fact the long exact sequence of the fibration.

[2] One should also prove that we can perform an equivalent construction to obtain a CW structure on the homotopy pullback.

1.2 Fiber bundles

Now it would be interesting to have some examples of fibrations. As a matter of fact we already saw two examples of fibrations in Example 1.1.0.4 but we have not proved this formally. To get more examples of fibrations we introduce another concept, the one of *fiber bundles*. Fiber bundles are special cases of fibrations, sometimes called locally trivial fibrations.

1.2.0.1 Definition (Trivial fiber bundles)**.** The *trivial fiber bundle* over a space B with fiber F is the product space $B \times F$ together with the projection $B \times F \to B$.

Maps of this kind have the homotopy lifting property with respect to cubes and thus they are fibrations. Fiber bundles over a space B with fiber F are maps $E \to B$ which are locally trivial in the following sense:

1.2.0.2 Definition (Fiber bundles)**.** A fiber bundle is a map $p : E \to B$ together with a space F such that B has an open cover $\{U_\alpha\}$ for which it exist a family of homeomorphisms $\varphi_\alpha : U_\alpha \times F \xrightarrow{\sim} p^{-1}U_\alpha$ that render commutative the following diagram:

$$\begin{array}{ccccc} U_\alpha \times F & \xrightarrow{\varphi_\alpha} & p^{-1}U_\alpha & \hookrightarrow & E \\ & \searrow^{\pi_1} & \downarrow p & & \downarrow p \\ & & U_\alpha & \hookrightarrow & B \end{array}$$

We call F the fiber and we call the U_α local trivializations. We denote such a fiber bundle as (E, B, p, F). We will frequently abuse notation and write E for the whole fiber bundle.

This local triviality permits to extend the homotopy lifting property with respect to cubes to fiber bundles (at least when the base is Haussdorf paracompact):

1.2.0.3 Proposition. *Fiber bundles with a Hausdorff paracompact base are fibrations.*

Proof. See [Spa94, Chapter 2, Section 7, Corollary 14]. ∎

Notice that the notion of map between fibrations and the notion of fiber homotopy equivalence applies to fiber bundles without change. It is time to give some examples of fiber bundles.

1.2.0.4 Example (Covering spaces)**.** It is easy to see that covering spaces are fiber bundles. Indeed covering spaces are fiber bundles in which the fiber is a discrete space. The coverings of the circle that we analyzed before are of course covering spaces.

Now let us see an example of a fiber bundle which is not a covering space.

1.2.0.5 Example (Real projective plane)**.** Consider the (real) projective plane constructed as a quotient of $\mathbb{R}^3 \setminus \{0\}$:

$$\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2.$$

Explicitly this map is the quotient map given by the relation $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for every $x, y, z \in \mathbb{R}^3 \setminus \{0\}$ and every $\lambda \in \mathbb{R}^\times$. See how the fiber of each point is the disjoint union of two lines. From this observation is easy to check the local triviality of the map.

As a matter of fact a construction similar to the one in the above example works for real and complex projective spaces of any dimension as we now show. Here we will make use of some very basic differential topology.

1.2.0.6 Example (Real and complex projective spaces)**.** Let $\mathbb{K}$ stand for the real or the complex numbers with its usual norm. Define the subspace $F = \{\lambda \in \mathbb{K} \mid \|\lambda\| = 1\}$ and the subspace $S = \{x \in \mathbb{K}^{n+1} \mid \|x\| = 1\}$, the unit sphere of the normed vector space $\mathbb{K}^{n+1}$. Notice that F has a topological group structure inherited by the group structure of $\mathbb{K}^{\times}$, the multiplicative group of the units of the field $\mathbb{K}$. There is also a continuous action of F on S: $\lambda \cdot x = \lambda x$. As with any group action we can consider the induced equivalence relation on S given by $x \sim \lambda x$. We denote the quotient space by $\mathbb{K}P^n := S/\sim$ and we endow it with the quotient topology. This space is the projective n-dimensional space over $\mathbb{K}$. Let $q : S \to \mathbb{K}P^n$ be the quotient map. With these definitions we claim that the following is a fibration:

$$
\begin{array}{ccccc}
 & & F & & \lambda \\
 & & \big\downarrow & & \big\downarrow \\
x & & S & & (\lambda, 0, \ldots, 0) \\
\big\downarrow & & \big\downarrow q & & \\
[x]_\sim & & \mathbb{K}P^n & &
\end{array}
$$

Indeed we will prove that this is a fiber bundle. Let us construct local trivializations. For the real case this is easy: Notice that in this case the fiber F can be identified with the discrete topological group $\mathbb{Z}_2$, and that its action on the unit sphere S is the antipode action. Since the fiber is discrete it is simple to construct a trivializing open cover that shows that the above map is locally trivial, and as a matter of fact a *covering map*. One can check that this is indeed the Möbius covering of Example 1.1.0.4.

The complex case is not so immediate. We will give two arguments, the first one is interesting because it generalizes to the case of a compact Lie group acting smoothly on a differentiable manifold, the second one is simpler and uses the classical covering of the projective spaces by affine spaces.

First notice that in this case the fiber F can be identified with the topological group $\mathbb{S}^1$. Now assume given a $[x] \in \mathbb{C}P^n$ and let us construct a trivializing neighborhood for $[x]$. Observe that the map q is an open map: To see this we have to check that the image of an open set is again open. Since the codomain has the final topology it is enough to show that the preimage of the image of an open set is open. Take an open set $U \subseteq S$ and notice that $p^{-1}(p(U)) = \cup_{\lambda \in \mathbb{S}^1} \lambda \cdot U$. Since $\mathbb{S}^1$ acts by homeomorphisms (it is a group action) all the sets in the union are open which proves what we wanted. It is important to notice that this argument works in the general case of a group acting on a space. Now that we know that q is open we can define the trivializing open neighborhood of $[x]$ as the image of an open subset of S that contains a representative $x \in S$.

Fixing this representative we see that the action of $\mathbb{S}^1$ induces a closed simple curve in S:

$$\begin{aligned}\gamma : \mathbb{S}^1 &\to S \\ \lambda &\mapsto \lambda \cdot x\end{aligned}$$

This is indeed a smooth curve and thus (remembering that S is a subspace of $\mathbb{C}^{n+1}$) we can consider the orthogonal space to the curve at each $\lambda \in \mathbb{S}^1$. Call this space T_λ. Each of these orthogonal spaces is a complex vector space of codimension 1. Let $N_\varepsilon(y) \subseteq S$ be the ball of radius ε centered at $y \in S$. Choosing a small enough $\varepsilon > 0$ we can make sure that the open set $N = \cup_{\lambda \in \mathbb{S}^1} \lambda \cdot N_\varepsilon(x)$ forms a tubular neighborhood of the curve γ in such a way that the intersection $N \cap T_\lambda$ is the ball contained in T_λ of radius ε and center λx. Call U to the image of N and observe that since N is a tubular neighborhood q restricts to a homeomorphism between each of the slices $N \cap T_\lambda$ and the open set U. Finally notice that N is homeomorphic to $\mathbb{S}^1 \times U$ as required.

A shorter construction of the trivializing neighborhood is the following. Let $[x] \in \mathbb{C}P^n$, then by definition the representative x must have some non-zero coordinate, say the i-th coordinate. Consider the open subset of $\mathbb{C}P^n$ given by the elements represented by elements in S with the i-th coordinate different from zero. It is straightforward to show that this is a trivializing neighborhood of $[x]$.

This last example yields a lot of interesting bundles.

1.2.0.7 Example (The Hopf fibration)**.** Specializing the last example to the case of the one dimensional complex projective space we observe that in this case S is the sphere $\mathbb{S}^3$ and $\mathbb{C}P^1$ is homeomorphic to the sphere $\mathbb{S}^2$. To prove this last statement recall that $\mathbb{S}^2$ is homeomorphic to the Riemann sphere $\mathbb{C}_\infty$ (by the stereographic projection). Thus it suffices to check that the following map is well defined and it is an homeomorphism:

$$\begin{aligned}\mathbb{C}P^1 &\to \mathbb{C}_\infty \\ [(x_0, x_1)] &\mapsto x_1/x_0\end{aligned}$$

which is a straightforward check.

This defines a fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ called the *Hopf fibration*. From this fibration and the long exact sequence induced by a fibration one can deduce at once some non-trivial results such as the isomorphisms $\pi_n(\mathbb{S}^3) \simeq \pi_n(\mathbb{S}^2)$ for $n \geq 3$.

Notice that applying the long exact sequence of a fibration to the fibrations defined in the Example 1.2.0.6 also yields a lot of interesting relations between the homotopy groups of the spheres and the homotopy groups of the projective spaces.

The same idea of Example 1.2.0.7 applies to a couple more cases.

1.2.0.8 Example (More Hopf fibrations)**.** Recall the construction of Example 1.2.0.6 and now let $\mathbb{K}$ be a division algebra over $\mathbb{R}$. Thus we consider the cases where $\mathbb{K}$ are the real numbers, the complex numbers, the quaternions and the octonions. A simple check shows that the same reasoning done in Example 1.2.0.7 yields fibrations in these four cases. The real case yields a fibration $\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \to \mathbb{S}^1$, which is the Möbius covering of the circle of Example 1.1.0.4. The complex case is the classical Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ covered in Example 1.2.0.7. The quaternionic case yields a fibration $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$ and the octonionic case yields a fibration $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{S}^8$.

1.2.0.9 Remark (Proof strategy for bundles)**.** The reader will notice that many proofs of results about fiber bundles have the following structure: First prove the result in the case of trivial bundles, and then prove it for arbitrary bundles using some sort of "gluing" to put together a construction for the whole bundle out of constructions for each local trivialization. This is exactly the reason why fiber bundles are useful: Locally they are easy to describe but they also leave enough room as to let us glue constructions made for each local trivialization. An example of the usage this strategy is Definition 1.3.1.1.

Observe that a fiber bundle over a space B with fiber F is also a fibration with fiber F and thus we have a natural definition of the concept of F-bundles.

1.2.0.10 Definition (F-bundles)**.** Given a base space B and a space F we can consider fiber bundles over B with F as its fiber. When B is implicit we call such fiber bundles *F-bundles*. Notice that F-bundles are special cases of F-fibrations.

1.3 Vector bundles

Vector bundles are fiber bundles with some extra structure:

1.3.0.1 Definition (Vector bundles)**.** A real vector bundle is a fiber bundle (E, B, p, V) such that V is a finite dimensional real vector space (with its usual Euclidean topology), and for each $b \in B$ the fiber E_b has a structure of real vector space such that each φ_α restricts to a linear isomorphism $\varphi_\alpha : \{b\} \times V \to E_b$. The dimension of the vector bundle is by definition the dimension of V as a real vector space. Many times, when there is no risk of confusion we will call vector bundles just bundles.

Under the fiber point of view vector bundles should be regarded as vector spaces parametrized by a base space B. Informally, just as a fibration $E \to B$ induces an functor $E : \pi_1(B) \to \mathsf{hSpace}$, a vector bundle $E \to B$ induces a functor $\pi_1(B) \to \mathsf{vec}_{\mathbb{R}}$. Following the idea given in Remark 1.1.0.2 this means that a vector bundle over B is in some sense a "linear" representation of the homotopy type of the space B.

The theory of vector bundles generalizes the theory of vector spaces in a very concrete sense. Consider the one point space $\{*\}$. It is immediate to see that having a vector bundle over $\{*\}$ is the same as having a vector space. Vector bundles over $\{*\}$ are examples of *trivial vector bundles*.

1.3.0.2 Definition (Trivial vector bundles)**.** Given any space B and an natural number n we define the trivial vector bundle of dimension n over B as $\pi_1 : B \times \mathbb{R}^n \to B$.

When B is implicit we denote this vector bundle simply as n.

By abuse of language we will call trivial bundle to any bundle which is isomorphic to a trivial bundle, usually omitting the actual isomorphism.

1.3.0.3 Example (The cylinder and the Möbius strip)**.** To further exemplify take the base space to be $\mathbb{S}^1$. Then the total space of the one dimensional trivial vector bundle over B is a cylinder (without boundary).

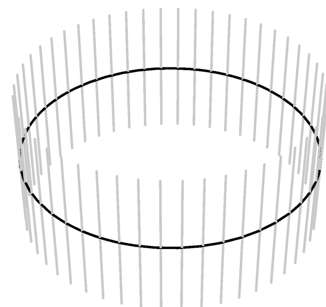


Figure 1.4: Cylinder bundle over $\mathbb{S}^1$.

Now consider another one dimensional vector bundle over $\mathbb{S}^1$. One can describe this bundle formally by taking an open cover $\{U_\alpha\}$ of $\mathbb{S}^1$ and by giving explicit maps $\{\varphi_\alpha\}$, but the following picture should be just as good:

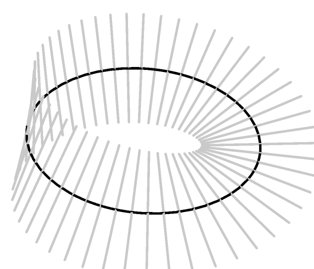


Figure 1.5: Möbius bundle over $\mathbb{S}^1$.

The total space of this bundle is a Möbius strip (without boundary). Notice that, locally, we cannot distinguish this vector bundle from the previous one, since locally they both look like an interval $U \subseteq \mathbb{S}^1$ times $\mathbb{R}$. But globally they are quite different, for example the Möbius strip is not orientable while the cylinder is.

Because vector bundles are fiber bundles with extra structure, morphisms between vector bundles are defined as morphisms of the underlying fiber bundles that preserve this structure.

1.3.0.4 Definition (Vector bundle morphisms)**.** A morphism m between vector bundles $p : E \to B$ and $p' : E' \to B$ over the same base is a morphism of fiber bundles such that each $m_b : E_b \to E'_b$ is linear.

This definition determines a category structure on the vector bundles over a fixed base space.

1.3.0.5 Notation. Given a space B the category of real vector bundles over B will be denoted $\mathsf{Vec}(X)$.

Although this automatically gives us the definition of isomorphism between vector bundles we state an equivalent definition explicitly.

1.3.0.6 Definition (Isomorphism of vector bundles)**.** A morphism between vector bundles is called an isomorphism if it restricts to linear isomorphisms in each fiber.

Now that we defined the morphisms we have the problem of describing the morphisms between two vector bundles. In the case of trivial bundles there is a simple and extremely useful characterization.

1.3.0.7 Remark (Morphisms between trivial vector bundles)**.** Recall that $\mathsf{vec}_{\mathbb{R}}$ is the category of real finite dimensional vector spaces. Observe that given two real finite dimensional vector spaces V and W there is a natural choice for the topology of $\mathsf{vec}_{\mathbb{R}}(V, W)$, since this is again a real finite dimensional vector space. Then, given two trivial vector bundles $B \times V$ and $B \times W$ there is a bijection between morphisms of vector bundles $B \times V \to B \times W$ and maps $B \to \mathsf{vec}_{\mathbb{R}}(V, W)$:

$$\frac{B \times V \to B \times W}{B \to \mathsf{vec}_{\mathbb{R}}(V, W)}$$

This follows at once from the definition of morphism between vector bundles: A commutative diagram of the form:

$$\begin{array}{ccc} B \times V & \xrightarrow{\quad g \quad} & B \times W \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & B & \end{array}$$

where g is fiberwise linear, is simply a map from B to the linear morphisms between V and W.

A priory $\mathsf{Vec}(-)$ is just a function that assigns to each space X a category $\mathsf{Vec}(X)$, but one could expect it to be functorial in some sense. For this we will assign to each map $f : A \to B$ a functor $\mathsf{Vec}(B) \to \mathsf{Vec}(A)$, and thus we will get a contravariant functor $\mathsf{Vec}(-) : \mathsf{Space}^{op} \to \mathsf{Cat}$.

1.3.0.8 Definition (Pullback of vector bundles)**.** Given a vector bundle $p : E \to B$ and a map $f : A \to B$ we can form the pullback bundle f^*E by taking the following pullback:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle f^*p} \downarrow & & \downarrow {\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

This construction yields a functor $f^* : \mathsf{Vec}(B) \to \mathsf{Vec}(A)$ for each map $f : A \to B$.

Notice that from the fiber point of view, by construction of pullbacks, we are taking the fibration E and precomposing it with f in the sense that $(f^*E)_a = E_{f(a)}$.

It is easy to see that this correspondence is indeed functorial: It is deduced from the universal property of pullbacks, for composition one can use the pasting pullback lemma. The reader interested in category theory might notice that this construction is entirely analogous to the change of basis functor that given a category (with pullbacks) $\mathcal{C}$ and a morphism $f : c \to d$ yields a functor $f^* : \mathcal{C}_{/d} \to \mathcal{C}_{/c}$.

We did not prove the fact that the above pullback exists but the construction is simple: Define $f^*(E) = \{(a, e) \in A \times E \mid f(a) = p(e)\}$ and take the topology to be the subspace topology of $A \times E$. To check the local triviality of this bundle one must use the local triviality of $p : E \to B$.

1.3.0.9 Remark. Notice how the pullback bundle does not change the fiber: The fiber of an element $a \in A$ is defined as the fiber of $f(a) \in B$.

A very important property of the pullback bundle construction is that for compact base spaces it is invariant under homotopy, as stated in the following lemma. This lemma implies that vector bundles over a space are representations of the homotopy type of the space (see Lemma 1.3.1.12).

1.3.0.10 Lemma (Pullback over homotopic maps)**.** *Let A be a compact space and suppose given a homotopy $H : A \times I \to B$ and a vector bundle $E \to B$. Then the vector bundles $H_0^*(E)$ and $H_1^*(E)$ over A are isomorphic.*

Proof. Consider the pullback bundle $H^*(E)$ over $A \times I$. On the other hand consider the projection $\pi : A \times I \to A$ and for each $t \in I$ the pullback bundle $\pi^* H_t^*(E)$ over $A \times I$. Restrict these two bundles to the subspace $A \times \{t\}$ where they are clearly isomorphic. Using the compactness of $A \times I$ and Lemma 1.3.0.11 we deduce that the two bundles are isomorphic when restricted to an open strip $A \times (t - \varepsilon, t + \varepsilon)$. This means that the isomorphism class of $H_t^*(E)$ is locally constant. Since the interval I is connected the restrictions to each endpoint $H_0^*(E)$ and $H_1^*(E)$ are isomorphic as required. ∎

1.3.0.11 Lemma. *Let X be a compact space and let $Y \subset X$ be a closed subspace. If two vector bundles E and F over X are isomorphic when restricted to Y then there exists an open set U containing Y to which this isomorphism can be extended.*

Proof. Call $E|Y$ and $F|Y$ the restrictions of the bundles E and F. Let $f : E|Y \to F|Y$ be the isomorphism of the hypothesis. By an application of Tietze extension theorem we can extend f to an open neighborhood of each $y \in Y$: Choosing a neighborhood U'' of y that trivializes both bundles we can use the characterization of the morphisms between trivial bundles (Remark 1.3.0.7) to deduce that $f|U''$ corresponds to a map $m : Y \cap U'' \to \mathsf{vec}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathbb{R}^{n \times n}$. Using Tietze in each coordinate of m we can extend it to a neighborhood U_y of y, open in X. Now by the compactness of X (and thus of Y) a partition of unity argument implies that we can glue the extensions to obtain an extension $\bar{f} : E|U' \to F|U'$ for U' an open set containing Y. Let U be the set for which $\bar{f}$ is an isomorphism. Notice that $Y \subseteq U$ and also that U is an open set since being an isomorphism is an open condition. Then U is the required open set and the restriction of $\bar{f}$ to U is the required isomorphism. ∎

1.3.1 Operations

Since vector bundles are parametrized vector spaces it would be nice to be able to make the same constructions we can carry on with vector spaces. For example, we should be able to take the direct sum of two vector bundles and also to take their tensor product. From the fiber point of view there is an obvious choice. If we have two vector bundles E and E' over B, their direct sum should satisfy:

$$(E \oplus E')_b = E_b \oplus E'_b.$$

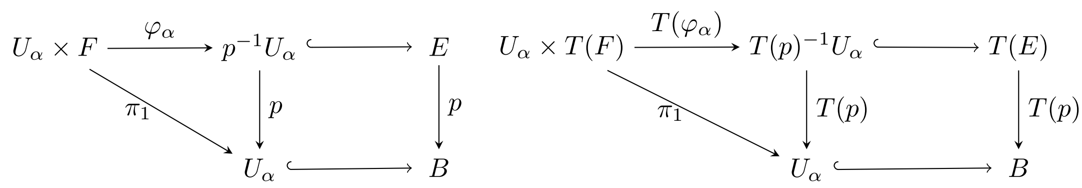
But this is not enough since we also need to topologize $E \oplus E'$. For this we use the following construction which is a reformulation of the one given in [Ati94, Section 1.2]. This is a very general method for constructing operations between vector bundles out of operations between vector spaces taken fiberwise. As a matter of fact this same idea generalizes to operations between other kinds of bundles as commented in Remark 1.5.0.9 when defining the join of orthogonal spherical bundles.

1.3.1.1 Definition (Operations between vector bundles)**.** Let $T : \mathsf{vec}_{\mathbb{R}} \to \mathsf{vec}_{\mathbb{R}}$ be a functor. We say that T is continuous if for any two vector spaces V, W the induced function $\mathsf{vec}_{\mathbb{R}}(V, W) \to \mathsf{vec}_{\mathbb{R}}(T(V), T(W))$ is continuous (here $\mathsf{vec}_{\mathbb{R}}(V, W)$ has the usual finite-dimensional $\mathbb{R}$-vector space topology). Now given a vector bundle $E \to B$ we can apply the functor fiberwise and define the set $T(E) := \coprod_{b \in B} T(E_b) = \{(b, e) \mid b \in B, e \in T(E_b)\}$. Our goal is to topologize this set so that the obvious projection $T(E) \to B$ is a vector bundle. Notice that the application of T is already functorial at the set level, since if we have a morphism of vector bundles $m : E \to E'$ we can define $T(m) : T(E) \to T(F)$ by the fiberwise application of T: $T(m)_b = T(m_b)$. And thus we will also want to show that the functions $T(m)$ are indeed bundle morphisms.

To give a topology to $T(E)$ we start with the case in which E is a trivial bundle and then we extend it to arbitrary bundles using the local triviality. Suppose then that $E = B \times V$ with $V \in \mathsf{vec}_{\mathbb{R}}$. Then by construction $T(E) = B \times T(V)$ as a set, but we have also a nice choice for the topology of $T(E)$, namely the product topology. Let us see that with this topology the application of T on a bundle morphism yields a bundle morphism. For this let $E' = B \times W$ be another trivial bundle and let $m : E \to E'$ be a bundle morphism. By Remark 1.3.0.7 this corresponds to exactly one map $m : B \to \mathsf{vec}_{\mathbb{R}}(V, W)$ and, under the same correspondence $T(m) : T(E) \to T(E')$ corresponds to $T(m) = T \circ m : B \to \mathsf{vec}_{\mathbb{R}}(T(V), T(W))$. Since $T : \mathsf{vec}_{\mathbb{R}}(V, W) \to \mathsf{vec}_{\mathbb{R}}(T(V), T(W))$ is continuous by hypothesis, $T(m)$ is indeed a bundle morphism in the case of trivial bundles *with a given choice of trivialization*.

To check that any choice of trivialization yields the same topology is straightforward: Given a trivial bundle E choose a trivialization for it and endow $T(E)$ with the topology induced by that trivialization as we did in the previous paragraph. If we have two trivializations $m : E \xrightarrow{\sim} B \times V$ and $m' : E \xrightarrow{\sim} B \times W$, consider $m' \circ m^{-1} : B \times V \xrightarrow{\sim} B \times W$. Since T is functorial on trivial bundles we have $T(m' \circ m^{-1}) = T(m') \circ T(m^{-1})$ which gives a homeomorphism between the topologies induced by m and m'.

Finally notice that a trivialization of E yields a trivialization for $T(E)$:



So for an arbitrary vector bundle E topologize $T(E)$ with the same local trivializations and declare a set $S \subseteq T(E)$ to be open if and only if it is open when restricted to each local trivialization. Using the same ideas as above one deduces at once that this construction yields a functor $T : \mathsf{Vec}(B) \to \mathsf{Vec}(B)$ that behaves like T in each fiber.

One should notice that this discussion extends to the case when T can take a finite number of (covariant or contravariant) arguments and thus, given two vector bundles E, E' over the same base we can construct the bundles: direct sum $E \oplus E'$, tensor product $E \otimes E'$, mapping space $\hom(E, F)$, dual bundle E^*, etc. We stress the fact that by construction we have the fiberwise equalities $(E \oplus E')_b = E_b \oplus E'_b$, $(E \otimes E')_b = E_b \otimes E'_b$, $\hom(E, F)_b = \hom(E_b, F_b)$, $(E^*)_b = E^*_b$, etc.

The formalism of this approach also makes it easy to translate properties of operations on vector spaces to those same properties of the corresponding operations on vector bundles. For example we have the following remark.

1.3.1.2 Remark (Distributivity of tensor product)**.** Recall that in $\mathsf{vec}_{\mathbb{R}}$ the tensor product distributes over the direct sum in the sense that for any three vector spaces U, V and W there is a natural isomorphism $U \otimes (V \oplus W) \simeq (U \otimes V) \oplus (U \otimes W)$. Using the functoriality of our previous construction one can deduce at once that the (natural) distributivity of tensor products over direct sums in the case of vector spaces translates to a (natural) distributivity of tensor products over direct sums in the case of vector bundles. By the fiberwise construction it is obvious that this holds for the underlying sets.

We also have the following commutativity (or functoriality) relation.

1.3.1.3 Lemma (Functoriality of operations)**.** *For any map $f : A \to B$ and any continuous functor $T : \mathsf{vec}_{\mathbb{R}} \to \mathsf{vec}_{\mathbb{R}}$ there is a natural isomorphism $f^*T(E) \simeq Tf^*(E)$.*

To prove this fact just look carefully at the definitions. For a fiberwise intuitive argument observe that $f^*T(E)_b = T(E)_{f(b)} = T(E_{f(b)}) = T(f^*(E))$.

Our next goal is to use the sum and product operations in $\mathsf{Vec}(B)$ to form a ring. This construction will respect maps between spaces and thus we will construct a functor $KO : \mathsf{Space}^{op} \to \mathsf{Ring}$. Until now we have constructed a functorial assignment $\mathsf{Space}^{op} \to \mathsf{Cat}$ that takes a space B to the category of all its vector bundles. Now notice that for any space the set[3] of isomorphism classes of $\mathsf{Vec}(B)$ has the structure of a commutative semigroup using the direct sum operation. This is because the direct sum is commutative (up to isomorphism). Moreover this commutative semigroup is in fact a commutative monoid since the zero dimensional trivial bundle acts as a

[3]It might not be obvious that the isomorphism classes of vector bundles over a space form a (small) set. This essentially follows from the fact that a finite dimensional vector space is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and the fact that the open sets of a topology form a set. These two facts combined with a trivializing open cover prove the statement.

unit (up to isomorphism). Let $\mathsf{Vec}(B)_\sim$ be the set of isomorphism classes of vector bundles over B. Using the direct sum operation and Remark 1.3.1.2 we get a semiring structure on $\mathsf{Vec}(B)_\sim$. Since by Lemma 1.3.1.3 the operations of direct sum and tensor product commute with taking pullback bundle, the semiring structure that we defined for each space B is functorial, in the sense that maps $f : A \to B$ induce semiring morphisms $f^* : \mathsf{Vec}(B)_\sim \to \mathsf{Vec}(A)_\sim$. In other words we have a functor:

$$\mathsf{Vec}(-)_\sim : \mathsf{Space}^{op} \to \mathsf{sRing}\,.$$

Luckily for us there is a formal method to construct a ring out of a semiring. In fact, this works for commutative semigroups. Given a commutative semigroup S we can construct what is usually called the *Grothendieck group* or the *groupification* of S.

1.3.1.4 Definition (Groupification)**.** Given a commutative semigroup S we define its groupification as the set of cosets of the diagonal subsemigroup $\Delta = \{(s,s) \in S \times S\}$ included in the product semigroup $\Delta \subseteq S \times S$. Call this set $G(S)$ and notice that it inherits a commutative semigroup structure. Let us see why this semigroup structure is in fact an abelian group structure. Explicitly we are considering elements in $S \times S$ up to elements in the diagonal. Since S is only a semigroup it doesn't make sense to subtract two elements $s, s' \in S$. But notice that the class of (s, s') in $G(S)$ behaves like $s - s'$: For example the class of (s,s) is zero, because it is in the diagonal. But also the class of (s,t) is the same as the class of $(s+u, t+u)$ since they differ in (u,u). Finally observe that (t,s) serves as an additive inverse for (s,t) since their sum is in the diagonal (here we need the commutativity of S).

Observe that we have a semigroup morphism $S \to G(S)$ that sends s to the class of $(s,0)$. This morphism is universal in the sense that any morphism from S to a group factors through this morphism. It is easy to check that this is indeed the counit of the adjuction of Remark 1.3.1.6.

1.3.1.5 Example (Groupification of the natural numbers)**.** The standard example of groupification is to consider the semigroup $\mathbb{N}$ of non-negative integers. One can show at once that the semigroup morphism:

$$\begin{aligned} G(\mathbb{N}) &\to \mathbb{Z} \\ (n,m) &\mapsto n-m \end{aligned}$$

is well defined and that it is in fact an isomorphism.

The following lemma is a simple exercise in category theory.

1.3.1.6 Remark (Universal property of groupification)**.** This last construction is the solution to a universal property problem. Namely, this last construction gives a left adjoint to the inclusion functor $\iota : \mathsf{Ab} \to \mathsf{csGrp}$.

1.3.1.7 Remark (Groupification of a group)**.** Notice that ι is full and faithful and thus the groupification of a commutative semigroup which is already an abelian group yields and isomorphic group. One can also express this fact by saying that being an abelian group is a *property* that a particular semigroup might have, a given semigroup can be an abelian group in at most one way. Having inverses in this case is a *property* in contrast of being *additional structure*. In Section 2.1.2 we discuss the notions of property and additional structure in more detail.

1.3.1.8 Remark (Groupification of semirings)**.** One good thing about groupification is that its construction is so formal that it behaves well with respect to additional structure on the semigroup. For example it is easy to check that if we start with a semiring the product operation induces a well behaved product operation in the groupification of the underlying semigroup, and thus in this case the groupification produces a ring.

Finally we construct the real K-theory functor composing $\mathsf{Vec}(-)_\sim$ with the groupification and using Remark 1.3.1.8 to get a ring.

1.3.1.9 Definition (Real K-theory functor)**.** Using the definitions in this section we have constructed a functor $KO : \mathsf{Space}^{op} \to \mathsf{Ring}$ that assigns to each space the isomorphism classes of vector bundles over it, modulo an equivalence relation. This equivalence relation is given by the groupification of the semigroup formed by the isomorphism classes of vector bundles over the space. The direct sum of vector bundles corresponds to the sum operation of the ring and the tensor product of bundles corresponds to the product operation of the ring (recall Remark 1.3.1.2).

1.3.1.10 Notation. Given a (base) space B, when referring to elements in $KO(B)$ or elements in $\mathsf{Vec}(B)_\sim$ we will often denote them $[E]$, where E is a vector bundle over B. By abuse of notation we might denote them simply by E if there is no risk of confusion.

Let us analyze a very simple case of this ring. Namely the K-theory of the one point space $\{*\}$.

1.3.1.11 Example (K-theory of $\{*\}$)**.** Recall that vector bundles over $\{*\}$ are the same as vector spaces. Indeed we can say this formally as: The category $\mathsf{vec}_\mathbb{R}$ is equivalent to the category $\mathsf{Vec}(\{*\})$. To see this send a vector space V to the vector bundle $\{*\} \times V$, it is easy to check this is an equivalence (and in fact an isomorphism) of categories. Now let us compute $KO(\{*\})$. Since the functor $\mathsf{Vec}_\sim(-)$ only distinguishes isomorphism classes of vector bundles, we deduce that $\mathsf{Vec}_\sim(\{*\})$ is in bijection with $\mathbb{N}$, because (up to isomorphism) a vector space is completely determined by its dimension. Concretely, the bijection is given by sending an isomorphism class of vector bundles over $\{*\}$ to its dimension (the dimension is invariant under isomorphism). Moreover this bijection respects sums and products since the dimension of a direct sum of vector spaces is the sum of the dimensions, and the same is true if we replace direct sum with tensor product and sum with product. Thus using Example 1.3.1.5 we have established $KO(\{*\}) \simeq \mathbb{Z}$. This isomorphism takes the class of a vector bundle to its dimension.

Although this previous discussion works for non-compact spaces, from now on we will assume that all our base spaces are compact since this is a necessary condition for some results that we will state (homotopy invariance and representability).

Next we state the homotopical invariance of the functor KO. This means that KO does not distinguish homotopically equivalent maps and in particular, that it does not distinguish homotopically equivalent spaces.

1.3.1.12 Lemma (Homotopy invariance of KO)**.** *The functor KO that we defined factors through the category* hSpace*:*

$$\begin{array}{ccc} \mathsf{cSpace}^{op} & \xrightarrow{KO} & \mathsf{Ring} \\ \big\downarrow & \nearrow & \\ \mathsf{hcSpace}^{op} & & \end{array}$$

Here the vertical arrow takes a map to its homotopy class, cSpace *is the category of compact spaces and* hcSpace *is the homotopy category of compact spaces.*

Proof. This is an immediate application of Lemma 1.3.0.10 ■

As an application of the theorem we deduce that the K-theory of a contractible space is $\mathbb{Z}$. Next we motivate the reduced (real) K-theory functor. This is a contravariant functor from spaces to groups. As a matter of fact this functor can be already defined from the functor KO, but we will give an explicit definition since it will be useful to understand some constructions. But first the motivation.

1.3.1.13 Remark (Suspending a vector bundle)**.** As we saw above, the K-theory of a point is $\mathbb{Z}$. On the other hand every vector bundle over $\{*\}$ is a trivial vector bundle, thus it would probably be better if the K-theory of a point was the trivial group. For this we introduce the suspension operation[4] between vector bundles. For any base space B we have the functor:

$$\begin{gathered} \mathsf{Vec}(B) \to \mathsf{Vec}(B) \\ E \mapsto E \oplus \mathbb{R} \end{gathered}$$

Since $E \oplus \mathbb{R}$ is obtained from E by adding a trivial bundle it would be nice to regard both vector bundles as the same bundle. A potential solution is to regard trivial vector bundles as zero, that is to consider the group $KO(B)$ modulo trivial vector bundles.

This situation is analogous to when one defines the Homology functor and then the reduced Homology functor so that a contractible space has trivial homology groups.

1.3.1.14 Definition (Stable equivalence)**.** Two vector bundles E, E' over the same space are said to be *stably equivalent* if they become isomorphic after adding trivial bundles. Concretely, if there exist natural numbers n, m such that $E \oplus n \simeq E' \oplus m$.

1.3.1.15 Definition (Reduced real K-theory functor)**.** Given a space B consider the set $\mathsf{Vec}(B)_\sim$ and quotient it by the equivalence relation given by stable equivalence. Observe that this equivalence relation respects the sum of vector bundles, and thus it yields a commutative semigroup. We denote reduced real K-theory of a space B as $\widetilde{KO}(B)$.

At this point one might expect that the next step is to take the groupification of this commutative semigroup, but it turns out that this is not necessary since the reduced real K-theory as we defined is already an abelian group (at least when the base is compact), so by Remark 1.3.1.7 the groupification would make no difference. The following discussion shows that given a vector

[4] The usage of the term *suspension* will be justified once we introduce orthogonal spherical bundles which relate vector bundles and spherical bundles.

bundle E *over a compact space* there exists another vector bundle E' such that $E \oplus E'$ is a trivial vector bundle. This implies that reduced K-theory takes values in commutative groups. In the rest of this subsection we will study the necessary lemmas to prove this fact.

1.3.1.16 Definition (Section of a bundle)**.** Given a bundle $p : E \to B$ a section for it is a map $m : B \to E$ such that $p \circ m$ is the identity of B.

1.3.1.17 Example (Zero section)**.** If we have a vector bundle $p : E \to B$ there is always an obvious section, namely the map $b \mapsto 0 \in E_b$.

Under the fiber point of view a section is a continuous choice of an element for each fiber E_b. Once again, if we consider vector bundles over $\{*\}$ the notion of section coincides with the familiar notion of element of a vector space.

We now define what it means to have a metric or inner product on a bundle.

1.3.1.18 Definition (Metric on a bundle)**.** Using the construction carried on in Definition 1.3.1.1, given a vector bundle $E \to B$ we can define a new vector bundle $Sym(E)$ such that each fiber $Sym(E)_b$ is the space of symmetric forms on E_b. For this one just has to check that the functor $V \mapsto Sym(V)$ is continuous.

A *metric* on a bundle $E \to B$ is then a section $s : B \to Sym(E)$ of the vector bundle $Sym(E) \to B$, such that for any $b \in B$ the symmetric form $s(b)$ is positive definite.

Metrics on vector spaces are very useful, for example they give a canonical choice for a complement of a subspace. As the reader must be guessing this translates into the world of vector bundles. For this one must translate first the notion of subspace.

1.3.1.19 Definition (Subbundle)**.** A subbundle $F \to B$ of a vector bundle $E \to B$ is a choice of a (vector) subspace $F_b \subseteq E_b$ for every b, in such a way that $F \to B$ is a vector bundle with the subspace topology.

1.3.1.20 Lemma (Orthogonal complement of subbundles)**.** *Given a vector bundle E endowed with a metric, for any subbundle F we can take its (fiberwise) orthogonal complement $F^\perp$. Then $F^\perp$ is a subbundle of E and moreover the inclusions yield an isomorphism $E \simeq F \oplus F^\perp$.*

Proof. To prove this one has to check that taking the complement in each fiber is continuous. This is a local condition and thus the proof is reduced to the case of trivial vector bundles which is immediate. ■

1.3.1.21 Example (Metrics on a trivial bundle)**.** Since any real finite dimensional vector space is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ we deduce that any trivial vector bundle can be endowed with a metric.

Now, given an arbitrary vector bundle over a compact base can we endow it with a metric? The answer is yes, and the proof consists of straightforward application of the previous example, the partition of unity and the compactness of the base. So we have:

1.3.1.22 Lemma (Existence of metrics)**.** *Given a bundle, there exists a metric over it.*

This construction makes it possible to embed any bundle as a subbundle of a trivial bundle, although there are some technicalities involved.

1.3.1.23 Lemma (Embedding of bundle in trivial bundle)**.** *Any bundle over a compact base is isomorphic to a subbundle of a trivial bundle.*

Proof. See [Ati94, Corollary 1.4.11 and Corollary 1.4.13]. ■

Finally, using the results stated in this section one can prove:

1.3.1.24 Lemma (Inverse bundle)**.** *Given a vector bundle E there exists a vector bundle E' such that $E \oplus E'$ is trivial.*

Proof. Using Lemma 1.3.1.23 we can embed E in a trivial bundle F. Applying Lemma 1.3.1.20 we get a complement $E^{\perp}$ of E. ■

And thus we deduce the fact that, under stable equivalence, any bundle has an inverse for the direct sum operation. This means that reduced K-theory takes values in abelian groups and not just commutative semigroups. Finally notice that by definition of reduced K-theory we have an obvious function $\mathsf{Vec}(B)_{\sim} \to \widetilde{KO}(B)$. We summarize this in the following remark.

1.3.1.25 Remark (K-theory to reduced K-theory)**.** Since the group operation of the reduced K-theory is the direct sum of bundles, by the universal property of groupification we get a group morphism $KO(B) \to \widetilde{KO}(B)$.

1.4 Classifying spaces

In category theory there is the notion of representable functor. Given a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathsf{Set}$ we say that F is representable if there exists an object $c \in \mathcal{C}$ such that F is naturally isomorphic to $[c, -]$. Notice that the same idea applies to contravariant functors: A contravariant functor is called representable or sometimes corepresentable, if it is naturally isomorphic to a functor of the form $[-, c]$.

This definition is important because it says that a given functor is already encoded in the structure of a category. Many times representable functors do not have Set as its codomain. For example if we have a functor $F : \mathcal{C} \to \mathsf{Grp}$ we can still ask ourselves if F is representable. How can a functor of the form $[c, -]$ take values in Grp? One possibility is to take the representing object c to be a cogroup object. The following is a standard example of this phenomenon.

1.4.0.1 Example (Cogroup structure on $\mathbb{S}^1$)**.** Consider the category $\mathsf{hSpace}_{\bullet}$ of pointed nice topological spaces as objects and maps up to homotopy as arrows. Notice that the coproduct in this category is the wedge operation $\vee$ that takes two pointed spaces and glues their base points together. Consider the space $\mathbb{S}^1$, the one dimensional sphere. Now notice that up to homotopy $\mathbb{S}^1$ has a cogroup structure since we have a comultiplication $\mu : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$, a counit $\eta : \mathbb{S}^1 \to \{*\}$ and a coinverse $\iota : \mathbb{S}^1 \to \mathbb{S}^1$. Such that μ is coassociative, η is a counit for the comultiplication, and ι is the inverse for the comultiplication. Explicitly if we model $\mathbb{S}^1$ as the unit vectors of the complex plane, the comultiplication glues 1 and -1 and ι sends z to $-z$.

The comultiplication can be visualized in the following way:

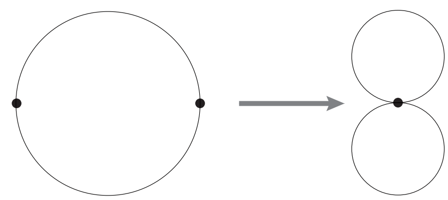


Figure 1.6: Comultiplication of $\mathbb{S}^1$.

The functor represented by $\mathbb{S}^1$ is just the fundamental group functor:

$$[\mathbb{S}^1, -]_\bullet \simeq \pi_1(-) : \mathsf{hSpace}_\bullet \to \mathsf{Grp}\,.$$

Notice how the comultiplication induces the composition of paths. This discussion extends to higher dimensional spheres and thus we get the representability of the homotopy groups functors: $[\mathbb{S}^n, -]_\bullet \simeq \pi_n(-) : \mathsf{hSpace}_\bullet \to \mathsf{Grp}$.

A *classifying space* often comes in the form of a contravariant representable functor. The notion of classifying space is a very broad one. In full generality a classifying space of some data on objects of a category $\mathcal{C}$ is an object $c \in \mathcal{C}$ such that for every $X \in \mathcal{C}$, arrows $X \to c$ correspond to data on X. In other words, the contravariant functor "data on $-$" is represented by c.

1.4.1 The classifying space of F-fibrations

A special case of this concept is the following. Suppose that we want to classify F-fibrations over a space B. The notion of classifying space then suggests that we might want to study some particular space $\mathscr{C}F$ such that maps $B \to \mathscr{C}F$ are in one-to-one correspondence with F-fibrations over B. It turns out that we don't have such a space in the category Space. Just like in the example about the cogroup structure on $\mathbb{S}^1$ we need to pass to hSpace to get a space that classifies F-fibrations. As a matter of fact the classification works *up to fiber homotopy equivalence* of F-fibrations. In this case the functor "data on $-$" that we want to represent is "F-fibrations over $-$ up to fiber homotopy equivalence".

Since the abstract concept of *classifying space* is not too precise and this very concept is used in many different situations, one can find many different definitions in the literature. The following discussion relates our (yet not given) definition of the *classifying space of F-fibrations* to the definition of the *classifying space of a topological group*, and more generally the *classifying space of an associative H-space* [5]. Some references to the history of this theory are given.

Probably the most familiar notion of classifying space is the one of the classifying space of a topological group. To explain what does a classifying space of a topological group G classify we would have to introduce the notion of principal G-bundles. Principal bundles are a very interesting topic on their own but it would be too much of a digression to discuss them here. We refer the reader to the introductory exposition of [MLM94, Chapter 8, Part 1]. For a modern discussion see also the introduction of [Lur09]. The relation between principal G-bundles, classifying spaces and F-fibrations with structure group G is explained in [Mit01] and in [Hus94, Chapter 4, Part 5] (see also the discussion in [May99, Chapter 23, Section 8]). Although we won't discuss the theory of principal bundles we must give a definition of classifying space of a topological group to justify some notation. One way to do this is the following: A classifying space of a topological group G is a space $\mathscr{B}G$ such that $\Omega\mathscr{B}G$ has the homotopy type of G, and such that composition of paths in $\Omega\mathscr{B}G$ corresponds to multiplication in G. For obvious reasons the classifying space $\mathscr{B}G$ is also called the *delooping* of G. The existence and uniqueness of this object is far from obvious, the classical reference for this topic is Milnor's seminal paper [Mil56] in which this space is constructed using the *join construction* (for a shorter exposition see [Hus94, Chapter 4, Part 11]). It is important to keep in mind that this construction can be made in a functorial way and thus a morphism of topological groups induces a map of the classifying spaces (see [Mit01, Theorem 11.1 and Theorem 11.3]).

A generalization of this result to the case of H-spaces is made in [DL59]. Notice that for any space F, the space $\mathrm{hAut}(F)$ carries an obvious structure of H-space. We use the composition of maps for the H-space operation and the identity for the unit. Since composition is associative (on the nose) this is indeed an H-space structure for our definition of H-space (recall Definition 0.0.0.9). Now as explained in [DL59, Definition 7.2, Theorem 6.2 and Corollary 7.4] a classifying space of the H-space $\mathrm{hAut}(F)$, that we denote for now by $\mathscr{B}\,\mathrm{hAut}(F)$, serves to distinguish F-bundles (over a base space B) up to fiber homotopy equivalence in the following sense: Each F-bundle determines a unique map $B \to \mathscr{B}\,\mathrm{hAut}(F)$ (up to homotopy). And two such bundles are fiber homotopy equivalent if and only if the induced maps to the classifying space are homotopic.

Although the result that we just mentioned is sufficient to develop the theory of F-bundles needed in the rest of this chapter it is probably more enlightening to take a look at the more general classification in Stasheff's article [Sta63a], a generalization of the one by Dold and Lashof [DL59]. This is also a good idea because we will study an analogous result in HoTT but using a much simpler construction (see Theorem 2.5.0.2). Before stating the result let us simplify the notation.

1.4.1.1 Notation (Classifying space of F-bundles)**.** We will denote the classifying space $\mathscr{B}\,\mathrm{hAut}(F)$ by $\mathscr{C}F$.

Now we can state the main result of [Sta63a], namely the correspondence between F-bundles and maps to the classifying space of F-bundles.

[5] For this construction the translation by elements of the H-space must induce homotopy equivalences, but notice that if the H-space is connected this follows from the existence of a unit. More details are given in the references.

1.4.1.2 Theorem (Classifying space of F-bundles)**.** *For any compact space F there exists a space $\mathscr{C}F$ such that for every other space B the homotopy equivalence classes of maps $B \to \mathscr{C}F$ are in one-to-one correspondence with the fiber homotopy equivalence classes of F-fibrations over B. The space $\mathscr{C}F$ is in fact the classifying space $\mathscr{B}\,\mathrm{hAut}(F)$ defined in [DL59].*

Moreover there exists a universal map $u : \mathcal{E} \to \mathscr{C}F$ such that the function that takes a homotopy class of maps $f : B \to \mathscr{C}F$ and sends it to the corresponding fiber homotopy class of F-fibrations is given by taking the pullback of this universal map along f, as depicted in the following pullback diagram:

$$\begin{array}{ccc} f^*(\mathcal{E}) & \longrightarrow & \mathcal{E} \\ {\scriptstyle f^*u}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow[f]{} & \mathscr{C}F \end{array}$$

We can restate this as:

$$\frac{F \hookrightarrow E \to B \text{ up to fiber homotopy equivalence}}{[B, \mathscr{C}F]}$$

Next we state a very important theorem about K-theory, the representability of the K-theory functors that we defined. The reader can compare this with Example 1.4.0.1 in which we briefly discussed the representability of the fundamental group functor. This is also a classical example of classifying spaces.

1.4.1.3 Theorem (Representability of KO and $\widetilde{KO}$)**.** *For any (compact) space B we have a natural isomorphism:*

$$\widetilde{KO}(B) \simeq [B, \mathscr{B}O]$$

Here $\mathscr{B}O$ denotes the colimit space of the sequential diagram $\cdots \to \mathscr{B}O(n) \to \mathscr{B}O(n+1) \to \cdots$, where $O(n)$ is the n-dimensional orthogonal group, $\mathscr{B}O(n)$ denotes its classifying space (or delooping) and the inclusions $\mathscr{B}O(n) \to \mathscr{B}O(n+1)$ are given by applying the functor $\mathscr{B}$ to the group morphism $i_n : O(n) \to O(n+1)$ that completes an n-by-n matrix to an $(n+1)$-by-$(n+1)$ one by adding a 1 in the right bottom corner.

We have an analogous statement for non-reduced K-theory:

$$KO(B) \simeq [B, \mathscr{B}O \times \mathbb{Z}]$$

Proof. See [Kar08, Section 1, Proposition 1.32 and Theorem 1.33]. Notice that the cited proof uses the infinite Grassmannian as a model for $\mathscr{B}O(n)$. For a proof of the fact that infinite Grassmannians serve as deloopings of the orthogonal groups see the discussion at the beginning [May99, Chapter 23]. The idea is to prove that we have a fibration $O(n) \hookrightarrow E \to G_n$ where G_n is the Grassmannian of n-dimensional subspaces of $\mathbb{R}^\infty$. This fibration is easy to construct if one knows just a little bit about Grassmannians: The fibration is given by the quotient map $V_n \to G_n$ where V_n is the Stiefel manifold of orthogonal n-frames in $\mathbb{R}^\infty$. This is one of the typical definition of Grassmanians. Of course one must also prove that V_n is contractible. As a side note observe that Grassmanians and orthogonal groups are manifolds and thus have a CW-complex structure, so they are indeed spaces for our definition of space. In [Hat09, Section 1.2, Cell Structures on Grassmannians] it is given a quite explicit characterization of a cell structure on Grassmannians. ■

1.4.1.4 Remark (Vector bundles over the circle)**.** As a matter of fact the proof of the above theorem first shows that the n-dimensional Grassmannian represents n-dimensional vector bundes. See for example [Hat09, Theorem 1.16]. Having this in mind we see that $[\mathbb{S}^1, G_n]$ is in bijection with the isomorphism classes of n-dimensional vector bundles over the circle. Notice that we have $\Omega G_n = \Omega \mathscr{B}O(n) \simeq O(n)$ and it is easy to see that $O(n)$ has exactly two connected components, the orientation-preserving maps and the non-orientation-preserving ones. This implies that $\pi_1(G_n) \simeq \mathbb{Z}_2$ which is abelian. So the action $\pi_1(G_n) \curvearrowright \pi_1(G_n)$ is trivial (this action is fact given by conjugation). Then by Lemma 0.0.0.21 there is a canonical identification $[\mathbb{S}^1, G_n] \simeq [\mathbb{S}^1, G_n]_\bullet$. This means that $\pi_1(G_n)$ is in bijection with the n-dimensional vector bundles over the circle. Thus, up to isomorphism, there are only two n-dimensional vector bundles over the circle. When $n = 1$ these are the vector bundles described in Example 1.3.0.3.

1.4.1.5 Remark (Compact objects)**.** Throughout this section we required the base space to be compact a number of times. Categorically the reason is the following. Consider for example the characterization of the homotopy groups given in Example 1.4.0.1. Now let $\{A_i\}$ be a filtered diagram of spaces. It is usually the case that the homotopy groups of the colimit of the diagram are can be computed as the colimit of the diagram formed by computing the homotopy groups of each A_i. The proof of this fact depends on the model of space one is using.

In the case of CW-complexes one can prove the statement by noticing that a class in the n-th homotopy group of a space A is represented by a (pointed) map $\mathbb{S}^n \to A$. Moreover a homotopy between two representatives is given by a map $\mathbb{S}^n \times I \to A$. Notice that both $\mathbb{S}^n$ and $\mathbb{S}^n \times I$ are compact spaces and thus their image can touch only a finite number of cells of A. In the case of a filtered diagram of spaces this implies two things: That a class in the n-th homotopy group of the filtered colimit is represented by a map from $\mathbb{S}^n$ to one of the spaces A_i of the diagram and that two such classes are equal if and only if their representatives are homotopical already in some A_i of the diagram.

We can state this in the following way. Let B be a compact space and let $\{A_i\}$ be a filtered diagram of spaces. Then the induced morphism $\operatorname{colim}_i[B, A_i] \to [B, \operatorname{colim}_i A_i]$ is an isomorphism. In the language of category theory we say that B is a compact object. This discussion is continued in the second chapter (see Section 2.3 and Section 2.2).

1.5 Spherical fibrations and orthogonal spherical bundles

1.5.0.1 Definition (Spherical fibrations and spherical bundles)**.** Given a space B, a *spherical fibration* over it is a fibration over B with a sphere as its (essentially unique) fiber. We will sometimes call a spherical fibration with an n-dimensional sphere as fiber an n-spherical fibration. A *spherical bundle* over B is a bundle with a sphere as fiber.

1.5.0.2 Example (Spherical bundle induced by a vector bundle with a metric)**.** Given a vector bundle $p : E \to B$ endowed with a metric we can restrict the map to the unit sphere in each fiber, yielding a spherical fibration $S(p) : S(E) \to B$. Under the fiber point of view we denote such a fiber bundle by $S(E)$. Explicitly $S(E)_b$ consists of the unit vectors of E_b with the subspace topology. Notice that the local triviality of E implies local triviality of $S(E)$ and thus the spherical fibration $S(E)$ turns out to be a spherical bundle. Spherical bundles defined in this way are called *orthogonal spherical bundles*.

As a particular case of this last example we have:

1.5.0.3 Example (Spherical bundles over the circle)**.** Applying the construction on the Cylinder and Möbius vector bundles described in Example 1.3.0.3 (we can endow them with the Euclidean metric induced by an embedding in $\mathbb{R}^3$) we obtain the two double coverings of the circle described in Example 1.1.0.4:

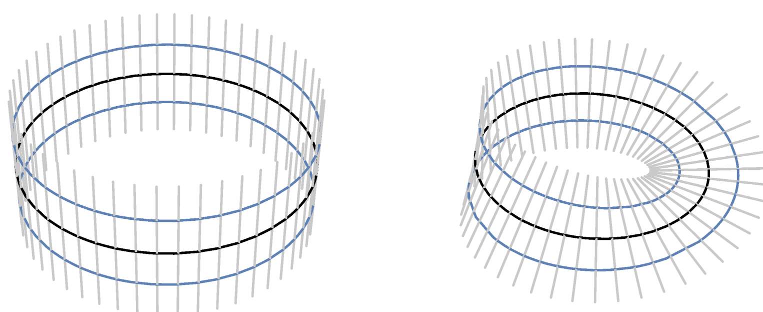


Figure 1.7: Orthogonal spherical bundles over $\mathbb{S}^1$

We learned how to construct spherical bundles out of vector bundles *endowed with a metric*, but if we are given just a vector bundle there is a priori no canonical choice for a metric on it. *But* if we are only interested in the fiber homotopy type of spherical bundles then the choice of the metric makes no difference as stated in the following lemma.

1.5.0.4 Lemma (Homotopy type of spherical bundle induced by a vector bundle)**.** *Let $E \to B$ be a vector bundle and assume given two choices of metric. Call $S \to B$ and $S' \to B$ to the spherical bundles induced by each of the metrics. Then S and S' have the same fiber homotopy type.*

Proof. Start by proving the fact for trivial bundles and then glue the homotopies using a partition of unity. The proof for trivial bundles essentially uses the fact that $GL(n)$ deformation retracts to $O(n)$. See for example [Hir12, Chapter 4, Lemma 2.3]. ■

1.5.0.5 Definition (Stable homotopy type)**.** Two orthogonal spherical bundles $S(E_1), S(E_2)$ are said to have the same stable homotopy type if there exist integers n_1, n_2 such that $S(E_1 \oplus n_1)$ and $S(E_2 \oplus n_2)$ have the same fiber homotopy type.

1.5.0.6 Definition (The set $J(B)$)**.** We define $J(B)$ as the set of stable fiber homotopy types of orthogonal spherical bundles over B. For a spherical bundle $E \to B$ its class in $J(B)$ will be denoted by $J(E)$.

Now we will define a group structure on the set $J(B)$. Recall that in the case of vector bundles we used operations between vector spaces to define operations between vector bundles. When we wanted to define a group structure on vector bundles we used a binary operation on vector spaces, namely the direct sum. In this case the same idea applies. An operation that can be carried on with two spheres to get a new sphere is the join.

1.5.0.7 Definition (Join of two spaces)**.** The underlying set of the join of two spaces X and Y is defined as the set of formal sums $rx + sy$ with $x \in X, y \in Y$, $r, s \in [0, 1]$ such that $r + s = 1$. Now one considers the four projection functions $rx + sy \mapsto r$, $rx + sy \mapsto s$, $rx + sy \mapsto x$ and $rx + sy \mapsto y$, and gives $X * Y$ the initial topology with respect to these functions.

It is natural to interpret the join construction as actually joining every point of one space to every point of the other in a continuous way. With this definition it is easy to prove that the join is an associative operation. A proof of this fact can be found in [Bro06, Section 5.7]. As we said before we can characterize the join of two spheres.

1.5.0.8 Lemma (Join of two spheres)**.** *Let n and m be natural numbers, then the join $\mathbb{S}^n * \mathbb{S}^m$ is homeomorphic to $\mathbb{S}^{n+m+1}$.*

Proof. Notice that for a space A the join $A * \mathbb{S}^0$ is just the (non reduced) suspension of A. Thus a sphere $\mathbb{S}^n$ can be written as $\mathbb{S}^0 * \ldots * \mathbb{S}^0$, joining $n + 1$ times. Using this and the associativity of the join operation we are finished. ■

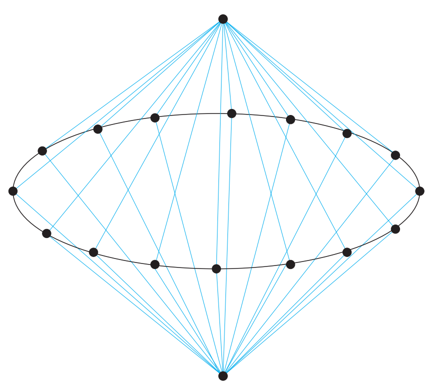


Figure 1.8: The join of $\mathbb{S}^0$ and $\mathbb{S}^1$ is $\mathbb{S}^2$.

1.5.0.9 Remark (Fiberwise join)**.** Making an argument analogous to the one explained in Definition 1.3.1.1 one can indeed define an operation between fiber homotopy classes of spherical fibrations that behaves like the join in each fiber. Moreover, by the Yoneda lemma this induces an H-space structure on the spherical fibrations classifier to be defined in Definition 1.6.0.2.

But we can also make the following remark that will lead us to a simpler definition, since for now we are only interested in orthogonal spherical bundles.

1.5.0.10 Remark (Unit sphere of a direct sum)**.** From the pictures one might infer that the unit sphere of a sum of two vector spaces is the join of the unit spheres of each space, at least up to homotopy. This is indeed true and just like in Lemma 1.5.0.4 it does not depend on the choice of metric used to define the spherical bundle.

So we will define the join operation directly on *orthogonal spherical bundles* instead of making a general definition for spherical fibration. For this we will use the sum operation of vector bundles. An advantage of this approach is that once we define the operation on $J(B)$ it will be fairly obvious that there is a group morphism $KO(B) \to J(B)$. Nonetheless the reader should keep in mind the topological idea that this operation corresponds to the fiberwise join.

1.5.0.11 Definition (Operation on $J(B)$)**.** Given two classes of orthogonal spherical bundles $J(E), J(E')$ over the same space B we define their sum in $J(B)$ as $J(E) + J(E') := J(E \oplus E')$.

Although this definition simplifies some arguments it is not clear that it is well defined since it is defined on representatives. And thus we have to prove the following proposition.

1.5.0.12 Proposition. *The operation defined on $J(B)$ is well defined.*

Proof. We must show that the definition does not depend on the choice of the representatives. Concretely we have to show that if $J(E_1) = J(E_2)$ then $J(E_1 \oplus E') = J(E_2 \oplus E')$. First we reduce it to the case in which E_1 and E_2 do not have just the same *stable* fiber homotopy type but the same fiber homotopy type. For this we can add trivial bundles on both sides. So we assume that E_1 and E_2 have the same fiber homotopy type. Now assume given f, f', h, h' that constitute the fiber homotopy equivalence. Using these we can define F, F', H, H' that constitute a fiber homotopy equivalence between $E_1 \oplus E'$ and $E_2 \oplus E'$. First notice that an element of $S(E_1 \oplus E')$ can be written in a unique way as $u \cos\theta \oplus v \sin\theta$ with $u \in E_1$ and $v \in E'$, and thus we can define:

$$\begin{aligned} F &: S(E_1 \oplus E') \to S(E_2 \oplus E') \\ u \cos\theta &\oplus v \sin\theta \mapsto f(u) \cos\theta \oplus v \sin\theta \end{aligned}$$

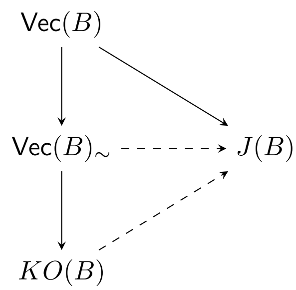
By understanding what we did with F here the rest of the proof follows at once. We simply plugged the maps we were given by hypothesis in the first coordinate. It is then immediate that if we do this for F', H and H' we will get the desired fiber homotopy equivalence. ■

1.5.0.13 Proposition (Group structure on $J(B)$)**.** *The operation defined on $J(B)$ makes it an abelian group. In particular for every bundle E over a fixed space B there exists another orthogonal spherical bundle E' such that $J(E) + J(E')$ is trivial.*

Proof. By definition of the operation we deduce at once that $J(B)$ is at least a commutative semigroup. It remains to check the existence of additive inverses. For this, given $J(E)$ we use Lemma 1.3.1.24 to construct a vector bundle E' such that $E \oplus E'$ is trivial. Then $J(E) + J(E') = J(E \oplus E') = 0$, since we are using spherical bundles up to *stable* fiber homotopy equivalence. ■

Finally observe that $J : \mathsf{Vec}(B) \to J(B)$ factors through $\mathsf{Vec}(B)_\sim$ because it does not distinguish isomorphic bundles. Moreover, since $KO(B)$ is the groupification of $\mathsf{Vec}(B)_\sim$ and $J(B)$

is a group, by the universal property of groupification we get a group morphism $J : KO(B) \to J(B)$. For this we have to use that J is a semigroup morphism and this is immediate by the definition of sum in $J(B)$. Diagrammatically we have:



As a side remark we note that it is easy to prove that the last morphism in the diagram factors through the morphism mentioned in Remark 1.3.1.25 yielding a morphism $\widetilde{KO}(B) \to J(B)$. For this just use that we are considering spherical bundles up to *stable* fiber homotopy equivalence.

By the previous argument we define:

1.5.0.14 Definition (First J homomorphism)**.** The morphism we just constructed is called the J-homomorphism $J : KO(B) \to J(B)$.

Explicitly the homomorphism takes a real vector bundle to the orthogonal spherical bundle induced by endowing the vector bundle with an arbitrary metric. This yields a spherical bundle well defined up to fiber homotopy equivalence and thus is well defined in $J(B)$.

1.6 The homotopy groups of the classifying space of spherical fibrations

Now we will use some concepts introduced in the section about classifying spaces.

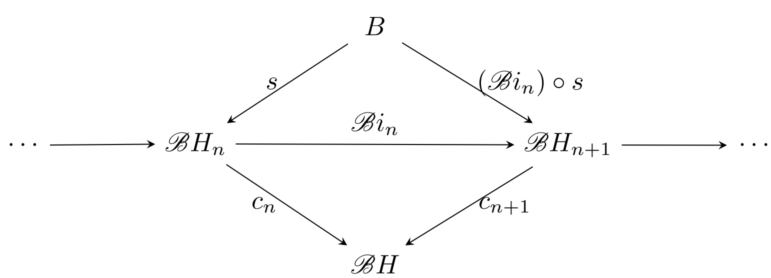
1.6.0.1 Definition. We define the space H_n as the space of self homotopy equivalences of the sphere $\mathbb{S}^{n-1}$ with the compact open topology. Using the notation introduced in Notation 0.0.0.17 we have $H_n := \mathrm{hAut}(\mathbb{S}^{n-1})$.

Now we observe that there is an obvious map $i_n : H_n \to H_{n+1}$ that takes a homotopy equivalence $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ to $\Sigma f : \mathbb{S}^n \to \mathbb{S}^n$ (recall Definition 0.0.0.18). It is easy to see that i_n is in fact continuous.

1.6.0.2 Definition (Classifying space of spherical fibrations)**.** Using Theorem 1.4.1.2 we define the classifying space of $(n-1)$-dimensional spherical fibrations as $\mathscr{B}H_n = \mathscr{B}\,\mathrm{hAut}(S^{n-1})$. Using the notation introduced in Notation 1.4.1.1 we have $\mathscr{B}H_n = \mathscr{C}\mathbb{S}^{n-1}$. Since $\mathscr{B}$ is functorial we also get maps $\mathscr{B}i_n : \mathscr{B}H_n \to \mathscr{B}H_{n+1}$ by the application of $\mathscr{B}$ to the maps defined above. We define the classifying space of spherical fibrations as the colimit of this diagram $\mathscr{B}H := \operatorname*{colim}_n \mathscr{B}H_n$.

The following remark discusses the meaning of these definitions.

1.6.0.3 Remark. Fix a base space B. By Theorem 1.4.1.2 it is clear that $\mathscr{B}H_n$ classifies $\mathbb{S}^{n-1}$-fibrations. But what does $\mathscr{B}H$ classify? By the definition of $\mathscr{B}H$ it is immediate that a map $B \to \mathscr{B}H_n$ induces a map $B \to \mathscr{B}H$ by postcomposition with the inclusion of $\mathscr{B}H_n$ in the colimit $c_n : \mathscr{B}H_n \to \mathscr{B}H$. This is true for every $n \in \mathbb{N}$ and thus it might seem to be the case that $\mathscr{B}H$ classifies all spherical fibrations. This is not exactly true. Consider an $(n-1)$-spherical fibration classified by a map $s : B \to \mathscr{B}H_n$. Composing the classifying map s with the map $\mathscr{B}i_n$ we obtain a map $(\mathscr{B}i_n) \circ s : B \to \mathscr{B}H_{n+1}$ that classifies an n-spherical fibration. This n-spherical fibration can be regarded as the suspension of the original $(n-1)$-spherical fibration (one can check that it is indeed the fiberwise suspension). Notice that $c_n \circ s = c_{n+1} \circ (\mathscr{B}i_n) \circ s$ and thus the spherical fibrations classified by s and by $(\mathscr{B}i_n) \circ s$ get identified when seen as maps $B \to \mathscr{B}H$. The following commutative diagram depicts the described situation:



Since $\mathscr{B}H$ is a filtered colimit, for a compact base space B two classes $s \in [B, \mathscr{B}H_n]$ and $s' \in [B, \mathscr{B}H_m]$ are identified when passing to the colimit if and only if they are identified when suspending them a finite number of times. So for a compact base space $\mathscr{B}H$ classifies spherical fibrations *up to stable homotopy equivalence*. For this one also needs to justify that since we are requiring the base space to be compact then every map $B \to \mathscr{B}H$ factors through a map $B \to \mathscr{B}H_i$ for some i (recall Remark 1.4.1.5).

The reader is invited to compare this with the classifying space of reduced K-theory (see Theorem 1.4.1.3) which classifies vector bundles *up to stable equivalence*.

Although the space $\mathscr{B}H$ classifies spherical fibrations up to stable homotopy equivalence we call it the classifier of spherical fibrations for simplicity.

The proof of the next theorem follows closely the one given in [Ati61], but we fill in many arguments that are not explained in the reference. All of them are probably obvious to an experienced homotopy theorist but it is instructive to understand the constructions in this proof because it seems very plausible to make an analogous proof in HoTT and the proof uses many standard results in homotopy theory that would also be nice to formalize in HoTT (see Section 2.6.1 for a discussion about translating the proof into HoTT). Some of these results are stated and proved during the course of the proof and other are stated and proved as additional lemmas after the proof.

The theorem relates the homotopy groups of the space $\mathscr{B}H$ with the stable homotopy groups of the spheres (recall Definition 0.0.0.20).

1.6.0.4 Theorem (Homotopy groups of $\mathscr{B}H$)**.** *The following three statements characterize the homotopy groups of $\mathscr{B}H$: The π_0 of $\mathscr{B}H$ is trivial. The fundamental group of $\mathscr{B}H$ is isomorphic to $\mathbb{Z}_2$. The* *higher homotopy groups of $\mathscr{B}H$ are isomorphic to the stable homotopy groups of the spheres shifted by -1. Formally $\pi_r(\mathscr{B}H) \simeq \pi^{\mathbb{S}}_{r-1}$ when $r > 1$.*

Proof. By construction the space $\mathscr{B}H$ is connected (see [DL59, Section 7]) and thus its π_0 is trivial. This establishes the first statement.

Now we use the fact that $\Omega\mathscr{B}H_n$ is homotopy equivalent to H_n and thus we only have to deal with H_n since we are concerned with homotopy groups from 1 onwards. To see that $\Omega\mathscr{B}H_n$ is in fact homotopy equivalent to H_n use Lemma 1.6.0.5 and the universal bundle constructed in [DL59, Part 6] which has $\mathscr{B}H_n$ as its base space, H_n as its fiber and a contractible space as total space.

The rest of the proof is structured as follows: For the fundamental group case we will see that the group $\pi_0(H_n)$ is isomorphic to $\mathbb{Z}_2$. For the case of the higher homotopy groups we will see that $\pi_{r-1}(H_n)$ is isomorphic to $\pi_{n+r-2}(\mathbb{S}^{n-1})$ if $2 \leq r \leq n-2$. Moreover, the isomorphisms will render commutative the following square for every n and every r:

$$\begin{array}{ccc}
\pi_r(H_n) & \xrightarrow{\simeq} & \pi_{n+r-1}(\mathbb{S}^{n-1}) \\
{\scriptstyle \pi_r(i_n)}\downarrow & & \downarrow{\scriptstyle S} \\
\pi_r(H_{n+1}) & \xrightarrow{\simeq} & \pi_{n+r}(\mathbb{S}^{n})
\end{array}$$

Here S denotes the suspension morphism:

$$S : [\mathbb{S}^{n+r-1}, \mathbb{S}^{n-1}]_\bullet \to [\mathbb{S}^{n+r}, \mathbb{S}^{n}]_\bullet$$
$$f \mapsto \Sigma f$$

and $i_n : H_n \to H_{n+1}$ are the maps defined previously. The result then follows by taking the colimit in r on both sides of these squares, and by using the fact that the $\pi_n(-)$ commute with filtered colimits (recall Remark 1.4.1.5).

So let us now prove the second statement, the one about the fundamental group of $\mathscr{B}H$. It suffices to show that H_n has just two connected components and that the maps i_n respect them. To see this remember that H_n is the space of homotopy self equivalences of the $(n-1)$-dimensional sphere. But by Proposition 0.0.0.23 we know that maps $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ are completely determined up to homotopy by their degree. Since we are considering homotopy self *equivalences*, by the multiplicativity of the degree, the degree can be either 1 or -1 and thus H_n has exactly two connected components. Moreover the degree is preserved by the maps i_n because they are defined by taking the suspension of a homotopy self equivalence. Let us rename the two connected components of H_n as H_n^+ and H_n^- respectively, since we will need them for the proof of the third statement.

The proof of the third statement is more involved. Fix a distinguished point $b \in \mathbb{S}^{n-1}$ and let F_n be the space of *pointed* maps $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ with the compact open topology, formally $F_n := \mathsf{Space}_\bullet(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$. The same discussion about degrees applies here and thus F_n is the disjoint union of the connected components F_n^d that consist of the maps of degree $d \in \mathbb{Z}$. Now, notice that the component F_n^1 embeds in a obvious way in H_n^+ (a map of degree 1 is a homotopy equivalence). On the other hand we have a map $q : H_n^+ \to \mathbb{S}^{n-1}$ that evaluates a self homotopy equivalence at b and by Lemma 1.6.0.6 this map is in fact a fibration. Moreover, the fiber of q is by definition F_n^1 and thus we can apply the long exact sequence of homotopy groups to the fibration $F_n^1 \hookrightarrow H_n^+ \to \mathbb{S}^{n-1}$. To be precise we must choose a base point for H_n^+, and for this we pick the identity $\mathsf{Id} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$.

Now notice that if $r \leq n-3$ the groups $\pi_{r+1}(\mathbb{S}^{n-1})$ and $\pi_r(\mathbb{S}^{n-1})$ are trivial and so, from the long exact sequence we get natural isomorphisms:

$$\pi_r(H_n^+) \simeq \pi_r(F_n^1), \quad 1 \leq r \leq n-3$$

But the inclusion $H_n^+ \to H_n$ induces an isomorphism $\pi_r(H_n^+) \simeq \pi_r(H_n)$ because the functors $\pi_r(-)$ depend only on the connected component of the base point for $r \geq 1$ and we pointed H_n with the identity. This gives us $\pi_r(H_n) \simeq \pi_r(F_n^1)$.

We now use the fact that F_n has an H-space structure, since it is by definition the $(n-1)$-fold loop space of $\mathbb{S}^{n-1}$. Moreover the monoid structure induced by the H-space operation on the connected components of F_n is a group structure, since this monoid is exactly $\pi_{n-1}(\mathbb{S}^{n-1})$. Thus by Lemma 1.6.0.8 we deduce that we have isomorphisms $\pi_r(F_n^1) \simeq \pi_r(F_n^0)$.

Let us make a diagram reflecting our current situation. For $r \leq n-3$ we have:

$$\begin{array}{ccccc}
 & & \pi_{r+1}(\mathbb{S}^{n-1}) & & \\
 & & \downarrow & & \\
\pi_r(F_n^0) & \xrightarrow{\simeq} & \pi_r(F_n^1) & & \\
 & & \downarrow \simeq & & \\
 & & \pi_r(H_n^+) & \xrightarrow{\simeq} & \pi_r(H_n) \\
 & & \downarrow & & \\
 & & \pi_r(\mathbb{S}^{n-1}) & &
\end{array}$$

where the vertical maps are part of the long exact sequence.

Next we must show that we have isomorphisms $\pi_r(F_n^0) \simeq \pi_{n+r-1}(\mathbb{S}^{n-1})$ for this will give us isomorphisms $\pi_r(H_n) \simeq \pi_{n+r-1}(\mathbb{S}^{n-1})$. Notice that we have an equivalence:

$$\pi_r(F_n^0) = [\mathbb{S}^r, F_n^0]_\bullet \simeq [\mathbb{S}^r, \mathsf{Space}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})]_\bullet = [\mathbb{S}^r, F_n]_\bullet \tag{1.1}$$

Remember that the $\bullet$ means that we are considering pointed maps and the $[-,-]$ indicates that we are considering maps up to homotopy. The equivalence of this last statement is induced by the inclusion $F_n^0 \to \mathsf{Space}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$ that forgets the pointing. Why does the inclusion induce an equivalence? Observe that F_n is pointed by the function that takes all $\mathbb{S}^{n-1}$ to the standard base point and that map lies in the connected component F_n^0 for it has degree zero, since it is nullhomotopic. Since we are considering pointed maps that have $\mathbb{S}^r$ as its domain and $\mathbb{S}^r$ is connected we conclude that pointed maps $\mathbb{S}^r \to F_n$ are the same as pointed maps $\mathbb{S}^r \to F_n^0$ and thus the equivalence.

Finally in Eq. (1.1) we can use the adjunction of Proposition 0.0.0.19 between the exponential and the smash product in the category of pointed spaces and the fact that the smash product of two spheres is again a sphere to get isomorphisms:

$$\pi_r(F_n^0) = [\mathbb{S}^r, \mathsf{Space}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})]_\bullet \simeq [\mathbb{S}^{n-1} \wedge \mathbb{S}^r, \mathbb{S}^{n-1}]_\bullet \simeq [\mathbb{S}^{n+r-1}, \mathbb{S}^{n-1}]_\bullet = \pi_{n+r-1}(\mathbb{S}^n).$$

To conclude we must show that the square in the beginning of this proof commutes. To see this recall that the vertical arrow on the right side of the square is given by taking reduced suspension:

$$[\mathbb{S}^{n+r-1}, \mathbb{S}^{n-1}]_\bullet \to [\Sigma\mathbb{S}^{n+r-1}, \Sigma\mathbb{S}^{n-1}]_\bullet \simeq [\mathbb{S}^{n+r}, \mathbb{S}^n]_\bullet$$

On the other hand, the vertical arrow of the left side consists on composing a map $\pi_r(H_n) = [\mathbb{S}^r, H_n]_\bullet$ with the reduced suspension $i_n : H_n \to H_{n+1}$. A straightforward check shows that all the isomorphisms we defined respect this suspension. One must pay attention in the case of the adjunction between the $\hom$ space and the smash product. ■

1.6.0.5 Lemma. *A pointed fibration $F \hookrightarrow E \to B$ such that the total space E is contractible induces a homotopy equivalence between F and ΩB.*

Proof. Consider the connecting map $\Omega B \to F$ mentioned in the discussion of Proposition 1.1.0.14. This map induces the connecting morphisms in the long exact sequence of the fibration. Since E is contractible its homotopy groups are trivial and thus the connecting morphisms turn out to be isomorphisms in every degree. This proves that the connecting map is a weak homotopy equivalence but since we are working with CW-complexes this is indeed a homotopy equivalence by Whitehead's theorem. ■

1.6.0.6 Lemma. *The map $q : H_n^+ \to \mathbb{S}^{n-1}$ that evaluates at b is a fiber bundle with fiber F_n^1 and thus it is a fibration and the long exact sequence of homotopy groups applies.*

Proof. We must construct an open cover of $\mathbb{S}^{n-1}$ that trivializes the map q. For this, take $U_b := \mathbb{S}^{n-1} \setminus \{b\}$ and $U_{-b} := \mathbb{S}^{n-1} \setminus \{-b\}$ where $-b$ is the antipode of b. Now we must construct homeomorphisms φ_α with $\alpha \in \{b, -b\}$ that render commutative the triangles:

$$\begin{array}{ccc} U_\alpha \times F_n^1 & \xrightarrow{\varphi_\alpha} & q^{-1}U_\alpha \\ & \searrow^{\pi_1} & \downarrow q \\ & & U_\alpha \end{array}$$

Let us begin with the case $\alpha = b$. For each $y \in U_b$ we can consider the geodesic $p : b \rightsquigarrow y$ endowing $\mathbb{S}^{n-1}$ with the metric induced by $\mathbb{R}^n$, and we can rotate $\mathbb{S}^{n-1}$ following this geodesic. This yields a rotation $r_p : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ that sends b to y. Then define $\varphi_b(y, f) := r_p \circ f$. Notice that the paths $p = p(y)$ depend continuously on y. This is well defined since $-b \notin U_b$ and $-b$ is the only point for which there is more than one geodesic that connects it to b. The fact that we were able to make this choice in a continuous way implies that φ_b is continuous. Since rotations are homeomorphisms φ_b is a homeomorphism.

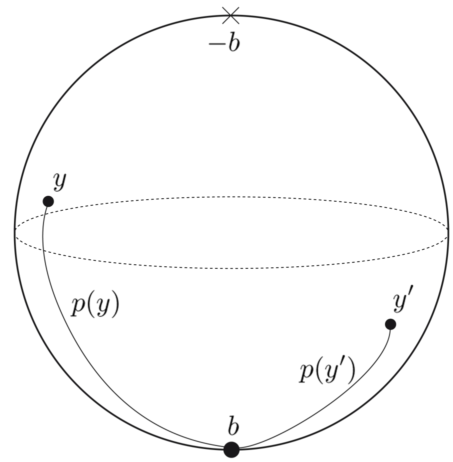


Figure 1.9: A path $p(y) : b \rightsquigarrow y$ for each $y \in U_b$.

It remains to deal with the case $\alpha = -b$. For this case we cannot use exactly the same idea as above, since we do not have a canonical choice for a geodesic connecting b and $-b$. We solve this problem by making a non canonical choice at the beginning and then using the same idea as above. Let $t : b \rightsquigarrow -b$ be a geodesic connecting b to $-b$ and let r_t be its associated rotation. Now define $\varphi_{-b}(y, f) := r_p \circ r_t \circ f$ where p is the geodesic $p : -b \rightsquigarrow y$ which is well defined since $y \in U_{-b}$. Again $p = p(y)$ is continuous in y. ■

Now to the proof of the lemma about H-spaces. For this we need:

1.6.0.7 Remark. An H-space structure on a space induces a monoid structure on the set of connected components. We will regard the set of connected components as a discrete space. Call X_x to the connected component of $x \in X$. If the H-space operation of X is called μ define the operation induced on the connected components as $\overline{\mu}(X_x, X_y) = X_{\mu(x,y)}$. To see that this is well defined notice that if $X_x = X_z$ then there is a path $p : x \rightsquigarrow z$ and thus a path $\mu(p, y) : \overline{\mu}(X_x, X_y) \rightsquigarrow \overline{\mu}(X_z, X_y)$. Since the space of connected components is discrete we have $\overline{\mu}(X_x, X_y) = \overline{\mu}(X_x, X_y)$. Since μ is associative it is immediate that $\overline{\mu}$ is also associative.

1.6.0.8 Lemma. *If we have an H-space such that the monoid structure induced on the connected components of the space is a group structure then all the connected components are homotopy equivalent.*

Proof. Suppose X has an H-space structure with operation $\mu : X \times X \to X$. Let X_e be the connected component of the unit of the H-space structure and let X_x be the connected component of an element $x \in X$. We will show that for any connected component X_x we have a homotopy equivalence $X_0 \simeq X_x$. For any connected component X_x we know that there exists a connected component X_y such that $\overline{\mu}(X_y, X_x) = X_e$ where $\overline{\mu}$ is the operation induced by μ on the connected components of X. This is a direct consequence of the fact that μ induces a group structure on the connected components. Now consider the maps $\mu(y, -) : X_x \to X_e$ and

$\mu(x,-): X_e \to X_x$. Then the following equivalence shows that these are homotopy inverses of each other:

$$\mu(x, \mu(y,-)) = \mu(\mu(x,y),-) \simeq \mu(e,-) = \mathsf{Id}\,.$$

The first equality is due to the associativity of μ, the second equivalence holds because $\mu(x,y) \in X_e$ and thus we can choose a path $p : \mu(x,y) \rightsquigarrow e$ that induces a homotopy $\mu(p,-) : \mu(\mu(x,y),-) \simeq \mu(e,-)$. The last equality is just the unit law of μ.

The proof that the other composition is also a homotopy equivalence is completely analogous. ∎

1.6.0.9 Remark. In the course of the previous proof we chose an element in each connected component of X. This cannot be done in a natural way and thus the homotopy equivalence between X_x and X_e that we constructed is not canonical. *But* the induced isomorphisms in the homotopy groups are indeed canonical: They do not depend on the choice of representatives for the connected components. This is because any two choices $x, x' \in X$ induce homotopical maps $\mu(x,-) \simeq \mu(x',-)$ and we know that homotopical maps induce the same map in the homotopy groups.

1.7 The J-homomorphism

In the last section we established the fact that the homotopy groups of the spherical fibrations classifier are isomorphic to the stable homotopy groups of the spheres. Recalling once again the representability of the homotopy groups functors this means that:

$$[\mathbb{S}^n, \mathscr{B}H]_\bullet = \pi_n(\mathscr{B}H) \simeq \pi^S_{n-1}$$

for $n > 1$.

Now the construction gets interesting. Notice that we have an obvious inclusion $j_n : O(n) \to H_n$ that takes an orthogonal transformation and restricts it to the unit sphere: An orthogonal transformation restricts to a homeomorphisms of the unit sphere and thus it is a self homotopy equivalence of the unit sphere. It is easy to see that these maps commute with the suspension maps we defined in Theorem 1.4.1.3 and Definition 1.6.0.1, and thus that the following diagram commutes:

$$\begin{array}{ccc} O(n) & \xrightarrow{\;j_n\;} & H_n \\ {\scriptstyle i_n}\big\downarrow & & \big\downarrow{\scriptstyle i_n} \\ O(n+1) & \xrightarrow{\;j_{n+1}\;} & H_{n+1} \end{array}$$

This follows from the fact that taking the suspension of a sphere adds a dimension but the suspension of a map is constant in this new dimension. The same occurs when taking the suspension of an orthogonal transformation by adding a one in the bottom right corner.

Now fix a pointed space B. By applying first the functor $\mathscr{B}$ and then the functor $[B,-]$ we get commutative squares:

$$\begin{array}{ccc} [B,\mathscr{B}O(n)] & \xrightarrow{[B,\mathscr{B}j_n]} & [B,\mathscr{B}H_n] \\ \Big\downarrow{\scriptstyle [B,\mathscr{B}i_n]} & & \Big\downarrow{\scriptstyle [B,\mathscr{B}i_n]} \\ [B,\mathscr{B}O(n+1)] & \xrightarrow{[B,\mathscr{B}j_{n+1}]} & [B,\mathscr{B}H_{n+1}] \end{array}$$

Now if we take B to be a compact space the sequential colimits formed by the vertical arrows of the squares commute with the functor $[B,-]$ yielding a morphism $\mathscr{J}:[B,\mathscr{B}O]\to[B,\mathscr{B}H]$.

Why is this construction useful? Because it gives an alternative way of understanding the group $J(B)$ that we defined in Definition 1.5.0.6: $J(B)$ turns out to be the image of this last morphism $\mathscr{J}$ (although to see this one must carefully follow the construction of the universal bundle made in [DL59]). Finally, taking B to be a sphere $\mathbb{S}^n$ and recalling from Theorem 1.4.1.3 that reduced real K-theory is represented by $\mathscr{B}O$ we have constructed a morphism from the reduced real K-theory of the spheres to the stable homotopy groups of the spheres:

$$\widetilde{KO}(\mathbb{S}^n)=[\mathbb{S}^n,\mathscr{B}O]\to[\mathbb{S}^n,\mathscr{B}H]$$

for $n>1$.

Although we did not prove this formally it is easy to check that the fiberwise join makes $\mathscr{B}H$ a homotopy associative H-space[6] and thus by Proposition 0.0.0.24 the fundamental group of $\mathscr{B}H$ acts trivially on its homotopy groups. This means that we can identify the homotopy classes of maps $[\mathbb{S}^n,\mathscr{B}H]$ with the homotopy classes of *pointed* maps $[\mathbb{S}^n,\mathscr{B}H]_\bullet$. Using the morphism that we just defined and Theorem 1.6.0.4 we can define the following morphism.

1.7.0.1 Definition (Second J-homomorphism)**.** For every $n\in\mathbb{N}$ there is a morphism $\widetilde{KO}(\mathbb{S}^n)\to\pi^{\mathbb{S}}_{n-1}$ called the *stable J-homomorphism*. The image of this morphism is isomorphic to the group $J(\mathbb{S}^n)$.

This morphism was first defined by Whitehead in [Whi42]. As we said before the reader can find many applications of this morphism, including computations of stable homotopy groups of spheres, in [Ada63], [Ati61] and [Hat09].

[6]Since $\mathscr{B}H$ classifies spherical bundles up to stable fiber homotopy equivalence, and the fiberwise join makes the stable fiber homotopy equivalences of spherical bundles over a base space B a monoid, one can use Yoneda to endow $\mathscr{B}H$ with an H-space structure.

2

Spherical fibrations in HoTT

To translate the basics of the theory of spherical fibrations into HoTT we will need to discuss some topics first. To classify spherical fibrations we will need *pointed connected types* and *classifying spaces* in HoTT. To classify spherical fibrations up to stability we will need *sequential diagrams*. And just like in the classical setting, this classification will work when the base space is a *compact type*. Moreover, to get an analog of Theorem 1.6.0.4 we will need the spheres to be compact types. These topics are studied in the first sections of this chapter. One should keep in mind that many results on these topics depend on conjectures. Finally we will apply these results to study the fundamental group of the classifying space of spherical fibrations, and here we will also study the degree of maps $\mathbb{S}^n \to \mathbb{S}^n$ and the action $\pi_1(B) \curvearrowright \pi_n(B)$ for a pointed space B.

But first, as promised at the beginning of the first chapter, let us explain in which way the language of HoTT lets us identify fibrations over a type and representations of this type. Fix a type $B : \mathcal{U}$. In the same way that a presheaf $P : \mathscr{C}^{op} \to \mathsf{Set}$ over a category $\mathscr{C}$ is a representation of this category by sets, a map $P : B \to \mathcal{U}$ can be regarded as a representation of B by homotopy types: Each inhabitant $b : B$ is represented by a (homotopy) type $P(b) : \mathcal{U}$ and each path $p : b = c$ is represented by an equivalence $P(p) : P(b) \simeq P(c)$. Maps with the universe $\mathcal{U}$ as codomain are usually called *type families*, in analogy with the construction made in Remark 1.1.0.1. Continuing with this analogy given a map $f : E \to B$ one can consider the fiber of each $b : B$ defined as:

$$\begin{aligned} \mathsf{fib}_f &: B \to \mathcal{U} \\ &b \mapsto (e : E) \times (f(e) = b) \end{aligned}$$

It is nice to observe how, syntactically, this is just a translation of the usual fiber (or preimage) of a point through a function between sets, but homotopically this corresponds to taking the *homotopy* fiber of a map between spaces as we did in Remark 1.1.0.15. By the functoriality of all the constructions in HoTT this induces an application:

$$\frac{E \to B}{B \to \mathcal{U}}$$

Moreover this correspondence is an equivalence as proved in [Uni13, Theorem 4.8.3]:

$$\frac{(E : \mathcal{U}) \times (E \to B)}{\overline{} B \to \mathcal{U}}$$

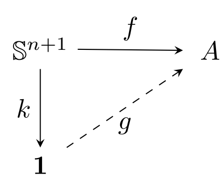
This is a formal version of the analogy explained in Remark 1.1.0.2. This is analogous to what happens when we restrict the study of presheafs to the study of sheaves: Sheaves can be characterized as functors satisfying a gluing condition or, equivalently, as fibrations satisfying a local triviality condition. We will use this result in Theorem 2.5.0.2 and in the last section of this chapter (Section 2.7) where we will refine the analogy between representations and fibrations to the case of pointed connected types.

2.1 Connected types

The goal of this section is to motivate the definition of connected types and to see that connected types enjoy many properties that one would expect. Some results in this section have generalizations to higher notions of connectedness but we won't need the more general statements.

The standard definition of being n-truncated is by induction: A type is (-2)-truncated if it is contractible. A type is $(n+1)$-truncated if its identity types are n-truncated. For n small n-truncated types have special names: (-2)-truncated types are the *contractible types*, (-1)-truncated types are called *mere propositions* (or just *propositions*), and 0-truncated types are called *sets* (or *discrete types*). An n-truncated type is also called an n-type.

An equivalent definition of n-truncated type is the following. A type A is n-truncated if it is (strongly) localized at the map $\mathbb{S}^{n+1} \to \mathbf{1}$. This means that for every map $f : \mathbb{S}^{n+1} \to A$ there exists a unique $g : \mathbf{1} \to A$ such that the following triangle commutes.



Here k is the only map $k : \mathbb{S}^{n+1} \to \mathbf{1}$. In HoTT we say this by considering the map $- \circ k : (\mathbf{1} \to A) \to (\mathbb{S}^{n-1} \to A)$ and defining A to be an n-type if $\mathsf{isEquiv}((- \circ k))$.

It is clear that being an n-type is a mere proposition and thus we can consider the universe $\mathcal{U}^n :\equiv (A : \mathcal{U}) \times \mathsf{isntype}(A)$ as a sub-universe of $\mathcal{U}$ in the sense that the inclusion/projection:

$$\begin{aligned} \iota : \mathcal{U}^n &\to \mathcal{U} \\ (A, d) &\mapsto A \end{aligned}$$

has a mere proposition as fiber.

The n-truncation of a type for $n \geq -1$ is the type $\|A\|_n$ obtained by localizing A at the map $\mathbb{S}^{n+1} \to \mathbf{1}$. Formally we can construct $\|A\|_n$ with a higher inductive type having constructors:

$$\begin{aligned} |-|_n : A &\to \|A\|_n \\ h : (\mathbb{S}^{n+1} \to \|A\|_n) &\to \|A\|_n \\ s : (r : \mathbb{S}^{n+1} \to \|A\|_n) &\to (x : \mathbb{S}^{n+1}) \to r(x) = h(r) \end{aligned}$$

2.1.0.1 Notation. The (-1)-truncation of a type $A : \mathcal{U}$ is often called just the *truncation* of A. Following this usage, the notation $\|A\|$ will mean $\|A\|_{-1}$. We will do the same for the truncation constructor $|-|$ which will denote the (-1)-truncation constructor $|-|_{-1}$.

2.1.0.2 Remark (Functoriality of truncation)**.** Lemma 7.3.1 in [Uni13] proves that $\|A\|_n$ is an n-type. Moreover, the map $\|-\|_n : \mathcal{U} \to \mathcal{U}^n$ is universal in the following sense. If $\mathsf{isntype}(B)$ then:

$$\frac{\|A\|_n \to B}{\overline{A \to B}}$$

and thus $\mathcal{U}^n$ is a reflective subuniverse of $\mathcal{U}$: This is proved in [Uni13, Lemma 7.3.3]. The left adjoint of the inclusion $\iota : \mathcal{U}^n \to \mathcal{U}$ is the truncation $|-|_n : \mathcal{U} \to \mathcal{U}^n$.

If n-types are types with trivial homotopy in dimensions grater than n, the connected types are the dual concept: Types with trivial homotopy in dimensions less or equal than n. Formally we say that a type is n-connected if its n-truncation is contractible. Why is this definition sensible? In the case of 0-connectedness a simple example can help to understand it:

2.1.0.3 Example. Suppose we have the type $\mathbb{S}^1 + \mathbf{1}$, the disjoint union of a circle and a point. This type should not be connected for it is a disjoint union of non-empty types. What happens when we take its 0-truncation? Every path space $x = y$ collapses to a proposition, and thus the inhabitants of the $\mathbb{S}^1$ component are all now (naturally) equal to each other. The space that we got after applying $\|-\|_0$ is not contractible and so $\mathbb{S}^1 + \mathbf{1}$ is not connected. But if we make the same construction but with $\mathbb{S}^1$ we get a contractible space, so $\mathbb{S}^1$ is connected.

Proving this assertion formally exemplifies the difference between equality and mere equality.

2.1.0.4 Example ($\mathbb{S}^1$ is connected)**.** Notice that it suffices to give a mere equality $\|\mathsf{base} = x\|_{-1}$ for every $x : \mathbb{S}^1$. We do this by induction on $\mathbb{S}^1$. For the base case we use the (truncated) reflexivity $|\mathsf{refl}_{\mathsf{base}}|_{-1}$. Then it remains to prove that loop respects this choice: We have to show that the transport of $|\mathsf{refl}_{\mathsf{base}}|_{-1}$ through loop is equal to itself. But $|\mathsf{refl}_{\mathsf{base}}|_{-1}$ lives in a mere proposition and thus this is immediate (see also Remark 2.1.0.13).

Since we won't need n-connectedness for $n > 0$ we state a more explicit definition of 0-connectedness.

2.1.0.5 Definition (Connected type)**.** A type A is said to be 0-connected (or just connected) if the type is merely inhabited any two inhabitants are merely equal. Formally define $\mathsf{isConn}(A) :\equiv \|A\| \times ((a, b : A) \to \|a = b\|_{-1})$.

A very important remark is the following.

2.1.0.6 Remark (Connectedness is a mere property)**.** A type can be connected in at most one way. This means that the space of proofs that a fixed type is connected is a mere proposition. The fact that $\mathsf{isConn}(A)$ is a mere proposition is a direct consequence of the fact that a product of mere propositions is again a mere proposition. The proof of this last statement is a simple use of function extensionality (see [Uni13, Example 3.6.2]).

If we try to draw a (homotopy) type we will probably picture it as a disjoint union of connected types. To understand a bit more why the given definition of connected type is a good notion of connectedness let us prove that any type is a disjoint union of connected types. For this we must define what a disjoint union is. Since sets are by definition 0-truncated types (discrete spaces) we can define a *disjoint union* to be a sigma type indexed by a set. Now consider the zero truncation map $|-|_0 : A \to \|A\|_0$. This map collapses every connected component to a point. Notice that for every $x : \|A\|_0$ the fiber is necessarily connected:

2.1.0.7 Proposition (Fiber of the 0-truncation)**.** *For any type A the fiber of an element $x : \|A\|_0$ through the map $|-|_0 : A \to \|A\|_0$ is connected.*

Proof. This is in fact a particular case of a more general statement: For any n the map $|-|_n : A \to \|A\|_n$ is n-connected. And this is [Uni13, Corollary 7.5.8]. ■

Using this we can prove that a type is indeed the disjoint union of connected components.

2.1.0.8 Proposition (A type as a disjoint sum)**.** *Any type can be written as a disjoint union of connected components.*

Proof. Using [Uni13, Lemma 4.8.2] we can write $A \simeq (x : \|A\|_0) \times |-|^{-1}(x)$ and the previous lemma implies that $|-|^{-1}(x)$ is connected. The type $\|A\|_0$ is a set by [Uni13, Lemma 7.3.1]. ■

The following definition gives a way of constructing a new type from a type and an element of this type. It is useful to think about this definition as the description of the *connected component* that a given element inhabits.

2.1.0.9 Definition (Connected component)**.** Given a type A and an inhabitant $a : A$ define the *connected component* of a to be the type $\mathscr{C}(A, a) :\equiv (b : A) \times \|a = b\|$. The type of elements of A that are merely equal to a.

The notation $\mathscr{C}$ will be justified in Section 2.5 where we will see how this construction lets us build a classifying space in the sense of Theorem 1.4.1.2. There is a natural inclusion of a connected component in the original type:

2.1.0.10 Definition (Inclusion of connected component)**.** Fix a type A and an $a : A$. By projecting the first component of the dependent sum we get a map $i : \mathscr{C}(A, a) \to A$.

Observer that this map is an embedding: The fibers are mere propositions. One can observe that we have two slightly different notions of connected component. One is to think about a connected component as the fiber of a point through the 0-truncation, as we did in Proposition 2.1.0.7. The other notion is the one of Definition 2.1.0.9 above which says the connected component *of a distinguished inhabitant* is the type of inhabitants merely equal to it. The following lemma serves to relate the two notions[1].

2.1.0.11 Lemma. *Given a type $A : \mathcal{U}$ and two inhabitants $a, b : A$ there is an equivalence:*

$$(|a|_0 =_{\|A\|_0} |b|_0) \simeq \|a = b\|_{-1}$$

[1]Notice that this is the zero case of [Uni13, Theorem 7.3.12].

Proof. Since both types are mere propositions it suffices to show that they are logically equivalent. A proof of the LHS implies that b is in the fiber of a through the map $\| - \|_0$. By Proposition 2.1.0.7 this fiber is connected and thus we get a mere path connecting a and b. For the converse recall that the path spaces of a 0-truncation are mere propositions and thus given a mere path connecting a and b we can assume it comes from an actual path $p : a = b$ by induction on the (-1)-truncation. Now use the functoriality of the 0-truncation on p to finish the proof. ■

2.1.0.12 Remark (Characterization of connected component)**.** With this last result it is immediate to prove that the connected component of an inhabitant $a : A$ in the sense of Definition 2.1.0.9 is equivalent to the fiber of $|a|_0$ through the map $| - |_0 : A \to \|A\|_0$.

Before passing to the study of pointed connected types we make a remark on proving mere propositions about connected types and about higher inductive types.

2.1.0.13 Remark (Proving mere propositions)**.** Suppose we want to prove a *mere* proposition $P : A \to \mathcal{U}$ over some type A. If the type A happens to be connected then it suffices to prove the proposition just for some $a : A$. This is because any other $b : A$ is merely equal to $a : A$, so by induction on truncation we can prove the proposition $P(b)$ by assuming that b is equal to a. This means that to prove a proposition about a type A it is enough to show it for an inhabitant of each connected component of A.

In the case of higher inductive types this means that to prove a mere proposition about a higher inductive type it suffices to prove it only for the point constructors. This can be justified by induction on the type: The higher inductive steps of the induction will be automatic, since they will ask us to show that the transport of a proof that we already gave (for example, a proof for a point constructor) is equal to another proof that we already gave, but since we are proving a mere proposition this will be immediate.

2.1.1 Pointed connected types

A pointed type is a type together with an inhabitant of that type. When working with pointed types it is useful to think about connected types as types together with a map that proves that any inhabitant of the type is merely equal to the distinguished point.

2.1.1.1 Definition (Pointed and pointed connected types)**.** We define the universe of pointed types as $\mathcal{U}_\bullet :\equiv (A : \mathcal{U}) \times A$. We will usually denote pointed types as (A, a) or just by A if the pointing is implicit. The universe of pointed connected types is defined by:

$$\mathcal{U}_{\bullet c} :\equiv (A : \mathcal{U}) \times (a : A) \times ((b : A) \to \|a = b\|_{-1}).$$

We will also write (A, a) for a pointed connected type, leaving the fact that the type is connected implicit. This is not harmful: Being connected is a mere proposition and thus a particular proof of this fact is no different than any other proof.

2.1.1.2 Definition (Pointed maps)**.** For two pointed types (A, a) and (B, b) define the type of pointed maps by: $A \to_\bullet B :\equiv (f : A \to B) \times (f(a) = b)$.

Once we are working in $\mathcal{U}_\bullet$ the fiber of a map is defined without having to explicitly give a point in the codomain of the map. Moreover, when we have a map with a pointed connected type as its codomain its fiber is essentially unique in the following sense (one can compare this fact with Remark 1.1.0.8).

2.1.1.3 Proposition (Uniqueness of the fiber)**.** *Let A be any type and let (B, b) be a pointed connected type. Given a map $f : A \to B$, for any $b' : B$ its fiber is merely equal to the fiber of $b : B$.*

Proof. Just use the fact that we have a map $\mathsf{fib}_f : B \to \mathcal{U}$ that takes a point in B to its fiber. Since any $b' : B$ is merely equal to b, by functoriality of truncation (Remark 2.1.0.2) we are done. ∎

Given a type and an element in it we defined the connected component of this element. Let us show that the connected component of an inhabitant is indeed connected and moreover naturally pointed.

2.1.1.4 Remark. Given $A : \mathcal{U}$ and $a : A$ the type $\mathscr{C}(A, a)$ is by definition $(b : A) \times \|a = b\|$. We can point this type with $(a, |\mathsf{refl}_a|)$. To show that it is connected we use the fact that the first coordinate of every element of type $\mathscr{C}(A, a)$ is merely equal to a by definition, and the fact that any two inhabitants of a mere proposition are equal.

2.1.1.5 Remark (Loop space of connected component)**.** Given a pointed type (A, a) it is immediate to see that that the loop space of (A, a) is (definitionally) equal to the loop space of $\mathscr{C}(A, a)$.

A generalization of this last remark is the following.

2.1.1.6 Remark (Factorization through connected component)**.** Given a pointed connected type (A, a), a pointed type (B, b) and a pointed map $f : A \to_\bullet B$ there is a natural factorization:

$$\begin{array}{ccc} A & \overset{\overline{f}}{\dashrightarrow} & \mathscr{C}(B, b) \\ & \underset{f}{\searrow} & \downarrow i \\ & & B \end{array}$$

Where $i : \mathscr{C}(B, b) \to B$ is the embedding defined in Definition 2.1.0.10 and f is the underlying map of the pointed map $f : A \to_\bullet B$. To construct $\overline{f}$ we only have to notice that the connectedness of A implies that any element in the image of f is merely equal to b.

Moreover, this construction induces an equivalence:

$$\frac{A \to_\bullet B}{A \to_\bullet \mathscr{C}(B, b)}$$

where $\mathscr{C}(B, b)$ is pointed as in Remark 2.1.1.4.

2.1.2 A remark on mere propositions vs. additional data

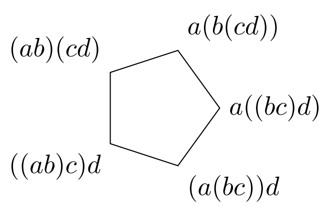
When doing classical mathematics most of the time there's not a clear distinction between (mere) properties and additional data. For example, when we say that a category has all colimits we can be saying that there *exists* some colimit for each diagram, or that we already have a *choice* for each one of these colimits. In this case the distinction is subtle since the axiom of choice makes both statements equivalent. But in a constructive setting having a choice function is a stronger condition.

But we don't have to work with constructive mathematics to have a taste of the difference between properties and additional data. Think about what additional data *forces* us to do. For example, a pointed space is in particular a space. But being pointed is not just a property of the space, it is an additional datum: If we want to define a morphism between pointed spaces we should define it as a morphism between the spaces *that respects the pointing*. In the sense that the map takes the point of the first to the point in the second. So a morphism between pointed spaces isn't just a map in the usual sense.

Now think about being connected. Being connected is a property: Any two proofs that a particular space is connected are equal. Notice that to define what a map between connected types is we don't have to do anything new, just use a standard map between spaces. Since a property is not an additional datum we didn't have to make sure to be coherent when mapping this property.

To give a more algebraic example think about monoids. A monoid is a set with additional data: a distinguished element (the unit), and an operation. But also with some properties: unit law and associativity. What about monoid morphisms? They must respect the unit and they must respect the operation. But we don't have to explicitly state anything regarding associativity or the unit law.

In category theory a monoid object is an object together with some morphisms (unit and the operation) *such that* some diagrams commute (associativity and unit law). In the 1-categorical setting a diagram can commute in at most one way: Either it commutes or it doesn't, and thus in the 1-categorical setting associativity is a mere property of the operation, and so is the unit law. This is not the case in a higher categorical setting: The possibilities for a diagram to commute are now a space that might have many different connected components. So, to prove that a that a triangle commutes one must make a choice regarding which 2-cell one is using to make it commute. Here just requiring associativity of the monoid might not be enough: Consider the classical situation in which we have a so-called *associator*, that for each triple of objects a, b, c gives an isomorphism between the two possible ways in which we can multiply them: $\alpha_{a,b,c}$: $(a(bc)) \xrightarrow{\sim} ((ab)c)$. The associator $\alpha_{a,b,c}$ can be regarded as a 1-cell. But using these 1-cells we can construct the following diagram:



This shows that the α's give us at least two isomorphisms between $(a(bc))$ and $((ab)c)$. So if we want the associator to be coherent we need to have a 2-cell for each triple a, b, c making the pentagon commute. But again these 2-cells would need to be coherent. There is an analogous discussion for the unit law. In the case of monoidal categories (which can be regarded as 2-categories with just one object, in the same way that a monoid is a 1-category with just one object), a well known theorem by Mac Lane says that it is sufficient to make the associator and the unit coherent in order to achieve full (higher) coherence. A related result states that every 2-category is equivalent to a strict 2-category. For a formal exposition of these subjects see for example [Lei98]. But these results do not generalize to higher dimensions. This is a typical complication that arises when passing from a yes-or-no situation (mere propositions) to a space-of-choices one (additional data): Now our morphisms must respect this new additional data, and there can be highly non-trivial interaction between this data. Moreover the compatibility between higher cells is often very complicated even to state. These are known as coherence problems.

2.2 Sequential diagrams

Let us first remember the categorical definition of compact object. In full generality, a compact object in a category is an object K such that for every small filtered diagram $\{A_i\}_I$ the natural map $\mathrm{colim}\,[K, A_i] \to [K, \mathrm{colim}\ A_i]$ is an isomorphism. To define this map one uses the universal property of $\mathrm{colim}\,[K, A_i]$: The inclusions in the colimit $i_n : A_n \to \mathrm{colim}\ A_i$ induce morphisms $i_{n*} : [K, A_n] \to [K, \mathrm{colim}\ A_i]$ by postcomposition, which form a cocone. From this we get the natural morphism by the universal property of $\mathrm{colim}\,[K, A_i]$.

Let us adapt this definition to our case. We will use just one kind of directed diagram, namely the diagrams of the form:

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots$$

We call these diagrams *sequential diagrams*.

The reader is invited to take a look at [Rij12, Section 3.6] where these diagrams are introduced as *directed diagrams* and some of their properties are proved.

2.2.0.1 Notation. Since in this section we will deal with many dependent types over the natural numbers we will usually denote them using subscripts. To be clear, whenever we have a dependent type $T : (n : \mathbb{N}) \to B(n)$ we will denote its evaluation at some particular $n : \mathbb{N}$ as T_n instead of $T(n)$.

2.2.0.2 Notation. Following the classical notation for families of sets, we will usually denote type families $T : \mathbb{N} \to \mathcal{U}$ as $\{T_n\}_{n:\mathbb{N}}$. When there is only one free variable inside the curly braces and its type is implicit we will drop the indexing type and write $\{T_n\}$.

2.2.0.3 Definition (Sequential diagrams)**.** Define the type $\mathsf{seqDiag} : \mathcal{U}$ as the type of maps $A : \mathbb{N} \to \mathcal{U}$ together with maps $\alpha : (n : \mathbb{N}) \to (A_n \to A_{n+1})$.

By abuse of notation we might write $\{A_n\}$ for a directed diagram with types A_n, leaving the maps implicit. When we want the maps to be explicit we will write $\{(A_n, \alpha_n)\}$. Sometimes we will refer to a sequential diagram simply by $D : \mathsf{seqDiag}$ leaving the types and the maps implicit.

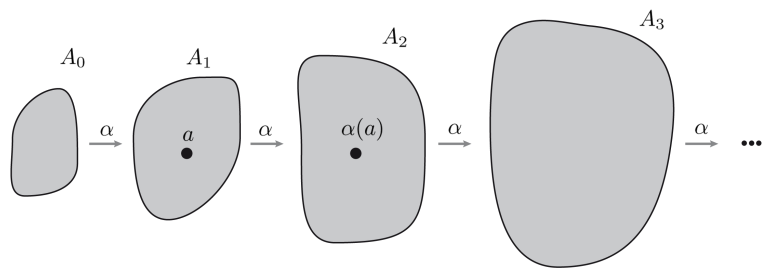


Figure 2.1: A sequential diagram.

Given a sequential diagram we can form its colimit. There are many (equivalent) ways of defining this colimit, but all of them use some kind of higher inductive type. Since sequential diagrams are a particular case of graph-indexed diagrams, the colimit of this kind of diagrams is a special case of the colimit of graph-indexed diagrams (see Definition 2.4.0.2 and Remark 2.4.0.5) but for now we define it by an explicit construction.

2.2.0.4 Definition (Colimit of a sequential diagram)**.** Given a sequential diagram $\{A_n\}$ with maps α_n we define its colimit $\operatorname{colim}_n A_n$ or more succinctly A_∞, as the higher inductive type with constructors:

$$\begin{aligned} i &: (n : \mathbb{N}) \to A_n \to A_\infty \\ \mathsf{glue} &: (n : \mathbb{N}) \to (a : A_n) \to i_n a = i_{n+1}(\alpha_n(a)) \end{aligned}$$

We can picture this as:

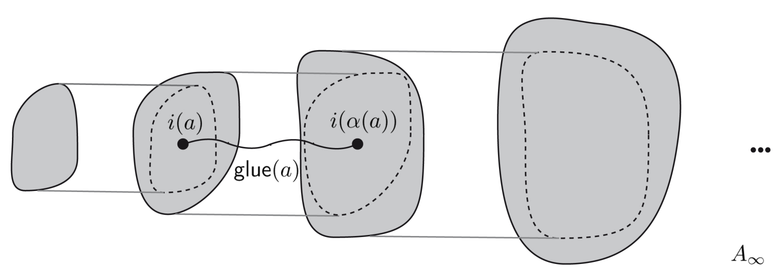


Figure 2.2: The colimit of a sequential diagram.

Now is a good time to give some examples of sequential diagrams and their colimits.

2.2.0.5 Example ($\mathbb{N}$)**.** Take the type family $[\![-]\!] : \mathbb{N} \to \mathcal{U}$ defined inductively as $[\![0]\!] :\equiv \mathbf{0}$ and $[\![n+1]\!] :\equiv [\![n]\!] + \mathbf{1}$. There are obvious maps $[\![n]\!] \to [\![n+1]\!]$ that are just the inclusion in the coproduct: $i : [\![n]\!] \to [\![n+1]\!] :\equiv [\![n]\!] + \mathbf{1}$. It is straightforward but instructive to check that the colimit of this sequential diagram is the type $\mathbb{N}$.

2.2.0.6 Example ($\mathbb{S}^\infty$)**.** Consider the spheres type family $\mathbb{S} : \mathbb{N} \to \mathcal{U}$ defined inductively by iterating the suspension construction. If we want the indices to match the actual dimension of the sphere we should start from the zero dimensional sphere which is equivalent to the type $[\![2]\!]$. Then we can include $[\![2]\!]$ in its suspension by sending one point to the north pole and the other to the south pole. By functoriality of suspension we get a sequential diagram. The colimit of this diagram is usually denoted by $\mathbb{S}^\infty$. It is easy to prove that $\mathbb{S}^\infty$ is contractible just like in the classical setting. For a direct proof of this fact the reader can take a look at the solutions of the exercises of [Uni13][2] or at [Rij12, Theorem 3.6.9].

2.2.0.7 Example (Colimit of contractible types)**.** The colimit of a sequential diagram in which every type is contractible is again contractible. One way to show this is to use the invariance under homotopy proved in [Rij12, Theorem 3.6.4]. For this one must prove that a sequential diagram of contractible types is homotopic to the sequential diagram consisting only on the type **1**, which can be done by noticing that every map between contractible types is an equivalence.

2.2.0.8 Notation. Given a sequential diagram $\{(A_n, \alpha_n)\}$ it is convenient to have some notation for the iterated application of the maps α_n. We will write $\alpha_n^{n+k} : A_n \to A_{n+k}$ for the k-fold application of successive α_i's starting from α_n and ending at α_{n+k-1}.

Maps between sequential diagrams are maps between the underlying types, together with homotopies rendering all the squares commutative. In categorical language this is nothing more than a natural transformation.

2.2.0.9 Definition (Map between sequential diagrams)**.** Given two sequential diagrams $\{(A_n, \alpha_n)\}$ and $\{(B_n, \beta_n)\}$ the space of maps between them is the type:

$$(n : \mathbb{N}) \to (f_n : A_n \to B_n) \times (f_{n+1} \circ \alpha_n = \beta_n \circ f_n).$$

We will usually denote such a map by $\{(f_n, H_n)\}_n$ or simply by $\{f_n\}$ leaving the homotopies implicit.

Diagrammatically we can picture a map between two sequential diagrams $\{(A_n, \alpha_n)\}$ and $\{(B_n, \beta_n)\}$ as follows:

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots \\
{\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\
B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots
\end{array}$$

where the squares are filled by the homotopies H_n.

Just like in the classical setting a natural transformation between two functors induces a morphism between the colimits of the functors.

2.2.0.10 Remark (Induced map between colimits)**.** Given two sequential diagrams $\{(A_n, \alpha_n)\}$ and $\{(B_n, \beta_n)\}$ and a map between them $\{(f_n, H_n)\}$ there is a natural induced map $f_\infty : A_\infty \to$

[2]See `https://github.com/HoTT/book` file `exercise_solutions.tex`.

B_∞. We construct it by induction on A_∞: For a point constructor $i_m(a) : A_\infty$ for some $m : \mathbb{N}$ use $i_m(f_m(a)) : B_m$. For a path constructor $\mathsf{glue}_m(a)$ use $\mathsf{glue}_m(f_m(a)) : i_m(f_m(a)) = i_{m+1}(\beta_m(f_m(a)))$ and the homotopy $H_m : f_{m+1} \circ \alpha_m = \beta_m \circ f_m$ to get:

$$\mathsf{glue}_m(f_m(a)) \cdot i_{m+1}(H_m(a)^{-1}) : i_m(f_m(a)) = i_{m+1}(f_{m+1}(\alpha_m(a))).$$

The following lemma gives us more intuition about sequential diagrams and connected types.

2.2.0.11 Lemma (Sequential colimit of connected types is connected)**.** *The colimit of a sequential diagram $\{A_i\}$ in which every type A_i is connected is again connected.*

Proof. We prove this by induction on the colimit A_∞. Notice that we are proving a mere property and thus it suffices to give the proof only for point constructors as noted in Remark 2.1.0.13. Given $n, m : \mathbb{N}$ and $i_n(a), i_m(b) : A_\infty$ we have to construct a mere equality between them. We know that the order of the natural numbers is computable and thus we can consider the cases in which $n \geq m$ and $n < m$. Both are analogous so suppose that $n \geq m$. Then $a, \alpha_m^n(b) : A_n$, which gives us a mere path between them by the connectedness of A_n. Now we only have to map this mere path to a mere path connecting $i_n(a)$ and $i_m(b)$ in A_∞. This is done using the path constructors of A_∞ that say that A_∞ is a cocone, and the functoriality of truncation. ■

2.2.1 Fibrations over sequential diagrams

We will now define what it means to have a fibration over a sequential diagram. Essentially it is a fibration over each type together with transition maps.

2.2.1.1 Definition (Fibration over a digram)**.** Given a sequential diagram $\{(A_n, \alpha_n)\}$ a fibration over it is a sequence of fibrations $P : (n : \mathbb{N}) \to A_n \to \mathcal{U}$ together with maps:

$$\rho : (n : \mathbb{N}) \to (a : A_n) \to (P_n(a) \to P_{n+1}(\alpha_n(a))).$$

We will write $\{(P_n, \rho_n)\}$ when we want to make the maps explicit.

Given a fibration over a sequential diagram we are naturally led to consider its colimit, which should induce a (standard) fibration over the colimit of the sequential diagram. To give an informal explanation of this construction let us define it first on the point constructors of A_∞. So given an inhabitant of a sequential colimit $a : A_\infty$ we assume that it comes from an $a : A_n$ for some n. Then we can consider the sequential diagram given by the transition maps:

$$P_n(a) \to P_{n+1}(\alpha_n(a)) \to P_{n+2}(\alpha_{n+1} \circ \alpha_n(a)) \to \ldots \to P_{n+k}(\alpha_n^{n+k}(a)) \to \cdots$$

and define the colimit fibration evaluated at a as the colimit of that diagram. So we make the following definition.

2.2.1.2 Definition (Diagram induced by a fibration)**.** If we have a sequential diagram $\{(A_n, \alpha_n)\}$, a fibration $\{(P_n, \rho_n)\}$ over it and for a specific $m : \mathbb{N}$ an inhabitant $a : A_m$, we form a sequential diagram with types $\{P_{n+k}(\alpha_n^{n+k}(a))\}_k$ and maps:

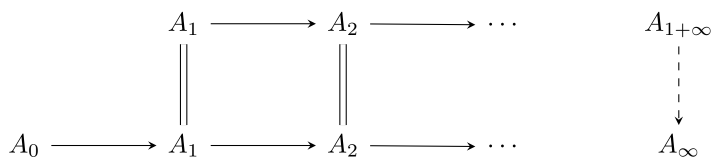
$$\rho_{n+k}(\alpha_n^{n+k}(a)) : P_{n+k}(\alpha_n^{n+k}(a)) \to P_{n+k+1}(\alpha_n^{n+k+1}(a)).$$

Notice that here n is fixed and the indexing variable is k. We will denote this sequential type as E_n^a and thus E can be regarded as a dependent sequential diagram:

$$E : (n : \mathbb{N}) \to (a : A_n) \to \mathsf{seqDiag}\,.$$

We defined the colimit fibration only on the point constructors so it remains to show that this definition extends to the path constructors. Stated otherwise, we need to show that this construction respects the gluing. For this we will use the following lemma.

2.2.1.3 Lemma (Invariance under traslation)**.** *Given a sequential diagram* $\{(A_n, \alpha_n)\}$ *we can consider the (left) shifted diagram* $A[-1] :\equiv \{(A_{1+k}, \alpha_{1+k})\}_k$ *and write* $A_{1+\infty}$ *for its colimit. The identity maps* $\mathsf{Id} : A_{1+k} \to A_{1+k}$ *induce a map* $A_{1+\infty} \to A_\infty$ *by induction on* $A_{1+\infty}$*. In diagrammatic form:*



The induced map is an equivalence. We denote its inverse by $tr^A : A_\infty \simeq A_{1+\infty}$*.*

Proof. We define the maps:

$$\begin{aligned} A_{1+\infty} &\to A_\infty \\ i_n(a) &\mapsto i_{n+1}(a) \\ \mathsf{glue}_n(a) &\mapsto \mathsf{glue}_{n+1}(a) \end{aligned}$$

and:

$$\begin{aligned} A_\infty &\to A_{1+\infty} \\ i_n(a) &\mapsto i_n(\alpha_n(a)) \\ \mathsf{glue}_n(a) &\mapsto \mathsf{glue}_n(\alpha_n(a)) \end{aligned}$$

And we must prove that the compositions:

$$\begin{aligned} A_\infty &\to A_\infty \\ i_n(a) &\mapsto i_{n+1}(\alpha_n(a)) \\ \mathsf{glue}_n(a) &\mapsto \mathsf{glue}_{n+1}(\alpha_n(a)) \end{aligned}$$

and:

$$\begin{aligned} A_{1+\infty} &\to A_{1+\infty} \\ i_n(a) &\mapsto i_{n+1}(\alpha_n(a)) \\ \mathsf{glue}_n(a) &\mapsto \mathsf{glue}_{n+1}(\alpha_n(a)) \end{aligned}$$

are homotopic to the identity $\mathsf{Id} : A_\infty \to A_\infty$ and $\mathsf{Id} : A_{1+\infty} \to A_{1+\infty}$ respectively. Both homotopies are analogous so let us prove only the case of A_∞. By induction on A_∞ we give the equalities $\mathsf{glue}_n(a) : i_n(a) = i_{n+1}(\alpha_n(a))$. And then we must show that this respects the gluing. This reduces to show that we have (filled) squares:

$$\begin{array}{ccc} a & \overset{\mathsf{glue}_n(a)}{=\!=\!=} & \alpha_n(a) \\ \mathsf{glue}_n(a) \Big\| & & \Big\| \mathsf{glue}_{n+1}(\alpha_n(a)) \\ \alpha_n(a) & \underset{\mathsf{glue}_{n+1}(\alpha_n(a))}{=\!=\!=} & \alpha_{n+1}(\alpha_n(a)) \end{array}$$

To fill these squares notice that for composable paths $p : x = y$ and $q : y = z$ there is always a filling for the square:

$$\begin{array}{ccc} x & \overset{p}{=\!=\!=} & y \\ p \Big\| & & \Big\| q \\ y & \underset{q}{=\!=\!=} & z \end{array}$$

constructed by path induction. Using this construction we conclude the proof.

We formalized the above argument and others definitions in this section in `cubicaltt`[3]. ■

With this last fact we can define a (standard) fibration over the colimit from a fibration over the sequential diagram. To do this notice that, using the notation introduced in Definition 2.2.1.2 the above lemma proves $tr^{E_n^a} : \mathrm{colim}\ E_n^a \simeq \mathrm{colim}\ E_n^a[-1]$. But there is also an obvious equivalence $e_n^a : E_n^a[-1] \simeq E_{n+1}^{\alpha_n(a)}$: Just unroll the definition of E in both cases and use the commutativity of addition:

$$\begin{aligned} E_{n+1}^{\alpha_n(a)} &\equiv (m \mapsto P_{n+1+m}(\alpha_{n+1}^{n+1+m}(\alpha_n(a))), \\ &\qquad m \mapsto \rho_{n+1+m}(\alpha_{n+1}^{n+1+m}(\alpha_n(a)))) \\ E_n^a[-1] &\equiv (m \mapsto P_{n+m+1}(\alpha_n^{n+m+1}(a)), \\ &\qquad m \mapsto \rho_{n+m+1}(\alpha_n^{n+m+1}(a))) \end{aligned}$$

And of course equivalent diagrams induce equivalent colimits: $\mathrm{colim}\ e_n^a : \mathrm{colim}\ E_n^a[-1] \simeq \mathrm{colim}\ E_{n+1}^{\alpha_n(a)}$.

2.2.1.4 Definition (Colimit fibration)**.** Given a sequential diagram $\{A_n\}$ and a fibration $\{P_n\}$ over it we define the fibration $P_\infty : A_\infty \to \mathcal{U}$ by induction on A_∞:

$$\begin{aligned} P_\infty : A_\infty &\to \mathcal{U} \\ i_n(a) &\mapsto \mathrm{colim}\ E_n^a \\ \mathsf{glue}_n(a) &\mapsto tr^{E_n^a} \cdot \mathrm{colim}\ e_n^a \end{aligned}$$

Now that we can define fibrations over sequential colimits as colimits of fibrations we wonder if we can describe the total space of this colimit fibration as a colimit of each total space. To make sense out of this we must construct a sequential diagram out of the total spaces of the fibrations P_n.

[3] See `https://github.com/LuisScoccola/cubicaltt.git`, file `seqcolim.ctt`, function `invcolim`.

2.2.1.5 Definition (Fibration out of the total spaces)**.** For a sequential diagram $\{(A_n, \alpha_n)\}$ and a fibration $\{(P_n, \rho_n)\}$ we can form a sequential diagram that has as types the total spaces $\Sigma P_n :\equiv (a : A_n) \times P_n(a)$ of the fibrations P_n. For the maps between these types take:

$$\begin{aligned} \Sigma P_n &\to \Sigma P_{n+1} \\ (a, p) &\mapsto (\alpha_n(a), \rho_n(a)(p)) \end{aligned}$$

2.2.1.6 Remark. One can prove that the above construction gives, for every sequential diagram $\{A_i\}$, an equivalence between fibrations of sequential diagrams over $\{A_i\}$ and sequential diagrams $\{B_i\}$ together with a sequential diagram map from $\{B_i\}$ to $\{A_i\}$. This follows from the equivalence between representations and fibrations stated at the beginning of this chapter.

The following conjecture can be regarded as a flattening lemma for sequential colimits in the same way that [Uni13, Lemma 6.12.2] is a flattening lemma for coequalizers. The author learned this idea from Rijke.

2.2.1.7 Conjecture (Commutativity of sums and colimits)**.** Given a sequential diagram $\{A_n\}$ and a fibration $\{P_n\}$ over it we can characterize the total space of P_∞ as:

$$\operatorname*{colim}_n \Sigma P_n \simeq \Sigma P_\infty$$

The equivalence should be given by the map:

$$\begin{aligned} \text{colim } \Sigma P_n &\to \Sigma P_\infty \\ i_n(a, d) &\mapsto (i_n(a), i_n(d)) \\ \mathsf{glue}_n(a, d) &\mapsto (\mathsf{glue}_n(a), \mathsf{glue}_n(d)) \end{aligned}$$

where $\mathsf{glue}_n(a, d) : i_n(a, d) = i_{n+1}(\alpha_n(a), \rho_n(d))$.

We will now use this conjecture to describe the identity types of a sequential colimit. To do this consider the Yoneda fibration.

2.2.1.8 Definition (Yoneda fibration)**.** Given a pointed type (T, t) define the Yoneda fibration to as:

$$\begin{aligned} \mathcal{Y}_t : T &\to \mathcal{U} \\ s &\to t = s \end{aligned}$$

Notice that $\mathcal{Y}_t(t)$ is by definition the loop space of (T, t). For a sequential diagram this fibrations yield a fibration over the whole diagram in a very natural way:

2.2.1.9 Definition (Colimit Yoneda fibration)**.** Given a sequential diagram $\{(A_n, \alpha_n)\}$ and an inhabitant $a : A_m$ for a fixed $m : \mathbb{N}$ we form a fibration over the shifted diagram $\{A_{m+k}\}_k$ using the Yoneda fibration in the following way. For the fibrations in each degree k take:

$$\begin{aligned} P_k :\equiv \mathcal{Y}_{\alpha_m^{m+k}(a)} : A_{m+k} &\to \mathcal{U} \\ b &\mapsto \alpha_m^{m+k}(a) = b \end{aligned}$$

For the transition maps take:

$$\begin{aligned} \rho_k : (b : A_{m+k}) \to P_k(b) \to P_{k+1}(\alpha_{m+k}(b)) \\ b \mapsto (p \mapsto \alpha_{m+k}(p)) \end{aligned}$$

Call the colimit of this fibration $\mathcal{Y}_a^\infty : A_\infty \to \mathcal{U}$. Notice that strictly speaking this construction yields a fibration over the shifted colimit $A_{m+\infty}$ but by Lemma 2.2.1.3 we can transport this fibration to a fibration over A_∞.

Now, assume given an $a : A_m$. Then we can consider the pointed type $(A_\infty, i_m(a))$. By abuse of notation we write (A_∞, a) for this type. Having a pointed type we can consider its Yoneda fibration $\mathcal{Y}_a : A_\infty \to \mathcal{U}$. The question is whether we can relate this standard Yoneda fibration with the colimit Yoneda fibration defined in Definition 2.2.1.9. Using Conjecture 2.2.1.7 we can do this with the following argument due to Rijke, which serves as a characterization of the identity types of a sequential colimit.

2.2.1.10 Conjecture (Identity types of sequential colimits)**.** For any sequential diagram $\{A_n\}$ together with an inhabitant $a : A_m$ for some $m : \mathbb{N}$ there is an equivalence of fibrations:

$$\begin{array}{ccc} \Sigma\mathcal{Y}_a^\infty & \xrightarrow{\simeq} & \Sigma\mathcal{Y}_a \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & A_\infty & \end{array}$$

Idea. By [Rij12, Corollary 2.4.20] it suffices to show that we have a fiberwise map $\mathcal{Y}_a^\infty \to \mathcal{Y}_a$ and that the total space of $\mathcal{Y}_a^\infty$ is contractible. The fiberwise map definition should be straightforward by induction on sigma types and the colimit $\operatorname{colim}_n \mathcal{Y}_a^n$, getting us a map:

$$\Sigma \operatorname{colim}_n \mathcal{Y}_a^n \to \Sigma\mathcal{Y}_a$$

To show that the total space of $\mathcal{Y}_a^\infty$ is contractible we can use Conjecture 2.2.1.7 to get:

$$\Sigma\mathcal{Y}_a^\infty \equiv \Sigma \operatorname{colim}_k \mathcal{Y}_{\alpha_m^{m+k}(a)} \simeq \operatorname{colim}_k \Sigma\mathcal{Y}_{\alpha_m^{m+k}(a)}$$

Notice that inside the colimit in the RHS we have a total space of a Yoneda fibration which we know is contractible by [Rij12, Lemma 2.3.16]. Thus we are taking a colimit of contractible types which is contractible by Example 2.2.0.7. □

2.2.1.11 Remark. The previous result implies the following equivalence. Assume given a sequential diagram $\{A_i\}$ and two inhabitants $a : A_n$ and $b : A_{n+k}$. Then there is an equivalence:

$$(i_n(a) =_{A_\infty} i_{n+k}(b)) \simeq \operatorname{colim}_l(\alpha_n^{n+k+l}(a) =_{A_{n+k+l}} \alpha_{n+k}^{n+k+l}(b)).$$

which characterizes identity types of sequential colimits.

We will see some consequences of this theorem later but as an example let us prove a proposition about sequential colimits of n-types.

2.2.1.12 Proposition† (Preservation of truncatedness)**.** *If we have a sequential diagram such that each of its types is n-truncated then its colimit is also n-truncated.*

Proof. The proof goes by induction on the truncatedness n. For $n = -2$ all types are contractible and thus the colimit is contractible by Example 2.2.0.7. For the inductive case assume given a sequential diagram $\{A_i\}$ of n types. Observe that by Remark 2.2.1.11 the identity types *of the point constructors* of a sequential colimit are sequential colimits of identity types. By hypothesis each of these identity types is $(n-1)$-truncated since each A_i is n-truncated. And thus, by inductive hypothesis, the colimit of the identity types *of the point constructors* is $(n-1)$-truncated. Since being n-truncated is a mere property, we can use Remark 2.1.0.13 to deduce that all the identity types of A_∞ are $(n-1)$-truncates and thus A_∞ is n-truncated. ∎

We also state the following related conjecture.

2.2.1.13 Conjecture (Conmutativity of truncation and sequential colimits)**.** Given a sequential diagram $\{A_n\}$ and a fixed $k : \mathbb{N}$ we can use the functoriality of truncation to construct a diagram $\{\|A_n\|_k\}$. By induction on the colimit of this diagram we can construct a map $\mathrm{colim}_n \|A_n\|_k \to \|A_\infty\|_k$, and this map is an equivalence.

2.3 Compact types

Just like in the category theoretical setting, for every type T and every sequential diagram $D :\equiv \{A_n\}$ we have a canonical map $cl_D : \mathrm{colim}\,[K, A_n] \to [K, \mathrm{colim}\ A_n] \equiv [K \to A_\infty]$. This map can be defined by induction on $\mathrm{colim}\,(K \to A_n)$ in the following way. Since we are dealing with two sequential colimits we will use an overline for the constructors of $\mathrm{colim}\,(K \to A_n)$. By induction on $\mathrm{colim}\,(K \to A_n)$ we define:

$$\overline{i_n}(f) \mapsto i_n \circ f$$
$$\overline{\mathsf{glue}}_n(f) \mapsto \mathsf{glue}_n \circ f$$

2.3.0.1 Definition. Given a type $T : \mathcal{U}$ and a sequential diagram $D :\equiv \{A_n\}$ we call the defined map $cl_D : \mathrm{colim}\,(T \to A_n) \to (T \to A_\infty)$.

With the discussion of Remark 1.4.1.5 in mind we now define in HoTT what it means to be a *sequentially compact* type (also called ω-compact type or simply compact type).

2.3.0.2 Definition (Sequentially compact type)**.** We say that a type K is compact if the canonical map cl_D is an equivalence for every sequential diagram D. We form the type:

$$\mathsf{isComp}(K) :\equiv (D : \mathsf{seqDiag}) \to \mathsf{isEquiv}(cl_D).$$

Since we do not consider any other kind of directed diagram we call these types simply compact types. A very nice property of this definition is that being compact is *not* an additional datum: A type can be compact in at most one way. We state this as a lemma.

2.3.0.3 Lemma (Compactness is proposition)**.** *For every type $A : \mathcal{U}$ the type $\mathsf{isComp}(A)$ is a mere proposition.*

Proof. Since being an equivalence is a mere proposition and dependent products of mere propositions are again mere propositions (just like in the case of connected types) we deduce the desired result. ∎

Let us give some examples of compact types. Observe that the non-trivial ones depend on yet unproven results discussed in the next section.

2.3.0.4 Example (**0** is compact)**.** The empty type **0** is compact. One way to prove this is to show that for any sequential diagram $\{A_n\}$ both $\mathrm{colim}\,(\mathbf{0} \to A_n)$ and $\mathbf{0} \to A_\infty$ are contractible since any map between contractible types is an equivalence. The second type is obviously contractible since **0** is initial. For the first one use Example 2.2.0.7 (a sequential colimit of contractible types is contractible) and again the initiality of **0**.

2.3.0.5 Example (**1** is compact)**.** This follows at once from the fact that for any type A there is a natural equivalence $(\mathbf{1} \to A) \simeq A$.

Using this two examples one can show inductively that the types $[\![n]\!]$ are compact (recall that these types were defined in Example 2.2.0.5).

2.3.0.6 Example† (Finite sets are compact)**.** The type **0** is compact by the above example. For the inductive case it suffices to show that the disjoint union of compact types is again compact which is a consequence of Proposition† 2.4.0.6 and Lemma 2.4.0.9 which will be discussed later.

We can generalize this last example a little bit using the fact that compactness is a mere proposition.

2.3.0.7 Example† (Sets with finite cardinality are compact)**.** A type T is a set *with finite cardinality* provided it is merely equal to a type of the form $[\![n]\!]$ for some $n : \mathbb{N}$. Since compactness is a mere proposition, to prove the compactness of T we can assume that T is actually equal to $[\![n]\!]$ which is compact by the previous example.

2.3.0.8 Example† (Spheres are compact)**.** Spheres are compact types. The empty space is compact, and the spheres are constructed by suspending a finite number of times. Since the suspension of a type is a particular case of a pushout we get the desired result by Corollary† 2.4.0.11.

There are many other spaces that can be constructed using pushouts of compact types. For example cell complexes (with a finite number of cells) as presented in [Uni13, Section 6.6]. We now make a digression to discuss the commutativity of finite limits and filtered colimits and the results on which the above examples depend.

2.4 Finite limits and filtered colimits

A very important property of the category of sets is that finite limits commute with filtered colimits. This implies that many other categories enjoy this property, for example presheaf categories and even more, Grothendieck topoi (see for example [Bor94, Proposition 3.4.5, Section 3.4]).

To define in which sense finite limits commute with filtered colimits consider a functor $F : P \times J \to \mathcal{C}$ such that P is a finite category and J is a small filtered category. Assume that all the limits and colimits that we will want to take exist in $\mathcal{C}$ (this is not a restriction in the case of Set). Then we can form the following diagram:

$$\begin{array}{ccccc} F(p,j) & \longleftarrow & \lim_p F(p,j) & \longrightarrow & \operatorname{colim}_j \lim_p F(p,j) \\ \downarrow & & \vdots & & \vdots\, cl \\ \operatorname{colim}_j F(p,j) & \longleftarrow & \lim_p \operatorname{colim}_j F(p,j) & = & \lim_p \operatorname{colim}_j F(p,j) \end{array}$$

The solid arrows are the universal cones of the limits and colimits. The vertical arrow in the middle of the diagram exists by the universal property of $\lim_p \operatorname{colim}_j F(p,j)$. Now this arrow gives us a cone $\lim_p F(p,j) \to \lim_p \operatorname{colim}_j F(p,j)$ that together with the universal property of $\operatorname{colim}_j \lim_p F(p,j)$ induces the arrow cl.

We say that finite limits commute with filtered colimits in the category $\mathcal{C}$ if the map cl is an isomorphism for every choice of P, J and F. This is true in the category of sets and a proof can be found in [ML98, Theorem 1, Section IX 2]. As one can expect the argument uses the explicit characterization of limits and filtered colimits in the category of sets, thus it might not be clear why this statement should hold in HoTT. One can argue that since HoTT is conjectured to be the internal language of ∞-topoi it should be the case that finite limits commute with filtered colimits. A reference for this result in the case of ∞-categories is [Lur09, Proposition 5.3.3.3]. The basic theory of filtered colimits and compact objects in the ∞-categorical setting is done in [Lur09, Sections 5.3.3 and 5.3.4]. Notice that in the reference the *universe of spaces* is denoted by a calligraphic $\mathcal{S}$ and it is by definition the simplicial nerve of the category of Kan complexes.

To study an analogous problem HoTT we will restrict our attention to *finite graph-indexed* limits, and sequential colimits. Although we won't use graph-indexed diagrams explicitly, we state the definition to be able to state some conjectures regarding limits and colimits. A very nice article that studies limits in HoTT is [AKL15], and the following definitions are taken from there.

2.4.0.1 Definition (Graph)**.** A graph is a type of vertices:

$$\mathcal{G}_0 : \mathcal{U}$$

together with a dependent type of arrows:

$$\mathcal{G}_1 : (i, j : \mathcal{G}_0) \to \mathcal{U}.$$

We usually denote a graph by $\mathcal{G}$.

2.4.0.2 Definition (Graph-indexed diagram)**.** Given a graph $\mathcal{G}$ a $\mathcal{G}$*-indexed* diagram (or diagram over $\mathcal{G}$) is a representation of the graph $\mathcal{G}$ by types. Concretely a $\mathcal{G}$-indexed diagram D is a map representing the vertices:

$$D_0 : \mathcal{G}_0 \to \mathcal{U}$$

together with a dependent map representing the edges:

$$D_1 : (i, j : \mathcal{G}_0) \to \mathcal{G}_1(i, j) \to (D_0(i) \to D_0(j)).$$

Now that we have diagrams we can define cones and cocones.

2.4.0.3 Definition (Cone of a diagram)**.** If D is a $\mathcal{G}$-indexed diagram, the type of cones of D is the type of types $C : \mathcal{U}$ together with cone maps:

$$c : (i : \mathcal{G}_0) \to C \to D_0(i)$$

and homotopies making the triangles commute:

$$h : (i, j : \mathcal{G}_0) \to (g : \mathcal{G}_1(i, j)) \to D_1(i, j)(g) \circ c(i) = c(j).$$

The type C is called the *vertex* of the cone.

The type of cones of a given $\mathcal{G}$-indexed diagram with a fixed type C as vertex of the cone is denoted by $\mathsf{Cone}(C; D)$ and its inhabitants by (c, h) where c and h have the types of the definition above. The definition of cocone $\mathsf{coCone}(C; D)$ is dual. Now we can define the type of limits of a diagram.

2.4.0.4 Definition (Limit of a diagram)**.** Given a $\mathcal{G}$-indexed diagram D, a cone $(c, h) : \mathsf{Cone}(C; D)$ is a limit for D if for every type $A : \mathcal{U}$ the map $(c \circ -) : (A \to C) \to \mathsf{Cone}(A; D)$ is an equivalence.

The definition of colimit is dual.

2.4.0.5 Remark (Colimits)**.** In Definition 2.2.0.4 we defined the colimit of a sequential diagram using an explicit higher inductive type. One can prove that this type satisfies the universal property of the colimit of the diagram by induction on this higher inductive type. Recent work by Boulier, Quirin, Tabareau and Rijke includes the necessary definitions for the formalization of this fact in Coq (see [BQTR16]).

In [AKL15] it is proved that the limit of a graph-indexed diagram exists and that it is unique. Then it is shown that many 1-categorical arguments apply for graph-indexed diagrams. For example [AKL15, Example 3.2.11] shows that any graph-indexed limit can be constructed as an equalizer of two arrows between two products. A very similar argument should prove that every finite graph-indexed limit can be constructed using pullbacks and the final object. This implies that to show that sequential colimits commute with finite graph-indexed limits we can show the commutativity between pullbacks and sequential colimits plus the fact that a sequential colimit of contractible types is contractible. Having said that, it seems pretty non-trivial to prove the commutativity of pullbacks and sequential colimits. For example, even to prove the commutativity of binary products and sequential colimits is not immediate. We can deduce this fact from the commutativity of sequential colimits and sigma types of Conjecture 2.2.1.7.

2.4.0.6 Proposition† (Binary products commute with sequential colimits)**.** *For two sequential diagrams $\{(A_i, \alpha_i)\}$ and $\{(B_i, \beta_i)\}$ we have $\operatorname{colim}_i(A_i \times B_i) \simeq A_\infty \times B_\infty$, where the map is defined by induction on $\operatorname{colim}_i(A_i \times B_i)$.*

Proof. Consider the fibration over $\{A_i\}$ given by the degree-wise (constant) fibrations:

$$\begin{aligned} A_i &\to \mathcal{U} \\ a &\mapsto B_i \end{aligned}$$

and with transition maps given by $\beta_i : B_i \to B_{i+1}$. It is clear that the degree-wise total spaces are $A_i \times B_i$. It is also clear that the colimit fibration over A_∞ is given by:

$$\begin{aligned} A_\infty &\to \mathcal{U} \\ a &\mapsto B_\infty \end{aligned}$$

And thus the total space of the colimit fibration is $A_\infty \times B_\infty$. The result then follows by applying Conjecture 2.2.1.7. ∎

The following discussion is about pullbacks, but a very similar discussion applies in the case of equalizers. Suppose we have three directed diagrams $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ that form a cospan. This situation can also be thought as a directed diagram of cospans $\{A_n \to C_n \leftarrow B_n\}$. We can depict the situation as follows:

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots \\
{\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\
C_0 & \xrightarrow{\gamma_0} & C_1 & \xrightarrow{\gamma_1} & C_2 & \xrightarrow{\gamma_2} & \cdots \\
{\scriptstyle g_0}\uparrow & & {\scriptstyle g_1}\uparrow & & {\scriptstyle g_2}\uparrow & & \\
B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots
\end{array}$$

Where the commutativity of the diagram is given by homotopies:

$$\begin{aligned} F_n &: f_{n+1} \circ \alpha_n = \gamma_n \circ f_n, \\ G_n &: g_{n+1} \circ \beta_n = \gamma_n \circ g_n \end{aligned}$$

for each n. Given this context we can take the colimits of the directed diagrams and get A_∞, B_∞ and C_∞. By Remark 2.2.0.10 we get a cospan: $A_\infty \to B_\infty \leftarrow C_\infty$. Call the limit of this cospan $\mathcal{L}$.

On the other hand we can take the pullbacks of each cospan, call this pullbacks L_n one for each $n : \mathbb{N}$. These types form a directed diagram in a natural way. The diagram can be constructed using the characterization of pullbacks of [AKL15, Example 3.2.10] that says that the limit of a pullback diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ is equivalent to $(x : X) \times (z : Z) \times (f(x) = g(z))$. Then we map:

$$\begin{aligned} L_n &\to L_{n+1} \\ (a, b, p) &\mapsto (\alpha_n(a), \beta_n(b), F(a) \cdot \gamma_n(p) \cdot G(b)^{-1}) \end{aligned}$$

where $p : f_n(a) = g_n(b)$. Taking the colimit of this diagram we get L_∞, and by induction on L_∞ we can construct a map $l : L_\infty \to \mathcal{L}$. Notice that this construction is just an instance of the general argument at the beginning of this section and that a proof of Remark 2.4.0.5 would give us a completely formal way to construct this map.

The commutativity of pullbacks and sequential colimits can then be stated as:

2.4.0.7 Conjecture (Pullbacks commute with sequential colimits)**.** In the above context the map $l : L_\infty \to \mathcal{L}$ is an equivalence.

The proof of this fact might be pretty involved. It seems that the characterization of identity types of sequential colimits (Conjecture 2.2.1.10) is a key ingredient in the proof. Moreover, when trying to prove this one necessarily has study equalities in pullbacks: The limit $\mathcal{L}$ is the pullback of a diagram of sequential colimits, which have point constructors *and* path constructors. And as we already stated, an inhabitant of the pullback of a given diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ is a pair of points $x : X$ and $z : Z$ together with a path $p : f(x) = g(z)$. Then a *path* between two such inhabitants (x, z, p) and (x', z', p') is given by two paths $r : x = x'$ and $s : z = z'$ and a square in Y with sides $p, f(r), p', g(s)$. Then the result should follow from a characterization analogous to the one of Remark 2.2.1.11 but for *squares* instead of *paths*.

To be able to apply this conjectures to get results about compact types we need the following lemma. Its proof should be analogous to the 1-categorical one[4].

2.4.0.8 Lemma (Right exatness of representable map)**.** *Given a type $A : \mathcal{U}$ and a graph $\mathcal{G}$, the map:*

$$\begin{aligned} (- \to A) : \mathcal{U} &\to \mathcal{U} \\ T &\mapsto (T \to A) \end{aligned}$$

sends $\mathcal{G}$-shaped colimits to $\mathcal{G}$-shaped limits.

Proof. Given a $\mathcal{G}$-shaped diagram D call its colimit C. The universal property of C says that for every type K there is an equivalence (given by composition with the cone of C):

$$(C \to K) \simeq \mathsf{coCone}(K; D).$$

On the other hand we have to prove that for every type L there is an equivalence (given by composition with the cone of $(C \to A)$):

$$(L \to (C \to A)) \simeq \mathsf{Cone}(L, [D, A])$$

where $[D, A]$ is the $\mathcal{G}$-shaped diagram obtained by applying $(- \to A)$ on each object an each map of the diagram D. Notice that by the universal property of C this is equivalent to proving:

$$(L \to \mathsf{coCone}(A; D)) \simeq \mathsf{Cone}(L, [D, A]).$$

This last equivalence follows from the fact that both types are equivalent to having, for each $l : L$, a family of maps $g_i : D_i \to A$ together with homotopies $H_{i,j} : g_i \circ D(i, j) = g_j$, that make the g's compatible with the diagram. To do this formally one must also keep track of the fact that the equivalences must be given by composition with the cones. We formalized this argument in Coq[5]. ■

Then we can translate the classical 1-categorical argument used to prove that a finite colimit of compact objects is again compact:

2.4.0.9 Lemma. *Assume given a graph-indexed diagram D such that every type in the diagram is compact. If sequential colimits commute with limits of the shape of the diagram D, then the colimit of D is compact.*

[4]One can prove the dual result that states that the map $(A \to -)$ is left exact.

[5]See `https://github.com/LuisScoccola/limandcolim.git`, file `Colim2Lim.v`, function `homisexact`.

Proof. Suppose that we have a diagram D_i and a sequential diagram $\{F_n\}$ with colimit F_∞. Then there are canonical maps:

$$\begin{aligned} \operatorname{colim}_n[\operatorname{colim}_i D_i, F_n] &\to \operatorname{colim}_n \lim_i[D_i, F_n] \\ &\to \lim_i \operatorname{colim}_n[D_i, F_n] \\ &\to \lim_i[D_i, \operatorname{colim}_n F_n] \\ &\to [\operatorname{colim}_i D_i, \operatorname{colim}_n F_n] \\ &\equiv [\operatorname{colim}_i D, F_\infty] \end{aligned}$$

and all of them are equivalences. To justify this we apply Lemma 2.4.0.8 to the first and the last map. For the second map we use the commutativity of the limit and sequential colimits assumed by hypothesis. The third equivalence holds by the compactness of the types of the diagram. To conclude the proof one must observe that this composition is equal to the canonical map $\operatorname{colim}_n[P, F_n] \to [P, F_\infty]$. This is not conceptually complicated, but is its complicated to do formally since there are many identifications involved. ■

Using this last result and the conjectured commutativity of pushouts and sequential colimits we have:

2.4.0.10 Corollary†. *A finite graph-indexed colimit of compact types is compact.*

And in particular:

2.4.0.11 Corollary†. *A pushout of compact types is compact.*

2.5 Classifying spaces

Given a space F we want to define a new type $\mathscr{C}F$ such that for every other space A, maps $A \to \mathscr{C}F$ classify fibrations $B \to A$ with fiber equivalent to F. To make sense out of this potential definition we must make some assumptions about A. First, since we mentioned the *fiber* of a map with codomain A we need a point in A to be able to define this fiber. So suppose given an A and an inhabitant $a : A$. If we know that the fiber (of a) through a map $B \to A$ is equivalent to F we can use Proposition 2.1.1.3 to deduce that this is (merely) true for every other $a' : A$ by requiring A to be connected.

This leads us to define the classifying space of F fibrations as[6]:

2.5.0.1 Definition (Classifying space of F-fibrations)**.** For a fixed space F we define the space $\mathscr{C}F :\equiv (A : \mathcal{U}) \times \|A = F\|$. We call it the classifying space of F-fibrations.

Notice that this is just the connected component of F in the universe $\mathcal{U}$ in the sense of Definition 2.1.0.9. Having this in mind we can use Remark 2.1.1.4 to deduce that $\mathscr{C}F$ is pointed and connected.

Next we state in which sense the defined space is the classifying space of F-fibrations (the reader can compare with Theorem 1.4.1.2).

[6]The author learned these ideas from [Shu15b].

2.5.0.2 Theorem (Classifying space of F-fibrations)**.** *Fix a space F and a pointed connected space (T, t). There is an equivalence between pointed maps $p : T \to_\bullet \mathscr{C}F$ and spaces E together with a map $E \to T$ with fiber equivalent to F. Formally this last type is $(E : \mathcal{U}) \times (f : E \to T) \times (f^{-1}(t) = F)$.*

We can restate this as:

$$\frac{(E : \mathcal{U}) \times (f : E \to T) \times (f^{-1}(t) = F)}{T \to_\bullet \mathscr{C}F}$$

Proof. Notice the similarity with [Uni13, Theorem 4.8.3] that essentially states that the universe $\mathcal{U}$ classifies maps. As a matter of fact we can prove the theorem by applying the cited theorem and by noticing just a couple of things. To see that the equivalence of [Uni13, Theorem 4.8.3] "restricts" to the stated equivalence notice that an element of type $(E : \mathcal{U}) \times (f : E \to T) \times (f^{-1}(t) = F)$ is the same that a fibration $P :\equiv f^{-1} : T \to \mathcal{U}$ *such that* $P(t) = F$, since $f^{-1}(t) = F$. And thus we have an equivalence:

$$\frac{(E : \mathcal{U}) \times (f : E \to T) \times (f^{-1}(t) = F)}{T \to_\bullet (\mathcal{U}, F)}$$

But since T is connected we can use the equivalence stated in Remark 2.1.1.6 deduce the desired result. ■

Notice that we have a projection $F : \mathscr{C}F \to \mathcal{U}$ that only remembers the first component, which is just the inclusion defined in Definition 2.1.0.10. Then for a map $p : T \to_\bullet \mathscr{C}F$ that classifies some F-fibration, the composition $F \circ p : T \to \mathcal{U}$ is a standard fibration with fiber F.

From Remark 2.1.1.5 we deduce the following lemma.

2.5.0.3 Lemma. *For any type F the loop space $\Omega\mathscr{C}F$ is (definitionally) equal to $F = F$.*

2.6 The classifying space of spherical fibrations

2.6.0.1 Definition (Classifying spaces of spherical fibrations)**.** Specializing this definition in the case of spheres we define for each n the space $Sp_n :\equiv \mathscr{C}\mathbb{S}^n$.

Notice that these spaces are the $\mathscr{B}H_n$ of Definition 1.6.0.1. As in the classical case these spaces form a directed diagram in a natural way. To make this construction notice that if a type A is merely equal to $\mathbb{S}^n$ then the suspension ΣA is merely equal to $\mathbb{S}^{n+1}$, simply by the functoriality of the truncation map.

2.6.0.2 Definition. Define for each n a map $i_n : Sp_n \to Sp_{n+1}$ mapping $(A, q) \mapsto (\Sigma A, \Sigma q)$. Here Σq is notation for the proof that ΣA is merely equal to $\mathbb{S}_{n+1}$ obtained from the proof q that A is merely equal to $\mathbb{S}_n$.

Recall that we have a natural pointing for the types Sp_n, since they are defined as connected components. Moreover it is clear that the maps i_n preserve this pointing because $(\mathbb{S}^n, q)$ maps to $(\mathbb{S}^{n+1}, \Sigma q)$ and these are exactly the inhabitants used to point Sp_n and Sp_{n+1} respectively. We define the space Sp as the colimit of the diagram.

2.6.0.3 Definition (Classifier of spherical fibrations)**.** Define the *classifier of spherical fibrations* Sp to be the colimit of the sequential diagram constructed above:

$$Sp_{-1} \to Sp_0 \to Sp_1 \to Sp_2 \to \cdots .$$

Notice that the type Sp corresponds to $\mathscr{B}H$ in the same sense that the types Sp_n correspond to $\mathscr{B}H_n$. Notice also that Sp is naturally pointed, since the maps in the diagram are all pointed maps. Moreover from Lemma 2.2.0.11 we deduce that Sp is connected.

Just like in the classical setting, Sp seems to classify spherical fibrations only up to stability: For a pointed type T and an n-dimensional spherical fibration classified by a map to $T \to_\bullet Sp_n$ we get an induced map $T \to_\bullet Sp$ by postcomposition with the inclusion in the colimit j_n : $Sp_n \to Sp$. This discussion is analogous to the one made in Remark 1.6.0.3. Again, to get an equivalence class of spherical fibrations (up to stability) out of a map to Sp we require T to be compact since by definition this means that $\operatorname{colim}_n(T \to Sp_n) \simeq (T \to Sp)$.

2.6.0.4 Proposition† (The loop space of the classifier)**.** *As with any pointed space, we can consider the loop space $\Omega\, Sp$. Because we constructed Sp as the colimit of the Sp_n, we would like to relate the loop spaces with some kind of colimit. For this we use the characterization of Conjecture 2.2.1.10 and the fact that $\Omega\, Sp_n \simeq (\mathbb{S}^n = \mathbb{S}^n)$. Then $\Omega\, Sp$ is equivalent to the colimit of the diagram:*

$$(\mathbb{S}_{-1} = \mathbb{S}_{-1}) \to (\mathbb{S}_0 = \mathbb{S}_0) \to (\mathbb{S}_1 = \mathbb{S}_1) \to (\mathbb{S}_2 = \mathbb{S}_2) \to \dots$$

and the maps are given by:

$$\mathsf{ap}_\Sigma : (\mathbb{S}^n = \mathbb{S}^n) \to (\mathbb{S}^{n+1} = \mathbb{S}^{n+1})$$
$$f \mapsto \Sigma f$$

By [Uni13, Lemmas 8.5.9 and 8.5.10] we know that that the join of two spheres is again a sphere: $\mathbb{S}^n * \mathbb{S}^m = \mathbb{S}^{n+m+1}$. This induces a homotopy associative operation:

$$* : Sp_n \to Sp_m \to Sp_{n+m+1}.$$

There is a unit for this operation, namely $\mathbb{S}^{-1}$. Recalling Remark 1.5.0.9 we might wonder:

2.6.0.5 Question (H-space structure on Sp)**.** Can we extend this operation to Sp to get an H-space structure?

As we proved in Proposition 1.5.0.13 the operation for *orthogonal spherical bundles* already has (homotopy) inverses. This means that when considering orthogonal spherical bundles up to stable fiber homotopy equivalence the join operation is a group operation. This leads us to the question of whether the H-space that we should be able to define on Sp has homotopy inverses:

2.6.0.6 Question. Does the conjectured H-space structure of Sp have homotopy inverses?

This might be pretty non-trivial to answer, for example in the classical case it depended on the orthogonal complement construction for vector bundles (see Lemma 1.3.1.20). Moreover, this question applies to the classical setting as well: In the classical setting we know that orthogonal spherical bundles have inverses (up to stable fiber homotopy equivalence) but does this hold for general spherical fibrations?

2.6.1 Its homotopy groups

Let us conclude this section with a discussion about some aspects of a possible translation of Theorem 1.6.0.4 into HoTT.

2.6.1.1 Proposition† (The fundamental group of Sp). *We have an equivalence $\pi_1(Sp) \simeq \mathbb{Z}_2$.*

Proof. Using Proposition† 2.6.0.4 we need to classify equivalences $\mathbb{S}^n \to \mathbb{S}^n$ up to homotopy: We have to show that the set $\|\mathbb{S}^n \simeq \mathbb{S}^n\|_0$ has exactly two elements and a group structure, and that the suspension maps in the diagram respect this structure. To do this we observe three facts. First that there is an equivalence $\pi_n(\mathbb{S}^n) \simeq \|\mathbb{S}^n \to \mathbb{S}^n\|_0$ and that $\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$. Secondly that composition in $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$ corresponds to multiplication in $\mathbb{Z}$. Assuming these two facts we deduce that $\|\mathbb{S}^n = \mathbb{S}^n\|_0$ with composition is isomorphic to the units of the integers $\mathbb{Z}^\times$ with multiplication. And one should prove also that this last group is isomorphic to $\mathbb{Z}_2$. Thirdly, the suspension map in Proposition† 2.6.0.4 respects this equivalences. Then using Conjecture 2.2.1.13 we deduce that $\pi_1(Sp)$ is equivalent to $\mathbb{Z}_2$.

For the first observation we need to study the action of the fundamental group on the higher homotopy groups of a space, to be able to prove $\pi_n(\mathbb{S}^n) \simeq \|\mathbb{S}^n \to \mathbb{S}^n\|_0$, which we do later. One must keep in mind that this equivalence will respect composition of endomaps of $\mathbb{S}^n$.

For the second observation we must justify why is that composition in $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$ corresponds to multiplication in $\mathbb{Z}$. This can be regarded as the multiplicativity of the degree that we will define later. One way to do this is to use representatives for each class in $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$: It is enough to show that composition of representatives $[k] : \|\mathbb{S}^n \to \mathbb{S}^n\|_0$ for each $k : \mathbb{Z}$ corresponds to multiplication in $\mathbb{Z}$. When $n = 1$ using the proof that $\Omega(\mathbb{S}^1) \simeq \mathbb{Z}$ we can find these representatives:

$$\begin{aligned} [k] : \mathbb{S}^1 &\to \mathbb{S}^1 \\ \mathsf{base} &\mapsto \mathsf{base} \\ \mathsf{loop} &\mapsto \mathsf{loop}^k \end{aligned}$$

Moreover, it is clear that $[k \circ m] = [k] \times [m]$. This representatives should yield representatives in the case of $\mathbb{S}^n$ for $n > 1$ by suspending them: $[k]_n :\equiv \Sigma^{n-1}([k]) : \mathbb{S}^n \to \mathbb{S}^n$. As a matter of fact one can prove by induction on k that $\Sigma([k]_n) : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to the map:

$$\begin{aligned} \mathbb{S}^n &\to \mathbb{S}^n \\ N &\mapsto N \\ S &\mapsto S \\ \mathsf{merid}(s) &\mapsto \mathsf{merid}([k]_{n-1}(s)) \end{aligned}$$

and this last fact makes it clear that $[k \circ m]_n = [k]_n \times [m]_n$ for every n. So the problem reduces to show that the $[k]_n$ are representatives for the type $\pi_n(\mathbb{S}^n)$. The problem is that to use the proof of [Uni13, Theorem 8.6.17] to get representatives for $n \geq 2$ we need representatives for $n = 2$, since this is the base case of the induction in the cited theorem.

But we can also use the proof of $\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ given in [LB13]. There it is shown that the Freudenthal map $\mathbb{S}^n \to \Omega\Sigma\mathbb{S}^n$ induces the equivalence $\pi_n(\mathbb{S}^n) \simeq \pi_{n+1}(\mathbb{S}^{n+1})$ for every $n \geq 1$.

On the other hand, using the adjunction between the suspension and the loop space, this map corresponds to the suspension map $(\mathbb{S}^n \to \mathbb{S}^n) \to (\mathbb{S}^{n+1} \to \mathbb{S}^{n+1})$ given by functoriality of suspension. So under this correspondence we have:

$$\begin{aligned} \pi_n(\mathbb{S}^n) &\simeq \mathbb{Z} \\ [k]_n &\mapsto k \end{aligned}$$

which proves the third statement.

■

We saw that to classify homotopy classes of maps $\mathbb{S}^n \to \mathbb{S}^n$ by its degree using $\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ (proved in [Uni13, Theorem 8.6.17]) we can use the canonical identification $\pi_n(\mathbb{S}^n) \simeq \|\mathbb{S}^n \to \mathbb{S}^n\|_0$ given by the fact that the fundamental group of spheres $\mathbb{S}^n$, $n > 1$ is trivial and thus the action $\pi_1(\mathbb{S}^n) \curvearrowright \pi_n(\mathbb{S}^n)$ is also trivial. We can prove directly the equivalence $\pi_n(\mathbb{S}^n) \simeq \|\mathbb{S}^n \to \mathbb{S}^n\|_0$ which we do in Lemma 2.6.1.4. But let us also study a bit the definitions needed for a more general statement. We want to define the action of the fundamental group of a space on its homotopy groups. The following is an idea on how to do this. Recall that a very important consequence of [Uni13, Lemma 6.5.4] is that the n-fold loop space of a pointed space (B, b) is equivalent to the space of pointed maps $\mathbb{S}^n \to_\bullet B$. Seen this way an element of $\Omega^n(B)$ can be regarded as a pair (f, p) such that $f : \mathbb{S}^n \to B$ and $p : f(\mathsf{base}) = b$.

2.6.1.2 Definition (Action of fundamental group)**.** Given an $l : \Omega(B) \equiv (b = b)$ we can make it act on (f, p) by:

$$l \cdot (f, p) :\equiv (f, p \cdot l)$$

which defines a map $-\cdot- : \Omega(B) \times \Omega^n(B) \to \Omega^n(B)$. This map induces a *right* action $\pi_1(B) \curvearrowright \pi_n(B)$ by functoriality of the 0-truncation and the commutativity of truncation and products (see [Uni13, Theorem 7.3.8]).

It is easy to check that it is a group *right* action. For example –omitting the truncation maps– we have :

$$(l \cdot l') \cdot (f, p) \equiv (f, p \cdot (l \cdot l')) = (f, (p \cdot l) \cdot l') \equiv l' \cdot (f, p \cdot l) \equiv l' \cdot (l \cdot (f, p)).$$

To argue that this is the right definition notice that in the case of the fundamental group acting on itself one can check that the action is given by conjugation since $(f, p) : \mathbb{S}^1 \to_\bullet B$ corresponds to $p \cdot f \cdot p^{-1} : \Omega B$. We say that $\pi_1(B)$ acts trivially on $\pi_n(B)$ if we have a map:

$$(l : \pi(B)) \to (e : \pi_n(B)) \to (l \cdot e = e).$$

The next step is to show that when the fundamental group acts trivially on the n-th homotopy group, we have an equivalence $\|\mathbb{S}^n \to_\bullet B\|_0 \to \|\mathbb{S}^n \to B\|_0$.

2.6.1.3 Proposition. *For a pointed connected space B and an $n : \mathbb{N}$ such that the action $\pi_1(B) \curvearrowright \pi_n(B)$ is trivial we have an equivalence $\pi_n(B) \simeq \|\mathbb{S}^n \to B\|_0$.*

Before proving this result let us prove a weaker statement that is already enough for our purposes:

2.6.1.4 Lemma. *For $n > 1$ there is an equivalence $\|\mathbb{S}^n \to \mathbb{S}^n\|_0 \simeq \pi_n(\mathbb{S}^n)$.*

Proof. By [Uni13, Theorem 7.3.9] we have an equivalence:

$$\|\mathbb{S}^n \to_\bullet \mathbb{S}^n\|_0 \equiv \|(f : \mathbb{S}^n \to \mathbb{S}^n) \times (f(\mathsf{base}) = b)\|_0 \simeq \|(f : \mathbb{S}^n \to \mathbb{S}^n) \times \|f(\mathsf{base}) = b\|_0\|_0.$$

But since for $n > 1$ the spheres are simply connected the RHS is equivalent to $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$ as required. ■

Proof of Proposition 2.6.1.3. Just like in the previous proof we have:

$$\|\mathbb{S}^n \to_\bullet B\|_0 \equiv \|(f : \mathbb{S}^n \to B) \times (f(\mathsf{base}) = b)\|_0 \simeq \|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_0\|_0$$

by [Uni13, Theorem 7.3.9]. Using the fact that $\pi_1(B)$ acts trivially on $\pi_n(B)$ we will show next that:

$$\|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_0\|_0 \simeq \|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_{-1}\|_0 \tag{2.1}$$

But by connectedness of B the type $\|f(\mathsf{base}) = b\|_{-1}$ is contractible, yielding an equivalence:

$$\|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_0\|_0 \simeq \|\mathbb{S}^n \to B\|$$

as required. So let us prove that we have an equivalence Eq. (2.1).

The map that induces the equivalence is constructed by the functoriality of the 0-truncation:

$$\begin{gathered}\|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_0\|_0 \to \|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_{-1}\|_0 \\ |(f, |p|_0)_0| \mapsto |(f, |p|_{-1})|_0\end{gathered}$$

We prove that this map is an equivalence by showing that each fiber is contractible, and to do this we first show that each fiber is a mere proposition and only then we exhibit an inhabitant for each fiber. Since being a mere proposition is a mere proposition (by induction on 0-truncation) it is enough to show that the fiber of inhabitants of the form:

$$|(f, m)|_0 : \|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_{-1}\|_0$$

are mere propositions. Now assume given two inhabitants of the fiber which by induction on 0-truncation can be assumed to be of the form (we are proving a mere proposition):

$$|(g, |r|_0)|_0, |(h, |s|_0)|_0 : \|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_0\|_0$$

together with equalities $u : |(g, |r|_{-1})|_0 = |(f, m)|_0$ and $v : |(h, |s|_{-1})|_0 = |(f, m)|_0$. Notice that it is enough to show that we have an equality $e : |(g, |r|_0)|_0 = |(h, |s|_0)|_0$ since u and v inhabit mere propositions and thus the compatibility $\mathsf{transport}(e, u) = v$ will be immediate. Using u and v to transport (and 0-truncation induction) we obtain $|(g, |r|_0)|_0 = |(f, |\gamma|_0)|_0$ and $|(h, |s|_0)|_0 = |(f, |\lambda|_0)|_0$, and so it is enough to show $|(f, |\gamma|_0)|_0 = |(f, |\lambda|_0)|_0$. Now observe how the loop $l :\equiv |\gamma^{-1} \cdot \lambda|_0 : \pi_1(B)$ acts:

$$l \cdot |(f, |\gamma|_0)|_0 \equiv |(f, |\gamma \cdot \gamma^{-1} \cdot \lambda|_0)|_0 = |(f, |\lambda|_0)|_0.$$

But since the action $\pi_1(B) \curvearrowright \pi_n(B)$ is trivial we also have $l \cdot |(f, |\gamma|_0)|_0 = |(f, |\gamma|_0)|_0$ which proves that the fiber of the map is a mere proposition.

Now we have to show that each fiber is inhabited, and here comes the interesting part: Since we proved that each fiber is a mere proposition we can use induction on the (-1)-truncation to get an inhabitant $|(f, |p|_0)|_0 : \|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_0\|_0$ out of an inhabitant $|(f, |p|_{-1})|_0 : \|(f : \mathbb{S}^n \to B) \times \|f(\mathsf{base}) = b\|_{-1}\|_0$, finishing the proof. ∎

The previous discussion allows us to define the degree of an endomap of an n-dimensional sphere for $n > 1$.

2.6.1.5 Definition (Degree)**.** By Lemma 2.6.1.4 we have an equivalence $\|\mathbb{S}^n \to \mathbb{S}^n\|_0 \simeq \pi_n(\mathbb{S}^n)$. Composing this with the equivalence $\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ we get a map $deg' : \|\mathbb{S}^n \to \mathbb{S}^n\|_0 \to \mathbb{Z}$. Define the degree of $f : \mathbb{S}^n \to \mathbb{S}^n$ as $deg(f) :\equiv deg'(|f|_0)$.

2.6.1.6 Remark (Degree is property)**.** Notice that by definition the degree takes values in a set and thus having degree a fixed number $k : \mathbb{N}$ is *mere* property.

To be able to translate the rest of the proof of Theorem 1.6.0.4 we need to prove analogues to Lemma 1.6.0.6 and Lemma 1.6.0.8. The first one is fairly easy as we will now see. Define F_n as the type of pointed maps $\mathbb{S}^{n-1} \to_\bullet \mathbb{S}^{n-1}$. Define F_n^k as the type of pointed maps $\mathbb{S}^{n-1} \to_\bullet \mathbb{S}^{n-1}$ of degree k: $F_n^k :\equiv (f : \mathbb{S}^{n-1} \to_\bullet \mathbb{S}^{n-1}) \times (deg(f) = k)$.

2.6.1.7 Lemma. *There is a fibration $F_n^1 \hookrightarrow H_n^+ \to \mathbb{S}^n$ given by:*

$$\begin{aligned} H_n &\to \mathbb{S}^n \\ f &\mapsto f(\mathit{base}) \end{aligned}$$

Proof. We have to show that the homotopy fiber of $\mathsf{base} : \mathbb{S}^n$ through this map is equivalent to F_n^1. For this notice that by definition the fiber is $(f : H_n^+) \times (f(\mathsf{base}) = b)$. Remember that H_n^+ is the connected component $\mathsf{Id} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ of the space of self equivalences of the $(n-1)$-dimensional sphere. Recall that having degree 1 is a mere proposition and being in the connected component of Id is a mere proposition. Being an equivalence is also a mere proposition and thus we have equivalences:

$$\begin{aligned} (f : H_n^+) \times (f(\mathsf{base}) = b) &\simeq (f : \mathbb{S}^{n-1} \to_\bullet \mathbb{S}^{n-1}) \times \|f = \mathsf{Id}\| \\ &\simeq (f : \mathbb{S}^{n-1} \to_\bullet \mathbb{S}^{n-1}) \times (deg(f) = 1) \\ &\equiv F_n^1 \end{aligned}$$

∎

For the case of Lemma 1.6.0.8 observe that given an H-space X there is an induced operation on the connected components $\|X\|_0$ by functoriality of truncation. Using this definition the translation of the lemma into HoTT is straightforward by arguments of connectedness.

The main theorem then follows by the long exact sequence argument, using the adjunction between exponentiation and the smash product. However we must keep in mind that we also need the characterization of identity types of sequential colimits to derive the commutativity of homotopy groups and sequential colimits.

2.7 ∞-Groups

The idea of this section[7] is to motivate the fact that pointed connected spaces can be regarded as generalized groups. Many ideas in this section are due to Michael Shulman and Urs Schreiber. The generalization comes in the form of a higher notion of group: ∞-groups. The first (and in a sense the only) example of such an object is the loop space of a pointed connected space. A loop space is again a space, but with the operation given by path composition. This operation is usually not associative on the nose but it is associative up to homotopy. The homotopies used for associativity satisfy higher coherence laws, that come again in the form of (higher) homotopies. This higher homotopies also satisfy coherence laws, and this holds for every dimension. All these coherence laws can be packed up nicely using operads (see [MSS07, Part 1, Chapter 1, Section 1.6]). In the case of CW-complexes what is needed is Stasheff's associahedra, a family of polyhedra used to define what it means for an operation on a space to be coherently associative in all dimensions. The first non-trivial polyhedron is the associativity pentagon commented in Section 2.1.2. Stasheff proved that a space together with an operation and homotopies that satisfy the coherence laws described by the associahedra is indeed equivalent to the loop space of a pointed connected space, where the operation corresponds to path composition (the standard reference is [Sta63b]). There are other ways to define what ∞-groups are, but for all the standard definitions there is an analogous result, sometimes called the delooping hypothesis, which states that every ∞-group G has a delooping. This delooping is a pointed connected space such that its loop space is equivalent to G as ∞-groups. Having this in mind we can define the universe of ∞-groups in HoTT as follows.

2.7.0.1 Definition (∞-groups)**.** Define the universe $\infty\,\mathsf{Grp} :\equiv \mathcal{U}_{\bullet c}$.

2.7.0.2 Remark (Groups are ∞-groups)**.** If ∞-groups are generalized groups we should be able to assign an ∞-group to each standard (discrete) group. In the classical setting this is done by assigning to a group G its classifying space (also known as Eilenberg-MacLane space) $\mathscr{B}G$ (also written $K(G,1)$). When the group is not discrete but a topological group this can be done using the join construction mentioned in Section 1.4.1. The construction of Eilenberg-MacLane spaces has been formalized in HoTT in [LF14]. This implies that ∞-groups in HoTT as defined above are indeed a generalization of the notion of (discrete) group.

2.7.0.3 Notation. For an ∞-group $\mathscr{B}G$ we will usually denote its distinguished point as $\bullet$.

An ∞-group should be in particular an H-space as defined in [Uni13, Definition 8.5.4]. For this we define the *underlying type* of an ∞-group.

2.7.0.4 Definition (Underlying type of an ∞-group)**.** The underlying type of an ∞-group $\mathscr{B}G$: $\infty\,\mathsf{Grp}$ is defined as the space $G :\equiv \Omega\mathscr{B}G \equiv (\bullet =_{\mathscr{B}G} \bullet)$.

See that there is an obvious operation defined on G, namely path composition and this induces an H-space structure on G. Moreover, the unit of the group is the constant path $\mathsf{refl}_\bullet$: $G :\equiv (\bullet = \bullet)$. Let us give some examples of ∞-groups.

[7] Formalizations of some constructions in this section can be found in `https://github.com/LuisScoccola/ooActions.git`.

2.7.0.5 Example (The integers)**.** As proved in [Uni13, Section 8.1] we have an equivalence $\Omega(\mathbb{S}^1) \simeq \mathbb{Z}$, where path composition corresponds to addition of integers. In the language of this section this means that the group $\mathbb{Z}$ is the underlying type of the ∞-group $\mathbb{S}^1$. Stated otherwise, $\mathbb{S}^1$ is the classifying space of the group $\mathbb{Z}$.

Given a category and an object in the category one can consider the automorphism group of the object. One of the most basics theorems in algebra, Cayley's theorem, says that any group acts regularly on itself and that this action induces an inclusion of the group on the automorphisms group of the underlying set of the group. We will prove an analogous result for ∞-groups, for this we need the automorphisms ∞-group of a space.

2.7.0.6 Example (Automorphisms ∞-group)**.** Given a space $A : \mathcal{U}$ define its automorphisms ∞-group to be $\mathscr{C}A$. This definition makes sense since by Remark 2.1.1.5 the underlying type of $\mathscr{C}A$ is $A = A$.

As an instance of the previous example we have:

2.7.0.7 Example ($\mathscr{B}\mathbb{Z}_2$)**.** Consider the type $\mathscr{B}\mathbb{Z}_2 :\equiv \mathscr{C}[\![2]\!]$, the space of types merely equal to $[\![2]\!]$. This space is the delooping of $\mathbb{Z}_2$ because its loop space is $[\![2]\!] = [\![2]\!]$ which is *discrete* and in fact equivalent to $[\![2]\!]$. This can be proved using the univalence axiom to get equivalences out of equalities $[\![2]\!] = [\![2]\!]$ and then by induction on $[\![2]\!]$ to conclude that there are exactly two such equivalences. Moreover this identification takes composition of equivalences to a (group) product in $[\![2]\!]$.

Although $[\![2]\!] = [\![2]\!]$ is equivalent to $[\![2]\!]$ it might not be such a good idea to refer to this space as $[\![2]\!]$, since $[\![2]\!] = [\![2]\!]$ comes with a natural operation given by path composition. Together with this operation this type is in fact a group and thus it makes more sense to call it $\mathbb{Z}_2$.

Having defined ∞-groups we must look for a suitable definition of ∞-group morphism. The definition is simple.

2.7.0.8 Definition (∞-morphisms)**.** Given two ∞-groups G and H we define the space of ∞-morphisms between G and H as the space of *pointed* maps $\mathscr{B}G \to_{\bullet} \mathscr{B}H$. Morphisms between ∞-groups can be denoted as $\mathscr{B}\varphi : \mathscr{B}G \to \mathscr{B}H$ is one wishes to regard $\varphi : G \to H$ as the actual group morphism.

2.7.0.9 Example. A simple example of ∞-group morphism is the classifying map for the multiplication-by-two morphism:

$$\begin{aligned}(\times 2) : \mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto 2n\end{aligned}$$

Consider the Möbius covering of the circle, which can be defined in HoTT by circle induction:

$$\begin{aligned}\mathbb{S}^1 &\to \mathbb{S}^1 \\ \mathsf{base} &\mapsto \mathsf{base} \\ \mathsf{loop} &\mapsto \mathsf{loop}^2\end{aligned}$$

It straightforward to show that the induced map $\mathbb{Z} \simeq \Omega\mathbb{S}^1 \to \Omega\mathbb{S}^1 \simeq \mathbb{Z}$ is the morphism $(\times 2)$ defined above.

2.7.0.10 Example. A less trivial example is the quotient morphism:

$$q : \mathbb{Z} \to \mathbb{Z}_2$$
$$n \mapsto n \mod 2$$

Consider the self equivalence $[\![2]\!] \simeq [\![2]\!]$ given by the only possible transposition. Call this equivalence -1. We define a map $\mathbb{S}^1 \to \mathscr{B}\mathbb{Z}_2$ by circle induction (recall the definition of $\mathscr{B}\mathbb{Z}_2$ given above):

$$\mathbb{S}^1 \to \mathscr{B}\mathbb{Z}_2$$
$$\mathsf{base} \mapsto [\![2]\!]$$
$$\mathsf{loop} \mapsto \mathsf{ua}(-1)$$

Here ua is the map of the univalence axiom that gives us a path in $\mathcal{U}$ out of an equivalence of types.

The induced map $\mathbb{Z} \simeq \Omega\mathbb{S}^1 \to \Omega(\mathscr{B}\mathbb{Z}_2) \simeq \mathbb{Z}_2$ takes n to l^n and then to $\mathsf{ua}(-1)^n$. It is clear that permuting the elements of $[\![2]\!]$ an even number of times is the identity and an odd number of times is the equivalence $-1 : [\![2]\!] \simeq [\![2]\!]$. And thus the defined map classifies the group morphism q defined above.

Notice that the last example classifies a group morphism with non-trivial kernel. It would be nice to find a proper definition for the kernel of an ∞-group morphism in such a way that gives us the correct kernel in the above example. To make an attempt to do so we need some basic theory about ∞-group actions.

2.7.0.11 Definition (∞-action)**.** An ∞-action of an ∞-group $\mathscr{B}G$ is a fibration $X : \mathscr{B}G \to \mathcal{U}$. We can denote such an action as $G \curvearrowright X(\bullet)$.

The following notation makes the action of G on the space $X(\bullet)$ more explicit.

2.7.0.12 Notation (Action notation)**.** Write $\mathcal{X}$ for the space $X(\bullet)$. Then one can think that G is acting on the space $\mathcal{X}$: Each element $g : G$ yields an equivalence between $\mathcal{X}$ and $\mathcal{X}$. By transporting along a given $g : G$ we can recover the usual action notation $g \cdot x$. Concretely given $g : G$ and $x : \mathcal{X}$ we write $g \cdot x : \mathcal{X}$ for $\mathsf{transport}(g, x) : \mathcal{X}$.

One can verify that this is an H-space action. The following remark lets us translate many definitions of the classical setting to our setting.

2.7.0.13 Remark. Notice that by the characterization of the path spaces of sigma types given in [Uni13, Theorem 2.7.2], for any two elements $x, y : \mathcal{X}$ the path space $(\bullet, x) = (\bullet, y)$ is equivalent to $(g : G) \times (g \cdot x = y)$. In particular we have the equivalence $((\bullet, x) = (\bullet, x)) \simeq (g : G) \times (g \cdot x = x)$.

2.7.0.14 Definition (Homotopy quotient of ∞-action)**.** Given an ∞-action $X : \mathscr{B}G \to \mathcal{U}$ the total space ΣX is sometimes called the homotopy quotient of the action since ΣX can be seen as a quotient of $\mathcal{X}$ by regarding the map:

$$\mathcal{X} \to \Sigma X$$
$$x \mapsto (\bullet, x)$$

as a quotient map.

Let us refine the analogy between fibrations and actions.

2.7.0.15 Remark (Fibrations = Representations = Actions)**.** Recall Theorem 2.5.0.2. Notice how using the language of this section it implies that, just like in the classical setting, actions of an ∞-group G on a type $\mathcal{X}$ are equivalent to a group morphisms $G \to (\mathcal{X} = \mathcal{X})$, and both are equivalent to fibrations over $\mathscr{B}G$. In other words, the following types are equivalent:

$$\frac{\dfrac{\dfrac{G \curvearrowright \mathcal{X}}{\mathscr{B}G \to \mathcal{U} \text{ such that } \bullet \mapsto \mathcal{X}}}{\mathscr{B}G \to \mathscr{C}\mathcal{X}}}{G \to \mathcal{X} = \mathcal{X} \text{ (morphism)}}$$

Now we will translate some classical properties that group actions might enjoy into our setting. By means of Notation 2.7.0.12 we have at least a naïve translation for *transitive*, *free* and *regular* actions. Moreover, the reader should notice that when the ∞-group is acting on a *set*, the definitions reduce to the classical definitions. Notice that, since being a set is a mere property, ∞-group actions on sets are equivalent to fibrations $\mathscr{B}G \to \mathsf{Set}$, where Set is the type of discrete types. First a remark on group actions.

2.7.0.16 Remark (Group actions are ∞-group actions)**.** Say G is a standard (discrete) group and we have a set $\mathcal{X}$ and an action $G \curvearrowright \mathcal{X}$. In the same way that –as mentioned in Remark 2.7.0.2– we can construct an ∞-group $\mathscr{B}G$, the classifying space of G, we should be able to construct a fibration $X : \mathscr{B}G \to \mathsf{Set}$ such that $X(\bullet) = \mathcal{X}$ and such that the action given by the notation of Remark 2.7.0.13 is the original action $G \curvearrowright \mathcal{X}$. This construction has been carried on in HoTT in [Hou15]. The article develops the basic of the theory of covering spaces in HoTT and proves the classical equivalence between covering spaces and actions of the fundamental group on sets. Notice that by definition a covering space over a type A is a fibration $A \to \mathsf{Set}$, a fibration with discrete fiber. The equivalence then reads:

$$\frac{A \to \mathsf{Set}}{\pi_1(A) - \mathsf{Set}}$$

This gives us a way to construct an ∞-action $X : \mathscr{B}G \to \mathsf{Set}$ out of a classical group action $G \curvearrowright \mathcal{X}$.

2.7.0.17 Definition (Non-empty ∞-action)**.** A non-empty action is an action $X : \mathscr{B}G \to \mathcal{U}$ such that we have $\|\mathcal{X}\|$.

2.7.0.18 Definition (Transitive ∞-action)**.** An action $X : \mathscr{B}G \to \mathcal{U}$ is transitive if the total space ΣX is connected (0-connected).

The following characterization of transitive action is closer to the classical definition and motivates the above definition[8].

2.7.0.19 Proposition (Equivalent definition of transitive action)**.** *An action $X : \mathscr{B}G \to \mathcal{U}$ is transitive if and only if it is non-empty and we have a map $(x, y : \mathcal{X}) \to \|(g : G) \times (g \cdot x = y)\|$.*

[8]Although the definition gives the correct notion when instantiated with 0-truncated types one might choose to say that such an action is *merely* transitive, instead of transitive.

Proof. Notice that both definitions are mere propositions. To prove that a connected total space implies the existence of a map of the described type just use Remark 2.7.0.13 and the fact that both definitions are mere propositions to be able to get "real" paths out of truncated paths: If ΣX is connected, given $x, y : \mathcal{X}$ we have a *mere* path $p : (\bullet, x) = (\bullet, y)$, but since we want to prove a proposition, we can assume that it is an actual path. But a path p is nothing but a path $g : (\bullet = \bullet) \equiv G$ together with a path $g \cdot x = y$, which is just what we needed.

For the converse use Remark 2.7.0.13 again and the fact that $\mathscr{B}G$ is connected to reduce the proof to the case in which all the elements live in $\mathcal{X}$.

With this argument one proves that both definitions are logically equivalent, but since both are mere proposition the definitions turn up to be equivalent as types. We formalized this proof in Coq[9]. ■

2.7.0.20 Definition (Free ∞-action)**.** An action $X : \mathscr{B}G \to \mathcal{U}$ is free if the total space is discrete (0-truncated).

Again, we have a characterization that resembles a lot the classical definition of free action.

2.7.0.21 Proposition (Equivalent definition of free action)**.** *An action $X : \mathscr{B}G \to \mathcal{U}$ is free if and only if we have a map $(x : \mathcal{X}) \to \mathsf{isContr}((g : G) \times (g \cdot x = x))$.*

Proof. Again both definitions are mere propositions. Since $(g : G) \times (g \cdot x = x)$ is equivalent to $(\bullet, x) = (\bullet, x)$ having a map of type $(x : \mathcal{X}) \to \mathsf{isContr}((g : G) \times (g \cdot x = x))$ implies that the connected component of every $(\bullet, x)$ for $x : \mathcal{X}$ is contractible. Since $\mathscr{B}G$ is connected this implies that every connected component of ΣX is contractible and thus ΣX is discrete.

On the other hand if ΣX is discrete, then $(\bullet, x) = (\bullet, x)$ is contractible for each $x : \mathcal{X}$ which implies that we have a map of type $(x : \mathcal{X}) \to \mathsf{isContr}((g : G) \times (g \cdot x = x))$ since $((\bullet, x) = (\bullet, x)) \simeq (g : G) \times (g \cdot x = x)$. We formalized this proof in Coq[10]. ■

See how if X is a set then $g \cdot x = y$ is a mere proposition and thus, if G is a discrete group this reduces to the classical definition of free action of a (discrete) group on a set.

We can combine both these definitions to get the definition of regular action.

2.7.0.22 Definition (Regular ∞-action)**.** An action $X : \mathscr{B}G \to \mathcal{U}$ is regular if it is both transitive and free.

Similarly combining both characterizations we get the following result.

2.7.0.23 Proposition (Equivalent definition of regular action)**.** *An action $X : \mathscr{B}G \to \mathcal{U}$ is regular if and only if the total space ΣX is contractible.*

Proof. Once again both definitions are mere propositions. This proof follows at once from the fact that a type is contractible if and only if it is connected and 0-truncated. ■

A space together with a regular action of an ∞-group $\mathscr{B}G$ is usually called a principal homogeneous space. One should also notice that an action is regular if and only if we have $(x, y : \mathcal{X}) \to \mathsf{isContr}((g : G) \times (g \cdot x = y))$.

[9]See `https://github.com/LuisScoccola/ooActions.git`, file `ooAction2.v`, function `transitiveequivtransitive'`.

[10]See `https://github.com/LuisScoccola/ooActions.git`, file `ooAction2.v`, function `freeequivfree'`.

2.7.0.24 Example. It is interesting to interpret these definitions in the case of the proof that the classifying ∞-group of the integers is the circle given in [Uni13, Section 8.1.4]. The proof can be interpreted as constructing a regular action of the circle on the integers, thus proving that the integers are a principal homogeneous $\Omega\mathbb{S}^1$ space. Together with a choice of inhabitant of the integers, for example $0 : \mathbb{N}$, we get the equivalence $\Omega\mathbb{S}^1 \simeq \mathbb{Z}$.

Now we can state the appropiate version of Cayley's theorem. Notice that, just like in the classical setting, this can be regarded as an application of Yoneda's lemma. Recall the Yoneda fibration of Definition 2.2.1.8.

2.7.0.25 Lemma (Cayley's theorem for ∞-groups)**.** *Given an ∞-group $\mathscr{B}G$ the Yoneda fibration $\mathcal{Y}_\bullet : \mathscr{B}G \to \mathcal{U}$ induces an ∞-group morphism $\mathscr{B}G \to \mathscr{C}G$ which is nothing but an ∞-group action $G \curvearrowright G$. Moreover this action is regular.*

Proof. The action is regular since the total space of the Yoneda fibration is contractible. The ∞-group morphism is induced by the connectedness of $\mathscr{B}G$ using Remark 2.1.1.6. ■

To define the kernel of an ∞-group morphism we first define the more general notion of stabilizer of an element in an ∞-group action. For this we study the orbit of an element. Given an action $G \curvearrowright \mathcal{X}$ the orbit of $x : \mathcal{X}$ should be $(y : \mathcal{X}) \times \|(g : G) \times (g \cdot x = y)\|$. Moreover, given an action and an element in the acted space we can define the action *restricted* to the orbit of that element.

2.7.0.26 Definition (Restriction of action)**.** Let $X : \mathscr{B}G \to \mathcal{U}$ be an action and let $x : \mathcal{X}$. We define the action restricted to the orbit of x as:

$$\begin{aligned} X_x &: \mathscr{B}G \to \mathcal{U} \\ & b \mapsto (y : X(b)) \times \|(\bullet, x) =_{\Sigma X} (b, y)\| \end{aligned}$$

We call this the *restricted action* or the *orbit action*.

2.7.0.27 Remark. This is just the restriction of the fibration X to the connected component of $(\bullet, x)$ in ΣX. Moreover, we have a canonical identification $\Sigma X_x \simeq \mathscr{C}(\Sigma X, (\bullet, x))$.

By projecting the first coordinate we get a fibration map $X_x \to X$ which can be regarded as the inclusion of the orbit of x in the space $\mathcal{X}$. Notice that $X_x(\bullet)$ is the orbit of x and thus we define:

2.7.0.28 Definition (∞-orbit)**.** Given an action $X : \mathscr{B}G \to \mathcal{U}$ and an element $x : \mathcal{X}$ define its orbit as $\mathcal{O}^x :\equiv X_x(\bullet)$.

The stabilizer of an element must be an ∞-group and thus to define it we must define its classifying space.

2.7.0.29 Definition (∞-stabilizer)**.** Given an action $X : \mathscr{B}G \to \mathcal{U}$ define $\mathscr{B}Stab_x :\equiv \Sigma X_x$, the total space of the orbit fibration. By the identification of Remark 2.7.0.27 it is clear that this space is pointed and connected.

This definition might seem strange at first but notice that:

$$Stab_x :\equiv \Omega\mathscr{B}Stab_x \simeq (g : G) \times (g \cdot x = x)$$

Naturally we must have a morphism from the stabilizer of an element to the group that is acting.

2.7.0.30 Definition (Map from stabilizer to group)**.** For this recall that the classifying space of the stabilizer is defined as a total space of a fibration over $\mathscr{B}G$ and thus we can just project the first coordinate of this total space:

$$\mathscr{B}Stab_x :\equiv \Sigma X_x \equiv (b : \mathscr{B}G) \times (y : X(b)) \times \|(\bullet, x) = (b, y)\|$$

so we define:

$$\begin{aligned} \mathscr{B}st_x : \mathscr{B}Stab_x &\to \mathscr{B}G \\ (b, y, p) &\mapsto b \end{aligned}$$

With this definition we can construct the kernel of an ∞-group morphism by noticing that a morphism $G \to H$ induces an action $G \curvearrowright H$ and then taking the stabilizer of the unit of H in this action.

2.7.0.31 Remark. Given an ∞-group morphism $m : \mathscr{B}G \to \mathscr{B}H$ we can postcompose it with the Yoneda fibration to get an action of G on H:

$$\begin{aligned} \tilde{m} :\equiv \mathcal{Y}_\bullet \circ m : \mathscr{B}G &\to \mathcal{U} \\ \bullet &\mapsto H \end{aligned}$$

Now we can define the kernel of an ∞-group morphism.

2.7.0.32 Definition (∞-kernel)**.** We define the kernel of a morphism $m : \mathscr{B}G \to \mathscr{B}H$ as the stabilizer of the unit of H in the induced action $\tilde{m} : G \curvearrowright H$. Recall that unit of H is the element $\mathsf{refl}_\bullet : \Omega\mathscr{B}H$.

Let us consider the kernel of the ∞-morphisms that we constructed above.

2.7.0.33 Example (Kernel of $\times 2$)**.** The kernel of multiplying by two should be trivial. To convince ourselves of this fact we take the classifying map $\mathbb{S}^1 \to \mathbb{S}^1$ given by the Möbius covering and we postcompose it with the Yoneda fibration to get the fibration:

$$\begin{aligned} X &: \mathbb{S}^1 \to \mathcal{U} \\ &\mathsf{base} \mapsto \Omega\mathbb{S}^1 \\ &\mathsf{loop} \mapsto (- \cdot \mathsf{loop}^2) : \Omega\mathbb{S}^1 \simeq \Omega\mathbb{S}^1 \end{aligned}$$

We now have to compute the stabilizer of $\mathsf{refl}_b : b = b$. We know that the delooping of the stabilizer is equivalent to the connected component of $(\bullet, \mathsf{refl}_b) : \Sigma X$, so to prove that the kernel is trivial it suffices to show that this connected component is contractible. Now notice that this fibration is very similar to the fibration used to prove that the circle is the delooping of the integers. Only that this time the ∞-group $\Omega\mathbb{S}^1$ is acting (regularly) on *two* copies of $\mathbb{Z}$: The even numbers and the odd numbers. This is reflected by the fact that the total space of this fibration consists of two connected components, and both this connected components are contractible: The action is a disjoint union of two Yoneda fibrations, and thus ΣX_x which is the total space of one of this fibrations, is contractible.

2.7.0.34 Example (Kernel of q)**.** Consider the group $\mathbb{Z}_2$ as a multiplicative group with elements 1 and -1. Then we have the bijection $(\times(-1)) : \mathbb{Z}_2 \to \mathbb{Z}_2$ that multiplies by -1. To compute the kernel we first postcompose q with the Yoneda map $\mathcal{Y}_\bullet : \mathscr{B}\mathbb{Z}_2 \to \mathscr{C}\mathbb{Z}_2$. With the notation above introduced this gets us the fibration:

$$\begin{aligned} X : \mathbb{S}^1 &\to \mathcal{U} \\ \mathsf{base} &\mapsto \mathbb{Z}_2 \\ \mathsf{loop} &\mapsto (\times(-1)) \end{aligned}$$

Then we have $\Omega(\Sigma X) \simeq ((b,1) = (b,1)) \simeq (n : \mathbb{Z}) \times ((-1)^n = 1) \simeq \mathbb{Z}$. Where the first equivalence is the characterization of Remark 2.7.0.13, the second one is an application of the fact that $\Omega\mathbb{S}^1 \simeq \mathbb{Z}$, and the third one follows from the fact that $\mathbb{Z}_2$ is a set and thus $(-1)^n = 1$ is a mere proposition. Then the loop space of the classifying space of the kernel is $\mathbb{Z}$, so the classifying space of the kernel is indeed $\mathbb{S}^1$, since it is connected by definition.

We now see how our naïve translation approach works for the classical result that reads: *There is an equivalence between transitive pointed actions and subgroups of the acting group*. In our setting a transitive pointed action is an action $X : \mathscr{B}G \to \mathcal{U}$ together with an element $x : \mathcal{X}$ and such that the homotopy quotient ΣX is connected.

2.7.0.35 Remark (Pointed transitive actions)**.** There is an equivalence:

$$\frac{(X : \mathscr{B}G \to \mathcal{U}) \times (x : \mathcal{X}) \times \mathsf{isConn}(\Sigma X)}{(\mathscr{B}H : \infty\,\mathsf{Grp}) \times (\mathscr{B}H \to \mathscr{B}G)}$$

This is again a restriction the equivalence between fibrations and actions. But in this case it says that an ∞-group morphism is the same that a pointed transitive action of the codomain.

Conclusions

We gave a minimal introduction to the theory of spherical fibrations and we related it to real K-theory by means of the J-homomorphism. We proved Theorem 1.6.0.4 that characterizes the homotopy groups of the classifier of spherical fibrations and together with the J-homomorphism we constructed a group morphism from the K-theory of the spheres to the stable homotopy groups of the spheres. K-theory has not been defined in HoTT although it might be possible to do so at least for compact types, since it is a homotopy-invariant notion. Maybe real-cohesive homotopy type theory ([Shu15a]) can be used for this.

We translated the very basics of the theory of spherical fibrations into HoTT but to deduce fairly basic results we used some conjectures. Some of these are not new and have already appeared in other contexts. The most important ones imply that filtered colimits, or in our case sequential colimits, behave as expected if one believes that HoTT is the internal logic of ∞-topoi: The total space of a filtered colimit fibration is the colimit of the degree-wise total spaces (Conjecture 2.2.1.7), and sequential colimits commute with pullbacks (Conjecture 2.4.0.7). It is clear that a good HoTT theory on limits and colimits –with theorems relating both constructions– is essential. The basics of limits and colimits are studied in [AKL15] and [BQTR16]. Here we formalized Lemma 2.4.0.8 that states that the map $(- \to A) : \mathcal{U} \to \mathcal{U}$ sends colimits to limits. As we saw in Section 2.2 sequential colimits are of particular importance as they permit a more controlled construction of higher inductive types. They are also essential in the definition of classifying spaces for stabilized concepts, such as spherical fibrations up to stable fiber homotopy equivalence. Here we formalized Lemma 2.2.1.3 that gives an equivalence between the colimit of a sequential diagram and the colimit of the (-1)-shifed diagram.

On top of the strong results about sequential diagrams commented above, to translate the characterization of the homotopy groups of the classifier of spherical fibration (Theorem 1.6.0.4) we needed basic concepts from homotopy theory, such as the degree of endomaps of spheres, the multiplicativity of the degree, the action of the fundamental group on the homotopy groups of a space, and the long exact sequence of homotopy groups (the long exact sequence has already been studied in HoTT see [Uni13, Section 8.4] and [AKL15, Section 3.3]). In Section 2.6.1 we studied these concepts and we proved some lemmas and results needed for the translation of the proof of Theorem 1.6.0.4 into HoTT.

Finally, in Section 2.7 we tried to convey that a development of a theory of ∞-groups in HoTT could be useful to understand HoTT from the point of view of representation theory. Here we formalized the (very basic) theory that we developed.

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