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Sobre la conjetura de la p-curvatura de Grothendieck-Katz

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Introducción

La comparación entre la información local y global es una idea que permea toda la matemática. En geometría, los fenómenos locales son aquellos que suceden en entornos infinitesimales de cada punto, o más precisamente, en el germen de la estructura estudiada en el punto, mientras que los fenómenos globales son aquellos que involucran al objeto en su totalidad. El objetivo de esta tesis es presentar y comprender la teoría que rodea a la conjetura de la p-curvatura de Grothendieck-Katz. Esta conjetura es un principio local-global para ecuaciones diferenciales en esquemas.

Supongamos que tenemos un sistema lineal y homogéneo de ecuaciones diferenciales

$$\frac{\mathrm{d}y}{\mathrm{d}z} = A(z)y(z)$$

donde $A(z)$ es una matriz cuadrada de $n \times n$ cuyas entradas son funciones racionales con coeficientes en un cuerpo de números K. Tiene sentido entonces, para casi todo primo $\mathfrak{p}$ de K, reducir la ecuación módulo $\mathfrak{p}$ y así obtener una ecuación diferencial en $\mathbb{F}_q(z)$. Supongamos que para casi todos los primos $\mathfrak{p}$ la ecuación reducida tiene una base de soluciones (es decir, n soluciones en $\mathbb{F}_q(z)^n$ que son linealmente independientes sobre $\mathbb{F}_q(z)$). La pregunta de Grothendieck es si, bajo estas condiciones, la ecuación original posee una base de soluciones algebraicas.

Un sistema lineal homogéneo de primer orden como en el párrafo anterior puede pensarse como una conexión ∇_A en $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{x_1, \ldots, x_m\}$ donde $\{x_1, \ldots, x_m\}$ es el conjunto de polos de las entradas de A. Las soluciones de la ecuación diferencial se traducen en este nuevo contexto como las secciones horizontales de la conexión, y por ende pueden ser entendidas geométricamente a través de la monodromía. La existencia de una base de soluciones algebraicas está controlada, en el caso clásico, por la finitud del grupo de Galois diferencial de la ecuación, y en el caso en que las singularidades sean regulares esto es a su vez equivalente a la finitud del grupo de monodromía. Si definimos una *ecuación diferencial algebraica* como una conexión integrable en un $\mathbb{C}$-esquema suave de tipo finito, diremos que tiene una base de soluciones algebraicas si tiene grupo de monodromía finito. La conjetura de Grothendieck en este contexto relaciona la finitud del grupo de monodromía de una conexión integrable en un esquema con un cierto invariante local de la reducción módulo p de la conexión: la p-curvatura.

En una serie de artículos, Katz estudió este problema en el contexto de ecuaciones diferenciales algebraicas (i.e., conexiones integrables) y probó la conjetura en el caso de la conexión de Gauss-Manin. Probó además que la conjetura de Grothendieck es equivalente a una caracterización conjetural del álgebra de Lie del grupo de Galois diferencial como la más chica que contiene a la p-curvatura para casi todo primo. A pesar de que esta conjetura sigue abierta, ha impulsado muchos avances y ha sido probada en varios contextos por los Chudnovsky, André, Bost y más recientemente por Kisin y Esnault.

La tesis está organizada de la siguiente forma.

El capítulo 1 contiene los preliminares básicos de categorías, haces y esquemas que necesitaremos más adelante.

El capítulo 2 presenta los aspectos geométricos de las conexiones, su caracterización a través del haz de secciones horizontales y la relación con la monodromía. Probamos la correspondencia de Riemann-Hilbert y discutimos el vínculo entre el problema veintiuno de Hilbert y la formulación geométrica de ecuaciones diferenciales en términos de sus conexiones asociadas.

El capítulo 3 habla acerca de la teoría de esquemas grupo y categorías Tannakianas. En este marco, podemos extender la teoría de Galois diferencial al contexto de conexiones como un grupo fundamental Tannakiano y establecer de este modo la igualdad con el grupo de monodromía en el caso de singularidades regulares.

El capítulo 4 trata los aspectos aritméticos de conexiones en esquemas y sus propiedades en el contexto de característica $p > 0$. Presentamos la demostración del teorema de Katz que establece que la nulidad de la p-curvatura para casi todo primo implica la finitud de la monodromía local. Probamos luego la conjetura de Grothendieck-Katz en el caso de que el grupo de monodromía es abeliano.

Introduction

The comparison between the local and the global information is an idea that pervades all of mathematics. In geometry, the local phenomena are those that take place around infinitesimal neighborhoods of each points, or more precisely the germ of the structure in the point, whereas global phenomena are those that involve the object in its entirety. The aim of this thesis is to present and understand the theory surrounding the Grothendieck-Katz p-curvature conjecture. This conjecture is a local to global principle for differential equations on schemes.

Grothendieck raised a question on linear homogeneous systems of first-order differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}z} = A(z)y(z)$$

where $A(z)$ is a square $n \times n$ matrix of rational functions on z with coefficients in some number field. Then, for almost all primes $\mathfrak{p}$ of K it makes sense to reduce this equation modulo $\mathfrak{p}$ obtaining a differential equation over $\mathbb{F}_q(z)$. Suppose that for almost all primes $\mathfrak{p}$ the reduced equation has a full set of solutions (that is, n solutions in $\mathbb{F}_q(z)^n$ which are linearly independent over $\mathbb{F}_q(z)$). Grothendieck's question is then if, under these conditions, the original equation admits a full set of algebraic solutions.

A linear homogeneous system of first-order differential equation as above can be thought of as a connection ∇_A on $\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \ldots, x_m\}$ where $\{x_1, \ldots, x_m\}$ is the set of poles of the entries of A. The solutions of the differential equation are translated into this new context as the horizontal sections of the connection and hence they can be understood geometrically via the monodromy. The existence of a full set of algebraic solutions is handled in the classical case by the finiteness of the differential Galois group of the equation, and in the case that the singularities are regular this is also equivalent to the finiteness of the monodromy group. If we define an *algebraic differential equation* as an integrable connection on a smooth $\mathbb{C}$-scheme of finite type, we thus say that it has a full set of algebraic solutions if it has finite monodromy group. Grothendieck's conjecture in this context relates the finiteness of the monodromy group of an integrable connection on a scheme to a certain local invariant of the reductions modulo p of the connection: the p-curvature.

In a series of articles, Katz studied this problem in the context of algebraic differential equations (i.e., integrable connections) and proved the conjecture in the case of the Gauss-Manin connection. He proved that the conjecture of Grothendieck is equivalent to characterizing the Lie algebra of the differential Galois group as the smallest to contain the p-curvatures for almost every prime. Although this conjecture is still open, it has lead to many interesting developments and it has been proved in several more situations by the Chudnovsky's, André, Bost and more recently by Kisin and Esnault.

The thesis is organized in the following way.

Chapter 1 contains the basic preliminaries on categories, sheaves and schemes that we will need further on.

Chapter 2 presents the geometric aspects of connections, their characterization by the sheaf of horizontal sections and the relationship with monodromy. We prove the Riemann-Hilbert correspondence and discuss the link between Hilbert's twenty-first problem and the geometric formulation of differential equations with their associated connections.

Chapter 3 discusses the theory of group schemes and Tannakian categories. In this framework we can extend the differential Galois theory to the context of connections as a Tannakian fundamental group and establish the equality with the monodromy group in the case of regular singularities.

Chapter 4 deals with the arithmetic aspects of connections on schemes and their properties in the characteristic $p > 0$ setting. We present a proof of the theorem of Katz which says that the vanishing of the p-curvatures implies the finiteness of the local monodromy. The conjecture of Grothendieck-Katz is then proved in the case with abelian monodromy.

Chapter 1

Preliminaries

In this chapter we will give, in the way of a glossary, some basic definitions and results that will be needed later on.

1.1 Categories

We will not delve into the set-theoretic aspects of categories. For a more rigorous exposition on categories, see [ML98] and [Fre64].

Definition 1.1.1. For us, a category will consist of a class of *objects* and a set of *morphisms* between pairs of objects. We denote the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(A, B)$ (or simply $\mathrm{Hom}(A, B)$ if the category is clear from context) between two objects A and B. We shall denote $\phi : A \to B$ for an element $\phi \in \mathrm{Hom}(A, B)$. Moreover, given objects A, B, C we have a binary operation

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

where $\phi \times \psi \mapsto \psi \circ \phi$ called composition such that it is associative and that for every object A there exists an identity morphism $1_A : A \to A$ such that $1_A \circ \phi = \phi$ and $\psi \circ 1_A = \psi$ for any given maps ϕ, ψ. We will say that a morphism $\phi : A \to B$ is an *isomorphism* if there exists ψ $B \to A$ such that $\psi \circ \phi = 1_A$ and $\phi \circ \psi = 1_B$.

If $\mathscr{C}$ is a category, the *opposite category* is the category $\mathscr{C}^{\mathrm{op}}$ which has the same objects of $\mathscr{C}$ but the arrows are reversed, or in other words, we define $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathscr{C}}(B, A)$. The *product* of two categories $\mathscr{C}_1, \mathscr{C}_2$ is the category $\mathscr{C}_1 \times \mathscr{C}_2$ whose objects are pairs (C_1, C_2) with $C_1 \in \mathscr{C}_1$, $C_2 \in \mathscr{C}_2$, and whose morphisms are pairs (ϕ_1, ϕ_2) with $\phi_1 \in \mathrm{Hom}_{\mathscr{C}_1}(C_1, C_1')$ and $\phi_2 \in \mathrm{Hom}_{\mathscr{C}_2}(C_2, C_2')$. A *subcategory* of a category $\mathscr{C}$ is a category $\mathscr{D}$ consisting of some objects and some morphisms of $\mathscr{C}$. A subcategory $\mathscr{D}$ of $\mathscr{C}$ is said to be *full* if $\mathrm{Hom}_{\mathscr{D}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$.

Example 1.1.2. Some examples of categories are Sets the category of sets whose morphisms are set-theoretic functions, Grp , Ab the category of abelian groups whose morphisms are group homomorphisms, Top the category of topological spaces whose morphisms are continuous functions. Both Ab and Top are subcategories of Sets but they are not full subcategories. On the other hand Ab is a full subcategory of Grp.

Definition 1.1.3. A (covariant) *functor* $F : \mathscr{C}_1 \to \mathscr{C}_2$ is a rule $A \mapsto F(A)$ on objects and a map on sets of morphisms $F_{AB} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ which sends identity morphisms to identity morphisms and preserves composition. A *contravariant* functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is a covariant functor $F : \mathscr{C}_1 \to \mathscr{C}_2^{\mathrm{op}}$. The *identity functor* $1_{\mathscr{C}}$ on a category $\mathscr{C}$ is defined by leaving everything fixed. If we fix an object A we have a covariant functor $\mathrm{Hom}(A, -)$ sending an object B to the set $\mathrm{Hom}(A, B)$ and a morphism $\phi : B \to C$ to the morphism $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ induced by composing with ϕ. Similarly, we have the contravariant functor $\mathrm{Hom}(-, A)$.

Definition 1.1.4. If $F, G : \mathscr{C}_1 \to \mathscr{C}_2$ are two functors, a *morphism of functors* (also called *natural transformation*) $\Phi : F \to G$ is a collection of morphisms $\Phi_A : F(A) \to G(A)$ such that for each morphism $\phi : A \to B$ the following diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\Phi_A} & G(A) \\ {\scriptstyle F(\phi)}\downarrow & & \downarrow{\scriptstyle G(\phi)} \\ F(B) & \xrightarrow{\Phi_B} & G(B) \end{array}$$

commutes. The morphism Φ is an *isomorphism of functors* if each Φ_A is an isomorphism.

Definition 1.1.5. A functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is an *equivalence of categories* if there exists a functor $G : \mathscr{C}_2 \to \mathscr{C}_1$ and isomorphisms of functors $\Phi : G \circ F \to 1_{\mathscr{C}_1}$, $\Psi : F \circ G \to 1_{\mathscr{C}_2}$. With this notation, we say that G is a *quasi-inverse* for F. If there exists an equivalence of categories between two categories we say that they are *equivalent*. In the case that we actually have $G \circ F = 1_{\mathscr{C}_1}$ and $F \circ G = 1_{\mathscr{C}_2}$ we say that the categories are *isomorphic*. A contravariant functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is an *anti-equivalence of categories* if the corresponding functor $F : \mathscr{C}_1 \to \mathscr{C}_2^{\mathrm{op}}$ is an equivalence of categories.

Definition 1.1.6. A functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is *faithful* if for any two objects A and B of $\mathscr{C}_1$ the map of sets $F_{AB} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is injective. In the case that F_{AB} is bijective for every two objects A and B we say that F is *fully faithful*. A functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is *essentially surjective* if every object of $\mathscr{C}_2$ is isomorphic to some object of the form $F(A)$.

In general, if we wish to prove that a functor F is an equivalence of categories, it is very hard to construct a quasi-inverse. The following proposition comes in to help and makes an intrinsic characterization of F as an equivalence of categories.

Proposition 1.1.7. *A functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Proof. See [ML98]. ∎

Definition 1.1.8. Let $\mathscr{C}$ be a category. A functor $\mathscr{F} : \mathscr{C} \to \mathrm{Sets}$ is *representable* if there is an object $C \in \mathscr{C}$ and an isomorphism of functors $F \simeq \mathrm{Hom}(C, -)$.

Notice that if $D \to C$ is a morphism, this induces a morphism of functors $\mathrm{Hom}(C, -) \to \mathrm{Hom}(D, -)$ via composition. The following lemma is a sort of converse for this idea.

Lemma 1.1.9 (Yoneda)**.** *Let $F, G : \mathrm{C} \to \mathrm{Sets}$ functors represented by objects C, D respectively. Every morphism of functors $\Phi : F \to G$ is induced by a unique morphism $D \to C$.*

Proof. Because of the representability of the functors, we may rewrite the morphism $\Phi_C : F(C) \to G(C)$ as a map $\mathrm{Hom}(C, C) \to \mathrm{Hom}(D, C)$. The image of the identity morphism $1_C \in \mathrm{Hom}(C, C)$ by Φ_C is the morphism $\rho : D \to C$ inducing Φ. Indeed, for an object A each element of $F(A) \simeq \mathrm{Hom}(C, A)$ identifies with a morphism $\phi : C \to A$ and ϕ as an element of $F(A)$ is none other but the image of $1_C \in \mathrm{Hom}(C, C) \simeq F(C)$ via $F(\phi)$. Since Φ is a morphism of functors we see that $\Phi_A(\phi) = G(\phi)(\rho)$ and via the isomorphism $G(A) \simeq \mathrm{Hom}(D, A)$ this corresponds precisely to $\phi \circ \rho$. ∎

Corollary 1.1.10. *The representing object of a representable functor is unique up to unique isomorphism.*

Proof. If both C and D represent a functor F, applying Yoneda lemma to the identity map $F \to F$ we obtain the desired result. ∎

We want to treat categories that have a bit more of structure. The type of categories that we will be intrested now are those which we obtain by axiomatizing some properties of the category of abelian groups.

Definition 1.1.11. Let $\mathscr{C}$ be a category and A_1, A_2 objects in $\mathscr{C}$. A *product* of A_1, A_2 (in the case that it exists) is an object $A_1 \times A_2$ together with morphisms $p_1 : A_1 \times A_2 \to A_1$ and $p_2 : A_1 \times A_2 \to A_2$ satisfying the universal property: for every object C and morphisms $\phi_1 : C \to A_1$, $\phi_2 : C \to A_2$ there exists a unique map $\phi : C \to A_1 \times A_2$ such that $\phi_1 = p_1\phi$ and $\phi_2 = p_2\phi$. In other words $A_1 \times A_2$ represents the functor

$$C \mapsto \mathrm{Hom}(C, A_1) \times \mathrm{Hom}(C, A_2).$$

Dually, we define a *coproduct* $A_1 \coprod A_2$ as an object representing the functor

$$C \mapsto \mathrm{Hom}(A_1, C) \times \mathrm{Hom}(A_2, C).$$

Moreover, if A_1, A_2 are provided with morphisms $\psi_1 : A_1 \to A$, $\psi_2 : A_2 \to A$ for an object A, the *fiber product* $A_1 \times_A A_2$, if it exists, is the object representing the set-valued functor

$$C \mapsto \{(\phi_1, \phi_2) \in \mathrm{Hom}(C, A_1) \times \mathrm{Hom}(C, A_2) : \psi_1 \circ \phi_1 = \psi_2 \circ \phi_2\}.$$

Definition 1.1.12. A category $\mathscr{C}$ is *additive* if every pair of objects has a product and the sets $\mathrm{Hom}(A, B)$ carry the structure of an abelian group so that the composition map $(\phi, \psi) \mapsto \psi \circ \phi$ is $\mathbb{Z}$-bilinear. Moreover, if $\mathrm{Hom}(A, B)$ have k-vector space structure over a fixed field k and the composition is k-bilinear, we say that the category is *k-linear additive*.

In an additive category, there is a zero element in every $\mathrm{Hom}(A, B)$. Namely, there is a distinguished map $0 : A \to B$ such that the composite with any other morphism is again 0. This allows us to define the notions of kernel and cokernel.

Definition 1.1.13. Suppose that $\phi : A \to B$ is a morphism between two objects A, B of an additive category. The *kernel* of ϕ is defined as the fiber product of

ϕ and 0 over B (if this fiber product exists) and is denoted by $\ker\phi$. We define cokernels dually.

If $\phi : A \to B$ is a morphism and $\ker\phi$ exists, we can define the *coimage* of ϕ as the cokernel of the natural map $\ker\phi \to A$ (if it exists) and will be denoted $\mathrm{coim}\phi$. Analogously, assuming the existence of $\mathrm{coker}\phi$ we define the image of ϕ as the kernel of the natural map $B \to \mathrm{coker}\phi$ (if it exists) and will be denoted $\mathrm{im}\phi$. Note that if ϕ has both an image and a coimage, it induces a natural map $\mathrm{coim}\phi \to \mathrm{im}\phi$. In the case that every morphism has both kernel and cokernel, any morphism must have both image and coimage.

Definition 1.1.14. An *abelian category* is an additive category in which every morphism ϕ has a kernel and a cokernel and the natural map $\mathrm{coim}\phi \to \mathrm{im}\phi$ is an isomorphism. An abelian category is k-linear for some field k if it is k-linear as an additive category.

Example 1.1.15. The categories of abelian groups, modules over a fixed ring or sheaves with coefficients in an abelian category over a topological space are abelian. The category of vector spaces over a field k is a k-linear abelian category.

Definition 1.1.16. Let A, B be objects of an abelian category and $\phi : A \to B$ a morphism. We say that ϕ is a *monomorphism* if the morphism $\ker\phi \to A$ is the zero map. Analogously, we say that ϕ is a *epimorphism* if the morphism $B \to \mathrm{coker}\phi$ is the zero map. We say that B is a *subobject* of A if there exists a monomorphism $B \to A$, and we say that C is a *quotient* of A if there exists an epimorphism $A \to C$. For a subobject $\phi : B \to A$, we define the quotient A/B as the cokernel of the map $\ker\phi \to B$. An object A is *simple* if for each subobject $\phi : B \to A$ the map ϕ is either the zero map or an isomorphism.

If A is an object in an abelian category, we can consider the full subcategory spanned by the objects isomorphic to a subquotient of a finite direct sum of copies of A. This subcategory will be denoted $\langle A \rangle$.

Definition 1.1.17. Let A be an object of an abelian category. A *composition series* of A is, if it exists, a descending series

$$A = F^0 \supset F^1 \supset F^2 \supset \cdots$$

of subobjects such that the quotients F^i/F^{i+1} are simple. We say that A has *finite length* if there is a finite composition series.

One can prove as in the case of abelian groups that every chain of subobjects in an object A of finite length can be refined to a composition series. Hence, if A has finite length all ascending and descending chains of subobjects in A stabilize. Moreover, all composition series of A are finite of the same length, and the finite set of the isomorphism classes of the F^i/F^{i+1} is the same up to permutation.

Definition 1.1.18. Let A, B, C be objects of an abelian category and $f : A \to B$, $g : B \to C$ morphisms. We say that

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $\mathrm{im} f = \ker g$. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

or more concretely, f is a monomorphism, g an epimorphism and $\mathrm{im} f = \ker g$.

Definition 1.1.19. Let $\mathscr{C}_1, \mathscr{C}_2$ be abelian categories. A functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is said to be *left exact* if for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have that

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is an exact sequence. Analogously F is *right exact* if

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is an exact sequence. Finally F is *exact* if it is both left and right exact.

Definition 1.1.20. An object P of an abelian category is *projective* if the functor $\mathrm{Hom}(P, -)$ is exact. Equivalently, given an epimorphism $A \to B$, each map $P \to B$ can be lifted to a map $P \to A$. Dually, one may define *injective* objects.

Definition 1.1.21. An object G of an abelian category is a *generator* if the functor $\mathrm{Hom}(G, -)$ is faithful. Equivalently, for each nonzero morphism $\phi : A \to B$ there is a morphism $G \to A$ such that the composite $G \to A \xrightarrow{\phi} B$ is again nonzero.

In the case that G is projective, the condition of being a generator can be stated as $\mathrm{Hom}(G, A) \neq 0$ for every $A \neq 0$. Indeed, the projectivity of G allows us to lift a nonzero morphism $G \to \mathrm{im}\phi$ to a morphism $G \to A$.

To conclude this section, we state a variant of the Freyd-Mitchell embedding theorem

Theorem 1.1.22. *Let $\mathscr{C}$ be an abelian category such that every object has finite length. If P is a projective generator then the functor $\mathrm{Hom}(P, -)$ induces an equivalence of $\mathscr{C}$ with the category $\mathrm{Modf}_{\mathrm{End}(P)}$ of finitely generated right $\mathrm{End}(P)$-modules.*

1.2 Sheaves

In this section we give some basic notions of sheaf theory. Our main references will be [Har77], [MLM94] and [Sza09].

Definition 1.2.1. Let X be a topological space. A *presheaf* $\mathscr{F}$ on X consists of the data

- For every open subset $U \subseteq X$, a set $\mathscr{F}(U)$.
- For every inclusion $V \subseteq U$ of open subsets of X, a map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$.

such that $\rho_{UU} : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity map and if we have $W \subseteq V \subseteq U$ three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. We refer to the elements of $\mathscr{F}(U)$ as the *sections* of the presheaf $\mathscr{F}$ over the open set U, and we call the maps ρ_{UV} *restriction maps* and for that reason we write $s|_V$ instead of $\rho_{UV}(s)$ for $s \in \mathscr{F}(U)$.

Remark 1.2.2. We can phrase the definition of a presheaf in a more fancy manner. Let X be a topological space and X_{Top} be a category whose objects are the open subsets of X, and where the only morphisms are the inclusion maps. A presheaf is just a contravariant functor from the category X_{Top} to the category Sets. This allows us to generalize the concept to presheaves on X with values in any category (such as sets, abelian groups, rings or finite dimensional vector spaces) as a contravariant functor from X_{Top} to the category of our choice. Moreover, it allows us to define a notion of morphism of presheaves as a natural transformation between these two functors. That is, if $\mathscr{F}, \mathscr{G}$ are presheaves, a morphism of presheaves $\Phi : \mathscr{F} \to \mathscr{G}$ is a collection of morphisms $\Phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that for each inclusion $V \subseteq U$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\Phi_V} & \mathscr{G}(V) \\ {\scriptstyle \rho^{\mathscr{F}}_{UV}}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho^{\mathscr{G}}_{UV}} \\ \mathscr{F}(U) & \xrightarrow{\Phi_U} & \mathscr{G}(U) \end{array}$$

Definition 1.2.3. A presheaf $\mathscr{F}$ on a topological space X is a *sheaf* if it satisfy the following two axioms:

- If U is an open set, $\{V_i\}_{i\in I}$ is an open covering of U, and if $s, t \in \mathscr{F}(U)$ are such that $s|_{V_i} = t|_{V_i}$ for all $i \in I$, then $s = t$.
- If U is an open set, $\{V_i\}_{i\in I}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each $i \in I$ with the property that for each i, j, $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each $i \in I$. (Note that the previous condition implies that s is unique).

We have two natural and relevant examples of sheaves:

Example 1.2.4. If x is a complex manifold, we define the sheaf of holomorphic functions on X to be the sheaf of rings $\mathcal{O}_X$ whose sections over some open subset $U \subseteq X$ are the complex functions holomorphic on U.

Example 1.2.5. Let S be a topological space and X another topological space. Define a sheaf $\mathscr{F}_S$ on X by taking $\mathscr{F}_S(U)$ to be the set of continuous functions $U \to S$ for all nonempty set $U \subseteq X$. If S is discrete, we say that $\mathscr{F}_S$ is the *constant sheaf* on X with value S. The name comes from the fact that over a connected open subset U the sections of $\mathscr{F}_S$ are just constant functions, that is, $\mathscr{F}_S(U) = S$.

Definition 1.2.6. Let X be a topological space. A *sheaf of $\mathcal{O}$-modules* is a sheaf of abelian groups $\mathscr{F}$ on X such that for each open $U \subseteq X$ the group $\mathscr{F}(U)$ is equipped with an $\mathcal{O}(U)$-module structure $\mathcal{O}(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ making the following diagram commute for each inclusion of open sets $V \subseteq U$

$$\begin{array}{ccc} \mathcal{O}(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{O}(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \end{array}$$

Moreover, we say that a sheaf of $\mathcal{O}$-modules $\mathscr{F}$ is *locally free* if every point of X has an open neighbourhood $U \subseteq X$ such that $\mathscr{F}|_U \simeq \mathcal{O}^n|_U$ for some $n > 0$, where $\mathcal{O}^n$ denotes the n-fold direct sum of $\mathcal{O}$. The integer n is called the *rank* of $\mathscr{F}$. We say that $\mathscr{F}$ is *free* of rank n is there is actually an isomorphism $\mathscr{F} \simeq \mathcal{O}^n$ on the whole X.

Definition 1.2.7. A sheaf $\mathscr{F}$ on a topological space X is *locally constant* if each point of X has an open neighbourhood U such that the restriction of $\mathscr{F}$ to U is isomorphic (in the category of sheaves on U) to a constant sheaf.

Let $p : Y \to X$ be a continuous mapping of locally connected topological spaces. If $U \subseteq X$ is an open set, a *section* of p over U is a continuous map $s : U \to Y$ such that $p \circ s = \mathrm{id}_U$. We can consider the set $\mathscr{F}_Y(U)$ of all sections of p over U, and for each inclusion $V \subseteq U$ we have the restriction map $\mathscr{F}_Y(U) \to \mathscr{F}_Y(V)$ by restricting sections to V. This defines a presheaf of *local sections*. In fact, we have the following proposition.

Proposition 1.2.8. *The presheaf $\mathscr{F}_Y$ is a sheaf. If $p : Y \to X$ is a cover map, then $\mathscr{F}_Y$ is locally constant. It is constant if and only if the cover is trivial.*

Proof. See [Sza09]. ■

Definition 1.2.9. Let $\mathscr{F}$ be a presheaf on a topological space X, and let x be a point of X. The *stalk* $\mathscr{F}_x$ of $\mathscr{F}$ at x is defined as the direct limit $\varinjlim_{U \ni x} \mathscr{F}(U)$ indexed over all open sets of X containing x ordered by the reverse inclusion. In more down to earth terms, we can describe the stalk as the disjoint union of the sets $\mathscr{F}(U)$ for all open neighbourhoods U of x in X modulo the following equivalence relation: $s \in \mathscr{F}(U)$ and $t \in \mathscr{F}(V)$ are equivalent if there exists an open neighbourhood $W \subseteq U \cap V$ of x with $s|_W = t|_W$.

Definition 1.2.10. Let X a topological space and $\mathscr{F}$ a presheaf of sets on X. We define a topological space $X_{\mathscr{F}}$ (together with a local homeomorphism $p_{\mathscr{F}} : X_{\mathscr{F}} \to$ X) called the *espace étalé* of $\mathscr{F}$ as follows:

- As a set it is the disjoint union of the stalks $\mathscr{F}_x$ for all $x \in X$.
- The projection map is defined as $p_{\mathscr{F}}(s) = x$ if $s \in \mathscr{F}_x$.
- For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$ we obtain a map $\bar{s} : U \to X_{\mathscr{F}}$ by $\bar{s}(x) = s_x$ its germ at x. The topology on $X_{\mathscr{F}}$ is the final topology respect to the maps $\bar{s} : U \to X_{\mathscr{F}}$ for all $s \in \mathscr{F}(U)$ and $U \subseteq X$ open sets.

Remark 1.2.11. In the case where $\mathscr{F}$ is a locally constant sheaf, the espace étalé $X_{\mathscr{F}}$ is a cover of X. To see this, let U a connected open subset of X such that $\mathscr{F}|_U$ is isomorphic to the constant sheaf defined by a set F. Then, we must have $\mathscr{F}_x = F$ for all $x \in U$ and hence $p_{\mathscr{F}}^{-1}(U)$ is isomorphic to $U \times F$, where F carries the discrete topology.

A morphism $\phi : \mathscr{F} \to \mathscr{G}$ of presheaves induces maps on the stalks $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$ for each $x \in X$, and thus a map $\Phi : X_{\mathscr{F}} \to X_{\mathscr{G}}$ compatible with the projections onto X. One can easily see that the map Φ is a morphism of spaces over X. Hence, we have defined a functor $\mathscr{F} \mapsto X_{\mathscr{F}}$ from the category of sheaves on X to that of spaces over X. On the full subcategory of locally constant sheaves it takes values in the category of covers of X, and the stalk $\mathscr{F}_x$ at a point x equals the fibre of $X_{\mathscr{F}}$ over x.

Let $p : Y \to X$ and $q : Z \to X$ be covers of a locally connected topological space X. Suppose we are given a morphism $\phi : Y \to Z$ of covers of X. We have a morphism of locally constant sheaves $\mathscr{F}_Y \to \mathscr{F}_Z$ by mapping a local section $s : U \to Y$ of $p : Y \to X$ to the precomposition $\phi \circ s$, which is easy to verify that is a local section of $q : Z \to X$. Thus, we have a functor $Y \mapsto \mathscr{F}_Y$. In fact, this functor induces an equivalence of categories between the category of covers of X and that of locally constant sheaves on X.

Theorem 1.2.12. *Let X be a connected and locally simply connected topological spaces, and let x be a point in X. The category of covers of X and that of locally constant sheaves on X are equivalent.*

Proof. The main idea is to see that the functors $\mathscr{F} \mapsto X_{\mathscr{F}}$ and $Y \mapsto \mathscr{F}_Y$ are mutually inverses. For details see [MLM94]. ∎

1.3 Schemes

In this section we will give the basic definitions and properties of schemes and analytification, following [Har77].

Definition 1.3.1. Let A be a commutative ring with unit. We can provide the set of prime ideals $\mathrm{Spec}(A)$ with the Zariski topology in which the sets

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subseteq \mathfrak{p}\}$$

are the closed sets. The sets

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\}$$

form a basis for the Zariski topology and over that basis we may define a (unique) sheaf of rings $\mathcal{O}_{\mathrm{Spec}(A)}$ satisfying $\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = A_f$ for all nonzero $f \in A$. This sheaf will be called the *structure sheaf*. An *affine scheme* is a pair $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ for some commutative ring A.

For the correspondence $A \mapsto (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ which maps a ring into its prime spectrum to be functorial, we need the notion of *locally ringed spaces*.

Definition 1.3.2. A *ringed space* is a pair $(X, \mathscr{A})$ consisting of a topological space X and a sheaf of rings $\mathscr{A}$ on X. A *morphism* of ringed spaces $(X, \mathscr{A}) \to (Y, \mathscr{B})$ is a pair $(f, f^\sharp)$ of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\sharp : \mathscr{B} \to f_*\mathscr{A}$. We say that a ringed space $(X, \mathscr{A})$ is a *locally ringed space* if for each point $x \in X$ the stalk $\mathscr{A}_x$ is a local ring. A *morphism* of locally ringed spaces $(f, f^\sharp) : (X, \mathscr{A}) \to (Y, \mathscr{B})$ is a morphism of ringed spaces such that for each $x \in X$ the induced map of local rings $f^\sharp_x : \mathscr{B}_{f(x)} \to \mathscr{A}_x$ is a local homomorphism of local rings.

Notice that every affine scheme is a locally ringed space since the stalk of the structure sheaf at each point $\mathfrak{p} \in \mathrm{Spec}(A)$ is simply $A_\mathfrak{p}$. We can thus define the category of affine schemes as the full subcategory of the category of locally ringed spaces whose objects are affine schemes. In more concrete terms, the objects are affine schemes $(A, \mathcal{O}_{\mathrm{Spec}(A)})$ and morphisms are morphisms of locally ringed spaces. Any morphism of rings $\phi : A \to B$ induces a morphism of locally ringed spaces $(f, f^\sharp) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ via $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$ and $f^\sharp$ is given by precomposition with the induced morphism of local rings $\phi_\mathfrak{p} : A_{\phi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$. Furthermore, the converse holds: if we have a morphism of locally ringed spaces $(\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ it must be induced by a ring morphism $\phi : A \to B$. In other words, the functor $A \mapsto (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is an anti-equivalence of categories between the category of commutative rings and that of affine schemes.

Definition 1.3.3. A *scheme* is a locally ringed space $(X, \mathscr{A})$ such that there is an open cover $(U_i)_{i \in I}$ such that for all i the locally ringed spaces $(U_i, \mathscr{A}|_{U_i})$ are isomorphic (as locally ringed spaces) to affine schemes. A morphism of schemes is a morphism of locally ringed spaces between to schemes. If S is a scheme, we say that X is an S-scheme if we have a morphism of schemes $X \to S$ and a morphism of S-schemes $X \to Y$ is a morphism of schemes compatible with the morphisms $X \to S$, $Y \to S$.

Notice that taking A the coordinate ring of an affine variety, the construction of the affine scheme $\mathrm{Spec}(A)$ gives all of the points of the variety (which correspond to maximal ideals) and a new set of points which are the prime ideals.

Now, let us discuss some basic properties of schemes together with the notions of open and closed subschemes, products and constructible schemes.

Definition 1.3.4. Given a scheme X and an open subset $U \subseteq X$, the ringed space given by U and $\mathcal{O}_U := (\mathcal{O}_X)|_U$ is also a scheme, the *open subscheme* associated with U. The morphism of schemes defined by the topological inclusion $U \hookrightarrow X$ and the morphism of sheaves $\mathcal{O}_X \to j_*\mathcal{O}_U$ is called an *open immersion*.

A morphism $Z \to X$ of affine schemes is a *closed immersion* if it corresponds to a quotient map $A \to A/I$ for some ideal $I \subseteq A$ in the anti-equivalence of categories with commutative rings. A morphism of schemes is a closed immersion if it is injective with closed image and its restrictions to elements of an affine open covering yield closed immersions in the affine sense.

Definition 1.3.5. A scheme X is *noetherian* if it admits a finite covering by open affine subschemes $U_i = \mathrm{Spec}(A_i)$ where A_i are noetherian rings. More generally, a scheme is locally noetherian if it is covered by the spectra of noetherian rings.

Definition 1.3.6. Given a scheme S and two morphisms $p : X \to S$, $q : Y \to S$, the contravariant functor

$$Z \mapsto \{(\phi, \psi) \in \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) : p \circ \phi = q \circ \psi\}$$

on the category of S-schemes is representable by a scheme $X \times_S Y$ over S which we will call the *fiber product of X and Y over S*.

The representability of the functor is proved in the affine scheme case by passing to the commutative rings and taking the tensor product. For the global case of schemes we must perform a glueing procedure that can be found in [Har77]. Using the fiber product, we can define the fiber of a morphism at a point and the diagonal map.

Definition 1.3.7. Given a morphism $\phi : Y \to X$ and a point $x \in X$, the *fiber* of ϕ at x is the scheme $Y_x := Y \times_X \mathrm{Spec}(\kappa(x))$ where $\kappa(x)$ is the residue field of $\mathcal{O}_{X,x}$.

Definition 1.3.8. If X is an S-scheme, the morphism induced (because of the universal property of pullbacks) by the identity $X \to X$ on both components gives us a map $\Delta : X \to X \times_S X$ called the *diagonal map*.

Although the topology of schemes is almost never Hausdorff, we can define an analogue property in terms of the diagonal. Intuitively the following property will say that the fibers of a morphism $Y \to X$ are Hausdorff.

Definition 1.3.9. A morphism of schemes $Y \to X$ is *separated* if the diagonal map $\Delta : Y \to Y \times_X Y$ is a closed immersion.

Definition 1.3.10. A morphism $\phi : Y \to X$ is *affine* if Y has a covering by affine open subsets $U_i = \mathrm{Spec}(A_i)$ such that for each i the open subscheme $\phi^{-1}(U_i)$ of X is affine as well.

Definition 1.3.11. A morphism $\phi : Y \to X$ is *locally of finite type* if X has an affine open covering by subsets $U_i = \mathrm{Spec}(A_i)$ so that $\phi^{-1}(U_i)$ has an open covering $V_{ij} = \mathrm{Spec}(B_{ij})$ with finitely generated A_i-algebras B_{ij}. We say that ϕ is of *finite type* if there is such an open covering with finitely many V_{ij} for each i.

Just as separatedness, the following property will say that the fibers of a morphism $Y \to X$ are compact.

Definition 1.3.12. A separated morphism $Y \to X$ is *proper* if it is of finite type and for every morphism $Z \to X$ the base change map $Y \times_X Z \to Z$ is a closed map.

If A is a ring and M an A-module, then there is a unique $\mathcal{O}_{\mathrm{Spec}(A)}$-module $\widetilde{M}$ satisfying $\widetilde{M}(D(f)) = M \otimes_A A_f$ over each basic open set $D(f) \subseteq \mathrm{Spec}(A)$.

Definition 1.3.13. Let $(X, \mathcal{O}_X)$ be a ringed space. A *quasi-coherent sheaf* on X is an $\mathcal{O}_X$-module $\mathscr{F}$ for which there is an open affine cover $(U_i)_{i \in I}$ of X such that the restriction of $\mathscr{F}$ of each $U_i = \mathrm{Spec}(A_i)$ is isomorphic to an $\mathcal{O}_{U_i}$-module of the form $\widetilde{M_i}$ with some A_i-module M_i. If moreover each M_i is finitely generated over A_i, then $\mathscr{F}$ is a *coherent sheaf*.

For an affine scheme $X = \mathrm{Spec}(A)$, the functor $M \mapsto \widetilde{M}$ is an equivalence of categories between the category of A-modules and that of quasi-coherent sheaves on X. Moreover, if $\phi : Y \to X$ is affine, then $\phi_* \mathcal{O}_Y$ is a sheaf of $\mathcal{O}_X$-algebras and the functor $Y \mapsto \phi_* \mathcal{O}_Y$ is an anti-equivalence of categories between affine morphisms $Y \to X$ and quasi-coherent sheaves of $\mathcal{O}_X$-algebras.

Definition 1.3.14. If M is an A-module, we say that M is *flat* if the functor $M \otimes_A -$ is exact, and we say that M is *faithfully flat* if $M \otimes_A -$ is exact and faithful. A ring map $A \to B$ is flat (resp. faithfully flat) if the B is a flat (resp. faithfully flat) A-module (where ϕ gives B the A-module structure).

Proposition 1.3.15. *Let $\phi : A \to B$ a ring map. Then ϕ is faithfully flat if and only if ϕ is flat and the induced map* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is surjective.*

This motivates the following definition.

Definition 1.3.16. A map of schemes $f : Y \to X$ is *flat* at $y \in Y$ if the induced map of rings $f^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat. We call a map of schemes *faithfully flat* if it is both flat and surjective.

The following two theorems are key results that we will need for the *spreading out*.

Theorem 1.3.17 (Generic freeness for rings)**.** *Let A be a noetherian integral domain and B a finite A-algebra. For any finitely generated B-module M, there exists a non-zero $f \in A$ such that M_f is a free A_f-module.*

Proof. See [Gro65, Lemme 6.9.2]. ∎

Theorem 1.3.18 (Generic flatness for schemes)**.** *Let $f : Y \to X$ be a finite type morphism between noetherian schemes and $\mathscr{F}$ a coherent $\mathcal{O}_Y$-module. The flat locus of $\mathscr{F}$ (that is, the points $y \in Y$ such that $\mathscr{F}_y$ is a flat $\mathcal{O}_{X,f(y)}$-module) is open.*

Proof. See [Gro65, Théorème 6.9.1]. ∎

Definition 1.3.19. Given a morphism $A \to B$ of rings, we define the B-module $\Omega^1_{B|A}$ of *relative differential forms* (also known as the module of Kahler differentials) as the free B-module generated by the symbols $(\mathrm{d}b)_{b \in B}$ modulo the submodule spanned by the elements of the form $\mathrm{d}a$, $\mathrm{d}(b_1 + b_2) - \mathrm{d}b_1 - \mathrm{d}b_2$ and $\mathrm{d}(b_1 b_2) - b_1\,\mathrm{d}b_2 - \mathrm{d}b_1 b_2$ for some $a \in A$, $b_1, b_2 \in B$.

The following intrinsic characterization of the module of differentials will allow us to generalize it to the context of schemes.

Proposition 1.3.20. *Let I be the kernel of the multiplication map $B \otimes_A B \to B$ sending $b_1 \otimes b_2$ to $b_1 b_2$. There is a canonical isomorphism of B-modules $\Omega^1_{B|A} \simeq I/I^2$.*

Definition 1.3.21. Let $\phi : Y \to X$ be a separated morphism of schemes and let $\Delta : Y \to Y \times_X Y$ the diagonal morphism. Let $\mathscr{I} \subseteq \mathcal{O}_{Y\times_X Y}$ be the kernel of the morphism of structure sheaves $\Delta^\sharp : \mathcal{O}_{Y\times_X Y} \to \Delta_*\mathcal{O}_Y$. It defines a closed subscheme $\Delta(Y) \subseteq Y \times_X Y$. The sheaf of *relative differentials* $\Omega^1_{Y|X}$ is the $\mathcal{O}_Y$-module defined by pulling back the $\mathcal{O}_{\Delta(Y)}$-module $\mathscr{I}/\mathscr{I}^2$ via the isomorphism $Y \to \Delta(Y)$.

Definition 1.3.22. A morphism of schemes $Y \to X$ is *smooth* if it is flat and the sheaf of relative differentials $\Omega^1_{Y|X}$ is free of rank equal to the relative dimension of $Y|X$. A smooth morphism with finite fibers is said to be *étale*.

Definition 1.3.23. Let X be a topological space. A subset $A \subseteq X$ is *locally closed* if it is the intersection of an open subset and a closed subset. A *constructible subset* is a subset which belongs to the smallest family of subsets such that every open set is in the family, a finite intersection of family members is in the family and the complement of family members is also in the family.

Proposition 1.3.24. *A subset $A \subseteq X$ of a topological space X is constructible if and only if it is a finite disjoint union of locally closed subsets.*

Theorem 1.3.25 (Chevalley's constructible image theorem). *Let $f : Y \to X$ be a morphism of noetherian schemes of finite type. The image of any constructible set is constructible. In particular, the image of f is constructible.*

Proof. See [Gro64, Théorème 1.8.4]. ∎

Combining the theorem of generic flatness and Chevalley's constructible image theorem we obtain the following corollary.

Corollary 1.3.26. *A flat morphism $f : Y \to X$ of finite type of noetherian schemes is open.*

The topology of schemes is awkward: it does not enjoy many properties that the geometric object we picture should have such as being Hausdorff. If X is a smooth scheme locally of finite type over $\mathbb{C}$, we can consider the set of closed points $X(\mathbb{C})$ of X and give this set a different topology called the *analytic topology*. Indeed, if we look at $X = \operatorname{Spec}(\mathbb{C}[z_1, \ldots, z_n]/(f_1, \ldots, f_r))$ the set of complex points $X(\mathbb{C})$ is simply the subset of $\mathbb{C}^n$ defined as the locus of zeros of the polynomials $f_1, \ldots, f_r$. Since X is smooth, there is no point p on $X(\mathbb{C})$ such that every partial derivative $\partial_i f(p)$ vanish at the same time. Consider the open sets $U_i = \{p \in X : \partial_i f(p) \neq 0\}$. For any point in U_i consider the projection in all of the other $n-1$ variables. By the inverse function theorem and the holomorphic version of the implicit function theorem (cf. [GH94, p. 18,19]) we get holomorphic charts U_i with holomorphic transition functions. This defines a structure of complex manifold on the set of closed points $X(\mathbb{C})$. If X is a smooth $\mathbb{C}$-scheme of finite type, it is locally of that form, and thus we can define the analytic topology on $X(\mathbb{C})$ locally and the sheaf of holomorphic functions $\mathcal{O}_X^{\mathrm{an}}$ also locally. We will denote X^{an} the set of closed points with the analytic topology. One may verify that the natural map $X^{\mathrm{an}} \to X$ is continuous (that is, the Zariski topology is coarser than the analytic topology).

Furthermore, one may prove that $(X, \mathcal{O}_X) \mapsto (X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ defines a functor from the category of smooth $\mathbb{C}$-schemes of finite type to the category of complex manifolds. We will call this functor the *analytification functor*. This functor preserves many properties of functions. For example, a morphism $f : Y \to X$ of smooth $\mathbb{C}$-schemes of finite type is flat if and only if the analytification $f^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is flat. The same holds changing flat for étale, smooth or proper (see [Gro03, Exposé XII, Proposition 3.1, 3.2]). To each sheaf $\mathscr{F}$ on X we may consider the pullback sheaf $\mathscr{F}^{\mathrm{an}}$ on X^{an} via the canonical map $X^{\mathrm{an}} \to X$. The main theorem of analytification is the following.

Theorem 1.3.27 (GAGA)**.** *Let X be a smooth, proper $\mathbb{C}$-scheme of finite type. The analytification functor $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}}$ from the category of coherent sheaves on X to the category of coherent sheaves on X^{an} is an equivalence of categories.*

Proof. See [Gro03, Exposé XII, Théorème 4.4]. ∎

Chapter 2

The Geometry of Connections

This chapter focuses on the geometrical aspects of connections. We begin by treating holomorphic connections from an elementary point of view with many computations that help to illustrate how they work. The Ehresmann viewpoint of connections of characterizing them by the horizontal bundle will allow us, together with Frobenius integrability theorem, to prove that the sheaf of horizontal sections contains all of the relevant information of an integrable connection. Finally, we focus on the monodromy of connections and treat the case of connections with regular singularities. The most remarkable result of the Chapter is the Riemann-Hilbert correspondence in the context of connections with regular singularities.

2.1 Holomorphic connections

As usual, $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on a complex manifold X. We can define a holomorphic connection in the following manner.

Definition 2.1.1. Let X be a complex manifold. A *holomorphic connection* on X is a pair $(\mathscr{E}, \nabla)$, where $\mathscr{E}$ is a locally free sheaf on X and $\nabla : \mathscr{E} \to \mathscr{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ is a morphism of sheaves of $\mathbb{C}$-vector spaces satisfying the Leibniz rule

$$\nabla(fs) = s \otimes \mathrm{d}f + f\nabla(s)$$

for all $U \subseteq X$, $f \in \mathcal{O}(U)$ and $s \in \mathscr{E}(U)$. We call ∇ the *connection map*.

Let us do some local computations, following [Con15]. Suppose that X is a complex manifold and $(\mathscr{E}, \nabla)$ is a connection on X. Take an open set $U \subseteq X$

sufficiently small so that $\mathscr{E}|_U$ is free with a basis of local sections $\{e_1, \ldots, e_r\}$ and there are local coordinates $\{z_1, \ldots, z_n\}$ for U. Therefore, we can write

$$\nabla(e_j) = \sum_{i,k} \Gamma_{ij}^k e_k \otimes \mathrm{d}z_i$$

for certain holomorphic functions Γ_{ij}^k over U, called the *Christoffel symbols* of ∇. Furthermore, if σ is a local section of $\mathscr{E}|_U$, it is of the form $\sigma = \sum_{j=1}^{n} \sigma_j e_j$ for certain holomorphic functions σ_j over U, and then we have

$$\nabla(\sigma) = \sum_{i,k} \left(\frac{\partial \sigma_k}{\partial z_i} + \sum_{j=1}^{n} \Gamma_{ij}^k \sigma_j \right) e_k \otimes \mathrm{d}z_i.$$

We say that a local section σ over U is a *horizontal section* if $\nabla(\sigma) = 0$. Notice that this is equivalent to σ satisfying the system of partial differential equations

$$\frac{\partial \sigma_k}{\partial z_i} + \sum_{j=1}^{n} \Gamma_{ij}^k \sigma_j = 0.$$

So, we can think of a connection on a complex manifold as a first order partial differential equation and its horizontal sections will be the solutions.

We can also think of a connection as a *covariant derivative*, that is, a way of differentiating sections along vector fields. This is done by considering the composition

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\nabla"] \arrow[dr, "\nabla_\theta"'] & & \mathscr{E} \otimes_{\mathcal{O}_X} \Omega_X^1 \arrow[dl, "c_\theta"] \\ & \mathscr{E} & \end{tikzcd}$$

where $c_\theta : \mathscr{E} \otimes_{\mathcal{O}_X} \Omega_X^1 \to \mathscr{E}$ is the contraction $\sigma \otimes \omega \mapsto \omega(\theta)\sigma$ with the vector field θ. One may verify that ∇_θ satisfies

$$\nabla_\theta(f\sigma) = \theta(f)\sigma + f\nabla_\theta(\sigma)$$

for local sections σ of $\mathscr{E}$ and f of $\mathcal{O}_X$. If we consider the covariant derivative along the coordinate vector fields $\partial/\partial z_i$, we obtain

$$\nabla_{\partial/\partial z_i}(e_j) = \sum_{k=1}^{r} \Gamma_{ij}^{k} e_k.$$

We say that a connection $(\mathscr{E}, \nabla)$ on X is *integrable* or *flat* if for any pair of vector fields θ_1, θ_2 we have

$$\nabla_{[\theta_1,\theta_2]} = [\nabla_{\theta_1}, \nabla_{\theta_2}].$$

Notice that this condition is local. Consider local coordinates as before. In view of the Leibniz rule, to check that ∇ is integrable it is enough to verify that condition for the coordinate vector fields $\theta_1 = \partial/\partial z_i$ and $\theta_2 = \partial/\partial z_j$. Since the coordinate fields commute, this boils down to computing

$$[\nabla_{\partial/\partial z_i} e_k, \nabla_{\partial/\partial z_j} e_k] = \nabla_{\partial/\partial z_i} \nabla_{\partial/\partial z_j} e_k - \nabla_{\partial/\partial z_j} \nabla_{\partial/\partial z_i} e_k.$$

By a straightforward computation, this is equivalent to

$$[\nabla_{\partial/\partial z_i} e_k, \nabla_{\partial/\partial z_j} e_k] = \sum_{\ell=1}^{r} \left(\sum_{s=1}^{n} (\Gamma_{jk}^{s} \Gamma_{is}^{\ell} - \Gamma_{ik}^{s} \Gamma_{js}^{\ell}) + \left(\frac{\partial \Gamma_{jk}^{\ell}}{\partial z_i} - \frac{\partial \Gamma_{ik}^{\ell}}{\partial z_j} \right) \right) e_\ell.$$

If we define R_{ijk}^{ℓ} as the coefficient of e_ℓ in the previous formula, then the integrability of ∇ is equivalent to the vanishing of these coefficients R_{ijk}^{ℓ}, which is a system of partial differential equations. This is reminiscent of the situation of the Levi-Civita connection in a Riemannian manifold, where R_{ijk}^{ℓ} are the coefficients of the curvature tensor, which motivates the name *flat* connection.

2.2 The Horizontal Bundle and Integrability

In this section we prove a correspondence between local systems and vector bundles with integrable connections on a complex manifold. This essentially says that we can reconstruct a vector bundle together with an integrable connection from its horizontal sections. We further explore this result and extend it to integrable connections on complex manifolds with regular singularities along a divisor with normal crossings, following [Del70] and [BGK+87].

Let M an m-dimensional real manifold and $1 \leq r \leq m$ a natural number.

Recall that an r-dimensional *distribution* $\mathscr{D}$ on M is a choice of an r-dimensional subspace $\mathscr{D}(p)$ of T_pM for each $p \in M$. We say that a distribution $\mathscr{D}$ is *smooth* if for each $p \in M$ there is a neighborhood U of p and there are r vector fields $X_1, \dots, X_r$ of class $\mathscr{C}^\infty$ on U which span $\mathscr{D}$ at each point of U. This is equivalent to saying that a smooth distribution is a vector subbundle of the tangent bundle of M of rank r.

A vector field X on M is said to belong to the distribution $\mathscr{D}$ if $X_p \in \mathscr{D}(p)$ for each $p \in M$. Finally, a smooth distribution $\mathscr{D}$ is called *involutive* if $[X, Y] \in \mathscr{D}$ whenever X, Y are smooth vector fields belonging to $\mathscr{D}$.

Given a manifold M together with a distribution $\mathscr{D}$ on M, a submanifold $N \subseteq M$ is an *integral manifold of* $\mathscr{D}$ if $T_pN = \mathscr{D}(p)$ for each $p \in N$. If an integral manifold exists through each point of M, $\mathscr{D}$ is said to be completely integrable. The following theorem explains the relationship between these concepts.

Theorem 2.2.1 (Frobenius)**.** *A smooth distribution is completely integrable if and only if it is involutive.*

Proof. See [War83] for a proof. ∎

This theorem still holds when extended to the complex scenario. Indeed, on a complex manifold M one may define a *holomorphic distribution* of rank r as a holomorphic vector subbundle of rank r of the holomorphic tangent bundle. Now, integrability in the holomorphic sense will mean that there is a complex submanifold such that its holomorphic tangent bundle is the given holomorphic distribution. The holomorphic Frobenius theorem can be proved by a reduction to the real differentiable case, the details can be found in [Voi07, p. 51].

Let X be a complex manifold and $\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_M$ an integrable connection. Recall that we have an equivalence of categories between the category of locally free sheaves on a complex manifold X and that of vector bundles over X. That is, if we have a locally free sheaf $\mathscr{E}$ on X we can associate to it a vector bundle $V_{\mathscr{E}} \to X$ such that the sheaf of sections of that bundle is $\mathscr{E}$. Choose a basis of local sections $\{e_1, \dots, e_r\}$ on $\mathscr{E}|_U$ and consider $\{v_1, \dots, v_r\}$ coordinates on $V_{\mathscr{E}}|_U$ over U that are dual to the chosen basis of $\mathscr{E}|_U$. If $\{z_1, \dots, z_n\}$ are local coordinates for U, the tangent bundle $TV_{\mathscr{E}}$ is free over $V_{\mathscr{E}}|_U$ on the basis of the

$\partial/\partial z_i$ and $\partial/\partial v_j$, for $i = 1, \dots, n$, $j = 1, \dots, r$. Consider the vector fields

$$X_i = \frac{\partial}{\partial z_i} - \sum_{k=1}^{r} \left(\sum_{j=1}^{r} v_j \Gamma_{ij}^{\ k}(z) \right) \frac{\partial}{\partial v_k}$$

on $V_{\mathscr{E}}|_U$, where $\Gamma_{ij}^{\ k}$ are the Christoffel symbols of the connection $(\mathscr{E}, \nabla)$ over U. A straightforward computation shows that

$$[X_i, X_j] = -\sum_{k,\ell=1}^{r} R_{ijk}^{\ell} v_k \frac{\partial}{\partial v_\ell}$$

and thus the vanishing of the Lie brackets of the vector fields $X_1, \dots, X_n$ on $V_{\mathscr{E}}|_U$ is precisely the integrability condition. Furthermore, notice that the only term of $\{\partial/\partial z_1, \dots, \partial/\partial z_n\}$ that appears in X_i is $\partial/\partial z_i$ and so these vector fields $\{X_1, \dots, X_n\}$ freely generate a vector bundle W_∇ over $V_{\mathscr{E}}|_U$ that is a subbundle of $TV_{\mathscr{E}}$ over $V_{\mathscr{E}}|_U$. This will be called the *horizontal bundle of the connection*. Since $\partial/\partial z_k$ does not appear in the formula for $[X_i, X_j]$, the only way for $[X_i, X_j]$ to lie in W is if $[X_i, X_j] = 0$. Hence, we have proved that the integrability of $(\mathscr{E}, \nabla)$ over U is equivalent for a subbundle W_∇ of $TV_{\mathscr{E}}|_U$ to be stable under the bracket on vector fields.

It is natural to consider then the relationship between the horizontal sections of the connection $(\mathscr{E}, \nabla)$ over U and the vector bundle W_∇ over $V_{\mathscr{E}}|_U$. Let $\sigma : U \to V_{\mathscr{E}}|_U$ be a section in $\mathscr{E}(U)$. The differential $\mathrm{d}\sigma : TX|_U \to TV_{\mathscr{E}}|_U$ is characterized by

$$\frac{\partial}{\partial z_i} \mapsto \frac{\partial}{\partial z_i} + \sum_{k=1}^{r} \frac{\partial \sigma_k}{\partial z_i} \frac{\partial}{\partial v_k}$$

where $\sigma_k = v_k \circ \sigma$. This implies that the only possible way for $\mathrm{d}\sigma$ to factor through the subbundle W_∇ is if we can write

$$\mathrm{d}\sigma\left(\frac{\partial}{\partial z_i}\right) = \sum_{j=1}^{n} a_{ij} X_j$$

which can only occur if $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = 1$ because the the only element of $\{\partial/\partial z_1, \dots, \partial/\partial z_n\}$ appearing in $\mathrm{d}\sigma(\partial/\partial z_i)$ is $\partial/\partial z_i$. In conclusion, $\mathrm{d}\sigma$ factors through W_∇ if and only if $\mathrm{d}\sigma(\partial/\partial z_i) = X_i$, or in other words

$$\frac{\partial \sigma_k}{\partial z_i} = -\sum_{j=1}^{n} \sigma_j \Gamma_{ij}^{\ k}$$

which is precisely the differential equation that a horizontal section σ must verify. The Ehresmann viewpoint of the connection is precisely understanding the connection by its horizontal bundle. We are now able to prove the Riemann-Hilbert correspondence in our first setting:

Theorem 2.2.2 (Riemann-Hilbert I)**.** *Let X be a complex manifold. There is a bijective correspondence between isomorphism classes of holomorphic vector bundles on X equipped with an integrable connection and isomorphism classes of local systems of complex vector spaces on X.*

Proof. The correspondence is defined by taking a holomorphic vector bundle with integrable connection $(\mathscr{E}, \nabla)$ to the sheaf of horizontal sections $\mathscr{E}^{\nabla}$, and the inverse functor will be given by sending a local system Λ of complex vector spaces to the sheaf $\mathscr{E}_\Lambda = \mathcal{O}_X \otimes_{\mathbb{C}_X} \Lambda$ provided with the connection $\nabla_\Lambda = \mathrm{d} \otimes 1$. It is clear that the horizontal sections of $\mathscr{E}_\Lambda$ are canonically isomorphic to Λ. On the other hand, to see that the map $\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathscr{E}^{\nabla} \to \mathscr{E}$ is an isomorphism we must prove that it induces isomorphisms $\mathscr{E}^{\nabla}_{x_0} \to \mathscr{E}(x_0)$ on fibers over the points $x_0 \in X$. But this means that for each point $e_0 \in \mathscr{E}(x_0)$ of the fiber of $\mathscr{E}$ over x_0, there is a unique holomorphic flat section σ near x_0 such that $\sigma(x_0) = e_0$, and in turn this means that the system of partial differential equations $\nabla(\sigma) = 0$ with initial condition $\sigma(x_0) = e_0$ has unique solution.

Since $(\mathscr{E}, \nabla)$ is an integrable connection, the bundle W_∇ constructed in the previous discussion is integrable and hence Frobenius integrability theorem tells us that there exist an integral manifold M through $e_0 \in \mathscr{E}(x_0)$ in a neighborhood of e_0 so that $T_e M = W_\nabla(e)$ for every $e \in M$. The differential of the projection $M \to U$ at e_0 sends the basis $\{X_1(e_0), \ldots, X_n(e_0)\}$ of $W_\nabla(e_0) = T_{e_0}M$ to the basis $\{\frac{\partial}{\partial z_1}(x_0), \ldots, \frac{\partial}{\partial z_n}(x_0)\}$ of $T_{x_0}U$ and this implies that the projection is a local isomorphism. After shrinking U around x_0 if necessary, the analytic inverse of the projection $M \to U$ near x_0 provides a local section $\sigma : U \to M$ satisfying $\sigma(x_0) = e_0$, and the differential of the composite $U \to M \to V_{\mathscr{E}}|_U$ factors through W_∇ since $T_e M = W_\nabla(e)$ for all $e \in M$. The previous discussion allows us to conclude that this is the horizontal section that we are looking for. The uniqueness is handled by restricting the section σ to curves and applying the uniqueness theorem for ordinary differential equations. ∎

This correspondence gives us a complete understanding of holomorphic connections in terms of topological data of the complex manifold. However, it will be fruitful further on to allow the connection to have singularities. If these singularities are mild enough, we will still have a Riemann-Hilbert correspondence. We will come back to this in Section 2.4.

To conclude this section, let us apply the discussion on horizontal sections to the pullback of connections. If we have a holomorphic map $f : Y \to X$ between complex manifolds X, Y and a holomorphic connection $(\mathscr{E}, \nabla)$ on X, we have by functoriality a map

$$f^*\nabla : f^*\mathscr{E} \to f^*\left(\mathscr{E} \otimes_{\mathcal{O}_X} \Omega^1_X\right) \simeq f^*\mathscr{E} \otimes_{\mathcal{O}_Y} f^*\Omega^1_X \to f^*\mathscr{E} \otimes_{\mathcal{O}_Y} \Omega^1_Y$$

which satisfies the Leibniz rule, as one can easily verify. This defines a pullback connection $f^*(\mathscr{E}, \nabla) = (f^*\mathscr{E}, f^*\nabla)$. Let $V_{\mathscr{E}} \to X$ be the holomorphic vector bundle associated to $\mathscr{E}$. By the equivalence of categories, the pullback of vector bundles $f^*V_{\mathscr{E}} \to Y$ must be the vector bundle associated to $f^*\mathscr{E}$. Furthermore, by the expressions in local coordinates obtained, we can see that the vector bundle $W_{f^*\nabla}$ must be the pullback of the vector bundle W_∇. Hence, the following commutative square

$$\begin{tikzcd} W_{f^*\nabla} \arrow[r, "p"] \arrow[d] & W_\nabla \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

is cartesian and this implies that if f is a finite covering map, then so is p. Restricting to open sets $U \subseteq X$, $V \subseteq Y$ such that $f : V \to U$ is a diffeomorphism and $p : W_{f^*\nabla}|_V \to W_\nabla|_U$ is n-to-1 then the sections of the projection $W_{f^*\nabla}|_V \to V$ are in n-to-1 correspondence with the sections of $W_\nabla|_U \to U$. So we have an isomorphism of sheaves $(f^*\mathscr{E})^{f^*\nabla} \simeq \left(\mathscr{E}^\nabla\right)^{\oplus n}$ in the case that $f : Y \to X$ is a finite étale cover.

2.3 The Monodromy Action

Suppose that X is a topological space which is connected and locally connected. For a point $x \in X$, and two loops $\gamma_1, \gamma_2 : [0,1] \to X$ based at x, we shall denote its composition by $(\gamma_1 \cdot \gamma_2)(t) = \gamma_2(2t)$ if $0 \le t \le 1/2$ and $\gamma_1(2t)$ if $1/2 \le t \le 1$ (this convention differs from some textbooks). Fix a point $x \in X$ and consider a covering map $p : \widetilde{X} \to X$. The *monodromy action* is the action of $\pi_1(X, x)$ on the fiber $p^{-1}(x)$ defined by

$$\pi_1(X,x) \times p^{-1}(x) \to p^{-1}(x), \quad ([\gamma], e) \mapsto \widetilde{\gamma}(1)$$

where $\gamma : [0,1] \to X$ is a loop on X and $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ is the unique lifting of γ starting at e. The advantage of using the above convention for composition of loops is that the group operation of the fundamental group is opposite and the monodromy action is actually a left group action.

Given a locally constant sheaf of sets $\mathscr{F}$ over X and a point $x \in X$, we may consider the monodromy action on the stalk $\mathscr{F}_x$ which is simply the monodromy associated to the fiber of x via the *éspace étalé* $X_{\mathscr{F}} \to X$ of the sheaf. However, we may describe the action on $\mathscr{F}_x$ without appealing to this covering, just in terms of the sheaf. Indeed, if $\gamma : [0,1] \to X$ is a path, consider the pullback sheaf $\gamma^*\mathscr{F}$ on $[0,1]$. Since this sheaf is locally constant and $[0,1]$ is simply connected, we conclude that $\gamma^*\mathscr{F}$ is constant. This gives a canonical identification between $\gamma^*\mathscr{F}_0 = \mathscr{F}_{\gamma(0)}$ and $\gamma^*\mathscr{F}_1 = \mathscr{F}_{\gamma(1)}$, which moreover is invariant under homotopies. By taking γ a loop based on $x \in X$, we obtain an endomorphism of $\mathscr{F}_x$ and this describes the monodromy action.

We have thus described a functor $\mathscr{F} \mapsto \mathscr{F}_x$ from the category of locally constant sheaves of sets to that of left $\pi_1(X,x)$-sets. If our topological space X is locally simply connected, this functor is an equivalence of categories. Furthermore, if we add an extra structure to $\mathscr{F}$ of sheaf of complex vector spaces, the stalk $\mathscr{F}_x$ is naturally a complex vector space and the monodromy action of $\pi_1(X,x)$ on $\mathscr{F}_x$ is compatible with the vector space structure. Namely, we have the following result.

Theorem 2.3.1. *Let X be a connected and locally simply connected topological space, and x a point in X. The category of complex local systems on X is equivalent to the category of finite dimensional left representations of $\pi_1(X,x)$.*

Proof. For a proof, see [Sza09, Corollary 2.6.2]. ∎

This theorem, together with our first form of the Riemann-Hilbert correspondence, tells us that the category of holomorphic flat connections on a complex manifold X is equivalent to the category of finite dimensional left represetations of $\pi_1(X,x)$ via the monodromy representation of the sheaf of horizontal sections of the connection. The monodromy of the sheaf of horizontal sections of a connection will simply be called the monodromy of the connection. Hence, the isomorphism class of a holomorphic flat connection is determined by its monodromy.

Example 2.3.2. Consider the punctured open disc $D_R = \{z \in \mathbb{C} : 0 < |z| < R\}$ centered at 0 and of radius $R > 1$. If we set 1 as the base point for the fundamental group of D_R, we know that $\gamma : t \mapsto e^{2\pi \mathbf{i} t}$ is a generator for $\pi_1(D_R, 1)$. If f is a holomorphic function on D_R that extends meromorphically into 0, the solutions of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}z} = f(z)y(z)$$

amount to the horizontal sections of the holomorphic connection with simple poles on the origin defined by $\nabla(s) = \mathrm{d}s - fs$. The monodromy representation of this connection will be defined by the image of the loop γ which will be a real number m. To compute this number we must perform analytic continuation of germs of solutions of the differential equation along γ and hence via standard techniques of complex analysis we find $m = \exp(2\pi \mathbf{i} \mathrm{Res}(f, 0))$. If we take $f(z) = \mu z^{-1}$ where $\mu \in \mathbb{C}$ is such that $\exp(2\pi \mathbf{i} \mu) = m$ we obtain a differential equation with only a simple pole at the origin and prescribed monodromy m.

More generally, an n-dimensional monodromy representation of D_R is determined by a matrix $M \in \mathrm{GL}_n(\mathbb{C})$ which is the image of the loop γ. In order to find a holomorphic connection on D_R except for a single pole at the origin, reasoning like in the previous paragraph, we consider the connection $\nabla(s) = \mathrm{d}s - Az^{-1}s$ where $A \in \mathrm{GL}_n(\mathbb{C})$ is such that $\exp 2\pi \mathbf{i} A = M$ (such a matrix A exists by Jordan normal form considerations). This gives a differential equation $\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{A}{z} y(z)$ with prescribed monodromy M.

2.4 Riemann-Hilbert Correspondence

Now that we understand the monodromy of connections in the holomorphic case, let us exploit this fact to extend the Riemann-Hilbert correspondence to the case with *mild* singularities. The following discussion intends to give a precise meaning to this.

Definition 2.4.1. Let X be a complex manifold and $D \subseteq X$ a hypersurface (that means that D is locally defined by the vanishing of single a holomorphic equation). We say that D is a *normal crossing divisor* if for each $x \in X$ there exist local coordinates $z_1, \dots, z_n$ around x such that D is defined by the equation $z_1 \cdot \ldots \cdot z_r = 0$ for $r \leq n$.

If X is a complex manifold with $D \subseteq X$ a normal crossing divisors, we may consider holomorphic k-forms on $X^* := X \smallsetminus D$ with logarithmic singularities along D. This sheaf will be denoted $\Omega^k_X(\log D)$. Given local coordinates $z_1, \dots, z_n$ on an open set $U \subseteq X$ such that $U \cap D = \{z_1 \cdot \ldots \cdot z_r = 0\}$, we can give explicitly a basis for $\Omega^k_X(\log D)\big|_U$ as an $\mathcal{O}_U$-module by

$$\frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{\mathrm{d}z_{i_\ell}}{z_{i_\ell}} \wedge \mathrm{d}z_{j_1} \wedge \ldots \wedge \mathrm{d}z_{j_m}$$

for $i_s \leq r$, $j_s > r$ and $\ell + m = r$.

We are now able to define the concept of connections with regular singularities.

Definition 2.4.2. Let X be a complex manifold, $D \subseteq X$ a divisor with normal crossings. Let $(\mathscr{E}^*, \nabla^*)$ be a connection on X^*. We say that this connection has *regular singularities* along D if there exists a locally free sheaf $\mathscr{E}$ on X such that the map $\nabla^* : \mathscr{E}^* \to \mathscr{E}^* \otimes_{\mathcal{O}_{X^*}} \Omega^1_{X^*}$ extends to a map $\nabla : \mathscr{E} \to \mathscr{E} \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$.

In more concrete terms, this condition states that for any singular point $p \in D$ there exists a punctured neighborhood of p such that the connection matrix has entries that are meromorphic at p with at worst a simple pole. A key fact about this kind of connections with regular singularities is the following extension property.

Proposition 2.4.3. *Let X be a complex manifold and $D \subseteq X$ be a divisor with normal crossings. Suppose that $(\mathscr{E}^*, \nabla^*)$ is a holomorphic connection on $X^* = X \smallsetminus D$. There exists an integrable connection $(\mathscr{E}, \nabla)$ on X with regular singularities along D, and the flat sections of $\mathscr{E}^*$ on puncured neighborhoods of any $p \in D$ are meromorphic at p.*

Proof. The idea is to use the local monodromy of the connection to construct an extension of $\mathscr{E}^*$ locally along D in such a way that these local extensions are compatible and patch together to give the global extension $\mathscr{E}$. It is pertinent to note that in the one-dimensional case, D consists of isolated points and in that case the extension problem is local around each point of D and there is no patching problem. However, in several variables we can slide along D from one point to another so there is a patching problem.

To construct the local extension, take local coordinates $z_1, \dots, z_n$ on an open U such that $U \cap D = \{z_1 \cdots z_r = 0\}$ for some $r \leq n$ and let x_0 be the point where the divisors cross (i.e., the zero of this coordinate system). In a small coordinate polydisc V around x_0 defined by $|z_j| < \varepsilon$ for $j = 1, \dots, n$ the open manifold $U \cap V$ is the product of r punctured discs $0 < |z_j| < \varepsilon$, $j = 1, \dots, r$ and of $n - r$ discs $|z_j| < \varepsilon$ for $j = r+1, \dots, n$. Because of the Riemann-Hilbert correspondence I, the restriction of $(\mathscr{E}^*, \nabla^*)$ to $V \cap U$ corresponds to a representation ρ of $\pi_1(V \cap U, *)$ in a finite dimensional complex vector space L. Since the fundamental group of $V \cap U$ is the free abelian group on the r generators given by the loops γ_i defined by turning once counterclockwise around the divisor $z_i = 0$, the monodromy representation ρ is determined by the r commuting automorphisms $\rho([\gamma_i])$ of L. Just as in Example 2.3.2, a Jordan normal form consideration shows that there are unique mutually commutative endomorphisms B_j of the representation space L such that $\exp(2\pi \mathbf{i} B_j) = \rho([\gamma_j])$ for each $j = 1, \dots, r$ and such that the eigenvalues of each B_j have real parts in the strip $-1 \leq \mathrm{Re}(\lambda) \leq 0$. The condition on the eigenvalues of B_j is simply to make a canonical choice for every matrix, since the monodromy is already determined by the fact that $\exp(2\pi \mathbf{i} B_j) = \rho([\gamma_j])$.

Now, we shall extend $(\mathscr{E}^*, \nabla^*)$ to the whole polydisc V. Consider the sheaf $L \otimes_{\mathbb{C}} \mathcal{O}_V$ and the connection with logarithmic poles by defining

$$\nabla(\ell \otimes f) = f\left(-\sum_{i=1}^{r} B_i \ell \otimes \frac{\mathrm{d}x_i}{x_i}\right) + \ell \otimes \mathrm{d}f.$$

This extension is called the *canonical extension*. To see that this connection extends $(\mathscr{E}^*, \nabla^*)$ we only need to check that its restriction to $V \cap U$ has the same monodromy, but this is evident by looking at the connection matrix and the computation of monodromy with respect to the connection matrix seen in Example 2.3.2. The advantage of constructing the extensions in this manner is that it can be done along any point of D and the data will turn out to patch well. The key for this glueing will be that the horizontal sections of a connection with regular singularities satisfy a condition of *moderate growth*. For the proof of this, see the wonderful exposition of Deligne [Del70, Théorème 1.19, Proposition 5.7]. ■

Combining this last proposition with the Riemann-Hilbert correspondence I, we obtain the following.

Theorem 2.4.4 (Riemann-Hilbert II). *Let X be a complex manifold and $D \subseteq X$ be a divisor with normal crossings. The category of holomorphic connections on X*

with regular singularities along D is equivalent to the category of local systems on $X^* = X \setminus D$.

The Riemann-Hilbert correspondence II allows us to conclude that any connection with regular singularities along a divisor with normal crossings is uniquely determined (up to isomorphism) by its monodromy. As a consequence of this, we have a better understanding of those connections which have finite monodromy.

Proposition 2.4.5. *Let X be a complex manifold and let $D \subseteq X$ be a divisor with normal crossings. If $(\mathscr{E}, \nabla)$ is a connection with regular singularities along D has finite monodromy group, then there is a holomorphic finite covering map $f : Y \to X$ (or in other words, a finite étale cover) such that $f^*(\mathscr{E}, \nabla)$ is trivial.*

Proof. Consider the restriction $(\mathscr{E}^*, \nabla^*)$ of $(\mathscr{E}, \nabla)$ to $X^* = X \setminus D$. Because of the Riemann-Hilbert correspondence II, the monodromy of the restriction $(\mathscr{E}^*, \nabla^*)$ on X^* characterizes the connection $(\mathscr{E}, \nabla)$. Let $p : \widetilde{X}_\nabla \to X$ be the *espace étalé* of the sheaf of flat sections of the restriction and fix $x_0 \in X^*$ and also fix a point $\widetilde{x_0} \in p^{-1}(x_0)$. Let $H \subseteq \pi_1(X^*, x_0)$ be the stabilizer of $\widetilde{x_0}$ in the monodromy action. Since the monodromy group is finite, the orbit of $\widetilde{x_0}$ is finite and therefore H is a subgroup of finite index in $\pi_1(X^*, x_0)$. The Galois correspondence between subgroups of the fundamental group and covering maps gives us a finite covering map $f : Y \to X^*$. The espace étalé of the pullback connection $f^*(\mathscr{E}^*, \nabla^*)$ is the pullback of the map $p : \widetilde{X}_\nabla \to X$ via $f : Y \to X^*$. Since Y is the covering corresponding to the stabilizer H, the monodromy action of the pullback connection on Y must be trivial. As a connection is determined by its monodromy, this implies that the connection itself must be trivial. ∎

Remark 2.4.6. Given the open, connected set $X = \mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \ldots, x_n\}$ we may consider ordinary differential equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}z} = A(z)y(z)$ where $A(z)$ is a matrix whose entries are holomorphic functions on X that have at most simple poles in $\{x_1, \ldots, x_n\}$, namely, the connection associated to this differential equation has regular singularities (this kind of equation is called *Fuchsian*). The question whether every representation $\rho : \pi_1(D, *) \to \mathrm{GL}_n(\mathbb{C})$ arises as the monodromy representation of a Fuchsian system of linear equations is the renowned *Hilbert's twenty-first problem*. One could ask the same question without casting any restriction on the poles of the entries of $A(z)$; however there are many simple examples such as the differential equation over $\mathbb{C} = \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$ with

$$\frac{\mathrm{d}y}{\mathrm{d}z} = p(z)y(z)$$

for $p \in \mathbb{C}[z]$ a polynomial. The solution $y(z) = \exp\left(\int_0^z p(\xi)\,\mathrm{d}\xi\right)$ of this equation has trivial monodromy, for it is entire. But as differential equations, varying p we obtain many different, non-isomorphic equations which proves that the differential equation is not determined by its monodromy if we do not ask for regular singularities. The only solution that has a regular singularity at the point of infinity is that with $p = 0$. On the other hand, the problem is settled in the affirmative in the case of Fuchsian equations, and was proved several times, for instance by Birkhoff, Plemelj and Röhrl, and also by Deligne using the Riemann-Hilbert correspondence. For a historical account and exposition on Deligne's work on Hilbert's twenty-first problem see the survey of Katz [Kat76].

Chapter 3

The Tannakian Formalism

In this chapter we develop the basic theory of Tannakian duality. This theory concentrates on the problem of understanding the interplay between a group and its category of representations, generalizing classical theorems of Pontryajin and Tannaka-Krein in the context of compact topological groups. However, it was not until Grothendieck and his student Saavedra-Rivano that the categorical ideas came into play. The main theorem states that one may reconstruct an affine group scheme from the tensor structure of the category of its finite dimensional representations by looking at the automorphisms of the fiber functor, in a way reminiscent of Grothendieck's Galois theory. The theory of Tannakian Fundamental groups will allow us to define differential Galois theory in the context of connections. These definitions fit the ideas of the relationship between Galois groups and Fundamental groups. Finally, we relate the monodromy group of an integrable connection with regular singularities to the differential Galois group. We follow mostly the expositions of Deligne [Del90] and [Sza09].

3.1 Algebraic Group Schemes

In this section we shall study algebraic group schemes and their intimate relationship with Hopf algebras, and we develop the basic theory that we will need following [Wat79].

Definition 3.1.1. A *group scheme* over a field k is a group object in the category of k-schemes. More explicitly, it is a k-scheme G provided with k-morphisms $m : G \times G \to G$, $e : \mathrm{Spec}(k) \to G$ and $i : G \to G$ called multiplication, unit and inverse respectively, which are subject to the following commutative diagrams

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "1 \times m"] \arrow[d, "m \times 1"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "1 \times e"] \arrow[d, "e \times 1"'] \arrow[dr, "1"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "1 \times i"] \arrow[d, "i \times 1"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G.
\end{tikzcd}
$$

If G is a group scheme the set $G(S) = \mathrm{Hom}_k(S, G)$ of k-morphisms has group structure for any k-scheme S. Restricting this functor to the full subcategory of affine k-schemes we obtain a covariant functor $R \mapsto \mathrm{Hom}_k(\mathrm{Spec}(R), G)$. In the case that G is itself affine, $G = \mathrm{Spec}(A)$ for a certain k-algebra A, and the antiequivalence of categories between commutative rings and affine schemes implies that $\mathrm{Hom}_k(\mathrm{Spec}(R), \mathrm{Spec}(A)) = \mathrm{Hom}_k(A, R)$. Therefore, G can be identified canonically with the functor $R \mapsto \mathrm{Hom}_k(A, R)$. Because of this, will think of an affine group scheme as a functor from the category of k-algebras to the category of groups which are representable (viewed as a set-valued functor) by some k-algebra A which will be called the *coordinate ring* of G.

The group structure on G provides its coordinate ring A with some extra structure. Indeed, notice that the functor $R \mapsto G(R) \times G(R)$ is representable by the tensor product $A \otimes_k A$ and so Yoneda lemma shows that the morphism of functors $m : G \times G \to G$ must arise from a unique k-algebra morphism $\Delta : A \to A \otimes_k A$ which is called a *comultiplication*. In a similar fashion, the unit and inverse maps are induced by Yoneda lemma by unique maps $\varepsilon : A \to k$ and $\iota : A \to A$ called *counit* and *antipode* respectively. Dualizing the group axioms motivates the following definition.

Definition 3.1.2. A k-algebra A is a called a *Hopf algebra* if it is provided of maps $\Delta : A \to A \otimes_k A$, $\varepsilon : A \to k$ and $\iota : A \to A$ called comultiplication, counit and antipode respectively which are subject to the following commutative diagrams

$$
\begin{tikzcd}
A \otimes_k A \otimes_k A & A \otimes_k A \arrow[l, "1 \otimes \Delta"'] \\
A \otimes_k A \arrow[u, "\Delta \otimes 1"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A & A \otimes_k A \arrow[l, "1 \otimes \varepsilon"'] \\
A \otimes_k A \arrow[u, "\varepsilon \otimes 1"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A & A \otimes_k A \arrow[l, "m \circ (1 \otimes \iota)"'] \\
A \otimes_k A \arrow[u, "m \circ (\iota \otimes 1)"] & A. \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "\varepsilon"']
\end{tikzcd}
$$

The previous discussion shows that we have an antiequivalence of categories between the category of affine group schemes over k and that of commutative Hopf algebras over k.

Example 3.1.3. The *multiplicative group* $\mathbb{G}_m$ is the functor $R \mapsto \mathbb{G}_m(R) = R^\times$. It is an affine group scheme represented by $k[X, Y]/(XY - 1)$ since any invertible element in R corresponds to a k-algebra morphism $k[X, X^{-1}] \to R$. The comultiplication map is induced by $\Delta(x) = x \otimes x$, the counit sends x to 1 and the antipode is induced by $x \mapsto x^{-1}$.

More generally, the functor $R \mapsto \mathrm{GL}_n(R)$ sending a k-algebra R to the group of invertible matrices with entries in R defines an affine group scheme. The coordinate ring A of this affine group scheme is

$$A = k[X_{11}, X_{12}, \ldots, X_{nn}, X]/(X \det(X_{ij}) - 1)$$

since a morphism $\phi : A \to R$ gives a matrix $(\phi(X_{ij})_{ij}$ with invertible determinant, that is an element of $\mathrm{GL}_n(R)$. The comultiplication is induced by $X_{ij} \mapsto \sum_{\ell=1}^{n} X_{i\ell} \otimes X_{\ell j}$, the counit sends X_{ij} to the Kronecker delta δ_{ij} and the antipode comes from the explicit formula for the inverse matrix.

We would like to understand the representation theory of an affine group scheme G over a field k. Given V a k-vector space, for any k-algebra R, consider the set $\mathrm{Aut}_R(V \otimes_k R)$ of R-linear automorphisms of $V \otimes_k R$. It is a group under composition and we may define in this fashion a functor Aut_V from the category of k-algebras to that of groups via the rule $R \mapsto \mathrm{Aut}_R(V \otimes_k R)$. A *linear representation* is a collection of representations $\rho_R : G(R) \to \mathrm{Aut}_R(V \otimes_k R)$ for each k-algebra R such that for every k-algebra morphism $R \to S$ the diagram

$$\begin{array}{ccc} G(R) & \xrightarrow{\rho_R} & \mathrm{Aut}_R(V \otimes_k R) \\ \downarrow & & \downarrow \\ G(S) & \xrightarrow{\rho_S} & \mathrm{Aut}_S(V \otimes_k S) \end{array}$$

commutes. In categorical terms, it is a natural transformation of group-valued functors $\rho : G \to \mathrm{Aut}_V$. We say that a linear representation $\rho : G \to \mathrm{Aut}_V$ is *finite dimensional* if $\dim_k V < \infty$ and that it is *faithful* every morphism ρ_R is injective. If $W \subseteq V$ is a k-linear subspace, then $W \otimes_k R \subseteq V \otimes_k R$ is an R-linear subspace for every k-algebra R and we say that W is a *subrepresentation* of V if $\rho_R(g)(W \otimes_k R) \subseteq W \otimes_k R$ for every $g \in G(R)$ and every k-algebra R. Suppose that (V_1, ρ_1) and (V_2, ρ_2) are two linear representations of an affine group scheme G. We define a *morphism of linear representations* as a k-linear map $u : V_1 \to V_2$ such that the diagram

$$\begin{tikzcd}
V_1 \otimes_k R \arrow[r, "u \otimes 1"] \arrow[d, "(\rho_1)_R(g)"'] & V_2 \otimes_k R \arrow[d, "(\rho_2)_R(g)"] \\
V_1 \otimes_k R \arrow[r, "u \otimes 1"] & V_2 \otimes_k R
\end{tikzcd}$$

commutes for all $g \in G(R)$ and every k-algebra R. This allows us to consider the category Rep_G of finite dimensional linear representations of G, which will have many wonderful properties as we will see further on.

A natural question is whether we can understand the representation theory of an affine group scheme in terms of its Hopf algebra. To answer that question, we must first define the notion of *comodule* over a Hopf algebra.

Definition 3.1.4. Let A be a Hopf algebra over a field k. A right A-comodule is a k-vector space M together with a k-linear map $\rho : M \to M \otimes_k A$ so that the diagrams

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes_k A \arrow[d, "1 \otimes \Delta"] \\
M \otimes_k A \arrow[r, "\rho \otimes 1"] & M \otimes_k A \otimes_k A
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[dr, "\simeq"] & M \otimes_k A \arrow[d, "1 \otimes \varepsilon"] \\
 & M \otimes_k k
\end{tikzcd}$$

commute.

Given an A-comodule M we may construct a representation of the corresponding affine group scheme G represented by A. For a k-algebra R, an element of $G(R)$ is a k-algebra morphism $\lambda : A \to R$ and the composition $\rho_\lambda : M \to M \otimes_k A \xrightarrow{1 \otimes \lambda} M \otimes_k R$ induces an R-linear map $M \otimes_k R \to M \otimes_k R$ that depends on R in a functorial way because of the comodule axioms. Hence we obtain $G(R) \to \mathrm{Aut}_R(M \otimes_k R)$ sending $\lambda \mapsto \rho_\lambda$ a linear representation of G. For the details, see [Wat79, 3.2]. A simple example of an A-comodule is to take $M = A$ and $\rho = \Delta$ the comultiplication. The associated representation of G is called the *regular representation*. The previous question is settled by the following proposition.

Proposition 3.1.5. *There is a bijection between right comodules over the commutative Hopf algebra A and left representations of the corresponding affine group scheme G.*

Proof. We have already seen how to obtain a representation from a comodule. Conversely, given a representation $G \to \mathrm{Aut}_V$ consider the image of the identity element in $G(A)$ which is an A-linear map $V \otimes_k A \to V \otimes_k A$ and by composition with the natural map $V \to V \otimes_k A$, $v \mapsto v \otimes 1$ we obtain an A-comodule structure $V \to V \otimes_k A$ on V. Applying Yoneda lemma one may check that these constructions are one the inverse of the other. ■

The following finiteness property of comodules will allow us to understand any representation of an affine group scheme G in terms of its finite dimensional representations.

Lemma 3.1.6. *Let A be a Hopf algebra over k and (V, ρ) an A-comodule. Every finite subset of V is contained in a subcomodule of V having finite dimension over k.*

Proof. Let $\{a_i\}_{i \in I}$ be a basis for A over k. If $\{v_1, \dots, v_m\} \subseteq V$ is a finite subset, write

$$\rho(v_j) = \sum_{i \in I} v_{ji} \otimes c_i$$

(the sum being finite) and consider the k-vector space generated by the v_i and the v_{ij}. This is the desired subcomodule. ■

Proposition 3.1.7. *Every linear representation of an affine group scheme is a directed union of finite dimensional subrepresentations.*

Proof. The set of all subcomodules of a comodule V which are finite dimensional over k is partially ordered by inclusion and directed (by the previous Lemma 3.1.6) and the union is V. Therefore, applying Proposition 3.1.5 we are done. ■

Suppose that G is an affine group scheme and (V, ρ) a finite dimensional representation. Let G_V be the image of ρ in Aut_V. The coordinate ring of G_V corresponds to a subcomodule of the coordinate ring A of G. Therefore, proceeding as in the previous Proposition 3.1.7 and applying the Spec functor we obtain the following result.

Corollary 3.1.8. *Every affine group scheme G over k is a directed inverse limit*

$$G = \varprojlim_{V \in \mathrm{Rep}_k(G)} G_V.$$

The previous Lemma 3.1.6 also allows us to establish a classical theorem by Chevalley. We say that an affine group scheme G is *algebraic* if its coordinate ring A is finitely generated as a k-algebra.

Theorem 3.1.9 (Chevalley)**.** *Let G be an algebraic affine group scheme. Every subgroup H of G (that is H is a subfunctor of G) is the stabilizer of a line D in some finite dimensional representation of G.*

Proof. Let A_G, A_H be the coordinate rings of G and H respectively. The inclusion $H \subseteq G$ induces a map $A_G \to A_H$ whose kernel is the ideal I of regular functions on G which vanish along H. Clearly H is the stabilizer of I in the regular representation of G on A_G. Since A is noetherian, the ideal I must be finitely generated and therefore by Lemma 3.1.6 we have a finite dimensional subspace V of A_G that is G-stable and contains a generating set for I. Then, H is the stabilizer of $I \cap V$ in V and taking $D = \wedge^d(I \cap V)$ in $\wedge^d V$ for $d = \dim_k(I \cap V)$ we obtain the desired. ∎

If G is an affine group scheme, $H \subseteq G$ is a *closed subgroup scheme* if H is a subfunctor of G such that the induced map on Hopf algebras is surjective.

Example 3.1.10. Consider the affine variety $\mathrm{GL}_n(k)$ and let $G(k) \subseteq \mathrm{GL}_n(k)$ be a Zariski closed subgroup. This inclusion induces, by the antiequivalence of categories between affine varieties and finitely generated reduced k-algebras, a surjective map $k[X_{11}, \ldots, X_{nn}, 1/\det(X_{ij})] \to A_G$ which equips the coordinate ring A_G of $G(k)$ with the structure with a Hopf algebra. The affine group scheme associated to A_G is a functor whose k-valued points are precisely $G(k)$. It is obvious that $G \subseteq \mathrm{GL}_n$ is a closed subgroup scheme.

Proposition 3.1.11. *Every algebraic affine group scheme G over a field k is isomorphic to a closed subgroup of some GL_n.*

Proof. Let A be the Hopf algebra of G and let V be a finite dimensional subcomodule of A containing algebra generators. Let $\{v_i\}_{i \in I}$ be a basis of V, and write $\Delta(v_j) = \sum_i v_i \otimes a_{ij}$. The image of $k[X_{11}, \ldots, X_{nn}, 1/\det(X_{ij})] \to A$ contains the a_{ij}, the images of X_{ij}. But $v_j = (\varepsilon \otimes 1)\Delta(v_j) = \sum_i \varepsilon(v_i) a_{ij}$, so the image contains V and hence is all of A. ∎

Notice that if $G \subseteq \mathrm{GL}_n$ is a closed subgroup scheme we have a canonical representation of G on k^n.

Lemma 3.1.12. *Let G be a closed subgroup scheme of GL_n over k and let W be the canonical representation of G on k^n. Consider the strictly full subcategory $\langle W \rangle_\otimes$ of $\mathrm{Rep}_k(G)$ whose objects are subquotients of finite direct sums of objects of the form $W^{\otimes r} \otimes (W^*)^{\otimes s}$ for $r, s \geq 0$. Then $\langle W \rangle_\otimes = \mathrm{Rep}_k(G)$.*

Proof. For a proof see [Sza09, Lemma 6.5.16]. ∎

We are now in conditions of proving the following characterization of subgroups in terms of their fixed tensors, following the argument of Deligne in [DMOS82, 3.2].

Theorem 3.1.13 (Characterizing subgroups by their fixed tensors)**.** *Let G be an algebraic affine group scheme and $H \subseteq G$ a subgroup. Let H' be the group fixing all tensors fixed by H occurring in any representation of G. Then $H = H'$.*

Proof. By Proposition 3.1.11 we can view G as a closed subgroup of GL_n and thus Lemma 3.1.12 every representation of G is a subquotient of a tensor of the canonical representation $G \to \mathrm{GL}_n$. Finally, by Chevalley's Theorem 3.1.9 H is the stabilizer of a line D in a representation of G. Since there is no other element fixing D other than those of H, this finishes the proof. ∎

3.2 Monoidal Categories

Let $\mathscr{C}$ be a category and let

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}, \quad (X, Y) \mapsto X \otimes Y$$

be a functor. An *associator* for $(\mathscr{C}, \otimes)$ is a natural isomorphism

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

that satisfies the *pentagon axiom*, that is, for all objects X, Y, Z, T the following diagram

$$\begin{tikzcd}
 & X \otimes (Y \otimes (Z \otimes T)) \arrow[dl, "1 \otimes \phi"'] \arrow[dr, "\phi"] & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[d, "\phi"] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d, "\phi"] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi \otimes 1"] & & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd}$$

is commutative. A *braiding* for $(\mathscr{C}, \otimes)$ is a natural isomorphism

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X$$

such that for all objects X, Y the composition $\psi_{Y,X}\psi_{X,Y} : X \otimes Y \to X \otimes Y$ is the identity morphism on $X \otimes Y$. An associativity constraint ϕ and a commutativity constraint ψ are compatible if, for all objects X, Y, Z the *hexagon axiom* holds, that is, the following diagram

$$\begin{tikzcd}[column sep=small]
 & X \otimes (Y \otimes Z) \arrow[r, "\phi"] \arrow[dl, "1 \otimes \psi"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi"] & & & Z \otimes (X \otimes Y) \arrow[dl, "\phi"] \\
 & (X \otimes Z) \otimes Y \arrow[r, "\psi \otimes 1"] & (Z \otimes X) \otimes Y &
\end{tikzcd}$$

commutes for every triple (X, Y, Z) of objects in $\mathscr{C}$. A *unit object* in $(\mathscr{C}, \otimes)$ will be an object 1 together with an isomorphism $\nu : 1 \to 1 \otimes 1$ so that moreover the functors $X \mapsto 1 \otimes X$ and $X \mapsto X \otimes 1$ are fully faithful. One may prove that a unit object is unique up to unique isomorphism.

Definition 3.2.1. A category $\mathscr{C}$ together with a tensor product $\otimes$, a unit object 1, an associator ϕ and a braiding ψ which are compatible are said to form a *braided monoidal category*.

Definition 3.2.2. A *monoidal functor* (F, Λ) between braided monoidal categories $\mathscr{C}$ and $\mathscr{C}'$ is a functor $F : \mathscr{C} \to \mathscr{C}'$ together with an isomorphism Λ of functors from $\mathscr{C} \times \mathscr{C}$ to $\mathscr{C}'$ given on a pair (X, Y) of objects of $\mathscr{C}$ by

$$\Lambda_{X,Y} : F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y)$$

such that for any triple (X, Y, Z) of objects of $\mathscr{C}$ the diagram

$$\begin{tikzcd}
F((X \otimes Y) \otimes Z) \arrow[r, "\Lambda"] \arrow[d, "F\phi"] & F(X \otimes Y) \otimes F(Z) \arrow[r, "\Lambda \otimes 1"] & (F(X) \otimes F(Y)) \otimes F(Z) \arrow[d, "\phi"] \\
F(X \otimes (Y \otimes Z)) \arrow[r, "\Lambda"] & F(X) \otimes F(Y \otimes Z) \arrow[r, "1 \otimes \Lambda"] & F(X) \otimes (F(Y) \otimes F(Z))
\end{tikzcd}$$

commutes. Moreover, if 1 and $1'$ denote the unit objects of $\mathscr{C}$ and $\mathscr{C}'$, respectively, we require that $F(1) = 1'$, where the isomorphism $1' \to 1' \otimes 1'$ is $\Lambda_{1,1}F(\nu)$.

Definition 3.2.3. We define a braided monoidal category to be *rigid* if each object X has a dual X^*. More precisely, the functor $T \mapsto \mathrm{Hom}(T \otimes X, 1)$ is representable by an object X^* (that is, $X^* = \underline{\mathrm{Hom}}(X, 1)$ is the *internal hom*) and we have isomorphisms $\epsilon : X \otimes X^* \to 1$ and $\delta : 1 \to X^* \otimes X$ so that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & 1 \otimes X \\ \downarrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \epsilon \otimes 1} \\ X \otimes 1 & \xrightarrow{1 \otimes \delta} & X \otimes X^* \otimes X \end{array} \qquad \begin{array}{ccc} X^* & \xrightarrow{\simeq} & X^* \otimes 1 \\ \downarrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle 1 \otimes \epsilon} \\ 1 \otimes X^* & \xrightarrow{\delta \otimes 1} & X^* \otimes X \otimes X^* \end{array}$$

commute. The dual X^* of an object X is uniquely determined up to isomorphism and furthermore if we fix one of the maps ϵ or δ, then this isomorphism will be unique.

Given a rigid braided monoidal category $(\mathscr{C}, \otimes)$ every morphism $f : X \to Y$ has a *transpose*, namely a morphism $f^t : Y^* \to X^*$ defined as the composite

$$Y^* \xrightarrow{\sim} 1 \otimes Y^* \xrightarrow{\delta \otimes 1} X^* \otimes X \otimes Y^* \xrightarrow{1 \otimes f \otimes 1} X^* \otimes Y \otimes Y^* \xrightarrow{1 \otimes \epsilon} X^* \otimes 1 \xrightarrow{\sim} X^*.$$

This induces a functor $\mathscr{C}^{\mathrm{opp}} \to \mathscr{C}$ given by $X \mapsto X^*$, $f \mapsto f^t$ which is an equivalence of categories (by noticing that $X^{**} \simeq X$ canonically). In particular, the map $f \mapsto f^t$ induces an isomorphism $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y^*, X^*)$.

Definition 3.2.4. Let F and G be monoidal functors $\mathscr{C} \to \mathscr{C}'$. A *morphism of monoidal functors* $F \to G$ is a morphism of functors such that for every finite set $X_1, \dots, X_n$ of objects in $\mathscr{C}$, the diagram

$$\begin{array}{ccc} F(X_1) \otimes \cdots \otimes F(X_n) & \longrightarrow & F(X_1 \otimes \cdots \otimes X_n) \\ \downarrow & & \downarrow \\ G(X_1) \otimes \cdots \otimes G(X_n) & \longrightarrow & G(X_1 \otimes \cdots \otimes X_n) \end{array}$$

is commutative.

Lemma 3.2.5. *A morphism of monoidal functors between rigid braided monoidal categories is always an isomorphism.*

Proof. If $\lambda : F \to G$ is an isomorphism of monoidal functors, for any object X the naturality of dualization tells us that the following diagram

$$\begin{tikzcd}
F(X^*) \arrow[r, "\lambda_{X^*}"] \arrow[d, "\simeq"'] & G(X^*) \arrow[d, "\simeq"] \\
F(X)^* \arrow[r, dashed, "(\mu_X)^t"] & G(X)^*
\end{tikzcd}$$

commutes and the map $F(X)^* \to G(X)^*$ must be the transpose of a certain map $\mu_X : G(X) \to F(X)$. This defines a morphism of functors $\mu : G \to F$. It is now straightforward to verify that μ is a morphism of monoidal functors and that it is inverse to λ on both sides. ∎

An example of a braided monoidal category is the category mod_R of finitely generated modules over a commutative ring R furnished with the usual tensor product and the obvious associativity and commutativity constraints. For the unit object we may take any free R-module U of rank 1 and $U \to U \otimes U$, $u \mapsto u \otimes u$. Clearly this category is also rigid since every R-module M has a dual object $M^* = \mathrm{Hom}_R(M, R)$ for the adjointness of the Hom and $\otimes$ functors prove that the internal Hom is just the usual Hom.

The main example which we will be interested in is the category $\mathrm{Rep}_k(G)$ of finite-dimensional representations of an affine group scheme G over a field k. Indeed, we may define the *tensor product of linear representations* by taking two representations (V_1, ρ_1), (V_2, ρ_2) and considering $(V_1 \otimes_k V_2, \rho_1 \otimes \rho_2)$ defined by $(\rho_1 \otimes \rho_2)_R(x) = (\rho_1)_R(x)(\rho_2)_R(x)$ for every $x \in G(R)$ and every k-algebra R. The unit object is the trivial representation $V = k$ with trivial G-action and the dual representation of (V, ρ) is defined by (V^*, ρ^*) where V^* is the dual space of V and $(\rho^*)_R(g) = (\rho_R(g^{-1}))^t$ as one may verify.

The category of $\mathrm{Rep}_k(G)$ finite dimensional linear representations of a group scheme is naturally provided with a forgetful functor $\omega : (V, \rho) \mapsto V$ to the category of vector spaces over k. Notice that for every commutative k-algebra R we have a monoidal functor $\omega \otimes R : \mathrm{Rep}_k(G) \to \mathrm{mod}_R$ via $(\omega \otimes R)(V, \rho) = V \otimes_k R$. We may define the set-valued functor $\mathbf{End}^{\otimes}(\omega)$ from the category of commutative k-algebras by sending each k-algebra R to the set of monoidal functor morphisms $\omega \otimes R \to \omega \otimes R$. Analogously, we may define $\mathbf{Aut}^{\otimes}$ as the monoidal functor isomorphisms $\omega \otimes R \to \omega \otimes R$. By virtue of Lemma 3.2.5, the natural map $\mathbf{Aut}^{\otimes} \to \mathbf{End}^{\otimes}$ of functors is an isomorphism. The functor $\mathbf{Aut}^{\otimes}(\omega)$ is moreover group-valued.

Recall that Pontryagin duality gives us a way to recover a locally compact commutative group from its one-dimensional unitary representations via the

Fourier transform and Tannaka-Krein duality is the natural generalization of this to non-commutative groups. It states that we may obtain a compact non-commutative group from its category of finite-dimensional representations (See [HR70]). The following proposition shows us that we may extend this to the algebrogeometric setting.

Proposition 3.2.6. *Let G a group scheme over field k and ω the forgetful functor on* Rep_G*. There is a canonical isomorphism of group-valued functors* $G \xrightarrow{\sim} \mathbf{Aut}^{\otimes}(\omega)$*.*

Proof. Notice that a monoidal automorphism of $\omega \otimes R$ is a family of morphisms $(\lambda_V)_{V \in \mathrm{Rep}_k(G)}$ where $\lambda_V : V \otimes_k R \to V \otimes_k R$ is an R-linear automorphism and such that:

- For every pair $(V, \rho), (W, \tau)$ of linear representations, $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$.
- If 1 is the trivial representation, $\lambda_1 = 1$.
- For any morphism of linear representations $u : (V, \rho) \to (W, \tau)$ the following diagram commutes

$$\begin{tikzcd}
V \otimes_k R \arrow[r, "\lambda_V"] \arrow[d, "u \otimes 1"'] & V \otimes_k R \arrow[d, "u \otimes 1"] \\
W \otimes_k R \arrow[r, "\lambda_W"] & W \otimes_k R.
\end{tikzcd}$$

With this description, any element of $G(R)$ defines an element of $\mathbf{Aut}^{\otimes}(\omega)$ since for any representation (V, ρ) we have maps $\rho_R : G(R) \to \mathrm{Aut}(V \otimes_k R)$ and thus we take $\lambda_V = \rho_R(g) : V \otimes_k R \to V \otimes_k R$ is an automorphism. It is straightforward to check that the family of maps $(\lambda_V)_{V \in \mathrm{Rep}_k(G)}$ verifies the conditions to be an element of $\mathbf{Aut}^{\otimes}(\omega)(R)$. Therefore, we have a canonical map $G \to \mathbf{Aut}^{\otimes}(\omega)$.

Consider the strictly full subcategory $\langle V \rangle_{\otimes}$ of $\mathrm{Rep}_k(G)$ of objects isomorphic to a subquotient of $P(V, V^*)$ where $P \in \mathbb{N}[t, s]$ and multiplication is interpreted as $\otimes$ and addition as $\oplus$. Notice that we may view $\mathbf{Aut}^{\otimes}\left(\omega|_{\langle V \rangle_{\otimes}}\right)(R)$ as a subgroup of $\mathrm{GL}(V \otimes R)$ since the family of maps $(\lambda_X)_{X \in \langle V \rangle_{\otimes}}$ is determined by λ_V. The image G_V of the representation $G \to \mathrm{GL}_V$ defines a closed algebraic subgroup of GL_V and we have the chain of inclusions

$$G_V(R) \subseteq \mathbf{Aut}^{\otimes}\left(\omega|_{\langle V \rangle_{\otimes}}\right)(R) \subseteq \mathrm{GL}(V \otimes R).$$

If $X \in \langle V \rangle_{\otimes}$ and $x \in X$ is fixed by G (that is, $x \otimes 1 \in X \otimes_k R$ is fixed by $G(R)$ for every k-algebra R) then the map $u : k \to X$, $t \mapsto tx$ defines a morphism of linear representations of G. Therefore, by the naturality of λ_V the following diagram

$$\begin{tikzcd}
k \otimes_k R \arrow[r, "1"] \arrow[d, "u \otimes 1"'] & k \otimes_k R \arrow[d, "u \otimes 1"] \\
V \otimes_k R \arrow[r, "\lambda_V"] & V \otimes_k R
\end{tikzcd}$$

commutes. Hence $\lambda_V(x \otimes 1) = \lambda_V((u \otimes 1)(1 \otimes 1)) = (u \otimes 1)(1 \otimes 1) = x \otimes 1$ and thus x is fixed by $\mathbf{Aut}^{\otimes}\left(\omega|_{\langle V \rangle_{\otimes}}\right)$. Applying Theorem 3.1.13 we must have that $G_V = \mathbf{Aut}^{\otimes}\left(\omega|_{\langle V \rangle_{\otimes}}\right)$.

The rest of the argument is just a *passage to the limit*. If $V' = V \oplus W$ for some representation W, then $\langle V \rangle_{\otimes} \subseteq \langle V' \rangle_{\otimes}$ and the following diagram

$$\begin{tikzcd}
G_{V'} \arrow[r] \arrow[d] & \mathbf{Aut}^{\otimes}\left(\omega|_{\langle V' \rangle_{\otimes}}\right) \arrow[d] \\
G_V \arrow[r] & \mathbf{Aut}^{\otimes}\left(\omega|_{\langle V \rangle_{\otimes}}\right)
\end{tikzcd}$$

commutes. Because $G = \varprojlim_{V \in \mathrm{Rep}_k(G)} G_V$ and $\mathbf{Aut}^{\otimes}(\omega) = \varprojlim_V \mathbf{Aut}^{\otimes}\left(\omega|_{\langle V \rangle_{\otimes}}\right)$ the desired isomorphism $G \xrightarrow{\simeq} \mathbf{Aut}^{\otimes}(\omega)$ follows. ∎

If G_1, G_2 are affine group schemes and $\varphi : G_1 \to G_2$ is a group scheme morphism, then every finite dimensional representation of G_2 induces a representation of G_1 via composition with φ. Therefore, we have a monoidal functor $\varphi^* : \mathrm{Rep}_k(G_2) \to \mathrm{Rep}_k(G_1)$ satisfying $\omega_1 \varphi^* = \omega_2$ where ω_i is the forgetful functor on $\mathrm{Rep}_k(G_i)$ for each $i = 1, 2$.

Corollary 3.2.7. *The rule $\varphi \mapsto \varphi^*$ induces a bijection between group scheme morphisms $G_1 \to G_2$ and monoidal functors $F : \mathrm{Rep}_k(G_2) \to \mathrm{Rep}_k(G_1)$ satisfying that $\omega_1 F = \omega_2$.*

Proof. If we have $F : \mathrm{Rep}_k(G_2) \to \mathrm{Rep}_k(G_1)$ a monoidal functor then every monoidal automorphism of ω_1 yields a monoidal automorphism of ω_2 via composition with F and the same holds for $\omega \otimes R$ where R is any k-algebra. Therefore, this gives us a morphism of group-valued functors $\mathbf{Aut}^{\otimes}(\omega_1) \to \mathbf{Aut}^{\otimes}(\omega_2)$, and by the previous Proposition 3.2.6 this is identified canonically with a group scheme homomorphism $G_1 \to G_2$. It is straightforward to verify that this construction is inverse to the map $\varphi \mapsto \varphi^*$. ∎

3.3 Neutral Tannakian Categories

Definition 3.3.1. A *neutral Tannakian category* over a field k is a rigid k-linear abelian braided monoidal category $\mathscr{C}$ whose unit 1 satisfies $\mathrm{End}(1) \simeq k$, and is moreover equipped with an exact faithful monoidal functor $\omega : \mathscr{C} \to \mathrm{Vecf}_k$ into the category of finite-dimensional k-vector spaces. The functor ω is called a (neutral) *fiber functor*.

A nuance in the previous definition is that when we talk about a k-linear abelian braided monoidal category there is a compatibility condition relating the monoidal and abelian category structures on $\mathscr{C}$. The tensor operation

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Z,W) \to \mathrm{Hom}(X \otimes Z, Y \otimes W), \quad (\phi, \psi) \mapsto \phi \otimes \psi$$

should be k-bilinear with respect to the k-vector space structures on the Hom-sets involved. This also explains the presence of the condition $\mathrm{End}(1) \simeq k$. That is, the isomorphism $1 \otimes X \xrightarrow{\simeq} X$ induces a map $\mathrm{End}(1) \to \mathrm{End}(X)$ that provides the group $\mathrm{Hom}(X,Y)$ with the structure of and $\mathrm{End}(1)$-module via composition, and similarly for endomorphisms of Y. We require that the isomorphism $\mathrm{End}(1) \simeq k$ transforms these to the k-linear structure on $\mathrm{Hom}(X,Y)$. These requirements are satisfied in braided monoidal categories of k-vector spaces.

If G is an affine group scheme over a field k, the category $\mathrm{Rep}_k(G)$ of finite dimensional linear representations of G with its usual monoidal structure and the forgetful functor taken as the fiber functor is a neutral Tannakian category. The main theorem states that the converse holds.

Theorem 3.3.2 (Main Theorem of Tannakian Formalism)**.** *Every neutral Tannakian category $(\mathscr{C}, \omega)$ over k is equivalent to the category $\mathrm{Rep}_k(G)$ of finite dimensional representations of an affine group scheme G over k.*

Proof. We shall outline the proof provided in [Del90]. It will suffice to prove that if $\mathscr{C}$ is a k-linear abelian category equipped with an exact faithful k-linear functor $\omega : \mathscr{C} \to \mathrm{Vecf}_k$ then there exists a k-coalgebra $A_{\mathscr{C}}$ so that each $\omega(X)$ carries a natural right $A_{\mathscr{C}}$-comodule structure for X in $\mathscr{C}$ and that moreover ω induces an equivalence of categories between $\mathscr{C}$ and the category $\mathrm{Comodf}_{A_{\mathscr{C}}}$ of finite dimensional right $A_{\mathscr{C}}$-comodules.

Let X be an object in $\mathscr{C}$. First we should prove that there exist a finite dimensional k-algebra R and an equivalence of categories between $\langle X \rangle$ and the category mod_R of finitely generated right R-modules. This is achieved by a theorem of Gabber on the existence of projective generators for $\langle X \rangle$. If P is a projective generator of $\langle X \rangle$, and we take $R = \mathrm{End}(P)$ then $A \mapsto \mathrm{Hom}(P, A)$ induces an equivalence of categories between $\langle X \rangle$ and the category of finitely generated $\mathrm{End}(P)$-modules $\mathrm{mod}_{\mathrm{End}(P)}$. Moreover, via this equivalence of categories, the functor ω becomes isomorphic to the functor from the category of right $\mathrm{End}(P)$-modules to that of k-vector spaces mapping $M \mapsto M \otimes_{\mathrm{End}(P)} \omega(P)$. There is a canonical k-coalgebra structure on $A_X := \omega(P)^\vee \otimes_{\mathrm{End}(P)} \omega(P)$ and a right A_X-comodule structure on $M \otimes_{\mathrm{End}(P)} \omega(P)$ for each right $\mathrm{End}(P)$-module so that the functor $M \mapsto M \otimes_{\mathrm{End}(P)} \omega(P)$ induces an equivalence of categories between right $\mathrm{End}(P)$-modules and finite dimensional right A_X-comodules. Finally, writing $\mathscr{C}$ as a directed union of subcategories of the form $\langle X \rangle$ and a passage to the limit argument finish the proof. ∎

We can determine the group scheme G as Proposition 3.2.6 tells us that $\mathbf{Aut}^{\otimes}(\omega)$ is isomorphic to G. The group scheme $G = \mathbf{Aut}^{\otimes}(\omega)$ determined by the neutral Tannakian category $(\mathscr{C}, \omega)$ is called the *Tannakian fundamental group* of $(\mathscr{C}, \omega)$.

It is natural to ask how properties of the group scheme G over k are reflected in $\mathrm{Rep}_k(G)$ and viceversa. The following proposition is an example of this.

Proposition 3.3.3. *Let G be an affine group scheme over k.*

- *G is finite if and only if there exists an object X of $\mathrm{Rep}_k(G)$ such that every object of $\mathrm{Rep}_k(G)$ is isomorphic to a subquotient of X^n for some $n \geq 0$.*
- *G is algebraic if and only if $\mathrm{Rep}_k(G)$ has a tensor generator X (i.e., there exists a representation X such that $\mathrm{Rep}_k(G) = \langle X \rangle_{\otimes}$).*

Proof. For the first claim, if G is finite, the regular representation X of G is finite-dimensional and has the required property. Conversely, if $\mathrm{Rep}_k(G) = \langle X \rangle$ then $G = \mathrm{Spec}(B)$ where B is the linear dual of the finite k-algebra A_X.

For the second claim, if G is algebraic then it has a finite-dimensional faithful representation X and we can see that $X \oplus X^\vee$ is a tensor generator for $\mathrm{Rep}_k(G)$. Conversely, if X is a tensor generator for $\mathrm{Rep}_k(G)$, then it is a faithful representation of G. ∎

3.4 Differential Galois Theory

Given a connected and locally simply connected topological space X with a fixed base point $x \in X$, the category of local systems of complex vector spaces on X is a $\mathbb{C}$-linear abelian category. This category has a rigid braided monoidal structure induced by the usual tensor product and dual space constructions of linear algebra and considering the stalk of a local system at x gives us a fiber functor with values in $\mathrm{Vecf}_{\mathbb{C}}$. Therefore it is a neutral Tannakian category.

Let X be a complex manifold. Consider the category $\mathrm{Conn}(X)$ of holomorphic connections on X. By the Riemann-Hilbert correspondence I (Theorem 2.2.2) we know that $\mathrm{Conn}(X)$ is equivalent to the category of local systems on X and hence it is a $\mathbb{C}$-linear abelian category. The tensor product of two connections $(\mathscr{E}_1, \nabla_1)$ and $(\mathscr{E}_2, \nabla_2)$ is defined as $(\mathscr{E}_1 \otimes_{\mathcal{O}_X} \mathscr{E}_2, \nabla_1 \otimes \nabla_2)$ where

$$(\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2)$$

and the dual connection of $(\mathscr{E}, \nabla)$ is defined as $(\mathscr{E}^\vee, \nabla^\vee)$ where the locally free sheaf $\mathscr{E}^\vee$ is given by $U \mapsto \mathrm{Hom}(\mathscr{E}|_U, \mathcal{O}_X|_U)$ and

$$\nabla^\vee(\phi)(s) = 1 \otimes \mathrm{d}\phi(s) - \left(\phi \otimes \mathrm{id}_{\Omega^1_X}\right)(\nabla(s)).$$

The fiber functor is given by $(\mathscr{E}, \nabla) \mapsto \mathscr{E}_x^\nabla$. The equivalence of categories given by Theorem 2.2.2 is compatible with the tensor structures and fiber functor.

Definition 3.4.1. Let X be a complex manifold and $D \subseteq X$ a divisor with normal crossings and $(\mathscr{E}, \nabla)$ be a connection with regular singularities along D. The *differential Galois group scheme* of a connection $(\mathscr{E}, \nabla)$ is the Tannakian fundamental group of the full Tannakian subcategory $\langle(\mathscr{E}^*, \nabla^*)\rangle_\otimes$ of $\mathrm{Conn}(X)$ where $(\mathscr{E}^*, \nabla^*)$ is the extension of Proposition 2.4.3 and will be denoted $\mathrm{Gal}(\mathscr{E}, \nabla)$.

The differential Galois group is intimately related to the monodromy of the connection around the singular points. Indeed, due to Theorem 2.3.1 the local systems on X are in correspondence to the finite dimensional representations of $\pi_1(X, x)$. Let us prove a more general result first. For an arbitrary group Γ, the category $\mathrm{Rep}_k(\Gamma)$ of finite dimensional k-linear representations of Γ is equipped with the usual tensor product, dual operations and the forgetful functor which make it a Tannakian category.

Lemma 3.4.2. *Let Γ be an arbitrary group and $\rho : \Gamma \to \mathrm{GL}(V)$ be a finite dimensional k-linear representation of Γ. The Tannakian fundamental group of the full Tannakian subcategory $\langle (V, \rho) \rangle_{\otimes}$ of $\mathrm{Rep}_k(\Gamma)$ is canonically isomorphic to the Zariski closure of the image of ρ in $\mathrm{GL}(V)$.*

Remark 3.4.3. We view the closure $\overline{\mathrm{im}(\rho)}$ as an affine group scheme by means of Example 3.1.10.

Proof. The representation $\rho : \Gamma \to \mathrm{GL}(V)$ gives rise to a representation $\overline{V}$ of the affine group scheme G defined by $\overline{\mathrm{im}\,\rho}$ in a natural way and moreover we have an equivalence of categories between $\langle V \rangle_{\otimes}$ and $\langle \overline{V} \rangle_{\otimes}$, but Lemma 3.1.12 shows that $\langle \overline{V} \rangle_{\otimes}$ is the whole $\mathrm{Rep}_k(G)$. Since these categories must have isomorphic fundamental groups the result follows. ∎

If $(\mathscr{E}, \nabla)$ is a holomorphic connection on X and consider the local system of horizontal sections $\mathscr{E}^{\nabla}$ we get that $\mathrm{Gal}(\mathscr{E}, \nabla)$ is the Tannakian fundamental group of the full Tannakian subcategory $\langle \mathscr{E}^{\nabla} \rangle_{\otimes}$. This local system corresponds to the monodromy representation $\rho_{(\mathscr{E},\nabla),x}$ of the fundamental group $\pi_1(X, x)$ in $\mathscr{E}_x^{\nabla}$ which is a finite dimensional complex vector space and thus by Lemma 3.4.2 the Tannakian fundamental group of $\langle \mathscr{E}^{\nabla} \rangle_{\otimes}$ must be the Zariski closure of the image of the monodromy representation. If we denote by $\mathrm{Mono}(\mathscr{E}, \nabla)$ the Zariski closure of the image of the monodromy representation, using the Riemann-Hilbert correspondence II (Theorem 2.4.4) we have the following result.

Theorem 3.4.4. *Let X be a complex manifold and $D \subseteq X$ a divisor with normal crossings. If $(\mathscr{E}, \nabla)$ is a connection with regular singularities along D, then*

$$\mathrm{Gal}(\mathscr{E}, \nabla) = \mathrm{Mono}(\mathscr{E}, \nabla).$$

Putting everything together, we have the following theorem.

Theorem 3.4.5. *Let X be a complex manifold and $D \subseteq X$ a divisor with normal crossings. If $(\mathscr{E}, \nabla)$ is a connection with regular singularities along D, the following are equivalent:*

1. *$(\mathscr{E}, \nabla)$ becomes trivial on a finite étale cover of X.*

2. *The algebraic monodromy group* $\mathrm{Mono}(\mathscr{E}, \nabla)$ *is finite.*

3. *The differential Galois group* $\mathrm{Gal}(\mathscr{E}, \nabla)$ *is finite.*

Proof. If $f : Y \to X$ is a finite étale covering, we know from the discussion in Chapter 2, Section 2.2 that the sheaf of horizontal sections of the pullback of a connection is isomorphic to a finite number n of copies of the sheaf of horizontal sections of the connection. Consider the full Tannakian subcategories spanned by $(\mathscr{E}, \nabla)$ and $f^*(\mathscr{E}, \nabla)$. The fiber functors of each of these categories is just taking the stalk at a point of the sheaf of horizontal sections. If $f^*(\mathscr{E}, \nabla)$ is the trivial connection, then $\mathrm{Gal}(f^*(\mathscr{E}, \nabla))$ must be the trivial group and hence the cardinality of the set of maps $\mathrm{Gal}(f^*(\mathscr{E}, \nabla)) \to \mathrm{Gal}(\mathscr{E}, \nabla)$ is precisely the cardinality of $\mathrm{Gal}(\mathscr{E}, \nabla)$. Using Corollary 3.2.7 as we have n functors $\langle(\mathscr{E}, \nabla)\rangle_{\otimes} \to \langle f^*(\mathscr{E}, \nabla)\rangle_{\otimes}$ that permute the fibers there is a finite number of maps $\mathrm{Gal}(f^*(\mathscr{E}, \nabla)) \to \mathrm{Gal}(\mathscr{E}, \nabla)$ and thus $\mathrm{Gal}(\mathscr{E}, \nabla)$ must be finite.

The equivalence of the assertions 2 and 3 is clear due to Theorem 3.4.4. Finally, if the monodromy group is finite, by applying Proposition 2.4.5 we obtain the desired finite étale cover. ∎

Remark 3.4.6. This extends classical differential Galois theory as in [Kap76]. A *differential extension* $L \supseteq \mathbb{C}(z)$ is a field extension L together with a derivation $\partial : L \to L$ that restricts to the canonical derivation on $\mathbb{C}(z)$. Consider an ordinary linear homogeneous differential equation

$$\mathcal{L} : y^{(n)} + a_{n-1}(z)y^{(n-1)} + \ldots + a_1(z)y' + a_0(z)y = 0.$$

In classical Galois theory we know that the splitting field of a polynomial exists and is unique up to isomorphism. The Picard-Vessiot extension for $\mathcal{L}$ will be the differential analogue of the splitting field: it will be the differential extension spanned by the (germs at a regular point of) solutions and all of its derivatives. The differential Galois group of $\mathcal{L}$ is then, in analogy to the usual Galois group, the group of differential $\mathbb{C}(z)$-automorphisms of L. This group is an algebraic group variety over $\mathbb{C}$ and the study of questions of solvability of differential equations by quadratures lead to pioneering work in the theory of algebraic groups by Kolchin and others.

To relate this definition to the Tannakian viewpoint, recall that the differential equation can be put as a matrix system $\frac{\mathrm{d}y}{\mathrm{d}z} = A(z)y(z)$ where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} & -a_n \end{pmatrix}.$$

Consider $X = \mathbb{P}^1_{\mathbb{C}}$ the Riemann sphere and take the connection $\nabla_A = \mathrm{d} - A$ on $\mathcal{O}_X^{\oplus n}$. The horizontal sections of this connection correspond to the solutions of the original differential equation. When k is algebraically closed, the k-points of the differential Galois group scheme Gal $\left(\mathcal{O}_X^{\oplus n}, \nabla_A\right)(k)$ coincide with the usual differential Galois group of the differential equation $\mathcal{L}$, see for instance [Sza09, Proposition 6.6.8].

The Theorem 3.4.5 is well understood in the classical framework of Remark 3.4.6. Indeed, suppose that we have an homogeneous ordinary differential equation

$$\mathcal{L} : y^{(n)} + a_{n-1}(z)y^{(n-1)} + \ldots + a_1(z)y' + a_0(z)y = 0$$

with coefficients $a_i \in \mathbb{C}(z)$ rational functions in one variable. Let L be the Picard-Vessiot extension of $\mathbb{C}(z)$ corresponding to the differential equation $\mathcal{L}$. If the solutions are algebraic over $\mathbb{C}(z)$, then $L|\mathbb{C}(z)$ is an algebraic extension. The differential Galois group in the classical sense is the group of differential automorphisms of $L|\mathbb{C}(z)$. We know from a classical theorem in differential Galois theory (see for instance [vdPS03, Corollary 1.30]) that the transcendence degree of $L|\mathbb{C}(z)$ is equal to the dimension of the differential Galois group of $\mathcal{L}$, and hence it has a full set of algebraic solutions if and only if its differential Galois group is finite. Moreover, if $L|\mathbb{C}(z)$ is an algebraic extension, the theory of Riemann surfaces tells us that there exists a Riemann surface X and a finite holomorphic covering map $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ such that the field of meromorphic functions of X is L. Pulling back the connection associated to the differential equation on $\mathbb{P}^1_{\mathbb{C}}$ to X we obtain the trivial connection, as L has a full set of solutions.

Chapter 4

The Arithmetic of Connections

The goal of this Chapter is two-fold. First we want to generalize the concept of differential equations to schemes, and to do so we think of a differential equation as a connection. Particularly relevant are the properties of connections in characteristic p, which are in principle richer in structure than those of characteristic 0 since we also have a p-structure: the p-th power of a derivation is also a derivation. The second goal is to understand the relationship between the geometry of a connection in characteristic 0 and the arithmetic of connections in characteristic p. In order to do this, we must *thicken* a smooth $\mathbb{C}$-scheme X of finite type to a family $\mathbb{X} \to \mathrm{Spec}(R)$ where R is a finitely generated $\mathbb{Z}$-algebra which has fiber X over the point corresponding to $R \hookrightarrow \mathbb{C}$. This process, called spreading out, will allow us to reduce modulo p by taking the pullback with the canonical map $R \to R/pR$ for any prime p not invertible in R. Katz proved that if the reductions modulo p (for almost every prime) are such that the resulting connection is *arithmetically flat* then the original connection must have finite local monodromy. We conclude by using this theorem to prove, following Katz, the Grothendieck-Katz p-curvature conjecture in the case of abelian monodromy. The main references for this Chapter are the seminal papers [Kat70], [Kat82] of Katz.

4.1 Connections on schemes

In this section we will give the basic definitions of connections on schemes. We will do so from two different points of view: a classical one similar to its holomorphic counterpart and a *crystalline* one as in [Del70].

Definition 4.1.1. Let X be a smooth S-scheme of finite type. An *S-connection* on X consists of a pair (M, ∇) where M is a locally free coherent sheaf on X and $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_{X|S}$ is an $\mathcal{O}_S$-linear map satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f$$

for local sections $s \in M(U)$ and $f \in \mathcal{O}_X(U)$. Moreover, we say that a connection (M, ∇) is *integrable* if for local sections $D_1, D_2 \in \mathrm{Der}_S(\mathcal{O}_X, \mathcal{O}_X)$ we have that

$$[\nabla_{D_1}, \nabla_{D_2}] = \nabla_{[D_1,D_2]}.$$

Suppose that (M, ∇) is an S-connection on X. Just as we uniquely extend the universal $\mathcal{O}_S$-derivation $\mathrm{d} : \mathcal{O}_X \to \Omega^1_{X|S}$ to a complex $\Omega^\bullet_{X|S}$ on the exterior powers of $\Omega^1_{X|S}$ such that the connecting maps satisfy the Leibniz rule, we may extend the map $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_{X|S}$ to obtain maps

$$\nabla^i : M \otimes_{\mathcal{O}_S} \Omega^i_{X|S} \to M \otimes_{\mathcal{O}_S} \Omega^{i+1}_{X|S}$$

given by

$$\nabla^i(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^i \omega \wedge \nabla(s).$$

However, there is now a non-trivial obstruction to obtain a complex. The composites $\nabla^{j+1}\nabla^j$ vanish for all j if and only if $\nabla^1\nabla = 0$. If we define the *curvature* of ∇ as the map $K = \nabla^1\nabla : M \to M \otimes_{\mathcal{O}_S} \Omega^2_{X|S}$ then we obtain a complex $(M \otimes_{\mathcal{O}_X} \Omega^\bullet_{X|S}, \nabla^\bullet)$ if and only if $K = 0$. Moreover, local computations as done in Chapter 2 show that (M, ∇) is integrable if and only if its curvature is 0.

Let X be a smooth $\mathbb{C}$-scheme of finite type. Thanks to Hironaka's resolution of singularities we can find $\overline{X}$ a proper, smooth $\mathbb{C}$-scheme of finite type and a divisor with normal crossings D such that $X = \overline{X} \smallsetminus D$. We say that an integrable connection (M, ∇) on X has *regular singularities at infinity* if for any compactification $\overline{X}$ as above we can extend (M, ∇) to a connection $(\overline{M}, \overline{\nabla})$ on $\overline{X}$ which has regular singularities along D. This notion does not depend on the compactification chosen because if $\overline{X}_1, \overline{X}_2$ are two such compactifications, there exists a third one $\overline{X}_3$ that dominates both. Furthermore, GAGA shows that $(\overline{M}, \overline{\nabla})$ has regular singularities along D if and only if the analytification $(\overline{M}^{\mathrm{an}}, \overline{\nabla}^{\mathrm{an}})$ on X^{an} has regular singularities along D.

Apart from this definition of connections on a scheme $X|S$ which is completely analogous to that of holomorphic connections, we may give another definition in the spirit of Grothendieck. Let X be a smooth S-scheme and consider the diagonal immersion $\Delta : X \to X \times_S X$. If $\mathscr{I}$ is the ideal sheaf of the diagonal map, we may consider the structure sheaf of the first infinitesimal neighborhood $\mathscr{P}_{X|S} = \mathcal{O}_{X \otimes_S X}/\mathscr{I}^2$. This structure sheaf has two natural $\mathcal{O}_X$-module structures which are induced by the pullback along the two projections $p_1, p_2 : X \times_S X \to X$. By definition, the kernel of the map $\mathscr{P}_{X|S} \to \Delta_* \mathcal{O}_X$ is the sheaf of differentials $\Omega^1_{X|S}$ and the two $\mathcal{O}_X$-module structures on $\mathscr{P}_{X|S}$ induce the same $\mathcal{O}_X$-module structure on $\Omega^1_{X|S}$. A local computation shows that the universal derivation $\mathrm{d} : \mathcal{O}_X \to \Omega^1_{X|S}$ satisfies $\mathrm{d}(f) = p_2^*(f) - p_1^*(f)$ for local sections f of $\mathcal{O}_X$ since the derivation is defined by $\mathrm{d}(b) = 1 \otimes b - b \otimes 1$ in the module of Kahler differentials of a commutative ring.

Definition 4.1.2. An S-connection on a locally free sheaf M on a smooth S-scheme X is a $\mathscr{P}_{X|S}$-linear isomorphism

$$\xi : \mathscr{P}_{X|S} \otimes_{\mathcal{O}_X} M \to M \otimes_{\mathcal{O}_X} \mathscr{P}_{X|S}$$

that lifts to the identity on M (modulo the augmentation ideal $\Omega^1_{X|S}$). Equivalently, an S-connection on M is a linear isomorphism $p_2^* M \to p_1^* M$.

Indeed, notice that the lifting property of ξ to the identity on M means that $\xi(1 \otimes s) = s \otimes 1 + \nabla(s)$ for a unique section $\nabla(s)$ of the subsheaf $M \otimes_{\mathcal{O}_X} \Omega^1_{X|S}$ The $\mathscr{P}_{X|S}$-linearity says that $p_2^*(f)\xi(1 \otimes s) = \xi(1 \otimes fs)$ or equivalently

$$s \otimes p_2^*(f) + f\nabla(s) = s \otimes p_1^*(f) + \nabla(fs)$$

and since $\mathrm{d}(f) = p_2^*(f) - p_1^*(f)$ this is precisely the Leibniz rule and the two notions of connections coincide.

Furthermore, in this context, the integrability condition can be thought of as a cocycle condition. Suppose that $\xi : p_2^* M \to p_1^* M$ is a linear isomorphism. Let p_{ij} be the projections from the first infinitesimal neighborhood of the triple diagonal in $X \times_S X \times_S X$ to the first infinitesimal neighborhood of the diagonal in $X \times_S X$. Pulling back ξ along the q_{ij} defines linear isomorphisms $\xi_{ij} : q_j^* M \to q_i^* M$ where $q_1, q_2, q_3 : X \times_S X \times_S X \to X$ are the projections. The integrability of the connection is equivalent for the ξ_{ij} to satisfy the cocycle condition $\xi_{12}\xi_{23}\xi_{31} = 1$. It can be proved that this notion of integrability coincides with the usual one.

4.2 Connections in characteristic p

Suppose that R is an $\mathbb{F}_p$-algebra and X a smooth R-scheme. Given M a locally free, coherent sheaf on X and $\nabla : M \to M \otimes \Omega^1_{X|R}$ an integrable connection, recall that ∇ can be viewed as a way of differentiating sections. Namely as the $\mathcal{O}_X$-linear map $\mathrm{Der}_R(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{End}_R(M)$ defined by $D \mapsto \nabla_D$. The Leibniz rule implies that

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} D^i(f) D^{n-i}(g)$$

for D a section of $\mathrm{Der}_R(\mathcal{O}_X, \mathcal{O}_X)$ and f, g sections of $\mathcal{O}_X$. Hence, in characteristic $p > 0$ if we set $n = p$ we find that D^p also satisfies the Leibniz rule. Therefore, both $\mathrm{Der}_R(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathrm{End}_R(M)$ are provided with a p-Lie algebra structure so we may ask whether ∇ is compatible with these structures or not. The obvious measure of the failure of compatibility is the *p-curvature* of ∇, defined as

$$\psi_p(D) = \nabla_D^p - \nabla_{D^p}$$

for any section D of $\mathrm{Der}_R(\mathcal{O}_X, \mathcal{O}_X)$ and one may easily verify that this defines an $\mathcal{O}_X$-linear mapping $\psi_p : \mathrm{Der}_R(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{End}_R(M)$. Moreover, a somewhat more involved computation shows that this map ψ_p is also p-linear, which means that

$$\psi_p(f_1 D_1 + f_2 D_2) = f_1^p \psi_p(D_1) + f_2^p \psi_p(D_2)$$

for f_1, f_2 local sections of $\mathcal{O}_X$ and D_1, D_2 local sections of $\mathrm{Der}_R(\mathcal{O}_X, \mathcal{O}_X)$.

The following theorem by Cartier is, in some sense, an analogue of the Riemann-Hilbert correspondence in characteristic $p > 0$. It shows that the p-curvature is an obstruction to having enough horizontal sections.

Theorem 4.2.1 (Cartier). *Let R be a $\mathbb{F}_p$-algebra and X be a smooth R-scheme. Let $F_{\mathrm{abs}} : R \to R$ the absolute Frobenius (that is, the p-th power mapping on $\mathcal{O}_R$) and $X^{(p)} = X \times_{F_{\mathrm{abs}}} R$ the fiber product of $F_{\mathrm{abs}} : R \to R$ and $X \to R$. Let $F : X \to X^{(p)}$ be the relative Frobenius (that is the elevation of vertical coordinates to the p-th power). There is an equivalence of categories between the category of quasi-coherent sheaves on $X^{(p)}$ and the full subcategory of the category of integrable R-connections on X whose p-curvature is zero.*

Proof. The idea for the proof is to take the same functors as in Theorem 2.2.2. Indeed, suppose that we are given a quasi-coherent sheaf $\mathscr{F}$ on $X^{(p)}$ then we have a canonical R-connection $(F^*\mathscr{F}, \nabla_{\mathrm{can}})$ on X defined by $f \otimes s \mapsto s \otimes \mathrm{d}f$.

Clearly, ∇^{can} has p-curvature zero and the canonical map $\mathscr{F} \to (F^*(\mathscr{F}))^{\nabla_{\mathrm{can}}}$ is an isomorphism. The functor in the reverse direction is given by taking the horizontal sections. In other words, if (M, ∇) is an integrable R-connection on X we consider the map $(M, \nabla) \mapsto M^{\nabla}$. To check that the canonical map of sheaves $F^*(M^{\nabla}) \to M$ is an isomorphism, since the question is local on X, we may suppose that X is affine and étale over $\mathbb{A}^r_R$ with $\Omega^1_{X|R}$ free on $\{\mathrm{d}x_1, \dots, \mathrm{d}x_r\}$. By looking at the Taylor expansion of the horizontal sections in this local basis, we can check that the canonical map is an isomorphism. ■

If (M, ∇) is an integrable R-connection on X such that the p-curvature is zero, then there exists a covering of X by affine open subsets U, and on each U sections $u_1, \dots, u_r$ of $\mathcal{O}_X$ over U such that $\Omega^1_{U|R}$ is free on $\mathrm{d}u_1, \dots, \mathrm{d}u_r$ and

$$\nabla\left(\frac{\partial}{\partial u_i}\right)^p = 0$$

for every $i = 1, \dots, r$. This is clear because $\psi_p\left(\frac{\partial}{\partial u_i}\right) = \left(\nabla\left(\frac{\partial}{\partial u_i}\right)\right)^p$ since $\left(\frac{\partial}{\partial u_i}\right)^p = 0$. Moreover, the converse also holds: if we take any derivation $D \in \mathrm{Der}(U|R)$ we can write $D = \sum_{i=1}^{r} a_i \frac{\partial}{\partial u_i}$ and so

$$\psi_p(D) = \sum_{i=1}^{r} a_i^p \psi_p\left(\frac{\partial}{\partial u_i}\right) = \sum_{i=1}^{r} a_i^p \left(\nabla\left(\frac{\partial}{\partial u_i}\right)\right)^p = 0.$$

With the same flow of ideas we can prove the following.

Proposition 4.2.2. *Let R be an $\mathbb{F}_p$-algebra and X be a smooth R-scheme. If (M, ∇) is an integrable R-connection on X and n is a positive integer, the following conditions are equivalent:*

- *If $D_1, \dots, D_n$ are sections of* $\mathrm{Der}(X|R)$ *over on open subset of X, then*

$$\psi_p(D_1) \cdots \psi_p(D_n) = 0.$$

- *There exists a covering of X by affine open subsets U and on each U, sections $u_1, \dots, u_r \in \mathcal{O}_X(U)$ such that $\Omega^1_{U|R}$ is free on* $\mathrm{d}u_1, \dots, \mathrm{d}u_r$ *and for every r-*

tuple $(w_1, \dots, w_r)$ of integers with $\sum_{i=1}^{r} w_i = n$ we have

$$\left(\nabla\left(\frac{\partial}{\partial u_1}\right)\right)^{pw_1} \cdots \left(\nabla\left(\frac{\partial}{\partial u_r}\right)\right)^{pw_r} = 0.$$

If these conditions hold, we say that (M, ∇) is nilpotent of exponent at most n. Clearly, connections of exponent at most 1 are precisely those which have p-curvature zero.

We can also characterize nilpotency of a connection $(\mathscr{E}, \nabla)$ of exponent at most n in terms of filtrations. One can show this is equivalent to the existence of a filtration

$$0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = \mathscr{E}$$

such that ∇ induces a connection ∇^i on each graded piece F^i/F^{i+1} and each ∇^i has vanishing p-curvature. This is helpful for inductive arguments since any connection $(\mathscr{E}, \nabla)$ nilpotent of exponent at most n fits into a short exact sequence

$$0 \longrightarrow (\mathscr{E}', \nabla') \longrightarrow (\mathscr{E}, \nabla) \longrightarrow (\mathscr{E}'', \nabla'') \longrightarrow 0$$

where $(\mathscr{E}', \nabla')$ has zero p-curvature and $(\mathscr{E}'', \nabla'')$ is nilpotent of exponent at most $n - 1$.

The following proposition describes the behaviour of the p-curvature (and more generally, nilpotency) under pullbacks.

Proposition 4.2.3. *Let R be a $\mathbb{F}_p$-algebra and X, Y be smooth R-schemes. If $f : Y \to X$ is a morphism of R-schemes then for any integrable R-connection (M, ∇) on X which is nilponent of exponent at most n, the pullback $f^*(M, \nabla)$ is an integrable R-connection on Y which is also nilpotent of exponent at most n. In particular, if (M, ∇) has p-curvature zero, so does $f^*(M, \nabla)$.*

Proof. Let us first prove the assertion regarding the p-curvature, that is, the case of $n = 1$. Because of Cartier's Theorem 4.2.1 we know that any integrable R-connection (M, ∇) on X with p-curvature zero is isomorphic to $(F^*(\mathscr{F}), \nabla_{\text{can}})$ where $\mathscr{F} = M^{\nabla}$ is a quasi-coherent $X^{(p)}$-module. If F_X, F_Y are the relative Frobenius on X and Y respectively, it is clear that

$$f^*(F_X^*(\mathscr{F}), \nabla_{\text{can}}) = (F_Y^*(f^{(p)^*}(\mathscr{F})), \nabla_{\text{can}})$$

which has p-curvature zero. The general case is handled by induction on n:

write the short exact sequence of the discussion above and recall that the inverse image is a right exact functor. ■

4.3 Spreading out

Given X a connected, smooth $\mathbb{C}$-scheme of finite type we would like to understand, in some sense, its arithmetic. The main technique that we will use for this is the so-called *spreading out*. We would like to find a subring $R \subseteq \mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra, and a connected, smooth R-scheme $\mathbb{X}$ of finite type with geometrically connected fibers, from which we recover X by making the extension of scalars $R \hookrightarrow \mathbb{C}$.

Indeed, first consider the local case, namely when

$$X = \mathrm{Spec}(\mathbb{C}[X_1, \dots, X_n]/(f_1, \dots, f_r))$$

is the coordinate ring of a connected, smooth, affine variety. The spread-out in this case will be to consider R as the $\mathbb{Z}$-algebra generated by the coefficients of the f_i and $\mathbb{X} = \mathrm{Spec}(R[X_1, \dots, X_n]/(f_1, \dots, f_r))$. Clearly this R-scheme is connected, smooth and the extension of scalars $R \hookrightarrow \mathbb{C}$ gives back X since its simply the tensor product $\mathrm{Spec}(R[X_1, \dots, X_n]/(f_1, \dots f_r) \otimes_R \mathbb{C})$.

The Nagata compactification theorem ([Con07]) shows that if X is a connected, smooth $\mathbb{C}$-scheme of finite type, there exists a proper $\mathbb{C}$-scheme $\overline{X}$ and a dense open immersion $j : X \to \overline{X}$ over $\mathbb{C}$. Due to the properness, $\overline{X}$ is a finite union of coordinate rings of affine varieties and we may consider R the $\mathbb{Z}$-algebra generated by the coefficients of all the polynomials defining these varieties. If we spread-out each variety over R as in the previous paragraph, then a glueing procedure gives us a proper R-scheme $\overline{\mathbb{X}}$ such that extension of scalars $R \hookrightarrow \mathbb{C}$ gives back $\overline{X}$. So we have the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "j"] \arrow[dr] & & \overline{X} \arrow[r] \arrow[dl] & \overline{\mathbb{X}} \arrow[dl] \\
& \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(R). &
\end{tikzcd}$$

For technical purposes, we will need $\overline{\mathbb{X}} \to \mathrm{Spec}(R)$ to be flat. By generic flatness there must be an open of $\mathrm{Spec}(R)$ over which $\overline{\mathbb{X}}$ is flat. By further shrinking to an open affine, we may replace R with a larger ring $R[1/f]$ for certain $f \in R$ such that $\overline{\mathbb{X}} \to \mathrm{Spec}(R)$ is flat. The properness still holds since it is stable by base change. Hence we have a ring R and a flat, proper R-scheme $\overline{\mathbb{X}}$ from which we recover $\overline{X}$ by extension of scalars $R \hookrightarrow \mathbb{C}$.

Finally, define $\mathbb{X}$ as the image of X via the composition map $X \xrightarrow{j} \overline{X} \to \overline{\mathbb{X}}$, which is clearly a connected, smooth R-scheme of finite type. The assertion regarding the geometrically connectedness of the fibers follows from [Gro66, Corollaire 15.5.4]. Indeed, $\overline{\mathbb{X}}$ is a flat, proper R-scheme and the generic fiber is geometrically connected (since further extending the generic fiber to $\mathbb{C}$ we obtain $\overline{X}$) and hence every closed fiber is geometrically connected. Because $\mathbb{X}$ is an open subscheme of $\overline{\mathbb{X}}$, the connectedness of the fibers follows.

To summarize, given X a connected, smooth $\mathbb{C}$-scheme of finite type we have constructed a subring $R \subseteq \mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra, and a connected smooth R-scheme $\mathbb{X}$ of finite type, with geometrically connected fibers such that the extension of scalars $R \hookrightarrow \mathbb{C}$ gives back X. A pair $(\mathbb{X}, R)$ satisfying this is called a *spread out* of X. This allows us to *reduce modulo p* for $p \in \mathbb{Z}$ any prime not invertible in R since we may extend to scalars via $R \hookrightarrow R/pR = R \otimes_{\mathbb{Z}} \mathbb{F}_p$ and hopefully the arithmetic of this family of schemes will allow us to conclude something regarding our original scheme X.

Notice that the choice of a spread-out is certainly non-unique. However, given two spread-outs $(\mathbb{X}_1, R_1)$ and $(\mathbb{X}_2, R_2)$ there exists a third spread-out $(\mathbb{X}_3, R_3)$ such that $R_1, R_2 \subseteq R_3$ and base-changing $\mathbb{X}_i$ along $R_i \hookrightarrow R_3$ gives $\mathbb{X}_3$. Indeed, one may take R_3 as the $\mathbb{Z}$-algebra generated by R_1 and R_2 simultaneously and then base-changing $\mathbb{X}_1$ or $\mathbb{X}_2$ give the same result by using the pullback lemma and that base-changing to $\mathbb{C}$ both give X. As we are only interested in spreading-out to reduce modulo p, the burden of choice is lightened by the following technical lemma.

Lemma 4.3.1. *Let R_1, R_2 be two integral domains with fraction fields of characteristic 0 which are both finitely generated as $\mathbb{Z}$-algebras. Suppose that $R_1 \hookrightarrow R_2$. Then for all but finitely many primes p, the induced map between their reductions modulo p*

$$R_1/pR_1 \hookrightarrow R_2/pR_2$$

is injective.

Proof. We may assume that R_2 is flat over R_1. Indeed, by generic flatness for rings we may assume that $R_2[1/f]$ is flat over R_1 for some $f \in R_2$, $f \neq 0$

and if the theorem is true for $R_1 \subseteq R_2[1/f]$ it is certainly true for $R_1 \subseteq R_2$ with at worst the same set of exceptional primes. If R_2 is flat over R_1, then by Chevalley's constructible image theorem (or more precisely by Corollary 1.3.26) there exists an element $g \in R_1$, $g \neq 0$ such that $R_2[1/g]$ is faithfully flat over $R_1[1/g]$. Therefore, $R_1[1/g] \to R_2[1/g]$ is universally injective and in particular it must be injective modulo p for every p. Hence, we are reduced to the case $R_1 \subseteq R_1[1/g]$. The induced map between the reductions modulo p

$$R_1 \otimes \mathbb{F}_p \to (R_1[1/g]) \otimes \mathbb{F}_p = (R_1 \otimes \mathbb{F}_p)[1/g]$$

is injective if and only if the endomorphism of $R_1 \otimes \mathbb{F}_p$ given by multiplication by g is injective. Consider the short exact sequence

$$0 \longrightarrow R_1 \xrightarrow{\cdot g} R_1 \longrightarrow R_1/gR_1 \longrightarrow 0.$$

The quotient R_1/gR_1 is a finitely generated $\mathbb{Z}$-algebra and hence after inverting some integer $N \geq 1$ the algebra $(R_1/gR_1)[1/N]$ will be flat over $\mathbb{Z}[1/N]$. For such N, we have a short exact sequence

$$0 \longrightarrow R_1[1/N] \xrightarrow{\cdot g} R_1[1/N] \longrightarrow (R_1/gR_1)[1/N] \longrightarrow 0$$

whose last term is $\mathbb{Z}[1/N]$-flat. Thus, that sequence remains exact after tensoring over $\mathbb{Z}[1/N]$ with any $\mathbb{Z}[1/N]$-module. In particular, if p is a prime not dividing N we may tensor with $\mathbb{F}_p$ to obtain the short exact sequence

$$0 \longrightarrow R_1 \otimes \mathbb{F}_p \xrightarrow{\cdot g} R_1 \otimes \mathbb{F}_p \longrightarrow (R_1/gR_1) \otimes \mathbb{F}_p \longrightarrow 0$$

which implies the injectivity of the multiplication by g map on $R_1 \otimes \mathbb{F}_p$. ■

In the same fashion as in the previous discussion, if X is a connected, smooth $\mathbb{C}$-scheme of finite type and (M, ∇) is an algebraic differential equation, we may spread out so that there exists a locally free coherent sheaf $\mathbb{M}$ together with an integrable connection ∇ so that we recover (M, ∇) by making the extension of scalars $R \hookrightarrow \mathbb{C}$. Indeed, this is just the same argument of glueing local data also adding the coefficients of the connection matrix.

We shall say that a finitely generated $\mathbb{Z}$-algebra whose field of fractions is of characteristic 0 is a *global affine variety*. Therefore, by spreading out, we have reduced to studying certain aspects of integrable connections on smooth $\mathbb{C}$-schemes to that of integrable connections on schemes over a global affine variety. We are mainly interested in the local to global interplay for connections on schemes over global affine varieties and how we can extrapolate this interplay to the connection we had before spreading out.

Definition 4.3.2. Let X be a smooth connected $\mathbb{C}$-scheme and (M, ∇) an integrable connection on X. We say that (M, ∇) is *globally nilpotent of exponent at most* n if for any spreading out $(\mathbb{X}, R)$ the reduction modulo p (for non-invertible primes on R) of the connection $(\mathbb{M}, \nabla)$ is nilpotent of exponent at most n. We say that (M, ∇) is an *arithmetically flat connection* if any spreading out is globally nilpotent of exponent at most 1 (that is, the p-curvature of the reduction modulo p is zero for any prime p that is not invertible in R).

Notice that the concept of globally nilpotent connections is well defined because of the Lemma 4.3.1. Moreover, the following Proposition is a straightforward consequence of the Proposition 4.2.3.

Proposition 4.3.3. *Let R_1, R_2 be global affine varieties and X_1, X_2 be smooth schemes over R_1 and R_2 respectively. If we have maps $f : X_2 \to X_1$ and $g : R_2 \to R_1$ such that the diagram*

$$\begin{array}{ccc} X_2 & \xrightarrow{f} & X_1 \\ \downarrow & & \downarrow \\ R_2 & \xrightarrow{g} & R_1 \end{array}$$

commutes, and (M, ∇) is a globally nilpotent connection on X_1 of exponent at most n, then so is $(f, g)^*(M, \nabla)$.

More interestingly, in the case that the base $R_1 = R_2$ is the same, the previous proposition admits a certain converse provided that the map f is good enough.

Proposition 4.3.4. *Let R be a global affine variety and X, Y be smooth schemes over R. If $f : Y \to X$ is a proper étale map and (M, ∇) is an integrable connection on X, then (M, ∇) is globally nilpotent of exponent at most n if and only if $f^*(M, \nabla)$ is globally nilpotent of exponent at most n. In particular, (M, ∇) is arithmetically flat if and only if $f^*(M, \nabla)$ is arithmetically flat.*

Proof. As in Proposition 4.2.3, we shall prove only the case of arithmetically flat connections, since the general case will follow by induction on n. The key is again to use Cartier's Theorem 4.2.1. As f is étale and proper, the sheaf of horizontal sections of an integrable connection over X is quasi-coherent over $X^{(p)}$ if and only if its pullback connection via $f^{(p)}$ is quasi-coherent over $Y^{(p)}$. ■

These base-change properties will allow us, via a *dévissage* argument, to reduce the study of arbitrary connections to the case of relative dimension 1. The advantage of studying curves is that we can understand the algebraic and arithmetic aspects more easily: instead of working with the curves, we shall work with their function fields. If X is a smooth connected algebraic variety over $\mathbb{C}$, we will say that a connection (M, ∇) has finite local monodromy at infinity if, for every smooth connected complete complex curve C, every finite subset $S \subseteq C$ and every morphism $f : C \smallsetminus S \to X$, the pullback connection $f^*(M, \nabla)$ on $C \smallsetminus S$ has finite local monodromy around every point $s \in S$. We may likewise change "finite" for "quasi-unipotent" and "regular singularities". These notions are in accordance with what we would obtain via analytification.

4.4 Connections with regular singularities in curves

Let k be a field of characteristic 0 and K a function field in one variable over k. As we mentioned in the previous section, this is the same as saying that K is the function field of a projective, smooth, absolutely irreducible curve over k. Let W be a finite dimensional vector space over K. A k-connection ∇ on W is an additive mapping

$$\nabla : W \to \Omega^1_{K|k} \otimes_K W$$

which satisfies the Leibniz rule

$$\nabla(fw) = \mathrm{d}f \otimes w + f\nabla(w)$$

for every $f \in K, w \in W$. Equivalently, ∇ is a K-linear mapping

$$\nabla : \mathrm{Der}(K|k) \to \mathrm{End}_k(W)$$

such that $\nabla(D)(fw) = D(f)w + f(\nabla(D))(w)$ for $D \in \mathrm{Der}(K|k), f \in K$ and $w \in W$. Notice that every k-connection on $K|k$ is automatically integrable since $\Omega^2_{K|k} = 0$.

The category of k-connections on $K|k$ has objects are k-connections (W, ∇) and morphisms are $\phi : (W_1, \nabla_1) \to (W_1, \nabla_2)$ where $\phi : W_1 \to W_2$ is a K-linear map compatible with the connections, namely the diagam

$$\begin{tikzcd} W_1 \arrow[r, "\phi"] \arrow[d, "\nabla_1"'] & W_2 \arrow[d, "\nabla_2"] \\ W_1 \otimes_K \Omega^1_{K|k} \arrow[r, "\phi \otimes 1"'] & W_2 \otimes_K \Omega^1_{K|k} \end{tikzcd}$$

commutes. It is easy to verify, as in chapter 3 section 3.4, that it is a neutral Tannakian category.

Suppose that (W, ∇) is a k-connection on $K|k$. If every entry of the connection matrix has at most a single pole at $\mathfrak{p}$, we say that $\mathfrak{p}$ is a *regular singular point* of (W, ∇). It is clear that this definition coincides with the one given at Definition 2.4.2 in the case of a smooth curve. A straightforward computation shows that $\mathfrak{p}$ is a regular singular point if and only if

$$\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{e} = B\mathbf{e}$$

for a certain matrix $B \in \mathrm{M}_n(\mathcal{O}_\mathfrak{p})$ and $\mathbf{e} = (e_1, \ldots, e_n)$ a basis of W as a K-vector space.

The following result is very useful for inductive arguments.

Proposition 4.4.1. *Let k be a field of characteristic 0 and $K|k$ a function field in one variable over k. Suppose that*

$$0 \longrightarrow (W_2, \nabla_2) \longrightarrow (W_1, \nabla_1) \longrightarrow (W_3, \nabla_3) \longrightarrow 0$$

is an exact sequence in the category of k-connections on $K|k$. Then (W_1, ∇_1) has a regular singular point at $\mathfrak{p}$ if and only if both (W_2, ∇_2) and (W_3, ∇_3) have a regular singular point at $\mathfrak{p}$.

Proof. Take t a uniformizing parameter at $\mathfrak{p}$. Let $\mathbf{e} = (e_1, \ldots, e_n)$ be a basis of W_2 and $\mathbf{f} = (f_1, \ldots, f_m)$ a basis of W_3 such that $\nabla_2\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)(\mathbf{e}) = A\mathbf{e}$ and $\nabla_3\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)(\mathbf{f}) = C\mathbf{f}$ such that $A \in \mathrm{M}_n(\mathcal{O}_\mathfrak{p})$ and $C \in \mathrm{M}_m(\mathcal{O}_\mathfrak{p})$. Then, we have that

$$\nabla_1\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix} = \begin{pmatrix}A & 0\\ B & C\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix}$$

for some $B \in \mathrm{M}(m \times n, K)$. The problem is that the entries of B may not be regular. However, for any ν sufficiently large we have that the entries of $t^\nu B$ are regular and since

$$\nabla_1\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\begin{pmatrix}\mathbf{e}\\ h^\nu\mathbf{f}\end{pmatrix} = \begin{pmatrix}A & 0\\ h^\nu B & C+\nu\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ h^\nu\mathbf{f}\end{pmatrix}$$

which implies that (W_1, ∇_1) has a regular singular point at $\mathfrak{p}$.

Conversely, suppose that $(e_1, \ldots, e_n, f_1, \ldots, f_m)$ is a basis for W_1 which makes $\nabla_1\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\binom{\mathbf{e}}{\mathbf{f}} = B\binom{\mathbf{e}}{\mathbf{f}}$. Hence, the basis $\mathbf{e}$ and $\mathbf{f}$ of W_2 and W_3 respectively make $\mathfrak{p}$ a regular singular point of (W_2, ∇_2) and (W_3, ∇_3). ■

We say that a k-connection (W, ∇) is *cyclic* if there is a vector $w \in W$ such that for some non-zero derivation $D \in \mathrm{Der}(K|k)$ the vectors

$$w, \nabla(D)(w), (\nabla(D))^2(w), (\nabla(D))^3(w), \ldots$$

span W over K. A direct consequence of Proposition 4.4.1 is the following.

Corollary 4.4.2. *Let (W, ∇) be a k-connection on $K|k$. Then (W, ∇) has a regular singular point at $\mathfrak{p}$ if and only if every cyclic subobject of (W, ∇) has a regular singular point at $\mathfrak{p}$.*

Proof. If every cyclic subobject has a regular singular point at $\mathfrak{p}$, since (W, ∇) is a quotient of a direct sum of finitely many of its cyclic subobjects, Proposition 4.4.1 gives the claim. Conversely, if (W, ∇) has a regular singular point at $\mathfrak{p}$, Proposition 4.4.1 tells us that every subobject has a regular singular point at $\mathfrak{p}$. ■

Let (W, ∇) be a k-connection on $K|k$. An $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ in W is a subgroup of W which is a free $\mathcal{O}_\mathfrak{p}$-module of rank equal to $\dim_K(W)$. We say that (W, ∇) satisfies *Jurkat's estimate* at $\mathfrak{p}$ for the lattice $W_\mathfrak{p}$ if there exists an integer ν such that for every integer $j \geq 1$ and every j-tuple $D_1, \ldots, D_j \in \mathrm{Der}_\mathfrak{p}(K|k)$ we have

$$\nabla(D_1)\cdots\nabla(D_j)(W_\mathfrak{p}) \subseteq t^\nu(W_\mathfrak{p})$$

for a uniformizing parameter t at $\mathfrak{p}$. In this form, this condition is hard to work with. However, we can restate this in more manageable terms. Let D_0 be an

$\mathcal{O}_\mathfrak{p}$-basis of $\mathrm{Der}_\mathfrak{p}(K|k)$, then for any $D_1, \ldots, D_j \in \mathrm{Der}_\mathfrak{p}(K|k)$ we can write

$$\nabla(D_1) \cdots \nabla(D_j) = \sum_{i=0}^{j} a_i (\nabla(D_0))^i$$

for certain $a_0, \ldots, a_j \in \mathcal{O}_\mathfrak{p}$. Hence, Jurkat's estimate holds if and only if, for some $\mathcal{O}_\mathfrak{p}$-basis D_0 of $\mathrm{Der}_\mathfrak{p}(K|k)$ one has

$$\nabla(D_0)^j(W_\mathfrak{p}) \subseteq t^\nu(W_\mathfrak{p})$$

for every $j \geq 1$. Moreover, if we choose a basis $\mathbf{e}$ for $W_\mathfrak{p}$, for each $j \geq 1$ we can define a matrix $B_j \in \mathrm{M}_n(K)$ by

$$(\nabla(D_0))^j \mathbf{e} = B_j \mathbf{e}$$

and thus Jurkat's estimate boils down to $\mathrm{ord}_\mathfrak{p}(B_j) \geq \nu$ for every $j \geq 1$. Reminding that for a uniformizing parameter t at $\mathfrak{p}$, we can take $t\frac{\mathrm{d}}{\mathrm{d}t}$ as an $\mathcal{O}_\mathfrak{p}$-basis of $\mathrm{Der}_\mathfrak{p}(K|k)$, Jurkat's estimate is now a fairly concrete condition. One may easily verify that Jurkat's estimate does not depend of the basis chosen.

Proposition 4.4.3. *If (W, ∇) has a regular singular point at $\mathfrak{p}$, then it satisfies Jurkat's estimate at $\mathfrak{p}$.*

Proof. If $\mathbf{e} = (e_1, \ldots, e_n)$ is a basis such that $\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{e} = B\mathbf{e}$ for $B \in \mathrm{M}_n(\mathcal{O}_\mathfrak{p})$ for a uniformizing parameter t at $\mathfrak{p}$, then the matrix B_j such that

$$\left(\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)^j \mathbf{e} = B_j \mathbf{e}$$

verifies the recurrence relation $B_{j+1} = t\frac{\mathrm{d}}{\mathrm{d}t}(B_j) + B_j B$. Hence, each entry of B_j is regular at $\mathfrak{p}$ or in other words $\mathrm{ord}_\mathfrak{p}(B_j) \geq 0$. This implies that Jurkat's estimate holds. ■

The following theorem of Turritin gives us a characterization for regular singular points in the case of curves.

Theorem 4.4.4 (Turritin)**.** *Let k be a field of characteristic 0 and $K|k$ a function field in one variable over k. Let (W, ∇) be a k-connection on $K|k$, $\mathfrak{p}$ a place of $K|k$ and t be a uniformizing parameter at $\mathfrak{p}$. If $n = \dim_K(W)$, the following are equivalent:*

1. *(W, ∇) does not have a regular singular point at $\mathfrak{p}$.*

2. *Jurkat's estimate is not satisfied at* $\mathfrak{p}$.

3. *For every integer a multiple of* $n!$ *there exists a basis* $\mathbf{e}$ *of* $W \otimes_K K(t^{1/a})$ *in terms of which the connection is expressed as*

$$\nabla\left(h\frac{\mathrm{d}}{\mathrm{d}h}\right)\mathbf{e} = B\mathbf{e}$$

where $h = t^{1/a}$ *and* $B = h^{-\nu}B_{-\nu}$ *with* $\nu \geq 1$ *and* $B_{-\nu} \in \mathrm{M}_n(\mathcal{O}_{\mathfrak{p}^{1/a}})$ *has non-nilpotent image in* $\mathrm{M}_n(\kappa(\mathfrak{p}))$.

Proof. The idea of the proof will be to proceed by induction on n invoking Proposition 4.4.1. If (W, ∇) has no proper non-zero subobjects it is cyclic and hence by Corollary 4.4.2 we are done. If not, there is a subobject (W', ∇') and we hace a short exact sequence

$$0 \longrightarrow (W', \nabla') \longrightarrow (W, \nabla) \longrightarrow (W'', \nabla'') \longrightarrow 0$$

with $\dim_K W', \dim_K W'' < n$. Now both (W', ∇') and (W'', ∇'') admit an expresion as in 3. For the details of the rest of the proof, see [Kat70]. ∎

4.5 Monodromy revisited

In this section, we will study monodromy in the algebraic setting and provide the proof a criterion that relates local monodromy to the p-curvature given at [Kat70]. Let $K|k$ be a function field in one variable over a field k of characteristic 0. Suppose that $\mathfrak{p}$ is a place of $K|k$ which is rational, or in other words, $\kappa(\mathfrak{p}) = k$ and suppose that (W, ∇) is a k-connection for which $\mathfrak{p}$ is a regular singularity. In terms of a uniformizing parameter t at $\mathfrak{p}$, and a basis $\mathbf{e}$ of an $\mathcal{O}_{\mathfrak{p}}$-lattice $W_{\mathfrak{p}}$ of W which is stable under $\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)$ we express the connection as

$$\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{e} = B\mathbf{e}$$

for $B \in \mathrm{M}_n(\mathcal{O}_{\mathfrak{p}})$. The conjugacy class of the matrix $B(\mathfrak{p}) \in \mathrm{M}_n(k)$ depends only on the lattice $W_{\mathfrak{p}}$ but not on the particular choice of a basis for $W_{\mathfrak{p}}$ or on the choice of the uniformizing parameter t. Indeed, if $\mathbf{e}$ and $\mathbf{f}$ are $\mathcal{O}_{\mathfrak{p}}$-bases for $W_{\mathfrak{p}}$, we can write $\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{e} = B\mathbf{e}$ and $\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{f} = C\mathbf{f}$ with $B, C \in \mathrm{M}_n(\mathcal{O}_{\mathfrak{p}})$. If

$A \in \mathrm{M}_n(\mathcal{O}_\mathfrak{p})$ is the base-change matrix $A\mathbf{e} = \mathbf{f}$, then it follows that

$$CA\mathbf{e} = \nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)(A\mathbf{e}) = \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)(A)\mathbf{e} + A\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{e} = \left(t\frac{\mathrm{d}A}{\mathrm{d}t}\right)\mathbf{e} + AB\mathbf{e}.$$

After reducing modulo $\mathfrak{m}_\mathfrak{p}$, we have $t\frac{\mathrm{d}A}{\mathrm{d}t} = 0$ and, since $\mathfrak{p}$ is a rational place we have $\kappa(\mathfrak{p}) = \mathcal{O}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} = k$ and therefore $C(\mathfrak{p})A(\mathfrak{p})\mathbf{e} = A(\mathfrak{p})B(\mathfrak{p})\mathbf{e}$ in $\mathrm{M}_n(k)$. As the conjugacy class of $B(\mathfrak{p})$ does not depend on the basis chosen, the eigenvalues of $B(\mathfrak{p})$ are always the same. Moreover, if we assume that k is algebraically closed then all of the eigenvalues are in k and the set of images in the additive group $k^+/\mathbb{Z}$ of the eigenvalues of $B(\mathfrak{p})$ is independent of the choice of the $\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)$-stable $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ in W. These images are called the *exponents* of (W, ∇) at $\mathfrak{p}$. Fix a set-theoretic section $\phi : k^+/\mathbb{Z} \to k^+$ of the projection mapping $k^+ \to k^+/\mathbb{Z}$ (for example, in the case $k = \mathbb{C}$ we could choose $0 \leq \mathrm{Re}(\phi) < 1$). A theorem of Manin (see [Man65]) says that there exists a unique $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ of W stable under $\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)$ in terms of a basis $\mathbf{e}$ of which the connection is expressed as

$$\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{e} = C\mathbf{e}$$

with $C \in \mathrm{M}_n(\mathcal{O}_\mathfrak{p})$ and such that the eigenvalues of $C(\mathfrak{p}) \in \mathrm{M}_n(k)$ are all fixed by the composition $k^+ \to k^+/\mathbb{Z} \xrightarrow{\phi} k^+$. The point is that non-equal eigenvalues of $C(\mathfrak{p})$ do not differ by integers.

If $k = \mathbb{C}$, then consider X the compact Riemann surface such that the field of meromorphic functions of X is K. If x is the point that corresponds to the place $\mathfrak{p}$, a multivalued fundamental matrix of horizontal sections over a small punctured disc around x is given by $t^{-C(\mathfrak{p})}$ (contrast this to Example 2.3.2). Thus, when t turns around x counterclockwise, $\log(t)$ becomes $\log(t) + 2\pi\mathbf{i}$ and the fundamental matrix becomes $\exp(-2\pi\mathbf{i}C(\mathfrak{p}))t^{-C(\mathfrak{p})}$. Therefore, the *local monodromy matrix* is given by $\exp(-2\pi\mathbf{i}C(\mathfrak{p}))$. This motivates the following definition.

Definition 4.5.1. Let $K|k$ the function field in one variable over an algebraically closed field k of characteristic 0 and (W, ∇) a k-connection on $K|k$. If $\mathfrak{p}$ is a rational place and (W, ∇) has a regular singularity there, the *local monodromy* of (W, ∇) around $\mathfrak{p}$ is defined as $\exp(-2\pi\mathbf{i}C(\mathfrak{p}))$ where $C(\mathfrak{p})$ is chosen as in the discussion above.

As comment on the previous definition, if we were not to choose $C(\mathfrak{p})$ as before and we just fixed a basis $\mathbf{e}$ for a lattice $W_{\mathfrak{p}}$, took the matrix B such that $\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{e} = B\mathbf{e}$ and looked at the matrix $\exp(-2\pi\mathbf{i}B(\mathfrak{p}))$, the eigenvalues would have been the same. Moreover, if we denote $\sigma_1, \ldots, \sigma_n$ the exponents of (W, ∇) at $\mathfrak{p}$ it is easy to see, by looking at the Jordan normal form of the matrix $B(\mathfrak{p})$, that the eigenvalues of the monodromy are nothing else than $\exp(-2\pi\mathbf{i}\sigma_1), \ldots, \exp(-2\pi\mathbf{i}\sigma_n)$. Such matrix given by the theorem of Manin merely allows us to make a canonical choice when defining local monodromy.

Definition 4.5.2. Let $K|k$ be a function field in one variable, with k an algebraically closed field of characteristic 0. Let $\mathfrak{p}$ be a place of $K|k$ which is rational and (W, ∇) a k-connection on $K|k$ which has a regular singular point at $\mathfrak{p}$. We say that the local monodromy at $\mathfrak{p}$ is *quasi-unipotent* if the exponents at $\mathfrak{p}$ are rational numbers. If the local monodromy at $\mathfrak{p}$ is quasi-unipotent, we say that its exponent of nilpotence is at most ν if the nilpotent part N of the Jordan normal form of the matrix $C(\mathfrak{p})$ satisfies that $N^{\nu} = 0$.

Before stating and proving the theorem relating the p-curvature and local monodromy, we need one arithmetic fact. Since our intention is to understand the relationship between local invariants such as the p-curvature and the global aspects such as the existence of horizontal sections, it makes sense to think about other local to global principles. The following result, or more precisely its corollary, is one example of this and will be needed in order to prove the main theorem of this section.

Theorem 4.5.3 (Chebotarev density theorem)**.** *Let $L|K$ be a Galois extension extension with Galois group G. Let X be a subset of G that is stable under conjugation. The set of primes $\mathfrak{p}$ of K that are unramified in L and whose associated Frobenius conjugacy class* $\mathrm{Frob}_{\mathfrak{p}}$ *is contained in X has density $\sharp X/\sharp G$.*

Proof. See [Neu99, p. 545] for a proof. ∎

Corollary 4.5.4. *Let R be an integral domain of finite type over $\mathbb{Z}$ whose quotient field is of characteristic* 0*. Suppose that $a \in R$ is such that for every closed point $\mathfrak{p}$ of* $\mathrm{Spec}(R)$ *the image of a in the residue field $R/\mathfrak{p}$ at $\mathfrak{p}$ lies in the prime field. Then $a \in R \cap \mathbb{Q}$.*

Proof. Consider the splitting field L of a over $\mathbb{Q}$. Then, by Noether normalization lemma, $R \subseteq \mathcal{O}_L$ where $\mathcal{O}_L$ is the ring of integers of L. If the image of a lies in the prime field of the residue field $R/\mathfrak{p}$, by Dedekind's theorem the prime

$p = \mathbb{Z} \cap \mathfrak{p}$ must be unramified in L. It follows that the rational primes ramifying in L have null density. Appealing to Chebotarev density Theorem 4.5.3, this implies that $L = \mathbb{Q}$ and thus $a \in \mathbb{Q}$ as desired. ∎

We are now in conditions of proving the main theorem of this section.

Theorem 4.5.5 (Katz). *Let R be a global affine variety and X be a smooth one-dimensional R-scheme whose geometric fiber is geometrically connected. Let (M, ∇) be an integrable R-connection on X. If k is the function field of R and K is the function field of X. If (M, ∇) is globally nilpotent then the inverse image of (M, ∇) in the category of k-connections on $K|k$ has a regular singular point at every place $\mathfrak{p}$ of $K|k$ and has quasi-unipotent local monodromy at every place $\mathfrak{p}$ of $K|k$.*

Proof. Because of base-changing arguments and the local nature of the question, we may safely assume that X is a principal open subset of $\mathbb{A}^1_R$, or in other words, that $X = \operatorname{Spec}\left(R[t]\left[\frac{1}{g(t)}\right]\right)$ with $g(t) \in R[t]$. Moreover, we may assume that the place $\mathfrak{p}$ of $K = k(t)$ is defined by $t = 0$. We can write $g(t) = t^j h(t)$ with $h(t) \in R[t]$ and $j \geq 1$, since otherwise there would be no singularity at $t = 0$, and at the expense of localizing R at $h(0)$, we can also assume that $h(0)$ is invertible.

Suppose that (M, ∇) is globally nilpotent but that $t = 0$ is not a regular singular point of its restriction to the category of k-connections on $k(t)$. Let n be the rank of the free $R[t]\left[\frac{1}{g(t)}\right]$-module M. The change of variables $z = t^{1/n!}$ induces a map

$$R[t]\left[\frac{1}{g(t)}\right] \to R[z]\left[\frac{1}{g(z)}\right]$$

and due to Proposition 4.2.3 the inverse image of (M, ∇) on $\operatorname{Spec}\left(R[z]\left[\frac{1}{g(z)}\right]\right)$ is still globally nilpotent. Now, if $z = 0$ is not a regular singular point, Turritin's Theorem 4.4.4 implies that there exists a basis $\mathbf{m}$ of M (over an open subset of $\operatorname{Spec}\left(R[z]\left[\frac{1}{g(z)}\right]\right)$, which by enlarging g we may suppose to be all of $\operatorname{Spec}\left(R[z]\left[\frac{1}{g(z)}\right]\right)$) and in terms of this basis the connection is expressed as

$$\nabla\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\mathbf{m} = z^{-\mu}(A + zB)\mathbf{m}$$

for $\mu \geq 1$, $A \in \mathrm{M}_n(R)$ non-nilpotent and $B \in \mathrm{M}_n\left(R[z]\left[\frac{1}{h(z)}\right]\right)$. A straightforward computation shows that, for each integer $j \geq 1$, we have

$$\left(\nabla\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)^{j}\mathbf{m} = z^{-\mu j}(A^{j} + zB_{j})\mathbf{m}$$

with $B_j \in \mathrm{M}_n\left(R[z]\left[\frac{1}{h(z)}\right]\right)$.

Let p be a prime number. In $\mathrm{Der}(\mathbb{F}_p[z]|\mathbb{F}_p)$ we have $\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{p} = z\frac{\mathrm{d}}{\mathrm{d}z}$ and therefore the hypothesis of global nilpotence is that, for every prime number p, there is an integer $\alpha(p)$ such that

$$\left(\left(\nabla\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)^{p} - \nabla\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)^{\alpha(p)} M \subseteq pM.$$

However, we can rewrite this condition as

$$(z^{-\mu p}(A^{p} + zB_{p}) - z^{-\mu}(A + zB))^{\alpha(p)} \in p\mathrm{M}_n\left(R[z]\left[\frac{1}{g(z)}\right]\right).$$

Hence, looking at the most polar term we conclude that

$$A^{p\alpha(p)} \in \mathrm{M}_n(R)$$

for every prime p. In other terms, this means that the characteristic polynomial $\det(\xi - A) \in R[\xi]$ of A at every closed point of $\mathrm{Spec}(R)$ is ξ^n and this implies that $\det(\xi - A) = \xi^n$. But this in turn implies that A must be nilpotent. Therefore, $t = 0$ was a regular singular point of the inverse image of (M, ∇) in the category of k-connections on $k(t)$.

We now turn to proving the quasi-unipotence of the local monodromy at $t = 0$. Since $t = 0$ is a regular singular point, there is a basis $\mathbf{m}$ of M (again, over an open subset of $\mathrm{Spec}\left(R[t]\left[\frac{1}{g(t)}\right]\right)$ which by enlarging g, we may suppose to be all of $\mathrm{Spec}\left(R[t]\left[\frac{1}{g(t)}\right]\right)$) and in terms of which, the connection is expressed as

$$\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{m} = (A + tB)\mathbf{m}$$

with $A \in \mathrm{M}_n(R)$ and $B \in \mathrm{M}_n\left(R[t]\left[\frac{1}{h(t)}\right]\right)$. By adjoining the eigenvalues of A to R (and perhaps localizing the resulting ring) we can safely assume that the Jordan normal form of A is defined over R. Write $A = D + N$ where D is the diagonal part and N the nilpotent one. If (M, ∇) is globally nilpotent, for each prime number p we have that

$$\left(\left(\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)^p - \nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)^{\alpha(p)} M \subseteq pM.$$

A straightforward computation shows that for each integer $j \geq 1$ we have

$$\left(\nabla\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)^j \mathbf{m} = (A^j + tB_j)\mathbf{m}$$

for some matrix $B_j \in \mathrm{M}_n\left(R[t]\left[\frac{1}{h(t)}\right]\right)$. Looking at the constant term of that matricial expression we find that

$$(A^p - A)^{\alpha(p)} \in p\mathrm{M}_n(R)$$

for every prime number p. But looking at the Jordan normal form of A in that expression, it follows that $(D^p - D)^{\alpha(p)} \in p\mathrm{M}_n(R)$. Hence, if d is an eigenvalue of D, then it is a quantity in an integral domain R of finite type over $\mathbb{Z}$ with field of fractions of characteristic 0 and such that at every closed point p of $\mathrm{Spec}(R)$, the image of d in the residue field $R/\mathfrak{p}$ at $\mathfrak{p}$ lies in the prime field. Now, Corollary 4.5.4 implies that $d \in \mathbb{Q}$ and thus the connection must have quasi-unipotent monodromy. This finishes the proof. ∎

What is more, we can estimate the exponent of nilpotence of the local monodromy with the exponent ν of globally nilpotence of the connection. In the notations of the proof of the previous theorem, if $A = D + N$, at a closed point $\mathfrak{p}$ of R of residue characteristic p, we have that $D^p \equiv D \pmod{\mathfrak{p}}$. Since we can take $\alpha(p) = \nu$ for every prime p, it follows that $(N^p - N)^\nu \equiv 0 \pmod{\mathfrak{p}}$. However, if we write

$$(N^p - N)^\nu = (-1)^\nu N^\nu (1 - N^{p-1})^\nu$$

and notice that, since N is nilpotent, $(1 - N^{p-1})^\nu$ is invertible in $\mathrm{M}_n(R)$. Therefore, we have that $N^\nu \equiv 0 \pmod{\mathfrak{p}}$ for every closed point $\mathfrak{p}$. Therefore, $N^\nu = 0$ in $\mathrm{M}_n(R)$ and thus the exponent of nilpotence of the local monodromy is at most ν. In particular, taking $\nu = 1$ the nilpotent part must be trivial and we have the following important corollary.

Corollary 4.5.6. *Let X be a smooth, connected $\mathbb{C}$-scheme of finite type. If $(\mathscr{E}, \nabla)$ is an arithmetically flat connection on X it must have finite local monodromy around every singularity.*

4.6 The Grothendieck-Katz conjecture

In this section we want to understand the influence of the arithmetic in the geometry of the connections. More precisely, we wish to understand what effect does the p-curvature have over the geometric properties of an integrable connection $(\mathscr{E}, \nabla)$ on a smooth connected $\mathbb{C}$-scheme. The conjecture of Grothendieck-Katz serves this purpose: it is a local to global principle relating the arithmetic and the geometry of the connection.

Conjecture 4.6.1 (Grothendieck-Katz)**.** Let X be a smooth, connected $\mathbb{C}$-scheme of finite type and let $(\mathscr{E}, \nabla)$ be an integrable connection on X. If the connection is arithmetically flat then it becomes trivial on a finite étale cover of X.

The converse of the conjecture is easy to prove. Indeed, due to Proposition 4.3.4 we know that the p-curvature is compatible with étale localization and thus arithmetical flatness is preserved. Since the trivial connection is clearly arithmetically flat, any connection that becomes trivial on a finite étale covering must be arithmetically flat.

A nice illustration of the power of the techniques developed so far is that we can prove fairly easily that the Grothendieck-Katz conjecture is true in the case of an integrable connection on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus S$ which has abelian monodromy. Let us do so.

Theorem 4.6.2. *Let $(\mathscr{E}, \nabla)$ be an integrable connection on an open set $\mathbb{P}^1_{\mathbb{C}} \smallsetminus S$ of $\mathbb{P}^1_{\mathbb{C}}$. Suppose that $(\mathscr{E}, \nabla)$ has abelian monodromy. Then $(\mathscr{E}, \nabla)$ becomes trivial on a finite étale covering of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus S$. That is, Conjecture 4.6.1 holds for $(\mathscr{E}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus S$.*

Proof. If the connection is arithmetically flat, thanks to Theorem 4.5.5 every singularity must be regular and thanks to Corollary 4.5.6, the local monodromy around each point is finite. Since the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \smallsetminus S, *)$ is generated by the local monodromies around the missing points S and since these local monodromies commute (since the monodromy is abelian) we must have a finite monodromy group. Finally, by applying the analytification functor, Theorem 3.4.5 tells us that the connection must become trivial on a finite étale cover of X and we are done. ∎

The following elegant example illustrates the interplay between all of the ideas involved.

Example 4.6.3. Suppose that we have a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \frac{1}{z}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

for some complex constants $\lambda_1, \ldots, \lambda_n$. By taking the canonical connection $(\mathcal{O}^n_{\mathbb{C}}, \nabla)$ on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, \infty\}$ associated to the differential equation the monodromy group must be abelian. Indeed, this is because $\pi_1(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{0, \infty\}, *) = \mathbb{Z}$ is abelian. Therefore the conjecture holds and we know that this equation has a full set of algebraic solutions if and only if the p-curvature is 0 for almost every prime p. Notice that $(z^{\lambda_1}, 0, \ldots, 0), (0, z^{\lambda_2}, 0, \ldots, 0), \ldots, (0, \ldots, 0, z^{\lambda_n})$ is a full set of solutions of this equation and these solutions are algebraic if and only if $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$. The monodromy group is generated by the matrix

$$\exp\begin{pmatrix} 2\pi\mathbf{i}\lambda_1 & & \\ & \ddots & \\ & & 2\pi\mathbf{i}\lambda_n \end{pmatrix}$$

and thus (since it has trivial nilpotent part) it has finite monodromy group if and only if it has quasi-unipotent monodromy, which is, by definition, precisely when $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$. Now, let us compute directly the p-curvature for any prime p. For this, consider the obvious spreading out $R = \mathbb{Z}[\lambda_1, \ldots, \lambda_n] \subseteq \mathbb{C}$, over which the p-curvature can be computed explicitly as

$$\begin{aligned}
\psi_p\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) &= \left(\nabla\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)^p - \nabla\left(\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^p\right) \\
&= \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^p - \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1^p - \lambda_1 & & \\ & \ddots & \\ & & \lambda_n^p - \lambda_n \end{pmatrix}.
\end{aligned}$$

Therefore, the vanishing of the p-curvature amounts to having $\lambda_i^p = \lambda_i$ in R/pR for each $i = 1, \ldots, n$. Hence, we see in this way that the local to global principle Corollary 4.5.4 arises naturally in this example and indeed the conjecture may be viewed as a local to global principle for differential equations.

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