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Quillen's Theorem A through homology with local coefficients and Grothendieck spectral sequence

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Introducción

En [33], Daniel Quillen probó el siguiente resultado,

Theorem (Terema A de Quillen). *Sea $F : \mathcal{C} \to \mathcal{D}$ un funtor entre categorías pequeñas. Si las fibras F/d son contráctiles, F es una equivalencia homotópica.*

La demostración involucraba conjuntos bisimpliciales y un argumento diagonal. Más tarde, en [34], Quillen dió una demostración diferente de este teorema, usando la sucesión espectral de Grothendieck y homología con coeficientes locales. Aunque en esta segunda prueba, se asumía que $\mathcal{C}$ y $\mathcal{D}$ eran posets, Quillen observó que todo se podía generalizar a categorías pequeñas sin cambios esenciales.

En este trabajo extendemos la segunda prueba al caso general donde $\mathcal{C}$ y $\mathcal{D}$ son categorías pequeñas. De hecho usando las mismas ideas, mostramos que el siguiente resultado es cierto

Theorem. *Sea $F : \mathcal{C} \to \mathcal{D}$ un funtor entre categorías pequeñas. Para todo $n \geq 1$, si cada fibra F/d es n-conexa entonces F es una $(n+1)$-equivalencia.*

Esto había sido probado en el contexto de posets por Bjorner [4] y Barmak [3]. También exhibimos una versión homológica de este resultado.

En el Capítulo 1, damos una breve introducción sobre conjuntos simpliciales y espacios clasificantes de categorías pequeñas. También explicamos algunas ideas debidas a K. Brown acerca de como simplificar la estructura celular de conjuntos simpliciales y espacios clasificantes. Usando estas ideas damos una demostración distinta de un resultado debido a Dwyer y Kan [12], que describe el tipo homotópico del producto libre de dos categorías . Finalmente revisamos la descripción de Quillen del grupo fundamental de una categoría pequeña y explicamos como obtener una presentación explícita del mismo cuando la categoría está presentada por un grafo y relaciones.

El Capítulo 2 contiene una introducción básica a las sucesiones espectrales. El objetivo del mismo es probar la sucesión espectral de Grothendieck, que precisaremos para probar el Teorema A.

El Capítulo 3 trata acerca de homología con coeficientes locales. La misma generaliza la homología con coeficientes en un grupo fijo G, y brinda una manera de tratar con problemas que puedan surgir de espacios con grupo fundamental no trivial. Definimos también homología con coeficientes locales para una categoría pequeña. Sorprendentemente, estos grupos de homología son los mismos que se obtienen al derivar a izquierda el funtor colímite. Mostramos que un espacio conexo con homología con coeficientes locales trivial es contráctil. Esto es una consecuencia de un resultado puramente algebraico: si $\mathbb{Z}$ es $\mathbb{Z}[G]$ playo, $G = 1$. Usando homología con coeficientes locales y la sucesión espectral de Grothendieck describimos la sucesión espectral de André que usamos en la demostración del Teorema A.

Finalmente en el Capítulo 4, ya con todos los lemas previos a nuestra disposición, probamos los resultados principales de este trabajo.

Un apéndice contiene teoría básica sobre categorías y álgebra homológica.

Introduction

In [33], Daniel Quillen proved the following result

Theorem (Quillen's Theorem A). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If every fiber F/d is contractible, F is a homotopy equivalence.*

The proof involves bisimplicial sets and a diagonal argument. Later, in [34], Quillen gave a different proof of this theorem, using Grothendieck spectral sequence and homology with local coefficients. Though in this second proof, $\mathcal{C}$ and $\mathcal{D}$ were assumed to be posets, Quillen observed that everything could be generalized to small categories without essential change.

In this work we extend this second proof to the general case where $\mathcal{C}$ and $\mathcal{D}$ are small categories. In fact we show that by using the same ideas the following more general result is true,

Theorem. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. For any $n \geq 1$, if every fiber F/d is n-connected then F is an $(n+1)$-equivalence.*

This has been proved in the particular context of posets by Bjorner [4] and Barmak [3]. We also exhibit an homological version of this result.

In Chapter 1, we give a brief introduction to simplicial sets and classifying spaces of small categories. Some ideas due to K. Brown on how to simplify the cell structure of simplicial sets and classifying spaces are also explained. Using these ideas we give a different proof of a simple result by Dwyer and Kan [12], describing the homotopy type of the free product of two categories . Finally, we recall Quillen's description of the fundamental group of a small category and explain how to obtain an explicit presentation of the fundamental group when the category is presented by a graph and relations.

Chapter 2 contains the most basic theory of spectral sequences. The aim is to prove Grothendieck spectral sequence, which will be used to prove Theorem A.

Chapter 3 deals with homology with local coefficients. This generalizes homology with coefficients in a fixed group G, and provides a way to deal with problems arising from non-simply connected spaces. We define homology with local coefficients for small categories. Interestingly, this is the same as left deriving the colimit functor. We show that a connected space that has zero homology groups with coefficients in any local system is contractible. This is a consequence of a purely algebraic result: if $\mathbb{Z}$ is $\mathbb{Z}[G]$ flat, $G = 1$. Using homology with local coefficients and Grothendieck spectral sequence we describe André spectral sequence, used in the proof of Theorem A.

Finally in Chapter 4 we put all the pieces together and prove the main results of this work.

There is also an appendix, which contains some basic category theory and homological algebra.

Chapter 1

Simplicial Sets and Classifying Spaces of Small Categories

This chapter begins by recalling the notion of simplicial sets. These are combinatorial models of topological spaces, that provide greater flexibility than other structures (such as simplicial complexes), at the cost of an apparent greater complexity. We will associate to a small category its nerve, which is a simplicial set. The geometric realization of the nerve is called the classifying space of the category. Then, we will describe some methods to simplify the homotopy type of these spaces. Finally, we will recall how to describe the fundamental group of a classifying space algebraically. General references for this chapter are [28] and [19].

1.1 Simplicial Sets

A simplicial set is a collection of simplices together with maps that show how these simplices are glued to each other. Formally,

Definition 1. *A simplicial set K is defined by providing the following information:*

A collection $\{K_n\}_{n\geq 0}$ of sets

Maps between these sets, $d_i : K_n \to K_{n-1}$ and $s_i : K_n \to K_{n+1}$ for $0 \leq i \leq n$, satisfying the so called simplicial identities,

$d_i d_j = d_{j-1} d_i$ *for* $i < j$

$s_i s_j = s_{j+1} s_i$ *for* $i \leq j$

$d_i s_j = id$ *if* $i = j$ *or* $i = j+1$

$d_i s_j = s_{j-1} d_i$ *for* $i < j$

$d_i s_j = s_j d_{i-1}$ *for* $i > j+1$

The maps d_i are called faces, and the maps s_j degeneracies. Elements of K_n are refered to as n-simplices. There is a natural notion of morphism,

Definition 2. *A morphism $f : K \to L$ of simplicial sets is a collection $\{f_n\}_{n\geq 0}$ of maps, $f_n : K_n \to L_n$, commuting with faces and degeneracies.*

To better understand this definition let us present an example. Given L a simplicial complex whose vertices are totally ordered, we can form a simplicial set $ss(L)$ in the following manner. Take $ss(L)_n = \{(v_0, \cdots, v_n)/\{v_i\}$ is a simplex of L and $v_i \leq v_{i+1}\}$ and define face and degeneracies by,

$$d_i((v_0,\cdots,v_n)) = (v_0,\cdots,\hat{v_i},\cdots,v_n)$$

(omit vertex i), and

$$s_j((v_0,\cdots,v_n)) = (v_0,\cdots,v_j,v_j,\cdots,v_n)$$

(repeat vertex j).

With this example in mind, it is easy to interpret the simplicial maps and identities. The i-face of an n-simplex is the $(n-1)$-simplex that doesn't include vertex i (the face opposed to vertex i). Degeneracies are simply a way of thinking an n-simplex as an $(n+1)$-simplex. They are useful because they allow to consider morphisms that map simplices of higher dimensions to lower dimensions.

Simplicial sets with morphisms so defined form the category $sSet$. There is an alternative definition of simplicial set: a contravariant functor from the simplex category Δ to Set.

Definition 3. *The simplex category Δ, has as its objects the non-empty finite totally ordered sets, and as its morphisms, ordered preserving maps.*

Many algebraic structures can be presented by generators and relations. The analogue for categories is to give a graph and relations between paths. A brief explanation of what presenting a category means is given in the appendix.

Δ admits a presentation from which it is immediate that $sSet = Set^{\Delta^{op}}$. Denote by n the totally ordered set with $n+1$ elements and the coface and codegeneracy maps in Δ, $\delta^i : n-1 \to n$ and $\sigma^j : n+1 \to n$

$$\delta^i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \geq i \end{cases}$$

$$\sigma^j(x) = \begin{cases} x & \text{if } x \leq j \\ x-1 & \text{if } x > j \end{cases}$$

The simplex categroy is presented by the coface and codegeneracy maps subject to the cosimplicial identities, the opposites of the simplicial identities given in the definition of simplicial set. A proof of this assertion can be found in [26].

It is then clear that a simplicial set is just a contravariant functor from Δ to Set, and that a morphism of simplicial sets is a natural transformation between functors. So we have $sSet = Set^{\Delta^{op}}$.

This is convenient for many reasons. First, as $sSet$ is a functor category over Set, it has all small limits and colimits and they are are computed pointwise. Second, take now the simplicial complex consisting of the nonempty subsets of n with the obvious ordering on the vertices. Interpreted as a simplicial set it is the contravariant functor $[-,n]$. So by Yoneda (see section A.1.3) defining a morphism from it to a simplicial set K is equivalent to choosing an n-simplex in K.

Minor modifications in the definition provide useful notions. For example, instead of Δ consider now the subcategory generated by injective morphisms Δ_{inj}. An element of $Set^{\Delta_{inj}^{op}}$ is referred to as a semi-simplicial set. It is somehow a midpoint between simplicial complexes and simplicial sets.

Replacing Set by any other category $\mathcal{C}$ and defining $s\mathcal{C}$ as $\mathcal{C}^{\Delta^{op}}$ gives the definition of simplicial object in $\mathcal{C}$. So there are such things as simplicial topological spaces, simplicial groups or even simplicial simplicial sets.

Geometric Realization

Simplicial sets model topological spaces. The idea is to think of an n-simplex as an n-cell, and glue simplices together according to the face and degeneracy maps.

First recall that there is a standard functor $\theta : \Delta \to Top$, that maps n to the standard n-simplex Δ^n (the convex hull of the standard basis in $\mathbb{R}^{n+1}$), the coface δ^i to the map that inserts a zero on place i, and the codegeneracy σ^j to the map that sums the components j and $j+1$, that is,

$$\theta(\delta^i)(v_0, \cdots, v_{n-1}) = (v_o, \cdots, \underset{place i}{0}, \cdots, v_{n-1})$$

and

$$\theta(\sigma^j)(v_0, \cdots, v_{n+1}) = (v_0, \cdots, v_j + v_{j+1}, \cdots, v_{n+1}).$$

By Yoneda (see section A.1.3), a simplicial set $K : \Delta^{op} \to Set$ is colimit of representable functors,

$$X = \underset{\Gamma_X}{colim}[-, n]$$

As said before $[-, n]$ is the n-simplex thought of as a simplicial set. Therefore its geometric realization should equal Δ^n, the standard n-simplex. It is then natural to define the geometric realization of X, noted $|X|$, in the following way,

$$|X| = \underset{\Gamma_X}{colim}\Delta^n$$

This is the colimit of the functor from Γ_X to Top that maps (n, x) to Δ^n and a morphism f to $\theta(f)$.

Taking geometric realization defines a functor $|\cdot| : sSet \to Top$. Colimits in Top are easy to describe so the geometric realization of X is exactly

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where X_n is given the discrete topology, and $\sim$ is the equivalence relation generated by

$$(d_i x_n, u_{n-1}) \sim (x_n, \theta(\delta^i) u_{n-1})$$
$$(s_j x_n, u_{n+1}) \sim (x_n, \theta(\sigma^j) u_{n+1})$$

for all x_n, u_{n-1}, u_{n+1}.

There is another approach to the geometric realization functor: it is the left adjoint to the singular set functor $S : Top \to sSet$ defined by taking $S(X)_n = \{\alpha : \Delta^n \to X : \alpha$ continuous$\}$ and as expected $d_i(\alpha) = \alpha\theta(\delta^i)$, $s_j(\alpha) = \alpha\theta(\sigma^j)$. As a consequence $|\cdot|$ preserves colimits and then it is relatively easy to show that $|X|$ admits a CW structure having one cell in dimension n for each nondegenerate simplex (a simplex is called degenerate if it is in the image of some degeneracy, and nondegenerate otherwise). For a reference consider, [19].

1.2 Nerves of Small Categories

To each small category $\mathcal{C}$ corresponds a simplicial set NC called its nerve. This was first defined by Segal in [39], though he attributed the idea to Grothendieck.

Definition 4. *Given $\mathcal{C}$ a small category, $N\mathcal{C}$ is the simplicial set having as 0-simplices $Ob(\mathcal{C})$, as n-simplices the n-tuples of composable arrows, $N\mathcal{C}_n = \{(f_1, \cdots, f_n)$:such that $cod(f_i) = Dom(f_{i+1})\}$, the faces and degeneracy maps,*

$$d_i(f_1, \cdots, f_n) = \begin{cases} (f_2, \cdots, f_n) & \text{if } i = 0\ n > 1 \\ cod(f_1) & \text{if } i = 0\ n = 1 \\ (f_1, \cdots, f_{n-1}) & \text{if } i = n\ n > 1 \\ Dom(f_1) & \text{if } i = n\ n = 1 \\ (f_1, \cdots, f_i f_{i+1}, \cdots, f_n) & \text{if } i > 0\ n > 1 \end{cases}$$

If $n > 0$,

$$s_j(f_1, \cdots, f_n) = (f_1, \cdots, f_j, Id_{cod(f_j)}, \cdots, f_n)$$

and $s_0(a) = (id_a)$.

The nerve is a functor $N : Cat \to sSet$ in an obvious way, if $F : \mathcal{C} \to \mathcal{D}$ is in Cat, $N(F)(f_1, \cdots, f_n) = (F(f_1), \cdots, F(f_n))$ defines a morphism between simplicial sets. The classifying space is defined as the composition of the nerve and the geometric realization, $B = |\cdot| \circ N : Cat \to Top$

For example if $\mathcal{C}$ is the following category,

$$\begin{tikzcd} A \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & B \end{tikzcd}$$

there are 2 0-simplices, 2 non-degenerate 2-simplices and no other nondegenerate simplices. The classifying space of this category is S^1.

If we now consider,

$$\begin{tikzcd} A \arrow[r, bend left, "\alpha"] & B \arrow[l, bend left, "\alpha^{-1}"] \end{tikzcd}$$

it is clear that the n-skeleton of this space is S^n. Thus its classifying space is the colimit of the sequence of inclusions $S^n \subset S^{n+1}$, i.e. S^∞.

In general the nerve of a category has many non-degenerate simplices. In fact it will have infinitely many unless the category has a finite number of objects and morphisms, and is acyclic (i.e. there are no non identity endomorphisms, and given two distinct objects there are morphisms in only one direction). It would be convenient to have some tools to simplify this construction, preserving the homotopy type of the space.

1.3 Discrete Morse Theory for Simplicial Sets

In [8] K.S. Brown, based on previous work by himself and R. Geoghegan [7], showed how to produce from a simplicial set K, together with what he called a *collapsing scheme* on

K a CW complex homotopy equivalent to $|K|$, but having less cells. He then proceeded to show that any monoid presented by a *complete rewritting system* had an associated collapsing scheme in a natural way. In [10] this notion was generalized for categories by M.G. Citterio, as Brown noted it could be.

This was completely previous to the more known work of Forman [14] and Chari [9] on discrete morse theory. However, collapsing schemes and acyclic matchings are essentially the same thing.

In this section we follow [10] and [8]. We begin by giving an intuitive idea of what a collapsing scheme is by means of a concrete example. Basically, while constructing the geometric realization of a simplicial set, we will change the order in which we glue cells and make deformations throughout the process, leaving the homotopy type of the final space unchanged. A collapsing scheme will indicate how to do this in an orderly fashion.

Suppose we were constructing the geometric realization of the simplicial set K associated to the simplicial complex

$$\{0, 1, 2, 3, 01, 02, 12, 13, 23, 012, 123\}.$$

We start with a point representing the 0-simplex 0, which will be called *essential.* Now we adjoin the 0-simplex 1, the 1-simplex 01, and contract the segment joining 0 and 1 to the point 0. This space is homotopy equivalent to the realization of the subcomplex

$$\{0, 1, 01\}.$$

We mark 1 as a *redundant* 0-simplex, and 01 as a *collapsible* 1-simplex: 01 collapses and allows us to forget about 1, making it redundant. We do this again with 2 and 02. Now, we adjoin the point 3, the segment 13 (remembering that 1 has been contracted to 0), and collapse 13 to 1. Note that the order was important here: first we collapsed 01, and then 13 (later we will say that 1 has lower height than 3). In order not to get confused through all the deformations, we keep track of them: formally associate 1 to 01, 2 to 02, and 3 to 13 by means of a function c that will map redundant to collapsible simplices. So far we have a space homotopy equivalent to the realization of

$$\{0, 1, 2, 3, 01, 02, 13\}.$$

We have depleted all 0-simplices and there remains two 1-simplices: 12 and 23. We adjoin 12 (noting that both 1 and 2 are identified with 0) and 012. Now, we can clearly deform the resulting space (which is a 2-disk), to the point 0 (in the general case we would be deforming a 2-simplex into one of its horns). Finally, do the same for 23 and 123 (noting again that $1, 2$ and 3 have been identified with a point). Set 12 and 23 as redundant, 012 and 123 as collapsible, and let $c(12) = 012$, and $c(23) = 123$. The resulting space is a point (as there was only one essential simplex), and it is homotopy equivalent to the geometric realization of K.

For the general case, take K a simplicial set and begin to form its geometric realization by putting together some of the 0-simplices of K: the *essential* 0-simplices. Then choose a subset of the rest: the *redundant* 0-simplices (those we will say to have height 1), and for every simplex x there, pick a 1-simplex, say $c(x)$ (the *collapsible* of x), such that $c(x)$ has x as one of its faces and the other face an essential 0-simplex. Contract all these 1-simplices to essential 0-simplices (note that we need for c to be injective so as not to change the homotopy type). Repeat this process by selecting a subset of the rest of the

0-simplices (those which we will call of height 2), in a way that for every x of height 2 the faces of $c(x)$, are essential, or redundant of lower height. Repeat again this process for every possible height, exhausting every simplex. Then go on to the 1-skeleton, some of the simplices there will be essential, some already collapsed, for the rest choose collapsible 2-simplex and proceed as before. Repeat the process for every dimension, the final space will be homotopy equivalent to $|K|$ and have one n-cell for every essential n-simplex.

The next definition organizes all the information needed to do this.

Definition 5. *A pre-collapsing scheme on a simplicial set K is defined by providing the following, for every $n \geq 0$,*

A partition of the nondegenerate n-simplices of K in three sets: E_n, R_n and C_n, such that C_0 is empty.

A bijective function $c_n : R_n \to C_{n+1}$.

A function $\iota_n : R_n \to n+1$, such that $d_{\iota_n(x)} = x$ for every element of R_n.

The elements of E_n, R_n and C_n are respectively called the essential, redudant and collapsible $n-$simplices. The function c_n assigns to a redundant simplex, a collapsible simplex, and ι specifies which face of $c(x)$ is x.

If x, y are redundant n-simplices such that $d_i c(x) = y$ for $i \neq \iota_n(x)$, y needs to be glued before x. Define then a relation by setting $x > y$. The height of a redundant n-simplex is the supremum over all the lengths of chains starting at x, $x > y_1 > \cdots > y_n$ (we understand that the length of $z_0 > z_1 > \cdots > z_n$ is $n+1$). If the height of every redundant simplex is finite the process essentially described above works, and we call the pre-collapsing scheme a collapsing scheme.

Note that asking for the height to be finite is the same as asking that there are no infinite descending chains $x > y > z > \cdots$.

In our first example, we considered K to be the simplicial set associated to the complex

$$\{0, 1, 2, 3, 01, 02, 12, 13, 23, 012, 123\}.$$

We had set $E_0 = \{0\}$, $R_0 = \{1, 2, 3\}$, $R_1 = \{12, 23\}$, $C_1 = \{01, 02, 13\}$, $C_2 = \{012, 123\}$, $c_0(1) = 01, c_0(2) = 02, c_0(3) = 13$, $c_1(12) = 012$, $c_1(23) = 123$, $\iota_0 = \iota_1 = 0$. In this case, $3 > 1$ and $23 > 12$.

Theorem 6 (Brown's theorem). *Let K be a simplicial set together with a collapsing scheme, as specified above. Then its geometric realization $X = |K|$ admits a quotient CW-complex Y, with as many n-cells as essential n-simplices. The quotient map $q : X \to Y$ is a homotopy equivalence. It maps each open essential cell of X homeomorphically onto the corresponding cell of Y, and it maps each collapsible $(n+1)$-cell into the n-skeleton of Y.*

The proof is elementary, and is essentialy a verification that the definition of collapsing scheme gives us a coherent way of collapsing simplices.

Proof. Define X_0^e as the essential 0-cells, X_n^+ as the space obtained from X_n^e by gluing the redundant n-cells and the collapsible $(n+1)$-cells, and X_{n+1}^e as the one obtained from X_n^+ by adjoining the essential $(n+1)$-cells.

The idea is to show that $j_n : X_n^e \to X_n^+$ is a strong deformation retract. First, choose strong deformation retractions from Δ^n to Λ_i^n for every n, i (where Λ_i^n is the geometric realization of the smallest subsimplicial set of $[-, n]$ containing all the faces $d_j(id_n)$ except the ith one)

Factor the inclusion $X_n^e \hookrightarrow X_n^+$, in the following manner,

$$X_n^e \hookrightarrow X_n^{e,1} \hookrightarrow X_n^{e,2} \hookrightarrow X_n^{e,3} \cdots \hookrightarrow X_n^+$$

Where we first adjoin redundant simplices of height 1, with their respective collapsible $(n+1)$-simplices, and we go on, adjoining redundants of higher height with their respective collapsibles.

X_n^+ is the colimit of this sequence of closed cofibrations. If we show that every $X_n^{e,k} \hookrightarrow X_n^{e,k+1}$ is a strong deformation retract, then it is standard that the colimit deformation retracts on X_n^e (see [17] $A.5.8$)

The following square is how $X_n^{e,k+1}$ is obtained from $X_n^{e,k}$ as an adjunction space. As the top horizontal arrow is a strong deformation retract, the bottom one is a strong deformation retract too, by basic properties of adjunction spaces.

$$\begin{tikzcd}
\bigoplus\limits_{\tau \in R_{n,k+1}} \Lambda^n_{\iota_n(\tau)} \arrow[r] \arrow[d] & \bigoplus\limits_{\tau \in R_{n,k+1}} \Delta^n \arrow[d] \\
X_n^{e,k} \arrow[r, "j_{n,k}"] & X_n^{e,k+1}
\end{tikzcd} \tag{1.1}$$

So, $j_n : X_n^e \hookrightarrow X_n^+$ is a strong deformation retract (and also a cofibration, as it is an inclusion of a subcomplex). Let r_n be the respective retraction. Now, consider the diagram,

$$\begin{tikzcd}
X_0^e \arrow[r, "j_0"] \arrow[d, "id"] & X_0^+ \arrow[r, "i_0"] \arrow[d, "r_0"] & X_1^e \arrow[r, "j_1"] \arrow[d, "\overline{r_0}"] & X_1^+ \arrow[r, "i_1"] \arrow[d, "\overline{r_0}r_1"] & X_2^e \arrow[r, "j_2"] \arrow[d, "\overline{\overline{r_0}r_1}"] & X_2^+ \arrow[r] \arrow[d, "\overline{\overline{r_0}r_1}r_2"] & \cdots \\
Y_0 \arrow[r, "id"] & Y_0 \arrow[r, "\overline{i_0}"] & Y_1 \arrow[r, "id"] & Y_1 \arrow[r, "\overline{i_1}"] & Y_2 \arrow[r, "id"] & Y_2 \arrow[r] & \cdots
\end{tikzcd}$$

Where Y_0 is X_0^e, and Y_n is defined as the pushout of the square it first appears in. Call Y the colimit of the bottom row.

The top row is a sequence of closed cofibrations, and the bottom row too, as closed cofibrations are stable under pushouts. The vertical maps are homotopy equivalences, as pushouts of homotopy equivalences along cofibrations are homotopy equivalences. As in this case the topological colimit equals the homotopy colimit, the induced map of X on Y is a homotopy equivalence (see [45])

Every vertical map is a quotient. Just note that retractions are quotient maps, composition of quotients is a quotient, and if

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"] & B \arrow[d, "c"] \\
C \arrow[r, "d"] & D
\end{tikzcd} \tag{1.2}$$

is an adjunction space (with A a closed subspace), if b is a quotient, then so is c.

So, as every vertical arrow is a quotient, the induced map on the colimits is a quotient too.

The structure for Y as a CW complex is immediate: Each Y_{n+1} is obtained from Y_n by adjoining $n+1$ cells, because X^e_{n+1} is obtained from X^+_n by adjoining $n+1$ cells and composition of pushouts is a pushout. $\square$

As an example, we see that a category with a terminal object is contractible, i.e. has a contractible classifying space. Let $\mathcal{C}$ be a small category, c a terminal object in it. Define E_0 as consisting only of c. There will be no other essential simplices. Take $R_n = \{(f_1, \cdots, f_n) : cod(f_n) \neq c\}$ and $C_n = \{(f_1, \cdots, f_n) : cod(f_n) = c\}$. Define $c_n(f_1, \cdots, f_n) = (f_1, \cdots, f_n, cod(f_n) \to c)$. If x is in R_n, and $i \neq n+1$, $d_i(c(x))$ is either collapsible or degenerate, so this is in fact a collapsing scheme, with only one essential 0-simplex.

In order to produce more interesting examples we need to introduce rewriting systems. But first, we state the relation between collapsing schemes and acyclic matchings.

Acyclic Matchings = Collapsing Schemes

Now we show how a collapsing scheme is exactly an acyclic matching.

We will understand that a matching M in a graph G (see the appendix) is a collection of disjoint arrows from G, where two arrows are disjoint if they don't share source nor target. A path in a graph is a sequence of composable arrows. A matching M in a graph G will be called acyclic, if in the graph formed from G by reversing the arrows in M, there are no infinite paths.

Now, take K a simplicial set and consider the graph $G(K)$, whose vertices are all nondegenerate simplices, where for each n-simplex x, and each $0 \leq i \leq n$, we draw an arrow from x to $d_i(x)$ if $d_i(x)$ is nondegenerate.

A collapsing scheme determines an acyclic matching. For each redundant n-simplex x, pick the arrow from $c_n(x)$ to x determined by $d_{\iota_n(x)}$. The resulting collection of arrows is a matching, because redundant and collapsible simplices were disjoint, and every c_n was a well defined injective function. The fact that height is well defined means exactly that the matching is acyclic.

This assignment from collapsing schemes to acyclic matchings is clearly bijective.

The proof of theorem 6 is really constructive, it tells us how to attach the essential simplices together. For example if x is an essential 1-simplex, to attach it we need to specify two points. Look at x in $G(K)$, and follow the arrows from x. If $d_0(x)$ is essential, we are done, one of the points is $d_0(x)$. If not, it is redundant, and has an arrow going away from it. Follow that arrow, until you get to an essential 0-simplex. That is the point to consider. For simplices of higher dimensions, just follow the arrows from that simplex until you arrive to an essential simplex (degenerate faces should also be considered), the information carried in the middle (especially the chosen retractions into horns used in the proof), specifies how to attach the simplex.

Rewriting Systems

In section A.1.4 we give some necessary definitions that we will use in this section.

Let G be a (directed) graph, and R a relation on $Free(G)$. That is, given a, b objects of G, $R_{a,b}$ is a relation on the set of words from a to b. We will think of elements (w_1, w_2)

of $R_{a,b}$ as rules for rewriting words, and call them rewriting rules. Usually we shall note (w_1, w_2) by $w_1 \to w_2$. More generally we will write $w_1 \to w_2$, if we can go from w_1 to w_2 by applying a rewriting rule on some subword of w_1, i.e. if there are words (some possibly empty) s, t, u, v such that $w_1 = sut$, $w_2 = svt$ and $u \to v$. We then say that w_2 is obtained from w_1 by rewriting or reduction.

A word is call reducible if some rewriting rule can be applied to it, and irreducible otherwise. If we can apply a finite sequence of reductions starting at word w_1 and ending at word w_2, we write $w_1 \Rightarrow w_2$.

Definition 7. *A complete rewriting system (G, R) is a relation R on a graph G such that, there are no infinite reduction sequences*

$$w_1 \to w_2 \to w_3 \to \cdots$$

and whenever $w \Rightarrow u$ and $w \Rightarrow v$, there is a word z such that $u \Rightarrow z$ and $v \Rightarrow z$.

Under these conditions, starting from any word w, and no matter which rewriting rules we apply, we will eventually arrive to the same irreducible word, $r(w)$.

Moreover, if we define a relation on $Hom(a, b)$ in $Free(G)$ by setting $w_1 \sim w_2$ iff $r(w_1) = r(w_2)$, then this relation is the smallest congruence containing R. So, in the category presented by G and R, we can take $r(w)$ as a representative of the class of w, that is, the arrows of the category presented by G and R, $(G|R)$, can be taken to be the irreducible words.

It is important to distinguish now that if w is a word from a to b, and v a word from b to c, we note wv their concatenation (the composition $v \circ w$ in $Free(G)$) and $v * w = r(wv)$, there composition in $(G|R)$.

We are now ready to show how to associate a collapsing scheme to the nerve of $(G|R)$, when (G, R) is a complete rewriting system. For simplicity, we assume that the arrows of G, are themselves irreducible words (otherwise we could do without them).

Every object will be an essential 0-simplex. An n-simplex, $\tau = (w_1, \cdots, w_n)$ will be essential if,

1. w_1 has length one
2. $w_i w_{i+1}$ is reducible
3. Every proper initial subword of $w_i w_{i+1}$ is irreducible

If $\tau = (w_1, \cdots, w_n)$ is not essential, let k be the least such that $(w_1, \cdots, w_k)$ is essential, and call this $min(\tau)$.

Clearly, $k = 0$ iff w_1 has length greater than one. In that case, set τ as redundant. Write $w_1 = sw$ where s has length one. Observe that s, w are irreducible words, because they are subwords of an irreducible word. Set then $c_n(\tau) = (s, w, w_2, \cdots, w_n)$ and $\iota_n(\tau) = min(c(\tau)) = 1$.

Assume now that w_1 has length one. . Then $1 \leq k < n$.

If $w_k w_{k+1}$ is irreducible define τ to be collapsible. If not, some proper initial subword of $w_k w_{k+1}$ is reducible. As no subword of w_k is reducible, there must be a reducible subword $w_k u$ of $w_k w_{k+1}$. Take u to be minimal such that $w_k u$ is reducible. Write $w_{k+1} = uv$. Set τ as redundant, $c_n\tau = (w_1, \cdots, w_k, u, v, \cdots, w_n)$, and $\iota_n(\tau) = min(c(\tau)) = k + 1$.

Note that in every case, if τ is redundant, $min(c_n(\tau)) = \iota(\tau) = min(\tau) + 1$.

Proposition 8. *The data above give a collapsing scheme for the nerve of* $(G|R)$*.*

It is clear from how we defined things, that we are in presence of a pre-collapsing scheme. What we really have to check, is that there are no infinite descending chains of redundant simplices. The following lemma, will say exactly that.

If $\tau = (w_1, \cdots, w_n)$, define $word(\tau) = w_1 \cdots w_n$.

On the set of words of $Free(G)$ define the following relation, $w > v$ if v is a strict subword of w or if $w \Rightarrow v$. Note that there cannot be an infinite descending chain $w_1 > w_2 > w_3 > \cdots$, because length must eventually stabilize, and there cannot be an infinite reduction sequence of words, as the rewrite system is complete.

Lemma 9. *If* τ, τ' *are redundant* n*-simplices, and* $\tau > \tau'$*, then either* $word(\tau') < word(\tau)$ *or they are equal and* $\min(\tau') > min(\tau)$

This will prove our proposition as for any infinite descending sequence of redundant simplices

$$\tau_1 > \tau_2 > \tau_3 > \cdots$$

$word(\tau_i)$ must eventually stabilize, and then $min(\tau_i)$ must start to grow strictly. As this number is bounded by $n-1$, the sequence must be finite.

So let's prove the lemma.

Proof. Let $c(\tau) = (w_1, \cdots, w_{n+1})$, and $\iota(\tau) = i$. Observe that $\iota(\tau) = min(c(\tau)) = min(\tau) + 1$. From the description of redundant simplices, we note that $w_i w_{i+1}$ is an irreducible word. Then $\tau = (w_1, \cdots, w_i w_{i+1}, \cdots, w_n)$, as $w_{i+1} * w_i = r(w_i w_{i+1}) = w_i w_{i+1}$. Let $j = \iota(\tau')$. Note that $j \neq i$.

If $j = 0$ or $j = n$, then $word(\tau')$ is a strict subword of $word(\tau)$, so $word(\tau) > word(\tau')$.

If not, $j > i$. This is because $(w_1, \cdots, w_i)$ is essential, as $min(c(\tau)) = i$. In this case $word(\tau) = word(\tau')$, but $j - 1 > i - 1$, so $min(\tau') > min(\tau)$. ☐

As a particular nice application found in [10],

Lemma 10. *The classifying space of a free category* $(G|\emptyset)$ *is homotopy equivalent to the geometric realization of the graph,* $|G|$

In particular the classifying space of $\mathbb{N}_0 = (\circlearrowleft|\emptyset)$ is homotopy equivalent to S^1.

Proof. Clearly if the relation R is empty, the presentation is a rewriting system, where every arrow is irreducible. As no composition of nonidentity arrows is reducible, the essential simplices are exactly the objects and the arrows. ☐

Every small category $\mathcal{C}$ has an obvious presentation by a complete rewriting system. Take G as the underlying graph of $\mathcal{C}$, and direct every word $(f_1, \cdots, f_n)$ to $(f_n \cdots f_1)$. Call this presentation $(G(\mathcal{C})|R(\mathcal{C}))$. Unsurprisingly every nondegenerate simplex of the nerve of $\mathcal{C}$ turns out to be essential, under this rewriting system. So, by taking the obvious presentation, we simplify nothing.

Consider now two categories $\mathcal{C}$ and $\mathcal{D}$ with the same objects. We shall denote by $B\mathcal{C} \vee B\mathcal{D}$, the union of these spaces at the common points of the 0-skeleton. On the other hand, we define $\mathcal{C} * \mathcal{D}$ as the free product of these categories. Formally, $\mathcal{C} * \mathcal{D}$ is the category presented by $(G(\mathcal{C}) \vee G(\mathcal{D})|R(\mathcal{D}) \cup R(\mathcal{C}))$, where $G(\mathcal{C}) \vee G(\mathcal{D})$ denotes the union of the graphs that share the same vertices (take both sets of arrows, and the common set of vertices).

Then, we have the following result

Lemma 11. *If $\mathcal{C}, \mathcal{D}$ are small categories with the same objects, $B(\mathcal{C} * \mathcal{D})$ is homotopy equivalent to $B\mathcal{C} \vee B\mathcal{D}$*

Proof. Consider $(G(\mathcal{C}) \vee G(\mathcal{D}) | R(\mathcal{D}) \cup R(\mathcal{C}))$. This is a complete rewriting system for $\mathcal{C} * \mathcal{D}$. We claim that an essential simplex $(w_1, w_2, \cdots, w_n)$ has every w_i either an arrow from $\mathcal{C}$ or an arrow from $\mathcal{D}$. Suppose $w_1 = f$ is an arrow from $\mathcal{C}$ (we know it must be an arrow of the graph). As, fw_2 must be reducible, $w_2 = gu$ where g is a nonidentity arrow from $\mathcal{C}$ and u is a word (if g where in $\mathcal{D}$, fw_2 would be irreducible). As every proper subword of fw_2 must be irreducible w_2 must equal g (because fg is reducible).

So, every essential n-simplex, is either an essential n-simplex of $(G(\mathcal{C})|R(\mathcal{C}))$ or of $(G(\mathcal{D})|R(\mathcal{D}))$, and the way in which this simplices are glued is exactly how they are glued in $(G(\mathcal{C})|R(\mathcal{C}))$ and in $(G(\mathcal{D})|R(\mathcal{D}))$. Then, the space homotopy equivalent to $B(\mathcal{C} * \mathcal{D})$, is a CW whose cell structure is the of the cell structures of $\mathcal{C}$ and $\mathcal{D}$, with the same 0-skeleton. And this last space, is $B\mathcal{C} \vee B\mathcal{D}$ □

We've found this last result as proposition 3.8 from [12], though the proof is different.

For a different example, where this technique fails, consider the monoid $M = (a|a^2 = a)$. If we think of this as a presentation of a category, with $a^2 \to a$, we have a complete rewriting system. In the collapsing scheme associated, every nondegenerate n-simplex is essential, but the monoid is contractible!

To see this, first note the following,

Proposition 12. *If $F, G : \mathcal{C} \to \mathcal{D}$ are functors between small categories, and $\eta : F \to G$ is a natural transformation between them, then, $B(F)$ is homotopic to $B(G)$*

Proof. Let I be the category with two objects and one morphism between them $(0 \to 1)$. A natural transformation form F to G is exactly a functor $H : \mathcal{C} \times I \to \mathcal{D}$,such that $H(-, 0) = F$ and $H(-, 1) = G$. Taking classifying spaces we have an homotopy between $B(F)$ and $B(G)$, as $B(I)$ is the unit interval. The technical detail here, is that $|N(\mathcal{C} \times I)| = |N\mathcal{C}| \times |NI|$ because the nerve commutes with products and as $|NI| = [0, 1]$ is locally compact. See [18]. □

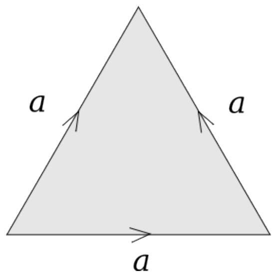
Following [35], we'll say that a monoid M has a black hole if there is an element z in M such that for every y in M, $zy = z$. Note that a monoid has a black hole iff there is natural transformation between id_M and e, where we think of M as a category with only one object, id_M as the identity functor of this category, and e as the functor that maps every arrow to the identity e of the monoid,

$$
\begin{tikzcd}
* \arrow[r, "z"] \arrow[d, "y"] & * \arrow[d, "e"] \\
* \arrow[r, "z"] & *
\end{tikzcd}
\tag{1.3}
$$

Then, in a monoid with a black hole, the identity is homotopic to a constant, so BM is contractible.

In our example $M = (a|a^2 = a)$, a is a black hole, so M is contractible. Our rewriting system fails to simplify this, and in fact there are no nonempty acyclic matchings, as in the graph $G(M)$ no arrow can be reversed without creating a loop.

Note that the 2-skeleton of BM is the dunce cap, the simplest example of a contractible space that is not collapsible.



The even dimensional skeletons of BM have been considered as higher dimensional dunce hats, as they are also contractible but not collapsible. See [1],[37] and [35]. BM could be thought of as an infinite dimensional dunce hat: it is contractible but it admits no collapsing scheme.

As a final example, consider the monoid $M = (a, b|ab)$. This is the so called bicyclic monoid. it is exactly the category presented as,

$$\begin{tikzcd} * \arrow[loop above, "a"] \arrow[loop below, "b"] \end{tikzcd}$$

With $(b, a) \to 1$ (remember we read words from left to right). In this case, the complete rewriting system yields 4 essential simplices, the point $*$, (a), (b), and $((b),(a))$. We can enlarge the acylic matching, by choosing the arrow from $((b),(a))$ to (a), and end up having only one essential 1-simplex, and one essential 0-simplex, so $|M|$ is homotopic to S^1.

1.4 The Fundamental Group of a Small Category

The fundamental group of a small category can be computed algebraically, within the category. This was first done in [33]. A friendly exposition can be found in [11], and we will follow it closely.

Definition 13. *A local system of sets in a small category $\mathcal{C}$ is a functor $F : \mathcal{C} \to Set$ such that $F(f)$ is an isomorphism for every arrow f in $\mathcal{C}$.*

Local systems of sets are a full subcategory of $Set^{\mathcal{C}}$, that we denote by $Cov(\mathcal{C})$. Analogously, replacing Set by Ab we get the definition of local system of abelian groups, that we will need later.

Recall that if X is a topological space, the coverings $q : E \to X$ of X constitute a category $Cov(X)$, where a morphism from q_1 to q_2, is a continuous function $\sigma : E_1 \to E_2$, such that $q_1 = q_2\sigma$.

There is an appropiate definition of covering in the simplicial setting, which we will need in this section.

Definition 14. *A map $f : E \to X$ of simplicial sets is called a simplicial covering iff $f : E(0) \to X(0)$ is onto and for every n-simplex τ in X, every $\iota : 0 \to n$ in Δ and every 0-simplex v in E such that $X(\iota)(\tau) = f(v)$, there exists a unique n-simplex $\tilde{\tau}$ in E, such that $f(\tilde{\tau}) = \tau$ and $E(\iota)(\tilde{\tau}) = v$.*

Basically, this says that given τ an n-simplex of X, we can choose a unique lifting of τ provided we first produce a lifting of a 0 face of τ.

The following result can be found in [18] or in [41].

Lemma 15. *The geometric realization functor transforms simplicial converings into topological coverings.*

The next theorem is the most relevant result in this section.

Theorem 16. *Let $\mathcal{C}$ be a small category. Then, there is a natural equivalence of categories,*

$$Cov(B\mathcal{C}) \simeq Cov(\mathcal{C})$$

Proof. We will define explicit functors between these categories and show that they give an equivalence between them. For simplicity assume $\mathcal{C}$ is connected.

Given $q : E \to B\mathcal{C}$, define $F(q) : \mathcal{C} \to Set$ as follows,

$$F(q)(c) = q^{-1}(c)$$

where we think of the object c of $\mathcal{C}$ as a point in $B\mathcal{C}$. If $f : c \to d$ is a morphism in $\mathcal{D}$, and x is a point in $F(q)(c)$, consider the lifting γ of the path defined by f in $B\mathcal{C}$ starting at x, and define $F(f)(x)$ as $\gamma(1)$. This is clearly a point in $F(d)$. $F(f)$ is a bijective function, because paths are reversible. This defines a functor $F : Cov(B\mathcal{C}) \to Cov(\mathcal{C})$.

Given now $T : \mathcal{C} \to Set$ that inverts morphisms, consider the category Γ_T whose objects are pairs (c, x), where c is an object of $\mathcal{C}$ and x an element of $T(c)$, and a morphism from (c, x) to (d, y) is an arrow $h : c \to d$ in $\mathcal{C}$ such that $T(h)(x) = y$. There is a projection functor $p_T : \Gamma_T \to \mathcal{C}$, that maps (c, x) to c. We will define $G : Cov(\mathcal{C}) \to Cov(B\mathcal{C})$ as $G(T) = Bp_T$. This is a well defined functor provided $B(p_T)$ is a covering. In fact, $Np_T : N\Gamma_T \to N\mathcal{C}$ is a simplicial covering, as we now show. Let τ be an n-simplex of $N\mathcal{C}$, that is,

$$\tau = c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \cdots \xrightarrow{f_n} c_n$$

Suppose we are given (c_i, x_i) in $Np_T^{-1}(c_i)$. Then, as T inverts morphisms the lifting $\tilde{\tau}$ is uniquely determined,

$$\tilde{\tau} = \cdots (c_{i-1}, (Tf_i)^{-1}(x_i)) \xrightarrow{f_i} (c_i, x_i) \xrightarrow{f_{i+1}} (c_{i+1}, Tf_{i+1}(x_i)) \cdots$$

As the geometric realization of a simplicial covering is a covering, $B(p_T)$ is a covering.

Finally the pair F, G provides an equivalence of categories. That $FG \simeq id_{Cov(\mathcal{C})}$ is immediate. FG maps a morphism inverting functor T to the functor that assigns to the object c of $\mathcal{C}$, the set $\{c\} \times F(c)$ and this set is naturally isomorphic to $F(c)$.

Showing that $GF \simeq id_{Cov(B\mathcal{C})}$ requires more work. Let $q : E \to B\mathcal{C}$ be a covering, and assume for simplicity that E is connected. We want to show that this covering is naturally isomorphic to $Bp_{F(q)} : B\Gamma_{F(q)} \to B\mathcal{C}$. Point the spaces by choosing c in $\mathcal{C}$, x in $q^{-1}(c)$, and (c, x) in $\Gamma_{F(q)}$

Consider the diagram

$$\begin{tikzcd} B\Gamma_{F(q)} \arrow[dr, "Bp_{F(q)}"'] & E \arrow[d, "q"] \\ & B\mathcal{C} \end{tikzcd}$$

If we show that $B\Gamma_{F(q)}$ is connected, and that $Im\pi_1(Bp_{F(q)}) = Im\pi_1(q)$, then, as the spaces are connected and locally path connected, we would get an isomorphism by general properties of coverings. We need the following result, which we prove at the end of this proof.

Lemma 17. *Let X be a CW complex, γ a path in X from the 0-cell a to the 0-cell b. Then γ is path homotopic to $\alpha_1 * \cdots * \alpha_n$ where α_i is β_i or $\tilde{\beta_i}$, and β_i is the characteristic function of a certain 1-cell of X.*

$B\Gamma_{F(q)}$ is connected. Take $(c, x), (d, y)$ in $\Gamma_{F(q)}$. As E is connected, we can take γ a path in E from x to y, as in the previous lemma (a covering of a CW complex admits a natural CW structure, by lifting cells).

So we have a diagram in $\mathcal{C}$ (where some arrows could be identities),

$$\begin{tikzcd}[column sep=small] c & & b_0 & & b_1 & \cdots & b_{n-1} & & d \\ & a_0 \arrow[ul, "f_0"] \arrow[ur, "g_0"'] & & a_1 \arrow[ul, "f_1"] \arrow[ur, "g_1"'] & & & & a_n \arrow[ul, "f_n"] \arrow[ur, "g_n"'] & \end{tikzcd}$$

And this diagram lifts to a diagram of paths in E, and connects x and y,

$$\begin{tikzcd}[column sep=small] x & & \tilde{b_0} & & \tilde{b_1} & \cdots & \tilde{b_{n-1}} & & y \\ & \tilde{a_0} \arrow[ul, "\tilde{f_0}"] \arrow[ur, "\tilde{g_0}"'] & & \tilde{a_1} \arrow[ul, "\tilde{f_1}"] \arrow[ur, "\tilde{g_1}"'] & & & & \tilde{a_n} \arrow[ul, "\tilde{f_n}"] \arrow[ur, "\tilde{g_n}"'] & \end{tikzcd}$$

But this means, that we have a similar diagram in $\Gamma_{F(q)}$, that connects (c, x) and (d, y).

$$\begin{tikzcd}[column sep=small] (c,x) & & (b_0, \tilde{b_0}) & & (b_1, \tilde{b_0}) & \cdots & (b_{n-1}, \tilde{b_{n-1}}) & & (d,y) \\ & (a_0, \tilde{a_o}) \arrow[ul, "f_0"] \arrow[ur, "g_0"'] & & (a_1, \tilde{a_1}) \arrow[ul, "f_1"] \arrow[ur, "g_1"'] & & & & (a_n, \tilde{a_n}) \arrow[ul, "f_n"] \arrow[ur, "g_n"'] & \end{tikzcd}$$

So $B\Gamma_{F(q)}$ is connected. To show that $Im\pi_1(Bp_{F(q)}) = Im\pi_1(q)$, choose $[w]$ in $\pi_1(B\mathcal{C}, c)$, and with the same idea as before it can be seen that the lifting of w to one covering is a closed path iff the lifting to the other covering is a closed path.

Finally, the isomorphism between the coverings is natural. First, any point (d, y) in $B\Gamma_{F(q)}$ is mapped to y. To verify this, take a path in $\mathcal{C}$ from c to d, lift it to E starting at x, and then take path in E from the endpoint of this lifting to y. Represent this path using liftings of arrows of $\mathcal{C}$ as before and by uniqueness of liftings of paths (d, y) is mapped to y. Naturallity is now obvious by using uniqueness of liftings.

For more details consult [11] □

Next is the proof of the lemma that was just used.

Proof. We will only sketch an idea, a detailed proof can be found in [11]. First, by cellular approximation, we can assume that X is 1-dimensional. Then, X is the geometric realization of a 1-dimensional simplicial complex K. If $\gamma : |I| \to |K|$ is the path, where I is a one simplex, take $\varphi : sd^N I \to K$ a simplicial approximation to γ, where sd^N is a good enough barycentric subvidision of I (see [40] chapter 3 sections 3 and 4) . Necessarily, $\varphi(0) = \gamma(0)$ and $\varphi(1) = \gamma(1)$, therefore, φ and γ are path homotopic ([40], chapter 3, section 4, lemma 2). The restriction of φ to each subinterval of the barycentric subdivision of I, is path homotopic to a characteristic map of a 1-cell of X or to the inverse of one of them. $\square$

Given $f : X \to Y$ a continuous function between topological spaces, f induces a functor $f_* : Cov(Y) \to Cov(X)$, by taking pullbacks of coverings along f. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, F induces another functor $F_* : Cov(\mathcal{D}) \to Cov(\mathcal{C})$, by precomposing with F. Under the previous equivalence F_* and $(BF)_*$ are identified.

By $\Pi_1(X)$ we will understand the fundamental grupoid of a topological space X. Recall that for a connected, locally path connected, and semi-locally simply connected space X, the functor

$$W : Cov(B) \to Set^{\Pi_1(X)}$$

defined, similarly as before, by taking fibers and lifting paths, is an equivalence of categories (see [46] Theorem 3.3.2). In particular this is true for CW complexes (eventually considering each connected component).

So by the previous result we have a chain of equivalences of categories,

$$Set^{\Pi_1(B\mathcal{C})} \simeq Cov(B\mathcal{C}) \simeq Cov(\mathcal{C}) \simeq Set^{S^{-1}\mathcal{C}},$$

where $S^{-1}\mathcal{C}$ is the localization of $\mathcal{C}$ (i.e. we add formal inverses to every arrow, see the appendix), and the last equivalence follows from the universal property that defines it (see section A.1.4)

Recall that every category is equivalent to a skeleton (a skeleton of a category $\mathcal{C}$ is a full subcategory, having exactly one object for each isomorphism type) . Assume that $\mathcal{C}$ is connected. If now we pick an object c in $\mathcal{C}$, the grupoid $S^{-1}\mathcal{C}$ is equivalent to the full subcategory generated by c, and that is the automorphism group of c considered as category with one object. We shall note this by $Aut(c, S^{-1}\mathcal{C})$. Similarly, the grupoid $\Pi_1(B\mathcal{C})$ is equivalent to $\pi_1(B\mathcal{C}, c)^{op}$ (in the fundamental groupoid composition is understood from right to left, and in the fundametal group, from left to right).

So, we get an equivalence

$$Set^{\pi_1(B\mathcal{C},c)^{op}} \simeq Set^{Aut(c,S^{-1}\mathcal{C})}$$

To recover the groups, we look at automorphisms of the forgetful functor. Remember that if G is a group, a functor from G to Set is exactly a left G-set.

Lemma 18. *Let G be a group. $U : Set^G \to Set$ the forgetful functor. Then, $Aut(U) \simeq G$, in a natural way.*

Proof. Given g in G, define $\ell_g : U \to U$ as left multiplication by g, i.e. given X G-set, x in X, $\ell_g(x) = gx$. If $f : X \to Y$ is a morphism of G-sets, $\ell_g f(x) = gf(x) = f(gx) = f\ell_g(x)$, so ℓ_g is natural, and an automorphism of U its inverse being $\ell_{g^{-1}}$.

This defines a map $\varphi : G \to Aut(U)$. φ is a morphism, since $\varphi(gh) = \ell_{gh} = \ell_g \ell_h = \varphi(g)\varphi(h)$.

It is clearly a monomorphism, since considering G as a left G set, $\ell_g(e) = g$. If η is an automorphism of U, let $g = \eta_G(e)$. Given X a left G-set, x in X, consider $r_x : G \to X$, defined as $r_x(g) = gx$. This is a morphism, and by naturality $\eta_X r_x(e) = r_x \eta_G(e)$, so $\eta_X(x) = gx = \ell_g(x)$.

For an easier proof (in fact, the same proof), just note that $U \simeq [G, -]$ and use Yoneda. $\square$

Note that the following diagram commutes,

$$\begin{tikzcd} Set^{Aut(c,S^{-1}\mathcal{C})} \arrow[r] \arrow[dr, "U"'] & Set^{\pi_1(B\mathcal{C},c)^{op}} \arrow[d, "U"] \\ & Set \end{tikzcd}$$

As the top arrow is an equivalence, this induces an isomorphism between the automorphism groups of the forgetful functors and we have the following theorem,

Theorem 19. *If $\mathcal{C}$ is a connected small category, its fundamental group is the automorphism group of any object in the localization of the category,*

$$\pi_1(B\mathcal{C}, c) \simeq Aut(c, S^{-1}\mathcal{C})^{op}$$

So to compute the fundamental group of a small category, formally invert every arrow and look at the automorphism group of any object.

Notice that what we did in this section is functorial on $\mathcal{C}$. Then we have the following result,

Lemma 20. *Let $F : \mathcal{C} \to \mathcal{D}$ a functor between small categories. Then, F induces an isomorphism on the fundamental grupoids iff*

$$F_* : Cov(\mathcal{D}) \to Cov(\mathcal{C})$$

is an equivalence of categories.

Some Applications

If $M = (G|R)$ is a presented monoid, to compute the fundamental groups of its classifying space, it is enough to consider $(G|R)$ as a presentation for a group. So the fundamental group of the classifying space of $\mathbb{N}_0 = (x|\emptyset)$ is $\mathbb{Z}$. But we saw in the previous section that $B\mathbb{N}_0 \simeq S^1$. We have then, the following known result,

Proposition 21.

$$\pi_1(S^1, x) = \mathbb{Z}$$

This last example can be generalized to presentations of small categories. Before stating this, an example is given.

Consider the category $\mathcal{C}$ presented by the following graph,

$$\begin{tikzcd}
b \arrow[loop left, "x_1"] \arrow[r, bend left, "x_4"] & c \arrow[l, "g"] \arrow[loop above, "x_2"] \arrow[r, "h"] & d \arrow[dl, "x_3"] \\
& a \arrow[ul, "f"] &
\end{tikzcd}$$

Subject to the relations

$$\begin{aligned}
hx_2^2 &= h \\
fx_3h &= g \\
x_1^4 f &= f \\
x_4x_1gx_2x_4x_1gx_2 &= id_c \\
hx_4g &= x_3
\end{aligned}$$

Pick the following maximal tree in the graph,

$$\begin{tikzcd}
b & c \arrow[l, "g"] \arrow[r, "h"] & d \\
& a \arrow[ul, "f"] &
\end{tikzcd}$$

Every arrow from the graph not in the tree will correspond to a generator of the fundamental group. So we have four generators x_1, x_2, x_3, x_4. The relations between them are just the relations given above, where we delete the arrows in the maximal tree.

$$\begin{aligned}
x_2^2 &= e \\
x_3 &= e \\
x_1^4 &= e \\
x_4x_1x_2x_4x_1x_2 &= e \\
x_4 &= x_3
\end{aligned}$$

And this group is D_4. So, from a presented category we immediately get a presentation of its fundamental group.

A graph is said to be connected if its geometric realization is connected. This means that any two vertices c, d can be joined by a sequence of arrows that may or may not be composable, i.e. $c \to x_0 \leftarrow x_1 \to x_2 \cdots \to d$. A graph is acyclic if it contains no cycles, where a cycle is a sequence of different arrows $c \to x_0 \leftarrow x_1 \to x_2 \cdots \to c$, that start and end at the same vertex. A tree is a connected acyclic graph. By Zorn's lemma every graph has a maximal tree. Moreover, a tree is maximal iff it contains every vertex. Note that in a tree there is exactly one path between any two vertices (since it is simply connected).

Proposition 22. *Let $\mathcal{C}$ be a connected small category, presented by a graph G and relations R. Take T a maximal tree in G. Then, if c is an object of G, $\pi_1(B\mathcal{C}, c)$ can* *be presented by generators $Arr(G) \setminus Arr(T)$, satisfying the following relations: for every $((f_1, \cdots, f_n), (g_1, \cdots, g_m))$ in R, delete the arrows in T, obtain $((f_{i_1}, \cdots, f_{i_k}), (g_{j_1}, \cdots, g_{j_l}))$, and impose $f_{i_1} \cdots f_{i_k} = g_{j_1} \cdots g_{j_l}$.*

Proof. We will show that the opposite of such presented group, call it G, is isomorphic to $Aut(c, S^{-1}\mathcal{C})$. This is obvious because of our definitions, we are just translating all the information of the presentation of the localization to point, in a canonical way, through the tree.

If a is any object of G, as we have formally inverted every arrow, the unique sequence of arrows in T from c to a, can now be composed, eventually inverting some of them. Call such morphism ℓ_a.

Take $x : a \to b$, in $Arr(G) \setminus Arr(T)$, and map it to $\ell_b^{-1} x \ell_a$. To verify that this defines a morphism from $\varphi : G \to Aut(c, S^{-1}\mathcal{C})$ it is enough to check that the relations used to present G are satisfied.

Suppose $((f_1, \cdots, f_n), (g_1, \cdots, g_m))$ is in R. Write

$$f_n \cdots f_1 = t_k x_k t_{k-1} \cdots t_1 x_1 t_0$$

where t_i is a composition of arrows of T (possibly empty), and x_i is an arrow not in the tree. Define $a_i = dom(t_i)$ and $a = cod(t_k)$. Similarly

$$g_m \cdots g_1 = u_l y_l u_{l-1} y_{l-1} \cdots u_1 y_1 u_0$$

where u_i is a composition of arrows in T, and y_i is an arrow not in T. Define too, $b_i = dom(u_i)$ and note that $a = cod(u_l)$ and $a_0 = b_0$

Note that $\varphi(x_i) = \ell_{a_i}^{-1} x_i t_{i-1} \ell_{a_{i-1}}$ since $t_{i-1}\ell_{a_{i-1}}$ is the unique path in T joining c and the domain of x_i. Also $\ell_{a_k}^{-1} = \ell_b^{-1} t_k$, for simliar reasons. Similarly $\varphi(y_i) = \ell_{a_i}^{-1} y_i u_{i-1} \ell_{b_{i-1}}$, and $\ell_{b_l}^{-1} = \ell_b^{-1} u_l$

$$\begin{aligned}
\varphi(x_k \cdots x_1) &= \ell_{a_k}^{-1} x_k t_{k-1} \ell_{a_{k-1}} \ell_{a_{k-1}}^{-1} x_{k-1} t_{k-2} \ell_{a_{k-2}} \cdots \ell_{a_1}^{-1} x_1 t_0 \ell_{a_0} \\
&= \ell_b^{-1} t_k x_k t_{k-1} \cdots t_1 x_1 t_0 \ell_{a_0} \\
&= \ell_b^{-1} f_n \cdots f_1 \ell_{a_0} \\
&= \ell_b^{-1} g_m \cdots g_1 \ell_{a_0} \\
&= \ell_b^{-1} u_l y_l u_{l-1} y_{l-1} \cdots u_1 y_1 u_0 \ell_{a_0} \\
&= \ell_{b_l}^{-1} y_l u_{l-1} \ell_{b_{l-1}} \ell_{b_{l-1}}^{-1} y_{l-1} u_{l-2} \ell_{b_{l-2}} \cdots \ell_{b_1}^{-1} y_1 u_0 \ell_{b_0} \\
&= \varphi(y_l \cdots y_1)
\end{aligned}$$

To see that φ is onto, note that any automorphism z of c can be thought of as a path $c \to a_0 \leftarrow a_1 \to a_2 \cdots \leftarrow c$, where some arrows will be inverted. Then it is easily checked (by the same methods as above) that z is the image of the arrows in this path not in the tree. It is also easy to see that φ is a monomorphism: if $\varphi(x_1 \cdots x_k) = id_c$ any reduction of the word $\varphi(x_1 \cdots x_k)$ translates into that $x_1 \cdots x_k$ is equal to different word. When through reductions we reach id_c we see that $x_1 \cdots x_k$ was actually equal to the empty word. $\square$

It is important to notice that all these ideas are extremely similar (if not the same) to those used for giving a characterization of the fundamental group of a simplicial complex, i.e. the edge-path grupoid. The following application is inspired by this thought (see [40], chapter 3 section 7, corollary 5).

Proposition 23. *Let K be a connected simplicial complex of dimension 1. If T is a maximal tree in K, then the fundamental group of K is a free group with as many generators as the edges of K not in T.*

Proof. When we say that a 1 dimensional simplicial complex T is a tree, we mean that given any orientation on its edges the resulting graph is a tree according to our definion (this turns out to be the same as asking for T to be simply connected).

Choose an orientation on the edges of K. Call G the resulting graph, and $\mathcal{C}$ the free category generated by G with no relations. Then, by the methods of the previous section, we know that $B\mathcal{C}$ is homotopy equivalent to the geometric realization of G which is by definition $|K|$. On the other hand, the fundamental group of $\mathcal{C}$ admits a presentation with one generator for each arrow of G not in T (that is, each edge of K not in T) and no relations. □

As another example, we can compute the fundamental group of Δ_{iny}, the subcategory of Δ having only injective morphisms quite easily. Recall that it is presented by the arrows $\delta_n^i : n-1 \to n$, where $0 \leq i \leq n$, subject to the relations

$$\delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{j-1}$$

for $i < j$. δ_n^i is just the map that omits element i.

Choose the maximal tree

$$0 \xrightarrow{\delta_1^0} 1 \xrightarrow{\delta_2^0} 2 \xrightarrow{\delta_3^0} 3 \xrightarrow{\delta_4^0} \cdots$$

The relation $\delta_{n+1}^j \delta_n^0 = \delta_{n+1}^0 \delta_n^{j-1}$, implies $\delta_{n+1}^j = \delta_n^{j-1}$. Then, if $j > 0$, $\delta_{n+1}^j = \delta_n^{j-1} = \cdots = \delta_{j-k}^{n-1-k}$. If $n+1 \geq j$, $\delta_{n+1}^j = 1$ and if $j = n+2$, $\delta_{n+1}^{n+2} = \delta_0^1$. But, the relation $\delta_{n+1}^{n+2} \delta_n^{n+1} = \delta_{n+1}^{n+1} \delta_n^{n+1}$ implies $\delta_0^1 \delta_0^1 = \delta_0^1$, so $\delta_0^1 = 1$, and Δ_{iny} has a trivial fundamental group. In chapter 3 we will see that in fact Δ_{iny} is contractible.

Finally, the next intuitive result will be used in the final chapter.

Lemma 24. *Let $F : \mathcal{J} \to \mathcal{C}$ be a morphism inverting functor between small categories such that $\mathcal{J}$ is connected and has trivial fundamental group. Then, colim$F$ is isomorphic to each object in the diagram, i.e. if*

$$\{F(j) \xrightarrow{\tau_j} colimF\}_{j \in \mathcal{J}}$$

is a univeresal cone, τ_j is an isomorphism for every j.

Proof. As F inverts morphisms, it factors through $S^{-1}\mathcal{J}$,

$$\begin{tikzcd} \mathcal{J} \arrow[r, "F"] \arrow[d, "q"'] & \mathcal{C} \\ S^{-1}\mathcal{J} \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

Fix x in $\mathcal{J}$. Since $S^{-1}\mathcal{J}$ is simply connected, given any of its objects j, there is a unique isomorphism $\alpha_j : j \to x$. Then,

$$\{\tilde{F}(j) \overset{\tilde{F}\alpha_j}{\to} \tilde{F}(x)\}_{j \in S^{-1}\mathcal{J}}$$

is a cone over $\tilde{F}$. Precomposing with q gives a cone over F,

$$\{F(j) \overset{\tilde{F}\alpha_j}{\to} F(x)\ \}_{j \in \mathcal{J}}$$

As every map $\tilde{F}(\alpha_j)$ is an isomorphism, the cone is trivially universal.

$\square$

Chapter 2

Spectral Sequences

Spectral sequences are a fundamental tool in both algebraic topology and homological algebra.

The most important functors in homological algebra, like Ext and Tor, involve resolutions of two variables, which lead to double complexes. There is a spectral sequence that tells us how to "compute" the homology of these double complexes.

On the other hand, given filtration of a topological space X,

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots X$$

there is a spectral sequence that says in which way the homology of the filtration relates to the homology of the total space.

In this chapter we will recall the very basic of n spectral sequences. Then we will describe Grothendieck's Spectral Sequence, that we will use in the proof of Quillen's theorem A. We will follow [29], [36], [22] and [20]. For the last section we will assume some familiarity with homological algebra.

2.1 Definitions and Basic Properties

We begin by defining bigraded modules over a ring.

Definition 25. *An R bigraded module M is a collection*

$$\{M_{(p,q)}\}_{(p,q)\in\mathbb{Z}^2},$$

of R modules. A morphism $f : M \to N$ of R bigraded modules of bidegree (a, b), is a collection

$$\{f_{(p,q)}\}_{(p,q)\in\mathbb{Z}^2}$$

of R module morphisms $f_{(p,q)} : M_{(p,q)} \to N_{(p+a,q+b)}$. Addition of morphisms with the same bidegree is done pointwise $(f + g)_{(p,q)} = f_{(p,q)} + g_{(p,q)}$. We define $deg(f)$ to be the bidegree of a morphism between bigraded modules.

With this structure it is clear that bigraded modules are an abelian category. The kernel, $Ker(f)$, of a morphism f of bidegree (a, b) is the bigraded module

$$Ker(f)_{(p,q)} = Ker(f_{(p,q)}),$$

its image $Im(f)$, is the bigraded module

$$Im(f)_{(p,q)} = Im(f_{(p-a,q-b)}).$$

If M is a bigraded module, we say that the bigraded module N is a submodule of M if $N_{(p,q)}$ is a submodule of $M_{(p,q)}$. In that case we can define the quotient M/N as $(M/N)_{(p,q)} = M_{(p,q)}/N_{(p,q)}$.

Recall that a differential object in an abelian category $\mathcal{C}$ is a pair (M, d), where M is an object of $\mathcal{C}$ and $d : M \to M$ satisfies $dd = 0$. Differential objects in an abelian category are again an abelian category. The homology of (M, d), is defined as $H(M, d) = Ker(d)/Im(d)$.

Definition 26. *A bigraded module with differential, is a pair (M, d), where M is a bigraded module, and $d : M \to M$ is a morphism, such that $dd = 0$. The homology of (M, d) is defined as,*

$$H(M, d) = Ker(d)/Im(d)$$

A spectral sequence is a sequence of bigraded modules with differential $(E^r, d^r)_{r\geq 1}$ such that $E^{r+1} = H(E^r, d^r)$.

In a similar manner as was done with R modules, spectral sequence can be defined in more general contexts. Starting with any abelian category $\mathcal{C}$, form the category of bigraded objects of $\mathcal{C}$, which will be an abelian category, and then consider sequences of differential objects in that category, such that every object is the homology of the previous one.

Definition 27. *A spectral sequence in $\mathcal{C}$ is a sequence of bigraded differential objects of $\mathcal{C}$, $E = ((E^r, d^r))_{r\geq 1}$, such that $H(E^r, d^r) = E^{r+1}$.*

Later we will need this level of generality.

One should think of a spectral sequence as a book with infinite pages. As pages are turned, letters (the modules) become more clear and precise, eventually converging to something. We will usually refer to E^r as the r-th page of the spectral sequence. Note for example that if $E^r_{(p,q)} = 0$, as a subquotient of zero is zero, $E^s_{(p,q)} = 0$ for $s > r$. It is also common to say that we have a first quadrant spectral sequence, if $E^1_{(p,q)} = 0$ unless $p \geq 0$ and $q \geq 0$.

If instead of bigraded modules, we considered graded modules (that is we index our set of modules by $\mathbb{Z}$), we get again an abelian category. The differential objects of this category (C, d), with $deg(d) = -1$ are exactly the chain complexes of R modules. We shall usually note a chain complex by C, and understand that d is implicit.

One of the slickest ways to construct spectral sequences are exact couples, invented by Massey in [27]. They arise naturally from filtrations of chain complexes.

Definition 28. *An exact couple is an exact triangle of bigraded modules. More precisely, it is a digram like the following,*

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

Where D, E are bigraded modules, α, β, γ are morphisms, $Im(\alpha) = Ker(\beta)$,$Im(\beta) = Ker(\gamma)$ and $Im(\gamma) = Ker(\alpha)$.

Notice that if $d = \beta\gamma : E \to E$, by exactness $dd = 0$. So (E, d) is a bigraded module with differential.

We now describe how exact couples arise from filtrations of chain complexes.

Definition 29. *A filtration F of an object M in an abelian category A, is a collection F^pM of nested subobjects of M, that is,*

$$\cdots \subseteq F^pM \subseteq F^{p+1}M \subseteq \cdots M$$

As an example, if X is a topological space, and

$$\cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots X$$

is a filtration indexed by $\mathbb{Z}$ of subspaces of X, we have an induced filtration on $S(X)$ the singular chain complex of X, by setting $F^nS(X) = S(X_n)$.

Lemma 30. *A filtration F of a chain complex C determines an exact couple*

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

Where $deg(\alpha) = (1, -1)$, $deg(\beta) = (0, 0)$ and $deg(\gamma) = (-1, 0)$.

Proof. Set $D_{(p,q)} = H_{p+q}(F^pC)$, and $E_{(p,q)} = H_{p+q}(F^pC/F^{p-1}C)$. The inclusion $F^pC \to F^{p+1}C$, and the quotient $F^p \to F^p/F^{p-1}$ induce maps $D_{(p,q)} \to D_{(p+1,q-1)}$ and $D_{(p,q)} \to E_{(p,q)}$ (remember that taking homology is functorial). These maps are α and β respectively. Finally, the short exact sequence,

$$0 \to F^{p-1} \to F^p \to F^p/F^{p+1} \to 0,$$

induces a connecting morphism $E_{(p,q)} \to D_{(p-1,q)}$. This morphism is γ.

Exactness follows immediately from the fact that a short exact sequence of chain complexes induces a long exact sequence in the homology.

□

Given an exact couple, we can arrive at a new one by a natural process called derivation,

Lemma 31. *Given an exact couple*

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

we have a new exact couple,

$$\begin{tikzcd} D' \arrow[rr, "\alpha'"] & & D' \arrow[dl, "\beta'"] \\ & E' \arrow[ul, "\gamma'"] & \end{tikzcd}$$

called the derived couple, where $D' = Im(\alpha)$, $E' = H(E, d)$, and α', β', γ' are to be defined below.

We shall also note $d' = \beta'\gamma'$.

Proof. Define $\beta' : D' \to E'$, as the unique morphism that completes this diagram,

$$\begin{tikzcd} D \arrow[r, "\beta"] \arrow[d, "\alpha"] & Ker(d) \arrow[r, "p"] & E' \\ D' \arrow[urr, dashed, "\beta'"'] & & \end{tikzcd}$$

That is, $\beta'(\alpha(x)) = p(\beta(x))$, where $p : Ker(d) \to Ker(d)/Im(d) = E'$ is the quotient morphism. Note that this is well defined by exactness, because $d\beta(x) = \beta\gamma\beta(x) = 0$, and $\beta(Ker(\alpha)) = \beta(Im(\gamma)) = 0$.

Similarly, γ' is defined as the unique morphism completing this diagram,

$$\begin{tikzcd} Ker(d) \arrow[r, "\gamma"] \arrow[d, "p"] & D' \\ E' \arrow[ur, dashed, "\gamma'"'] & \end{tikzcd}$$

So, $\gamma'(p(x)) = \gamma(x)$. Again, this is all well defined by exactness: If x is in $Ker(d)$, $\beta\gamma(x) = 0$, so $\gamma(x)$ is in $Ker(\beta) = Im(\alpha) = D'$, and if $p(x) = 0$, $x = \beta\gamma(z)$ for some z. But then $\gamma(x) = \gamma\beta\gamma(z) = 0$.

α' is just the corestriction of α.

Now we have to show exactness at each place. This is routine and somewhat tedious.

First note that $\alpha'\gamma'(p(x)) = \alpha\gamma(x) = 0$. If $x = \alpha(z)$ is in D', $\beta'\alpha'(x) = p(\beta(\alpha(z)) = p(d(z)) = 0$, and $\gamma'\beta'(x) = \gamma\beta(z) = 0$. So, $Im(\gamma') \subseteq Ker(\alpha')$, $Im(\alpha') \subseteq Ker(\beta')$, and $Im(\beta') \subseteq Ker(\gamma')$.

1. $Ker(\alpha') \subseteq Im(\gamma')$.

   Take x in $Ker(\alpha') \subseteq D'$, so $\alpha(x) = 0$, and $x = \alpha(z)$ for certain z. As $Ker(\alpha) = Im(\gamma)$, we get $x = \gamma(y)$ for some y. But $d(y) = \beta\gamma(y) = \beta(\alpha(z)) = 0$. So y is in $Ker(d)$. Then, $\gamma'(p(y)) = \gamma(y) = x$.

2. $Ker(\beta') \subseteq Im(\alpha')$

   Take x in $Ker(\beta') \subseteq D'$, so $x = \alpha(z)$ for some z, and $\beta'(x) = p\beta(z) = 0$, meaning that $\beta(z) = \beta\gamma(y)$ for certain y. Then $\beta(z - \gamma(y)) = 0$ so $z - \gamma(y)$ is in $Ker(\beta) = Im(\alpha)$, and we can get a w such that $\alpha(w) = z - \gamma(y)$. Finally, $x = \alpha(z) = \alpha\alpha(w) + \alpha\gamma(y) = \alpha\alpha(w)$, so $x = \alpha'(\alpha(w))$.

3. $Ker(\gamma') \subseteq Im(\beta')$

   Take $x = p(z)$ in $Ker(\gamma')$. So, $\gamma(z) = 0$, and z is in $Ker(\gamma) = Im(\beta)$. Take y such that $\beta(y) = z$. Then, $\beta'(\alpha y) = p\beta(y) = p(z) = x$.

Note that from the proof we know how bidegrees change, $deg(\alpha) = deg(\alpha')$, $deg(\gamma') = deg(\gamma)$ and $deg(\beta') = deg(\beta) - deg(\alpha)$.

□

Hence, if we are given an exact couple,

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

we immediately obtain a spectral sequence by taking successive derivatives, i.e. define $(E^1, d^1) = (E, d)$, and $(E^{r+1}, d^{r+1}) = (E^{(r)}, d^{(r)})$, where the superscript (r) stands for taking r derivatives.

If our exact couple comes from a filtration F of a chain complex C, we shall say that the spectral sequence is induced by the filtration F of C. Note that in this case

$$deg(\alpha^{(r)}) = (1, -1)$$

$$deg(\gamma^{(r)}) = (-1, 0)$$

$$deg(\beta^{(r)}) = -(r-1)deg(\alpha) = (1-r, r-1)$$

$$deg(d^r) = deg(\beta^{(r)}\gamma^{(r)}) = deg(\beta^{(r)}) + deg(\gamma^{(r)}) = (-r, r-1)$$

Moreover,

$$D^r_{(p,q)} = Im(H_{p+q}(F^{p-r+1}C) \to H_{p+q}(F^p)),$$

where the map is induced by the inclusion $F^{p-r+1}C \to F^pC$.

2.2 Convergence

We will try to understand now what could it mean for a spectral sequence to converge.

Definition 32. *If M is an R module, $N \subseteq N'$ submodules of M, we say that N'/N is a subquotient of M. That is, a subquotient of a module is a quotient of a submodule. Mutatis mutandis we get similar definitions for bigraded modules (or in fact, any abelian category)*

If (E^r, d^r) is a spectral sequence of bigraded R modules, E^{r+1} is a subquotient of E^r. In particular $E^2 = Z^2/B^2$ =cycles/boundaries, where $Z^2 = Ker(d^1)$ and $B^2 = Im(d^1)$ are submodules of E^1. E^3 is a subquotient of E^2, and by the correspondence theorem, there are unique submodules $B^3 \subseteq Z^3$ such that $Z^3 \subseteq Z^2$, and $B^2 \subseteq B^3$, and $(Z^3/B^2)/(B^3/B^2) = E^3$. Continuing like this, we get sequence of boundaries and cycles,

$$B^2 \subseteq \cdots \subseteq B^r \subseteq \cdots$$

$$\cdots Z^r \subseteq \cdots \subseteq Z^2$$

such that every B^i is a submodule of every Z^j, and there is an isomorphism between Z^r/B^r and E^r. Hence, the following definition is natural,

Definition 33. *Let (E^r, d^r) be a spectral sequence of bigraded R modules. Set $Z^\infty = \cap Z^r$ and $B^\infty = \cup B^r$. The quotient $E^\infty = Z^\infty / B^\infty$ is called the limit term of the spectral sequence*

For each p, q, $E^r_{(p,q)}$ gets closer to the limit $E^\infty_{(p,q)}$ as r grows. For example,

Lemma 34. *If $E^r_{(p,q)} = E^s_{(p,q)}$ for $s > r$, then $E^r_{(p,q)}$ is $E^\infty_{(p,q)}$.*

That is, if a point stabilizes, it is the limit term.

Proof. If $E^r_{(p,q)} = E^{r+1}_{(p,q)}$, the expression of $E^{r+1}_{(p,q)}$ as a subquotient of $E^1_{(p,q)}$ must be the same as the one of $E^r_{(p,q)}$. So, $Z^r_{(p,q)} = Z^{r+1}_{(p,q)}$ and $B^r_{(p,q)} = B^{r+1}_{(p,q)}$. But this is true for every $s > r$. So $Z^\infty_{(p,q)} = Z^r_{(p,q)}$ and $B^\infty_{(p,q)} = B^r_{(p,q)}$, and hence $E^\infty_{(p,q)} = E^r_{(p,q)}$. □

Our guiding example is the spectral sequence induced by a filtration F on a chain complex C. In that case, there is an obvious filtration on the graded module $\{H_n\}$ of homology modules of C: Take $(\Phi^p H)_n$ as the image of $\iota_p : F^p C \to C$ in the n-th homology group, that is, $Im(H_n(\iota_p))$.

Recall now that in this case, $E^1_{(p,q)} = H_{p+q}(F^p C / F^{p-1} C)$. If we set $n = p + q$, this "looks like" $(\Phi^p H)_n / (\Phi^{p-1} H)_n$, where we just commuted the operations filtration "$F \sim \Phi$" and homology. The idea will be that the spectral sequence will converge when we can commute homology and filtration at least in the limit. But first the following condition will be convenient, as it will discard trivial filtrations.

Definition 35. *A filtration Φ of a graded module H is called bounded if for each n there are integers $s(n)$ and $t(n)$ such that $\Phi_{s(n)} = 0$ and $\Phi_{t(n)} = H_n$.*

A weaker condition could be used, see [29]. Now, we can give a precise meaning to what we said.

Definition 36. *A spectral sequence (E^r, d^r) converges to a graded module H, if there is some bounded filtration Φ of H, such that $E^\infty_{(p,q)} \simeq \Phi^p H_n / \Phi^{p-1} H_n$, where $n = p + q$.*

This will be noted as $E^r_{(p,q)} \implies H_n$.

Hence, when the spectral sequence converges it will only determine factors $\Phi^p H_n / \Phi^{p-1} H_n$, and some extension problems could remain. But in some cases these problems can be avoided. Suppose for example that $E^2_{(p,q)} \implies H_n$, and the limit term E^∞ had all zeroes except maybe in line zero ($q = 0$). Then $\Phi^p H_n = \Phi^{p-1} H_n$ if $n \neq p$. As Φ is bounded, $\Phi^p H_n = 0$ if $p < n$ and $\Phi^p H_n = E^\infty_{(p,q)} = H_n$ if $p \geq n$, and no extension problems arise.

Our main convergence theorem is the following (theorem 10.14 of [36]),

Theorem 37. *If F is a bounded filtration on a chain complex of R modules C, and (E^r, d^r) is the spectral sequence induced by F,*

1. *For each p, q, $E^\infty_{(p,q)} = E^r_{(p,q)}$ for large r (depending on p and q).*
2. *$E^2_{(p,q)} \implies H_n(C)$, where the filtration on $H(C)$ is the one induced by F.*

Most homological spectral sequences arising in topology are first quadrant spectral sequences with bidegree $(-r, r-1)$. In these cases, for every position (p, q), there is a big enough r such that the differentials coming in and out of that position are zero, and thus $E^\infty_{(p,q)} = E^r_{(p,q)}$.

Proof. Given n, if $p < s(n)$ or $p > t(n)$ and $q = n - p$, $E^1_{(p,q)} = 0$, and consequently $E^r_{(p,q)} = 0$ for any r. Given now p, q, remember that the differential d^r has bidegree $(-r, r-1)$. So we have,

$$E^r_{(p+r,q-r+1)} \to E^r_{(p,q)} \to E^r_{(p-r,q+r-1)},$$

where the arrows are the respective differentials. So if r is large enough, the differentials coming in and out of the (p,q) position will be zero. More precisely, if $r > p - s(p+q-1)$, $E^r_{(p-r,q+r-1)} = 0$ and if $r > t(p+q+1) - p$, $E^r_{(p+r,q-r+1)} = 0$. Hence if r is big enough, $E^s_{(p,q)} = E^r_{(p,q)}$ for $s > r$ and by lemma 34, we have 1.

From the r-th page of the spectral sequence, consider the exact diagram

$$D^r_{(p+r-2,q-r+2)} \xrightarrow{\alpha^r} D^r_{(p+r-1,q-r+1)} \xrightarrow{\beta^r} E^r_{(p,q)} \xrightarrow{\gamma^r} D^r_{(p-1,q)}$$

Recall that

$$D^r_{(p,q)} = Im(H_{p+q}(F^{p-r+1}C) \to H_{p+q}(F^pC))$$

Therefore, if $n = p+q$, and $r > t(n)-p+2$, we have $(F^{p+r-2}C)_n = (F^{p+r-1}C)_n = C_n$, so,

$$D^r_{(p+r-2,q-r+2)} = Im(H_n(F^{p-1}C) \to H_n(C)) = (\Phi^{p-1}H)_n$$

$$D^r_{(p+r-1,q-r+1)} = Im(H_n(F^pC) \to H_n(C)) = (\Phi^pH)_n$$

On the other hand, if $r > p - s(n)$,

$$D^r_{(p-1,q)} = Im(H_n(F^{p-r}C \to H_n(F^{p-1}C)) = 0,$$

as $F^{p-r}C = 0$

And by exactness we arrive at 2 (α^r is an inclusion).

□

2.3 Double Complexes

We will need double complexes in the next section. Essentially they are chain complexes of chain complexes. Associated to them are two obvious filtrations that under some conditions converge to the homology of the "total complex".

Definition 38. *A double complex of R modules C is a collection*

$$\{C_{(p,q)}\}_{(p,q)\in\mathbb{Z}^2}$$

of R modules, together with maps

$$d^v_{(p,q)} : C_{(p,q)} \to C_{(p,q-1)},$$

$$d^h_{(p,q)} : C_{(p,q)} \to C_{(p-1,q)},$$

such that $d^vd^v = d^hd^h = d^vd^h + d^hd^v = 0$.

For simplicity we omit subscripts of differentials d^v, d^h.If M is a chain complex over the category of chain complexes of R modules, that is, we have a diagram,

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots & C_{(p-1,q+1)} \arrow[l] \arrow[d, "d^v"] & C_{(p,q+1)} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{(p+1,q+1)} \arrow[l, "d^h"'] \arrow[d, "d^v"] & \cdots \arrow[l] \\
\cdots & C_{(p-1,q)} \arrow[l] \arrow[d, "d^v"] & C_{(p,q)} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{(p+1,q)} \arrow[l, "d^h"'] \arrow[d, "d^v"] & \cdots \arrow[l] \\
\cdots & C_{(p-1,q-1)} \arrow[l] \arrow[d] & C_{(p,q-1)} \arrow[l, "d^h"'] \arrow[d] & C_{(p+1,q-1)} \arrow[l, "d^h"'] \arrow[d] & \cdots \arrow[l] \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

in order to make it into a double complex, we add a sign $(-1)d^v_{(p,q)}$ whenever p is odd. This process is reversible, and thus a double complex is the same as a chain complex of chain complexes. Homology groups don't change when we use this trick, as kernels and images remain the same.

If C is a double complex, we define its total complex as,

$$Tot(C)_n = \bigoplus_{p+q=n} C_{(p,q)}$$

and take $d = d^v + d^h$ as its differential. That is, we are adding all modules in every diagonal n, and applying d means moving left and down in each place. The relations $d^v d^v = d^h d^h = d^v d^h + d^h d^v = 0$ ensure that this is a chain complex.

If C is a double complex, by taking vertical homology we obtain a new chain complex: for every q, $(H_q(C, d^v), d^h)$ is a chain complex where $(H_q(C, d^v))_p$ is the q-th homology group of column p (vertical homology at the point (p, q)), and where the differentials are induced by d^h. Similarly, we could first take horizontal homology, and arrive at the chain complex $(H_p(C, d^h), d^v)$, where $(H_p(C, d^h))_q$ is the p-th homology group of row q.

Associated to a double complex we have two filtrations HV and VH of the total complex $Tot(C)$,

$$(HV^p Tot(C))_n = \bigoplus_{i \leq p} C_{(i,n-i)}$$

$$(VH^p Tot(C))_n = \bigoplus_{j \leq p} C_{(n-p,p)}$$

Differentials restrict well to these submodules so indeed we have filtrations of the chain complex $Tot(C)$. Note that if we have a first quadrant bicomplex, i.e. $C_{(p,q)} = 0$ unless $p \geq 0$ and $q \geq 0$, both of these filtrations are bounded so theorem 37 applies, and the spectral sequences induced by them will converge to the homology of the total complex.

Lemma 39. *Let C be a first quadrant double complex, HV the filtration described above. Then there is a second quadrant spectral sequence such that,*

1.
$$E^2_{(p,q)} = H_p(H_q(C, d^v), d^h) \implies H_{p+q}(Tot(C))$$
where the filtration is induced by HV

2. *For each p, q, $E^\infty_{(p,q)} = E^r_{(p,q)}$ for large r (depending on p and q)*

3. *$deg(d^r) = (-r, r-1)$*

Similarly, for the filtration VH,

1.
$$E^2_{(p,q)} = H_p(H_q(C, d^h), d^v) \implies H_{p+q}(Tot(C))$$
where the filtration is induced by VH

2. *For each p, q, $E^\infty_{(p,q)} = E^r_{(p,q)}$ for large r (depending on p and q)*

3. *$deg(d^r) = (-r, r-1)$*

Note that the (p, q) position of the second page of the spectral sequence induced by HV is just standing at position (p, q) of the double complex, and taking first vertical homology and then horizontal homology. For the filtration VH it will be first taking horizontal and then vertical homology, but at place (q, p), there is a "transposition" going on.

Proof. This is all a consequence of theorem 37, because as noted, both filtrations are bounded, C being a first quadrant bicomplex. To verify what the second page looks like, just note that $HV^pTot(C)/HV^{p-1}Tot(C)$ is the p-th column of C, and that $d = \beta\gamma$ is identified with d^v This shows that $E^1_(p, q) = (H_q(C, d^v))_p$, it is then immediate to see that $E^2_(p, q) = H_p(H_q(C, d^v), d^h)$. □

2.4 Grothendieck Spectral Sequence

Grothendieck spectral sequence can be understood as a chain rule for derived functors. Given $F : A \to B$ and $G : B \to C$ additive functors between abelian categories, it will relate the derived functors of GF with those of G and F. Some basic definitions and constructions of homological algebra are reviewed in section A.2, for a real treatment, consider [47] or [36].

We start with a previous definition,

Definition 40. *If B, C are abelian categories, B with enough projectives, $F : B \to C$ a right exact additive functor, an object b of B is said to be left F-acyclic if $L_nF(b) = 0$ for every $n > 0$.*

Theorem 41 (Grothendieck spectral sequence)**.** *Let $F : A \to B$, $G : B \to C$ be right exact additive functors between abelian categories, A, B with enough projectives, and such that F maps projective objects into left G-acyclic ones.*

Then, for any a in A, we have a first quadrant spectral sequence (E^r, d^r), such that

1. *$E^2_{(p,q)} = L_pGL_qF(a) \implies L_{p+q}GF(a)$*

2. *For each p, q, $E^\infty_{(p,q)} = E^r_{(p,q)}$ for large r (depending on p and q).*

3. $deg(d^r) = (-r, r-1)$

Proof. The strategy is the following: Take a in A and a projective resolution (P, α). Apply F to P, and construct a certain chain complex of chain complexes, that can be thought of as a projective resolution of $F(P)$. Then, consider the double complex associated to it, the two standard filtrations described before, and apply lemma 39 . One of the spectral sequences has the second page we want, and the other converges to what we are looking for. As both are converging to the homology of the total complex, we will be done.

Take a in A, and a projective resolution (P, α) of a. Apply F to P,

$$F(P_0) \leftarrow F(P_1) \leftarrow F(P_2) \longleftarrow \cdots$$

and factor it as

$$F(P_0) \hookleftarrow B_0 \twoheadleftarrow F(P_1) \hookleftarrow Z_1 \hookleftarrow B_1 \twoheadleftarrow F(P_2) \hookleftarrow Z_2 \hookleftarrow B_2 \twoheadleftarrow \cdots$$

where

$$B_n = Im(F(P_{n+1}) \to F(P_n))$$
$$Z_{n+1} = Ker(F(P_{n+1}) \to F(P_n)).$$

Now we will fill this diagram with projective resolutions. Consider first, projective resolutions $R_0 \to B_0$ and $T_0 \to F(P_0)/B_0$ and apply the horseshoe lemma to the diagram

$$\begin{tikzcd} T_0 \arrow[d, two heads] & & R_0 \arrow[d, two heads] \\ F(P_0)/B_0 & F(P_0) \arrow[l, two heads] & B_0 \arrow[l, hook'] \end{tikzcd}$$

thus obtaining a projective resolution $Q_0 \to F(P_0)$ as in lemma 78. Now, pick projective resolutions $R_1 \to B_1$ and $T_1 \to Z_1/B_1$, and by considering

$$\begin{tikzcd} T_1 \arrow[d, two heads] & & R_1 \arrow[d, two heads] \\ Z_1/B_1 & Z_1 \arrow[l, two heads] & B_1 \arrow[l, hook'] \end{tikzcd}$$

produce a projective resolution $K_1 \to Z_1$ by applying the horseshoe lemma. Then, obtain a projective resolution $Q_1 \to F(P_1)$ by applying the horseshoe lemma to

$$\begin{tikzcd} R_0 \arrow[d, two heads] & & K_1 \arrow[d, two heads] \\ B_0 & F(P_1) \arrow[l, two heads] & Z_1 \arrow[l, hook'] \end{tikzcd}$$

Proceeding inductively, we get

$$
\begin{tikzcd}
Q_0 \arrow[d, two heads] & R_0 \arrow[l, hook] \arrow[d, two heads] & Q_1 \arrow[l, two heads] \arrow[d, two heads] & K_1 \arrow[l, hook] \arrow[d, two heads] & R_1 \arrow[l, hook] \arrow[d, two heads] & Q_2 \arrow[l, two heads] \arrow[d, two heads] & K_2 \arrow[l, hook] \arrow[d, two heads] & R_2 \arrow[l, hook] \arrow[d, two heads] & \cdots \arrow[l, two heads] \\
F(P_0) & B_0 \arrow[l, hook] & F(P_1) \arrow[l, two heads] & Z_1 \arrow[l, hook] & B_1 \arrow[l, hook] & F(P_2) \arrow[l, two heads] & Z_2 \arrow[l, hook] & B_2 \arrow[l, hook] & \cdots \arrow[l, two heads]
\end{tikzcd}
$$

Now consider the double complex $G(Q)$, induced by applying G to the chain complex of chain complexes

$$Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots$$

Note that $G(Q)_{(p,q)} = G((Q_p)_q)$. This is a first quadrant bicomplex, so lemma 39 applies. As F maps projective objects to left G-acyclic ones, and $L_0G = G$, when we compute the second page of the spectral sequence induced by the filtration HV, we get

$$
\begin{array}{cccc}
\cdots & \cdots & \cdots & \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
GF(a) & L_1GF(a) & L_2GF(a) & \cdots
\end{array}
$$

Then $E^2 = E^\infty$, and we get that $H_n(TotG(Q)) = L_nGF(a)$.

Consider now the second filtration VH. To compute E^2, take first horizontal and then vertical homology and transpose.

Note that pointwise, every horizontal map we got from applying the horseshoe lemma was either an inclusion or a projection from a direct sum. As additive functors preserve direct sums, we get that G will preserve this inclusions and projections. So, we have the diagram of chain complexes

$$G(Q_0) \hookleftarrow G(R_0) \twoheadleftarrow G(Q_1) \hookleftarrow G(K_1) \hookleftarrow G(R_1) \twoheadleftarrow G(Q_2) \hookleftarrow \cdots$$

Hence $Ker(Q_p \to Q_{p-1}) = G(K_p)$ and $Im(Q_{p+1} \to Q_p) = G(R_p)$, so when taking horizontal homology we get $G(K_p)/G(R_p)$ and this is $G(T_p)$, because again G preserves direct sums. That is, as $(K_p)_q = (R_p)_q \bigoplus (T_p)_q$, and G preserves sums, $G((K_p)_q) = G((R_p)_q) \bigoplus G((T_p)_q)$.

But T_p was chosen as a projective resolution of $Z_p/B_p = L_pF(a)$. So starting from (p,q), when we take horizontal and then vertical homology we arrive at $L_qGL_pF(a)$. Transposing this we arrive at the desired description of the second page.

$\square$

As a consequence, we obtain the following results, which could have been proven in an elementary way.

Lemma 42. *If $F : A \to B$, and $G : B \to C$ are additive right exact functors between abelian categories, A, B with enough projectives,*

1. *If G is exact, $L_n(GF) \simeq GL_n(F)$*
2. *If F is exact and preserves projectives $L_n(GF) \simeq L_n(G)F$*

Chapter 3

Homology with Local Coefficients

Our aim in this chapter is to give a homological version of Whitehead's theorem. Recall that this result states that if $f : X \to Y$ is a continuous function between connected CW complexes, f is a homotopy equivalence iff $\pi_n(f)$ is an isomorphism for every n. If instead we know that $H_n(f)$ is an isomorphism, f need not be a homotopy equivalence: there are noncontractible CW complexes that have the homology of a point, these are called acyclic spaces, see example 2.38 of [21]. But if we assume that both X and Y are simply connected, the result holds: f is a homotopy equivalence iff $H_n(f)$ is an isomorphism for every n.

So spaces having non-trivial fundamental group pose a problem. To deal with them, we have homology with local coefficients. The homology version of Whitehead's theorem will state that f is a homotopy equivalence iff $\pi_1(f)$ is an isomorphism, and f induces and isomorphism on every homology group with any local coefficient system.

The classical reference for homology with local coefficients is [42]. We will also follow [48] and [13].

3.1 Definitions

In this section we will present two alternative but equivalent definitions of homology with local coefficients.

Definition 43. *A local coefficient system on a topological space X is a functor $F : \Pi_1(X) \to Ab$ from the fundamental grupoid of X to the category Ab of abelian groups.*

Associated to a local coefficient system F we have a chain complex $C(X, F)$ defined as

$$C(X, F)_p = \bigoplus_{\Delta^p \xrightarrow{\sigma} X} F(\sigma(e_1))$$

where $e_1 = (1, 0, \cdots, 0)$ is the first vector of the canonical base of $\mathbb{R}^{p+1}$, Δ^p is the convex hull of this basis, and the direct sum is taken over all continuous $\sigma : \Delta^p \to X$.

We denote an arbitrary element of the direct sum

$$\bigoplus_{\Delta^p \xrightarrow{\sigma} X} F(\sigma(e_1))$$

as $\sum_\sigma g[\sigma]$, where g is an element of the group $F(\sigma(e_1))$. If $\sigma : \Delta^p \to X$ is a continuous function, we define $d_i\sigma : \Delta^{p-1} \to X$ as $\sigma\theta(\delta^i)$ (see chapter 1), i.e. the restriction of σ to the i-th face of Δ^p.

The differential $\delta_p : C(X,F)_p \to C(X,F)_{p-1}$ is defined as

$$\delta_p(g[\sigma]) = F([\sigma([e_1\ e_2])])(g)[d_0\sigma] + \sum_{k\neq 1}(-1)^k g[d_k\sigma]$$

where $[e_1\ e_2]$ is the affine path from e_1 to e_2 in Δ^p. Note that this is well defined because $d_i\sigma(e_1) = \sigma(e_1)$ for $i \neq 0$, and $d_0\sigma(e_1) = \sigma(e_2) = \sigma([e_1\ e_2])(2)$. A standard computation shows that $\delta\delta = 0$, so $(C(X,F),\delta)$ is a chain complex of abelian groups.

Definition 44. *Given F a local coefficent system on a topological space X, we denote*

$$H_n(X,F)$$

the n-th homology group of the chain complex $(C(X,F),\delta)$, and refer to it as the n-th homology group of X with coefficient system F

Note that if $F(x) = G$ for any x in X and $F(\alpha) = id_G$ for any path α, then $H_n(X,F) = H_n(X,G)$, the classical homology of X with coefficients in the group G.

Moreover, we have functoriality in the following sense: if $f : X \to Y$ is a continuous map, and F a local coefficient system on Y, f induces an obvious morphism $H_n(X, f_*F) \to H_n(Y,F)$ where $f_*F = F\Pi_1(f)$ since,

$$g[\sigma] \to g[f\sigma]$$

is a chain map. The reader should be careful, since our notation is not standard, we note f_* instead of f^*, see section A.1.2 for further comments.

In some way the universal cover $\tilde{X} \to X$ of a space, untwists it: it has trivial fundamental group and the same higher homotopy groups as X. As this solves the problems arising from a non-trivial fundamental group, it may be expected that by somehow twitching the homology of $\tilde{X}$ we could arrive at the homology of X with local coefficients. The following alternative definition of homology with local coefficients makes this clear.

Start from a pointed space (X,x) with a universal cover $(\tilde{X},\tilde{x})$ (any locally path connected, semilocally simply connected space has one). Remember that the fundamental group $\pi_1(X,x)$ is isomorphic to the group of deck transformations of $\tilde{X}$ by mapping $[\alpha]$ in $\pi_1(X,x)$ to the unique deck transformation that sends $\tilde{x}$ to $\tilde{\alpha}(1)$, where $\tilde{\alpha}$ is the unique lifting of α starting from $\tilde{x}$. This means that $\pi_1(X,x)$ acts on the left of $\tilde{X}$ by automorphisms. This action is free as any deck transformation that fixes a point is the identity. Therefore, there is a natural left $\mathbb{Z}[\pi_1(X,x)]$ module structure on every $S(\tilde{X})_n$, and as the differential respect the action, $S(\tilde{X})$ turns out to be a chain complex of left $\mathbb{Z}[\pi_1(X,x)]$ modules.

On the other hand, note that the category $Ab^{\Pi_1(X)}$ is naturally equivalent to $Ab^{\pi_1(X,x)^{op}}$, since $\pi_1(X,x)^{op}$ is a skeleton of the grupoid $\Pi_1(X)$, and the equivalence is given by restricting functors. Then, F a local system of coefficients on X is the same as a right $\mathbb{Z}[\pi_1(X,x)]$ module, since $Ab^{\pi_1(X,x)^{op}}$ is naturally identified with the category of such structures: the abelian group $F(x)$ has a right $\pi_1(X,x)$ action, $g[\alpha] = F([\alpha])(g)$. Alternatively, if we are given a right $\mathbb{Z}[\pi_1(X,x)]$ module F, we can extend F to a system of local coefficients by previously choosing paths from x to every point in X. So we can go from local systems to right $\mathbb{Z}[\pi_1(X,x)]$ modules and back, and we will do so from now on.

Then, given F a right $\mathbb{Z}[\pi_1(X,x)]$, we can consider

$$F \underset{\mathbb{Z}[\pi_1(X,x)]}{\otimes} S_n(\tilde{X})$$

and these abelian groups will form a chain complex, since tensoring against a modules is functorial, i.e. $(id_F \otimes \delta)(id_F \otimes \delta) = 0$. Call $H'_n(X,F)$ the n-th homology group of this complex. It is isomorphic to $H_n(X,F)$ (so in particular, the second definition has no ambiguities), as shown by the next theorem by Eilenberg. For its proof we will follow [13] and [48].

Theorem 45 (Eilenberg)**.** *Let $F : \Pi_1(X) \to Ab$ be a system of local coefficients on X. Then, $H_n(X,F)$ is isomorphic to $H'_n(X,F)$*

Proof. We will show an isomorphism on the chain complex level. Assume that X is connected, and for every y in $\tilde{X}$ choose a path ℓ_y from $\tilde{x}$ to y. Note that up to path homotopy there is a unique choice.

Define

$$U : F \underset{\mathbb{Z}[\pi_1(X,x)]}{\otimes} S(\tilde{X})_n \longrightarrow C(X,F)_n$$

by mapping $g \otimes \sigma$ to $F([p\ell_{\sigma(e_1)}])(g)[p\sigma]$. Note that $p\ell_{\sigma(e_1)}$ is a path in X from x to $p\sigma(e_1)$ and g is in $F(x)$. Moreover, U is clearly bilinear, so to check that it is well defined it suffices to show that it respects the action, i.e. $U(g[\eta] \otimes \sigma) = U(g \otimes [\eta]\sigma)$ for $[\eta]$ in $\pi_1(X,x)$. Let h be the deck transformation induced by $[\eta]$, $\tilde{\eta}$ the lifting of η to a path starting at $\tilde{x}$, $y = \sigma(e_1)$ and $\tilde{y} = h(y)$.

First,

$$\begin{aligned} U(g[\eta] \otimes \sigma) &= U(F([\eta])(g) \otimes \sigma) = F([p\ell_y])(F([\eta])(g))[\sigma] \\ &= F([\eta p\ell_y])(g)[\sigma] \end{aligned}$$

and,

$$U(g \otimes [\eta]\sigma) = U(g \otimes h\sigma) = F([p\ell_{\tilde{y}}])(g)[ph\sigma] = F([p\ell_{\tilde{y}}])[\sigma]$$

as $ph = p$. Finally note that as $\tilde{X}$ is simply connected (the key step, we untwisted the space), $[\ell_{\tilde{y}}] = [\tilde{\eta}h(\ell_y)]$ as both are paths between the same points. Projecting this to X we get that $[\eta p\ell_y] = [p\ell_{\tilde{y}}]$. Then, U is a well defined group morphism.

We must verify that U commutes with the differential. Define $y = \sigma(e_1)$, and $z = \sigma(e_2)$.

$$\begin{aligned} U(\delta(g \otimes \sigma)) &= U(g \otimes \sum (-1)^k d_k\sigma) \\ &= F([p\ell_z])(g)[pd_0\sigma] + \sum_{k \neq 1} (-1)^k F([p\ell_y])(g)[pd_k\sigma] \end{aligned}$$

$$
\begin{aligned}
\delta(U(g \otimes \sigma)) &= \delta(F([p\ell_y])(g)[p\sigma]) \\
&= F([p\sigma([e_1\ e_2])])(F([p\ell_y]))(g)[d_o p\sigma] + \sum_{k \neq 1} (-1)^k F([p\ell_y])(g)[d_k p\sigma] \\
&= F([p\ell_y p\sigma([e_1\ e_2])])(g)[d_o p\sigma] + \sum_{k \neq 1} (-1)^k F([p\ell_y])(g)[d_k p\sigma]
\end{aligned}
$$

First note that $d_i p\sigma = p d_i \sigma$ for all i. And again, as $\tilde{X}$ is simply connected $[\ell_y \sigma([e_1\ e_2])] = [\ell_z]$ as both are paths between the same points. So their projection over X is the same, and we have that U is a chain map.

We will construct an inverse V to U. First, choose for each y in X a point $\tilde{y}$ in the fiber $p^{-1}(y)$. Define $y = \sigma(e_1)$, and consider

$$
V(g[\sigma]) = F([p\ell_{\tilde{y}}]^{-1})(g) \otimes \tilde{\sigma}
$$

where $\tilde{\sigma}$ is the unique lifting of σ such that $\tilde{\sigma}(e_1) = \tilde{y}$. Trivially, this is a well defined group morphism, and $UV = id$. Consider now VU. Define $z = \sigma(e_1)$, and $y = p(z)$. Then, it is immediate to check that $VU(g \otimes \sigma) = (g[\alpha] \otimes [\alpha]^{-1}\sigma) = (g \otimes \sigma)$, where $\alpha = p\ell_z (p\ell_{\tilde{y}})^{-1}$. □

Clearly, $\mathbb{Z}[\pi_1(X, x)]$ is a right module over itself. The singular classical homology of the universal cover can be recovered from the homology of X with this coefficient system.

Lemma 46. *$H(X, \mathbb{Z}[\pi_1(X, x)]) = H(\tilde{X})$*

Proof. This is immediate, since

$$
\mathbb{Z}[\pi_1(X, x)] \underset{\mathbb{Z}[\pi_1(X,x)]}{\otimes} S(\tilde{X}) \simeq S(\tilde{X})
$$

as $S(\tilde{X})$ is a free $\mathbb{Z}[\pi_1(X, x)]$ left module. □

Finally, note that if X is simply connected, a right $\mathbb{Z}[\pi_1(X, x)]$ module F, is just an abelian group, and $id_X : X \to X$ is a universal cover, so then

$$
F \underset{\mathbb{Z}[\pi_1(X,x)]}{\otimes} S(\tilde{X}) = F \underset{\mathbb{Z}}{\otimes} S(X)
$$

so the homology of X with coefficients in the local system F is the classical homology of X with coefficients in the group $F(x)$. In particular if X is n-connected, we know that for $0 < i \leq n$, $H_i(X) = 0$ for coefficients in any group, so

Lemma 47. *If X is n-connected, $n \geq 1$,*

$$
H_i(X, F) = 0
$$

for $0 < i \leq n$ and any local system of coefficients F.

3.2 Homology Whitehead Theorem with Local Coefficients

Before proving the promised result, we need to recall some basic theorems. Remember that a space is n-connected if it is nonempty and its first n homotopy groups are nonzero, -1-connected means nonempty. A pair (X, A) is n-connected if $\pi_i(X, A) = 0$ for $i \leq n$ and A intersects every path component of X. For further definitions, and a proof of the following result, see [20].

Theorem 48 (Hurewicz)**.** *If a space X is $(n-1)$-connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \simeq H_n(X)$. If a pair (X, A) is $(n-1)$-connected, $n \geq 2$, with A 1-connected, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \simeq H_n(X, A)$.*

This theorem tells us that the first nonzero homotopy and homology groups of a simply connected space occur in the same dimension and are isomorphic. A similar statement holds for a simply connected pair (X, A) with A 1-connected.

A map $f : X \to Y$ is said to be n-connected if $\pi_i(f)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$, equivalently (M_f, X) is n-connected, where M_f is the mapping cylinder of f. Recall that M_f is constructed as the following pushout,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle \iota_0} & & \Big\downarrow{\scriptstyle \overline{\iota_o}} \\ X \times I & \xrightarrow{\overline{f}} & M_f \end{array}$$

We define $r : M_f \to Y$ by mapping $[(x,t)]$ to $f(x)$ and $[y]$ to y, and $\iota : X \to M_f$ by $\iota(x) = [(x, 1)]$. It turns out that r is a homotopy equivalence (actually Y is a deformation retract of M_f), ι a closed cofibration, in particular a subspace, and

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & Y \\ & M_f \arrow[ur, "r"'] & \end{tikzcd}$$

commutes. So f can be replaced by ι. This is the standard way to replace a map by a closed cofibration.

Lemma 49. *If $f : X \to Y$ is a map between simply connected nonempty spaces, and $H_i(f)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$ then f is an n-equivalence.*

Proof. First, by replacing Y with M_f we can assume f is a closed subspace. From the long exact sequence of homotopy groups of the pair (Y, X) we see that (Y, X) is 1-connected. If $H_i(f)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$, from the long exact sequence of homology for the pair (Y, X), we have that $H_i(Y, X) = 0$ for $i \leq n$. By theorem 48, $\pi_i(Y, X) = 0$ for $i \leq n$. And then, using again the long exact sequence of homotopy groups, we have that f is an n-equivalence.

□

Theorem 50 (Homology Whitehead with Local Coefficients)**.** *Let $f : X \to Y$ be a continuous map, such that $\pi_1(f)$ is an isomorphism, and for any local coefficient system F on Y, f induces an isomorphism $H_i(X, f_*F) \to H_i(Y, F)$ for $i < n$ and an epimorphism at dimension n. Assume that X, Y have universal covers. Then, f is an n-equivalence.*

Proof. Consider the diagram,

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p"] & \tilde{Y} \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\tilde{f}$ is some lifting of f. Choose $\mathbb{Z}[\pi_1(Y, y)]$ as local coefficient system on Y. Then, since $\pi_1(f)$ is an isomorphism, $f_*\mathbb{Z}[\pi_1(Y, y)] = \mathbb{Z}[\pi_1(X, x)]$, and by lemma 46, $\tilde{f}$ induces an isomorphism

$$H_i(\tilde{X}) \simeq H_i(X, f_*\mathbb{Z}[\pi_1(Y, y)]) \simeq H_i(Y, \mathbb{Z}[\pi_1(Y, y)]) \simeq H_i(\tilde{Y})$$

for $i < n$ and an epimorphism for $i = n$. By lemma 49, applied to $\tilde{f}$, $\tilde{f}$ is n-connected. Considering that for $i \geq 2$, $\pi_i(p)$ and $\pi_i(q)$ are isomorphisms, the diagram

$$\begin{tikzcd} \pi_i(\tilde{X}) \arrow[r, "\pi_i(\tilde{f})"] \arrow[d, "\pi_i(p)"'] & \pi_i(\tilde{Y}) \arrow[d, "\pi_i(q)"] \\ \pi_i(X) \arrow[r, "\pi_i(f)"] & \pi_i(Y) \end{tikzcd}$$

shows that f is an n-equivalence.

□

3.3 Simplicial Homology with Local Coefficients

In the simplicial setting, we have analogous defintions. If $F : \mathcal{C} \to Ab$, is a functor we can define the following chain complex.

$$C(\mathcal{C}, F)_n = \bigoplus_{(f_1, \cdots, f_n)} F(d_1(f_1))$$

where the sum is taken over all n-simplices $\sigma = (f_1, \cdots, f_n)$ of the nerve of $\mathcal{C}$. An arbitraty element of this group will be written as $\sum_\sigma g[\sigma]$. The differential is defined as,

$$\delta_n(g[\sigma]) = F(f_1)(g)[d_0\sigma] + \sum_{k \neq 1} (-1)^k g[d_k\sigma]$$

This makes $(C(\mathcal{C}, F), \delta)$ into a chain complex, and its homology will be noted as $H(\mathcal{C}, F)$. Observe that for this definition F need not invert morphisms.

Note that as a consequence of the description of the fundamental group of a small category, there is an equivalence between the categories of local systems of abelian groups on $\mathcal{C}$, i.e. functors $F : \mathcal{C} \to Ab$ that invert morphisms, and systems of local coefficients on $B\mathcal{C}$. We will freely go from one to the other.

Theorem 51. *If F is a local coefficient system on $B\mathcal{C}$, then*

$$H(\mathcal{C}, F) \simeq H(B\mathcal{C}, F)$$

This is Milnor's theorem for homology with local coefficients. A careful modification of any of its proofs will work, we choose to omit it. See appendix 2 of [18], [33] or theorem 2.27 of [21].

There is an alternative algebraic description of the homology groups with local coefficients in small categories: they are the left derived functors of the colimit functor. See [18].

Theorem 52. *Let $\mathcal{C}$ be a small category, then*

$$\{H_n(\mathcal{C}, -)\}$$

$$\{L_n colim_{\mathcal{C}}\}$$

are isomorphic δ-functors from $Ab^{\mathcal{C}}$ to Ab.

It is crucial for this description that we consider the whole category $Ab^{\mathcal{C}}$ and not just the functors that invert morphisms, as we shall see later. Before proving it, we need to establish some lemmas.

Recall that $Ab^{\mathcal{C}}$ is an abelian category by computing everything pointwise.

Lemma 53. *Let $\mathcal{C}$ be a small category, and let $\mathcal{C}_0$ be the subcategory of $\mathcal{C}$ with the same set of objects and only the identity morphisms. Let $\iota : \mathcal{C}_0 \to \mathcal{C}$ be the inclusion. Then, for every N in $Ab^{\mathcal{C}_0}$, and evey $i > 0$ $H_i(\iota^*(N)) = 0$.*

Compare this lemma with the notion of acyclic object. In the appendix we defined ι^*, see section A.1.2.

Proof. Clearly, N is just a collection of abelian groups indexed by the objects of $\mathcal{C}$. Note that if $i > 0$, $H_i(\mathcal{C}_0, N) = \bigoplus_c H_i(c, N(c)) = 0$ as $H_i(c, N(c))$ is just the homology of a point because theorem 51 applies.

Moreover, from the definition of ι^*,

$$\iota^*(N)(c) = \bigoplus_{\alpha : a \to c} N(a)$$

where the sum is taken over all arrows with codomain c. And if $f : c \to d$ is a morphism, $\iota^*(N)(f)$ is defined by mapping the $N(a)$ factor associated to $\alpha : a \to c$, to the $N(a)$ factor associated to $f\alpha$ through $id_{N(a)}$. Then, if $\sigma = (f_1, \cdots, f_n)$ is an n-simplex, and $x_0 = dom(f_1)$,

$$C(\mathcal{C}, \iota^*(N))_n = \bigoplus_{\sigma} \bigoplus_{a \xrightarrow{\alpha} x_0} N(a) \simeq \bigoplus_{(\alpha,\sigma)} N(a) = C(\mathcal{C}_0, N)_{n+1}$$

This identification commutes with the differential, so $C(\mathcal{C}, \iota^*(N))$ is isomorphic to the chain complex obtained by shifting $C(\mathcal{C}_0, N)$ one position to the right, deleting $C(\mathcal{C}_0, N)_0$. Then, if $i > 0$, $H_i(\mathcal{C}, \iota^*(N)) = H_{i+1}(\mathcal{C}_0, N) = 0$.

□

As a consequence, we have the following lemma.

Lemma 54. *If $\mathcal{C}$ is a small category, Ab^C has enough projectives.*

Proof. Take $N : \mathcal{C} \to Ab$. It is clear that the counit of the adjunction $\iota^* \dashv \iota_*$ gives a map $\iota^*\iota_*(N) \to N$ that is an epimorphism, as for every a in $\mathcal{C}$,

$$\bigoplus_{\alpha : a \to c} N(a) \to N(a)$$

is an epimorphism by considering the arrow $\alpha = id_a$

Now, take an epimorphism $P \to \iota_*(N)$ with P projective in $Ab^{\mathcal{C}_0}$. This can be done because projective objects in $Ab^{\mathcal{C}_0}$ are those that are pointwise projective, and Ab has enough projectives.

Finally, consider the composition $\iota^*(P) \to \iota^*\iota_*(N) \to N$. The first arrow is an epimorphism since left adjoints are right exact and $P \to \iota_*(N)$ is an epimorphism. Therefore the composition is also an epimorphism. Finally, $\iota^*(P)$ is projective by lemma 73, since ι_* is exact (it is precomposition with ι and kernels and cokernels are computed pointwise). □

Proof of Theorem 52. The functors $\{H_n(\mathcal{C}, -)$ can be factored as

$$Ab^{\mathcal{C}} \to Ch(Ab)_{\geq 0} \to Ab$$

by first taking the chain complex $C(\mathcal{C}, -)$ and then homology groups. Since the first of this arrows is exact and the second a δ functor, the composition is a delta functor. In particular $H_0(\mathcal{C}, -)$ is exact.

Now we show that $H_0(\mathcal{C}, -)$ and $colim_{\mathcal{C}}$ are naturally isomorphic. Take $N : \mathcal{C} \to Ab$, $colim_{\mathcal{C}} N$ and $H_0(\mathcal{C}, N)$ are both the cokernel of the pair

$$d_0,\ d_1 : \bigoplus_f N(dom(f)) \rightrightarrows \bigoplus_c N(c).$$

This shows in particular that $colim_{\mathcal{C}}$ is right exact. As we saw before $Ab^{\mathcal{C}}$ has enough projectives, so $colim_{\mathcal{C}}$ can be derived, resulting in a universal δ-functor $\{L_n colim_{\mathcal{C}}\}$.

Finally, if we knew that $\{H_n(\mathcal{C}, -)\}$ was a universal δ functor, we would be done because universal δ-functors that are naturally isomorphic at dimension zero, are naturally isomorphic as δ-functors. By lemma 77, it is enough to show that the functors $H_n(\mathcal{C}, -)$ are coeffaceable. Take $N : \mathcal{C} \to Ab$. In the proof of lemma 53, we saw that $\iota^*\iota_*(N) \to N$ is an epimorphism, and by lemma 53, $H_n(\mathcal{C}, \iota^*\iota_*(N)) = 0$ for $n > 1$. □

So the homology groups of a small category with arbitrary coefficients measure by how much $colim_{\mathcal{C}}$ is not exact. It is clear now that considering every possible functor $N : \mathcal{C} \to Ab$, and not just those inverting morphisms was essential: a 1-connected not contractible category $\mathcal{C}$ has $H_1(\mathcal{C}, N) = 0$ for every local system N, but this does not imply that $L_1 colim_{\mathcal{C}} = 0$, otherwise $colim_{\mathcal{C}}$ would be exact and $H_i(\mathcal{C}, N) = 0$ for every N and $i \geq 1$, meaning by Hurewicz that $\mathcal{C}$ would be contractible.

A category $\mathcal{C}$ is said to be of homological dimension 0 if $colim_{\mathcal{C}}$ is exact or equivalently $H_1(\mathcal{C}, -) = 0$. For a review about homological dimension of small categories consider [23].

A contractible category, need not have homological dimension zero, as evidenced by the free category $\mathcal{C}$ generated by,

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "\beta"'] & b \\ c & \end{tikzcd}$$

and the functor $N : \mathcal{C} \to Ab$ defined as,

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "0"] \arrow[d, "0"'] & 0 \\ 0 & \end{tikzcd}$$

After some calculation it follows that $H_1(\mathcal{C}, N) = \mathbb{Z}$, and $B\mathcal{C}$ is homotopy equivalent to its graph, which is contractible.

It is true however that a connected category of homological dimension zero is contractible.

Theorem 55. *Let $\mathcal{C}$ be a connected category of homological dimension zero. Then $\mathcal{C}$ is contractible, i.e. BC is contractible.*

Proof. First, by lemma 46, $\tilde{BC}$ the universal cover of BC is acyclic. As it is also simply connected, it is contractible. Then, since $\pi_k(\tilde{BC}) \simeq \pi_k(BC)$ for $k \geq 2$, BC is a $K(G, 1)$, where $G = \pi_1(BC, x)$.

Consider now the chain complex $S(\tilde{BC})$, and augment it to a chain complex S by defining $S_{-1} = \mathbb{Z}$, and $\epsilon : S_0(\tilde{BC}) \to \mathbb{Z}$ as $\epsilon(\sum_x n_x x) = \sum_x n_x$. Regard $\mathbb{Z}$ as a left $\mathbb{Z}[G]$ module, and note that S is a complex of $\mathbb{Z}[G]$ modules, since $S(\tilde{BC})$ is one too. Moreover, since the fundamental group acts freely on $S(\tilde{BC})$ and $\tilde{BC}$ is connected and acyclic, S is in fact a free resolution of $\mathbb{Z}$ as a left $\mathbb{Z}[G]$ module. Therefore, if M is a right $\mathbb{Z}[G]$ module,

$$H(BC, M) = H(M \underset{\mathbb{Z}[G]}{\otimes} S(\tilde{BC})) = Tor^{\mathbb{Z}[G]}(M, \mathbb{Z})$$

Therefore, if $\mathcal{C}$ has homological dimension zero, $\mathbb{Z}$ is $\mathbb{Z}[G]$ flat. The following lemma shows that this implies that $G = 1$, so $\mathcal{C}$ is contractible. □

Lemma 56. *Let G be a group. If $\mathbb{Z}$ is $\mathbb{Z}[G]$ flat, then $G = 1$.*

Proof. Suppose first that G is abelian. Then, by considering BG, the same argument as in the last proof, and Hurewicz theorem,

$$G = H_1(BG) = H_1(BG, \mathbb{Z}) = Tor^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) = 1$$

For the general case, suppose G is not trivial and consider $1 < H < G$, with H abelian. Let F be a free resolution of $\mathbb{Z}$ as a left $\mathbb{Z}[G]$ module. By Lagrange theorem, every free $\mathbb{Z}[G]$ module is a projective $\mathbb{Z}[H]$ module: write G as a disjoint union $\cup_{g \in I} Hg$, then $\mathbb{Z}[G] \simeq \oplus_{g \in I} \mathbb{Z}[Hg]$ as left $\mathbb{Z}[H]$ modules, and each $\mathbb{Z}[Hg]$ is isomorphic to $\mathbb{Z}[H]$. So F may be considered as a projective resolution of $\mathbb{Z}$ as a left $\mathbb{Z}[H]$ module. Take now M a right $\mathbb{Z}[H]$ module.

$$M \underset{\mathbb{Z}[H]}{\otimes} F = M \underset{\mathbb{Z}[H]}{\otimes} (\mathbb{Z}[G] \underset{\mathbb{Z}[G]}{\otimes} F) = (M \underset{\mathbb{Z}[H]}{\otimes} \mathbb{Z}[G]) \underset{\mathbb{Z}[G]}{\otimes} F$$

So,

$$Tor^{\mathbb{Z}[H]}(M, \mathbb{Z}) = Tor^{\mathbb{Z}[G]}(M \underset{\mathbb{Z}[H]}{\otimes} \mathbb{Z}[G], \mathbb{Z}) = 1$$

Then, $\mathbb{Z}$ is $\mathbb{Z}[H]$ flat. Since H is abelian, $H = 1$, a contradiction.

□

A category $\mathcal{C}$ is called filtered if every finite diagram has a cone, that is

1. $\mathcal{C}$ is not empty.
2. Given a, b in $\mathcal{C}$ there is an object c and arrows $a \to c$, $b \to c$ in $\mathcal{C}$.
3. Given two arrows $\alpha_1, \alpha_2 : a \to b$, there is an arrow $\beta : c \to d$, such that $\beta\alpha_1 = \beta\alpha_2$.

It is well known that if $\mathcal{C}$ is a small filtered category, $colim_{\mathcal{C}} : Set^{\mathcal{C}} \to Set$ commutes with finite limits. This holds when we replace Set by Ab, and means that $\mathcal{C}$ has homologial dimension zero. So we have,

Proposition 57. *Every small filtered category is contractible*

The same result can be found in [33], though the proof is different.

In [24], J.R. Isbell gave a characterization of categories of homological dimension zero, and used it to prove a particular case of a conjecture by U. Oberst posed in [32]. This conjecture stated that a category had homological dimension zero iff each of its components was filtered. Later in [25], Isbell and B. Mitchell exhibited a counterexample: the category Δ_{iny} has homological dimension zero but it is not filtered.

Then, we have,

Proposition 58. Δ_{iny} *is contractible.*

To end this section, note that if X is a $K(G, 1)$ its homology with coefficients in a local system is the same as the group homology of G with coefficients. The classical reference for group cohomology is [6].

3.4 André Spectral Sequence

Having described the homology of a small category as derived functors, we can apply Grothendieck spectral sequence. We follow [18].

We begin with a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories. Now, note that the following triangle commutes

$$\begin{tikzcd} Ab^{\mathcal{D}} \arrow[rr, "F_*"] & & Ab^{\mathcal{C}} \\ & Ab \arrow[ul, "\Gamma_{\mathcal{D}}"] \arrow[ur, "\Gamma_{\mathcal{C}}"'] & \end{tikzcd}$$

where $\Gamma_{\mathcal{C}}$ and $\Gamma_{\mathcal{D}}$ are the diagonal functors described in section A.1.2. As $colim_{\mathcal{C}} \dashv \Gamma_{\mathcal{C}}$, $colim_{\mathcal{D}} \dashv \Gamma_{\mathcal{D}}$, and $F^* \dashv F_*$ we have that

$$colim_{\mathcal{C}} \simeq colim_{\mathcal{D}} \circ F^*$$

since composition of right adjoints is a right adjoint, and adjoints are unique up to natural isomorphism. By lemma 53 $Ab^{\mathcal{C}}$ and $Ab^{\mathcal{D}}$ have enough projectives, and lemma 73 implies that F^* preserves projectives, so in particular maps projectives into $colim_{\mathcal{D}}$ acyclic objects (projective objects are $colim_{\mathcal{D}}$ acyclic), and as it is a left adjoint it is right exact. So, we are able to apply Grothendieck spectral sequence.

Theorem 59 (André Spectral Sequence [2]). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. For any $N : \mathcal{C} \to Ab$, we have a first quadrant spectral sequence (E^r, d^r), such that*

1. $E^2_{(p,q)} = L_p colim_{\mathcal{D}} \circ L_q F^*(N) \implies L_{p+q} colim_{\mathcal{C}}(N)$
2. *For each p, q, $E^{\infty}_{(p,q)} = E^r_{(p,q)}$ for large r (depending on p and q).*
3. $deg(d^r) = (-r, r-1)$

Finally, we shall describe what the left derived functors of F^* look like. Take d in $\mathcal{D}$. Consider the commutative diagram,

$$\begin{tikzcd} Ab^{\mathcal{C}} \arrow[r, "F^*"] \arrow[d, "pr_*"] & Ab^{\mathcal{D}} \arrow[d, "ev_d"] \\ Ab^{F/d} \arrow[r, "colim_{F/d}"] & Ab \end{tikzcd}$$

where F/d is the category over d (the definition can be found in section A.1.2), $ev_d : Ab^{\mathcal{D}} \to Ab$ is the evaluation functor and $pr : F/a \to \mathcal{C}$ is the projection on the first component.

The functor ev_d is exact, so $L_n(ev_d \circ F^*) = ev_d \circ L_n(F^*)$. Since the following diagram commutes,

$$\begin{tikzcd} Ab^{\mathcal{C}_0} \arrow[r, "\iota^*"] \arrow[d, "pr_*"] & Ab^{\mathcal{C}} \arrow[d, "pr_*"] \\ Ab^{(F/d)_0} \arrow[r, "\iota^*"] & Ab^{F/d} \end{tikzcd}$$

(note that if F/d were any other arbitrary category, the diagram would not commute) $pr_* : Ab^{\mathcal{C}} \to Ab^{F/d}$ maps enough projectives into $colim_{F/d}$ acyclic objects (recall that the projectives we used where those in the image of ι^*), and it is also exact, so $L_n(colim_{F/d}) \circ pr_* = L_n(colim_{F/d} \circ pr_*)$. Then, we have

$$L_n(colim_{F/d}) \circ pr_* = ev_d \circ L_n(F^*)$$

Evaluating at a functor $N : \mathcal{C} \to Ab$ and using theorem 52

$$H_n(F/d, Npr) = L_n F^*(N)(d)$$

For simplicity, we state theorem 59 in terms of homology groups by applying theorem 52.

Theorem 60 (André Spectral Sequence [2]). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. For any $N : \mathcal{C} \to Ab$, we have a first quadrant spectral sequence (E^r, d^r), such that*

1. $E^2_{(p,q)} = H_p(\mathcal{D}, d \rightsquigarrow H_q(F/d, Npr)) \implies H_{p+q}(\mathcal{C}, N)$
2. *For each p, q, $E^\infty_{(p,q)} = E^r_{(p,q)}$ for large r (depending on p and q).*
3. $deg(d^r) = (-r, r-1)$

Chapter 4

Quillen's Theorem A

Quillen's original proof of Theorem A uses bisimplicial sets, i.e. simplicial objects in the category of simplicial sets. In [34] he gave a different proof of this theorem, using Grothendieck spectral sequence and homology with local coefficients. Though in this second proof, $\mathcal{C}$ and $\mathcal{D}$ were assumed to be posets, Quillen observed that everything could be generalized to small categories without essential change. In this chapter, we follow Quillen's approach, and generalize this second proof for categories. Moreover, we prove a stronger result concerning n-equivalences, theorem 63. This has been proved in the particular context of posets by Bjorner [4] and Barmak [3]. We also exhibit an homological version of the result.

We follow Quillen's approach closely, and begin with some previous definitions and lemmas.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories, d an object of D. We denote by F/d the category having as objects the pairs $(c, \alpha : F(c) \to d)$, where c is an object of $\mathcal{C}$, α a morphism in $\mathcal{D}$, and as morphisms

$$f : (c_1, \alpha_1 : F(c_1) \to d) \to (c_2, \alpha_2 : F(c_2) \to d)$$

where $f : c_1 \to c_2$ is a morphism in $\mathcal{C}$, satisfying $\alpha_2 F(f) = \alpha_1$. The categories F/d are called the fibers of F. We shall say that the fibers of F are n-connected if $B(F/d)$ is n-connected for every d in $\mathcal{D}$.

Lemma 61. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor betwen small categories. Let the fibers of F be n-connected.*

1. *If $n = -1$, $\mathcal{C}$ is nonempty iff $\mathcal{D}$ is nonempty.*
2. *If $n = 0$, F induces an isomorphism between $\pi_0(\mathcal{C})$ and $\pi_0(\mathcal{D})$.*
3. *If $n = 1$, F induces an isomorphism in the fundamental groupoids.*

Proof. If $n = -1$ this is obvious.

Suppose $n = 0$. It suffices to show that $\pi_0(F)$ is a bijection when we restrict F to the 1-skeletons. First take a_1, a_2 objects of $\mathcal{C}$, such that $F(a_1)$ can be connceted to $F(a_2)$ by morphisms in $\mathcal{D}$,

$$\begin{tikzcd}
 & b_1 & & b_2 & & \cdots & & b_n & \\
F(a_1) \arrow[ur] & & c_1 \arrow[ul] \arrow[ur] & & c_2 \arrow[ul] & & c_{n-1} \arrow[ur] & & F(a_2) \arrow[ul]
\end{tikzcd}$$

As F/c_i is 0-connected we can take d_i in $\mathcal{C}$, and maps $F(d_i) \to c_i$. Now, F/b_{i+1} is 0-connected so we can join the objects $(d_i, F(d_i) \to c_i \to b_{i+1})$ and $(d_{i+1}, F(d_{i+1}) \to c_{i+1} \to b_{i+1})$ by arrows in F/b_{i+1}, meaning in particular that d_i and d_{i+1} can be joined by arrows in $\mathcal{C}$. Similarly d_1 can be joined to a_1 and d_{n-1} to a_2. Then, a_1 can be connected to a_2 by a path of arrows in $\mathcal{C}$. Finally, if d is an object of $\mathcal{D}$, F/d is nonempty so F is onto on objects. So $\pi_0(F)$ is a bijection.

Consider now the case $n = 1$. We shall show that $F_* : Cov(\mathcal{D}) \to Cov(\mathcal{C})$ is an equivalence. Consider the diagram

$$\begin{tikzcd}
Set^{\mathcal{D}} \arrow[r, "F_*"] & Set^{\mathcal{C}} \\
Cov(\mathcal{D}) \arrow[u] \arrow[r, "F_*"] & Cov(\mathcal{C}) \arrow[u]
\end{tikzcd}$$

The idea will be to verify that the left adjoint of $F_* : Set^{\mathcal{D}} \to Set^{\mathcal{C}}$ coresctricts to the categories of coverings, i.e. $F^*(N)$ inverts morphisms if N does, and gives an equivalence between them.

Take $E : \mathcal{C} \to Set$ a morphism inverting functor, and $g : d \to d'$ an arrow in $\mathcal{D}$, and universal cones,

$$\{E(c) \overset{\tau_{(c,\alpha)}}{\to} F^*(E)(d)\}_{(c,\alpha)\in F/d}$$

$$\{E(c) \overset{\tau'_{(c,\alpha)}}{\to} F^*(E)(d')\}_{(c,\alpha)\in F/d'}$$

As F/d and F/d' are 1-connected and E inverts morphisms, by lemma 24 every map τ, τ' is an isomorphism. But for any (c, α) in F/d, $F^*(E)(g)$ satisfies $F^*(E)(g)\tau_{(c,\alpha)} = \tau'_{(c,g\alpha)}$, so $F^*(E)(g)$ is an isomorphism.

The fact that $F_*F^* \simeq id_{Cov(\mathcal{C})}$ and $F^*F_* \simeq id_{Cov(\mathcal{D})}$ is immediate from the definitions. $\square$

Note that in the last part of this proof, we could have replaced *Set* by any category having all small colimits, such as *Ab*. We obtain,

Lemma 62. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor between small categories with 1-connected fibers, then*

$$F^* : Ab^{\mathcal{C}} \to Ab^{\mathcal{D}}$$

preserves morphism inverting functors. Its restriction to the categories of local systems, gives an equivalence of categories, together wih F_*.

The following theorem was the reason of everything we have done so far. As noted, it is a minor improvement of Quillen's Theorem A.

Theorem 63. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories, with n-connected fibers, $n \geq 1$. Then, F induces an $(n+1)$-equivalence between the classifying spaces.*

Proof. By lemma 61 we can restrict to each connected component, and F induces an isomorphism between the fundamental groups of the classifying spaces in each of them. So we assume that $\mathcal{C}$ and $\mathcal{D}$ are connected. If we now show that F induces an isomorphism up to dimension n and an epimorphism in dimension $(n+1)$ for the homology groups of the classifying spaces with arbitrary local systems of coefficients we would be done by theorem 50.

Take then N a local system of coefficients on $\mathcal{C}$. By theorem 60, we have a first quadrant spectral sequence (E^r, d^r) such that

1. $E^2_{(p,q)} = H_p(\mathcal{D}, d \rightsquigarrow H_q(F/d, Npr)) \implies H_{p+q}(\mathcal{C}, N)$
2. For each p, q, $E^\infty_{(p,q)} = E^r_{(p,q)}$ for large r (depending on p and q).
3. $deg(d^r) = (-r, r-1)$

Since the fibers are n-connected, and Npr inverts morphisms, by theorem 51 and lemma 47, $E^2_{(p,q)} = 0$ if $1 \leq q \leq n$. Moreover, recall from the description in theorem 59 that

$$E^2_{(p,0)} = H_p(\mathcal{D}, L_0F^*(N)) = H_p(\mathcal{D}, F^*(N))$$

Noting the bidegree of the diferentials and the zeroes present, we have that for $k \leq n+1$, $E^2_{(k,0)} = E^\infty_{(k,0)}$. More precisely, stand at point $(k, 0)$. At sheet $r \geq 2$, this point receives a differential from a module located $(k + r, 1 - r)$ that is zero because it is outside the first quadrant, since $1 - r < 0$. It also sends a differential to a module M at position $(k - r, r - 1)$. If $r > k$ M is zero because again, it is not in the first quadrant. Suppose then that $r \leq k$. As $2 \leq r \leq k \leq n + 1$, we have that $1 \leq r - 1 \leq n$. This means that M is located in the horizontal band of zeroes.

This means we have isomorphisms

$$H_k(\mathcal{D}, F^*(N)) \simeq H_k(\mathcal{C}, N)$$

for $k \leq n$. If $k = n + 1$, the only nonzero elements in the $(n + 1)$ diagonal are $E^\infty_{(0,n+1)}$ and $H_{n+1}(\mathcal{D}, F^*(N))$. Recall that we had a bounded filtration $\Phi^p := \Phi^p H_{n+1}(\mathcal{C}, N)$, such that

$$\Phi^p/\Phi^{p-1} = E^\infty_{(p,n+1-p)}$$

By boundeness, this means that $\Phi^p = 0$ if $p < 0$, $\Phi^p = E^\infty_{(0,n+1)}$ if $0 \leq p \leq n$, and $\Phi^{n+1} = H_{n+1}(\mathcal{C}, N)$. Then, we have the extension problem,

$$0 \to E^\infty_{(0,n+1)} \to H_{n+1}(\mathcal{C}, N) \to H_{n+1}(\mathcal{D}, F^*(N)) \to 0$$

But this means that the map $H_{n+1}(\mathcal{C}, N) \to H_{n+1}(\mathcal{D}, F^*(N))$ is an epimorphism, and we are done.

□

There is an alternative to using Grothendieck spectral sequence, that may be more illuminating, specially since the isomorphisms arising from the spectral sequence seem obscure. We want to show that if the fibers of $F : \mathcal{C} \to \mathcal{D}$ are n-connected, and N is a local system of coefficients in $\mathcal{D}$, then the arrow $H_k(\mathcal{C}, NF) \to H_k(\mathcal{D}, N)$ induced by F is an isomorphism for $k \leq n$ and an epimorphism for $k = n + 1$. Consider the following commutative diagram,

$$\begin{array}{ccc} colim_{\mathcal{C}} \circ F_* & \longrightarrow & colim_{\mathcal{D}} \\ \downarrow & & \downarrow \\ H_0(\mathcal{C}, -) \circ F_* & \longrightarrow & H_0(\mathcal{D}, -) \end{array}$$

where the vertical arrows are isomorphisms, by theorem 52. Since F_* is exact, $H_k(\mathcal{C}, -) \circ F_*$ and $L_k(colim_{\mathcal{C}}) \circ F_*$ are δ-functors, $L_k(colim_{\mathcal{D}})$ and $H_k(\mathcal{D}, -)$ universal δ-functors, by universal property we have the following commutative diagram for every k,

$$\begin{array}{ccc} L_k(colim_{\mathcal{C}}) \circ F_* & \longrightarrow & L_k(colim_{\mathcal{D}}) \\ \downarrow & & \downarrow \\ H_k(\mathcal{C}, -) \circ F_* & \longrightarrow & H_k(\mathcal{D}, -) \end{array}$$

where the vertical arrows are isomorphisms. Then, it suffices to show that

$$L_k(colim_{\mathcal{C}}) \circ F_* \to L_k(colim_{\mathcal{D}})$$

is an isomorphism for $k \leq n$ and an epimorphism for $k = n + 1$ whenever we evaluate in a local system of coefficients. But F_*, F^* give an equivalence of categories, so it is enough to show that

$$L_k(colim_{\mathcal{C}}) \to L_k(colim_{\mathcal{D}}) \circ F^*$$

is an isomorphism for $k \leq n$ and an epimorphism for $k = n + 1$ whenever we evaluate in a local system of coefficients. In Chapter 3 we had shown that $colim_{\mathcal{C}} \simeq colim_{\mathcal{D}} \circ F^*$, so we consider

$$L_k(colim_{\mathcal{D}} \circ F^*) \to L_k(colim_{\mathcal{D}}) \circ F^*$$

With care it can be shown that this map is induced in the following manner: given $N : \mathcal{C} \to Ab$, take $P \to N$ a projective resolution. Consider now a projective resolution $Q \to F^*(N)$. There is a map $F^*(P) \to Q$, since F^* preserves projectives. Then, the morphism is the induced arrow from $L_k(colim_{\mathcal{D}} \circ F^*)(N) = H_k(colim_{\mathcal{D}} F^*(P))$ to $L_k(colim_{\mathcal{D}}) \circ F^*(N) = H_k(colim_{\mathcal{D}} Q)$.

Now, if the fibers are n-connected, and N is a local system of coefficients, $L_k(F^*)(N) = 0$ for $0 < k \leq n$.

If the fibers are contractible they are n-connected for every n, and we can choose $Q_k = F^*(P_k)$ for $0 \leq k \leq n + 1$, since this first section of the resolution is exact. This shows that the induced map is an isomorphisms for $k \leq n$ and it is rather immediate that it is also an epimorphism for $k = n + 1$.

Theorem 64 (Quillen's Theorem A)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories with contractible fibers. Then F induces a weak homotopy equivalence between the classifying spaces.*

For any small category, consider now the functor $\mathbb{Z}_{\mathcal{D}} : \mathcal{D} \to Ab$ that is constantly $\mathbb{Z}$ and maps arrows to $id_{\mathbb{Z}}$. One can prove a homological version of lemma 24:

Lemma 65. *Let $F : \mathcal{J} \to \mathcal{C}$ be a morphism inverting functor between small categories, such that $\mathcal{J}$ is connected and $H_1(\mathcal{J}, \mathbb{Z}_{\mathcal{J}}) = 0$. Then, if the induced functor on the localizaton $S^{-1}\mathcal{C}$ satisfies $\tilde{F}(\alpha\beta) = \tilde{F}(\beta\alpha)$ for every pair of endomorphisms α, β, we have that colimF is isomorphic to each object in the diagram, i.e. if*

$$\{F(j) \xrightarrow{\tau_j} colimF\}_{j\in\mathcal{J}}$$

is a univeresal cone, τ_j is an isomorphism for every j.

Proof. Proceed as in the proof of the original, now factoring F by $(S^{-1}\mathcal{C})_{ab}$, the abelianization of the localization obtained by taking the quotient generated by the relations $\alpha\beta = \beta\alpha$ for every pair of endomorphisms $\alpha\beta$. ☐

Using this it follows that $F^*(\mathbb{Z}_{\mathcal{C}}) \simeq \mathbb{Z}_{\mathcal{D}}$, and in a similar fashion as in the proof of theorem 63 we get

Theorem 66. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If the fibers are homologically n-connected, meaning that $H_k(F/d, \mathbb{Z}) = 0$ for $k \leq n$, then F is an homological $(n+1)$-equivalence, meaning it induces an isomorphism between integral homology for $k \leq n$ and an epimorphism for $k = (n+1)$.*

Appendix A

A.1 Some Basic Category Theory

In this section we present some basic definitions that we used throughout our exposition.

Definition 67. *A category $\mathcal{C}$ is defined by specifying a class $Ob(\mathcal{C})$ of objects, a set $Hom(a,b)$ of morphisms for each pair a,b in $Ob(\mathcal{C})$, and a composition $Hom(b,c) \times Hom(a,b) \to Hom(a,c)$ for each triple a,b,c. Composition is required to be associative and each object a to have an identity id_a in $Hom(a,a)$, that acts as the identity for the composition.*

Categories appear naturally and everywhere. We shall note Set for the category of Sets and maps of sets and Top for the category of topological spaces and continuous maps.

An element f of $Hom(a,b)$ is usually noted as $f : a \to b$. $Hom(a,b)$ is also noted as $[a,b]$.

If $\mathcal{C}$ is a category we can form a new category $\mathcal{C}^{op}$ by formally inverting all arrows, that is $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$, $Hom_{\mathcal{C}^{op}}(a,b) = Hom_{\mathcal{C}}(b,a)$, $f \underset{op}{\circ} g = gf$, and $id_{\mathcal{C}^{op}}(a) = id_{\mathcal{C}}(a)$.

An object a in a category $\mathcal{C}$ is called terminal if for every object b in $\mathcal{C}$, $[b,a]$ has a unique element. If a is terminal in $\mathcal{C}^{op}$ then we say that a is initial (in C).

A functor is a morphism between categories,

Definition 68. *A (covariant) functor $F; \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$, is a mapping that associates to each object c of $\mathcal{C}$ an object $F(c)$ of $\mathcal{D}$, and to every morphism $f : a \to b$ in $\mathcal{C}$, a morphism $F(f) : F(a) \to F(b)$, such that F respects identities and the composition operation.*

A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ will be just a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{D}$.

If c is an object of a category $\mathcal{C}$, it determines two functors $[-,c]$ and $[c,-]$, as follows: $[-,c]$ maps each object a in $\mathcal{C}$ to $[a,c]$, and a morphism $f : a \to b$ to the function that maps s in $[b,c]$ to sf in $[a,c]$. $[c,-]$ maps each object a in $\mathcal{C}$ to $[c,a]$, and a morphism $f : a \to b$ to the function that maps s in $[c,a]$ to fs in $[c,b]$

A category is called small if its objects and morphisms are sets. Cat will denote the category of small categories and functors between them.

A natural transformation is a morphism between functors,

Definition 69. *If $F,G : \mathcal{C} \to \mathcal{D}$ are functors, a natural transformation η from F to G, is specified by providing for each object a of $\mathcal{C}$, an arrow $\eta_a : F(a) \to G(a)$, such that for each morphism $f : a \to b$ in $\mathcal{C}$, the following square commutes,*

$$
\begin{array}{ccc}
F(a) & \xrightarrow{\eta_a} & G(a) \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\
F(b) & \xrightarrow{\eta_b} & G(b)
\end{array}
$$

If $\mathcal{C}, \mathcal{D}$ are categories, $\mathcal{C}$ small, $\mathcal{D}^{\mathcal{C}}$ shall denote the functor category, whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations between them.

If every η_a is an isomorphism, η is called a natural isomorphism (and is in fact an isomorphism in the functor category) and the functors are considered equal. A representable functor is one naturally isomorphic to a functor of the type $[c, -]$ or $[-, c]$.

Two categories $\mathcal{C}, \mathcal{D}$ are said to be equivalent, if there are functors $F : \mathcal{C} \to \mathcal{D}$, and $G : \mathcal{D} \to \mathcal{C}$, such that GF is naturally isomorphic to $id_{\mathcal{C}}$ and FG is naturally isomorphic to $id_{\mathcal{D}}$.

A.1.1 Limits and Colimits

If $F : \mathcal{J} \to \mathcal{C}$ is a functor, we shall say that $\{F(j) \xrightarrow{\alpha_j} c\}_{j \in \mathcal{J}}$, is a cone over F, if for every map $f : j \to k$ in $\mathcal{J}$, $\alpha_j = \alpha_k F(f)$.

If $\{F(j) \xrightarrow{\alpha_j} c\}_{j \in \mathcal{J}}$, $\{F(j) \xrightarrow{\beta_j} d\}_{j \in \mathcal{J}}$, are cones, a morphism from the former to the latter, is a morphism $g : c \to d$ in $\mathcal{C}$, such that $\beta_j = g\alpha_j$ for every j in $\mathcal{J}$.

So, we have a category of cones over F. We define the colimit of F (if it exists) to be the terminal object in this category. Dually, the limit is defined (if it exists) to be the initial object in the category of cocones.

We usually note it as,

$$
\underset{\mathcal{J}}{colim} F(j)
$$

.

As an example, the coproduct of two objects a, b in a category $\mathcal{C}$ is no more than the colimit of the functor $F : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ has two objects x, y and no nonidentity morphisms, and $F(x) = a$, $F(y) = b$.

A category $\mathcal{C}$ is called bicomplete if it has all small limits, that is every functor $F : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is a small category has a limit and a colimit. *Set* and *Top* are examples of bicomplete categories. In case $\mathcal{C}$ has all colimits indexed by $\mathcal{J}$, $colim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ is a functor, in an obvious way.

A.1.2 Adjoint Functors

Adjoint functors appear everywhere, including this appendix.

Definition 70. *If $F : \mathcal{C} \to \mathcal{D}$, and $G : \mathcal{D} \to \mathcal{C}$, are functors, an adjunction $F \dashv G$ between them is a collection bijections,*

$$
Hom_{\mathcal{D}}(F(c), d) \simeq Hom_{\mathcal{C}}(c, G(d))
$$

natural in c and d.

F is said to be a left adjoint to G, and G a right adjoint to F. For example, forgetful or underlying functors tend to have left adjoints which are free constructions.

Composing adjoints functors yields adjoint functos, in a natural way, if $F \dashv G$ and $H \dashv I$, $HF \dashv GI$. Moreover, adjoints are unique up to a natural isomorphism,if $F \dashv G$ and $F \dashv H$ then $H \simeq G$.

One useful property is that left adjoints preserve colimits and right adjoints preserve limits. Then, if we had additive adjoint functors between abelian categories, the left one would be right exact, and the right one left exact.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor between small categories, and $\mathcal{M}$ has all small colimits, the functor

$$F_* : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$$

given by precomposing with F, $F_*(T) = TF$ has a left adjoint,

$$F^* : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{D}}$$

defined by mapping a functor N to its left Kan extension along F, i.e.

$$F^*(N)(d) = \underset{(c,\alpha)\in F/d}{colim} N(c)$$

where F/d, called the fiber over d, is the category having as objects the pairs $(c, \alpha : F(c) \to d)$. A morphism $f : (c, \alpha : F(c) \to d) \to (c', \alpha' : F(c') \to d)$ is an arrow $f : c \to c'$ in $\mathcal{C}$ such that $\alpha = \alpha' F(f)$. If $g : d \to d'$ is in $\mathcal{D}$, $F^*(N)(g)$ is defined by universal property. Take

$$\{N(c) \overset{\tau_{(c,\alpha)}}{\to} F^*(N)(d)\}_{(c,\alpha)\in F/d}$$

$$\{N(c') \overset{\tau'_{(c',\alpha')}}{\to} F^*(N)(d')\}_{(c',\alpha')\in F/d'}$$

universal cones, then $F^*(N)(g)$ is the unique morphism making the following diagram commute for every (c, α) in F/d

$$\begin{tikzcd}
N(c) \arrow[r, "\tau_{(c,\alpha)}"] \arrow[dr, "\tau'_{(c,g\alpha)}"'] & F^*(N)(d) \arrow[d, "F^*(N)(g)"] \\
 & F^*(N)(d')
\end{tikzcd}$$

For a proof, see [26] or [5]. The reader should be careful, as our notation is not standard: generally F_* is noted as F^* and F^* as F_*, however we chose to follow the notation found in [18]

As an application, if $f : \mathcal{J} \to *$, where $*$ is the category with only one object and one morphism, and $\mathcal{M}$ has all colimits indexed by $\mathcal{J}$, we have the adjunction

$$f^* \dashv f_*$$

but it is immediate from the definitions that $f^* = colim_{\mathcal{J}}$, and $f_* : \mathcal{M} \to \mathcal{M}^{\mathcal{J}}$ is defined by $f_*(a)(b) = a$, $f_*(a)(f) = id_a$. We call $f_* = \Gamma_{\mathcal{J}}$ the diagonal functor.

A.1.3 Yoneda Lemma

The Yoneda lemma is a result that states how every category can be embedded in a category of functors. Let $F : \mathcal{C}^{op} \to Set$ be a functor, c an object of $\mathcal{C}$. Then there is a natural bijection between the set of natural transformations from $[-, c]$ to F and $F(c)$. Given a natural transformation $\theta : [-, c] \to F$, map it to $\theta_c(id_c)$. Now, given x in $F(c)$, and $f : d \to c$ in $\mathcal{C}$, define $\theta_d(f) = F(f)(x)$, θ is then a natural transformation.

This is a well defined bijection, and moreover it defines an embedding $h : \mathcal{C} \to Set^{\mathcal{C}^{op}}$.

Not all functors are representable, but every functor is a colimit of representable ones. This is an application (or a restatement) of Yoneda's lemma. Given $F : \mathcal{C}^{op} \to Set$ define Γ_F, as the category having as elements the pairs (c, x), with x in $F(c)$, and as morphisms $f : (c, x) \to (d, y)$, the arrows $f : c \to d$ in $\mathcal{C}$, such that $F(f)(y) = x$. F is then the colimit of the functor $\Diamond : \Gamma_F \to Set^{\mathcal{C}^{op}}$, defind by $\Diamond((c, x)) = [-, c]$, and $\Diamond(f : (c, x) \to (d, y))_a(g) = gf$.

Details can be found in [26].

A.1.4 Presentation of Categories

In this brief appendix a method of constructing categories, analogous to presentation of groups is shown. We will follow [26]

For a graph, we understand a collection of points (or 0-simplices), and directed arrows (or 1-simplices) between them. Formally,

Definition 71. *A graph G is defined by giving a pair of morphisms in Set,*

$$Arr(G) \overset{d_1}{\underset{d_0}{\rightrightarrows}} Ob(G)$$

$Arr(G)$ are the arrows of G, and $Ob(G)$, the objects or vertices of G
A morphism between graphs $D : G \to H$, is a pair of set maps,

$$D : Arr(G) \to Arr(H)$$

and

$$D : Ob(G) \to Ob(H)$$

such that $Dd_i f = d_i Df$ for $i = 0, 1$ and every f in $Arr(G)$.

So we have a category $Grph$ of graphs. Now, given a graph G, we form the free category induced by it, $Free(G)$, as follows: Objects of $Free(G)$ will be $Ob(G)$, and morphisms between a, b in $Ob(G)$ will be tuples $(f_0, \cdots, f_n)$ of composable arrows, beginning at a and ending at b (which we call words from a to b), that is $Hom(a, b) = \{(f_0, \cdots, f_n) : d_1(f_0) = a, d_0(f_n) = b \text{ and } d_0(f_i) = d_1(f_{i+1})\}$ ($d_1(f)$ is the domain of f, and $d_0(f)$ its codomain). Composition of morphisms (when defined) will be concatenation of tuples. That is, if $w = (f_1, \cdots, f_n)$ is a word from a to b, and $v = (g_1, \cdots, g_m)$ is a word from b to c, the composition $v \circ w$ is defined as the concatenation $wv = (f_1 \cdots, f_n, g_1, \cdots, g_m)$ We also add formal identities at each point (that should be regarded as empty words), which act as identities for concatenation.

If $w = (f_0, \cdots, f_n)$, is a word from a to b we shall say that a subword of w is a substring of w or id_a or id_b. We also define the length of w to be n.

Note that we have a functor $Free : Grph \to Cat$, and as expected this turs out to be the left adjoint to the forgetful functor $U : Cat \to Grph$, which assigns to a category its underlying graph.

Presenting a category will consist of taking a quotient of the free category generated by some graph.

Given a category $\mathcal{C}$, a relation R on that category will denote an assignment of for each pair of objects a, b in $\mathcal{C}$, a relation $R_{a,b}$ in $Hom(a, b)$. If each relation is an equivalence relation, such that whenever it makes sense, composition respects equivalence, we call R a congruence.

Now, if we have R a congruence on the category $\mathcal{C}$ we can form the quotient category $\mathcal{C}/R$. Its objects are the objects of $\mathcal{C}$ and its morphisms are the equivalence classes of morphisms of $\mathcal{C}$, that is,

$$Hom_{\mathcal{C}/R}(a, b) = Hom_{\mathcal{C}}(a, b)/R_{a,b}.$$

The compatibilty conditions means that composition is well defined in the obvious way.

If instead of having a congruence we had a relation R on $\mathcal{C}$, we can still take the quotient by previously enlarging the relation R to the smallest congruence containing it.

We are now ready to define what presenting a category means.

Definition 72. *A presentation of a category, is a graph G together with a relation R on $Free(G)$.*

A presentation (G, R), presents the category $(G|R) := Free(G)/R$, and has the expected universal property, that is, to define a functor from it to any category $\mathcal{D}$ it is enough to define a morphism $F : G \to U(\mathcal{D})$, sucht that F respects the relation R.

In the same way that every group admits a presentation, every category admits a presentation too. If $\mathcal{C}$ is small, let G be its underlying graph, and R the relation on G consisting of all valid identities, that is, $R_{a,b} = \{((f_1, \cdots, f_n), (g_1, \cdots, g_m))$ such that $f_n \cdots f_1 = g_m \cdots g_1$ and both are morphisms from a to $b\}$. Then, $(G|R) = \mathcal{C}$.

With this in mind it is easy to define what localizing a small category means, that is, formally inverting every morphism in it: just take a presentation, add one arrow for every arrow in the graph (but going in the other direction), and add the correspoing inverse relation. Given a category $\mathcal{C}$, we will denote its localization by $S^{-1}\mathcal{C}$. We have a natural functor $\mathcal{C} \to S^{-1}\mathcal{C}$, with the expected universal property: every functor from $\mathcal{C}$ that inverts every morphism factors by $S^{-1}\mathcal{C}$.

A.2 Some Homological Algebra

We will provide some definitions and state some general well known facts. We follow [47].

Recall that an object a in an abelian category $\mathcal{C}$ is called projective if given $f : b \to c$ an epimorphism, and $g : a \to c$, there exists $h : a \to b$ such that $fh = g$,

$$\begin{tikzcd} & a \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ b \arrow[r, "f"'] & c \arrow[r] & 0 \end{tikzcd}$$

An abelian category will be said to have enough projectives, if for every object a there is an epimorphism $p \to a$, with p projective.

A functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is additive if $F(f+g) = F(f)+F(g)$, for any f, g.

We say that the diagram,

$$A \xrightarrow{a} B \xrightarrow{b} C$$

is exact at B if $Im(a) = Ker(b)$ (as subobjects of B). An exact sequence of morphisms is a sequence of morphisms exact at each place. Hooks $\hookrightarrow$ will denote monomorphisms, and double headed arrows $\twoheadrightarrow$ epimorphisms.

An additive functor F is exact if given any short exact sequence,

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the sequence,

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

is exact (recall that if F is additive $F(0) = 0$). It is called right exact if instead

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

is exact.

The following lemma is quite useful,

Lemma 73. *If $F \dashv G$ are adjoint additive functors between abelian categories and G is exact, then F preserves projectives.*

Proof. Take P projective in $dom(F)$, and an exact diagram,

$$\begin{tikzcd} & F(P) \arrow[d, "g"] & \\ b \arrow[r, "f"'] & c \arrow[r] & 0 \end{tikzcd}$$

in $dom(G)$. Since G is right exact, the following diagram is exact,

$$\begin{tikzcd} & P \arrow[d, "g"] & \\ G(b) \arrow[r, "f"'] & G(c) \arrow[r] & 0 \end{tikzcd}$$

But now P is projective, and we can find $P \to G(b)$ making the diagram commute. By adjunction we have an arrow $F(P) \to b$ making the first diagram commute.

□

A projective resolution of an object a of $\mathcal{C}$, is a pair (C, α) such that C is a chain complex of projective objects, $C_n = 0$ if $n < 0$, $\alpha : C_0 \to a$, and the augmented complex

$$\cdots \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{\alpha} a$$

is exact. We will note the resolution (C, α) of a, by $C \stackrel{\alpha}{\to} a$

A category with enough projectives has projective resolutions for any object. If $F : \mathcal{C} \to \mathcal{D}$ is a right exact additive functor, its n-th left derived functor is obtained as follows: Given an object a, take a projective resolution (C, α), and define $L_nF(a) = H_n(F(C))$, where $F(C)$ is the chain complex, formed from C by applying F. This turns out to be well defined and functorial. Note that as F is right exact, L_0F is naturally isomorphic to F.

The natural setting to approach derived functors according to Grothendieck is that of δ-functors. See [44].

Definition 74. *A homological δ-functor between $\mathcal{C}$ and $\mathcal{D}$ is a collection of additive functors $T_n : \mathcal{C} \to \mathcal{D}$ for $n \geq 0$, together with morphisms*

$$\delta_n : T_n(c) \to T_{n-1}(a)$$

defined for each short exact sequence $0 \to a \to b \to c \to 0$, such that

$$\cdots \to T_{n+1}(c) \stackrel{\delta}{\to} T_n(a) \to T_n(b) \to T_n(c) \stackrel{\delta}{\to} T_{n-1}(a) \to \cdots \to T_0(c) \to 0$$

is a long exact sequence, and the δ's are natural, i.e. for each commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & a' \arrow[r] \arrow[d] & b' \arrow[r] \arrow[d] & c' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & a \arrow[r] & b \arrow[r] & c \arrow[r] & 0
\end{tikzcd}$$

we have commutative squares,

$$\begin{tikzcd}
T_n(c') \arrow[r] \arrow[d, "\delta"] & T_{n-1}(a') \arrow[d, "\delta"] \\
T_n(c) \arrow[r] & T_{n-1}(a)
\end{tikzcd}$$

For example the homology functors H_n from the category of positive chain complexes $Ch(Ab)_{\geq 0}$ (complexes where $C_k = 0$ if $k < 0$) to Ab form a δ-functor $\{H_n\}$.

A morphism of δ-functors $\{S_n\} \to \{T_n\}$ is a collection of natural transformation between them, that commute with the δ's. A delta functor $\{T_n\}$ is universal if for every δ-functor $\{S_n\}$, and every natural transformation $\eta : S_0 \to T_0$, there is a unique δ-functor morphism that extends it.

Theorem 75. *If $\mathcal{C}$ is an abelian category with enough projectives, the left derived functors of any right exact functor $L : \mathcal{C} \to \mathcal{D}$ form a universal δ-functor.*

That is, in categories with enough projectives, universal delta functors are determined in level 0. A certain condition guarantees that a δ-functor is universal.

Definition 76. *An additive functor $T : \mathcal{C} \to \mathcal{D}$ is coeffaceable if for every c in $\mathcal{C}$ there is an epimorphism $u : p \to c$, such that $T(u) = 0$. A δ-functor $\{T_n\}$is coeffaceable if T_n is coeffaceable for ever $n > 0$.*

Lemma 77 (see [44])**.** *A coeffaceable δ-functor is universal.*

The following is a useful lemma,

Lemma 78 (Horseshoe Lemma)**.** *Let A be an abelian category with enough projectives. If,*

$$0 \longrightarrow a' \xrightarrow{\iota_a} a \xrightarrow{\pi_a} a'' \longrightarrow 0$$

is exact and (P', α'), (P'', α'') are projective resolutions of a' and a'' respectively, then we have a projective resolution (P, α) that fits in the following diagram,

$$\begin{tikzcd}
 & P' \arrow[r, "\iota"] \arrow[d, "\alpha'"] & P \arrow[r, "\pi"] \arrow[d, "\alpha"] & P'' \arrow[d, "\alpha''"] & \\
0 \arrow[r] & a' \arrow[r, "\iota_a"] & a \arrow[r, "\pi_a"] & a'' \arrow[r] & 0
\end{tikzcd}$$

that is to say, ι and π are morphisms of chain complexes, $\alpha\iota_0 = \iota_a\alpha'$ and $\alpha''\pi_0 = \pi_a\alpha$.

Moreover, $P_n = P'_n \bigoplus P''_n$, and for each n, ι_n, π_n are the natural inclusion and projection maps.

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