UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Tesis de Licenciatura

Una Generalización de la Teoría de Ind-objetos de Grothendieck a 2-Categorías

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a Martín

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1 Introducción

1.1 Ideas y objetivos generales

El objetivo principal de este trabajo es extender la teoría de Ind-objetos de Grothendieck a 2-categorías. El interés de esta extensión radica en que, en un trabajo futuro, se estudiarán aplicaciones de este desarrollo a la teoría de homotopías y a la teoría de la forma fuerte (*strong shape theory*).

En este primer paso nos abocamos a entender la categoría de Ind-objetos de una categoría $\mathcal{C}$ y demostrar sus principales propiedades que son ser cerrada por colímites filtrantes (pequeños) y que estos son libres en el sentido dado por la propiedad universal que la caracteriza salvo equivalencia de categorías.

También estudiamos las nociones acerca de 2-categorías necesarias para encarar la generalización mencionada al comienzo.

Utilizando las construcciones conocidas de colímites filtrantes y de límites en la cateogría de conjuntos que denotaremos por $\mathcal{E}ns$, puede darse una descripción explícita de los morfismos de la categoría $Ind(\mathcal{C})$ que resulta muy útil a la hora de realizar ciertas demostraciones.

Los primeros resultados, inéditos, del Yoga de la teoría de 2-Ind-objetos son que, en el caso en que $\mathcal{C}$ es una 2-categoría, $Ind(\mathcal{C})$ resulta una 2-categoría y, por lo tanto, los morfismos entre dos Ind-objetos X e Y resultan ser objetos de una categoría que llamaremos $Hom(X, Y)$.

Con el propósito de describir esta categoría según los lineamientos del caso 1-categorías, repasamos (siguiendo [1]) en este trabajo los bilímites y bicolímites en la categoría $\mathcal{C}at$ indexados por una categoría y realizamos construcciones explícitas de los bicolímites filtrantes y de los bilímites.

Por último, introducimos la 2-categoría de 2-Ind-objetos de una 2-categoría $\mathcal{C}$. La diferencia entre este caso y el de Ind-objetos de una 2-categoría es que aquí el diagrama que describe al Ind-objeto ya no viene dado por una categoría filtrante de índices sino por una 2-categoría 2-filtrante de índices. Por lo tanto, para poder dar una descripción explícita de la categoría de morfismos entre dos 2-Ind-objetos fue necesario estudiar los bilímites y bicolímites en $\mathcal{C}at$ pero ahora indexados por una 2-categoría y construir los bicolímites 2-filtrantes y los bilímites.

Queda para un trabajo futuro probar propiedades análogas a las de la categoría $Ind(\mathcal{C})$ con $\mathcal{C}$ una categoría para las 2-categorías $Ind(\mathcal{C})$ y $2{-}Ind(\mathcal{C})$ con $\mathcal{C}$ una 2-categoría.

1.2 El contenido de este trabajo

Las secciones 6 y 8 de este trabajo contienen resultados inéditos. En ellas se introducen respectivamente las nociones de Ind-objeto y de 2-Ind-objeto de una 2-categoría $\mathcal{C}$. En las secciones 3, 4 y 5 repasamos la teoría necesaria para definir estos conceptos. Más precisamente:

- En la sección 3 explicamos la teoría de Ind-objetos en una categoría $\mathcal{C}$ desarrollada por A. Grothendieck en [2] y realizamos demostraciones explícitas de varios hechos, que no se encuentran en la literatura. Empezamos por definir las nociones de Ind-objeto y de morfismo de Ind-objetos que dan lugar a la categoría $Ind(\mathcal{C})$. En esta primera parte también damos una descripción explícita de los morfismos de Ind-objetos que facilita el posterior trabajo con los mismos. Luego mostramos dos propiedades importantes de la categoría $Ind(\mathcal{C})$, el hecho de que $\mathcal{C}$ es una subcategoría plena de $Ind(\mathcal{C})$ y que allí cada Ind-objeto es el colímite de su diagrama. También damos una equivalencia de categorías entre $Ind(\mathcal{C})$ y la subcategoría plena de la categoría de funtores contravariantes de $\mathcal{C}$ en $\mathcal{E}ns$ formada por aquellos que son colímites filtrantes de funtores representables. Esta equivalencia nos sirve para probar que $Ind(\mathcal{C})$ tiene todos los colímites filtrantes (pequeños). Por último, caracterizamos por una propiedad universal a la categoría $Ind(\mathcal{C})$, salvo equivalencia de categorías. Estas dos últimas propiedades nos permiten pensar a $Ind(\mathcal{C})$ como una completación de $\mathcal{C}$ por colímites filtrantes.

- Luego, en la sección 4 damos en primer lugar las definiciones básicas de la teoría de 2-Categorías para luego utilizarlas en las secciones posteriores. A continuación, estudiamos el caso particular de $\mathcal{C}at^{\mathcal{C}^{op}}$ definiendo explícitamente su estructura de 2-categoría. Esto será usado en la sección 6 para probar que $Ind(\mathcal{C})$ es una 2-categoría en el caso en que $\mathcal{C}$ es una 2-categoría. También enunciamos las definiciones relacionadas con la noción de categoría 2-filtrante, que aparecerá en las secciones 7 y 8. Por último definimos los pseudoconos asociados a un 2-funtor; esta noción generaliza el concepto de cono a 2-categorías y puede ser utilizada para definir los bilímites y bicolímites en la 2-categoría $\mathcal{C}at$.

- El objetivo de la sección 5 es definir y construir los bilímites y los bicolímites en $\mathcal{C}at$ indexados por una categoría (la construcción de los bicolímites se hace solamente para el caso en que la categoría de índices es filtrante). Aquí de nuevo desarrollamos demostraciones explícitas de muchos hechos tomados como ciertos sin demostración en la literatura. En este punto nos pareció oportuno e interesante hacer una comparación entre dos enfoques diferentes. Uno de ellos se basa en la teoría de fibraciones de Grothendieck expuesta en [1] que introducimos brevemente al principio de esta sección para facilitar su lectura. Este enfoque consiste en definir al bilímite como la categoría de secciones cartesianas de una fibración y al bicolímite como la categoría de fracciones de una categoría fibrada. La segunda generaliza de forma más visible las definiciones y construcciones en $\mathcal{E}ns$ de los límites y los colímites como conos universales, y consiste en definirlos como pseudoconos universales y construirlos con la misma "filosofía" con la que se construyen en $\mathcal{E}ns$ los límites y los colímites filtrantes. Al final de la sección probamos que ambas definiciones y construcciones son equivalentes.

- Los desarrollos de la sección 6 son inéditos. Aquí definimos los Ind-objetos de una 2-Categoría $\mathcal{C}$ y los morfismos entre ellos, y probamos que dan lugar a una 2-Categoría $Ind(\mathcal{C})$. Usando las construcciones hechas en la sección 5, damos una descripción explícita de la categoría $Hom(X,Y)$ de morfismos entre dos Ind-objetos.

- Análogamente a lo hecho en la sección 5, en la sección 7 repasamos y generalizamos la construcción de los bilímites y bicolímites en $\mathcal{C}at$ pero ahora indexados por una 2-categoría (la construcción de los bicolímites se hace solamente para el caso en que la 2-categoría de índices es 2-filtrante). En esta sección seguimos a [3]

- Finalmente, en la última sección de este trabajo, también inédita pero inconclusa, definimos la 2-categoría $2-Ind(\mathcal{C})$ cuyos objetos son los 2-Ind-objetos de una 2-categoría $\mathcal{C}$. Utilizando las construcciones hechas en 7, damos una descripción de la categoría de morfismos entre 2-Ind-objetos $Hom(X,Y)$.

2 Background

2.1 Previous definitions and concepts

In this work, we assume that the reader is familiar with the concepts of *category, functor, natural transformation, limit, colimit, filtered category, full and faithfulness, equivalence of categories* and *ends*.

We also take for granted the constructions of limits and filtered colimits in the category of sets and the Yoneda lemma. All these concepts from basic category theory can be found in [4].

Finally, we recall explicitly the following result:

Lemma 2.1. *Given a functor $F : \mathcal{C}^{op} \to \mathcal{E}ns$, we can consider its diagram Γ_F, which is a category whose objects are the pairs (x, C) with $x \in FC$, and an arrow f between (x, C) and (x', C') is given by an arrow $f : C \to C'$ in $\mathcal{C}$ such that $F(f)(x') = x$. F is the colimit of its diagram, in the sense* $F = \underset{(x,C)\in\Gamma_F}{colim}\ hom(-, C)$

2.2 Terminology

To avoid confusion, it is necessary to establish certain notations that will be used throughout this work:

- We are going to denote by $\mathcal{E}ns$ the category of all sets and functions between them.
- We are going to denote by $\mathcal{C}at$ the category of all small categories and functors between them. But we will also consider this category as a 2-category with 2-cells the natural transformations between functors.
- Since we are working with 2-categories, sometimes we will want to consider a set of morphisms between two objects as a category. To avoid confusion with the set of morphisms, we are going to adopt the notation $hom(-,-)$ for the set and $Hom(-,-)$ for the category.
- We are going to denote by $\mathcal{C}^{\mathcal{D}}$ the category of functors between $\mathcal{D}$ and $\mathcal{C}$.
- There is no standard terminology in the literature for the several higher-dimensional notions of limits and colimits. We use here "bilimit" and "bicolimit", notions that we define precisely, and which correspond to the concept denoted *Lim* (with capital L) in [1].

3 Ind-objects of a category $\mathcal{C}$

In this section, we take as reference [2] where Grothendieck develops his theory of Ind-objects, and the appendix of [5] where the authors set out the main definitions and properties about Ind-objects.

On the following pages, we do many of the proofs of the statements in [5] and others not found in detail in the literature. We conclude the section proving a characterization of the category $Ind(\mathcal{C})$ up to equivalence of categories.

3.1 Definition of the category $Ind(\mathcal{C})$

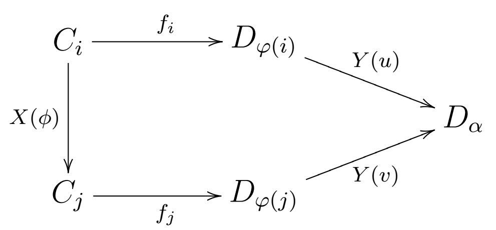
Definition 3.1. *Let $\mathcal{C}$ be any category. An Ind-object of $\mathcal{C}$ is a small filtered system $X = (C_i)_{i \in J}$, i.e. a functor $X : J \to \mathcal{C}$, with J a small filtered category.*

Definition 3.2. *Let $X = (C_i)_{i \in J}, Y = (D_\alpha)_{\alpha \in \Gamma}$ be two Ind-objects of $\mathcal{C}$. We define*

$$hom(X, Y) = \lim_{i \in J^{op}} \operatorname{colim}_{\alpha \in \Gamma} hom(C_i, D_\alpha).$$

Using the constructions of limits and filtered colimits in $\mathcal{E}ns$, we are going to give a description of the morphisms between two Ind-objects:

Proposition 3.3. *The morphisms of Ind-objects between X and Y as in definition 3.2 are pairs $(\varphi, (f_i)_{i \in J})$ quotient by an equivalence relation $\sim$, where $J \xrightarrow{\varphi} \Gamma$ is a function between the objects of J and the ones of Γ, and the f_i are morphisms $C_i \xrightarrow{f_i} D_{\varphi(i)}$ in $\mathcal{C}$ satisfying the following condition:*
For all $i \xrightarrow{\phi} j$ in J, $\exists \alpha \in \Gamma$ and arrows $\varphi(i) \xrightarrow{u} \alpha$, $\varphi(j) \xrightarrow{v} \alpha$ such that the following diagram commutes:



.

The relation $\sim$ is defined as $(\varphi, (f_i)_{i \in J}) \sim (\psi, (g_i)_{i \in J})$ if and only if for all $i \in J, \exists \alpha \in \Gamma$ and arrows $\varphi(i) \xrightarrow{u} \alpha$, $\psi(i) \xrightarrow{v} \alpha$ such that the following diagram commutes:

$$\begin{array}{ccc} C_i & \xrightarrow{f_i} & D_{\varphi(i)} \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle Y(u)} \\ D_{\psi(i)} & \xrightarrow[Y(v)]{} & D_\alpha \end{array} \quad .$$

Proof. Note that the limit and the colimit in definition 3.2 are computed in $\mathcal{E}ns$ and therefore we can make explicit constructions of them.

The diagram associated to the $\operatorname{colim}_{\alpha\in\Gamma} hom(C_i, D_\alpha)$ for $i \in J$ fixed is

$$\begin{aligned} \Gamma &\longrightarrow \mathcal{E}ns \\ \alpha &\longmapsto hom(C_i, D_\alpha) \\ \alpha \xrightarrow{u} \beta &\longmapsto Y(u)_* \end{aligned}$$

where $Y(u)_*$ is defined as $Y(u)_*(f) = Y(u) \circ f$.

Using the construction of filtered colimits in $\mathcal{E}ns$, we have that

$$\operatorname*{colim}_{\alpha\in\Gamma} hom(C_i, D_\alpha) = \coprod_{\alpha\in\Gamma} hom(C_i, D_\alpha) \Big/ \sim$$

where $(f,\alpha) \sim (g,\beta)$ if and only if $\exists\gamma \in \Gamma$ and arrows $\alpha \xrightarrow{u} \gamma$, $\alpha \xrightarrow{v} \gamma$ such that $Y(u) \circ f = Y(v) \circ g$. We will denote the equivalence classes by $[f,\alpha]$.

The diagram associated to the $\lim_{i\in J^{op}} \operatorname{colim}_{\alpha\in\Gamma} hom(C_i, D_\alpha)$ is

$$\begin{aligned} J^{op} &\longrightarrow \mathcal{E}ns \\ i &\longmapsto \coprod_{\alpha\in\Gamma} hom(C_i, D_\alpha) \Big/ \sim \\ j \xrightarrow{\phi^{op}} i &\longmapsto X(\phi)^* \end{aligned}$$

where $X(\phi)^*[f,\alpha] = [f \circ X(\phi), \alpha]$. It can be easily checked that $X(\phi)^*$ is well defined.

Now, using the construction of limits in $\mathcal{E}ns$, we have that

$$\lim_{i\in J^{op}} \operatorname*{colim}_{\alpha\in\Gamma} hom(C_i, D_\alpha) =$$

$$\Big\{ [f_i, \alpha_i]_{i\in J} \in \prod_{i\in J} \Big(\coprod_{\alpha\in\Gamma} hom(C_i, D_\alpha) \Big/ \sim \Big) \Big|$$

$$\forall i \xrightarrow{\phi} j \in J, [f_j \circ X(\phi), \alpha_j] = [f_i, \alpha_i]\Big\} =$$

$$\Big\{[f_i, \alpha_i]_{i\in J} \in \prod_{i\in J}\Big(\coprod_{\alpha\in\Gamma} hom(C_i, D_\alpha)\Big/\sim\Big)\Big| \forall i \xrightarrow{\phi} j \in J,$$
$$\exists \alpha \in \Gamma \text{ and arrows } \alpha_i \xrightarrow{u} \alpha, \alpha_j \xrightarrow{v} \alpha | Y(v) \circ f_j \circ X(\phi) = Y(u) \circ f_i\Big\}.$$

Then it's clear that the function φ of the statement is the one that sends i to α_i, the f_i are the ones above and the last relation means that the pentagon in the statement commutes. Furthermore, since they are elements of a product, two morphisms $[f_i, \alpha_i]_{i\in J}, [g_i, \beta_i]_{i\in J}$ are equal if and only if $[f_i, \alpha_i] = [g_i, \beta_i]$ $\forall i \in J \Leftrightarrow \forall i \in J$ $\exists \alpha \in \Gamma$ and morphisms $\alpha_i \xrightarrow{u} \alpha$, $\beta_i \xrightarrow{v} \alpha$ such that $Y(u) \circ f_i = Y(v) \circ g_i$, that is, the square in the statement commutes. □

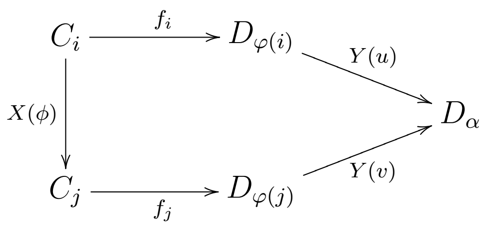
Remark 3.4. Note that it is not required that $J \xrightarrow{\varphi} \Gamma$ is a functor.

Definition 3.5. *Let $\mathcal{C}$ be any category. The category $Ind(\mathcal{C})$ is defined with objects the Ind-objects of $\mathcal{C}$, arrows the morphisms of Ind-objects defined in 3.2, identities $Id_X = (Id, (Id_{C_i})_{i\in J})$ and composition $(\psi, (g_i)_{i\in J}) \circ (\varphi, (f_i)_{i\in J}) = (\psi \circ \varphi, (g_i \circ f_i)_{i\in J})$ following the notation in 3.3. It can be easily checked that it is indeed a category.*

The following is the key fact of the construction of $Ind(\mathcal{C})$.

Corollary 3.6. *[of proposition 3.3] Every morphism of Ind-objects between $X = (C_i)_{i\in J}$ and $Y = (D_\alpha)_{\alpha\in\Gamma}$ induces a morphism between the colimits of the corresponding systems, wherever these colimits exist (this is, whenever we have a functor $\mathcal{C} \xrightarrow{f} \mathcal{E}$ with $\mathcal{E}$ having the colimits $\underset{i\in J}{colim}(f(C_i))$ and $\underset{\alpha\in\Gamma}{colim}(f(D_\alpha))$).*

Proof. To simplify the notation, we can omit the functor f. Let $(\varphi, (f_i)_{i\in J})$ be a morphism between X and Y. Because of the universal property of $\underset{i\in J}{\text{colim}}\, C_i$, we only need to prove that $\forall i \xrightarrow{\phi} j$, $\tilde{\lambda}_{\varphi(i)} \circ f_i = \tilde{\lambda}_{\varphi(j)} \circ f_j \circ X(\phi)$ where $\tilde{\lambda}_{\varphi(i)}$ and $\tilde{\lambda}_{\varphi(j)}$ are the inclusions of $\underset{\alpha\in\Gamma}{\text{colim}}\, D_\alpha$; but because of the definition of $(\varphi, (f_i)_{i\in J})$, we know that $\exists \alpha \in \Gamma$ and arrows $\varphi(i) \xrightarrow{u} \alpha, \varphi(j) \xrightarrow{v} \alpha$ such that the following diagram commutes:



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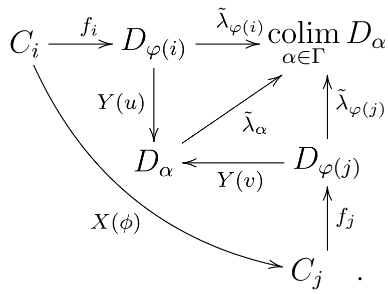
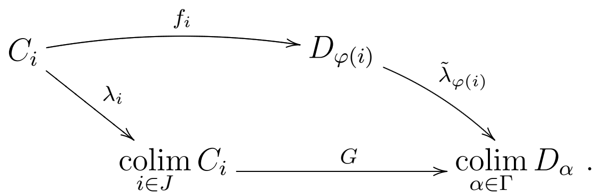
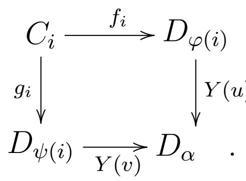
Then, using this and the definition of the $\underset{\alpha\in\Gamma}{\operatorname{colim}} D_\alpha$ we have that the following diagram commutes which is what we wanted to prove:



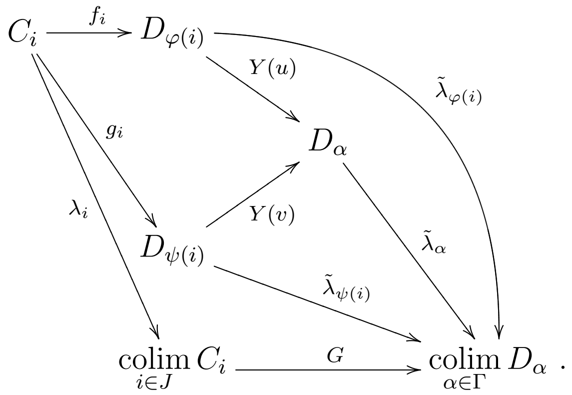
To verify the correct definition, let's suppose that $(\varphi, (f_i)_{i\in J}) \sim (\psi, (g_i)_{i\in J})$ and verify that the induced morphisms are equal. Because of the unicity of the arrow between $\underset{i\in J}{\operatorname{colim}} C_i$ and $\underset{\alpha\in\Gamma}{\operatorname{colim}} D_\alpha$ given by the first part of the proof, we only need to check that the morphism G induced by $(\psi, (g_i)_{i\in J})$ makes the following diagram commutative:



But $(\varphi, (f_i)_{i\in J}) \sim (\psi, (g_i)_{i\in J})$, so for all $i \in J$, $\exists \alpha \in \Gamma$ and arrows $\varphi(i) \xrightarrow{u} \alpha$, $\psi(i) \xrightarrow{v} \alpha$ such that the following diagram commutes:



Then the next diagram commutes which is what we wanted to prove:



□

Remark 3.7. This application is the value on arrows of the functor $\overline{f}$ to be defined in proposition 3.19.

Remark 3.8. In the particular case $\mathcal{E} = \mathcal{E}ns^{\mathcal{C}^{op}}$, we have a functor $Ind(\mathcal{C}) \xrightarrow{F} \mathcal{E}ns^{\mathcal{C}^{op}}$ which we will use in the next subsection.

3.2 $Ind(\mathcal{C})$ has all small filtered colimits

The fact that $Ind(\mathcal{C})$ has all small filtered colimits is taken for granted in the literature. In [2] Grothendieck gives the hint for a proof, which we follow.

Remark 3.9. Any object C of $\mathcal{C}$ can be considered as an Ind-object with the index category $\{*\}$. By abuse, we will write C to refer to this Ind-object.

□

Definition 3.10. *We define the functor $h : \mathcal{C} \to Ind(\mathcal{C})$ by the formulas:*

$$h(C) = C$$

$$h(C \xrightarrow{f} D) = (Id_{\{*\}}, f).$$

It can be checked that it is indeed a functor.

The following proposition is easy to prove:

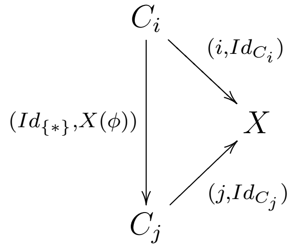
Proposition 3.11. *The functor h is full and faithful and injective on objects. Then, we can identify $\mathcal{C}$ with a full subcategory of $Ind(\mathcal{C})$.*

□

Proposition 3.12. *Given any Ind-object $X = (C_i)_{i \in J}$, the following formula holds in $Ind(\mathcal{C})$:*

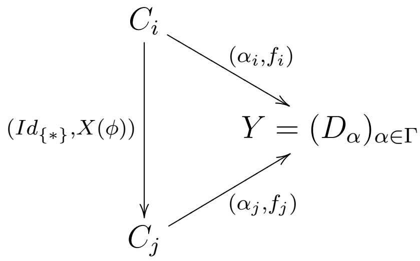
$$X = \underset{i \in J}{colim}\, C_i.$$

Proof. Let's begin by proving that X is a cone: Let $i \xrightarrow{\phi} j$ be a morphism of J, we want to see that the following diagram commutes in $hom(C_i, X)$



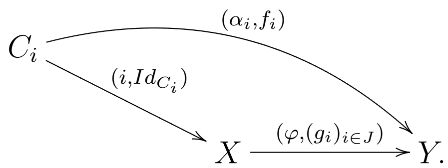
where i and j represent the functors from $\{*\}$ to J which send $*$ to i and j respectively. More specifically, we want to see that $(j \circ Id_{\{*\}}, Id_{C_j} \circ X(\phi)) \sim (i, Id_{C_i})$ i.e. $\exists k,\ j \xrightarrow{\phi_1} k$ and $i \xrightarrow{\phi_2} k \in J/\ X(\phi_1) \circ X(\phi) = X(\phi_2)$. Let's take $j = k, \phi_1 = Id_j, \phi_2 = \phi$, then the previous equality is clearly satisfied.

Now, to conclude the proof, let's see that it is universal: We want to see that if



(3.1)

is another cone, then $\exists!(\varphi, (g_i)_{i \in J}) : X \to Y$ such that the following diagram commutes:



(3.2)

Let's take $\varphi(i) = \alpha_i$ and $g_i = f_i\ \forall i \in J$. As (3.1) commutes, one has that $(\alpha_i, f_i) \sim (\alpha_i \circ Id_{\{*\}}, f_j \circ *)$ then $(\varphi, (g_i)_{i \in J})$ is a morphism of Ind-objects. The commutativity of (3.2) is clearly satisfied and the unicity can be easily checked. □

Definition 3.13. *Let $\mathcal{C}$ be a category and $C \in Ob(\mathcal{C})$. We define the contravariant functor which is representable by C (implicit in definition 3.2* *and proposition 3.3) as*

$$\mathcal{C}^{op} \xrightarrow{hom(-,C)} \mathcal{E}ns$$

$$D \longmapsto hom(D,C)$$

$$D \xrightarrow{f} E \longmapsto f^* \qquad .$$

Proposition 3.14. *Let X and Y be two Ind- objects of $\mathcal{C}$. Let's denote $F = \underset{i \in J}{colim}\, hom(-, C_i)$ and $G = \underset{\alpha \in \Gamma}{colim}\, hom(-, D_\alpha)$. Then there is a bijection between $hom(F, G)$ in $\mathcal{E}ns^{\mathcal{C}^{op}}$ and $hom(X, Y)$ in $Ind(\mathcal{C})$.*

Proof. We construct the bijection composing the bijections below (each horizontal arrow represents a bijection):

$$(\theta_i)_{i\in J} \in \lim_{i \in J^{op}} G(C_i)$$

limits in $\mathcal{E}ns$ ————————————————

$$(\theta_i)_{i\in J} \in \prod_{i\in J} G(C_i) \mid$$

$$\forall i \xrightarrow{\phi} j \in J,\ \underset{\alpha\in\Gamma}{\operatorname{colim}}\, hom(-, D_\alpha)(X(\phi))(\theta_j) = \theta_i$$

nat. of the Yoneda functor ————————————————

$$hom(-, C_i) \overset{\tilde{\theta}_i}{\Rightarrow} G \mid \forall i \xrightarrow{\phi} j \in J, \tilde{\theta}_i = \tilde{\theta}_j \circ X(\phi)_*$$

u.p. of $\underset{i\in J}{\operatorname{colim}}\, hom(-, C_i)$ ————————————————

$$F \overset{\tilde{\theta}}{\Rightarrow} G$$

□

Remark 3.15. In the previous bijection, the way down is exactly the application constructed in corollary 3.6.

□

Proposition 3.16. *$Ind(\mathcal{C})$ and $(\mathcal{E}ns^{\mathcal{C}^{op}})_f$ are equivalent categories, where $(\mathcal{E}ns^{\mathcal{C}^{op}})_f$ is the full subcategory of $\mathcal{E}ns^{\mathcal{C}^{op}}$ consisting of those functors which are filtered colimit of representable ones.*

Proof. Let $F : Ind(\mathcal{C}) \to (\mathcal{E}ns^{\mathcal{C}^{op}})_f$ be the functor of remark 3.8. More explicitly, F sends an Ind-object $X = (C_i)_{i\in J}$ into $\operatorname*{colim}_{i\in J} hom(-, C_i)$ and a morphism between X and Y to the corresponding natural transformation according to the bijection given in proposition 3.14. Then, it's enough to see that this functor is full and faithful and essentialy surjective on objects [4] p.91 Theorem 1. The full and faithfulness is inmediat from remark 3.14, and this functor is clearly surjective. Then $Ind(\mathcal{C})$ and $(\mathcal{E}ns^{\mathcal{C}^{op}})_f$ are equivalent categories. ☐

Proposition 3.17. *The category $Ind(\mathcal{C})$ has all small filtered colimits.*

Proof. By 3.16, it's enough to check that $(\mathcal{E}ns^{\mathcal{C}^{op}})_f$ has all small filtered colimits. Thus, we want to see that if $F = \operatorname*{colim}_{\alpha\in\Gamma} F_\alpha$ and $F_\alpha = \operatorname*{colim}_{i\in J_\alpha} hom(-, C_i^\alpha)$ $\forall\alpha \in \Gamma$ then F is a filtered colimit of representable functors. By lemma 2.1, it suffices to chech that the diagram Γ_F of F is filtered.

Let (x, C), (y, D) be two objects of Γ_F. Using the construction of filtered colimits in $\mathcal{E}ns$ and the fact that colimits of functors are taken pointwise, $x = [x', \alpha]$ with $x' \in F_\alpha C$ and $y = [y', \beta]$ with $y' \in F_\beta D$. Plus, for being Γ filtered, we have

$$\begin{array}{ccc} \alpha & \searrow^{u} & \\ & & \gamma \quad \text{in } \Gamma, \\ \beta & \nearrow_{v} & \end{array}$$

thus we have $x'' = (Fu)_C(x') \in F_\gamma C$, $y'' = (Fv)_D(y') \in F_\gamma D$ where $(Fu)_C$ and $(Fv)_D$ are notation for the transition morphisms of $\operatorname*{colim}_{\alpha\in\Gamma} F_\alpha(C)$ and $\operatorname*{colim}_{\alpha\in\Gamma} F_\alpha(D)$ and should not be confused with the functor F itself. Now, using again the construction of filtered colimits in $\mathcal{E}ns$, we have that $x'' = [x''', i]$ with $x''' \in hom(C, C_i^\gamma)$ and $y'' = [y''', j]$ with $y''' \in hom(C, C_j^\gamma)$. And for being J_γ filtered, we have

$$\begin{array}{ccc} i & \searrow^{\phi} & \\ & & k \quad \text{in } J_\gamma. \\ j & \nearrow_{\psi} & \end{array}$$

Then we have $x^{iv} = (F_\gamma\phi)_C(x''') \in [C, C_k^\gamma]$ and $y^{iv} = (F_\gamma\psi)_D(y''') \in hom(D, C_k^\gamma)$. Finally, by taking $E = C_k^\gamma$, $f = x^{iv}$, $g = y^{iv}$ and $z = [[id_E, k], \gamma]$, we have that the first axiom of filtered category is satisfied: we are going to check that $F(f)(z) = x$ ($F(g)(z) = y$ can be checked in a similar way). $F(f)(z)$ is defined by the universal property of FE as $F(f)(z) = [F_\gamma f[id_E, k], \gamma]$; and $F_\gamma f[id_E, k]$ is defined by the universal property of $F_\gamma E$ as $F_\gamma f[id_E, k] = [f, k]$. Then, $F(f)(z) = [[f, k], \gamma]$, thus we only have to check that $([f, k], \gamma) \sim (x', \alpha)$ in FC: But we have

$$\alpha \xrightarrow{u} \gamma \xleftarrow{id} \gamma \ ,$$

then we can conclude the proof by checking that $[x''', i] = x'' = (Fu)_C(x') = [f, k]$ in $G_\gamma C$, but this is satisfied because we have

$$i \xrightarrow{\phi} k \xleftarrow{id} k$$

where $(G_\gamma \phi)_C(x''') = x^{iv} = f$.

Now, let $(x, C) \underset{g}{\overset{f}{\rightrightarrows}} (y, D)$ be two morphisms of Γ_F. Since $y \in FD$, we have $y = [y', \alpha]$ with $y' \in F_\alpha D$. And since $Ff(y) = x = Fg(y)$, $[F_\alpha f(y'), \alpha] = [F_\alpha g(y'), \alpha]$ in FC. Then, $\exists$

$$\alpha \xrightarrow{u} \beta \xleftarrow{v} \alpha$$

such that $(F_u)_C(F_\alpha f(y')) = (F_v)_C(F_\alpha g(y'))$ in $F_\beta C$. For being Γ filtered, we have $\alpha \underset{v}{\overset{u}{\rightrightarrows}} \beta \overset{w}{\dashrightarrow} \gamma$ with $wu = wv$. Let's take $y'' = (F_{wu})_D(y')$ in $F_\gamma D = \operatorname{colim}_{i \in J_\gamma} hom(D, C_i^\gamma)$. Then $y'' = [y''', i]$ with $y''' \in hom(D, C_i^\gamma)$. Let's remark that $F_\gamma f(y'') = F_\gamma g(y'')$. In effect,

$$\begin{aligned}
F_\gamma f(y'') &= F_\gamma f((F_{wu})_D(y')) = \\
(F_{wu})_C(F_\alpha f(y')) &= (F_w)_C(F_u)_C(F_\alpha f(y')) = \\
(F_w)_C(F_v)_C(F_\alpha g(y')) &= (F_{wv})_C(F_\alpha g(y')) = \\
F_\gamma g((F_{wv})_D(y')) &= F_\gamma g(y'')
\end{aligned}$$

where the second and the sixth equalities hold because of the naturality of F_{wu} and F_{wv} respectively. Then, $[y''' \circ f, i] = [y''' \circ g, i]$ in $F_\gamma C = \underset{i \in J_\gamma}{\operatorname{colim}} hom(C, C_i^\gamma)$, thus $\exists$

$$\begin{array}{ccc} i & & \\ & \searrow^{\phi} & \\ & & j \\ & \nearrow_{\psi} & \\ i & & \end{array}$$

such that $(F_\gamma \phi)_*(y''' \circ f) = (F_\gamma \psi)_*(y''' \circ g)$. For being J_γ filtered, we have $i \overset{\phi}{\underset{\psi}{\rightrightarrows}} j \overset{\varphi}{\dashrightarrow} k$ with $\varphi\phi = \varphi\psi$. Let's take $y^{iv} = (F_\gamma \varphi\phi)_*(y''') \in hom(D, C_k^\gamma)$. Finally, by taking $E = C_k^\gamma$, $h = y^{iv}$ and $z = [[id_E, k], \gamma]$ we have that the second axiom of filtered category is satisfied: let's check first that $h \circ f = h \circ g$,

$$\begin{aligned} (F_\gamma \varphi\phi)_*(y''') \circ f &= F_\gamma \varphi\phi \circ y''' \circ f = \\ F_\gamma \varphi \circ F_\gamma \phi \circ y''' \circ f &= F_\gamma \varphi \circ F_\gamma \psi \circ y''' \circ g = \\ F_\gamma \varphi\psi \circ y''' \circ g &= h \circ g \end{aligned}$$

To conclude the proof, we have to verify that $Fh(z) = y$. In effect, since $Fh(z) = [F_\gamma h[id_E, k], \gamma] = [[h, k], \gamma]$, we need to check that $[[h, k], \gamma] = [y', \alpha]$ in FD. But we have

$$\begin{array}{ccc} \alpha & & \\ & \searrow^{wu} & \\ & & \gamma \ , \\ & \nearrow_{id} & \\ \gamma & & \end{array}$$

then it's enough to check that $(F_{wu})_D(y') = (F_{id})_D[h, k]$, which is the same as $[y''', i] = [h, k]$ in $F_\gamma D$. Let's consider for this

$$\begin{array}{ccc} i & & \\ & \searrow^{\varphi\phi} & \\ & & k \\ & \nearrow_{id} & \\ k & & \end{array}$$

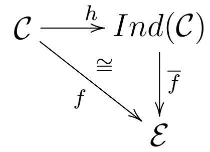
and note that $(F_\gamma \varphi\phi)_*(y''') = y^{iv} = h = (F_\gamma id)_* h$. $\square$

Corollary 3.18. *$Ind(Ind(\mathcal{C})) \cong Ind(\mathcal{C})$ i.e. there is an equivalence of categories between them.*

Proof. We proved in 3.11 that the functor $h : Ind(\mathcal{C}) \to Ind(Ind(\mathcal{C}))$ is full and faithful. In addition, it follows from propositions 3.17 and 3.12 that it is essentially suryective. Then it is an equivalence of categories. $\square$

3.3 Universal property of $h : \mathcal{C} \to Ind(\mathcal{C})$

Proposition 3.19. *The functor $h : \mathcal{C} \to Ind(\mathcal{C})$ defined in 3.10 has the following universal property:*



(3.3)

which means that given any functor $f : \mathcal{C} \to \mathcal{E}$ into a category having colimits of the form $\operatorname{colim}_{i \in J} f(C_i)$ for any small filtered system in $\mathcal{C}$, there is an extension (which preserves filtered colimits) $\overline{f} : Ind(\mathcal{C}) \to \mathcal{E}$.

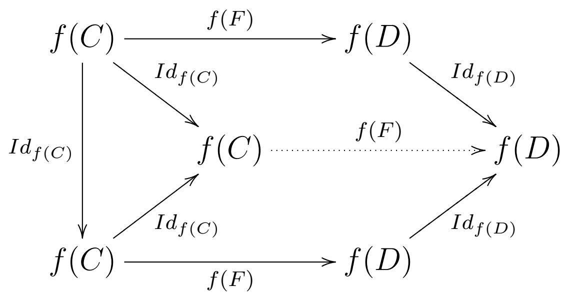
Proof. Let's define $\overline{f}$ on objects as $\overline{f}(X) = \operatorname{colim}_{i \in J} f(C_i)$ if $X = (C_i)_{i \in J}$ and on arrows as the application of proposition 3.6. It can be checked that $\overline{f}$ is indeed a functor.

This $\overline{f}$ makes the diagram (3.3) strictly commutative:

$$\overline{f} \circ h(C) = f(C)$$

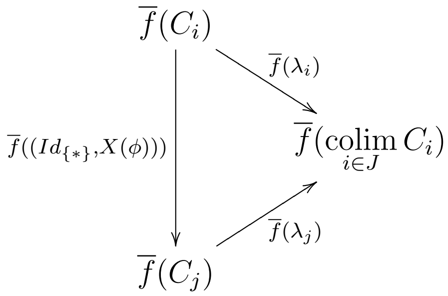
$$\overline{f} \circ h(C \xrightarrow{F} D) = \overline{f}((Id_{\{*\}}, F)) = f(F)$$

because f(F) makes the following diagram commutative:

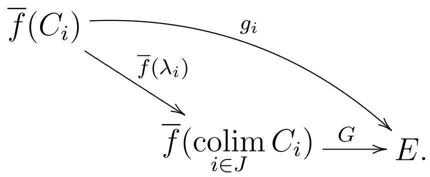


which is the definition of $\overline{f}(Id_{\{*\}}, F)$.

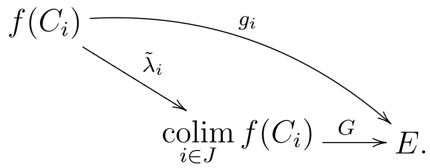
It only remains to prove that $\overline{f}$ preserves filtered colimits: we are going to see first that $\operatorname{colim}_{i \in J} \overline{f}(C_i) = \overline{f}(\operatorname{colim}_{i \in J} C_i)$ i.e. that $\overline{f}(\operatorname{colim}_{i \in J} C_i)$ is a universal cone for $(\overline{f}(C_i))_{i \in J}$. To prove that it is a cone, we can note that the following diagram commutes because $\overline{f}$ is a functor:



where λ_i, λ_j are the inclusions to $\operatorname{colim}_{i\in J} C_i$. Now, let's prove that it is universal: let E be another cone for $(\overline{f}(C_i))_{i\in J}$ with inclusions $(g_i)_{i\in J}$, then we want to see that $\exists! G : \overline{f}(\operatorname{colim}_{i\in J} C_i) \to E$ such that the following diagram commutes:



But because of the definition of $\overline{f}$, this diagram results:



And for this one it's clear that $\exists! G$ wich makes it commutative.

Now, using the first part and the proposition 3.17, we have that

$$\overline{f}(\operatorname{colim}_{\alpha\in\Gamma} X_\alpha) = \overline{f}(\operatorname{colim}_{i\in J} C_i) = \operatorname{colim}_{i\in J} f(C_i) = \operatorname{colim}_{\alpha\in\Gamma} \operatorname{colim}_{i\in J_\alpha} f(C_i^\alpha) = \operatorname{colim}_{\alpha\in\Gamma} \overline{f}(X_\alpha)$$

where the Ind-object $\operatorname{colim}_{i\in J} C_i$ is the filtered colimit of the Ind-objects X_α. $\square$

Proposition 3.20. *Let $\mathcal{E}$ be any category with small filtered colimits. Then composition with the functor $h : \mathcal{C} \to Ind(\mathcal{C})$ induces an equivalence of categories:*

$$h^* : Hom(Ind(\mathcal{C}), \mathcal{E})_+ \xrightarrow{\cong} Hom(\mathcal{C}, \mathcal{E}), \qquad g \mapsto g \circ h.$$

A quasi-inverse for this equivalence is given by the assignment $f \mapsto \overline{f}$ *(Here Hom stands for the category of functors and natural transformations, and "+" stands for the full subcategory of those functors that preserve filtered colimits).*

Proof. To prove that h^* is an equivalence of categories we can check that it is full and faithful and each object $f \in Hom(\mathcal{C}, \mathcal{E})$ is isomorphic to $h^*(g)$ for some object $g \in Hom(Ind(\mathcal{C}), \mathcal{E})_+$:

Full and faithful: Let $\theta : g \circ h \Rightarrow g' \circ h \ \in Hom(\mathcal{C}, \mathcal{E})$ be a natural transformation with $g, g' \in Hom(Ind(\mathcal{C}), \mathcal{E})_+$. We want to see that $\exists!\ \eta : g \Rightarrow g' / h^*(\eta) = \theta$: by definition, $(h^*(\eta))_C = \eta_{h(C)}$, then $\eta_{h(C)}$ has to be equal to θ_C. Now, if X is any Ind-object of $\mathcal{C}$, $X = \operatorname*{colim}_{i \in J} C_i$ and $\eta_X : \operatorname*{colim}_{i \in J} g \circ h(C_i) \to \operatorname*{colim}_{i \in J} g' \circ h(C_i)$ because g and g' preserve filtered colimits. And to be η natural, it has to make the following diagram commutative:

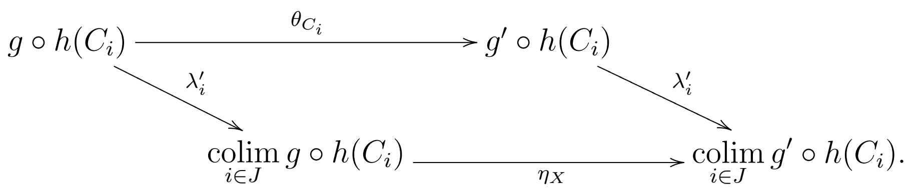
$$\begin{array}{ccc} \operatorname*{colim}_{i \in J} g \circ h(C_i) & \xrightarrow{\eta_X} & \operatorname*{colim}_{i \in J} g' \circ h(C_i) \\ \Big\downarrow {\scriptstyle g(\varphi, (f_i)_{i \in J})} & & \Big\downarrow {\scriptstyle g'(\varphi, (f_i)_{i \in J})} \\ \operatorname*{colim}_{\alpha \in \Gamma} g \circ h(D_\alpha) & \xrightarrow[\eta_Y]{} & \operatorname*{colim}_{\alpha \in \Gamma} g' \circ h(D_\alpha). \end{array} \tag{3.4}$$

But we have:

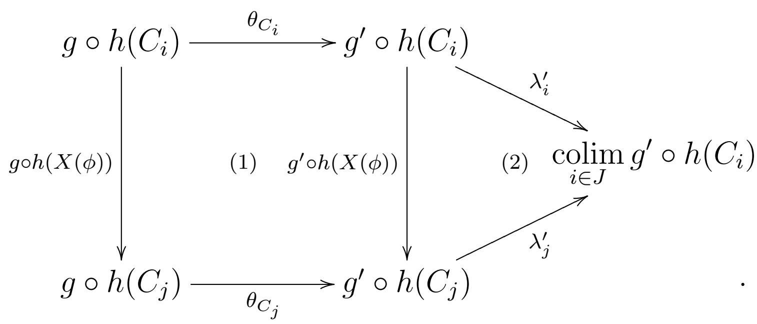
$$\begin{array}{ccccccc} \operatorname*{colim}_{i \in J} g \circ h(C_i) & & & \xrightarrow{\eta_X} & & & \operatorname*{colim}_{i \in J} g' \circ h(C_i) \\ & \nwarrow^{\lambda_i} & & (4) & & \nearrow^{\lambda'_i} & \\ \Big\downarrow {\scriptstyle g(\varphi, (f_i)_{i \in J})} & (2) & g \circ h(C_i) & \xrightarrow{\theta_{C_i}} & g' \circ h(C_i) & (3) & \Big\downarrow {\scriptstyle g'(\varphi, (f_i)_{i \in J})} \\ & & \Big\downarrow {\scriptstyle g \circ h(f_i)} & (1) & \Big\downarrow {\scriptstyle g' \circ h(f_i)} & & \\ & & g \circ h(D_{\varphi(i)}) & \xrightarrow[\theta_{D_{\varphi(i)}}]{} & g' \circ h(D_{\varphi(i)}) & & \\ & \swarrow_{\lambda_{\varphi(i)}} & & (5) & & \searrow_{\lambda'_{\varphi(i)}} & \\ \operatorname*{colim}_{\alpha \in \Gamma} g \circ h(D_\alpha) & & & \xrightarrow[\eta_Y]{} & & & \operatorname*{colim}_{\alpha \in \Gamma} g' \circ h(D_\alpha) \end{array}$$

where (1) commutes because of the naturality of θ and both (2) and (3) commute because of the construction of the induced morphism between the colimits made in remark 3.6 and the fact that g and g' are functors which preserve filtered colimits.

Then (3.4) commutes if and only if (4) and (5) both do, if and only if the following diagram commutes for every $X \in Ind(\mathcal{C})$



Then, because of the universal property of $\operatorname{colim}_{i \in J} g \circ h(C_i)$, to prove that η is unique, we only have to check that the following diagram commutes for all $i \xrightarrow{\phi} j$ in J



But (2) commutes because of the definition of $\operatorname{colim}_{i \in J} g' \circ h(C_i)$ and (1) commutes because of the naturality of θ.

Then, if there is an η such that $h^*(\eta) = \theta$, this η is unique; and defined as we said before, η results a natural transformation and $h^*(\eta) = \theta$ which is what we wanted to see.

Now, let's see that if $f \in Hom(\mathcal{C}, \mathcal{E}), \exists g \in Hom(Ind(\mathcal{C}), \mathcal{E})_+ / \ h^*(g) \cong f$: We only need to take $g = \overline{f}$ as in the proof of proposition 3.19.

□

Corollary 3.21. *While the extension* $\overline{f}$ *given in the diagram* (3.3) *is not unique, it is characterized by the equation* $\overline{f} \circ h \cong f$ *up to a unique natural isomorphism.*

Proof. It easily follows from the proof of the fact that h^* is full and faithful.

□

As usual, from proposition 3.20 it follows:

Corollary 3.22. *The category* $Ind(\mathcal{C})$ *is characterized by proposition 3.20 up to an equivalence of categories (and not up to an isomorphism).*

□

Corollary 3.23. *Given any functor $F : \mathcal{C} \to \mathcal{D}$, the formula $Ind(F)((C_i)_{i\in J}) = (F(C_i))_{i\in J}$ defines a filtered colimit preserving functor $Ind(F) : Ind(\mathcal{C}) \to Ind((D))$ commuting with the canonical functors h, i.e. $h \circ F = Ind(F) \circ h$. We have $Ind(Id_{\mathcal{C}}) = Id_{Ind(\mathcal{C})}$ and for any composition $F \circ G$, we have $Ind(F \circ G) = Ind(F) \circ Ind(G)$.*

Proof. Let's denote $f = h \circ F$. We see that $\overline{f} = Ind(F)$ ($Ind(F)$ is the canonical choice of $\overline{f}$ of proposition 3.19):

$$\overline{f}((C_i)_{i\in J}) = \operatorname*{colim}_{i\in J} f(C_i) = \operatorname*{colim}_{i\in J} h \circ F(C_i) = (F(C_i))_{i\in J}.$$

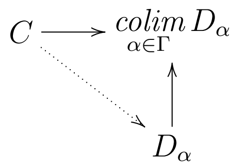
Then $Ind(F)$ is a functor and preserves filtered colimits. It is immediate that $h \circ F = Ind(F) \circ h$ and that the assignment is functorial in F. □

The following proposition generalizes the case $\mathcal{E} = \mathcal{E}ns^{\mathcal{C}^{op}}$ considered in proposition 3.16.

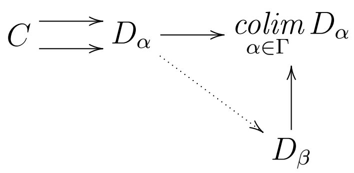
Proposition 3.24. *Let $\mathcal{C} \xrightarrow{f} \mathcal{E}$ be a full and faithful functor, where $\mathcal{E}$ is a category with colimits of the form $\operatorname*{colim}_{i\in J} f(C_i)$ for any small filtered system in $\mathcal{C}$ (to simplify notation, we omit to indicate f as if $\mathcal{C}$ were a full subcategory of $\mathcal{E}$). Consider the functor $Ind(\mathcal{C}) \xrightarrow{\overline{f}} \mathcal{E}$. Then:*

1. *$\overline{f}$ is full and faithful if $\forall C \in \mathcal{C}$ and $Y = (D_\alpha)_{\alpha\in\Gamma} \in Ind(\mathcal{C})$, the following two conditions are satisfied:*

   - *$\forall C \to \operatorname*{colim}_{\alpha\in\Gamma} D_\alpha$ in $\mathcal{E}$, there is a factorization:*



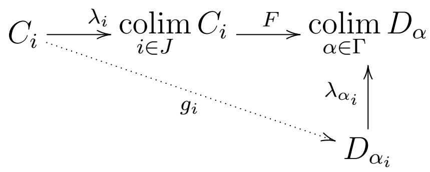
   - *$\forall\, C \rightrightarrows D_\alpha$ that become equal in the colimit, there is β and $\alpha \to \beta$ in Γ such that the two arrows become already equal in D_β:*



2. $\overline{f}$ *is an equivalence of categories if in addition every object in* $\mathcal{E}$ *is a small filtered colimit of objects* $D \in \mathcal{C}$

Under these conditions, $\mathcal{E}$ *has all small filtered colimits.*

Proof. 1. We want to see that $\forall \overline{f}(X) \xrightarrow{F} \overline{f}(Y)$ in $\mathcal{E}$, $\exists\ !\ X \xrightarrow{G} Y$ such that $\overline{f}(G) = F$: If $X = (C_i)_{i \in J}$ and $Y = (D_\alpha)_{\alpha \in \Gamma}$, we have $\forall i \in J$, $C_i \xrightarrow{\lambda_i} \operatorname{colim}_{i \in J} C_i \xrightarrow{F} \operatorname{colim}_{\alpha \in \Gamma} D_\alpha$, thus using the first condition, we have



and using that f is full and faithful, we know that $\exists!\ C_i \xrightarrow{f_i} D_{\alpha_i}$ such that $f(f_i) = g_i$. It is straightforward to check that $G = (\varphi, (f_i)_{i \in J})$ with $\varphi(i) = \alpha_i$ is the morphism that we are looking for.

2. Given $E \in \mathcal{E}$, we know that $E = \operatorname{colim}_{\alpha \in \Gamma} f(D_\alpha)$, then by taking $X = \operatorname{colim}_{\alpha \in \Gamma} D_\alpha$ we have that $E \cong \overline{f}(X)$. This proves that $\overline{f}$ is surjective on objects, but any full and faithful functor that is surjective on objects is an equivalence of categories.

□

Remark 3.25. This proposition is useful to recognize when a given category $\mathcal{E}$ is actually equivalent to $Ind(\mathcal{C})$.

4 2-Categories

In this section we give the definitions and set the notation of 2-categories that we will use in later sections.

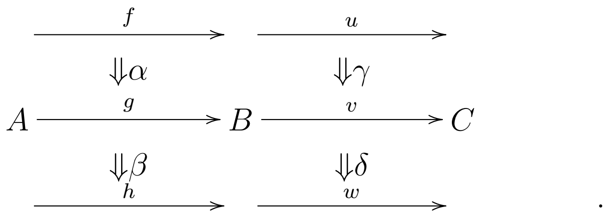
4.1 Basic definitions

Definition 4.1. *A 2-category $\mathcal{A}$ has objects A (0-cells). For every pair of objects A, B there is a category $Hom(A, B)$ whose objects are $f : A \to B$ (1-cells) and whose morphisms are $\alpha : f \Rightarrow g$ (2-cells). The composition in $Hom(A, B)$ is known as vertical composition. For each object A it has $Id_A : A \to A$ and $Id_{Id_A} : Id_A \Rightarrow Id_A$. For any objects A, B, C, there is a functor $* : Hom(B, C) \times Hom(A, B) \to Hom(A, C)$ known as horizontal composition. This composition is associative and has an identity Id_{Id_A}. We are going to denote the vertical composition as $\beta \circ \alpha$ and the horizontal composition as $\gamma\alpha$.*

Remark 4.2. For every 2-category $\mathcal{C}$, there is an underlying category with objects the 0-cells of $\mathcal{C}$, morphisms the 1-cells of $\mathcal{C}$, identities id_A and composition given by the functor $*$ evaluated on objects.

□

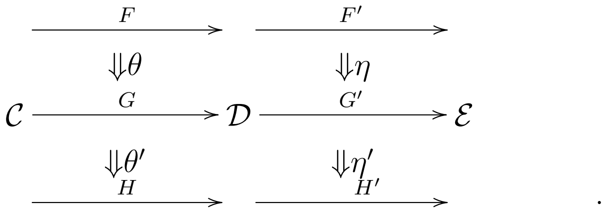
Remark 4.3. Given a configuration as follows,



using the functoriality of $*$, it's clear that $(\beta \circ \alpha)(\delta \circ \gamma) = (\delta\beta) \circ (\gamma\alpha)$

□

Remark 4.4. $\mathcal{C}at$ is a 2-category. Its objects are the categories. Given two categories $\mathcal{C}, \mathcal{D}$, the category $Hom(\mathcal{C}, \mathcal{D})$ has objects the functors between $\mathcal{C}$ and $\mathcal{D}$, and morphisms the natural transformations between those functors. The compositions are given by: if we have the following configuration



we define $(\theta' \circ \theta)_C = \theta'_C \circ \theta_C$ and $(\eta\theta)_C = G'(\theta_C) \circ \eta_{FC}$.

$\square$

Definition 4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ 2-categories. A 2-functor between $\mathcal{C}$ and $\mathcal{D}$ is a function $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$ and for each pair of objects A and B of $\mathcal{C}$, a functor $F_{AB} : Hom(A, B) \to Hom(FA, FB)$ preserving the horizontal composition, more specifically: $F_{AC}(\alpha\beta) = F_{BC}(\alpha)F_{AB}(\beta)$.*

$$
\begin{array}{ccccc}
 & \xrightarrow{f} & & \xrightarrow{u} & \\
 & \Downarrow\alpha & & \Downarrow\gamma & \\
A & \xrightarrow{g} & B & \xrightarrow{v} & C \\
 & \Downarrow\beta & & \Downarrow\delta & \\
 & \xrightarrow{h} & & \xrightarrow{w} &
\end{array}
\quad\longmapsto\quad
\begin{array}{ccccc}
 & \xrightarrow{Ff} & & \xrightarrow{Fu} & \\
 & \Downarrow F\alpha & & \Downarrow F\gamma & \\
FA & \xrightarrow{Fg} & FB & \xrightarrow{Fv} & FC \\
 & \Downarrow F\beta & & \Downarrow F\delta & \\
 & \xrightarrow{Fh} & & \xrightarrow{Fw} &
\end{array}
.
$$

Remark 4.6. The concepts of 2-category and 2-functor are those of $\mathcal{V}$-category and $\mathcal{V}$-functor in the case $\mathcal{V} = \mathcal{C}at$ (see [6] and [7]).

$\square$

Definition 4.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and F and G two 2-functors between them. A pseudonatural transformation $\theta : F \Rightarrow G$ is a family of 1-cells of $\mathcal{D}$ $(\theta_C : FC \to GC)_{c\in\mathcal{C}}$ and a family of invertible 2-cells of $\mathcal{D}$ $(\theta_f : \theta_D \circ F_{CD}(f) \to G_{CD}(f) \circ \theta_C)_{C\xrightarrow{f}D\in\mathcal{C}}$ satisfying the following conditions:*

PNT 0. $\theta_{id_C} = id\theta_C$.

PNT 1.

$$
\begin{array}{ccc}
FC & \xrightarrow{\theta_C} & GC \\
{\scriptstyle F_{CD}(f)}\downarrow & \Uparrow\theta_f & \downarrow{\scriptstyle G_{CD}(f)} \\
FD & \xrightarrow[\theta_D]{} & GD \\
{\scriptstyle F_{DE}(g)}\downarrow & \Uparrow\theta_g & \downarrow{\scriptstyle G_{DE}(g)} \\
FE & \xrightarrow[\theta_E]{} & GE
\end{array}
\quad = \quad
\begin{array}{ccc}
FC & \xrightarrow{\theta_C} & GC \\
{\scriptstyle F_{CD}(gf)}\downarrow & \Uparrow\theta_{gf} & \downarrow{\scriptstyle G_{CD}(gf)} \\
FE & \xrightarrow[\theta_E]{} & GE
\end{array}
$$

PNT2.

$$
\begin{array}{ccccc}
FC & & \xrightarrow{\theta_C} & & GC \\
{\scriptstyle F_{CD}(f)}\downarrow & \overset{F_{CD}(\gamma)}{\Rightarrow} & \downarrow{\scriptstyle F_{CD}(g)} & \Uparrow\theta_f & \downarrow{\scriptstyle G_{CD}(g)} \\
FD & & \xrightarrow[\theta_D]{} & & GD
\end{array}
\quad = \quad
\begin{array}{ccccc}
FC & & \xrightarrow{\theta_C} & & GC \\
{\scriptstyle F_{CD}(f)}\downarrow & \Uparrow\theta_f G_{CD}(f) & \downarrow & \overset{G_{CD}(\gamma)}{\Rightarrow} & \downarrow{\scriptstyle G_{CD}(g)} \\
FD & & \xrightarrow[\theta_D]{} & & GD
\end{array}
.
$$

Where the compositions in PNT1 and PNT2 are computed in $Hom(FC, GD)$.

4.2 $\mathcal{C}at^{\mathcal{C}^{op}}$ as a 2-category

The purpose of this subsection is to prove that $\mathcal{C}at^{\mathcal{C}^{op}}$ is a 2-category. Its 0-cells are the functors $F : \mathcal{C}^{op} \to \mathcal{C}at$, and the category $Hom(F, G)$ is the end $\int_C Hom(FC, GC)$ which is taken by considering the bifunctor $Hom(F-, G-) : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{C}at$.

Remark 4.8. This definition has as motivation the fact that for two functors $F, G : \mathcal{C} \to \mathcal{E}ns$, the end $\int_C hom(FC, GC)$ is the set of natural transformations between F and G.

Given a bifunctor $B : \mathcal{C}\times\mathcal{C}^{op} \to \mathcal{C}at$, one can construct the end $\int_C B(C, C)$ as the category with objects $(x_C)_{C\in\mathcal{C}}$ where $x_C \in B(C, C)$ verify that for every $C \xrightarrow{f} C'$ in $\mathcal{C}$, $B(id_C, f)(x_C) = B(f, id_{C'})(x_{C'})$; and arrows $(x_C \xrightarrow{\varphi_C} y_C)_{C\in\mathcal{C}}$ that verify that for every $C \xrightarrow{f} C'$ in $\mathcal{C}$, $B(id_C, f)(\varphi_C) = B(f, id_{C'})(\varphi_{C'})$.

In this particular case where $B = Hom(F-, G-)$, this construction implies that the objects of $Hom(F, G)$ are the natural transformations between F and G, and an arrow between two natural transformations θ and θ' is a family of natural transformations $\theta_C \overset{\psi_C}{\Rrightarrow} \theta'_C$ such that for every $C \xrightarrow{f} C'$ in $\mathcal{C}$, and $a \in FC'$

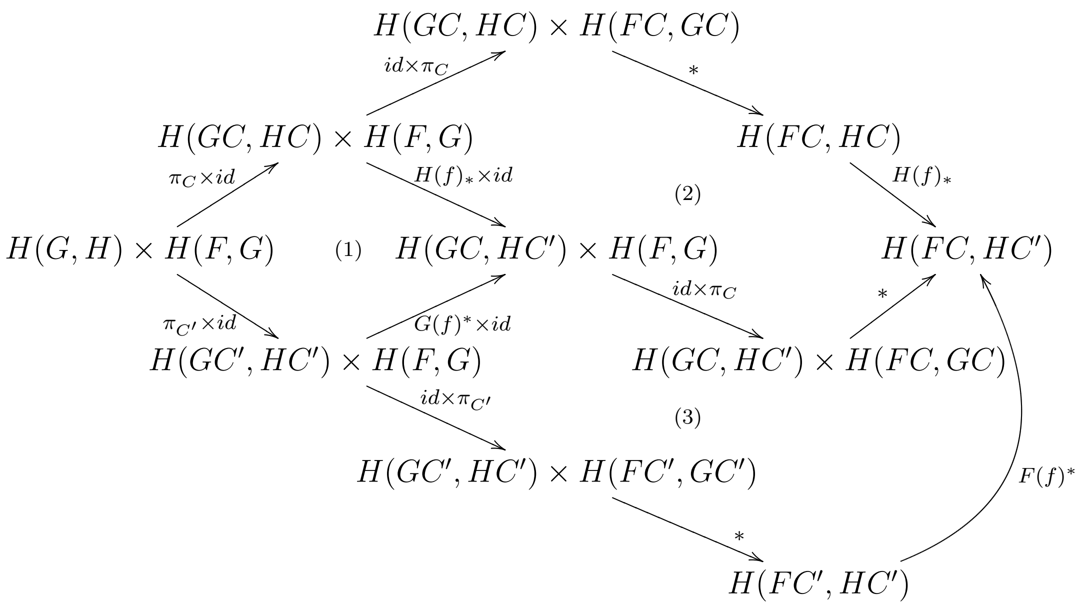
$$FC' \;\underset{\theta'_{C'}}{\overset{\theta_{C'}}{\rightrightarrows}}\; \Downarrow\psi_{C'} \; GC' \;\underset{Gf}{\overset{Gf}{\rightrightarrows}}\; \Downarrow id_{Gf} \; GC \quad = \quad FC' \;\underset{Ff}{\overset{Ff}{\rightrightarrows}}\; \Downarrow id_{Ff} \; FC \;\underset{\theta'_C}{\overset{\theta_C}{\rightrightarrows}}\; \Downarrow\psi_C \; GC$$

$i.e. Gf((\psi_{C'})_a) = (\psi_C)_{Ff(a)}$

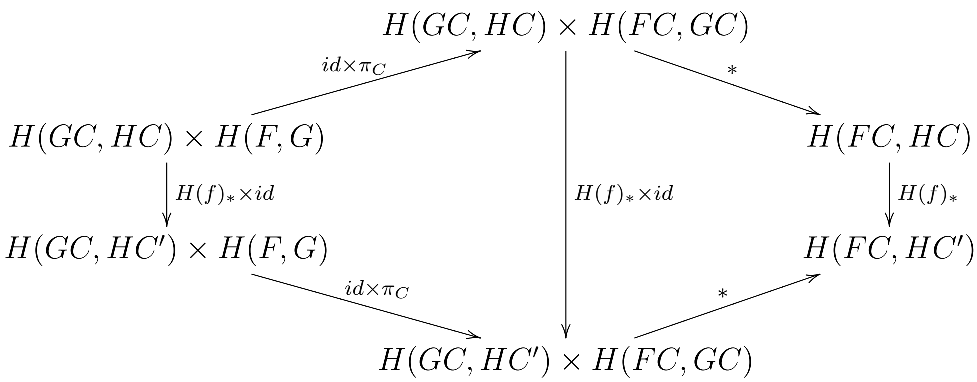
It only remains to construct the "composition" functor $Hom(G, H) \times Hom(F, G) \to Hom(F, H)$. This functor is a particular case of the composition that exists in $\mathcal{V}^{\mathcal{C}}$, since the functors from $\mathcal{C}$ to $\mathcal{V}$ form a $\mathcal{V}$-category when $\mathcal{C}$ is a $\mathcal{V}$-category. However, since in this work we deal only with the particular case of $\mathcal{C}at$-categories, we will explicitly construct the composition in this context. The functor is obtained by the universal property of the end $Hom(F, H)$: consider the following diagram (we replace Hom by H to make the diagrams smaller),

$$\begin{array}{ccccc}
 & H(GC,HC)\times H(FC,GC) & \xrightarrow{\;*\;} & H(FC,HC) & \\
 \nearrow^{\pi_C\times\pi_C} & & \nearrow^{\pi_C} & & \searrow^{H(F-,H-)(id_C,f)} \\
H(G,H)\times H(F,G) & \overset{\exists!}{\dashrightarrow} & H(F,H) & & H(FC,HC') \\
 \searrow_{\pi_{C'}\times\pi_{C'}} & & \searrow_{\pi_{C'}} & & \nearrow_{H(F-,H-)(f,id_{C'})} \\
 & H(GC',HC')\times H(FC',GC') & \xrightarrow{\;*\;} & H(FC',HC') &
\end{array}$$

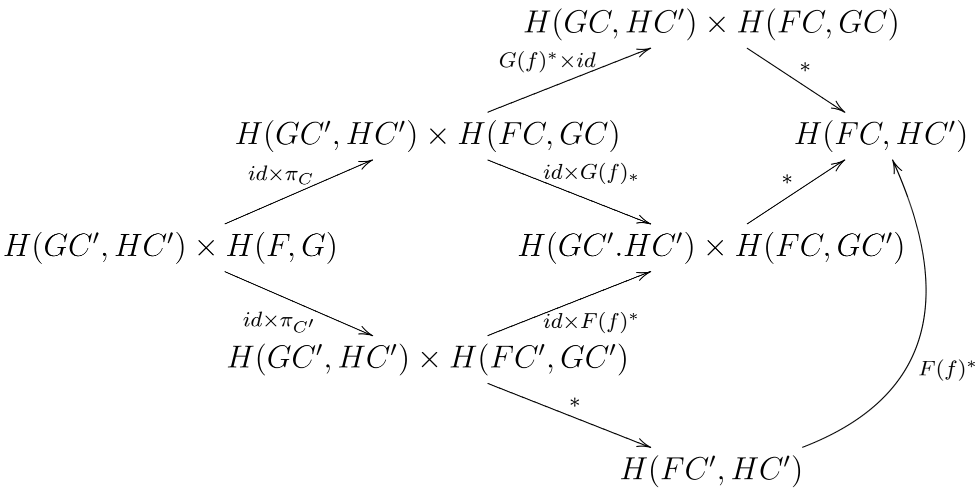
of which we have to prove that the exterior hexagon commmutes. To see that, we complete the hexagon to



Now (1) commutes because of the definition of the end $Hom(G, H)$, and we will see that (2) and (3) also commute. For (2), consider



where the left diagram commutes trivially and the commutativity of the right one is left to the reader. For (3), consider



where the commutativity of the small diagrams is also left to the reader.

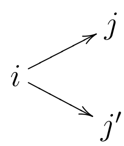
This concludes the proof of the fact that $\mathcal{C}at^{\mathcal{C}^{op}}$ is a 2-category and also gives a description of the horizontal composition there.

4.3 Notions of 2-filteredness

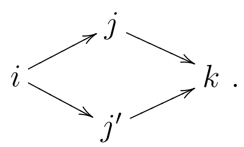
Let's start by recalling ([2] and [4]) the definition of pseudo-filteredness for a category.

Definition 4.9. *A category I is pseudo-filtered if it satisfies the following two axioms:*

PS1 Every diagram of the form



can be completed to a commutative one of the form



PS2 Every diagram of the form $i \overset{u}{\underset{v}{\rightrightarrows}} j$ *can be inserted on one of the form* $i \overset{u}{\underset{v}{\rightrightarrows}} j \xrightarrow{w} k$ *where* $wu{=}wv$.

For the next three definitions, we take as reference [3].

Definition 4.10. *Let $\mathcal{A}$ be a 2-category. $\mathcal{A}$ is pre-2-filtered if it satisfies the following axioms:*

F1.

Given $\begin{array}{ccc} & & A \\ & {}^{f}\nearrow & \\ E & & \\ & {}_{g}\searrow & \\ & & B \end{array}$ *there exists an invertible 2-cell* $\begin{array}{ccccc} & {}^{f}\nearrow & A & \searrow^{u} & \\ E & & \gamma \Downarrow & & C \\ & {}_{g}\searrow & B & \nearrow_{v} & \end{array}$.

F2.

Given any 2-cells $\begin{array}{ccccc} & {}^{f}\nearrow & A & \searrow^{u_1} & \\ E & & \gamma_1 \Downarrow & & C_1 \\ & {}_{g}\searrow & B & \nearrow_{v_1} & \end{array}$, $\begin{array}{ccccc} & {}^{f}\nearrow & A & \searrow^{u_2} & \\ E & & \gamma_2 \Downarrow & & C_2 \\ & {}_{g}\searrow & B & \nearrow_{v_2} & \end{array}$

there exists $\begin{array}{ccc} C_1 & & \\ & \searrow^{w_1} & \\ & & C \\ & \nearrow_{w_2} & \\ C_2 & & \end{array}$ *with invertible 2-cells α, β such that*

$$\begin{array}{ccccccc} & {}^{f}\nearrow & A & \searrow^{u_1} & & & \\ E & & \gamma_1 \Downarrow & & C_1 & \searrow^{w_1} & \\ & {}_{g}\searrow & B & \nearrow_{v_1} & \alpha \Downarrow & & C \\ & & & {}_{v_2}\searrow & C_2 & \nearrow_{w_2} & \end{array} \quad = \quad \begin{array}{ccccccc} & & & {}^{u_1}\nearrow & C_1 & \searrow^{w_1} & \\ & {}^{f}\nearrow & A & & u_2\ \beta \Downarrow & & C \\ E & & \gamma_2 \Downarrow & \searrow & C_2 & \nearrow_{w_2} & \\ & {}_{g}\searrow & B & \nearrow_{v_2} & & & \end{array} .$$

Where the compositions in F2 are computed in $Hom(E, C)$.

Remark 4.11. When $\mathcal{A}$ is a trivial 2-category (the only 2-cells are the identities), axiom F2 is vacuous and F1 corresponds to axiom PS1 in the definition of pseudo-filtered category, while axiom PS2 may not hold. Thus, a category which is pre-2-filtered as a trivial 2-category may not be pseudo-filtered.

□

Definition 4.12. *Let $\mathcal{A}$ be a 2-category. $\mathcal{A}$ is pseudo-2-filtered if it is pre-2-filtered and satisfies the stronger form of axiom F1:*

FF1.

Given $\begin{array}{ccc} & {}^{f_1}\nearrow & A \\ E_1 & & \\ & {}_{g_1}\searrow & B \end{array}$, $\begin{array}{ccc} & {}^{f_2}\nearrow & A \\ E_2 & & \\ & {}_{g_2}\searrow & B \end{array}$ *there exist* $\begin{array}{ccccc} & {}^{f_1}\nearrow & A & \searrow^{u} & \\ E_1 & & \gamma_1 \Downarrow & & C \\ & {}_{g_1}\searrow & B & \nearrow_{v} & \end{array}$, $\begin{array}{ccccc} & {}^{f_2}\nearrow & A & \searrow^{u} & \\ E_2 & & \gamma_2 \Downarrow & & C \\ & {}_{g_2}\searrow & B & \nearrow_{v} & \end{array}$,

with γ_1 and γ_2 invertible 2-cells (the same u and v for both pairs (f_1, g_1) and (f_2, g_2)).

Definition 4.13. *Let $\mathcal{A}$ be a 2-category. $\mathcal{A}$ is 2-filtered when it is pseudo-2-filtered, non empty and satisfies in addition the following axiom:*

F0.

Given two 0-cells A, B in $\mathcal{A}$, there exists

$$\begin{array}{ccc} A & & \\ & \searrow^{u} & \\ & & C \\ & \nearrow_{v} & \\ B & & \end{array} .$$

Remark 4.14. When $\mathcal{A}$ is a trivial 2-category, axiom F0 is the first of the usual axioms in the definition of filtered category, while axiom FF1 is equivalent to the conjunction of the two axioms PS1 and PS2 in the definition of pseudo-filtered category.

Proof. It is clear that F0 is the first of the usual axioms of filtered category. Now, let's see that FF1 is equivalent to the conjunction of PS1 and PS2: Suppose that $\mathcal{A}$ satisfies FF1, then PS1 is clearly satisfied. And if we have $i \overset{u}{\underset{v}{\rightrightarrows}} j$, we can apply FF1 to

$$\begin{array}{ccc} & & j \\ & \nearrow^{u} & \\ id & & \\ & \searrow_{i} & \\ & & i \end{array} \quad \text{and} \quad \begin{array}{ccc} & & j \\ & \nearrow^{v} & \\ id & & \\ & \searrow_{i} & \\ & & i \end{array}$$

thus we have

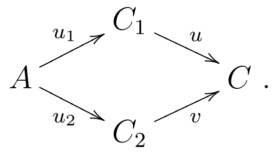
$$\begin{array}{ccccc} & & j & & \\ & \nearrow^{u} & & \searrow^{u'} & \\ id & & & & k \\ & \searrow_{i} & & \nearrow_{v'} & \\ & & i & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & j & & \\ & \nearrow^{v} & & \searrow^{u'} & \\ id & & & & k \\ & \searrow_{i} & & \nearrow_{v'} & \\ & & i & & \end{array} .$$

Then, we have $i \overset{u}{\underset{v}{\rightrightarrows}} j \xrightarrow{u'} k$ and $u'u = v'id = u'v$ which proves that PS2 is also satisfied. Now, suppose that $\mathcal{A}$ satisfies PS1 and PS2. Then, given

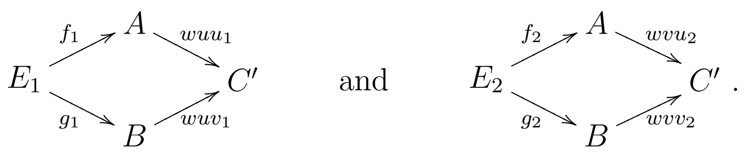
$$\begin{array}{ccc} & & A \\ & \nearrow^{f_1} & \\ E_1 & & \\ & \searrow_{g_1} & \\ & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} & & A \\ & \nearrow^{f_2} & \\ E_2 & & \\ & \searrow_{g_2} & \\ & & B \end{array}$$

by PS1, we have the following commutative diagrams

$$\begin{array}{ccccc} & & A & & \\ & \nearrow^{f_1} & & \searrow^{u_1} & \\ E_1 & & & & C_1 \\ & \searrow_{g_1} & & \nearrow_{v_1} & \\ & & B & & \end{array} \qquad \begin{array}{ccccc} & & A & & \\ & \nearrow^{f_2} & & \searrow^{u_2} & \\ E_2 & & & & C_2 \\ & \searrow_{g_2} & & \nearrow_{v_2} & \\ & & B & & \end{array}$$



And, by PS2, we have $B \overset{uv_1}{\underset{vv_2}{\rightrightarrows}} C \xrightarrow{w} C'$ where $wuv_1 = wvv_2$. Then we have the following commutative diagrams as we wanted:



□

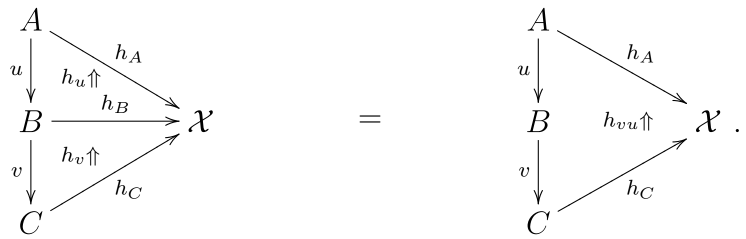
4.4 Pseudocones

For the next definition we take as reference [3].

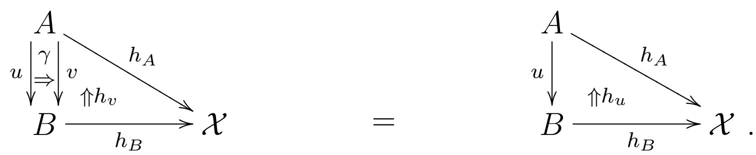
Definition 4.15. *A pseudoco-cone for a 2-functor $F : \mathcal{A} \to \mathcal{C}at$ with vertex the category $\mathcal{X}$ is a pseudonatural transformation $F \overset{h}{\Rightarrow} \mathcal{X}$ between F and the 2-functor which is constant at $\mathcal{X}$. More specifically: it is a family of functors $(h_A : FA \to \mathcal{X})_{A \in \mathcal{A}}$ and a family of invertible natural transformations $(h_u : h_B \circ Fu \to h_A)_{A \xrightarrow{u} B \in \mathcal{A}}$ satisfying the following equations:*

$PC0.$ $\quad h_{id_A} = id_{h_A}$.

$PC1.$



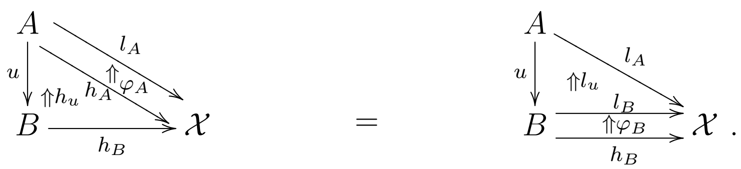
$PC2.$



Where the compositions in PC1 and PC2 are computed in $Hom(FA, \mathcal{X})$.

A morphism $h \overset{\varphi}{\Rightarrow} l$ of pseudoco-cones (with the same vertex) is a modification (i.e. a family of natural transformations $(h_A \overset{\varphi_A}{\Rightarrow} l_A)_{A \in \mathcal{A}}$) satisfying the following equation:

PCM.

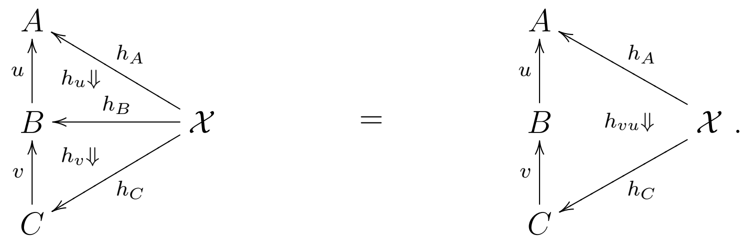


Where the compositions are computed in $Hom(FA, \mathcal{X})$.

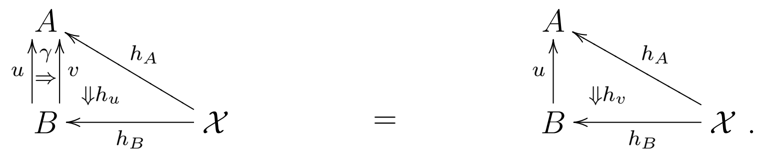
Remark 4.16. There is the dual concept of pseudocones: It is a pseudonatural transformation $\mathcal{X} \overset{h}{\Rightarrow} F$ between the 2-functor which is constant at $\mathcal{X}$ and F. More specifically: it is a family of functors $(h_A : \mathcal{X} \to FA)_{A\in\mathcal{A}}$ and a family of invertible natural transformations $(h_u : h_A \to Fu \circ h_B)_{A\overset{u}{\to}B\in\mathcal{A}}$ satisfying the following equations:

*PC*0. $h_{id_A} = id_{h_A}$.

*PC*1.



*PC*2.



For simplicity, we are going to call pseudoco-cones also pseudocones since this abuse does not cause any confusion.

5 Bilimits and bicolimits in $\mathcal{C}at$ indexed by a category

In this section, we construct the bilimits and the bicolimits in $\mathcal{C}at$ indexed by a category with two different approaches and then compare them to see that both give the same construction. The first one is based on [1] Exposé VI where Grothendieck constructs the bilimits and the bicolimits associated to a fibration. And the second one is more related with the usual definitions of limits and colimits as universal cones and consists on defining the bilimits and the bicolimits as universal pseudocones.

5.1 Basic definitions about Grothendieck fibrations

Definition 5.1. *Let $\mathcal{F}, \mathcal{G}, \mathcal{E}$ three categories, $\pi : \mathcal{F} \to \mathcal{E}$ and $\pi' : \mathcal{G} \to \mathcal{E}$ two functors. We define $Hom_{\mathcal{E}}(\mathcal{F}, \mathcal{G})$ as the category with objects the functors $u : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{u} & \mathcal{G} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & \mathcal{E} & \end{array}$$

and morphisms the $\mathcal{E}-$morphisms of functors, i.e. the morphisms of functors which are sent by π' in identity morphisms.

We will use the following terminology. Let $\mathcal{F}$ and $\mathcal{E}$ be two categories, $\pi : \mathcal{F} \to \mathcal{E}$ a functor, ξ and η objects of $\mathcal{E}$ and $f : \xi \to \eta$ a morphism of $\mathcal{E}$. We are going to say that an object x of $\mathcal{F}$ is over ξ if $\pi(x) = \xi$; and a morphism m of $\mathcal{F}$ is over f if $\pi(m) = f$. We say that f lifts to an arrow in $\mathcal{F}$ when there exists m over f.

Definition 5.2. *A morphism $m : x \to y$ of $\mathcal{F}$ over $\xi \xrightarrow{f} \eta$ is cartesian if $\forall p : z \to y$ over f $\exists! q : z \to x$ over id_ξ such that $mq = p$:*

$$\begin{array}{ccc} x & \xrightarrow{m} & y \\ {\scriptstyle \exists! q}\uparrow & /\!/\!/\!/ \nearrow{\scriptstyle p} & \\ z & & \end{array}$$

$$\xi \xrightarrow{f} \eta \,.$$

Definition 5.3. *A fibered category $\mathcal{F}$ over $\mathcal{E}$ is a functor $\pi : \mathcal{F} \to \mathcal{E}$ such that $\forall f : \xi \to \eta \in \mathcal{E}$ and y over η, there exists a cartesian mosphism $m : x \to y$ over f; and such that the composition of two cartesian morphisms of $\mathcal{F}$ is a cartesian morphism of $\mathcal{F}$. In this case, we say that $\pi : \mathcal{F} \to \mathcal{E}$ is a fibration.*

Definition 5.4. *A choice of x and $x \xrightarrow{m'} y$ for each f and each y over η as in the previous definition is called a cleavage.*

Definition 5.5. *Let ξ be an object of $\mathcal{E}$. We define the fiber of $\mathcal{F}$ in ξ as the subcategory of $\mathcal{F}$ whose objects are the ones of $\mathcal{F}$ over ξ and whose morphisms are the ones of $\mathcal{F}$ over id_ξ. We are going to denote this fiber $\mathcal{F}_\xi$.*

Remark 5.6. If $F : \Gamma^{op} \to \mathcal{C}at$ is a functor, and $u : \alpha \to \beta$ is an arrow of Γ, we will denote by $u^* = F(u)$ the action of F on u.

Remark 5.7. If $F : \Gamma^{op} \to \mathcal{C}at$ is a functor, there is a natural way to construct a fibered category Γ_F over Γ associated to F as follows:

Objects of Γ_F: (x, α) with $\alpha \in \Gamma$ and $x \in F\alpha$

Morphisms of Γ_F: A morphism between (x, α) and (y, β) is a pair (u, φ) where $\alpha \xrightarrow{u} \beta \in \Gamma$ and $x \xrightarrow{\varphi} u^*(y)$

Then we have the fibration: $\begin{array}{ccccc} \Gamma_F & & (x,\alpha) & & (u,\varphi) \\ \downarrow \diamond & & \downarrow & & \downarrow \\ \Gamma & & \alpha & & u \end{array}$.

Proof. Let's prove that $\begin{array}{c} \Gamma_F \\ \downarrow \diamond \\ \Gamma \end{array}$ is a fibration:

The composition in Γ_F is given by $(v, \psi) \circ (u, \varphi) = (v \circ u, u^*(\psi) \circ \varphi)$.

It can be checked that a morphism (u, φ) is cartesian if and only if φ is an isomorphism. Then it's clear that the composition of two cartesian morphisms is a cartesian morphism.

To conclude the proof, if we have $u : \alpha \to \beta$ in Γ and (y, β) over β, we can take the cartesian morphism $(u, id_{u^*(y)}) : (u^*(y), \alpha) \to (y, \beta)$ over u. □

Remark 5.8. When F takes values in $\mathcal{E}ns \hookrightarrow \mathcal{C}at$, we obtain exactly the diagram Γ_F of lemma 2.1, which is in this case a *discrete* fibration.

Remark 5.9. This construction has a canonical cleavage: given $\alpha \xrightarrow{u} \beta \in \Gamma$ and $(y, \beta) \in \Gamma_F$ over β, we take $(u^*y, \alpha) \xrightarrow{(u, id_{u^*y})} (y, \beta)$.

□

Remark 5.10. There are also the dual concepts: co-cartesian and co-fibered. And using them, one can make the corresponding construction in remark 5.7 for $F : \Gamma \to \mathcal{C}at$ a covariant functor.

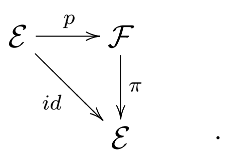
□

5.2 Bilimits

5.2.1 With Grothendieck fibrations

Definition 5.11. *Let $\pi : \mathcal{F} \to \mathcal{E}$ be a fibration. $Hom_{Cart/\mathcal{E}}(\mathcal{E}, \mathcal{F})$ is the category of cartesian sections of $\mathcal{F}$ which is described by:*

Objects: $p : \mathcal{E} \to \mathcal{F}$ which sends cartesian morphisms into cartesian morphisms and make the following diagram commutative:



Morphisms: $\theta : p \Rightarrow p'$ such that $\pi \circ \theta = id$.

Definition 5.12. *Let $\pi : \mathcal{F} \to \mathcal{E}$ be a fibration. Grothendieck defines $\underset{\overleftarrow{\mathcal{E}^{op}}}{Lim} \mathcal{F} = Hom_{Cart/\mathcal{E}}(\mathcal{E}, \mathcal{F})$.*

Remark 5.13. In general, $\underset{\overleftarrow{\mathcal{E}^{op}}}{\mathrm{Lim}} \mathcal{F}$ and $\lim_{\mathcal{E}^{op}} \mathcal{F}_\xi$ defined as a universal cone are not equivalent categories ([1], Exposé VI).

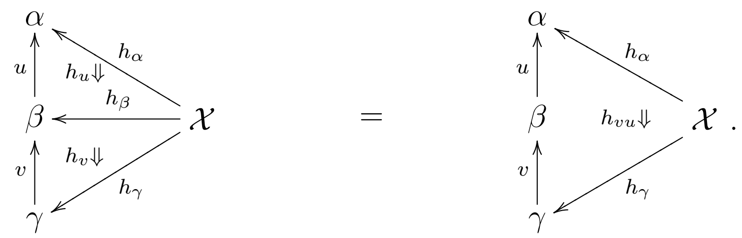
5.2.2 With pseudocones

When the 2-category $\mathcal{A}$ in remark 4.16 is trivial the definition of pseudocone yields:

Definition 5.14. *A pseudocone for a functor $F : \Gamma^{op} \to \mathcal{C}at$ with vertex the category $\mathcal{X}$ is a family of functors $(h_\alpha : \mathcal{X} \to F(\alpha))_{\alpha \in \Gamma}$ and a family of invertible natural transformations $(h_u : h_\alpha \to F(u) \circ h_\beta)_{\alpha \xrightarrow{u} \beta \in \Gamma}$ satisfying the following conditions:*

PC0. $h_{id_\alpha} = id_{h_\alpha}$.

PC1.



Definition 5.15. *Let $F : \Gamma^{op} \to \mathcal{C}at$ be a functor, we define $\underset{\alpha\in\Gamma^{op}}{bilim} F\alpha$ as the universal pseudocone associated to F i.e. a pseudocone $\underset{\alpha\in\Gamma^{op}}{bilim} F\alpha \overset{h}{\Rightarrow} F$ such that if $\mathcal{X} \overset{\lambda}{\Rightarrow} F$ is another pseudocone, then $\exists!\ \tilde{h} \mid h\tilde{\lambda} = \lambda$ i.e. $\tilde{\lambda}$ makes, for each $\alpha \in \Gamma$, the following diagram strictly commutative*

$$\begin{array}{ccc} \underset{\alpha\in\Gamma^{op}}{bilim} F\alpha & \xrightarrow{h_\alpha} & F\alpha \\ & \nwarrow^{\tilde{\lambda}} & \uparrow \lambda_\alpha \\ & & \mathcal{X} \end{array}$$

Proposition 5.16. *Given a functor $F : \Gamma^{op} \to \mathcal{C}$, $\underset{\alpha\in\Gamma^{op}}{bilim} F\alpha$ exists and can be constructed as follows:*

Objects: $(x_\alpha)_{\alpha\in\Gamma} |\ x_\alpha \in F\alpha$ and $\forall \alpha \overset{u}{\to} \beta \in \Gamma$, an isomorphism $x_\alpha \overset{\varphi_u}{\to} F(u)(x_\beta)$ in $F\alpha$ given in a functorial way.

Morphisms: A morphism $f : (x_\alpha)_{\alpha\in\Gamma} \to (y_\alpha)_{\alpha\in\Gamma}$ is a family $(x_\alpha \overset{f_\alpha}{\to} y_\alpha)_{\alpha\in\Gamma}$ such that $\forall \alpha \overset{u}{\to} \beta \in \Gamma$, the following diagram commutes:

$$\begin{array}{ccc} x_\alpha & \xrightarrow{f_\alpha} & y_\alpha \\ \varphi_u \downarrow & & \downarrow \psi_u \\ F(u)(x_\beta) & \xrightarrow[F(u)(f_\beta)]{} & F(u)(y_\beta) \end{array} .$$

And the composition is given by: $(f_\alpha)_{\alpha\in\Gamma} \circ (g_\alpha)_{\alpha\in\Gamma} = (f_\alpha \circ g_\alpha)_{\alpha\in\Gamma}$.

Proof. We define $h_\alpha((x_\alpha)_{\alpha\in\Gamma}) = x_\alpha$, $h_\alpha((x_\alpha \overset{f_\alpha}{\to} y_\alpha)_{\alpha\in\Gamma}) = f_\alpha$ and $(h_u)_{(x_\alpha)_{\alpha\in\Gamma}} = \varphi_u$. It's clear that $\underset{\alpha\in\Gamma^{op}}{\text{bilim}} F\alpha \overset{h}{\Rightarrow} F$ is a pseudocone. Let's check that it is universal: Let $\mathcal{X} \overset{\lambda}{\Rightarrow} F$ be another pseudocone. One can check the universality by taking $\tilde{h}(x) = (\lambda_\alpha(x))_{\alpha\in\Gamma}$ with $\varphi_u = (\lambda_u)_x$ and $\tilde{h}(f) = (\lambda_\alpha(f))_{\alpha\in\Gamma}$. □

5.2.3 The relation between the two definitions

We are going to prove that if we take the fibration associated to $F : \Gamma^{op} \to \mathcal{C}at$ as in 5.7 and make the Grothendieck construction of $\underset{\mathcal{E}^{op}}{\overleftarrow{\text{Lim}}} \mathcal{F}$, we obtain the universal pseudocone for F:

In $\begin{array}{c} \Gamma \\ \downarrow id \\ \Gamma \end{array}$ every morphism $f \in \Gamma$ is cartesian, thus the objects of $\underset{\mathcal{E}^{op}}{\overleftarrow{\text{Lim}}} \mathcal{F}$

must send every morphism in a cartesian one. Then they can be described as follows:

$$\begin{aligned} p:\Gamma &\longrightarrow \Gamma_F\\ \alpha &\longmapsto (x_\alpha,\alpha)\\ \alpha\xrightarrow{u}\beta &\longmapsto (u,\varphi_u)\end{aligned}$$

where φ_u is an isomorphism $\forall u \in \Gamma$. But this is exactly the same definition given in 5.16. Now, the morphisms of $\underset{\overleftarrow{\mathcal{E}^{op}}}{\mathrm{Lim}}\,\mathcal{F}$ are the natural transformations $\theta : p \Rightarrow p' \mid \diamond\,\theta = id$, i.e. $\forall\alpha \in \Gamma$ there is $p\alpha \overset{\theta_\alpha=(\theta^1_\alpha,\theta^2_\alpha)}{\longrightarrow} p'\alpha$ such that $\diamond\theta = id$ and $\forall\alpha \xrightarrow{u} \beta \in \Gamma$, the following diagram commutes:

$$\begin{array}{ccc} p\alpha & \xrightarrow{\theta_\alpha} & p'\alpha \\ {\scriptstyle p(u)}\downarrow & & \downarrow{\scriptstyle p'(u)} \\ p\beta & \xrightarrow[\theta_\beta]{} & p'\beta\ . \end{array}$$

But this means that $(\diamond\theta)_\alpha = \diamond(\theta_\alpha)\circ\diamond_{p(\alpha)} = \theta^1_\alpha\circ id_\alpha$, thus $\theta^1_\alpha = id_\alpha \forall\alpha\in\Gamma$ and $\varphi^{p'}_u\circ\theta^2_\alpha = u^*(\theta^2_\beta)\circ\varphi^p_u$, thus having a morphism of $\underset{\overleftarrow{\mathcal{E}^{op}}}{\mathrm{Lim}}\,\mathcal{F}$ is equivalent to having $\forall\alpha\in\Gamma, \theta_\alpha : x_\alpha \to x'_\alpha$ (it is θ^2_α) such that $\forall\alpha \xrightarrow{u} \beta \in \Gamma$, $\varphi^{p'}_u\circ\theta_\alpha = u^*(\theta_\beta)\circ\varphi^p_u$ which is also the same definition given in 5.16. It can be checked that the composition is the same too.

5.3 Bicolimits

5.3.1 With Grothendieck fibrations

For this subsection, we take as reference [1], Exposé VI and [8].

Definition 5.17. *[Category of fractions] Let $\mathcal{F}$ be a category and S a set of morphisms of $\mathcal{F}$. $\mathcal{F}[S^{-1}]$ is the category defined by the following universal property:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\rho} & \mathcal{F}[S^{-1}] \\ & {\scriptstyle\theta}\searrow \;\cong & \vdots\ {\scriptstyle\exists!\ \theta'} \\ & & \mathcal{D} \end{array} \tag{5.1}$$

where ρ is a functor that sends the morphisms in S to isomorphisms. More, specifically: $\forall \mathcal{F} \xrightarrow{\theta} \mathcal{D}$ *that sends the morphisms in S to isomorphisms,* $\exists!\ \mathcal{F}[S^{-1}] \xrightarrow{\theta'} \mathcal{D}$ *such that* $\theta'\rho \cong \theta$.

Remark 5.18. If S' is another set of morphisms of $\mathcal{F}$ which contains S as a subset and such that any morphism that are in S' but not in S is an isomorphism, then $\mathcal{F}[S^{-1}] = \mathcal{F}[S'^{-1}]$. Also, this equality holds if every morphism of S' is a composite of morphisms in S; that is, given S, we can close S under compositions and isomorphisms and obtain the same category of fractions.

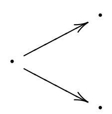
□

Definition 5.19. *If $\pi : \mathcal{F} \to \mathcal{E}$ is a co-fibration and S is the set of co-cartesian morphisms of $\mathcal{F}$, Grothendieck defines* $\underset{\vec{\mathcal{E}}}{\mathrm{Lim}}\, \mathcal{F} = \mathcal{F}[S^{-1}]$.

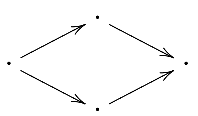
In the next pages, we are going to make an explicit construction of $\underset{\vec{\mathcal{E}}}{\mathrm{Lim}}\, \mathcal{F}$ in the case that the category $\mathcal{E}$ is pseudo-filtered.

Proposition 5.20. *Let $\mathcal{F}$ be a co-fibered category over $\mathcal{E}$ and suppose that $\mathcal{E}$ satisfies the following properties:*

L1. Every diagram of the form



can be completed to a commutative one:



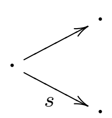
.

L2. For every pair of morphisms $u, v : \cdot \rightrightarrows \cdot$ *such that exists a morphism t satisfying $ut = vt$, there is a morphism w such that $wu = wv$.*

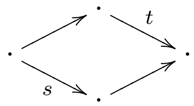
Then the set S of co-cartesian morphisms of $\mathcal{F}$ satisfies the following properties:

Fr1. The composition of two morphisms of S is a morphism of S.

Fr2. Every diagram of the form



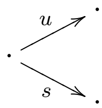
with $s \in S$, can be completed to a commutative one:



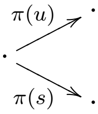
with $t \in S$.

Fr3. For every pair of morphisms $u, v : \cdot \rightrightarrows \cdot$ such that exists $s \in S \ \mid us = vs$, there is a morphism $t \in S$ such that $tu = tv$.

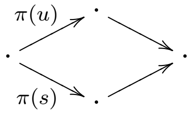
Proof. $\mathcal{F}$ is co-fibered over $\mathcal{E}$, thus, by definition, S satisfies Fr1. Let's check Fr2: Suppose that we have



with $s \in S$. Now, we can apply π and we have the following diagram in $\mathcal{E}$:

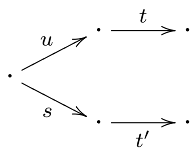


which, by L1, can be completed to a commutative diagram:

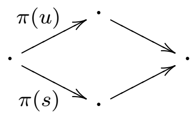


.

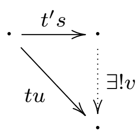
In addition, for being $\pi : \mathcal{F} \to \mathcal{E}$ a fibration, we have:



over



with $t, t' \in S$. Now, $t's$ is co-cartesian, thus, by definition, we have a commutative diagram:



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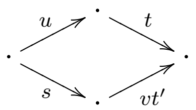
Then we have a commutative diagram as we wanted:



It remains to check Fr3: Suppose that we have $u, v : \cdot \rightrightarrows \cdot$ in $\mathcal{F}$ such that exists $s \in S \mid us = vs$. Then we have $\cdot \xrightarrow{\pi s} \cdot \underset{\pi v}{\overset{\pi u}{\rightrightarrows}} \cdot$ such that $\pi u \pi s = \pi v \pi s$ and L2 says that there exists $f \in \mathcal{E}$ such that $f \pi u = f \pi v$. Since $\mathcal{F}$ is fibered over $\mathcal{E}$, there exists $t \in \mathcal{F}$ co-cartesian over f and composable with u and v. Then $tus = tvs$, thus $tu = tv$ because s is co-cartesian.

□

Definition 5.21. *A set $S \subset \mathcal{F}$ which satisfies FR1, FR2 and FR3 is said to satisfy a* calculus of right fractions *([8])*

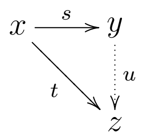
Remark 5.22. If $\mathcal{E}$ is pseudo-filtered, it satisfies L1 and L2.

□

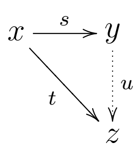
Definition 5.23. *Let $\mathcal{F}$ be a category and S a set of morphisms of $\mathcal{F}$ satisfying Fr1, Fr2 and Fr3. For each object x of $\mathcal{F}$, we define $S(x)$ as the category of morphisms of S with domain x.*

Remark 5.24. The category $S(x)$ is filtered.

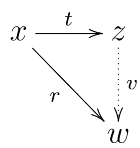
Proof. To begin the proof, let's explicit the morphisms and the composition of $S(x)$: a morphism between $x \xrightarrow{s} y$ and $x \xrightarrow{t} z$ is given by a commutative diagram of the form:



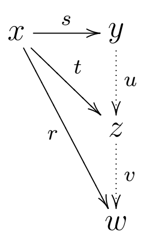
And the composition of two morphisms



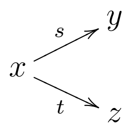
and



is given by

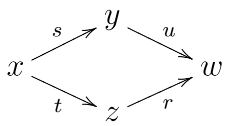


Now, let's prove that it is filtered: Let $x \xrightarrow{s} y, x \xrightarrow{t} z$ be two objects of $S(x)$. We have the following diagram in $\mathcal{F}$:



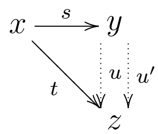
.

Since S satisfies Fr2, we have



with $r \in S$. Then we have

which proves that $S(x)$ satisfies the first axiom of filtered. To prove the second axiom, suppose that we have



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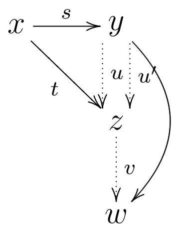
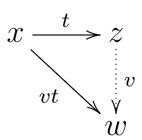
Since S satisfies Fr3, we have



with $v \in S$, and



is the co-equalizer that we need. □

Proposition 5.25. *[Construction of the category of fractions for a calculus of right fractions] Let $\mathcal{F}$ be a category and S a set of morphisms of $\mathcal{F}$ satisfying Fr1, Fr2 and Fr3. Then the category $\mathcal{F}[S^{-1}]$ can be described as follows:*

Objects: The objects are the ones of $\mathcal{F}$.

Morphisms: Given x, y objects of $\mathcal{F}$, $hom_{\mathcal{F}[S^{-1}]}(x,y) = \underset{S(y)}{colim}\, hom_{\mathcal{F}}(x,\cdot)$.

The composition is given by: let $f : x \to y$ and $g : y \to z$ be two morphisms of $\mathcal{F}[S^{-1}]$. Using the construction of filtered colimits in $\mathcal{E}ns$, we can think f and g as follows:

$$x \xrightarrow{f'} \cdot \xleftarrow{t} y\ , \qquad y \xrightarrow{g'} \cdot \xleftarrow{t'} z$$

with t and t' in S. Then we have:

$$\begin{array}{ccc} & & \cdot \\ & \overset{g'}{\nearrow} & \\ y & & \\ & \underset{t}{\searrow} & \\ & & \cdot \end{array} \qquad .$$

And, since S satisfies Fr2, we have

$$\begin{array}{ccccc} & & \cdot & & \\ & \overset{g'}{\nearrow} & & \overset{t''}{\searrow} & \\ y & & & & \cdot \\ & \underset{t}{\searrow} & & \underset{g''}{\nearrow} & \\ & & \cdot & & \end{array}$$

with $t'' \in S$.

We define gf by the following diagram:

$$x \xrightarrow{g''f'} \cdot \xleftarrow{t''t'} z \qquad .$$

Proof. First note that, because of remark 5.18, we can suppose that all the identities are in S. Now, we have to check that the category defined at the statement satisfies the universal property (5.1) of $\mathcal{F}[S^{-1}]$. We are going to denote this category also $\mathcal{F}[S^{-1}]$. We have $\rho : \mathcal{F} \to \mathcal{F}[S^{-1}]$ defined as $\rho(x) = x$ and $\rho(x \xrightarrow{f} y) = x \xrightarrow{f} y \xleftarrow{id} y$ And given θ as in definition 5.17, one can check the universality by taking $\theta'(x) = \theta(x)$ and $\theta'(x \xrightarrow{f} z \xleftarrow{s} y) = (\theta(s))^{-1} \circ \theta(f)$. ∎

Remark 5.26. In the previous proposition, we abuse notation and treat the elements of $\operatorname*{colim}_{S(y)} hom_{\mathcal{F}}(x, \cdot)$ not as equivalence classes, but just as its representatives. However, all the calculations that we have made are consistent with the quotient.

□

Remark 5.27. In general, $\operatorname*{Lim}_{\overrightarrow{\mathcal{E}}} \mathcal{F}$ and $\operatorname*{colim}_{\mathcal{E}} \mathcal{F}_\xi$ defined as universal cone are not equivalent categories. But they are equivalent when $\mathcal{E}$ is pseudo-filtered ([1] Exposé VI p. 272).

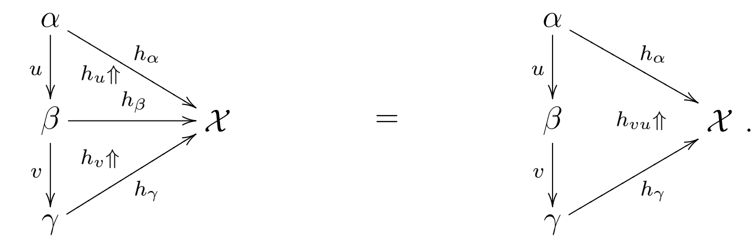
5.3.2 With pseudocones

The following definition makes explicit the dual case of definition 5.14.

Definition 5.28. *A pseudoco-cone for a functor $F : \Gamma \to \mathcal{C}at$ with vertex the category $\mathcal{X}$ is a family of functors $(h_\alpha : F(\alpha) \to \mathcal{X})_{\alpha \in \Gamma}$ and a family of invertible natural transformations $(h_u : h_\beta \circ F(u) \to h_\alpha)_{\alpha \xrightarrow{u} \beta \in \Gamma}$ satisfying the following conditions:*

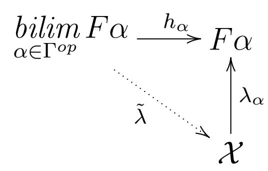
PC0. $h_{id_\alpha} = id_{h_\alpha}$.

PC1.



For simplicity, we are going to call pseudoco-cones also pseudocones since this abuse does not cause any confusion.

Definition 5.29. *Let $F : \Gamma \to \mathcal{C}at$ be a functor. We define $\operatorname*{bicolim}_{\alpha\in\Gamma} F\alpha$ as the universal pseudocone associated to F. More specifically, it is a pseudocone $F \overset{h}{\Rightarrow} \operatorname*{bicolim}_{\alpha\in\Gamma} F\alpha$ such that if $F \overset{\lambda}{\Rightarrow} \mathcal{X}$ is another pseudocone, then $\exists !\ \tilde{\lambda} : \operatorname*{bicolim}_{\alpha\in\Gamma} F\alpha \to \mathcal{X} \mid \tilde{\lambda} \circ h = \lambda$, i.e. $\tilde{\lambda}$ makes, for each $\alpha \in \Gamma$, the following diagram strictly commutative*



When Γ is a filtered category, we can make an explicit construction of $\operatorname{bicolim}_{\alpha\in\Gamma} F\alpha$:

Proposition 5.30. *If $F : \Gamma \to \mathcal{C}at$ is a functor and Γ is filtered, then $\operatorname{bicolim}_{\alpha\in\Gamma} F\alpha$ exists and can be constructed as follows:*

Objects: (C, α) where $C \in F\alpha$.

Morphisms: A morphism between (C, α) and (D, β) is a triplet (u, f, v) where $\alpha \xrightarrow{u} \gamma \xleftarrow{v} \beta$ and $F(u)(C) \overset{f}{\to} F(v)(D)$ quotient by $\sim$ where $(u, f, v) \sim (u', f', v')$ if and only if $\exists$ $\gamma \xrightarrow{\tilde{u}} \delta \xleftarrow{\tilde{v}} \gamma'$ such that $\tilde{u}u = \tilde{v}u', \tilde{u}v = \tilde{v}v'$ and $F(\tilde{u})(f) = F(\tilde{v})(f')$.

Proof. First, we are going to prove that $\sim$ is an equivalence relation: it's clear that is reflexive and symmetric. Let's check that it is transitive: suppose that $(u, f, v) \sim (u', f', v')$ and $(u', f', v') \sim (u'', f'', v'')$ Then, there is $\gamma \xrightarrow{\tilde{u}} \delta \xleftarrow{\tilde{v}} \gamma'$ and $\gamma' \xrightarrow{\tilde{\tilde{u}}} \varepsilon \xleftarrow{\tilde{\tilde{v}}} \gamma''$ such that $\tilde{u}u = \tilde{v}u', \tilde{u}v = \tilde{v}v', F(\tilde{u})(f) = F(\tilde{v})(g), \tilde{\tilde{u}}u' = \tilde{\tilde{v}}u'', \tilde{\tilde{u}}v' = \tilde{\tilde{v}}v''$ and $F(\tilde{\tilde{u}})(g) = F(\tilde{\tilde{v}})(h)$. And for being Γ filtered, there is $\delta \xrightarrow{a} \nu \xleftarrow{b} \varepsilon$, thus we have $\gamma' \underset{a\tilde{v}}{\overset{b\tilde{\tilde{u}}}{\rightrightarrows}} \nu$.

But, again for being Γ filtered, there is $\gamma' \underset{a\tilde{v}}{\overset{b\tilde{\tilde{u}}}{\rightrightarrows}} \nu \xrightarrow{c} \eta$. Then, we have $\gamma \xrightarrow{ca\tilde{u}} \eta \xleftarrow{cb\tilde{\tilde{v}}} \gamma''$.

Now $ca\tilde{u}u = ca\tilde{v}u' = cb\tilde{\tilde{u}}u' = cb\tilde{\tilde{v}}u'', ca\tilde{u}v = ca\tilde{v}v' = cb\tilde{\tilde{u}}v' = cb\tilde{\tilde{v}}v''$ and $F(ca\tilde{u})(f) = F(ca)F(\tilde{u})(f) = F(ca)F(\tilde{v})(g) = F(ca\tilde{v})(g) = F(cb\tilde{\tilde{u}})(g) = F(cb)F(\tilde{\tilde{u}})(g) = F(cb)F(\tilde{\tilde{v}})(h) = F(cb\tilde{\tilde{v}})(h)$ which concludes the demostration of $\sim$ is transitive.

Now, let's define the composition in $\operatorname{bicolim}_{\alpha\in\Gamma} F\alpha$: Suppose that we have $(C, \alpha) \overset{(u,f,v)}{\to} (D, \beta) \overset{(u',g,v')}{\to} (E, \delta)$. For being Γ filtered, there is $\gamma \xrightarrow{a} \nu \xleftarrow{b} \gamma'$ and $\beta \underset{bu'}{\overset{av}{\rightrightarrows}} \nu \xrightarrow{c} \eta$ such that $cav = cbu'$. Then the composition $(u', g, v') \circ (u, f, v)$ is $(cau, F(cb)(g) \circ F(ca)(f), cbv')$.

Let's see that it is a universal pseudocone: we define

$$\begin{aligned} h_\alpha :\ F\alpha &\longrightarrow \operatorname*{bicolim}_{\alpha\in\Gamma} F\alpha \\ C &\longmapsto (C,\alpha) \\ f &\longmapsto (id, f, id) \end{aligned}$$

and for $\alpha \xrightarrow{u} \beta, h_u : h_\beta \circ F(u) \to h_\alpha$ as follows: $(h_u)_C = (u, id_{F(u)(C)}, id)$. It is straightforward to check that is a pseudocone. If λ is another pseudocone for F, one can check the universality of $\operatorname*{bicolim}_{\alpha\in\Gamma} F\alpha$ by taking $\tilde{\lambda}$ as follows: $\tilde{\lambda}(C,\alpha) = \lambda_\alpha(C)$ and $\tilde{\lambda}((C,\alpha) \overset{(u,f,v)}{\to} (D,\beta)) = \lambda_\gamma f$.

□

Remark 5.31. This construction yields a category equivalent to a usual construction of filtered colimits of categories as the filtered colimit of the objects and then of the morphisms.

□

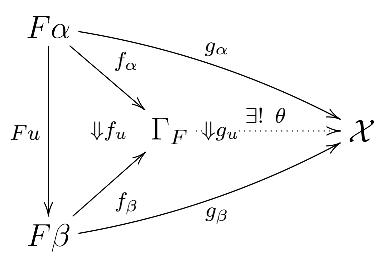
5.4 The relation between the two definitions

We are going to prove that if we take the co-fibration associated to $F : \Gamma \to \mathcal{C}at$ as in 5.7, and consider S the set of co-cartesian mosphisms of Γ_F, then $\underset{\overrightarrow{\Gamma}}{\mathrm{Lim}}\, \Gamma_F = \Gamma_F[S^{-1}]$ is the universal pseudocone for F. And we will do this in two different ways: using only the universal properties; and checking that both explicit constructions are the same one.

5.4.1 Using only universal properties

To check that $\Gamma_F[S^{-1}]$ is a universal pseudocone, let's observe first that Γ_F satisfies the following universal property:

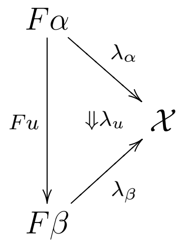
Proposition 5.32 (Universal propoerty of the Grothendieck construction Γ_F). *Γ_F as defined in remark 5.7 has the following universal property:*



where f_α is defined as $f_\alpha(x) = (x, \alpha)$ and $f_\alpha(a) = (id, a)$, f_u is defined as $(f_u)_x = (u, id_{u_*x})$. Moreover, if the arrows g_u are invertible, the arrow θ transforms the co-cartesian morphisms into invertible morphisms of $\mathcal{X}$.

Proof. θ has to be defined as $\theta(x, \alpha) = f_\alpha(x)$ and $\theta((u, \varphi)) = f_\beta(\varphi) \circ (g_u)_x$. □

Corollary 5.33. *It can be checked that $\Gamma_F[S^{-1}]$ is the universal pseudocone by taking $h_\alpha = \rho \circ f_\alpha$, $h_u = id_\rho \circ f_u$; and if*



is another pseudocone, we take the corresponding θ in the universal property of Γ_F and then the θ' given by the universal property of $\Gamma_F[S^{-1}]$.

5.4.2 Comparing both constructions explicitly

The co-fibration associated to F is given by:

Objects: (x, α) with $x \in F\alpha, \alpha \in \Gamma$.

Morphisms: A morphism between (x, α) and (y, β) is a pair (u, φ) where $\alpha \xrightarrow{u} \beta \in \Gamma$ and $u_*x \xrightarrow{\varphi} y \in F\beta$.

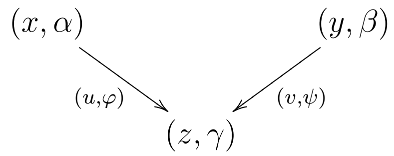
Composition: $(v, \psi) \circ (u, \varphi) = (v \circ u, \psi \circ v_*\varphi)$.

It is easy to check that the co-cartesian morphisms are the ones with φ isomorphism.

Now, let's describe $\Gamma_F[S^{-1}]$:

Objects: (x, α) with $x \in F\alpha, \alpha \in \Gamma$. These objects are the objects of $\underset{\alpha \in \Gamma}{\text{bicolim}}\, F\alpha$.

Morphisms: $hom_{\Gamma_F[S^{-1}]}((x, \alpha), (y, \beta)) = \underset{S((y,\beta))}{\text{colim}}\, hom_{\Gamma_F}((x, \alpha), \cdot)$. More specifically: are the equivalence classes of the relation $\sim$ between elements of the form



with ψ an isomorphism. Where the relation $\sim$ is given by

$$\begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \searrow^{(u,\varphi)} \quad \swarrow^{(v,\psi)} & \\ & (z,\gamma) & \end{array} \quad \sim \quad \begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \searrow^{(u',\varphi')} \quad \swarrow^{(v',\psi')} & \\ & (z',\gamma') & \end{array} \tag{5.2}$$

if and only if $\exists (v'',\psi'') : (y,\beta) \to (z'',\gamma'')$ with ψ'' isomorphism and arrows $(z,\gamma) \overset{(w,\theta)}{\to} (z'',\gamma''), (z',\gamma') \overset{(w',\theta')}{\to} (z'',\gamma'')$ such that $wv = v'' = w'v'$, $\theta w_*\psi = \psi'' = \theta' w'_*\psi'$, $wu = w'u'$ and $\theta w_*\varphi = \theta' w'_*\varphi'$. The composition is the one described in 5.25.

Let's see that there is a bijective correspondence between these morphisms and the ones of $\operatorname{bicolim}_{\alpha\in\Gamma} F\alpha$ given by:

$$\begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \searrow^{(u,\varphi)} \quad \swarrow^{(v,\psi)} & \\ & (z,\gamma) & \end{array} \quad \overset{a}{\longmapsto} \quad (x,\alpha) \xrightarrow{(u,\psi^{-1}\varphi,v)} (y,\beta)$$

$$\begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \searrow^{(u,f)} \quad \swarrow^{(v,id_{v_*y})} & \\ & (v_*y,\gamma) & \end{array} \quad \overset{b}{\longleftarrow\!\!\!\dashv} \quad (x,\alpha) \xrightarrow{(u,f,v)} (y,\beta)\ .$$

To check that a is well defined, suppose that two morphisms of $\Gamma_F[S^{-1}]$ are related as in (5.2). Then we have $\begin{array}{c} \gamma \searrow^{w} \\ \qquad \gamma'' \\ \gamma' \nearrow_{w'} \end{array}$ such that $wu = w'u'$, $wv = w'v'$ and (1) and (2) in the following diagram commute:

$$\begin{array}{ccc} w'_*u'_*x = w_*u_*x & \xrightarrow{w_*\varphi} & w_*z \\ \downarrow{\scriptstyle w'_*\varphi'} & (1) \quad \swarrow^{\theta} \ z'' \ \nwarrow^{\theta'} \quad (2) & \updownarrow{\scriptstyle w_*\psi} \\ w'_*z' & \xleftrightarrow{w'_*\psi'} & w_*v_*y = w'_*v'_*y \end{array} .$$

Then the exterior square commutes, thus $(u, \psi^{-1}, \varphi, v) \sim (u', \psi'^{-1}, \varphi', v')$ according to proposition 5.30.

To see the good definition of b, suppose that $(u, f, v) \sim (u', f', v')$, and we want to see that

$$\begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \overset{(u,f)}{\searrow} \quad \overset{(v,id_{v_*y})}{\swarrow} & \\ & (v_*y,\gamma) & \end{array} \quad \sim \quad \begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \overset{(u',f')}{\searrow} \quad \overset{(v',id_{v'_*y})}{\swarrow} & \\ & (v'_*y,\gamma') & \end{array} . \tag{5.3}$$

By proposition 5.30, we have $\gamma \xrightarrow{w} \gamma'' \xleftarrow{w'} \gamma'$ such that $wu = w'u'$, $wv = w'v'$ and $w_*f = w'_*f'$.

If we take these γ'', w and w'; $v'' = wv = w'v'$, $\theta = \theta' = \psi'' = id_{w_*v_*y}$ and $z'' = w_*v_*y = w'_*v'_*y$ we have (5.3).

It only remains to prove that the compositions ab and ba are identities. For $ba = id$, we have to check that

$$\begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \overset{(u,\varphi)}{\searrow} \quad \overset{(v,\psi)}{\swarrow} & \\ & (z,\gamma) & \end{array} \quad \sim \quad \begin{array}{ccc} (x,\alpha) & & (y,\beta) \\ & \overset{(u,\psi^{-1}\varphi)}{\searrow} \quad \overset{(v,id_{v_*y})}{\swarrow} & \\ & (v_*y,\gamma) & \end{array}$$

and we do it by taking $\gamma'' = \gamma$, $v'' = v$, $w = w' = id_\gamma$, $\psi'' = \psi = \theta'$, $\theta = id_z$ and $z'' = z$.

Finally, $ab = id$ is straightforward.

6 Ind-objects of a 2-category $\mathcal{C}$

In this section, we define an Ind-object of a 2-category and then we prove that, in this case, $Ind(\mathcal{C})$ results a 2-category. In the description of the category $Hom(X,Y)$ for X and Y Ind-objects we use the constructions of bilimits and bicolimits in $\mathcal{C}at$ made in section 5.

6.1 Definition of the 2-category $Ind(\mathcal{C})$

Definition 6.1. *Let $\mathcal{C}$ be any 2-category. An Ind-object of $\mathcal{C}$ is a small filtered system $X = (C_i)_{i\in J}$ i.e. a functor $X : J \to \mathcal{C}$ with J a small filtered category.*

Remark 6.2. An Ind-object of a 2-category $\mathcal{C}$ as in the previous definition is, in particular, an Ind-object of the underlying category of $\mathcal{C}$ in the sense of section 3.5.

Definition 6.3. *Let $\mathcal{C}$ be a 2-category and $X = (C_i)_{i\in J}$, $Y = (D_\alpha)_{\alpha\in\Gamma}$ two ind-objects of $\mathcal{C}$. We define the morphisms between X and Y as follows:*

$$Hom(X,Y) = \underset{i\in J^{op}}{bilim}\ \underset{\alpha\in\Gamma}{bicolim}\ Hom(C_i, D_\alpha).$$

Remark 6.4. $Hom(X,Y)$ results a category.

□

Using the construction of bilimits and bicolimits in $\mathcal{C}at$ indexed by a category given in section 5, we can give the following description of the category $Hom(X,Y)$ which is the corresponding to proposition 3.3 in the case of 2-categories:

Proposition 6.5. *An object of $Hom(X,Y)$ is a triplet $(\varphi, (f_i)_{i\in J}, (\varphi_\phi)_{i\overset{\phi}{\to} j\in J})$ where $\varphi : J \to \Gamma$ is a function between the objects of J and the ones of Γ, $f_i : C_i \to D_{\varphi(i)}$ morphisms of $\mathcal{C}$ and $\varphi_\phi : (f_i, \varphi(i)) \to \phi^*(f_j, \varphi(j))$ isomorphisms of $\underset{\alpha\in\Gamma}{bicolim}\ Hom(C_i, D_\alpha)$ given in a functorial way, i.e. $\forall i \overset{\phi}{\to} j$ $\exists$ $\varphi(i) \overset{u}{\longrightarrow} \alpha$, $\varphi(j) \overset{v}{\longrightarrow} \alpha$ and an isomorphism $Y(u) \circ f_i \overset{\varphi_\phi}{\to} Y(v) \circ f_j \circ X(\phi)$:*

$$\begin{array}{ccccc} C_i & \xrightarrow{f_i} & D_{\varphi(i)} & & \\ & & & \searrow^{Y(u)} & \\ {\scriptstyle X(\phi)}\downarrow & \cong \ \Downarrow \ \varphi_\phi & & & D_\alpha \\ & & & \nearrow_{Y(v)} & \\ C_j & \xrightarrow[f_j]{} & D_{\varphi(j)} & & \end{array} .$$

A morphism of $Hom(X,Y)$ *is the equivalence class according to the relation* $\sim$ *of a family* $(\theta_i)_{i\in J}$ *where* $\theta_i : (f_i, \varphi(i)) \to (g_i, \psi(i))$ *is a morphism of* $\underset{\alpha\in\Gamma}{\mathrm{bicolim}}\, Hom(C_i, D_\alpha)$ *such that* $\forall i \xrightarrow{\phi} j \in J$, $\psi_\phi \circ \theta_i = \phi^*(\theta_j) \circ \varphi_\phi$, *i.e.* $\exists \begin{array}{l}\varphi(i) \xrightarrow{u} \\ \psi(i) \xrightarrow[v]{} \end{array} \alpha$ *and a morphism* $Y(u) \circ f_i \xrightarrow{\theta_i} Y(v) \circ g_i$ *such that* $\forall i \xrightarrow{\phi} j \in J$ *the following diagram commutes:*

$$\begin{array}{ccc} Y(u) \circ f_i & \xrightarrow{\theta_i} & Y(v) \circ g_i \\ {\scriptstyle \varphi_\phi}\downarrow & & \downarrow{\scriptstyle \psi_\phi} \\ Y(v) \circ f_j \circ X(\phi) & \xrightarrow[X(\phi)^*(\theta_j)]{} & Y(v) \circ g_j \circ X(\phi) \end{array} .$$

The relation $\sim$ *is defined as* $(\theta_i)_{i\in J} \sim (\eta_i)_{i\in J}$ *if and only if* $\forall i \in J$, $\exists \begin{array}{l}\alpha_1 \xrightarrow{u} \\ \alpha_2 \xrightarrow[v]{} \end{array} \alpha$ *such that* $uu_1 = vu2$, $uv_1 = vv_2$ *and* $Y(u) \circ \theta_i = Y(v) \circ \eta_i$ *where* $(\theta_i)_{i\in J}$ *is given by* $\begin{array}{l}\varphi(i) \xrightarrow{u_1} \\ \psi(i) \xrightarrow[v_1]{} \end{array} \alpha_1$ *and* $Y(u_1) \circ f_i \xrightarrow{\theta_i} Y(v_1) \circ g_i$, $(\eta_i)_{i\in J}$ *is given by* $\begin{array}{l}\varphi(i) \xrightarrow{u_2} \\ \psi(i) \xrightarrow[v_2]{} \end{array} \alpha_2$ *and* $Y(u_2) \circ f_i \xrightarrow{\eta_i} Y(v_2) \circ g_i$.

Proof. First of all, let's calculate $\underset{\alpha\in\Gamma}{\mathrm{bicolim}}\, Hom(C_i, D_\alpha)$ for $i \in J$ fixed: the functor in question is

$$\begin{array}{rcl} \Gamma & \longrightarrow & \mathcal{C}at \\ \alpha & \longmapsto & Hom(C_i, D_\alpha) \\ \alpha \xrightarrow{u} \beta & \longmapsto & (Y(u))_* \end{array}$$

where $(Y(u))_*(f) = Y(u) \circ f$ and

$$(Y(u))_*(f \xrightarrow{\theta} g) = (Y(u) \circ f \overset{id_{Y(u)} \circ \theta}{\longrightarrow} Y(u) \circ g).$$

Then, we can describe $\operatorname*{bicolim}_{\alpha\in\Gamma} Hom(C_i, D_\alpha)$ as follows:

Objects: $(f, \alpha) \mid f : C_i \to D_\alpha$.

Morphisms: A morphism between (f, α) and (g, β) is the equivalence class according to the relation $\sim$ of a triplet (u, h, v) where $\alpha \xrightarrow{u} \gamma \xleftarrow{v} \beta$ and $Y(u) \circ f \xrightarrow{h} Y(v) \circ g$. The relation $\sim$ is defined as: $(u, h, v) \sim (u', h', v')$ if and only if $\exists\ \gamma \xrightarrow{\tilde{u}} \delta \xleftarrow{\tilde{v}} \gamma'$ / $\tilde{u}u = \tilde{v}u', \tilde{u}v = \tilde{v}v'$ and $Y(\tilde{u}) \circ h = Y(\tilde{v}) \circ h'$.

Now, let's calculate $\operatorname*{bilim}_{i\in J^{op}} \operatorname*{bicolim}_{\alpha\in\Gamma} Hom(C_i, D_\alpha)$: the functor in question is

$$G : J^{op} \longrightarrow \mathcal{C}at$$

$$i \longmapsto \operatorname*{bicolim}_{\alpha\in\Gamma} Hom(C_i, D_\alpha)$$

$$j \xrightarrow{\phi^{op}} i \longmapsto X(\phi)^*$$

where $X(\phi)^*(f, \alpha) = (f \circ X(\phi), \alpha)$ and $X(\phi)^*(u, h, v) = (u, h \circ id_{X(\phi)}, v)$. It can be checked that $X(\phi)^*$ is well defined in $\operatorname*{bicolim}_{\alpha\in\Gamma} Hom(C_j, D_\alpha)$.

Then, we can describe $\operatorname*{bilim}_{i\in J^{op}} \operatorname*{bicolim}_{\alpha\in\Gamma} Hom(C_i, D_\alpha)$ as follows:

Objects: $(f_i, \alpha_i)_{i\in J} |\ f_i : C_i \to D_{\alpha_i}$ and $\forall i \xrightarrow{\phi} j \in J$ an isomorphism $(f_i, \alpha_i) \xrightarrow{\varphi_\phi} \phi^*(f_j, \alpha_j)$ given in a functorial way. But, taking $\varphi(i) = \alpha_i$, by definition, this is exactly what it says in the statement.

Morphisms: To check that the characterization of the morphisms is the one in the statement, it is enough to follow the definitions. □

The definitions of pseudocone and bicolimit can be extended naturally to functors taking values in $\mathcal{C}at^{\mathcal{C}^{op}}$. Since bicolimits in $\mathcal{C}at^{\mathcal{C}^{op}}$ are computed by computing the corresponding bicolimit in $\mathcal{C}at$ for each $C \in \mathcal{C}$, we have the following lemma which is a consequence of Theorem 1.19 in [3].

Lemma 6.6. *Let $F : J \to \mathcal{C}at^{\mathcal{C}^{op}}$ a functor, and $X \in \mathcal{C}at^{\mathcal{C}^{op}}$. We have an isomorphism of categories*

$$Hom(\operatorname*{bicolim}_{i\in J} F(i), X) \xrightarrow{\cong} \mathcal{PC}(F, X) \xrightarrow{\cong} \operatorname*{bilim}_{i\in J^{op}} Hom(F(i), X).$$

where $\mathcal{PC}(F, X)$ stands for the category of pseudocones for the functor F with vertex X and the morphisms of pseudocones between them.

□

Here, a complete treatment generalizing the case of 1-categories developed in section 3 is in order. However, we leave it for future work, and develope here directly the generalization of proposition 3.14 to the 2-categories case:

Proposition 6.7. *Let X and Y be two Ind-objects of $\mathcal{C}$, and let $F = \underset{i\in J}{bicolim}\, Hom(-, C_i)$ and $G = \underset{\alpha\in\Gamma}{bicolim}\, Hom(-, D_\alpha)$. Then there is an isomorphism of categories between $Hom(F, G)$ in $\mathcal{C}at^{\mathcal{C}^{op}}$ and $Hom(X, Y)$ in $Ind(\mathcal{C})$.*

Proof. We construct the bijections between the objects and the arrows of both categories composing the ones below:
On objects:

$$(\theta_i)_{i\in J} \in \underset{i\in J^{op}}{\text{bilim}}\ \underset{\alpha\in\Gamma}{\text{bicolim}}\, Hom(C_i, D_\alpha)$$

bilimits in $\mathcal{C}at$

For each $i \in J$, $\theta_i \in G(C_i)$ and $\forall i \xrightarrow{\phi} j \in J$,
an isomorphism $\theta_i \xrightarrow{\varphi_\phi} G(X(\phi))(\theta_j)$ in a functorial way.

Yoneda

A family of natural transformations $Hom(-, C_i) \overset{\tilde{\theta}_i}{\Rightarrow} G$
and $\forall i \xrightarrow{\phi} j \in J$, an invertible 2-cell of $\mathcal{C}at^{\mathcal{C}^{op}}$

$$\begin{array}{ccc} Hom(-, C_i) & \overset{\tilde{\theta}_i}{\Longrightarrow} & G \\ X(\phi)_* \Downarrow & \cong \Uparrow \varphi^\phi & \nearrow \tilde{\theta}_j \\ Hom(-, X_j) & & \end{array}$$

given in a functorial way (i.e.. a pseudocone $\tilde{\theta}$
for the functor which sends $i \mapsto Hom(-, C_i)$).

u.p. of F

$$\underset{i\in J}{\text{bicolim}}\, Hom(-, C_i) \overset{\tilde{\theta}}{\Rightarrow} \underset{\alpha\in\Gamma}{\text{bicolim}}\, Hom(-, D_\alpha)$$

where, in the "Yoneda" biyection φ^ϕ is defined by $(\varphi^\phi_C)_f = Gf(\varphi^{-1}_\phi)$ for $C \in \mathcal{C}$ and $f : C \to C_i$; and in the way up φ_ϕ is defined as $\varphi_\phi = (\varphi^\phi_{C_i})_{id_{C_i}}$.

On arrows:

$$\theta \xrightarrow{\psi} \theta' \text{ in } \operatorname*{bilim}_{i \in J^{op}} \operatorname*{bicolim}_{\alpha \in \Gamma} Hom(C_i, D_\alpha)$$

bilimits in $\mathcal{C}at$ ____

For each $i \in J$, $\theta_i \xrightarrow{\psi_i} \theta'_i$ in GC_i such that
$\forall i \xrightarrow{\phi} j \in J$, the following diagram commutes

$$\begin{array}{ccc} \theta_i & \xrightarrow{\psi_i} & \theta'_i \\ \downarrow{\scriptstyle \varphi_\phi} & & \downarrow{\scriptstyle \varphi'_\phi} \\ G(X(\phi))(\theta_j) & \xrightarrow{G(X(\phi))(\psi_j)} & G(X(\phi))(\theta'_j) \end{array}$$

Yoneda ____

For each $i \in J$, a 2-cell of $\mathcal{C}at^{\mathcal{C}^{op}}$

$$Hom(-, C_i) \underset{\tilde{\theta}_j}{\overset{\tilde{\theta}_i}{\Longrightarrow}} \Downarrow \tilde{\psi}_i \; G$$

satisfying PCM (i.e.. a morphism of
pseudocones $\tilde{\psi}$ between $\tilde{\theta}$ and $\tilde{\theta}'$).

Lemma 6.6 ____

a 2-cell of $\mathcal{C}at^{\mathcal{C}^{op}}$

$$\operatorname*{bicolim}_{i \in J} Hom(-, C_i) \underset{\tilde{\theta}}{\overset{\tilde{\theta}}{\Longrightarrow}} \Downarrow \tilde{\psi} \; \operatorname*{bicolim}_{\alpha \in \Gamma} Hom(-, D_\alpha)$$

where, in the "Yoneda" biyection $\tilde{\psi_i}$ is defined by $((\tilde{\psi_i})_C)_f = Gf(\psi_i)$ for $C \in \mathcal{C}$ and $f : C \to C_i$; and in the way up ψ_i is defined as $\psi_i = ((\tilde{\psi_i})_{C_i})_{id_{C_i}}$. □

Corollary 6.8. *$Ind(\mathcal{C})$ is a 2-category.*

Proof. Its 0-cells are the Ind-objects of $\mathcal{C}$, and we have the category $Hom(X, Y)$ defined above (6.3). Finally, since $\mathcal{C}at^{\mathcal{C}^{op}}$ is a 2-category, we can compose there with its horizontal composition, using the isomorphism given in the previous proposition to go back and forth. □

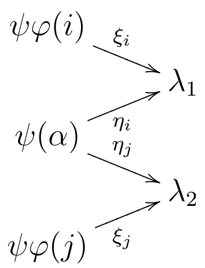
As an example, we are going to give an explicit formula for the vertical composition in $Hom(X,Y)$ and for horizontal composition on the objects using the description given in proposition 6.5.

The vertical composition is given by $(\theta_i)_{i\in J} \circ (\eta_i)_{i\in J} = (\theta_i \circ \eta_i)_{i\in J}$.

If $X = (C_i)_{i\in J}, Y = (D_\alpha)_{\alpha\in\Gamma}, Z = (E_\lambda)_{\lambda\in\Lambda}$, the horizontal composition on objects is given by: let $(\psi, (g_\alpha)_{\alpha\in\Gamma}, (\psi_u)_{\alpha \xrightarrow{u} \beta\in\Gamma})$ be an object of $Hom(Y,Z)$ and

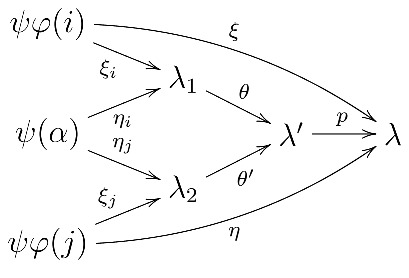
$(\varphi, (f_i)_{i\in J}, (\varphi_\phi)_{i\xrightarrow{\phi} j\in J})$ an object of $Hom(X,Y)$. Then, given $i \xrightarrow{\phi} j \in J$ we have $\varphi(i) \xrightarrow{u} \alpha \xleftarrow{v} \varphi(j)$ and an isomorphism $Y(u) \circ f_i \xrightarrow{\varphi_\phi} Y(v) \circ f_j \circ X(\phi)$.

Since $\varphi(i) \xrightarrow{u} \alpha$ and $\varphi(j) \xrightarrow{v} \alpha$ are in Γ, we have the corresponding ψ_u and ψ_v which give us



.

And, since Γ is filtered, we have



where $p\theta\eta_i = p\theta'\eta_j$.Then, we define

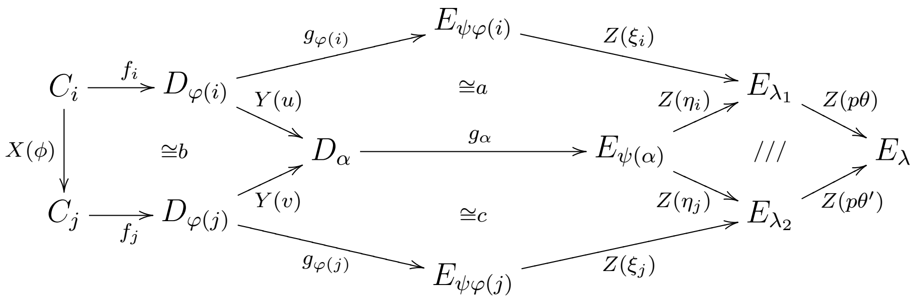
$$*((\psi, (g_\alpha)_{\alpha\in\Gamma}, (\psi_u)_{\alpha \xrightarrow{u} \beta\in\Gamma}), (\varphi, (f_i)_{i\in J}, (\varphi_\phi)_{i\xrightarrow{\phi} j\in J})) =$$

$$(\psi\circ\varphi, (g_{\varphi(i)} \circ f_i)_{i\in J}, ((\psi\circ\varphi)_\phi)_{i\xrightarrow{\phi} j\in J})$$

where $(\psi\circ\varphi)_\phi = (\xi, (\psi\circ\varphi)_\phi, \eta)$ with

$$(\psi\circ\varphi)_\phi : Z(p\theta\xi_i) \circ g_{\varphi(i)} \circ f_i \to Z(p\theta'\xi_j) \circ g_{\varphi(j)} \circ f_j \circ X(\phi)$$

given by the following diagram:



where the composition has to be made this way: first compose a with b in $Hom(C_i, E_{\lambda_1})$, and c with a in $Hom(C_i, E_{\lambda_2})$. And then compose in $Hom(C_i, E_\lambda)$ the second of those compositions, a^{-1} and the first one.

It would be very tedious to prove that this is indeed the horizontal composition that we mentioned before. But, after thinking about it a while, we are convinced that this is the only way to do it.

7 Bilimits and bicolimits in $\mathcal{C}at$ indexed by a 2-category

In this section we construct the bilimits and bicolimits in $\mathcal{C}at$ indexed by a 2-category. We consider the minimal developments necessary to achieve this aim. The definitions given are forced by the formalisms of 2-categories, but we have no reference except for the construction in [3] of bicolimits of 2-filtered diagrams of categories. A complete treatment of this constructions would need a generalization of Grothendieck's theory of fibered categories to a theory of 2-fibered 2-categories.

7.1 Bilimits

In this subsection, we are going to give an explicit characterization of the bilimits in $\mathcal{C}at$ associated to a 2-functor $F : \mathcal{A}^{op} \to \mathcal{C}at$. Throughout this subsection, $\mathcal{A}$ is going to be a 2-category and $\mathcal{A}^{op}$ is defined as follows:

0-cells: The 0-cells of $\mathcal{A}$.

1-cells: The 1-cells of $\mathcal{A}$ inverted.

2-cells: The 2-cells of $\mathcal{A}$.

More specifically, if the configuration of $\mathcal{A}$ is

$$\begin{array}{ccccc} & \xrightarrow{f} & & \xrightarrow{u} & \\ & \Downarrow\alpha & & \Downarrow\gamma & \\ A & \xrightarrow{g} & B & \xrightarrow{v} & C \\ & \Downarrow\beta & & \Downarrow\delta & \\ & \xrightarrow[h]{} & & \xrightarrow[w]{} & \end{array}$$

then the configuration of $\mathcal{A}^{op}$ is

$$\begin{array}{ccccc} & \xleftarrow{f} & & \xleftarrow{u} & \\ & \Downarrow\alpha & & \Downarrow\gamma & \\ A & \xleftarrow{g} & B & \xleftarrow{v} & C \\ & \Downarrow\beta & & \Downarrow\delta & \\ & \xleftarrow[h]{} & & \xleftarrow[w]{} & \end{array} .$$

Definition 7.1. *Let $F : \mathcal{A}^{op} \to \mathcal{C}at$ be a 2-functor. We define $\underset{A \in \mathcal{A}^{op}}{bilim} FA$ as a universal pseudocone for F. (see remark 4.16)*

Proposition 7.2. *$\underset{A \in \mathcal{A}^{op}}{bilim} FA$ can be described as follows:*

Objects: $(x_A)_{A\in\mathcal{A}}$ | $x_A \in FA$ *and* $\forall A \overset{u}{\to} B \in \mathcal{A}$, *an isomorphism* $x_A \overset{\varphi_u}{\to} u^*(x_B)$ *in* FB *given in a functorial way.*

Morphisms: *A morphism* $f : (x_A)_{A\in\mathcal{A}} \to (y_A)_{A\in\mathcal{A}}$ *is a family* $(x_A \overset{f_A}{\to} y_A)_{A\in\mathcal{A}}$ *such that* $\forall A \overset{u}{\to} B \in \mathcal{A}$,

$$\begin{array}{ccc} x_A & \xrightarrow{f_A} & y_A \\ \downarrow{\scriptstyle \varphi_u} & & \downarrow{\scriptstyle \psi_u} \\ F(u)(x_B) & \xrightarrow[F(u)(f_B)]{} & F(u)(y_B) \end{array}$$

commutes and

$$\forall u \overset{\gamma}{\Rightarrow} v, \quad \begin{array}{ccc} x_A & \xrightarrow{\varphi_u} & u^* x_B \\ \downarrow{\scriptstyle \varphi_v} & \swarrow{\scriptstyle (\gamma^*)_{x_B}} & \\ v^* x_B & & \end{array} \text{ commutes.} \tag{7.1}$$

And the composition is given by: $(f_A)_{A\in\mathcal{A}} \circ (g_A)_{A\in\mathcal{A}} = (f_A \circ g_A)_{A\in\mathcal{A}}$.

Proof. We have to prove that the category described in the statement is a universal pseudocone for F:

We define $h_A((x_A)_{A\in\mathcal{A}}) = x_A$, $h_A((x_A \overset{f_A}{\to} y_A)_{A\in\mathcal{A}}) = f_A$ and $(h_u)_{(x_A)_{A\in\mathcal{A}}} = \varphi_u$. It's clear that $\underset{A\in\mathcal{A}^{op}}{\text{bilim}} FA \overset{h}{\Rightarrow} F$ satisfies PC0 and PC1 of the definition of pseudocone; and PC2 is given by the condition (7.1). Let's check that it is universal: Let $\mathcal{X} \overset{\lambda}{\Rightarrow} F$ be another pseudocone. One can check the universality by taking $\tilde{h}(x) = (\lambda_A(x))_{A\in\mathcal{A}}$ with $\varphi_u = (\lambda_u)_x$ and $\tilde{h}(f) = (\lambda_A(f))_{A\in\mathcal{A}}$.

□

7.2 Bicolimits

In this subsection, we are going to give an explicit construction of the 2-filtered bicolimits in $\mathcal{C}at$ associated to a 2-functor $F : \mathcal{A} \to \mathcal{C}at$. For this purpose, we take as reference [3]. All the proofs that we omit in here are given in that article.

Definition 7.3. *Let* $F : \mathcal{A} \to \mathcal{C}at$ *be a 2-functor. We define* $\underset{A\in\mathcal{A}}{\text{bicolim}} FA$ *as a universal pseudocone for* F *(see definition 4.15).*

From now on $\mathcal{A}$ is going to be a pre-2-filtered 2-category.

Before describing the category $\operatorname*{bicolim}_{A\in\mathcal{A}} FA$, we are going to define a quasicategory $\mathcal{L}(F)$ (see [3]):[1]

Definition 7.4. . *We define $\mathcal{L}(F)$ as follows:*

Objects: (x, A) with $x \in FA$

Premorphisms: A premorphism between (x, A) and (y, B) is a triplet (u, ξ, v) where $A \xrightarrow{u} C$, $B \xrightarrow{v} C$ in $\mathcal{A}$ and $F(u)(x) \xrightarrow{\xi} F(v)(y)$ in FC.

Homotopies: A homotopy between two premorphisms (u_1, ξ_1, v_1) and (u_2, ξ_2, v_2) is a quadruple $(w_1, w_2, \alpha, \beta)$ where $C_1 \xrightarrow{w_1} C$, $C_2 \xrightarrow{w_2} C$ are 1-cells of $\mathcal{A}$ and $w_1v_1 \xrightarrow{\alpha} w_2v_2$, $w_1u_1 \xrightarrow{\beta} w_2u_2$ are invertible 2-cells of $\mathcal{A}$ such that the following diagram commutes in FC:

$$\begin{array}{ccc}
F(w_1)F(u_1)(x) = F(w_1u_1)(x) & \xrightarrow{F(\beta)x} & F(w_2)F(u_2)(x) = F(w_2u_2)(x) \\
\downarrow {\scriptstyle F(w_1)(\xi_1)} & & \downarrow {\scriptstyle F(w_2)(\xi_2)} \\
F(w_1)F(v_1)(x) = F(w_1v_1)(y) & \xrightarrow[F(\alpha)y]{} & F(w_2)F(v_2)(x) = F(w_2v_2)(y)
\end{array}$$

Definition 7.5. *We say that two premorphisms ξ_1, ξ_2 are equivalent if there is a homotopy between them. In that case, we write $\xi_1 \sim \xi_2$.*

The proof of the following proposition which is the corresponding to proposition 5.30 can be found in [3] with all the details.

Proposition 7.6. *$\operatorname*{bicolim}_{A\in\mathcal{A}} FA$ can be described as follows:*

Objects: (x, A) with $x \in FA$.

Morphisms: The equivalence classes of premorphisms of $\mathcal{L}(F)$.

Composition: Is defined by composing representative premorphisms.

□

[1] We use here the term *quasicategory* in a naive way. What we mean is explicitly described in definition 7.4. No claim is made here that this corresponds to the notion of quasicategory which is found in the literature.

8 2-Ind-Objects of a 2-category $\mathcal{C}$

In this section we lay the foundations for future work and introduce the notion of 2-Ind-object. We believe that it can be proved that $2 - Ind(\mathcal{C})$ is a 2-category in a similar way as it is proved that $Ind(\mathcal{C})$ is a 2-category in section 3.

Definition 8.1. *Let $\mathcal{C}$ be any 2-category. A 2-Ind-object of $\mathcal{C}$ is a 2-filtered system $X = (C_i)_{i \in J}$ i.e. a 2-functor $X : J \to \mathcal{C}$ with J a 2-filtered 2-category (In particular, J is pre-2-filtered).*

Definition 8.2. *Let $\mathcal{C}$ be a 2-category and $X = (C_i)_{i \in J}$, $Y = (D_\alpha)_{\alpha \in \Gamma}$ two 2-Ind-objects of $\mathcal{C}$. We define the morphisms between X and Y as follows:*

$$Hom(X, Y) = \underset{i \in J^{op}}{bilim}\ \underset{\alpha \in \Gamma}{bicolim}\, Hom(C_i, D_\alpha).$$

Remark 8.3. $Hom(X, Y)$ results a category.

□

Proposition 8.4. *An object of $Hom(X, Y)$ is a triplet $(\varphi, (f_i)_{i \in J}, (\varphi_\phi)_{i \xrightarrow{\phi} j \in J})$ where $\varphi : J \to \Gamma$ is a function between the objects of J and the ones of Γ, $f_i : C_i \to D_{\varphi(i)}$ morphisms of $\mathcal{C}$ and $\varphi_\phi : (f_i, \varphi(i)) \to \phi^*(f_j, \varphi(j))$ isomorphisms of $\underset{\alpha \in \Gamma}{bicolim}\, Hom(C_i, D_\alpha)$ given in a functorial way, i.e. $\forall i \xrightarrow{\phi} j$ $\exists\ \begin{array}{l} \varphi(i) \xrightarrow{u} \\ \varphi(j) \xrightarrow{v} \end{array} \alpha$ and an isomorphism $Y(u) \circ f_i \xrightarrow{\varphi_\phi} Y(v) \circ f_j \circ X(\phi)$:*

$$\begin{array}{ccccc} C_i & \xrightarrow{f_i} & D_{\varphi(i)} & & \\ & & & \searrow^{Y(u)} & \\ \downarrow{\scriptstyle X(\phi)} & \cong \Downarrow \varphi_\phi & & & D_\alpha \\ & & & \nearrow_{Y(v)} & \\ C_j & \xrightarrow[f_j]{} & D_{\varphi(j)} & & \end{array}.$$

A morphism of $Hom(X, Y)$ is an equivalence class according to the relation $\sim$ of a family $(\theta_i)_{i \in J}$ where $\theta_i : (f_i, \varphi(i)) \to (g_i, \psi(i))$ is a morphism of $\underset{\alpha \in \Gamma}{bicolim}\, Hom(C_i, D_\alpha)$ such that $\forall i \xrightarrow{\phi} j \in J$, $\psi_\phi \circ \theta_i = \phi^*(\theta_j) \circ \varphi_\phi$ and $\forall \phi \overset{\eta}{\Rightarrow} \phi', id_{f_j} \circ X(\eta) \circ \varphi_\phi = \varphi_{\phi'}$, i.e. $\exists\ \begin{array}{l} \varphi(i) \xrightarrow{u} \\ \psi(i) \xrightarrow{v} \end{array} \alpha$ and a morphism $Y(u) \circ f_i \xrightarrow{\theta_i} Y(v) \circ g_i$ such that $\forall i \xrightarrow{\phi} j \in J$ the following diagram commutes:

$$\begin{array}{ccc} Y(u)\circ f_i & \xrightarrow{\varphi_\phi} & Y(v)\circ f_j \circ X(\phi) \\ {\scriptstyle \theta_i}\downarrow & & \downarrow{\scriptstyle \phi^*(\theta_j)} \\ Y(v)\circ g_i & \xrightarrow[\psi_\phi]{} & Y(v)\circ g_j\circ X(\phi) \end{array}$$

and $\forall \phi \overset{\eta}{\Rightarrow} \phi'$ *the following diagram commutes:*

$$\begin{array}{ccc} (f_i, \varphi(i)) & \xrightarrow{\varphi_{\phi'}} & (f_j\circ X(\phi'), \varphi(j)) \\ {\scriptstyle \varphi_\phi}\downarrow & \nearrow{\scriptstyle id_{f_j}\circ X(\eta)} & \\ (f_j\circ X(\phi), \varphi(j)) & & \end{array}.$$

The relation $\sim$ *is defined as* $(\theta_i)_{i\in J} \sim (\eta_i)_{i\in J}$ *if and only if* $\forall i \in J$, $\exists\ \alpha_1 \xrightarrow{w_1} \alpha \xleftarrow{w_2} \alpha_2$ *and* $\theta : w_1v_1 \Rightarrow w_2v_2$, $\eta : w_1u_1 \Rightarrow w_2u_2$ *invertible 2-cells such that the following diagram commutes:*

$$\begin{array}{ccc} Y(w_1u_1)\circ f_i & \xrightarrow{Y(\eta)\circ id_{f_i}} & Y(w_2u_2)\circ f_i \\ {\scriptstyle Y(w_1)\circ\theta_i}\downarrow & & \downarrow{\scriptstyle Y(w_2)\circ\eta_i} \\ Y(w_1v_1)\circ f_j\circ X(\phi) & \xrightarrow[Y(\theta)\circ id_{f_j\circ X(\phi)}]{} & Y(w_2v_2)\circ f_j\circ X(\phi) \end{array}$$

where $(\theta_i)_{i\in J}$ *is given by* $\varphi(i) \xrightarrow{u_1} \alpha_1 \xleftarrow{v_1} \psi(i)$ *and* $Y(u_1)\circ f_i \xrightarrow{\theta_i} Y(v_1)\circ g_i$, $(\eta_i)_{i\in J}$ *is given by* $\varphi(i) \xrightarrow{u_2} \alpha_2 \xleftarrow{v_2} \psi(i)$ *and* $Y(u_2)\circ f_i \xrightarrow{\eta_i} Y(v_2)\circ g_i$.

Proof. First of all, let's calculate $\underset{\alpha\in\Gamma}{\text{bicolim}}\, Hom(C_i, D_\alpha)$ for $i \in J$ fixed: The functor in question is

$$\begin{array}{rcl} \Gamma & \longrightarrow & \mathcal{C}at \\ \alpha & \longmapsto & Hom(C_i, D_\alpha) \\ \alpha \xrightarrow{u} \beta & \longmapsto & (Y(u))_* \\ u \overset{\theta}{\Rightarrow} v & \longmapsto & (Y(\theta))_* \end{array}$$

where $((Y(\theta))_*)_f = Y(\theta)\circ id_f$.

Then, we can describe $\operatorname{bicolim}_{\alpha \in \Gamma} Hom(C_i, D_\alpha)$ as follows:

Objects: $(f, \alpha) \mid f : C_i \to D_\alpha$.

Morphisms: A morphism between (f, α) and (g, β) is a triplet (u, ξ, v) where $\begin{array}{l} \alpha \xrightarrow{u} \\ \beta \xrightarrow[v]{} \end{array} \gamma$ and $Y(u) \circ f \xrightarrow{\xi} Y(v) \circ g$, quotient by $\sim$ where $(u, \xi, v) \sim (u', \xi', v')$ if and only if $\exists$ a homotopy (w_1, w_2, θ, η) between them, i.e.

$$\begin{array}{l} \gamma \xrightarrow{w_1} \\ \gamma' \xrightarrow[w_2]{} \end{array} \delta$$

and invertible 2-cells $\theta : w_1 v \Rightarrow w_2 v', \eta : w_1 u \Rightarrow w_2 u'$ such that the following diagram commutes:

$$\begin{array}{ccc} Y(w_1 u) \circ f & \xrightarrow{Y(\eta) \circ id_f} & Y(w_2 u') \circ f \\ {\scriptstyle Y(w_1) \circ \xi} \downarrow & & \downarrow {\scriptstyle Y(w_2) \circ \xi'} \\ Y(w_1 v) \circ g & \xrightarrow[Y(\theta) \circ id_g]{} & Y(w_2 v') \circ g \end{array} .$$

Now, let's calculate $\operatorname{bilim}_{i \in J^{op}} \operatorname{bicolim}_{\alpha \in \Gamma} Hom(C_i, D_\alpha)$: The functor in question is

$$\begin{array}{rcl} G : J^{op} & \longrightarrow & \mathcal{C}at \\ i & \longmapsto & \operatorname{bicolim}_{\alpha \in \Gamma} Hom(C_i, D_\alpha) \\ j \xrightarrow{\phi^{op}} i & \longmapsto & X(\phi)^* \\ \phi^{op} \overset{\theta}{\Rightarrow} \phi'^{op} & \longmapsto & X(\theta)^* \end{array}$$

where $X(\phi)^*(f, \alpha) = (f \circ X(\phi), \alpha)$, $X(\phi)^*(u, \xi, v) = (u, \xi \circ id_{X(\phi)}, v)$ and $(X(\theta)^*)_{(f,\alpha)} = (id, id_f \circ X(\theta), id)$. It can be check that $X(\phi)^*$ is well defined in $\operatorname{bicolim}_{\alpha \in \Gamma} Hom(C_j, D_\alpha)$.

Then, we can describe $\operatorname{bilim}_{i \in J^{op}} \operatorname{bicolim}_{\alpha \in \Gamma} Hom(C_i, D_\alpha)$ as follows:

Objects: $(f_i, \alpha_i)_{i \in J} \mid f_i : C_i \to D_{\alpha_i}$ and $\forall i \xrightarrow{\phi} j \in J$ an isomorphism $(f_i, \alpha_i) \xrightarrow{\varphi_\phi} \phi^*(f_j, \alpha_j)$ given in a functorial way. But, taking $\varphi(i) = \alpha_i$, by definition, this is exactly what it says in the statement.

Morphisms: To check that the characterization of the morphisms is the one in the statement, one only has to follow the definitions. $\square$

We are going to define $2 - Ind(\mathcal{C})$ as the 2-category with 0-cells the 2-Ind-objects of $\mathcal{C}$, 1-cells the objects of $Hom(X, Y)$ for X, Y 2-Ind-objects and 2-cells the morphisms of $Hom(X, Y)$ for X, Y 2-Ind-objects. It remains to prove, in future work, that there is a horizontal composition in $2 - Ind(\mathcal{C})$ which would complete the proof of the fact that $2 - Ind(\mathcal{C})$ is a 2-category.

A question which arises naturally is to find the relationship between the underlying category of $Ind(\mathcal{C})$ and $2 - Ind(\mathcal{C})$ respectively and the category of Ind-objects of the underlying category of $\mathcal{C}$.

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