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Puntos fijos de acciones y funciones en 2–complejos

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Puntos fijos de acciones y funciones en 2-complejos

En esta tesis buscamos comprender cómo los invariantes algebraicos de un espacio X dan información acerca de la existencia de puntos fijos de una función o acción de un grupo en X, especialmente cuando X es de dimensión 2.

Probamos que el grupo fundamental de un complejo simplicial finito de dimensión 2 con la propiedad del punto fijo y característica de Euler par no puede ser abeliano o un subgrupo finito de $\mathrm{SO}(3)$. Damos un ejemplo de un complejo simplicial finito de dimensión 2 con la propiedad del punto fijo y característica de Euler igual a 2. También probamos que la propiedad del punto fijo no es un invariante homotópico para complejos simpliciales finitos de dimensión 2. Estos resultados responden dos preguntas que formuló R.H. Bing en 1969.

Un resultado de J.P. Serre dice que toda acción de un grupo finito en un árbol tiene un punto fijo. C. Casacuberta y W. Dicks conjeturaron que un grupo que actúa en un 2-complejo finito y contráctil tiene un punto fijo. La misma pregunta fue realizada independientemente por Aschbacher y Segev. Estudiamos la conjetura de Casacuberta-Dicks desde diferentes puntos de vista, partiendo de la clasificación dada por Oliver y Segev de los grupos finitos que actúan sin puntos fijos en un 2-complejo acíclico. Probamos que, módulo un caso particular de la conjetura de Kervaire–Laudenbach–Howie, si la conjetura de Casacuberta–Dicks resulta falsa, existe un contraejemplo de una forma particular. Utilizando un resultado de Brown que extiende la teoría de Bass-Serre a 2-complejos, traducimos la conjetura de Casacuberta–Dicks para A_5 en una pregunta de teoría combinatoria de grupos, cercana al *relation gap problem*. Probamos que algunos casos del problema obtenido se siguen del trabajo de Klyachko en ecuaciones sobre grupos. A través de experimentos computacionales analizamos los posibles grupos fundamentales de los G-complejos acíclicos de dimensión 2 sin puntos fijos que son potenciales contraejemplos. También probamos que ciertos grupos superperfectos π no aparecen de esta forma.

El complejo de curvas $\mathcal{C}(S_g)$ de una superficie orientada S_g de género g fue introducido por Harvey como un análogo de los Tits buildings para mapping class group $\mathrm{Mod}(S_g)$. Dado que hay una analogía entre $\mathrm{Aut}(F_n)$ y $\mathrm{Mod}(S_g)$, es natural buscar un análogo de $\mathcal{C}(S_g)$ en este contexto. Probamos que un posible análogo, el complejo simplicial $\mathrm{PB}(F_n)$ con símplices las bases parciales no vacías del grupo libre de rango n es Cohen-Macaulay y por lo tanto tiene el tipo homotópico de un wedge de $(n-1)$-esferas.

Palabras clave: 2-complejos, presentaciones de grupo, puntos fijos, acciones de grupo, propiedad del punto fijo.

Fixed points of maps and actions on 2-complexes

In this thesis we seek to understand how the algebraic invariants of a space X give information on the existence of fixed points of a mapping or group action on X, specially when X is 2-dimensional.

We prove the fundamental group of a finite 2-dimensional simplicial complex with the fixed point property and even Euler characteristic cannot be abelian or a finite subgroup of $\mathrm{SO}(3)$. We give an example of a finite 2-dimensional simplicial complex with the fixed point property and Euler characteristic 2. We also prove that the fixed point property is not a homotopy invariant for 2-dimensional finite simplicial complexes. These results answer two questions raised by R.H. Bing in 1969.

A result of J.P. Serre states that a finite group acting on a tree has a fixed point. C. Casacuberta and W. Dicks conjectured that a group acting on a finite contractible 2-complex has a fixed point. The same question was raised independently by Aschbacher and Segev. We study the Casacuberta-Dicks conjecture from different points of view, parting from Oliver and Segev's classification of the finite groups which act fixed point freely on a 2-dimensional acyclic complex. We prove that, modulo a special case of the Kervaire–Laudenbach–Howie conjecture, if the Casacuberta–Dicks conjecture fails, there is a counterexample of a particular form. Using a result of Brown which extends Bass–Serre theory to 2-complexes, we translate the Casacuberta–Dicks conjecture for the group A_5 into a question in combinatorial group theory, closely related to the relation gap problem. We prove that some cases of the resulting problem follow from the work of Klyachko on equations over groups. We use computer experimentation to analyze the possible fundamental groups of the 2-dimensional fixed point free acyclic G-complexes which are potential counterexamples. We also prove that certain superperfect groups π do not arise as fundamental groups in this way.

The curve complex $\mathcal{C}(S_g)$ of an oriented surface S_g of genus g was introduced by Harvey as an analogue of Tits buildings for the mapping class group $\mathrm{Mod}(S_g)$. Since there is an analogy between $\mathrm{Aut}(F_n)$ and $\mathrm{Mod}(S_g)$, it is natural to seek for an analogue of $\mathcal{C}(S_g)$ in this context. We prove that a possible analogue, the simplicial complex $\mathrm{PB}(F_n)$ with simplices the nonempty partial bases of the free group of rank n is Cohen-Macaulay and thus has the homotopy type of a wedge of $(n-1)$-spheres.

Keywords: 2-complexes, group presentations, fixed points, group actions, fixed point property.

Introducción

Los resultados de la teoría de homotopía en dimensión 2 y la teoría geométrica de grupos establecen conexiones profundas entre problemas algebraicos y problemas topológicos o geométricos. Con frecuencia, es posible traducir nociones de naturaleza geométrica – por ejemplo la idea de curvatura – a un contexto más topológico, combinatorio o algebraico, proveyendo herramientas poderosas para estudiar problemas que a priori no parecen geométricos. El funtor grupo fundamental, un invariante de los espacios topológicos punteados que toma valores en la categoría de grupos, provee una conexión entre topología y teoría de grupos. Este puente se manifiesta, por ejemplo, en la correspondencia entre tipos homotópicos de 2-complejos y presentaciones de grupo. Entre los problemas abiertos del área se destacan la conjetura de Whitehead [Whi41], la conjetura de Eilenberg-Ganea [EG57], la conjetura de Zeeman [Zee64], la conjetura de Andrews-Curtis [AC65], la conjetura de Kervaire–Laudenbach–Howie [How81], el $D(2)$-*problem* y el *relation gap problem* [Har18]. En esta tesis buscamos comprender cómo los invariantes algebraicos de un espacio X dan información sobre la existencia de puntos fijos de una función o una acción en X, especialmente cuando X es de dimensión 2.

La tesis se divide en tres capítulos. Los resultados principales del Capítulo 1, que se presentan en las Secciones 1.3 y 1.4, aparecieron en los artículos [BSC17] (escrito en colaboración con J.A. Barmak) y [SC17b] respectivamente. El contenido de las Secciones 1.1 a 1.6 también formó parte de la Tesis de Licenciatura [SC15]. El contenido de la Sección 1.7 es inédito y provee una demostración alternativa, no asistida por computadora, de los resultados de la Sección 1.4. Los resultados del Capítulo 2 son inéditos. En la Sección 2.3 se presentan los resultados del artículo en preparación [PSCV18], escrito en colaboración con K. Piterman y A. Viruel. Los resultados del Capítulo 3 aparecieron en [SC17a].

En el Capítulo 1 estudiamos la propiedad del punto fijo para 2-complejos. Recordemos que un espacio X tiene la *propiedad del punto fijo* si toda función continua $f\colon X \to X$ tiene un punto fijo. Por ejemplo, el teorema de punto fijo de Brouwer dice que los discos tienen la propiedad del punto fijo. Algunas preguntas básicas sobre la propiedad del punto fijo permanecieron abiertas por mucho tiempo. Kuratowski [Kur30] se preguntó si el producto de dos espacios con la propiedad del punto fijo necesariamente tiene la propiedad del punto fijo. Pasaron casi cuarenta años hasta que Lopez encontró un ejemplo de un poliedro compacto con la propiedad del punto fijo y característica de Euler par que le permitió responder la pregunta de Kuratowski por la negativa y probar que la propiedad del punto fijo no es un invariante homotópico de los poliedros compactos. Este ejemplo a su vez, motivó las siguientes dos preguntas, formuladas por R.H. Bing en su artículo "The elusive fixed point property" [Bin69], que estuvieron abiertas por 45 años [Hag07].

Pregunta (Pregunta 1 de Bing)**.** *¿Existe un poliedro compacto de dimensión* 2 *con la propiedad del punto fijo y característica de Euler par?*

Pregunta (Pregunta 8 de Bing)**.** *¿Cuál es el menor entero positivo n tal que la propiedad del punto fijo no es un invariante homotópico de los poliedros de dimensión menor o igual que n?*

La Pregunta 1 de Bing se trata de poliedros con característica de Euler par. Sin embargo hasta donde sabemos, todo ejemplo previamente conocido de un poliedro de dimensión 2 compacto con la propiedad del punto fijo era además $\mathbb{Q}$-acíclico y por lo tanto tenía característica de Euler igual a 1. En el Capítulo 1 presentamos resultados que responden estas preguntas.

Usando la clasificación de tipos homotópicos de 2-complejos compactos con grupo fundamental abeliano, un resultado que se debe a Browning, probamos el siguiente teorema.

Teorema 1.3.21 (Barmak–Sadofschi Costa)**.** *Un poliedro compacto de dimensión* 2 *con la propiedad del punto fijo y característica de Euler distinta de* 1 *no puede tener grupo fundamental abeliano.*

Con ideas similares probamos que el grupo fundamental de un tal espacio no puede ser un subgrupo finito de $\mathrm{SO}(3)$.

Teorema 1.3.22 (Barmak–Sadofschi Costa)**.** *Un poliedro compacto de dimensión* 2 *con la propiedad del punto fijo y característica de Euler distinta de* 1 *no puede tener grupo fundamental isomorfo a A_4, S_4, A_5 o D_n.*

Posteriormente construimos ejemplos de poliedros compactos de dimensión 2 con la propiedad del punto fijo y característica de Euler n para todo $n \geq 1$, dando una respuesta afirmativa a la Pregunta 1 de Bing. Si $n \leq 0$, de un resultado de Borsuk (Corollary 1.1.17) se sigue que no existe un tal ejemplo. El resultado de Borsuk en su formulación original habla de espacios más generales que los CW-complejos y la demostración es complicada. Aquí presentamos una demostración más simple utilizando un lenguaje más moderno.

Para responder estas preguntas introducimos la noción de *grupo de Bing*.

Definición 1.4.1. Sea G un grupo finitamente presentable tal que $H_1(G)$ es finito. Decimos que G es un *grupo de Bing* si o bien $H_2(G) = 0$ o llamando d_1 al primer factor invariante de $H_2(G)$, para todo endomorfismo $\phi\colon G \to G$ se tiene $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ en $\mathbb{Z}_{d_1}$.

El siguiente resultado permite producir ejemplos de 2-complejos con la propiedad del punto fijo.

Teorema 1.4.2. *Si $\mathcal{P}$ es una presentación eficiente de un grupo de Bing G entonces $X_{\mathcal{P}}$ tiene la propiedad del punto fijo.*

De esta forma reducimos el problema a encontrar ejemplos de grupos de Bing eficientes. Para un grupo finito simple G, dado que todo endomorfismo es o bien trivial o un automorfismo, es particularmente fácil chequear si es de Bing. Usando la clasificación de los grupos finitos simples probamos:

Teorema 1.5.1. *Los únicos grupos finitos simples de Bing G con $H_2(G) \neq 0$ son los grupos $D_{2m}(q)$con q impar y $m > 2$.*

De estos grupos $D_6(3)$ es el más chico y tiene orden 67628447006087702382529609728OO. Si alguno de estos grupos resultara eficiente daría un ejemplo de un poliedro de dimensión 2 con la propiedad del punto fijo y característica de Euler igual a 3. Con el fin de responder la Pregunta 1 de Bing originalmente usamos el software GAP para hallar un grupo de Bing de otra naturaleza.

Proposición 1.4.5. *El grupo G presentado por*

$$\mathcal{P} = \langle x,y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

es finito de orden 3^5. Se tiene $H_2(G) = \mathbb{Z}_3$, y por lo tanto $\mathcal{P}$ es eficiente. Más aún G es un grupo de Bing.

La demostración original de la Proposición 1.4.5 utiliza GAP. En cambio, en la Sección 1.7 presentamos una demostración alternativa y no asistida por computadora. Inmediatamente obtenemos:

Corolario 1.4.7. *El complejo $X_{\mathcal{P}}$ asociado a la presentación*

$$\mathcal{P} = \langle x,y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

tiene la propiedad del punto fijo. Más aún, $\chi(X_{\mathcal{P}}) = 2$.

En [Lop67], W. Lopez probó que la propiedad del punto fijo no es un invariante homotópico de los poliedros compactos. El espacio de Lopez da una cota superior de 17 para el entero n que aparece en la Pregunta 8 de Bing. Por otro lado, la propiedad del punto fijo es un invariante homotópico de los poliedros compactos de dimensión 1 (o sea, grafos finitos). Por lo tanto la respuesta a la Pregunta 8 de Bing es un número mayor o igual que 2. La siguiente consecuencia de Theorem 1.4.2 muestra que la respuesta a la Pregunta 8 de Bing es 2.

Teorema 1.4.12. *Existe un poliedro compacto Y de dimensión 2, sin la propiedad del punto fijo y tal que el poliedro X obtenido a partir de Y mediante un colapso elemental de dimensión 2 tiene la propiedad del punto fijo.*

Si un complejo X tiene la propiedad del punto fijo, toda acción de un grupo cíclico en X tiene un punto fijo (global). Los 2-complejos contráctiles y más generalmente, los 2-complejos racionalmente acíclicos tienen la propiedad del punto fijo. Los resultados del Capítulo 1 dan ejemplos de 2-complejos con esta propiedad que no son racionalmente acíclicos. Algunos 2-complejos tienen la propiedad del punto fijo con respecto a acciones de grupo; con esto nos referimos a que toda acción de un grupo tiene un punto fijo. Sin embargo hay 2-complejos acíclicos que no satisfacen esta propiedad. El grupo A_5 actúa simplicialmente y sin puntos fijos en la subdivisión baricéntrica X del 2-esqueleto de la esfera homológica de Poincaré, que es un 2-complejo acíclico. Un resultado famoso de J.P. Serre [Ser80] dice que toda acción de un grupo finito en un complejo contráctil de dimensión 1 (i.e. un árbol) tiene un punto fijo. Sin embargo los complejos contráctiles de dimensión 3 no comparten esta propiedad. Edwin E. Floyd y Roger W. Richardson [FR59] notaron que A_5 actúa simplicialmente y sin puntos fijos en el join $X * A_5$ del espacio mencionado anteriormente y el grupo discreto A_5, que resulta un complejo contráctil de dimensión 3. Más aún, viendo $X * A_5$ en $\mathbb{R}^{81}$ y tomando un entorno regular, probaron que A_5 actúa simplicialmente y sin puntos fijos en una triangulación del disco D^{81}. Este ejemplo era el único de su tipo hasta que Bob Oliver obtuvo una clasificación completa de los grupos que actúan sin puntos fijos en un disco D^n [Oli75]. El ejemplo de Floyd y Richardson muestra que el resultado de Serre no se puede extender a dimensión 3 y entonces resulta natural preguntarse qué ocurre en dimensión 2. Este problema abierto es el objeto de estudio del Capítulo 2. Carles Casacuberta y Warren Dicks formularon la siguiente conjetura [CD92].

Conjetura (Casacuberta–Dicks)**.** *Sea G un grupo. Si X es un G-complejo de dimensión 2, finito y contráctil entonces $X^G \neq \varnothing$.*

El enunciado original de la conjetura, dado por Casacuberta y Dicks, no requiere que X sea finito. En el caso finito, Aschbacher y Segev plantearon la misma pregunta [AS93a]. Aquí estudiamos la conjetura de Casacuberta–Dicks en el caso finito y por ese motivo la enunciamos de esta forma.

Casacuberta y Dicks probaron que la conjetura vale para los grupos resolubles. El mismo resultado fue probado independientemente por Yoav Segev [Seg93]. En [AS93a], Aschbacher y Segev usaron la clasificación de los grupos finitos simples para probar la conjetura para una clase amplia de grupos. En [Seg94] Segev probó que la conjetura vale cuando X es colapsable. En [OS02], Oliver y Segev dieron una clasificación completa de los grupos que pueden actuar sin puntos fijos en un 2-complejo acíclico. En [Cor01], Corson probó que toda acción de un grupo finito G en un 2-complejo simplemente conexo y diagramáticamente reducible (no necesariamente finito) tiene un punto fijo.

A lo largo del Capítulo 2 estudiamos esta conjetura desde diversos puntos de vista. Los resultados de [OS02] no solamente clasifican los grupos G que actúan sin puntos fijos en un 2-complejo acíclico sino que también dan una descripción de los 2-complejos acíclicos X donde G actúa sin puntos fijos. Usando estos resultados y asumiendo el siguiente caso particular de la conjetura de Kervaire–Laudenbach–Howie [How81, Conjecture] obtenemos una descripción de los posibles contraejemplos de la conjetura de Casacuberta–Dicks.

Conjetura 2.2.1. *Sea X un 2-complejo contráctil y finito. Si $A \subset X$ es un subcomplejo acíclico, entonces A es contráctil.*

Teorema 2.2.11. *Asumiendo la Conjetura 2.2.1, si la conjetura de Casacuberta–Dicks resulta falsa, debe existir un G-complejo X contráctil de dimensión 2, esencial y sin puntos fijos donde G es alguno de los siguientes grupos:*

(i) $\mathrm{PSL}_2(2^p)$ *con p primo.*

(ii) $\mathrm{PSL}_2(3^p)$ *con p un primo impar.*

(iii) $\mathrm{PSL}_2(q)$ *con $q > 3$ primo tal que* $q \equiv \pm 3 \text{ mód } 5$ *y* $q \equiv \pm 3 \text{ mód } 8$.

(iv) $\mathrm{Sz}(2^p)$ *con p un primo impar.*

Más aún, es posible tomar X obtenido a partir del grafo $\Gamma_{OS}(G)$ adjuntando $k \geq 0$ órbitas libres de 1-celdas y $k+1$ órbitas libres de 2-celdas.

El grafo $\Gamma_{OS}(G)$ que aparece en el Teorema 2.2.11 es cualquier elección posible del 1-esqueleto de un 2-complejo como los construidos por Oliver y Segev y resulta único salvo G-equivalencia homotópica.

Vale mencionar que la Conjetura 2.2.1 está probada cuando el grupo fundamental de A es localmente residualmente finito [GR62]. También se sabe que es verdadera si dicho grupo es hiperlineal [Tho12], y por lo tanto la conjetura se sigue de la conjetura de inmersión de grupos de Connes, que postula que todo grupo es hiperlineal [Pes08].

Otra conjetura relacionada con la de Casacuberta–Dicks es la conjetura de Quillen sobre el poset $\mathcal{S}_p(G)$ de p-subgrupos no triviales de un grupo finito G.

Conjetura (Quillen, [Qui78])**.** *Si el complejo de orden $\mathcal{K}(\mathcal{S}_p(G))$ es contráctil, la acción por conjugación de G en $\mathcal{S}_p(G)$ tiene un punto fijo.*

El siguiente resultado fue obtenido en colaboración con K. Piterman y A. Viruel [PSCV18].

Teorema 2.3.2 (Piterman – Sadofschi Costa – Viruel)**.** *La conjetura de Quillen vale para grupos de p-rango* 3.

Para probar este teorema aplicamos los resultados de [OS02] al subcomplejo $\mathcal{A}_p(G)$ de p-subgrupos abelianos elementales de G, que es de dimensión 2 si y solamente si el p-rango de G es 3. Por lo tanto el Teorema 2.3.2 puede ser visto como un caso especial de la conjetura de Casacuberta–Dicks.

El objeto clave que permite construir G-complejos acíclicos de dimensión 2 sin puntos fijos es el $\mathbb{Z}[G]$-módulo libre $H_1(\Gamma_{OS}(G))$. Notamos F_m al grupo libre de rango m. El G-grafo $\Gamma_{OS}(G)$ da un subgrupo $G \leq \mathrm{Out}(F_{|G|})$ que resulta ser el objeto análogo para intentar comprender si alguno de estos G-complejos acíclicos es contráctil. Usando esta idea obtenemos reformulaciones algebraicas de la conjetura de Casacuberta-Dicks, por ejemplo:

Conjetura 2.5.13. *Todo subgrupo permutacional G de* $\mathrm{Out}(F_m)$ *que actúa sin puntos fijos en una base normal de F_m se levanta a un subgrupo permutacional de* $\mathrm{Aut}(F_m)$.

Una *base normal* de F_m es un conjunto de m clases de conjugación que generan normalmente F_m. Un subgrupo finito de $\mathrm{Out}(F_m)$ es *permutacional* si actúa en una base normal y un subgrupo finito de $\mathrm{Aut}(F_m)$ es *permutacional* si actúa en una base.

La conjetura de Casacuberta–Dicks permanece abierta incluso cuando el grupo que actúa es $A_5 = \mathrm{PSL}_2(4)$. Nos concentramos especialmente en estudiar este caso particular. Utilizando un resultado de K.S. Brown [Bro84] mostramos que la siguiente conjetura se sigue de la conjetura de Casacuberta–Dicks:

Conjetura 2.6.7. *Sea $\phi\colon F(a,b,c,d,x) \to A_5$ el morfismo definido por $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ y $x \mapsto 1$. No existe una palabra $w \in \ker(\phi)$ tal que*

$$\langle a,b,c,d,x \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x^{-1}ax = d,w\rangle$$

es una presentación de A_5.

Con la misma técnica obtenemos la siguiente conjetura que, asumiendo la Conjetura 2.2.1 resulta equivalente al caso A_5 de la conjetura de Casacuberta–Dicks.

Conjetura 2.6.8. *Sea $\phi\colon F(a,b,c,d,x_0,\dots,x_k) \to A_5$ el morfismo definido por $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ y $x_i \mapsto 1$. No existe una presentación de A_5 de la forma*

$$\langle a,b,c,d,x_0,\dots,x_k \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x_0^{-1}ax_0 = d,w_0,\dots,w_k\rangle$$

con $w_0,\dots,w_k \in \ker(\phi)$.

Las Conjeturas 2.6.7 y 2.6.8 están profundamente relacionadas con otro famoso problema abierto, el relation gap problem (ver [Har18, Har15]). Podemos reformular la Conjetura 2.6.8 en términos de un relation gap.

Conjetura 2.7.2. *Sea* $E = \langle a,b,c,d,x_0,\dots,x_k \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x_0^{-1}ax_0 = d\rangle$ *y consideremos el morfismo* $\overline{\phi}\colon E \to A_5$ *definido por* $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ *y* $x_i \mapsto 1$. *Entonces si* $N = \ker(\overline{\phi})$ *la extensión*

$$1 \to N \to E \to A_5 \to 1$$

tiene un relation gap.

Usando un resultado de ecuaciones sobre grupos probado por Klyachko [Kly93] damos una demostración del siguiente caso particular de la Conjetura 2.6.7.

Teorema 2.6.12. *Para cualquier elección de una palabra* $w_0 \in F(a,b,c,d)$ *y* $k,l > 0$, *la Conjetura 2.6.7 vale para la palabra* $w = b(db)^k x^{-1} cd(acd)^l x w_0 x$.

En el Ejemplo 2.7.6 vemos como este resultado implica que infinitos de los potenciales contraejemplos de la conjetura de Casacuberta–Dicks son de hecho, no contráctiles.

También estudiamos la conjetura de Casacuberta–Dicks experimentalmente utilizando la computadora. Con este fin desarrollamos dos paquetes de GAP [GAP18]. El primero, G2Comp [SC18a] permite trabajar computacionalmente con G-complejos de dimensión 2. El segundo paquete, SmallCancellation [SC18b] implementa las condiciones clásicas de *small cancellation* [LS77]. Exhibimos ejemplos de A_5-complejos acíclicos de dimensión 2 con 1-esqueleto $\Gamma_{OS}(A_5)$ tales que el grupo fundamental es el *binary icosahedral group* A_5^*; es un producto libre de 6 o 7 copias de A_5^*; admite un epimorfismo a A_5 y además cumple la condición $C'(1/6)$ de small cancellation; no admite un epimorifsmo a A_5, pero satisface la condición $C(7)$ y por lo tanto es no trivial.

A partir de la evidencia experimental conjeturamos:

Conjetura 2.4.1. *Sea* X *un* A_5*-complejo acíclico finito y de dimensión* 2 *sin puntos fijos. Si* $\pi_1(X)$ *es finito entonces* $\pi_1(X) \simeq A_5^*$.

Los resultados experimentales y una sugerencia de Bob Oliver nos condujeron a los siguientes resultados.

Teorema 2.8.4. *Un* A_5*-complejo acíclico de dimensión* 2 *con* $X^{(1)} = \Gamma_{OS}(A_5)$ *no puede tener grupo fundamental* $\mathrm{PSL}_2(2^3)$.

Teorema 2.8.5. *Un* A_5*-complejo acíclico de dimensión* 2 *con* $X^{(1)} = \Gamma_{OS}(A_5)$ *no puede tener grupo fundamental* $\mathrm{PSL}_2(2^5)$.

Teorema 2.8.8. *Un* $\mathrm{PSL}_2(2^3)$*-complejo acíclico de dimensión* 2 *con* $X^{(1)} = \Gamma_{OS}(\mathrm{PSL}_2(2^3))$ *no puede tener grupo fundamental un producto libre de* $1 \leq n \leq 6$ *copias de* A_5^*.

El grupo $\mathrm{Out}(F_n)$ actúa en ciertos objetos geométricos similares a *buildings*. Uno de estos objetos es el complejo simplicial $\mathcal{B}(F_n)$ introducido en [DP13] que tiene por símplices las clases de conjugación de bases parciales del grupo libre F_n. Si pudiéramos probar que hay un símplex maximal de $\mathcal{B}(F_n)$ estable por la acción del subgrupo $G \leq \mathrm{Out}(F_m)$ inducido por el G-grafo $\Gamma_{OS}(G)$, entonces la conjetura de Casacuberta–Dicks sería falsa. Esta motivación es el punto de partida para los resultados del Capítulo 3.

El complejo de curvas $\mathcal{C}(S_g)$ de una superficie orientada S_g de género g fue introducido por Harvey [Har81] como un análogo de los Tits buildings para el mapping class group $\mathrm{Mod}(S_g)$. Harer probó que $\mathcal{C}(S_g)$ es homotópicamente equivalente a un wedge de $(g-1)$-esferas [Har85]. Masur y Minsky probaron que $\mathcal{C}(S_g)$ es hiperbólico [MM99]. Desde entonces, el complejo de curvas se volvió un objeto fundamental en el estudio de $\mathrm{Mod}(S_g)$. Dado que hay una analogía entre $\mathrm{Aut}(F_n)$ y $\mathrm{Mod}(S_g)$, es natural buscar un análogo de $\mathcal{C}(S_g)$ en este contexto. Hay varios candidatos que comparten propiedades con el complejo de curvas.

Uno de estos análogos es el poset $\mathrm{FC}(F_n)$ de factores libres propios de F_n. Hatcher y Vogtmann [HV98] probaron que el complejo simplicial asociado $\mathcal{K}(\mathrm{FC}(F_n))$ es Cohen-Macaulay (en particular es homotópicamente equivalente a un wedge de $(n-2)$-esferas). Bestvina y Feighn [BF14] probaron que $\mathcal{K}(\mathrm{FC}(F_n))$ es hiperbólico. Posteriormente, distintas demostraciones de este resultado aparecieron en [KR14] y [HH17].

Otros análogos naturales se definen en términos de bases parciales. Una *base parcial* de un grupo libre F es un subconjunto de una base de F. Day y Putman [DP13] definieron el complejo $\mathcal{B}(F_n)$ que tiene como símplices los conjuntos $\{C_0, \dots, C_k\}$ de clases de conjugación de F_n tales que existe una base parcial $\{v_0, \dots, v_k\}$ con $C_i = [\![v_i]\!]$ para $0 \leq i \leq k$. Ellos probaron que $\mathcal{B}(F_n)$ es 0-conexo si $n \geq 2$ y 1-conexo si $n \geq 3$ [DP13, Theorem A], que cierto cociente es $(n-2)$-conexo [DP13, Theorem B] y conjeturaron que $\mathcal{B}(F_n)$ es $(n-2)$-conexo [DP13, Conjecture 1.1]. Como aplicación, utilizaron $\mathcal{B}(F_n)$ para probar que el subgrupo de Torelli es finitamente generado.

En el Capítulo 3 estudiamos el complejo simplicial $\mathrm{PB}(F_n)$ que tiene por símplices las bases parciales no vacías de F_n. El resultado principal de este capítulo es el siguiente.

Teorema 3.4.5. *El complejo* $\mathrm{PB}(F_n)$ *es Cohen-Macaulay de dimensión* $n-1$.

En el proceso de probar este resultado utilizamos el método de McCool para obtener una presentación del grupo $\mathrm{SAut}(F_n, \{v_1, \dots, v_l\})$ de automorfismos especiales que fijan la base parcial $v_1, \dots, v_l$, que generaliza la presentación del grupo $\mathrm{SAut}(F_n)$ dada por Gersten. También probamos una versión de un resultado de Quillen [Qui78, Theorem 9.1]. En su versión original, este resultado produce una descomposición de $\widetilde{H}_n(X)$ a partir de un morfismo n-esférico de posets $f\colon X \to Y$. Nuestra versión, explicita una base del grupo de homología $\widetilde{H}_n(X)$.

Introduction

The results in the areas of two-dimensional homotopy theory and geometric group theory establish deep connections between algebraic and topological or geometric problems. Frequently, concepts from geometry, such as curvature, are translated into a more topological, combinatorial or algebraic setting, providing powerful tools to study problems which apparently are not of a geometrical nature. The fundamental group functor which is an invariant of (pointed) topological spaces taking values in groups provides a link between topology and group theory. This bridge appears, for example, in the correspondence between homotopy types of 2-complexes and group presentations. Among the open problems in these fields we may mention Whitehead's conjecture [Whi41], the Eilenberg-Ganea conjecture [EG57], Zeeman's conjecture [Zee64], the Andrews-Curtis conjecture [AC65], the Kervaire–Laudenbach–Howie conjecture [How81], the $D(2)$-problem and the relation gap problem [Har18]. In this thesis we seek to understand how the algebraic invariants of a space X give information on the existence of fixed points of a mapping or group action on X, specially when X is 2-dimensional.

The thesis is divided into three chapters. The main results of Chapter 1, which are presented in Sections 1.3 and 1.4, appeared in the articles [BSC17] (written in collaboration with J.A. Barmak) and [SC17b] respectively. The content of Sections 1.1 to 1.6 also appeared in the Licentiate Thesis [SC15]. The content of Section 1.7 is inedited and provides an alternative, non computer–assisted, proof for the results of Section 1.4. The results of Chapter 2 are inedited. Section 2.3 presents the results of our article in preparation [PSCV18], written in collaboration with K. Piterman and A. Viruel. The results of Chapter 3 appeared in [SC17a].

In Chapter 1 we study the fixed point property for 2-complexes. Recall that a space X has the *fixed point property* if any selfmap $f\colon X \to X$ has a fixed point. For example, Brouwer's fixed point theorem says the disks have the fixed point property. Some basic questions about the fixed point property remained unanswered for a long time. Kuratowski [Kur30] asked if the product of two spaces with the fixed point property has the fixed point property. Nearly forty years later, Lopez found an example of a compact polyhedron with the fixed point property and even Euler characteristic which allowed him to answer Kuratowski's question in the negative and to prove the fixed point property is not a homotopy invariant for compact polyhedra. This example, motivated the following two questions which were raised by R.H. Bing in his article

"The elusive fixed point property" [Bin69] and remained open for 45 years [Hag07].

Question (Bing's Question 1)**.** *Is there a compact two-dimensional polyhedron with the fixed point property which has even Euler characteristic?*

Question (Bing's Question 8)**.** *What is the least positive integer n such that the fixed point property is not a homotopy invariant for polyhedra of dimension at most n?*

Bing's Question 1 is about polyhedra with even Euler characteristic but, as far as we know, every previously known example of a 2-dimensional compact polyhedron with the fixed point property was $\mathbb{Q}$-acyclic, thus had Euler characteristic 1. In Chapter 1 we present results which give answer to these questions.

Using Browning's classification of homotopy types of compact 2-complexes with abelian fundamental group, we proved the following.

Theorem 1.3.21 (Barmak–Sadofschi Costa)**.** *A 2-dimensional compact polyhedron with Euler characteristic different from 1 and the fixed point property cannot have abelian fundamental group.*

With similar ideas, we also proved that the finite subgroups of $\mathrm{SO}(3)$ are not the fundamental group of such a space.

Theorem 1.3.22 (Barmak–Sadofschi Costa)**.** *A 2-dimensional compact polyhedron with Euler characteristic different from 1 and the fixed point property cannot have fundamental group A_4, S_4, A_5 or D_n.*

Posteriorly we constructed examples of 2-dimensional compact polyhedra with the fixed point property and Euler characteristic n for every $n \geq 1$, answering affirmatively Bing's Question 1. For $n \leq 0$, from a result by Borsuk (Corollary 1.1.17) it follows that there is no such example. Borsuk's original result concerns spaces more general than CW-complexes and its proof is involved. We present here a simpler proof using a more modern language.

To answer this questions we introduced the notion of *Bing group*.

Definition 1.4.1. Let G be a finitely presentable group such that $H_1(G)$ is finite. We say that G is a *Bing group* if either $H_2(G) = 0$ or, denoting the first invariant factor of $H_2(G)$ by d_1, we have $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ in $\mathbb{Z}_{d_1}$, for every endomorphism $\phi \colon G \to G$.

The following result yields examples of 2-complexes with the fixed point property.

Theorem 1.4.2. *If $\mathcal{P}$ is an efficient presentation of a Bing group G then $X_{\mathcal{P}}$ has the fixed point property.*

In this way, the problem is reduced to finding examples of efficient Bing groups. For a finite simple group G, every endomorphism being either trivial or an automorphism, it is particularly easy to check if it is Bing. Using the classification of the finite simple groups, we proved

Theorem 1.5.1. *The only finite simple Bing groups G such that $H_2(G) \neq 0$ are the groups $D_{2m}(q)$ for odd q and $m > 2$.*

Of these groups $D_6(3)$, which has order 6762844700608770238252960972800, is the smallest. If these groups turn out to be efficient, they would give examples of two dimensional polyhedra with the fixed point property and Euler characteristic equal to 3. To answer Bing's Question 1, we originally used the software GAP to obtain a Bing group of a different nature:

Proposition 1.4.5. *The group G presented by*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1}\rangle$$

is a finite group of order 3^5. We have $H_2(G) = \mathbb{Z}_3$, so $\mathcal{P}$ is efficient. Moreover G is a Bing group.

Our original proof of Proposition 1.4.5 uses GAP. In Section 1.7 we give an alternative proof of Proposition 1.4.5 which is not computer assisted. We immediatly deduce:

Corollary 1.4.7. *The complex $X_{\mathcal{P}}$ associated to the presentation*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1}\rangle$$

has the fixed point property. Moreover, $\chi(X_{\mathcal{P}}) = 2$.

In [Lop67], W. Lopez proves that the fixed point property is not a homotopy invariant for compact polyhedra. Lopez' example gives the upper bound 17 for the integer n in Bing's Question 8. On the other hand, for 1-dimensional compact polyhedra (finite graphs) the fixed point property is a homotopy invariant. Therefore the answer to Bing's Question 8 is at least 2. The following consequence of Theorem 1.4.2 shows the answer to Bing's Question 8 is 2.

Theorem 1.4.12. *There is a compact 2–dimensional polyhedron Y without the fixed point property and such that the polyhedron X, obtained from Y by an elementary collapse of dimension 2, has the fixed point property.*

If a complex X has the fixed point property, then every action of a cyclic group on X has a (global) fixed point. Contractible 2-complexes and, more generally, rationally acyclic 2-complexes have the fixed point property, and the results of Chapter 1 show that there are examples of 2-complexes with this property which are not rationally acyclic. Some 2-complexes have the fixed point property with respect to group actions, meaning that every action has a fixed point. There are, however, acyclic 2-complexes without this property. The group A_5 acts simplicially and fixed point freely on the barycentric subdivision X of the 2-skeleton of the Poincaré homology sphere which is an acyclic 2-complex. A famous result of J.P. Serre [Ser80] states that every action of a finite group on a 1-dimensional contractible complex (i.e.

a tree) has a fixed point. Contractible 3-dimensional complexes, however, do not have this property. Edwin E. Floyd and Roger W. Richardson [FR59] noted that A_5 acts simplicially and fixed point freely on the join $X * A_5$ of the space above and the discrete group A_5, which is a contractible 3-complex. Moreover, by embedding $X * A_5$ in $\mathbb{R}^{81}$ and taking a regular neighborhood they proved that A_5 acts simplicially and fixed point freely on a triangulation of the disk D^{81}. This was the only such example until Bob Oliver obtained a complete classification of the groups that act fixed point freely on a disk D^n [Oli75]. The example given by Floyd and Richardson makes clear that Serre's result cannot be extended to dimension 3, but does it hold for 2-complexes? This is an open problem and it is the object of study of Chapter 2. Carles Casacuberta and Warren Dicks made the following conjecture [CD92].

Conjecture (Casacuberta–Dicks). *Let G be a group. If X is a 2-dimensional finite contractible G-complex then $X^G \neq \varnothing$.*

We mention that in the original formulation by Casacuberta and Dicks X is not required to be finite. In the finite case, the same question was raised independently by Aschbacher and Segev [AS93a]. We study the Casacuberta–Dicks conjecture in the finite case and so we state it in this way.

Casacuberta and Dicks proved that the conjecture holds for a solvable group G. The same result was obtained independently by Yoav Segev [Seg93]. In [AS93a], Aschbacher and Segev used the classification of the finite simple groups to prove the conjecture for a vast class of groups. In [Seg94] Segev proves that it also holds when X is collapsible. In [OS02], Oliver and Segev give a complete classification of the groups that act fixed point freely on an acyclic 2-complex. In [Cor01], Corson proves that every action of a finite group G on a (not necessarily finite) simply connected and diagrammatically reducible 2-complex has a fixed point.

Along Chapter 2 we study this conjecture from different points of view. The results of [OS02] not only classify the groups G that act fixed point freely on an acyclic 2-complex X, but also give a description of the acyclic 2-complexes X where G acts fixed point freely. Using these results and assuming the following particular case of the Kervaire–Laudenbach–Howie conjecture [How81, Conjecture] we obtained a description of the possible counterexamples to the Casacuberta–Dicks conjecture.

Conjecture 2.2.1. *Let X be a finite contractible 2-complex. If $A \subset X$ is an acyclic subcomplex, then A is contractible.*

Theorem 2.2.11. *Assume Conjecture 2.2.1 holds. If the Casacuberta–Dicks conjecture 2.0.1 is false, then there is a 2-dimensional essential, fixed point free and contractible G-complex X where G is one of the following groups:*

(i) $\mathrm{PSL}_2(2^p)$ *for p prime.*

(ii) $\mathrm{PSL}_2(3^p)$ *for an odd prime* p.

(iii) $\mathrm{PSL}_2(q)$ *for a prime* $q > 3$ *such that* $q \equiv \pm 3 \mod 5$ *and* $q \equiv \pm 3 \mod 8$.

(iv) $\mathrm{Sz}(2^p)$ *for* p *an odd prime.*

Moreover X *can be taken so that it is obtained from the graph* $\Gamma_{OS}(G)$ *by attaching* $k \geq 0$ *free orbits of* 1*-cells and* $k+1$ *free orbits of* 2*-cells.*

The graph $\Gamma_{OS}(G)$ appearing in Theorem 2.2.11 is the 1-skeleton of any 2-complex of the type constructed by Oliver and Segev and is unique up to G-homotopy equivalence.

We mention that Conjecture 2.2.1 holds if the fundamental group of A is locally residually finite [GR62]. It also holds for hyperlinear groups [Tho12], and it thus follows from Connes' embedding conjecture for groups, which states that every group is hyperlinear [Pes08].

Another conjecture, related to the Casacuberta–Dicks Conjecture is Quillen's conjecture on the poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups of a finite group G.

Conjecture (Quillen, [Qui78])**.** *If the order complex* $\mathcal{K}(\mathcal{S}_p(G))$ *is contractible then the conjugation action of* G *on* $\mathcal{S}_p(G)$ *has a fixed point.*

The following result was obtained in collaboration with K. Piterman and A. Viruel [PSCV18].

Theorem 2.3.2 (Piterman – Sadofschi Costa – Viruel)**.** *Quillen's conjecture holds for groups of* p*-rank* 3.

To prove this theorem we apply the results of [OS02] to the subcomplex $\mathcal{A}_p(G)$ of elementary abelian p-subgroups, which is 2-dimensional if and only if the p-rank of G is 3. Thus Theorem 2.3.2 may be regarded as a special case of the Casacuberta–Dicks conjecture.

The key object to construct 2-dimensional acyclic fixed point free G-complexes is the free $\mathbb{Z}[G]$-module $H_1(\Gamma_{OS}(G))$. We denote by F_m the free group of rank m. The G-graph $\Gamma_{OS}(G)$ gives a subgroup $G \leq \mathrm{Out}(F_{|G|})$ which seems to be an analogue object to try to understand if any of these acyclic G-complexes is contractible. Using this idea we obtain algebraic reformulations of the Casacuberta–Dicks conjecture, for example:

Conjecture 2.5.13. *Any permutational subgroup of* $\mathrm{Out}(F_m)$ *that acts fixed point freely on a normal-basis of* F_m *lifts to a permutational subgroup of* $\mathrm{Aut}(F_m)$.

A *normal-basis* of F_m is a set of m conjugacy classes whose normal closure is F_m. A finite subgroup of $\mathrm{Out}(F_m)$ is *permutational* if it acts on a normal-basis and a finite subgroup of $\mathrm{Aut}(F_m)$ is *permutational* if it acts on a basis.

The Casacuberta–Dicks Conjecture is still open even when the group is $A_5 = \mathrm{PSL}_2(4)$. We focus specially on this case. Using a result of K.S. Brown [Bro84] we show the following follows from the Casacuberta–Dicks conjecture:

Conjecture 2.6.7. *Let $\phi\colon F(a,b,c,d,x) \to A_5$ be the map given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x \mapsto 1$. There is no word $w \in \ker(\phi)$ such that*

$$\langle a,b,c,d,x \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x^{-1}ax = d,w\rangle$$

is a presentation of A_5.

Using the same technique we obtain the following conjecture which under the assumption of Conjecture 2.2.1 is equivalent to the Casacuberta–Dicks conjecture for A_5.

Conjecture 2.6.8. *Let $\phi\colon F(a,b,c,d,x_0,\dots,x_k) \to A_5$ be the map given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x_i \mapsto 1$. There is no presentation of A_5 of the form*

$$\langle a,b,c,d,x_0,\dots,x_k \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x_0^{-1}ax_0 = d,w_0,\dots,w_k\rangle$$

with $w_0,\dots,w_k \in \ker(\phi)$.

Conjectures 2.6.7 and 2.6.8 are deeply related to other famous open problem, the relation gap problem (see [Har18, Har15]). We may restate Conjecture 2.6.8 in terms of a relation gap:

Conjecture 2.7.2. *Let $E = \langle a,b,c,d,x_0,\dots,x_k \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x_0^{-1}ax_0 = d\rangle$ and consider the morphism $\overline{\phi}\colon E \to A_5$ given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x_i \mapsto 1$. Then if $N = \ker(\overline{\phi})$ the extension*

$$1 \to N \to E \to A_5 \to 1$$

has a relation gap.

Using a result of Klyachko [Kly93] on equations over groups we prove the following case of Conjecture 2.6.7.

Theorem 2.6.12. *For any choice of a word $w_0 \in F(a,b,c,d)$ and $k,l > 0$, Conjecture 2.6.7 is satisfied for the word $w = b(db)^k x^{-1} cd(acd)^l x w_0 x$.*

In Example 2.7.6 we see how this result implies that infinitely many of the acyclic possible counterexamples to the Casacuberta–Dicks conjecture are in fact not contractible.

We also study the Casacuberta–Dicks conjecture by means of computer experimentation. To this end, we developed the GAP [GAP18] packages G2Comp [SC18a] which allows to work computationally with 2-dimensional G-complexes and SmallCancellation [SC18b] which implements the classical small cancellation conditions [LS77]. We exhibit examples of 2-dimensional acyclic A_5-complexes with 1-skeleton $\Gamma_{OS}(A_5)$ where the fundamental group is the binary icosahedral group A_5^*; is a free product of 6 or 7 copies of A_5^*; admits an epimorphism to A_5 and also satisfies the small cancellation condition $C'(1/6)$; does not admit an epimorphism onto A_5, but satisfies condition $C(7)$ and thus is nontrivial.

From this experimental evidence we conjecture

Conjecture 2.4.1. *Let X be a fixed point free 2-dimensional finite and acyclic A_5-complex. If $\pi_1(X)$ is finite then $\pi_1(X) \simeq A_5^*$.*

Computer experimentation and a suggestion by Bob Oliver lead to the following results.

Theorem 2.8.4. *An acyclic, 2-dimensional A_5-complex X with $X^{(1)} = \Gamma_{OS}(A_5)$ cannot have fundamental group* $\mathrm{PSL}_2(2^3)$.

Theorem 2.8.5. *An acyclic, 2-dimensional A_5-complex X with $X^{(1)} = \Gamma_{OS}(A_5)$ cannot have fundamental group* $\mathrm{PSL}_2(2^5)$.

Theorem 2.8.8. *An acyclic, 2-dimensional $\mathrm{PSL}_2(2^3)$-complex X with $X^{(1)} = \Gamma_{OS}(\mathrm{PSL}_2(2^3))$ cannot have fundamental group a free product of $1 \leq n \leq 6$ copies of A_5^*.*

The group $\mathrm{Out}(F_n)$ acts on certain geometric objects, similar to buildings. One of these objects is the simplicial complex $\mathcal{B}(F_n)$ introduced in [DP13] that has as simplices the conjugacy classes of partial bases of the free group F_n. If we could prove that there is a maximal simplex of $\mathcal{B}(F_n)$ stable by the action of the subgroup $G \leq \mathrm{Out}(F_m)$ induced by the G-graph $\Gamma_{OS}(G)$, then the Casacuberta–Dicks would be false. This motivation is the starting point for the results presented in Chapter 3.

The curve complex $\mathcal{C}(S_g)$ of an oriented surface S_g of genus g was introduced by Harvey [Har81] as an analogue of Tits buildings for the mapping class group $\mathrm{Mod}(S_g)$. Harer proved that $\mathcal{C}(S_g)$ is homotopy equivalent to a wedge of $(g-1)$-spheres [Har85]. Masur and Minsky proved that $\mathcal{C}(S_g)$ is hyperbolic [MM99]. The curve complex became a fundamental object in the study of $\mathrm{Mod}(S_g)$. Since there is an analogy between $\mathrm{Aut}(F_n)$ and $\mathrm{Mod}(S_g)$, it is natural to seek for an analogue of $\mathcal{C}(S_g)$ in this context. There are many candidates that share properties with the curve complex.

One of these analogues is the poset $\mathrm{FC}(F_n)$ of proper free factors of F_n. Hatcher and Vogtmann [HV98] proved that its order complex $\mathcal{K}(\mathrm{FC}(F_n))$ is Cohen-Macaulay (in particular, that it is homotopy equivalent to a wedge of $(n-2)$-spheres). Bestvina and Feighn [BF14] proved that $\mathcal{K}(\mathrm{FC}(F_n))$ is hyperbolic. Subsequently, different proofs of this fact appeared in [KR14] and [HH17].

Other natural analogues are defined in terms of partial bases. A *partial basis* of a free group F is a subset of a basis of F. Day and Putman [DP13] defined the complex $\mathcal{B}(F_n)$ whose simplices are sets $\{C_0, \dots, C_k\}$ of conjugacy classes of F_n such that there exists a partial basis $\{v_0, \dots, v_k\}$ with $C_i = [\![v_i]\!]$ for $0 \leq i \leq k$. They proved that $\mathcal{B}(F_n)$ is 0-connected for $n \geq 2$ and 1-connected for $n \geq 3$ [DP13, Theorem A], that a certain quotient is $(n-2)$-connected [DP13, Theorem B] and they conjectured that $\mathcal{B}(F_n)$ is $(n-2)$-connected [DP13, Conjecture 1.1]. As an application, they used $\mathcal{B}(F_n)$ to prove that the Torelli subgroup is finitely generated.

In Chapter 3 we study the simplicial complex $\mathrm{PB}(F_n)$ with simplices the nonempty partial bases of F_n. The main result of this chapter is the following.

Theorem 3.4.5. *The complex* $\mathrm{PB}(F_n)$ *is Cohen-Macaulay of dimension* $n-1$.

On the way to prove this result, we use McCool's method to obtain a presentation of the group $\mathrm{SAut}(F_n, \{v_1, \ldots, v_l\})$ of special automorphisms which fix the partial basis $v_1, \ldots, v_l$, which generalizes Gersten's presentation of $\mathrm{SAut}(F_n)$. We also prove a version of a result due to Quillen [Qui78, Theorem 9.1]. In its original version, this result produces a decomposition of $\widetilde{H}_n(X)$ from an n-spherical map of posets $f\colon X \to Y$. Our version produces an explicit basis of the homology group $\widetilde{H}_n(X)$.

Contents

Chapter 1

The fixed point property for 2-complexes

In this chapter we study the fixed point property for 2-dimensional polyhedra. Our main results give answer to two questions posed by R. H. Bing in 1969. Recall that a topological space has the fixed point property if every self-map has a fixed point. Brouwer's theorem says that the disk D^n has the fixed point property for every positive integer n. However, the antipodal map is a self-map of the sphere S^n without fixed points. Usually, it is not easy to decide whether a space has the fixed point property.

To each self-map $f: X \to X$ of a compact polyhedron we can associate an integer $L(f)$, the Lefschetz number of f. If this number is nonzero, the map has a fixed point. This is the Lefschetz fixed point theorem. If $L(f) = 0$ we cannot conclude that f does not have fixed points. Nielsen fixed point theory associates to f another integer, the Nielsen number $N(f)$ of f. If this number is zero, and some mild hypotheses are satisfied, there is a map g homotopic to f without fixed points.

The fixed point property is a topological invariant but it is not a homotopy invariant as shown by the spaces $\mathbb{R}$ and $*$. Even if we restrict ourselves to compact metric spaces, there is an example of Kinoshita [Kin53] of a compact contractible metric space that does not have the fixed point property. Surprisingly, the fixed point property is not a homotopy invariant for compact polyhedra (by *polyhedron* we mean the geometric realization of a simplicial complex). This was proved by W. Lopez in [Lop67], by constructing a 17-dimensional polyhedron with the fixed point property and such that, by attaching a disk along an arc we obtain a homotopy equivalent polyhedron without the fixed point property. A key step in Lopez' construction is to find a polyhedron X with the fixed point property and even Euler characteristic. Lopez' space X is 8-dimensional. In "The elusive fixed point property" [Bin69], R.H. Bing poses twelve questions regarding the fixed point property. As of 2014, eight of these questions had been answered. The following two questions, motivated by Lopez' example, remained open.

Question (Bing's Question 1)**.** *Is there a compact* 2*-dimensional polyhedron with the fixed point property which has even Euler characteristic?*

Question (Bing's Question 8)**.** *What is the least value of n so that there is an n-dimensional polyhedron X with the fixed point property and a disk D such that $D \cap X$ is an arc but $X \cup D$ does not have the fixed point property?*

In this chapter we present results that give answers to these questions. Section 1.1 consists mainly on preliminary results. In Section 1.2 we present some results of William Lopez and with this motivation we introduce Bing's Questions 1 and 8. In Section 1.3 we prove that a compact 2-dimensional polyhedron with the fixed point property and Euler characteristic different from 1 cannot have abelian fundamental group. In Section 1.4 we introduce the notion of Bing group which allows us to construct examples of 2-dimensional compact polyhedra with the fixed point property and Euler characteristic equal to any positive integer. We also use these ideas to show the answer to Bing's Question 8 is 2. The proofs provided in this section rely on some GAP computations. In Section 1.7 we present new, lengthy proofs for these results which are not computer assisted. In Section 1.5 we use the classification of the finite simple groups to find out which of these groups are Bing. Section 1.6 contains GAP code to decide if a group is Bing.

The main results of this chapter appeared in our articles [BSC17] and [SC17b] and in the Licentiate Thesis [SC15]. Some of the results we present do not appear in our articles and are only available in Spanish in [SC15]. For instance, Section 1.1.3 contains a proof of a result by Borsuk much simpler than the original which concerned Peano continua instead of simplicial complexes. The content of Section 1.7 was not published elsewhere and provides alternative and non computer-assisted proofs for the main results of [SC17b].

1.1 Preliminaries of fixed point theory

In this section, we introduce the tools that we will use along the chapter. In Section 1.1.1 we present some elementary results that involve the fixed point property. In Section 1.1.2, we introduce some local properties that will appear later. In Section 1.1.3 we present Borsuk's theorem characterizing spaces that retract to S^1. We give a modern proof of this result. In Section 1.1.4 we give a brief introduction to Nielsen theory, which is one of the main tools that we used to attack Bing's Questions 1 and 8.

1.1.1 The fixed point property

Definition 1.1.1. If X is a topological space, we say X has the *fixed point property* if every continuous map $f : X \to X$ has a fixed point.

Example 1.1.2. Brouwer's fixed point theorem says that the disk D^n has the fixed point property for every n. In opposition, the antipodal map $a\colon S^n \to S^n$ shows that the sphere S^n lacks the fixed point property for every n.

Remark 1.1.3. A space with the fixed point property is connected.

Usually it is not easy to decide whether a space has the fixed point property. Now we review some basic tools.

Lemma 1.1.4. *If X has the fixed point property and A is a retract of X, then A has the fixed point property.*

Proof. Let $i\colon A \hookrightarrow X$ be the inclusion and $r\colon X \to A$ a retraction. Let $f\colon A \to A$ be a map. Since X has the fixed point property there is $x_0 \in X$ such that $(i \circ f \circ r)(x_0) = x_0$. Then the point $r(x_0)$ is fixed by f. □

Lemma 1.1.5. *Let X_1, X_2 be topological spaces. Then $X_1 \vee X_2$ has the fixed point property if and only if both X_1 and X_2 have the fixed point property.*

Proof. Let $j_i\colon X_i \to X_1 \vee X_2$ the inclusions and let $x_0 \in X_1 \vee X_2$ be the base point of the wedge. There are retractions $r_i\colon X_1 \vee X_2 \to X_i$ given by

$$r_i(x) = \begin{cases} x & \text{if } x \in X_i \\ x_0 & \text{if } x \notin X_i \end{cases}$$

One implication follows immediately from Lemma 1.1.4. For the other implication, assume there is a fixed point free map $f\colon X_1 \vee X_2 \to X_1 \vee X_2$. Then if $f(x_0) \in X_i$, the composition $r_i \circ f \circ j_i : X_i \to X_i$ is fixed point free, contradiction. Thus $X_1 \vee X_2$ has the fixed point property. □

If a product $X_1 \times X_2$ has the fixed point property then both factors must have it. In 1930, Kuratowski asked if this is sufficient [Kur30]. Finally, in 1969 Lopez constructed a space X with the fixed point property and such that $X \times [0,1]$ lacks the fixed point property. We will review this example in Section 1.2. We will also prove that the fixed point property is not preserved by taking joins, suspensions and smash products.

1.1.2 Separating points

By *polyhedron* we mean a topological space homeomorphic to the geometric realization of a simplicial complex.

Definition 1.1.6. Let X be a connected polyhedron. We say that $x \in X$ is a *local separating point* if there is a connected open neighborhood $U \ni x$ such that $U - \{x\}$ is disconnected. We say $x \in X$ is a *global separating point* if $X - \{x\}$ is disconnected. Obviously every global separating point is a local separating point.

Recall that if X is a simplicial complex and $x \in X$ is a vertex, the *link* of x, $\mathrm{lk}(x,X)$ is the subcomplex of X having as simplices the simplices σ such that $\sigma \cup \{x\}$ is a simplex of X and $x \notin \sigma$.

Proposition 1.1.7. *Let X be a connected simplicial complex and let $x \in |X|$. The following are equivalent.*

(i) x is a local separating point.

(ii) x is a vertex of X such that $\mathrm{lk}(x,X)$ is disconnected or x lies in the interior of a maximal 1-simplex of X.

Proposition 1.1.8. *Let X be a connected simplicial complex other than the 1-simplex. The following are equivalent.*

(i) There is a global (resp. local) separating point $x \in X$.

(ii) There is a vertex $v \in X$ that is a global (resp. local) separating point.

If a vertex $v \in X$ is a global separating point, there are subcomplexes X_1 and X_2 of X whose union is X and whose intersection is $\{v\}$, that is $X = X_1 \vee X_2$.

We recall Whitehead's classical notion of simplicial expansion and collapse.

Definition 1.1.9. Let X be a simplicial complex and let σ, τ be simplices of X such that τ is the only simplex of X having σ as a proper face. Then $Y = X - \{\sigma, \tau\}$ is a simplicial complex homotopy equivalent to X. If $\dim \tau = n$, we say that Y is obtained from X by an *elementary n-collapse*. We also say that X is obtained from Y by an *elementary n-expansion*.

Proposition 1.1.10. *If X is a connected simplicial complex other than the 1-simplex, by doing repeated elementary 2-expansions it is always possible to obtain a simplicial complex without global separating points.*

Proof. If $u \in X$ is a vertex and a global separating point, there are vertices $v, w \in \mathrm{lk}(u,X)$ that lie in different connected components of $\mathrm{lk}(u,X)$. Then adding the simplices $\{v,w\}$ and $\{u,v,w\}$ is an elementary expansion. Since this expansion diminishes the rank of $\bigoplus_u \widetilde{H}_0(\mathrm{lk}(u,X))$, by doing this repeatedly we obtain a simplicial complex without global separating points. □

1.1.3 A theorem of Borsuk

Karol Borsuk proved that a *Peano continuum* (i.e. a compact connected and locally connected metric space) retracts to S^1 if and only if its first Betti number is nonzero. The proof goes by showing that both conditions are equivalent to X not being *unicoherent* and is divided in two papers (see [Bor31, 30. Théorème] and [Bor33, 11. Korollar]). From this result it follows that a Peano continuum with nontrivial first rational homology group lacks the fixed point property.

This subsection contains a proof of Borsuk's theorem for (not necessarily finite) simplicial complexes. Our proof is based on the proof given in [Kur68, §57, III, Theorem 4]. We will use Borsuk's result later in Sections 1.3 and 1.4, but the main reason to include a proof here is that we could not find an accesible proof in the literature. For another proof of this result see [SC15, Sección 1.3.2].

As usual, $K(G,n)$ denotes the Eilenberg-MacLane space. We denote the set of unpointed homotopy classes of maps $X \to Y$ by $[X,Y]$. Recall the following classical result.

Theorem 1.1.11 ([Hat02, Theorem 4.57])**.** *Let G be an abelian group and let $n > 0$. There are natural bijections $[X,K(G,n)] \to H^n(X;G)$ (defined for every CW-complex X) and given by $[f] \mapsto f^*(\omega)$, for certain class $\omega \in H^n(K(G,n);G)$ that does not depend on X.*

It is easy to see that the class of a null-homotopic map $X \to K(G,n)$ corresponds to $0 \in H^n(X;G)$.

Lemma 1.1.12 ([Kur68, §56, VI, Theorem 3])**.** *Let X be a simplicial complex and let A_1, A_2 be subcomplexes such that $X = A_1 \cup A_2$. Let $f\colon X \to S^1$ be a map. If the restrictions $f|_{A_j}$ are null-homotopic and $A_1 \cap A_2$ is connected, then f is null-homotopic.*

Proof. Since $S^1 = K(\mathbb{Z},1)$, we can use Theorem 1.1.11. By Mayer-Vietoris, there is an exact sequence

$$\widetilde{H}^0(A_1 \cap A_2;\mathbb{Z}) \xrightarrow{\partial} \widetilde{H}^1(X;\mathbb{Z}) \xrightarrow{(i_{A_1}^*,i_{A_2}^*)} \widetilde{H}^1(A_1;\mathbb{Z}) \oplus \widetilde{H}^1(A_2;\mathbb{Z})$$

We have $i_{A_j}^*(f^*(\omega)) = (f \circ i_{A_j})^*(\omega) = f|_{A_j}^*(\omega) = 0$. Therefore $f^*(\omega) \in \operatorname{Im}(\partial)$, and since $\widetilde{H}^0(A_1 \cap A_2;\mathbb{Z}) = 0$ we obtain $f^*(\omega) = 0$. Equivalently, f is null-homotopic. □

The idea of the following lemma may be found in the proof of [Kur68, §56, X, Theorem 6].

Lemma 1.1.13. *Let X be a connected CW-complex, $f\colon X \to Y$ a map and $B \subset X$ a subcomplex such that $f|_B$ is null-homotopic. Then there is a connected subcomplex $C \subset X$, with $B \subset C$ and such that $f|_C$ is null-homotopic.*

Proof. Write $B = \coprod_\alpha B_\alpha$ where B_α are connected subcomplexes. Consider the space Z obtained from X by shrinking each B_α to a point. The space Z is a CW-complex which has a k-cell for every k-cell of X that is not a cell of B and additionally a 0-cell for each α (see [FP90, Theorem 2.3.1]). Then the quotient map $q\colon X \to Z$ is cellular. Since Z is connected we can take a spanning tree $T \subset Z^{(1)}$. Then $C = q^{-1}(T)$ is a subcomplex of X and we have $B \subset C$. It is not difficult to prove that C is connected. Since $B \hookrightarrow X$ is a cofibration and $f|_B$ is null-homotopic, there is $g \simeq f$ such that $g|_B$ is a constant map. Let $\overline{g}\colon Z \to Y$ be the map obtained by passing to the quotient. Finally if $\iota\colon C \to X$ is the inclusion, we have

$$f|_C = f \circ \iota \simeq g \circ \iota = \overline{g} \circ q \circ \iota$$

which factors through T and thus is null-homotopic. □

Lemma 1.1.14 ([Kur68, §56, X, Theorem 7]). *Let $f\colon X \to S^1$ be a simplicial map for some triangulation of S^1. Then there are connected subcomplexes $C_1, C_2 \subset X$ such that $X = C_1 \cup C_2$ and the restrictions $f|_{C_i}$ are null-homotopic.*

Proof. There are subcomplexes A_1, A_2 of S^1, each one homeomorphic to an interval and such that $S^1 = A_1 \cup A_2$. Let $B_i = f^{-1}(A_i)$. Since f is simplicial, B_i is a subcomplex of X. Moreover $B_1 \cup B_2 = X$. Therefore $f|_{B_i}$ is null-homotopic. To conclude we obtain C_1 and C_2 from B_1 and B_2 using Lemma 1.1.13. □

Definition 1.1.15. We say that a connected simplicial complex is *unicoherent* if, for every subdivision of X and for every pair of connected subcomplexes C_1, C_2 such that $X = C_1 \cup C_2$, the intersection $C_1 \cap C_2$ is connected.

Theorem 1.1.16 (Borsuk). *Let X be a connected simplicial complex. The following are equivalent.*

(i) $\mathbb{Z}$ is a direct summand of $H_1(X)$.

(ii) X is not unicoherent.

(iii) S^1 is a retract of X.

Proof. First we will prove (i) implies (ii). There is an epimorphism $H_1(X) \to \mathbb{Z}$ and we thus have $\hom(H_1(X), \mathbb{Z}) \neq 0$. By the universal coefficient theorem, $H^1(X;\mathbb{Z}) \neq 0$. By Theorem 1.1.11 we have $[X, S^1] \approx H^1(X;\mathbb{Z})$ and thus there is a map $f\colon X \to S^1$ that is not null-homotopic. By the simplicial approximation theorem ([Mun84, Theorem 16.5]), subdividing X if necessary, we may assume f to be simplicial. By Lemma 1.1.14, we obtain connected subcomplexes C_1 and C_2 such that $X = C_1 \cup C_2$, and $f|_{C_i}$ are null-homotopic. Since f is not null-homotopic, by Lemma 1.1.12, $A = C_1 \cap C_2$ is disconnected. Therefore X is not unicoherent.

Now we prove (ii) implies (iii). Let C_1 and C_2 be connected subcomplexes of a subdivision such that $X = C_1 \cup C_2$ and $A = C_1 \cap C_2$ is disconnected. We can write $A = \coprod_{j \in J} A_j$, where A_j are connected subcomplexes and we have $\#J \geq 2$. For each $j \in J$ we take a maximal tree $S_j \subset A_j$. Let $F = \coprod_{j \in J} S_j$. Since C_i is connected, there is a maximal tree T_i of C_i such that $F \subset T_i$. It is easy to see that $F = T_1 \cap T_2$. We consider the graph $\Gamma = T_1 \cup T_2$. Clearly Γ is connected. We have the following Mayer-Vietoris exact sequence:

$$0 = \widetilde{H}_1(T_1) \oplus \widetilde{H}_1(T_2) \longrightarrow \widetilde{H}_1(\Gamma) \xrightarrow{\simeq} \widetilde{H}_0(F) \longrightarrow \widetilde{H}_0(T_1) \oplus \widetilde{H}_0(T_2) = 0$$

And since F is disconnected, $\widetilde{H}_1(\Gamma)$ is non-trivial, implying that Γ has a cycle. Now we will find a retraction $r_X\colon X \to \Gamma$ for the inclusion $\Gamma \hookrightarrow X$. Since S_j is contractible, there are retractions $r_{A_j}\colon A_j \to S_j$. This retractions glue to give a retraction $r_A\colon A \to F$. Since T_i is contractible, we can extend r_A to a retraction $r_{C_i}\colon C_i \to T_i$. By gluing r_{C_1} and r_{C_2} we obtain the retraction $r_X\colon X \to \Gamma$. Finally, since Γ has a cycle, S^1 is a retract of Γ and thus of X.

It is clear that (iii) implies (i), so we are done. □

If we drop the connection hypothesis we obtain the following.

Corollary 1.1.17 (Borsuk)**.** *Let X be a simplicial complex. The following are equivalent:*
(i) S^1 is a retract of X.
(ii) $\mathbb{Z}$ is a direct summand of $H_1(X)$.

Combining Corollary 1.1.17 and Lemma 1.1.4 we get:

Corollary 1.1.18 (Borsuk)**.** *If X is a polyhedron and $\mathbb{Z}$ is a direct summand of $H_1(X)$, then X does not have the fixed point property.*

In the proof, we have shown that X retracts to a subcomplex of a subdivision of X, homeomorphic to S^1. We do not known if it is really necessary to subdivide X. However there are examples of complexes for which it is not necessary to subdivide, but such that there is no simplicial retraction (e.g. a Möbius band triangulated using five vertices and five 2-simplices).

1.1.4 Nielsen theory

The original aim of Nielsen theory was to find the minimum number of fixed points $M(f)$ that a map g in the homotopy class of $f\colon X \to X$ can have. The first results by Jakob Nielsen go back to the 1920s. Nielsen defined the Nielsen number $N(f)$ and proved that it is a lower bound for the number of fixed points of f. Since it is a homotopy invariant, $N(f)$ gives a lower bound for $M(f)$ (Hopf proved that $M(f)$ is finite). In this section we introduce the basic notions relevant for Nielsen theory, namely fixed point classes and index and we state the classical results that will be required later. In particular we state here a result of Jiang (generalizing earlier results by Wecken and Shi) which says that, if the polyhedron X has no local separating points and is not a surface, we have $N(f) = M(f)$. From this result, we deduce Jiang's theorem on the homotopy invariance of the fixed point property for compact polyhedra without separating points.

The Lefschetz fixed point theorem

The Lefschetz fixed point theorem is our main tool to prove that a space has the fixed point property. The results presented here can be found in [JM06, Section 2.3].

Definition 1.1.19. Let R be a principal ideal domain. If M is a finite rank free R-module and $f\colon M \to M$ is an endomorphism, the trace $\mathrm{tr}_R(f) \in R$ is the trace of the matrix of f in any basis of M. More generally, if M is a finitely generated R-module, the trace is defined by $\mathrm{tr}_R(f\colon M \to M) = \mathrm{tr}_R(\bar{f}\colon M/T \to M/T)$, where T is the torsion of M. We will omit R from the notation when it is understood.

Definition 1.1.20. If X is a compact polyhedron and $f\colon X \to X$ is a map, the Lefschetz number of f is defined by $L(f) = \sum_k (-1)^k \mathrm{tr}_{\mathbb{Z}}(f_*\colon H_k(X) \to H_k(X))$. More generally we define $L(f_*;R) = \sum_k (-1)^k \mathrm{tr}_R(f_*\colon H_k(X;R) \to H_k(X;R))$ and $L(f^*;R) = \sum_k (-1)^k \mathrm{tr}_R(f^*\colon H^k(X;R) \to H^k(X;R))$.

Theorem 1.1.21 (Lefschetz fixed point theorem)**.** *Let X be a compact polyhedron and let $f\colon X \to X$ be a map. If $L(f) \neq 0$, then f has a fixed point.*

Lemma 1.1.22. *Let X be a compact polyhedron and let $f\colon X \to X$ be a map. Let R be a principal ideal domain and let $j\colon \mathbb{Z} \to R$ be the canonical morphism. Then $L(f_*;R) = L(f^*;R) = j(L(f))$.*

Corollary 1.1.23 (Lefschetz)**.** *Let X be a compact polyhedron and let $f\colon X \to X$ be a map. If R is a principal ideal domain and $L(f_*,R) \neq 0$ or $L(f^*,R) \neq 0$, then f has a fixed point.*

Example 1.1.24. By the Lefschetz fixed point theorem, every compact contractible polyhedron has the fixed point property. More generally, by the Lefschetz fixed point theorem, any compact $\mathbb{Q}$-acyclic polyhedron has the fixed point property (e.g. the real projective plane $\mathbb{RP}^2$). On the other hand, $\mathbb{R}$ is a contractible polyhedron without the fixed point property. Kinoshita [Kin53] gave an example of a compact and contractible metric space without the fixed point property.

Fixed point classes

There is a partition of the fixed point set of a map $f\colon X \to X$ into classes, called the *fixed point classes* of f. We give here two definitions of the fixed point classes. The first definition is more concrete. The advantage of the second definition is that it detects empty fixed point classes and is useful to prove the homotopy invariance of the fixed point classes of a map.

If X is a topological space, $U \subset X$ is open and $f\colon U \to X$ is a map, the *fixed point set of f* is given by $\mathrm{Fix}(f) = \{x \in U \,:\, f(x) = x\}$. If X is a compact polyhedron and $f\colon X \to X$, clearly $\mathrm{Fix}(f)$ is compact.

Definition 1.1.25 (Nonempty fixed point classes, [Jia83, Theorem 1.10])**.** Let X be a compact and connected polyhedron and let $f\colon X \to X$ be a map. The *nonempty fixed point classes of f* are the equivalence classes of the equivalence relation on $\mathrm{Fix}(f)$ given by $x \sim y$ if there is a path $c\colon I \to X$ such that $c(0) = x$, $c(1) = y$ and $f \circ c \simeq c$ (as paths).

Definition 1.1.26 (Fixed point classes, [Jia83, I, Definition 1.6])**.** Let X be a compact, connected polyhedron and let $p\colon \widetilde{X} \to X$ be its universal covering. Let $f\colon X \to X$ be a map. Two lifts of f to the universal covering $\widetilde{X}$ are *equivalent* if their are conjugated by a deck transformation. If $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ is a lift of f, we denote its equivalence class by $[\widetilde{f}]$. It is easy to see that $p(\mathrm{Fix}(\widetilde{f}))$ depends only on $[\widetilde{f}]$ but not on the representative $\widetilde{f}$.

Each equivalence class $[\widetilde{f}]$ of lifts of f has associated a *fixed point class* $p(\mathrm{Fix}(\widetilde{f})) \subset \mathrm{Fix}(f)$. The fixed point classes form a partition of $\mathrm{Fix}(f)$ indexed by the equivalence classes of lifts of f. Some fixed point classes may be empty and the nonempty fixed point classes are precisely the nonempty fixed point classes defined previously.

Theorem 1.1.27 ([Jia83, I, Theorem 1.12, Corollary 1.13])**.** *Let X be a compact and connected polyhedron and let $f: X \to X$ be a map. Then the number of nonempty fixed point classes of f is finite. The fixed point classes of f are compact and open in* $\mathrm{Fix}(f)$.

Definition 1.1.28. Let X be a compact and connected polyhedron. Let $f: X \to X$ be a map. We say that a subset $F \subset \mathrm{Fix}(f)$ is an *isolated set of fixed points* if it is open and closed.

By the previous theorem, every fixed point class is an isolated set of fixed points.

Fixed point index

The objective of this section is to define the index of an isolated set of fixed points and to state its properties. The index allows to count, with multiplicity, the number of fixed points of f in an open set. If $U \subset \mathbb{R}^n$ is open, $f: U \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$, assuming $f^{-1}(x)$ is compact, the degree of f over x can be thought as the number of points in $f^{-1}(x)$. If $i: U \hookrightarrow \mathbb{R}^n$ is the inclusion, the fixed point set of f is $(i-f)^{-1}(0)$. Thus if $\mathrm{Fix}(f)$ is compact, the degree of $i-f$ over 0 may be interpreted as the number of fixed points of f. This is the intuition for the definition of the fixed point index.

Definition 1.1.29. [Dol95, VII.5.1] We identify $S^n = \mathbb{R}^n \cup \{\infty\}$ and we fix a generator $\alpha \in H_n(S^n) = \mathbb{Z}$. If $U \subset \mathbb{R}^n$ is open and $K \subset U$ is compact, the *fundamental class* $\alpha_K \in H_n(U, U-K)$ *around* K is the image of α by $H_n(S^n) \to H_n(S^n, S^n - K) \simeq H_n(U, U-K)$ (the isomorphism is given by excision).

Definition 1.1.30 (Index in $\mathbb{R}^n$ [Dol95, VII.5.2])**.** Let $U \subset \mathbb{R}^n$ an open subset and let $f: U \to \mathbb{R}^n$ be a map such that $F = \mathrm{Fix}(f)$ is compact. Let $i: U \to \mathbb{R}^n$ be the inclusion. Consider the morphism

$$(i-f)_*: H_n(U, U-F) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$$

Then since $\alpha_{\{0\}} \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \simeq \mathbb{Z}$ is a generator, there is an integer $I(f)$, called the *index of f*, such that

$$(i-f)_*(\alpha_F) = I(f) \cdot \alpha_{\{0\}}$$

Clearly $I(f)$ does not depend on the choice of the generator $\alpha \in H_n(S^n)$.

Definition 1.1.31. We say that a topological space X is an ENR (*Euclidean Neighborhood Retract*) if X is a retract of an open subset of $\mathbb{R}^n$ for some n.

Every compact polyhedron is an ENR. An open subset of an ENR is an ENR. The product of two ENRs is an ENR.

Definition 1.1.32 (Index [Dol95, VII.5.10])**.** If X is an ENR and $U \subset X$ is an open subset, any map $f\colon U \to X$ can be factored as $f = \beta\alpha$, where $U \xrightarrow{\alpha} V \xrightarrow{\beta} X$ and $V \subset \mathbb{R}^n$ is open. If $\mathrm{Fix}(f)$ is compact, we define the *index of f*, by $I(f) = I(\alpha\beta\colon \beta^{-1}(U) \to V)$. The number $I(f)$ does not depend on the factorization. Moreover if $X = \mathbb{R}^n$, this definition of $I(f)$ coincides with Definition 1.1.30. Note that the index does not depend on the codomain of f in the following sense: if $X \subset X'$ and X' is an ENR, we have $i(f\colon U \to X) = i(f\colon U \to X')$.

The following properties for the index follow from analogue properties for the index in $\mathbb{R}^n$. Property (vi) motivates the previous definition.

Proposition 1.1.33 ([Dol95, VII.5.11-15])**.** *Let X be ENR, $U \subset X$ an open subset and $f\colon U \to X$ be a map such that $\mathrm{Fix}(f)$ is compact. The index satisfies the following properties.*

(i) If $W \subset U$ is open and $\mathrm{Fix}(f) \subset W$, then

$$I(f) = I(f|_W)$$

(ii) If f is the constant map x_0, then

$$I(f) = \begin{cases} 1 & \text{if } x_0 \in U \\ 0 & \text{if } x_0 \notin U \end{cases}$$

(iii) If $\{U_i\}$ is a finite open covering of U and $\{U_i \cap \mathrm{Fix}(f)\}$ is a partition of $\mathrm{Fix}(f)$, then

$$I(f) = \sum_i I(f|_{U_i})$$

(iv) If Y is an ENR, $V \subset Y$ an open subset and $g\colon V \to Y$, then

$$I(f \times g) = I(f)I(g)$$

(v) If $h_t\colon U \to X$ is a homotopy and $\bigcup_{t\in I} \mathrm{Fix}(h_t)$ is compact, then

$$I(h_0) = I(h_1)$$

(vi) Let Y be an ENR, $V \subset Y$ be an open subset and $f\colon U \to Y$, $g\colon V \to X$ two maps. Let $\widetilde{U} = f^{-1}(V)$ and $\widetilde{V} = g^{-1}(U)$. Then $\mathrm{Fix}(g \circ f|_{\widetilde{U}})$ and $\mathrm{Fix}(f \circ g|_{\widetilde{V}})$ are homeomorphic and if they are compact then

$$I(g \circ f|_{\widetilde{U}}) = I(f \circ g|_{\widetilde{V}})$$

Remark 1.1.34. The fixed point index is completely characterized by the properties of Proposition 1.1.33 (for a proof, see [JM06, Theorem 2.2.22]). The definition of the index given in [Jia83, I, Definition 3.4] is equivalent to Definition 1.1.32.

The following result intuitively says that $L(f)$ measures the number of fixed points of f.

Theorem 1.1.35 (Lefschetz-Hopf, [Dol95, VII, Proposition 6.6])**.** *If X is a compact connected polyhedron and $f\colon X \to X$ is a map, then $I(f) = L(f)$.*

Definition 1.1.36 (Index of an isolated set of fixed points [Jia83, I, Definition 3.8])**.** Let X be a compact connected polyhedron and let $f\colon X \to X$ be a map. If F is an isolated set of fixed points of f, the *index of F* is given by

$$i(f,F) = I(f|_U)$$

where U is an open subset of X such that $F = U \cap \mathrm{Fix}(f)$. From Proposition 1.1.33 it follows that $i(f,F)$ does not depend on the choice of U.

The index $I(f|_U)$ depends only on $f|_U$, however the index of an isolated set of fixed point F also depends on how f behaves in an open set $U \supset F$.

Using Theorem 1.1.27 and Proposition 1.1.33, we may rephrase the Lefschetz-Hopf theorem.

Theorem 1.1.37 (Lefschetz-Hopf)**.** *Let X be a compact connected polyhedron and let $f\colon X \to X$ be a map. Then*

$$L(f) = \sum_F i(f,F)$$

where the sum is over the fixed point classes of f.

The following result can be used to compute the index of an isolated fix point contained in the interior of a maximal simplex.

Proposition 1.1.38 ([Jia83, I, 3.2])**.** *Let $U \subset \mathbb{R}^n$ be an open subset, $f\colon U \to \mathbb{R}^n$ a differentiable map and $x_0 \in \mathrm{Fix}(f)$. If $\det(1 - Df_{x_0}) \neq 0$, then x_0 is an isolated fixed point and $i(f,\{x_0\}) = \operatorname{sgn}\det(1 - Df_{x_0})$.*

The following lemma is original and is used in the proof of Theorem 1.3.22. We mentioned previously that the index does not depend only on F, but imposing strong hypotheses on F this is the case.

Lemma 1.1.39 ([BSC17, Lemma 4.7])**.** *Let X be a compact and connected polyhedron, $f\colon X \to X$ a map and F an isolated set of fixed points of f. Suppose there is a subspace $K \subseteq X$ which is itself a compact polyhedron that satisfies:*

- *$f(K) \subseteq K$.*
- *K deformation retracts to F.*

- *$F \subseteq K^\circ$, the interior of K.*
- *$F = K \cap \mathrm{Fix}(f)$.*

Then $i(f,F) = \chi(F)$.

Proof. Let $U = K^\circ$. We have $i(f,F) = I(f|_U : U \to X)$. Now

$$I(f|_U : U \to X) = I(f|_U : U \to K) = I(f|_K : K \to K) = L(f|_K) = L(1_F) = \chi(F).$$

The first equality follows from the definition of the fixed point index. The second equality follows from part (i) of Proposition 1.1.33, the third from the Lefschetz-Hopf theorem (Theorem 1.1.35) and the fourth from the fact that $F \hookrightarrow K$ induces isomorphisms in homology. □

Nielsen number

Definition 1.1.40 (Nielsen number, [Jia83, I. Definitions 4.1-2])**.** Let X be a compact connected polyhedron and let $f : X \to X$ be a map. A fixed point class F is *essential* if $i(f,F) \neq 0$. The *Nielsen number of* f, $N(f) \in \mathbb{Z}$ is the number of essential fixed point classes of f. The index of an empty fixed point class is 0 and since there are finite nonempty fixed point classes this number is finite. Directly from the definition we have $\#\mathrm{Fix}(f) \geq N(f)$. The minimum number $\min\{\#\mathrm{Fix}(g) : g \simeq f\}$ of fixed points in the homotopy class of f is denoted by $M(f)$.

Let X be a compact connected polyhedron and $f,g : X \to X$ be two maps. An homotopy H between f and g induces an index preserving bijection between the fixed point classes of f and the fixed point classes of g (for more details see ([Jia83, I, Theorem 2.7], [Jia83, I, Definition 3.3] and [Jia83, I, Theorem 4.5]). Under this bijection some nonempty classes may become empty.

Theorem 1.1.41 (Homotopy invariance [Jia83, I, Theorem 4.6])**.** *Let X be a compact connected polyhedron and let $f,g : X \to X$ be two maps. If $f \simeq g$, then $N(f) = N(g)$.*

From the previous result, since the index of an empty fixed point class is 0, we deduce the fundamental theorem of Nielsen theory

Theorem 1.1.42 (Nielsen, [Jia83, I, Theorem 4.3])**.** *Let X be a compact and connected polyhedron and let $f : X \to X$ be a map. Then $M(f) \geq N(f)$.*

We may ask ourselves how good is the bound given by the previous theorem. The following result, due to Jiang, generalizes earlier results by Shi [Shi66] and Wecken [Wec42].

Theorem 1.1.43 (Jiang, [Jia80, Main Theorem])**.** *Let X be a compact and connected polyhedron and let $f : X \to X$ be a map. If X has no local separating points and X is not a surface (with or without boundary), then $M(f) = N(f)$.*

Proposition 1.1.44 (Commutativity, [Jia83, I, Theorem 5.2])**.** *Let X,Y be compact connected polyhedra and let $f\colon X \to Y$, $g\colon Y \to X$ be two maps. Then $N(fg) = N(gf)$.*

Theorem 1.1.45 (Jiang, [Jia80, Theorem 7.1])**.** *In the category of compact connected polyhedra without global separating points, the fixed point property is a homotopy type invariant.*

Moreover, if $X \simeq Y$ are compact connected polyhedra such that Y lacks the fixed point property and X does not have global separating points, then X lacks the fixed point property.

Proof. Let $X \simeq Y$ be compact connected polyhedra such that Y lacks the fixed point property and X has no global separating points. We will prove X lacks the fixed point property.

If X is a surface, we use the classification of compact surfaces. By Corollary 1.1.18, the only surfaces with the fixed point property are D^2 and $\mathbb{RP}^2$. But X cannot be homeomorphic to D^2 or $\mathbb{RP}^2$ for the Lefschetz fixed point theorem would imply that Y has the fixed point property. Thus we may assume that X is not a surface.

If X has a local separating point, since it is not a global separating point, X retracts to S^1 and thus lacks the fixed point property. Thus we may also assume that X has no local separating points.

Consider a fixed point free map $g\colon Y \to Y$. By Theorem 1.1.42, we have $N(g) = 0$. Let $\alpha\colon X \to Y$ and $\beta\colon Y \to X$ be homotopy inverses and let $f = \beta g \alpha$. By Proposition 1.1.44, we have $N(f) = N(\beta g \alpha) = N(g\alpha\beta) = N(g) = 0$ and by Theorem 1.1.43, there is a fixed point free map homotopic to f. □

The following theorem was proved by F. Wecken with more restrictive hypotheses [Wec42].

Theorem 1.1.46 (Wecken)**.** *Let X be a compact connected polyhedron without global separating points. If $\chi(X) = 0$, then X lacks the fixed point property. Moreover, if X has no local separating points, there is a fixed point free map homotopic to the identity.*

Proof. If X has a local separating point then S^1 is a retract of X thus X lacks the fixed point property. Thus we may assume that X has no local separating points. There are only four compact surfaces with Euler characteristic 0, namely $S^1 \times S^1$, $S^1 \times I$, the Möbius band and Klein's bottle. For each of them there is a fixed point free map homotopic to the identity. Thus we may also assume that X is not a surface.

By Theorem 1.1.43 it is enough to show that $N(1_X) = 0$. Since the identity has a unique nonempty fixed point class and $L(1_X) = \chi(X) = 0$, by the Lefschetz-Hopf theorem we have $N(1_X) = 0$. □

1.2 An example by Lopez

The objective of this section is to present the results obtained by W. Lopez on the fixed point property for polyhedra [Lop67]. The presentation of these results differs slightly from that of

Lopez' paper. Finally we will explain how these results were the motivation for Bing to ask Questions 1 and 8, the main object of study of this chapter.

Among other things, Lopez proved the fixed point property is not a homotopy invariant for compact polyhedra [Lop67]. Obviously, this result precedes Theorem 1.1.45 which was proved by Jiang in 1980. Lopez knew that $\Sigma\mathbb{CP}^{2n}$ has Euler characteristic $1-2n$ and the fixed point property for every n (the proof of this fact involves Steenrod squares). Thus, if one had an example of a polyhedron with the fixed point property and even Euler characteristic, by wedging with one of these spaces one would obtain a space X with the fixed point property and Euler characteristic 0. The space X would have a global separating point and by Theorem 1.1.46, by means of an elementary expansion, one would obtain a polyhedron $Y \simeq X$ without the fixed point property. Lopez gave the first example of a compact polyhedron with the fixed point property and even Euler characteristic, a space X_L with a specific rational cohomology algebra. Later it was noticed by Bredon that the spaces $\mathbb{HP}^{2n+1}$, with $n \geq 1$ provide easier examples of polyhedra with even Euler characteristic and the fixed point property (to prove this Steenrod powers are required).

1.2.1 Steenrod operations and applications

In [Lop67] it is mentioned that from a simple computation using Steenrod squares it follows that $\Sigma\mathbb{CP}^{2n}$ has the fixed point property. Here we write this computation.

Theorem 1.2.1 (Steenrod squares, [Hat02, Section 4.L])**.** *There are homomorphisms*

$$\mathrm{Sq}^i\colon H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$$

defined for every topological space X and $n, i \geq 0$ with the following properties:

(i) If $f\colon X \to Y$, then $\mathrm{Sq}^i \circ f^* = f^* \circ \mathrm{Sq}^i$.

(ii) $\mathrm{Sq}^i(\alpha+\beta) = \mathrm{Sq}^i(\alpha) + \mathrm{Sq}^i(\beta)$.

(iii) $\mathrm{Sq}^i(\alpha \cup \beta) = \sum_{j+k=i} \mathrm{Sq}^j(\alpha) \cup \mathrm{Sq}^k(\beta)$.

(iv) $\mathrm{Sq}^i \circ \sigma = \sigma \circ \mathrm{Sq}^i$ where $\sigma\colon H^n(X;\mathbb{Z}_2) \to H^{n+1}(\Sigma X;\mathbb{Z}_2)$ is the suspension homomorphism.

(v) $\mathrm{Sq}^i(\alpha) = \alpha \cup \alpha$ if $i = |\alpha|$ and $\mathrm{Sq}^i(\alpha) = 0$ if $i > |\alpha|$.

(vi) Sq^0 is the identity.

By these properties, $\mathrm{Sq} = \sum_{i=0}^{\infty} \mathrm{Sq}^i$ is a ring homomorphism $\mathrm{Sq}\colon H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$.

Complex projective space of dimension n, denoted by $\mathbb{CP}^n$, has a CW structure having one $2i$-cell for $i = 0, \dots, n$. If F is a field, there is an isomorphism of graded F-algebras $H^*(\mathbb{CP}^n;F) \simeq F[\alpha]/(\alpha^{n+1})$ where $|\alpha| = 2$. The following describes the Steenrod squares of $\mathbb{CP}^n$.

Lemma 1.2.2. *Let α be the generator of $H^2(\mathbb{CP}^n;\mathbb{Z}_2)$. Then $\mathrm{Sq}^{2i}(\alpha^k) = \binom{k}{i}\alpha^{k+i}$.*

Proof. We have

$$\begin{aligned}\mathrm{Sq}(\alpha^k) &= \mathrm{Sq}(\alpha)^k \\ &= (\mathrm{Sq}^0(\alpha)+\mathrm{Sq}^1(\alpha)+\mathrm{Sq}^2(\alpha))^k \\ &= (\alpha+0+\alpha^2)^k \\ &= \alpha^k(1+\alpha)^k \\ &= \sum_j \binom{k}{j}\alpha^{k+j}\end{aligned}$$

Since Sq^{2i} shifts degree by $2i$, we obtain $\mathrm{Sq}^{2i}(\alpha^k) = \binom{k}{i}\alpha^{k+i}$. □

Theorem 1.2.3. *If m is even then $\Sigma\mathbb{CP}^m$ has the fixed point property.*

Proof. If k is odd, by Lemma 1.2.2

$$\mathrm{Sq}^2\colon H^{2k}(\mathbb{CP}^m;\mathbb{Z}_2)\to H^{2k+2}(\mathbb{CP}^m;\mathbb{Z}_2)$$

is an isomorphism. By part (iv) of Theorem 1.2.1,

$$\mathrm{Sq}^2\colon H^{2k+1}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2)\to H^{2k+3}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2)$$

is an isomorphism for odd k. Now let $f\colon \Sigma\mathbb{CP}^m\to\Sigma\mathbb{CP}^m$. In the following diagram horizontal arrows are isomorphisms

$$\begin{tikzcd} H^{2k+1}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\sim"'] \arrow[d, "f^*"'] & H^{2k+3}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2) \arrow[d, "f^*"] \\ H^{2k+1}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2) \arrow[r, "\sim", "\mathrm{Sq}^2"'] & H^{2k+3}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2) \end{tikzcd}$$

thus, for odd k we have

$$\mathrm{tr}(f^*\colon H^{2k+1}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2)\to H^{2k+1}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2)) = \mathrm{tr}(f^*\colon H^{2k+3}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2)\to H^{2k+3}(\Sigma\mathbb{CP}^m;\mathbb{Z}_2)).$$

From this, it follows that $L(f^*;\mathbb{Z}_2)=1$. Finally, by the Lefschetz fixed point theorem, f has a fixed point. □

Clearly $\mathbb{HP}^1 = S^4$ does not have the fixed point property. The following proposition follows from a computation using Steenrod powers for the prime 3 (see [Hat02, Example 4.L.4]).

Proposition 1.2.4. *If $n\geq 2$, then $\mathbb{HP}^n$ has the fixed point property.*

As we mentioned earlier, the relevant property that the space X_L found by Lopez is that it has the fixed point property and positive even Euler characteristic. Later it was noticed by G. E. Bredon that, for $n>0$ the spaces $\mathbb{HP}^{2n+1}$ also have this property [Fad70].

1.2.2 Lopez' space

In this section, we construct Lopez' space X_L, having the fixed point property and even Euler characteristic. Our proof is mostly the same as Lopez' but is phrased so that it makes clear that any compact polyhedron with the same rational cohomology algebra as X_L has the fixed point property.

Theorem 1.2.5 (Lopez). *Let X be a compact polyhedron such that*

$$H^*(X;\mathbb{Q}) \simeq \mathbb{Q}[\alpha,\beta]/(\alpha^3,\alpha^2\beta,\alpha\beta^2,\beta^5)$$

as graded $\mathbb{Q}$-algebras, with $|\alpha| = |\beta| = 2$. Then X has the fixed point property.

Remark 1.2.6. A more concrete description of the cohomology ring is the following

$$\begin{aligned}
&H^0(X;\mathbb{Q}) = \mathbb{Q} && \text{with basis } 1\\
&H^2(X;\mathbb{Q}) = \mathbb{Q}\oplus\mathbb{Q} && \text{with basis } \alpha,\beta\\
&H^4(X;\mathbb{Q}) = \mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q} && \text{with basis } \alpha^2,\alpha\beta,\beta^2\\
&H^6(X;\mathbb{Q}) = \mathbb{Q} && \text{with basis } \beta^3\\
&H^8(X;\mathbb{Q}) = \mathbb{Q} && \text{with basis } \beta^4\\
&H^k(X;\mathbb{Q}) = 0 && \text{if } k \neq 0,2,4,6,8
\end{aligned}$$

and we have

$$\alpha^3 = \alpha^2\beta = \alpha\beta^2 = \beta^5 = 0$$

Since the cohomology is concentrated on even degrees the ring is commutative. Moreover, we have $\chi(X) = 8$.

Proof. Let $f\colon X \to X$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix of f^* in the basis $\{\alpha,\beta\}$ we have

$$0 = f^*(0) = f^*(\alpha^3) = f^*(\alpha)^3 = (a\alpha + c\beta)^3 = a^3\alpha^3 + 3a^2c\alpha^2\beta + 3ac^2\alpha\beta^2 + c^3\beta^3 = c^3\beta^3$$

thus $c = 0$. Now since f^* is a ring homomorphism, we can compute the trace of f

$$\begin{aligned}
&\mathrm{tr}(f^*\colon H^0(X;\mathbb{Q}) \to H^0(X;\mathbb{Q})) = 1\\
&\mathrm{tr}(f^*\colon H^2(X;\mathbb{Q}) \to H^2(X;\mathbb{Q})) = a+d\\
&\mathrm{tr}(f^*\colon H^4(X;\mathbb{Q}) \to H^4(X;\mathbb{Q})) = a^2+ad+d^2\\
&\mathrm{tr}(f^*\colon H^6(X;\mathbb{Q}) \to H^6(X;\mathbb{Q})) = d^3\\
&\mathrm{tr}(f^*\colon H^8(X;\mathbb{Q}) \to H^8(X;\mathbb{Q})) = d^4
\end{aligned}$$

and then

$$
\begin{aligned}
L(f^*,\mathbb{Q}) &= 1+a+d+a^2+ad+d^2+d^3+d^4 \\
&= \frac{1}{4}((2a+d+1)^2+(2d^2+d)^2+d^2+(d+1)^2+2) \\
&> 0
\end{aligned}
$$

Hence, by the Lefschetz fixed point theorem, f has a fixed point. $\square$

Recall that $\mathbb{CP}^n$ has a CW-structure with one k-cell of dimension $2k$ for $k = 0,\dots,n$. We may identify $S^2 = \mathbb{CP}^1$ and thus we have an inclusion $i\colon S^2 \hookrightarrow \mathbb{CP}^n$.

Definition 1.2.7. Lopez' space X_L is the colimit of the following diagram.

$$
\begin{tikzcd}
 & \mathbb{CP}^2 \\
S^2 \arrow[ur, hook, "i"'] \arrow[dr, hook, "(\mathbb{1}_{S^2},s_0)"'] & \\
 & S^2 \times S^2 \\
S^2 \arrow[ur, hook, "(s_0,\mathbb{1}_{S^2})"'] \arrow[dr, hook, "i"'] & \\
 & \mathbb{CP}^4
\end{tikzcd}
$$

where, $s_0\colon S^2 \to S^2$ denotes the constant map with image the unique 0-cell of S^2. Let $i_{S^2\times S^2}\colon S^2 \times S^2 \to X_L$, $i_{\mathbb{CP}^2}\colon \mathbb{CP}^2 \to X_L$, $i_{\mathbb{CP}^4}\colon \mathbb{CP}^4 \to X_L$ be the inclusions. The natural cell structure on X_L has one 0-cell e_1, two 2-cells e_α, e_β, three 4-cells $e_{\alpha^2}, e_{\alpha\beta}, e_{\beta^2}$, one 6-cell e_{β^3} and one 8-cell e_{β^4}. The cells $e_1, e_\alpha, e_{\alpha^2}$ form $\mathbb{CP}^2$, the cells $e_1, e_\alpha, e_\beta, e_{\alpha\beta}$ form $S^2 \times S^2$ and the cells $e_1, e_\beta, e_{\beta^2}, e_{\beta^3}, e_{\beta^4}$ form $\mathbb{CP}^4$.

Recall that $H^*(S^2 \times S^2;\mathbb{Q}) = \mathbb{Q}[\alpha,\beta]/(\alpha^2,\beta^2)$ where $|\alpha| = |\beta| = 2$.

Proposition 1.2.8 (Lopez)**.** *The cohomology with rational coefficients of X_L is given by*

$$H^*(X_L;\mathbb{Q}) = \mathbb{Q}[\alpha,\beta]/(\alpha^3,\alpha^2\beta,\alpha\beta^2,\beta^5)$$

where $|\alpha| = |\beta| = 2$. Thus X_L has the fixed point property.

Proof. Note that every cell of X_L has even dimension. Then we can identify $H^k(X_L;\mathbb{Q}) = \mathrm{Hom}(C_k(X_L),\mathbb{Q})$. It is easy to see that in each degree the dimensions of the two $\mathbb{Q}$-algebras coincide. Let $\alpha = e_\alpha^*$, $\beta = e_\beta^*$. We first prove $\alpha^2, \alpha\beta, \beta^2$ is linearly independent. If $p\alpha^2 +$

$q\alpha\beta + r\beta^2 = 0$, then pulling back with $i^*_{\mathbb{CP}^2}$, $i^*_{S^2\times S^2}$ and $i^*_{\mathbb{CP}^4}$ we obtain $p = 0$, $q = 0$, $r = 0$ respectively. To prove $\beta^3 \in H^6(X_L;\mathbb{Q})$ and $\beta^4 \in H^8(X_L;\mathbb{Q})$ are generators we use the same idea. To prove $\alpha^3 = \alpha\beta^2 = \alpha^2\beta = 0$, we note $i^*_{\mathbb{CP}^4} : H^6(X_L;\mathbb{Q}) \to H^6(\mathbb{CP}^4;\mathbb{Q})$ is an isomorphism and $i^*_{\mathbb{CP}^4}(\alpha) = 0$. This completes our computation of the cup product of X_L. □

1.2.3 Consequences

Now we will see some interesting consequences of the existence of X_L. Perhaps the most relevant of this consequences is that the fixed point property is not a homotopy invariant for compact polyhedra.

Theorem 1.2.9 (Lopez)**.** *The fixed point property is not a homotopy invariant in the category of compact polyhedra.*

Proof. Consider $X = X_L \vee \Sigma\mathbb{CP}^8$. Since it is a wedge of spaces with the fixed point property, X has the fixed point property. We have $\chi(X) = \chi(X_L) + \chi(\Sigma\mathbb{CP}^8) - 1 = 8 + (-7) - 1 = 0$. By Proposition 1.1.10, there is a compact polyhedron $Y \simeq X$ without global separating points (in effect, since X_L and $\Sigma\mathbb{CP}^8$ do not have global separating points, an elementary expansion is enough to attain this). Finally by Theorem 1.1.46, we conclude that Y lacks the fixed point property. □

For $A \times B$ to have the fixed point property it is necessary that A and B have it. In 1930, Kuratowski asked if this is sufficient [Kur30]. Using the space X_L, Lopez gave a negative answer to this question.

Theorem 1.2.10 (Lopez)**.** *There is a compact polyhedron X with the fixed point property and such that $X \times I$ and $X \times X$ do not have the fixed point property.*

Proof. We can take $X = X_L \vee \Sigma\mathbb{CP}^8$. We have $\chi(X) = 0$ thus $\chi(X \times I) = \chi(X \times X) = 0$. Moreover, $X \times I$ and $X \times X$ do not have global separating points. To conclude we use Theorem 1.1.46 as before. □

Theorem 1.2.11 (Lopez)**.** *There is a compact polyhedron with the fixed point property such that ΣX lacks the fixed point property.*

Proof. The space $X = X_L \vee \Sigma\mathbb{CP}^6$ has the fixed point property and $\chi(X) = 2$. Then $\chi(\Sigma X) = 0$ and ΣX has no global separating points. Then, by Theorem 1.1.46, ΣX lacks the fixed point property. □

Theorem 1.2.12 ([Fad70, Corollary 4.6])**.** *There is a compact polyhedron X with the fixed point property such that the join $X * X$ lacks the fixed point property.*

Proof. We consider $X = X_L \vee \Sigma\mathbb{CP}^8$. We know that X has the fixed point property and that $\chi(X) = 0$. Now, $\chi(X * X) = \chi(X) + \chi(X) - \chi(X)\chi(X) = 0$. Moreover $X * X$ has no global separating points and thus by Theorem 1.1.46, $X * X$ does not have the fixed point property. □

Theorem 1.2.13 ([Fad70, Corollary 4.6])**.** *There are compact polyhedra X, Y with the fixed point property such that the smash product $X \wedge Y$ lacks the fixed point property.*

Proof. Let $X = X_L \vee \Sigma\mathbb{CP}^8$ and $Y = X_L \vee \Sigma\mathbb{CP}^6$. We already know that X and Y have the fixed point property. We have $\chi(X \wedge Y) = \chi(X)\chi(Y) - \chi(X) - \chi(Y) + 2 = 0$. Since $X \wedge Y$ has no global separating points, by Theorem 1.1.46 we conclude that $X \wedge Y$ does not have the fixed point property. □

1.2.4 Bing's questions

In his 1969 article "The elusive fixed point property", R. H. Bing asked 12 questions regarding the fixed point property [Bin69]. As of 2014, eight of these questions had been answered [Hag07]. Questions 1, 8 and 11 concern polyhedra and were inspired by Lopez' results. Question 11 was answered by G. E. Bredon in 1971 [Bre71]. Questions 1 and 8, were answered recently in our article [SC17b]. Next we recall these questions.

As seen in the previous section, the results of Lopez and Fadell on the homotopy invariance of the fixed point property and its behaviour with respect to classical constructions rely on the existence of a space X_L with the fixed point property and even Euler characteristic. The space X_L has dimension 8. The first of Bing's questions concerns the existence of a similar example of smaller dimension.

Question 1.2.14 (Bing's Question 1)**.** *Is there a compact 2-dimensional polyhedron with the fixed point property which has even Euler characteristic?*

Note that even if the answer to this question is affirmative, it is not immediate that the fixed point property is not a homotopy invariant for 2-dimensional polyhedra (by Corollary 1.1.17 there is no 2-dimensional polyhedron with the fixed point property and negative Euler characteristic). We shall mention that, although Question 1.2.14 refers to polyhedra with even Euler characteristic, an example with odd Euler characteristic different from 1 was neither known.

Definition 1.2.15. A compact 2-dimensional polyhedron is a *Bing space* if it has the fixed point property and $\chi(X) \neq 1$.

By Corollary 1.1.18, if X is a Bing space then $H_1(X)$ must be torsion. Therefore, the cup product and Steenrod squares do not give any information. Waggoner studied the case of Question 1.2.14 where the fundamental group is trivial [Wag75]. The proof of Theorem 1.2.9 motivated the following question.

Question 1.2.16 (Bing's Question 8)**.** *What is the least value of n so that there is an n-dimensional polyhedron X with the fixed point property and a disk D such that $D \cap X$ is an arc but $X \cup D$ does not have the fixed point property?*

The answer to Question 1.2.16 is at least 2, since a 1-dimensional polyhedron X with the fixed point property is a tree, and $X \cup D$ would be contractible. If we ask the same question for wilder spaces, the answer is 1 [Bin69, Theorem 15].

Question 1.2.16 is related to the homotopy invariance of the fixed point property in a precise way. By Theorem 1.1.45, the least n such that the fixed point property is not a homotopy invariant for n-dimensional polyhedra is the answer to Question 1.2.16. According to Hagopian [Hag07], Bing conjectured that the answer to Question 1.2.16 is 2. The following question was also inspired by the work of Lopez.

Question 1.2.17 (Bing's Question 11)**.** *If X and Y are polyhedra with the fixed point property and without local separating points, must $X \times Y$ have the fixed point property?*

Question 1.2.17 has a negative answer as shown by Bredon [Bre71]. Husseini [Hus77] proved that the product of two manifolds with the fixed point property may lack the fixed point property. A detailed exposition on the relation between products and the fixed point property is given in [Bro82]. The following question remains open: Is there a closed manifold with the fixed point property such that $M \times M$ lacks the fixed point property? For a recent article on this topic see [KS17].

1.3 Two dimensional complexes with abelian fundamental group

In this section, following [BSC17], we prove that a Bing space cannot have abelian fundamental group. While doing this we will obtain some necessary conditions for a space to be Bing, for example we will show that the second homology group of its fundamental group cannot be trivial and that the Euler characteristic of such space must be minimum among 2-complexes that have the same fundamental group. Using similar ideas, we will prove that the fundamental group of a Bing space cannot be a finite subgroup of $\mathrm{SO}(3)$. The start point for these results is the correspondence between 2-complexes and group presentations (see Section A.3).

Example 1.3.1. If G is an abelian finite group with invariant factors $m_1 \mid m_2 \mid \ldots \mid m_n$. The presentations

$$\mathcal{T}_d = \langle a_1, \ldots, a_n | a_1^{m_1}, \ldots, a_n^{m_n}, [a_1^d, a_2], [a_i, a_j], i < j, (i,j) \neq (1,2) \rangle$$

of G for $(d, m_1) = 1$ are called *twisted* presentations. The complexes $X_{\mathcal{T}_d}$ will appear often in this section.

Definition 1.3.2. The *deficiency* of a presentation $\mathcal{P} = \langle a_1, \dots, a_n \mid r_1, \dots, r_k \rangle$ is defined by $\operatorname{def}(\mathcal{P}) = k - n$. Therefore, $\chi(X_{\mathcal{P}}) = \operatorname{def}(\mathcal{P}) + 1$. Given a finitely presented group G, its *deficiency* $\operatorname{def}(G)$ is the minimum possible deficiency of a presentation of G. Then, for any compact connected 2-complex X with fundamental group G, it follows that $\chi(X) \geq \operatorname{def}(G) + 1$. We say that X has *minimum Euler characteristic* if $\chi(X) = \operatorname{def}(G) + 1$.

Example 1.3.3. If $H_1(G)$ is torsion, then $\operatorname{def}(G) \geq 0$. The presentation $\langle a \mid a^m \rangle$ thus shows $\mathbb{Z}_m$ has deficiency 0. The presentation $\mathcal{P} = \langle a, b \mid a^2, b^4, [a,b] \rangle$ proves $\operatorname{def}(\mathbb{Z}_2 \oplus \mathbb{Z}_4) \leq 1$. Next, we will see that this presentation realizes the deficiency (i.e. that $X_{\mathcal{P}}$ has minimum Euler characteristic).

If X is path–connected, $\Sigma_2(X)$ denotes the image of the Hurewicz map $h\colon \pi_2(X, x_0) \to H_2(X)$. The elements of $\Sigma_2(X)$ are called *spherical elements*.

Theorem 1.3.4 (Hopf, [Bro94, II, Theorem 5.2])**.** *If X is path–connected there is a short exact sequence*

$$0 \to \Sigma_2(X) \to H_2(X) \to H_2(K(\pi_1(X, x_0), 1)) \to 0.$$

The map $i\colon \Sigma_2(X) \to H_2(X)$ is the inclusion. Attaching to X cells of dimension greater or equal than 3, we can form a space of type $K(\pi_1(X, x_0), 1)$. The morphism $j_*\colon H_2(X) \to H_2(K(\pi_1(X, x_0)))$ is induced by the inclusion $j\colon X \to K(\pi_1(X, x_0), 1)$. Exactness at the left and right terms is immediate. By naturality of the Hurewicz homomorphism, we have $j_* \circ i = 0$. We omit the proof of exactness at the middle term.

The short exact sequence in Theorem 1.3.4 can be used to obtain a lower bound on the deficiency of a finitely presented group. If $\mathcal{P} = \langle a_1, \dots, a_n \mid r_1, \dots, r_k \rangle$ is a presentation of G realizing the minimum deficiency, the existence of an epimorphism $H_2(X_{\mathcal{P}}) \to H_2(G) = H_2(K(G,1))$ implies $\operatorname{rk}(H_2(X_{\mathcal{P}}))$ is greater or equal than the number of invariant factors of $H_2(G)$. Then

$$\begin{aligned} \operatorname{def}(G) &= k - n \\ &= \chi(X_{\mathcal{P}}) - 1 \\ &= \operatorname{rk}(H_2(X_{\mathcal{P}})) - \operatorname{rk}(H_1(X_{\mathcal{P}})) \\ &= \operatorname{rk}(H_2(X_{\mathcal{P}})) - \operatorname{rk}(H_1(G)) \end{aligned}$$

where $H_1(G)$ is, of course, the abelianization of G. Then we have

$$\operatorname{def}(G) \geq \text{number of invariant factors of } H_2(G) - \operatorname{rk}(H_1(G)).$$

If in addition $H_1(G)$ is a torsion group we have

$$\operatorname{def}(G) \geq \text{number of invariant factors of } H_2(G).$$

Definition 1.3.5. Let G be a finitely presentable group. If equality is attained in the previous inequality, we say that G is *efficient*. If $\mathcal{P}$ is a presentation realizing the deficiency of an efficient group, we say $\mathcal{P}$ is *efficient*.

Example 1.3.6. From the Künneth formula it follows that the Schur multiplier $H_2(G)$ of an abelian finite group $G = \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_n}$ with $d_1 \mid \ldots \mid d_n$ has $\binom{n}{2}$ invariant factors. A twisted presentation $\mathcal{T}_d$ has deficiency $\binom{n}{2}$, thus is efficient.

B. H. Neumann asked if every group with trivial Schur multiplier is efficient [Neu56]. R.G. Swan proved that the answer is negative [Swa65]. Swan's examples are semidirect products $(\mathbb{Z}_7)^n \rtimes \mathbb{Z}_3$, where the generator of $\mathbb{Z}_3$ acts on $(\mathbb{Z}_7)^n$ as multiplication by 2. These groups have trivial Schur multiplier and are not efficient for sufficiently large values of n.

Primitive spherical elements and Waggoner's theorem

A strategy for proving that a space X without global separating points lacks the fixed point property is to show that there exists a space $Y \simeq X$ that has S^n as a retract. Waggoner used this idea to prove Theorem 1.3.9. We will need the following result from obstruction theory.

Theorem 1.3.7 ([Spa66, Theorem 8.4.1])**.** *Let (Y, y_0) be a pointed $(n-1)$-connected space where $n \geq 1$ and let (X, A) be a CW-pair such that $H^{q+1}(X, A; \pi_q(Y, yo)) = 0$ for $q > n$. Let $f\colon A \to Y$ be a map. If $\delta f^*\colon H^n(Y; \pi_n(Y, y_0)) \to H^{n+1}(X, A; \pi_n(Y, y_0))$ is the zero morphism, then f can be extended over X.*

The following lemma is due to Waggoner [Wag72]. We provide here a simpler proof.

Lemma 1.3.8 (Waggoner)**.** *Let (X, S^n) be a CW-pair with $\dim(X) \leq n+1$, $n \geq 1$. If $i_*\colon H_n(S^n) \to H_n(X)$ is a split monomorphism, then S^n is a retract of X.*

Proof. By the universal coefficient theorem we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), \pi_n(S^n)) \arrow[r] \arrow[d, "i^*"] & H^n(X; \pi_n(S^n)) \arrow[r] \arrow[d, "(i_*)^*"] & \mathrm{Hom}(H_n(X), \pi_n(S^n)) \arrow[r] \arrow[d, "(i_*)^*"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(S^n), \pi_n(S^n)) \arrow[r] & H^n(S^n; \pi_n(S^n)) \arrow[r] & \mathrm{Hom}(H_n(S^n), \pi_n(S^n)) \arrow[r] & 0
\end{tikzcd}$$

Since $H_n(i)$ and $H_{n-1}(i)$ are split monomorphisms, the vertical maps induced by them are epimorphisms. Thus by the five lemma ([Wei94, Exercise 1.3.3]), the map

$$i^*\colon H^n(X; \pi_n(S^n)) \to H^n(S^n; \pi_n(S^n))$$

is an epimorphism. Therefore, the connecting map

$$\delta\colon H^n(S^n; \pi_n(S^n)) \to H^{n+1}(X, S^n; \pi_n(S^n))$$

is trivial. Moreover, if $q > n$, we have $H^q(S^n;\pi_q(S^n)) = 0$ and $H^{q+1}(X;\pi_q(S^n)) = 0$. From the long exact sequence of cohomology groups for the pair (X,S^n), it follows that

$$H^{q+1}(X,S^n;\pi_q(S^n)) = 0$$

for $q > n$. Finally, by Theorem 1.3.7, $1_{S^n}\colon S^n \to S^n$ extends to X. $\square$

The following result for $n \neq 2,3$ is due to Waggoner [Wag72]. The case $n = 3$ was obtained later by [Jia80].

Theorem 1.3.9 (Waggoner)**.** *If X is a compact $(n-2)$-connected polyhedron of dimension $n \neq 2$ and $\widetilde{H}_*(X;\mathbb{Q}) \neq 0$, then X does not have the fixed point property.*

Proof. The cases $n = 0$ and $n = 1$ are evident. Suppose $n \geq 3$. First we will reduce to the case where X has no global separating points. We write $X = X_1 \vee \ldots \vee X_k$, where each X_i is a 1-simplex or or a polyhedron without global separating points. Then for some i we must have $\widetilde{H}_*(X;\mathbb{Q}) \neq 0$ and X_i must be $(n-2)$-connected (this is because X_i is a retract of X). Then it is enough to see that X_i lacks the fixed point property.

By the Hurewicz Theorem, we have $\widetilde{H}_k(X) = 0$ for $0 \leq k \leq n-2$, that $h_{n-1}\colon \pi_{n-1}(X) \to H_{n-1}(X)$ is an isomorphism and that $h_n\colon \pi_n(X) \to H_n(X)$ is an epimorphism. Then, since $\widetilde{H}_*(X;\mathbb{Q}) \neq 0$ for j equal to $n-1$ or n, there is a split monomorphism $\mathbb{Z} \hookrightarrow H_j(X)$ and since $h_j\colon \pi_j(X) \to H_j(X)$ is an epimorphism, from the definition of the Hurewicz homomorphism, there is $f\colon S^j \to X$ such that $f_*\colon H_j(S^j) \to H_j(X)$ is a split monomorphism. Subdividing if necessary, we may assume f is simplicial. Now applying Lemma 1.3.8 to the pair $(M(f),S^j)$, we conclude $M(f)$ lacks the fixed point property. Moreover, $M(f)$ is a polyhedron [Coh67, Proposition 9.8] and thus by Theorem 1.1.45, X lacks the fixed point property. $\square$

Theorem 1.3.9 can be seen as a higher dimensional analogue of Bing's Question 1. Whether this result also holds for $n = 2$ was not known previously. The same result does not hold for $n = 2$ as we will see in Section 1.4.

Next, we explore to what extent we can apply Waggoner's ideas. We will conclude that the Schur multiplier of the fundamental group of a Bing space cannot be trivial. We will also prove that, if the fundamental group G of a Bing space X is freely indecomposable, then G must be efficient and X must have minimum Euler characteristic.

Let F be a free abelian group. We say that $a \in F$ is *primitive* in F if the homomorphism $\mathbb{Z} \to F$ defined by $1 \mapsto a$ is a split monomorphism. This is equivalent to saying that $\{a\}$ can be extended to a basis of F.

Lemma 1.3.10. *Let*

$$0 \to S \to \mathbb{Z}^k \to A \to 0$$

be a short exact sequence of abelian groups. The following are equivalent *(i) The number of invariant factors of A is strictly smaller than k.*

(ii) There exists $a \in S$ primitive in $\mathbb{Z}^k$.

Proof. Suppose that A has $n < k$ invariant factors. There is an epimorphism $\mathbb{Z}^n \to A$. Since $\mathbb{Z}^k$ is projective, there is f such that the following diagram commutes:

$$\begin{tikzcd} & & & \mathbb{Z}^n \arrow[d, two heads] & \\ 0 \arrow[r] & S \arrow[r] & \mathbb{Z}^k \arrow[r] \arrow[ur, "f"] & A \arrow[r] & 0 \end{tikzcd}$$

Then it is enough to find an element $a \in \ker(f)$ primitive in $\mathbb{Z}^k$. There are bases $B = \{b_1, \ldots, b_k\}$ and $B' = \{b'_1, \ldots, b'_n\}$ of $\mathbb{Z}^k$ and $\mathbb{Z}^n$ respectively, such that the matrix of f in these bases is in Smith normal form and thus diagonal. The last $k-n$ columns of this matrix are 0 and then $b_k \in \ker(f) \subset S$ is primitive in $\mathbb{Z}^k$.

Conversely, suppose that $b_1 \in S$ is primitive in $\mathbb{Z}^k$. Then there is a basis $B = \{b_1, \ldots, b_k\}$ of $\mathbb{Z}^k$. Now letting $[x]$ be the class of $x \in \mathbb{Z}^k$ in A, the elements $\{[b_2], \ldots, [b_k]\}$ generate A, hence the number of invariant factors of A is at most $k-1$. □

Proposition 1.3.11. *Let X be a compact connected 2-dimensional polyhedron. The following are equivalent:*

(i) X is homotopy equivalent to a polyhedron Y having S^2 as a retract.

(ii) There is a map $f\colon S^2 \to X$ such that $H_2(f)$ is a split monomorphism.

(iii) There exists $a \in \Sigma_2(X)$ primitive in $H_2(X)$.

(iv) The number of invariant factors of $H_2(\pi_1(X))$ is strictly smaller than the rank of $H_2(X)$.

Proof. (i) $\Rightarrow$ (ii) is clear. For (ii) $\Rightarrow$ (i) we apply Lemma 1.3.8 to the pair $(M(f), S^2)$, where $M(f)$ is the mapping cylinder of f.

(ii) $\iff$ (iii) follows from the definition of the Hurewicz homomorphism.

(iii) $\iff$ (iv) follows from Lemma 1.3.10. □

Theorem 1.3.12. *If $H_2(G) = 0$ there are no Bing spaces with fundamental group G.*

Proof. Suppose X is a Bing space with $\pi_1(X) = G$. We write $X = X_1 \vee \ldots \vee X_m$, where each X_i is a polyhedron without global separating points or a 1-simplex (the base points of the wedges may not be the same). For some i, we must have $H_2(X_i) \neq 0$. Since X_i is a retract of X, it must have the fixed point property. On the other hand, $\pi_1(X_i)$ is a free factor of G, and then $H_2(G) = 0$ implies $H_2(\pi_1(X_i)) = 0$ ([Wei94, Corollary 6.2.10]). By Proposition 1.3.11 and Theorem 1.1.45, X_i lacks the fixed point property, a contradiction. □

The case $G = 0$ of Theorem 1.3.12 was studied previously by Waggoner [Wag75].

Corollary 1.3.13. *There are no Bing spaces with fundamental group isomorphic to the trivial group, cyclic groups, dihedral groups of order* $2 \pmod 4$*,* $\mathrm{SL}(n,\mathbb{F}_q)$ *(for* $(n,q) \neq (2,4)$*,* $(2,9)$*,* $(3,2)$*,* $(3,4)$*,* $(4,2)$*), deficiency-zero groups (e.g. the quaternion group), groups of square-free order (more generally, any group in which every Sylow subgroup has trivial Schur multiplier),* 13 *of the* 26 *sporadic simple groups and many infinite families of finite simple groups of Lie type.*

Proof. All these groups have trivial Schur multiplier. For cyclic groups, dihedral groups and $\mathrm{SL}(n,\mathbb{F}_q)$ this appears in [Wei94]. For deficiency-zero groups it is clear. For groups in which every Sylow subgroup has trivial Schur multiplier, it follows from [Bro94, Chapter III, Corollary 10.2 and Theorem 10.3]. For the statement about finite simple groups, see [GLS98, Section 6.1]. $\square$

Definition 1.3.14. A group G is said to be *freely indecomposable* if $G = H * K$ implies $H = 1$ or $K = 1$.

Finite groups and abelian groups clearly are freely indecomposable. The following reduction will be useful later in Section 1.3.

Proposition 1.3.15. *Let X be a Bing space with freely indecomposable fundamental group G. Then there is a Bing space $Y \simeq X$ without global separating points.*

Proof. Fix a triangulation of X. If X has a global separating point and is not a 1-simplex, then X is a wedge of two polyhedra X_1, X_2, each with fewer vertices than X. By van Kampen's theorem $G = \pi_1(X_1) * \pi_1(X_2)$ and since G is freely indecomposable, one of these two polyhedra, say X_2, is simply-connected. By Theorem 1.3.12 there are no simply-connected Bing spaces, so $\widetilde{H}_*(X_2) = 0$. Therefore X_2 is contractible and then $X = X_1 \vee X_2 \simeq X_1$. By induction there exists a Bing space $Y \simeq X_1$ without global separating points. $\square$

Proposition 1.3.16. *Let G be a freely indecomposable group. Suppose X is a Bing space with fundamental group G. Then G is efficient and X has minimum Euler characteristic.*

Proof. By Proposition 1.3.15, we may assume X has no global separating points. If the rank of $H_2(X)$ is strictly greater than the number of invariant factors of $H_2(G)$, by Proposition 1.3.11 and Theorem 1.1.45, we obtain a contradiction. $\square$

Using the Künneth formula it is easy to see that abelian finite groups are efficient and, excluding cyclic groups, have nontrivial Schur multiplier. To show these groups are not the fundamental group of a Bing space we will need other methods.

Bing spaces with abelian fundamental group

The following lemma is the central piece in the proof that there are no Bing spaces with abelian fundamental group.

Lemma 1.3.17. *Let $\mathcal{P} = \langle a, b \mid a^m, b^n, [a,b] \rangle$. Then $X_\mathcal{P}$ does not have the fixed point property.*

Proof. By Theorem 1.1.43 it suffices to find a map $f\colon X_\mathcal{P} \to X_\mathcal{P}$ such that $N(f) = 0$. Let $T = S^1 \times S^1 \subseteq \mathbb{C} \times \mathbb{C}$. The complex $X_\mathcal{P}$ can be identified with the following pushout:

$$\begin{tikzcd} S^1_a \coprod S^1_b \arrow[r, "(z^m,1)\coprod(1,z^n)"] \arrow[d, hook] & T \arrow[d, hook, "i_T"] \\ D^2_a \coprod D^2_b \arrow[r, "i_a \coprod i_b"'] & X_\mathcal{P} \end{tikzcd}$$

Here $S^1_a, S^1_b, D^2_a, D^2_b \subseteq \mathbb{C}$ denote copies of 1-dimensional spheres and 2-dimensional disks. We define $f_T : T \to X_\mathcal{P}$, $f_a\colon D^2_a \to X_\mathcal{P}$ y $f_b\colon D^2_b \to X_\mathcal{P}$ by

$$f_T(z,w) = i_T\,(-z, -\overline{w})$$

$$f_a(z) = \begin{cases} i_a(2z) & \text{if } 0 \le |z| \le \frac{1}{2} \\ i_T\left(\frac{z^m}{|z|^m}\exp(i\pi(2|z|-1)), \exp(i\pi(2|z|-1))\right) & \text{if } \frac{1}{2} \le |z| \le 1 \end{cases}$$

$$f_b(z) = \begin{cases} i_b(2\overline{z}) & \text{if } 0 \le |z| \le \frac{1}{2} \\ i_T\left(\exp(i\pi(2|z|-1)), \frac{\overline{z}^n}{|z|^n}\exp(i\pi(2|z|-1))\right) & \text{if } \frac{1}{2} \le |z| \le 1 \end{cases}$$

A simple verification shows that f_T, f_a and f_b are well-defined and continuous and that they determine a continuous map $f : X_\mathcal{P} \to X_\mathcal{P}$.

It is easy to see that the only fixed points of f are $i_a(0)$ and $i_b(0)$. We will show that they are equivalent. We exhibit a path c from $i_a(0)$ to $i_b(0)$ such that c and $f \circ c$ are homotopic. Consider the paths $\gamma_a, \delta_a, \delta_b, \gamma_b\colon [0,1] \to X_\mathcal{P}$ defined by

$$\begin{aligned} \gamma_a(t) &= i_a\,(t/2) \\ \delta_a(t) &= i_a\,(1/2 + t/2) \\ \delta_b(t) &= i_b\,(1 - t/2) \\ \gamma_b(t) &= i_b\,(1/2 - t/2) \end{aligned}$$

The concatenation $c = \gamma_a * \delta_a * \delta_b * \gamma_b$ is a well-defined path from $i_a(0)$ to $i_b(0)$. In order to prove

$$\gamma_a * \delta_a * \delta_b * \gamma_b \simeq f \circ (\gamma_a * \delta_a * \delta_b * \gamma_b)$$

it suffices to show that

$$\gamma_a * \delta_a = f \circ \gamma_a \tag{1.1}$$

$$\delta_b * \gamma_b = f \circ \gamma_b \tag{1.2}$$

$$e_{i_T(1,1)} \simeq (f \circ \delta_a) * (f \circ \delta_b), \tag{1.3}$$

where $e_{i_T(1,1)}$ denotes the constant loop at $i_T(1,1)$. Equalities 1.1 and 1.2 are clear, 1.3 follows from

$$(f \circ \delta_a)(t) = i_T(\exp(i\pi t), \exp(i\pi t)) = (f \circ \delta_b)(1-t).$$

Then f has a unique nonempty fixed point class. We must prove this class is not essential. One way to show this is noting that the fixed point indices of $i_a(0)$, $i_b(0)$ are 1, -1 respectively (by Proposition 1.1.38). Another way is by computing $L(f) = 0$ and invoking Theorem 1.1.37. □

If G is any finite group, above the minimum Euler characteristic all 2-complexes with fundamental group G are homotopy equivalent. That fact along with Theorem 1.3.18 below constitutes the classification of homotopy types of compact 2-complexes with finite abelian fundamental group. We refer to [HAMS93, Chapter III] and [GL91] for a detailed exposition of this topic.

Theorem 1.3.18 (Browning, [HAMS93, Chapter III, Theorem 2.11])**.** *Let G be a finite abelian group with invariant factors $m_1 \mid m_2 \mid \dots \mid m_n$. The number of homotopy types of compact connected 2-complexes with fundamental group G and minimum Euler characteristic is given by $\left|\mathbb{Z}_{m_1}^* / \pm (\mathbb{Z}_{m_1}^*)^{n-1}\right|$. Every such complex is homotopy equivalent to the presentation complex of a twisted presentation*

$$\mathcal{T}_d = \langle a_1, \dots, a_n \mid a_1^{m_1}, \dots, a_n^{m_n}, [a_1^d, a_2], [a_i, a_j], i < j, (i,j) \neq (1,2) \rangle$$

with $(d, m_1) = 1$.

In the previous theorem, we have $X_{\mathcal{T}_d} \simeq X_{\mathcal{T}_{d'}}$ if and only if $[d] = [d']$ in the obstruction group $\mathbb{Z}_{m_1}^* / \pm (\mathbb{Z}_{m_1}^*)^{n-1}$. Thus we have the following special case.

Corollary 1.3.19. *Let G be a finite abelian group with invariant factors $m_1 \mid m_2$ and let X be a compact connected 2-complex with $\pi_1(X) = G$. If X has minimum Euler characteristic, then $X \simeq X_{\mathcal{P}}$ where $\mathcal{P} = \langle a_1, a_2 \mid a_1^{m_1}, a_2^{m_2}, [a_1, a_2] \rangle$.*

The last ingredient needed to prove Theorem 1.3.21 is the following

Lemma 1.3.20. *Let*

$$\mathcal{T}_d = \langle a_1, \dots, a_n \mid a_1^{m_1}, \dots, a_n^{m_n}, [a_1^d, a_2], [a_i, a_j], i < j, (i,j) \neq (1,2) \rangle$$

with $n \geq 2$ and

$$\mathcal{R}_d = \langle a_1, a_2 \mid a_1^{m_1}, a_2^{m_2}, [a_1^d, a_2] \rangle.$$

Then $X_{\mathcal{R}_d}$ is a retract of $X_{\mathcal{T}_d}$.

Proof. Clearly $X_{\mathcal{R}_d}$ is a subcomplex of $X_{\mathcal{T}_d}$. We will define a cellular retraction $r\colon X_{\mathcal{T}_d} \to X_{\mathcal{R}_d}$. The unique 0-cell of $X_{\mathcal{T}_d}$ and the 1-cells a_1, a_2 are fixed by r. The remaining 1-cells $a_3,\dots,a_n$ are mapped to the 0-cell. In the 2-skeleton, r fixes the 2-cells $a_1^{m_1}$, $a_2^{m_2}$ and $[a_1^d, a_2]$, and we must extend r to the remaining 2-cells. This can be achieved since the composition of r with the attaching maps of those cells is null-homotopic. □

Theorem 1.3.21. *There are no Bing spaces with abelian fundamental group.*

Proof. Suppose there is a Bing space X with abelian fundamental group G. By Corollary 1.1.17, $H_1(X)$ is torsion. Since $H_1(X) = G$ is finitely generated and torsion, G is finite abelian. Let $m_1 \mid m_2 \mid \dots \mid m_n$ be its invariant factors.

Since G is freely indecomposable, by Proposition 1.3.15 we may assume X has no global separating points. By Proposition 1.3.16 we know that X has minimum Euler characteristic. By Theorem 1.3.12, G is not cyclic, so $n \geq 2$. From Theorem 1.3.18, there is a presentation

$$\mathcal{T}_d = \langle a_1,\dots,a_n \mid a_1^{m_1},\dots,a_n^{m_n},[a_1^d,a_2],[a_i,a_j], i<j,(i,j)\neq(1,2)\rangle$$

with $(d,m_1) = 1$ such that $X \simeq X_{\mathcal{T}_d}$. By Theorem 1.1.45, $X_{\mathcal{T}_d}$ has the fixed point property. Let

$$\mathcal{R}_d = \langle a_1,a_2 \mid a_1^{m_1},a_2^{m_2},[a_1^d,a_2]\rangle.$$

By Lemma 1.3.20, $X_{\mathcal{R}_d}$ is a retract of $X_{\mathcal{T}_d}$, thus by Lemma 1.1.4 $X_{\mathcal{R}_d}$ has the fixed point property. Finally consider

$$\mathcal{R}_1 = \langle a_1,a_2 \mid a_1^{m_1},a_2^{m_2},[a_1,a_2]\rangle.$$

By Corollary 1.3.19, $X_{\mathcal{R}_1} \simeq X_{\mathcal{R}_d}$, thus by Theorem 1.1.45, $X_{\mathcal{R}_1}$ has the fixed point property, which contradicts Lemma 1.3.17. □

The ideas used in the proof of the last result can be applied to other cases. The classification of 2-complexes has been achieved for a few finite groups, in addition to finite abelian groups. Our last theorem relies on a classification result of Hambleton and Kreck concerning the finite subgroups of $\mathrm{SO}(3)$.

Theorem 1.3.22. *A Bing space cannot have fundamental group A_4, S_4, A_5 or D_{2n}.*

Proof. By [HK93, Theorem 2.1] for these groups, the homotopy type of a 2-complex is determined by the Euler characteristic. Consider the following presentations with deficiency 1:

$$\begin{aligned}
A_4 &= \langle a,b,c \mid a^2,b^3,c^3,abc\rangle,\\
S_4 &= \langle a,b,c \mid a^2,b^3,c^4,abc\rangle,\\
A_5 &= \langle a,b,c \mid a^2,b^3,c^5,abc\rangle,
\end{aligned}$$

$$D_{2n} = \langle a,b,c \mid a^2, b^2, c^n, abc\rangle.$$

We only need to prove that the complexes associated to these presentations lack the fixed point property (we do not need to check whether these presentations have minimum deficiency).

Let $\mathcal{P} = \langle a,b,c \mid a^l, b^m, c^n, abc\rangle$. Consider the space $X = X(l,m,n)$ obtained by deleting three disjoint disks from S^2 and then gluing three 2-cells on the boundaries of these disks, with attaching maps of degrees l, m and n (Figure 1.1). We note that $X_{\mathcal{P}}$ is a quotient of X by a contractible subcomplex, therefore $X_{\mathcal{P}} \simeq X$. We will show X lacks the fixed point property.

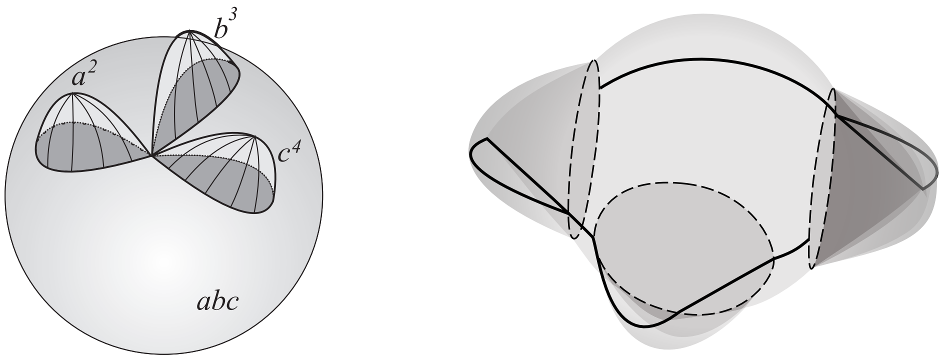

Figure 1.1: The space $X_{\mathcal{P}}$ at the left and the space $X(2,3,4)$ at the right, along with the fixed points of f.

Concretely, $X = X(l,m,n)$ is obtained from the surface

$$S = \{(x,y,z) : x^2+y^2+z^2 = 1 \text{ y } x \leq \frac{4}{5} \text{ y } -\frac{4}{5} \leq y \leq \frac{4}{5}\}$$

attaching three 2-cells where the attaching maps $\phi_a, \phi_b, \phi_c \colon S^1 \to S$ are given by

$$\phi_a(z) = \left(\frac{3}{5}\mathrm{Re}(z^l), -\frac{4}{5}, \frac{3}{5}\mathrm{Im}(z^l)\right)$$
$$\phi_b(z) = \left(\frac{4}{5}, \frac{3}{5}\mathrm{Re}(z^m), \frac{3}{5}\mathrm{Im}(z^m)\right)$$
$$\phi_c(z) = \left(-\frac{3}{5}\mathrm{Re}(z^n), \frac{4}{5}, \frac{3}{5}\mathrm{Im}(z^n)\right)$$

Let $i_a, i_b, i_c \colon D^2 \to X$ be the characteristic maps of these cells and let $i_S \colon S \hookrightarrow X$ be the inclusion.

The maps $f_S \colon S \to X$, $f_a, f_b, f_c \colon D^2 \to X$ given by

$$f_S(x,y,z) = i_S(x,y,-z)$$
$$f_a(w) = i_a(\overline{w})$$
$$f_b(w) = i_b(\overline{w})$$
$$f_c(w) = i_c(\overline{w})$$

glue to give a map $f\colon X \to X$.

Each connected component of $\mathrm{Fix}(f)$ is homotopy equivalent to S^1, as depicted in Figure 1.1. A fixed point class F of f is a union of connected components of $\mathrm{Fix}(f)$. Therefore, $\chi(F) = 0$ for every fixed point class F of f. An application of Lemma 1.1.39 yields $i(f, F) = 0$. Thus, $N(f) = 0$ and we are done. □

1.4 Bing groups

In this section, following [SC17b], we present results which answer Bing's Questions 1 and 8. Concretely, we exhibit 2-dimensional polyhedra with the fixed point property and arbitrary (positive) Euler characteristic. Finally, we prove the fixed point property is not a homotopy invariant for 2-dimensional polyhedra. To do this we use some basic results on group homology and we study a class of groups that we named *Bing groups*. Using this notion we will reduce the problem to finding a Bing group with certain properties.

Definition 1.4.1. Let G be a finitely presentable group such that $H_1(G)$ is finite. We say that G is a *Bing group* if either $H_2(G) = 0$ or, denoting the first invariant factor of $H_2(G)$ by d_1, we have $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ in $\mathbb{Z}_{d_1}$, for every endomorphism $\phi\colon G \to G$.

Theorem 1.4.2. *If $\mathcal{P}$ is an efficient presentation of a Bing group G then $X_{\mathcal{P}}$ has the fixed point property.*

Proof. Let $X = X_{\mathcal{P}}$ and $f\colon X \to X$ be a map. If $H_2(G) = 0$, X is rationally acyclic, so f has a fixed point. Therefore we may assume $H_2(G) \neq 0$. Let $d_1 \mid \ldots \mid d_k$ be the invariant factors of $H_2(G)$. There is a $K(G,1)$ space Y with $X = Y^{(2)}$. Let $i\colon X \hookrightarrow Y$ be the inclusion. Since $\pi_n(Y) = 0$ for $n \geq 2$, f extends to a map $\overline{f}\colon Y \to Y$.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Y \arrow[d, dashed, "\overline{f}"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

Since X is the 2-skeleton of Y the horizontal arrows in the following diagram are epimorphisms:

$$\begin{tikzcd} H_2(X) \arrow[r, two heads, "i_*"] \arrow[d, "f_*"'] & H_2(Y) \arrow[d, "\overline{f}_*"] \\ H_2(X) \arrow[r, two heads, "i_*"'] & H_2(Y) \end{tikzcd}$$

Since $\mathcal{P}$ is efficient, the rank of $H_2(X)$ equals the number of invariant factors of $H_2(Y)$ and thus $H_2(X) \otimes \mathbb{Z}_{d_1} \simeq H_2(Y) \otimes \mathbb{Z}_{d_1}$. By right exactness of the tensor product, the horizontal arrows in the following diagram are isomorphisms.

$$\begin{tikzcd}
H_2(X)\otimes\mathbb{Z}_{d_1} \arrow[r, "i_*\otimes\mathbb{1}_{\mathbb{Z}_{d_1}}", "\simeq"'] \arrow[d, "f_*\otimes\mathbb{1}_{\mathbb{Z}_{d_1}}"'] & H_2(Y)\otimes\mathbb{Z}_{d_1} \arrow[d, "\overline{f}_*\otimes\mathbb{1}_{\mathbb{Z}_{d_1}}"] \\
H_2(X)\otimes\mathbb{Z}_{d_1} \arrow[r, "\simeq", "i_*\otimes\mathbb{1}_{\mathbb{Z}_{d_1}}"'] & H_2(Y)\otimes\mathbb{Z}_{d_1}
\end{tikzcd}$$

Now $\mathrm{tr}(f_*\otimes\mathbb{1}_{\mathbb{Z}_{d_1}}) = \mathrm{tr}(\overline{f}_*\otimes\mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ in $\mathbb{Z}_{d_1}$, since G is a Bing group. In this step we used the natural isomorphism $H_2(BG)\approx H_2(G)$ (see [Ros94, Theorem 5.1.27]) and the fact that every map $BG\to BG$ comes, up to homotopy, from an endomorphism of G.

Finally, we obtain $\mathrm{tr}(f_*\colon H_2(X)\to H_2(X)) \neq -1$ in $\mathbb{Z}$, since $H_2(X)$ is free abelian and thus tensoring with $\mathbb{Z}_{d_1}$ reduces the trace modulo d_1. Since $H_1(X)$ is torsion, $L(f)\neq 0$ and by the Lefschetz fixed point theorem, f has a fixed point. □

Example 1.4.3. Efficient Bing groups with trivial second homology are easy to find (for example $\mathbb{Z}_n$ or any other finite group with deficiency zero). But the presentation complexes we get in this way are rationally acyclic, therefore have Euler characteristic 1. Aside from cyclic groups, abelian groups are not Bing groups (this follows from [BSC17, Theorem 4.6]).

Example 1.4.4. If G is a group, we consider the action $\mathrm{Aut}(G)\curvearrowright H_2(G)$. If $\phi\in\mathrm{Inn}(G)$ then $B\phi\colon BG\to BG$ is (freely) homotopic to the identity. Therefore $H_2(\phi)$ is the identity morphism. So there is an induced action $\mathrm{Out}(G)\curvearrowright H_2(G)$. When G is a finite simple group, every endomorphism $\phi\colon G\to G$ is either trivial or an automorphism. For the trivial morphism ϕ we have $\mathrm{tr}(H_2(\phi)\otimes\mathbb{1}_{\mathbb{Z}_{d_1}})=0$. Therefore for a finite simple group G, understanding the action $\mathrm{Out}(G)\curvearrowright H_2(G)$ suffices to determine if G is a Bing group. As we will see in Section 1.5, the only finite simple groups with nontrivial second homology that are also Bing groups are the groups $D_{2m}(q)$, for $m>2$ and q odd. The smallest of these groups is $D_6(3)$, a group of order 67628447006087702382529609728000. Simple groups of order at most 5000000 are efficient, except perhaps $C_2(4)$ [CHRR07, CHRR14]. However, it is not known if A_n is efficient for all n [CHRR07]. It is known that $D_{2m}(q)$ has deficiency at most 24 [GKKL11, Theorem 10.1]. Since $H_2(D_{2m}(q))=\mathbb{Z}_2\oplus\mathbb{Z}_2$, if these groups turn out to be efficient, they would give examples of two dimensional polyhedra with the fixed point property and Euler characteristic equal to 3. To answer Question 1.2.14 we will need another source of Bing groups.

Bing spaces with arbitrary Euler characteristic

In this section we obtain concrete examples of Bing spaces with arbitrary positive Euler characteristic. This answers affirmatively Question 1.2.14. We then use two different Bing spaces to answer Question 1.2.16. In this subsection we use the software GAP to prove that certain groups are Bing. It is clear that there is an algorithm that decides, for any finite group G, if G is Bing. In Section 1.6 there is a GAP function implementing this algorithm, similar to the one used to find the examples presented here.

Proposition 1.4.5. *The group G presented by*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

is a finite group of order 3^5. We have $H_2(G) = \mathbb{Z}_3$, so $\mathcal{P}$ is efficient. Moreover G is a Bing group.

Proof. We prove this using a GAP computation. We use the packages HAP [Ell13] and SONATA [ABE+12].

```
gap> LoadPackage("HAP");;
gap> LoadPackage("SONATA");;
gap> F:=FreeGroup("x","y");;
gap> AssignGeneratorVariables(F);;
#I  Assigned the global variables [ x, y ]
gap> G:= F/[x^3, x*y*x^-1*y*x*y^-1*x^-1*y^-1,
> x^-1*y^-4*x^-1*y^2*x^-1*y^-1];;
gap> Order(G);
243
gap> G:=Image(IsomorphismPermGroup(G));;
gap> R:=ResolutionFiniteGroup(G,3);;
gap> Homology(TensorWithIntegers(R),2);
[ 3 ]
gap> Set(List(Endomorphisms(G),
> f->Homology(TensorWithIntegers(EquivariantChainMap(R,R,f)),2)));
[ [ f1 ] -> [ <identity ...> ], [ f1 ] -> [ f1 ] ]
```

Therefore $|G| = 243$ and $H_2(G) = \mathbb{Z}_3$, so $\mathcal{P}$ is efficient. The last line of the output says that there are only two endomorphisms of $H_2(G)$ that are induced by an endomorphism of G. The former maps the generator `f1` of $H_2(G) = \mathbb{Z}_3$ to $0 \in H_2(G)$, so it is the zero morphism. The latter maps `f1` to `f1`, so it is the identity morphism of $H_2(G)$. From this we conclude that, after tensoring with $\mathbb{Z}_3$, the traces of these endomorphisms are 0 and 1, proving that G is a Bing group.

□

Remark 1.4.6. In Section 1.7 we give a lengthy proof of Proposition 1.4.5 without using GAP. In particular Proposition 1.7.16 gives a description of the group G as a semidirect product $(\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3$. The action of $\mathbb{Z}_3$ in $\mathbb{Z}_9 \times \mathbb{Z}_9$ is multiplication by $\begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix}$.

From Theorem 1.4.2 and Proposition 1.4.5 we have:

Corollary 1.4.7. *The complex $X_{\mathcal{P}}$ associated to the presentation*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

has the fixed point property. Moreover $\chi(X_{\mathcal{P}}) = 2$.

By Corollary 1.1.18, a two dimensional polyhedron with the fixed point property has positive Euler characteristic.

Corollary 1.4.8. *There are compact 2–dimensional polyhedra with the fixed point property and Euler characteristic equal to any positive integer n.*

Proof. For $n = 1$ this is immediate. For $n > 1$ take a wedge of $n - 1$ copies of the space $X_{\mathcal{P}}$ of Corollary 1.4.7. □

We have the following stronger version of Theorem 1.2.11.

Corollary 1.4.9. *There is a compact 2–dimensional polyhedron K with the fixed point property such that ΣK lacks the fixed point property.*

Proof. We take $K = X_{\mathcal{P}}$. Then $\chi(\Sigma K) = 0$ and since ΣK does not have global separating points, we can use Theorem 1.1.46. □

To answer Question 1.2.16 we will need another efficient Bing group.

Proposition 1.4.10. *The group H presented by $\mathcal{Q} = \langle x, y \mid x^4, y^4, (xy)^2, (x^{-1}y)^2 \rangle$ is a finite group of order 2^4. We have $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $\mathcal{Q}$ is efficient. Moreover H is a Bing group.*

Proof. As before, we use a GAP computation.

```
gap> LoadPackage("HAP");;
gap> LoadPackage("SONATA");;
gap> F:=FreeGroup("x","y");;
gap> AssignGeneratorVariables(F);;
#I  Assigned the global variables [ x, y ]
H:= F/[x^4, y^4, (x*y)^2, (x^-1*y)^2];;
gap> Order(H);
16
gap> H:=Image(IsomorphismPermGroup(H));;
gap> R:=ResolutionFiniteGroup(H,3);;
gap> Homology(TensorWithIntegers(R),2);
[ 2, 2 ]
gap> Set(List(Endomorphisms(H),
```

```
> f->Homology(TensorWithIntegers(EquivariantChainMap(R,R,f)),2)));
[ [ f1, f2 ] -> [ <identity ...>, <identity ...> ],
[ f1, f2 ] -> [ f1, f2 ],[ f1, f2 ] -> [ f1^-1*f2^-1, f2^-1 ] ]
```

Thus $|H| = 16$ and $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $\mathcal{Q}$ is efficient. The last two lines say that there are only three endomorphisms of $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ that are induced by an endomorphism of H. The first of these endomorphisms maps both generators `f1` and `f2` to $0 \in H_2(H)$, so it is the zero morphism. The second one maps `f1` to `f1` and `f2` to `f2`, so it is the identity morphism. The third endomorphism maps `f1` to `f1^-1*f2^-1=f1*f2` and `f2` to `f2^-1=f2`. So in the basis given by `f1` and `f2` it is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. From this we see that, after tensoring with $\mathbb{Z}_2$, the trace of each of these endomorphisms is 0. Therefore H is a Bing group. $\square$

Remark 1.4.11. In Section 1.7 we also prove Proposition 1.4.10 without using GAP. In particular Proposition 1.7.10 gives a description of the group H as a semidirect product $(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$. The action of $\mathbb{Z}_2$ in $\mathbb{Z}_2 \times \mathbb{Z}_4$ is given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Now we will show that the answer to Question 1.2.16 is 2:

Theorem 1.4.12. *There is a compact* 2*–dimensional polyhedron Y without the fixed point property and such that the polyhedron X, obtained from Y by an elementary collapse of dimension* 2*, has the fixed point property.*

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be the presentations of Propositions 1.4.5 and 1.4.10. By Theorem 1.4.2, $X_\mathcal{P}$ and $X_\mathcal{Q}$ have the fixed point property, so $X = X_\mathcal{P} \vee X_\mathcal{Q}$ also has the fixed point property. Since neither $X_\mathcal{P}$ nor $X_\mathcal{Q}$ have global separating points, by adding a 2–simplex, we can turn X into a polyhedron Y, without global separating points and such that, by collapsing that 2–simplex, we obtain X. By [Wei94, Corollary 6.2.10] we have

$$H_2(\pi_1(Y)) = H_2(\pi_1(X_\mathcal{P}) * \pi_1(X_\mathcal{Q})) = H_2(\pi_1(X_\mathcal{P})) \oplus H_2(\pi_1(X_\mathcal{Q})) = \mathbb{Z}_2 \oplus \mathbb{Z}_6$$

and we also have $\mathrm{rk}(H_2(Y)) = \mathrm{rk}(H_2(X_\mathcal{P})) + \mathrm{rk}(H_2(X_\mathcal{Q})) = 3$. By Proposition 1.3.11 there is $Z \simeq Y$ such that Z retracts to S^2 and therefore by Theorem 1.1.45, we conclude that Y does not have the fixed point property. $\square$

Towards a converse of Theorem 1.4.2

The following result is a first step in an attempt to obtain a converse for Theorem 1.4.2.

Lemma 1.4.13 ([HAMS93, III, Lemma 1.4])**.** *Let X and Y be compact, connected,* 2*–dimensional CW–complexes. If $f\colon X \to Y$ is a map and $\delta\colon H_2(X) \to \Sigma_2(Y)$ is any homomorphism, there is a map $g\colon X \to Y$ such that $\pi_1(f) = \pi_1(g)$ and $H_2(g) = H_2(f) + \delta$.*

Theorem 1.4.14. *Let X be a compact, connected, 2–dimensional polyhedron and let G be its fundamental group. Suppose that G is not Bing, or that G is not efficient or that X does not have minimum Euler characteristic. Then there is a map $f\colon X \to X$ with Lefschetz number* 0.

Proof. If $H_1(G)$ is not finite, X retracts to S^1, so X has a self–map f with Lefschetz number zero. Therefore we may assume that $H_1(G)$ is finite. Let $d_1,\dots,d_n$ be the invariant factors of $H_2(G)$. Consider the inclusion $\imath\colon \Sigma_2 X \to H_2(X)$. Let m be the rank of $H_2(X)$ and let k be the rank of $\Sigma_2(X)$. We consider the Smith normal form of $\imath$. Let $\alpha_1 \mid \dots \mid \alpha_k$ be the numbers on the diagonal and let $\{e_1,\dots,e_m\}$ be the basis of $H_2(X)$. Since $\imath$ is injective, α_i is nonzero for $i = 1,\dots,k$. By the short exact sequence above we have $H_2(G) = \mathbb{Z}_{\alpha_1} \oplus \dots \oplus \mathbb{Z}_{\alpha_k} \oplus \mathbb{Z}^{m-k}$. Note that the first ones of the α_i may be equal to 1. But in any case (if $k>0$) we have $\alpha_1 \mid d_1$.

Suppose G is not Bing. Then there is an endomorphism $\phi\colon G \to G$ such that $\mathrm{tr}(H_2(\phi)\otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) = -1$ in $\mathbb{Z}_{d_1}$. Let $\tilde{f}\colon X \to X$ be a map inducing ϕ on fundamental groups. We have $\mathrm{tr}(H_2(\tilde{f})) \equiv -1 \mod d_1$. If $d_1 = 0$ we are done. Otherwise, $k>0$ and since $\alpha_1 \mid d_1$ there is $c \in \mathbb{Z}$ such that $\mathrm{tr}(H_2(\tilde{f})) + c\alpha_1 = -1$. Define $\delta\colon H_2(X) \to \Sigma_2(X)$ by $\delta(e_1) = c\alpha_1 e_1$ and $\delta(e_j) = 0$ if $1 < j \le m$. Now using Lemma 1.4.13 we get a map $f\colon X \to X$ with $\mathrm{tr}(H_2(f)) = \mathrm{tr}(H_2(\tilde{f})) + \mathrm{tr}(\delta) = -1$, therefore $L(f) = 0$.

Now suppose G is not efficient or X does not have minimum Euler characteristic. Then $m > n$, so we must have $k > 0$ and $\alpha_1 = 1$. By the argument above we get a map $f\colon X \to X$ with $L(f) = 0$. Alternatively, in this case we could use Proposition 1.3.11. □

Notice that the existence of a map with Lefschetz number 0 is not enough to conclude that X does not have the fixed point property. To do that we would need to find a map f with Nielsen number 0.

Question 1.4.15. *Is it true that Theorem 1.4.14 still holds if we replace "Lefschetz number" by "Nielsen number"?*

If G is not efficient or X has non minimal Euler characteristic, the answer to the previous question is clearly yes. If G is not Bing, it seems to be a difficult question.

1.5 Finite simple Bing groups

In this section we prove the following result that is mentioned in Example 1.4.4. The proof of this result was only available in the Licentiate thesis [SC15].

Theorem 1.5.1. *The only finite simple Bing groups with nontrivial Schur multiplier are the groups $D_{2m}(q)$ for odd q and $m > 2$.*

The fundamental tool is obviously the classification of the finite simple groups [GLS00, Table I]. As we mentioned in Example 1.4.4, we only have to understand the action of $\mathrm{Out}(G)$

on the Schur multiplier $H_2(G)$ for each of the finite simple groups. Results describing completely this action appear in [GLS98]. We explain these results. There is a factorization $H_2(G) = M_c(G) \times M_e(G)$ [GLS98, Theorem 6.1.4]. The *canonical part* $M_c(G)$ is defined in [GLS98, Definition 6.1.1] and is computed in [GLS98, Theorem 6.1.2, Table 6.1.2]. The *exceptional part* is defined in [GLS98, Definition 6.1.3] and can be read from [GLS98, Table 6.1.3]. We remark that $M_c(G)$ and $M_e(G)$ do not only depend on the isomorphism type of the group G but also on the *guise* of the group (see [GLS98, Remark after Definition 6.1.1]). The subgroups $M_c(G)$ and $M_e(G)$ are stable by the action of $\mathrm{Out}(G)$ and thus it is enough to understand the action on each of these groups. This is the content of [GLS98, Theorem 6.3.1], which describes explicitly the action of $\mathrm{Out}(G)$ on $M_e(G)$ and says that $M_c(G)$ identifies with certain subgroup $\mathrm{Outdiag}(G) \triangleleft \mathrm{Out}(G)$, in such a way that the action of $\mathrm{Out}(G)$ on $M_c(G)$ is the conjugation action of $\mathrm{Out}(G) = \mathrm{Outdiag}(G) \rtimes \Phi_G\Gamma_G$ on $\mathrm{Outdiag}(G)$. Finally, this action is described in [GLS98, Theorem 2.5.12].

1.5.1 Cyclic groups Z_p

These groups have trivial Schur multiplier.

1.5.2 Alternating groups A_n

- If $n = 5$ or $n > 7$, the Schur multiplier is $\mathbb{Z}_2$. Since the identity of $\mathbb{Z}_2$ has trace -1, these groups are not Bing.
- If $n = 6$ or $n = 7$, the Schur multiplier is $\mathbb{Z}_6$ and there is an outer automorphism that acts as multiplication by -1.

1.5.3 Groups of Lie type

$A_n(q)$

The Schur multiplier is $\mathbb{Z}_{(n+1,q-1)}$, with the following exceptions:

- The Schur multiplier of $A_1(4)$ is $\mathbb{Z}_2$
- The Schur multiplier of $A_1(9)$ is $\mathbb{Z}_6$
- The Schur multiplier of $A_2(2)$ is $\mathbb{Z}_2$
- The Schur multiplier of $A_2(4)$ is $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$
- The Schur multiplier of $A_3(2)$ is $\mathbb{Z}_2$

In the general case, by [GLS98, Theorem 2.5.12 (i)], there is an outer automorphism that acts as multiplication by -1. When the Schur multiplier is $\mathbb{Z}_2$ it is clear. We have already addressed the case $A_1(9) = A_6$. In the case $A_2(4)$, we have $M_c = \mathbb{Z}_3$ and $M_e = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Moreover, $\mathrm{Out} = S_3 \times \mathbb{Z}_2$ and the action on M_e is faithful. An order 3 automorphism of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ has trace -1 (it is not necessary to understand the action on M_c).

$^2A_n(q)$

The Schur multiplier is $\mathbb{Z}_{(n+1,q+1)}$, with the following exceptions:

- The Schur multiplier of $^2A_3(2)$ is $\mathbb{Z}_2$.
- The Schur multiplier of $^2A_3(3)$ is $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$.
- The Schur multiplier of $^2A_5(2)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

First we analyze the general case. Let $k = (n+1, q+1)$.

- If $k = 1$ or $k = 2$ it is clear.
- If $k > 2$ and q is a power of a prime p, since $q \equiv -1 \pmod{k}$, $p \not\equiv 1 \pmod{k}$. We have $M_c = \mathbb{Z}_k$ and $M_e = 0$. We want to use [GLS98, Theorem 2.5.12]. In this case, $d = 2$. Since $k \nmid p-1$, the primitive k-th roots of unity are in $\mathbb{F}_{q^2} \smallsetminus \mathbb{F}_p$. Therefore, there is an automorphism of $\mathbb{F}_{q^2}$ sending each k-th root of unity to its inverse. Then by part (g) of [GLS98, Theorem 2.5.12], these groups are not Bing.

For $^2A_3(3)$, the identity has trace -1. In the case $^2A_5(2)$, $\mathrm{Out} = S_3$ acts faithfully on $M_e = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The order 3 automorphisms of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ have trace -1 (we do not need to understand the action on $M_c = \mathbb{Z}_3$).

$B_n(q)$

The Schur multiplier is trivial or $\mathbb{Z}_2$, save for $B_3(3)$. In this case, the Schur multiplier is $\mathbb{Z}_6$, $M_c = \mathbb{Z}_2$ and there is an outer automorphism that acts on $M_e = \mathbb{Z}_3$ as multiplication by -1.

$^2B_2(q)$

The Schur multiplier of $^2B_2(8)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and there is an outer automorphism with trace -1. In the remaining cases the Schur multiplier is trivial.

$C_n(q)$

The Schur multiplier is trivial or $\mathbb{Z}_2$.

$D_n(q)$

- If n is odd, the Schur multiplier is $\mathbb{Z}_{(4,q-1)}$. By [GLS98, Theorem 2.5.12 (i)], there is an outer automorphism that acts as multiplication by -1.

- If n is even and q is even, the Schur multiplier is trivial, save for $D_4(2)$. In this case, the Schur multiplier is $M_e = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the outer automorphism group is S_3 and acts faithfully, therefore there is an automorphism of trace -1.

- If n is even and q is odd, the Schur multiplier is $M_c = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If $n = 4$, by [GLS98, Theorem 2.5.12 (j)] S_3 is a subgroup of Out and acts faithfully, therefore there are automorphisms of trace -1.

  We look carefully at the case $n > 4$, using [GLS98, Theorem 2.5.12]. We have $M_e = 0$ and $M_c = \text{Outdiag} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We also have $\text{Out} \simeq \text{Outdiag} \rtimes \Phi\Gamma$. The action of Out on M_c is the action of Out on Outdiag by conjugation. The action of Outdiag on Outdiag is clearly trivial. By [GLS98, Theorem 2.5.12 (h)], Φ centralizes Outdiag, therefore the action of Φ is also trivial. Finally by [GLS98, Theorem 2.5.12 (j)], $\Gamma = \mathbb{Z}_2$ acts faithfully on M_c. The order 2 automorphisms of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ have trace 0. Therefore these groups are Bing.

${}^2D_n(q)$

- If n is even, the Schur multiplier is trivial or $\mathbb{Z}_2$.

- If n is odd, the Schur multiplier is $\mathbb{Z}_{(4,q+1)}$.

  - If $4 \nmid q+1$, the Schur multiplier is trivial or $\mathbb{Z}_2$.
  - If $4 \mid q+1$, the Schur multiplier is $M_c = \text{Outdiag} = \mathbb{Z}_4$. If q is a power of a prime p, since $q \equiv 3 \pmod 4$, we have $p \equiv 3 \pmod 4$. We use [GLS98, Theorem 2.5.12]. In this case $d = 2$. Since $4 \nmid p-1$, the 4-th primitive roots of unity are in $\mathbb{F}_{q^2} \smallsetminus \mathbb{F}_p$. Therefore, there is an automorphism of $\mathbb{F}_{q^2}$ sending each fourth root of unity to its inverse. Then by part (g) of [GLS98, Theorem 2.5.12], these groups are not Bing.

${}^3D_4(q)$

These groups have trivial Schur multiplier.

$G_2(q)$

The Schur multiplier is trivial or $\mathbb{Z}_2$, save for $G_2(3)$. In this case, the Schur multiplier is $\mathbb{Z}_3$. There is an outer automorphism that acts as multiplication by -1.

$^2G_2(q)$

These groups have trivial Schur multiplier.

$F_4(q)$

The Schur multiplier is trivial or $\mathbb{Z}_2$.

$^2F_4(q)$ and $^2F_4(2)'$

These groups have trivial Schur multiplier.

$E_6(q)$

The Schur multiplier is trivial or $\mathbb{Z}_3$. When it is $\mathbb{Z}_3$, by [GLS98, Theorem 2.5.12 (i)] there is an outer automorphism that acts as multiplication by -1.

$^2E_6(q)$

The Schur multiplier is $\mathbb{Z}_{(3,q+1)}$, save for $^2E_6(2)$.

Fist we analyze the general case.

- If $3 \nmid q+1$, the Schur multiplier is trivial.
- If $3 \mid q+1$ and q is a power of a prime p, we have $p \equiv 2 \pmod 3$. We use [GLS98, Theorem 2.5.12]. We have $d = 2$. Since $3 \nmid p-1$, the primitive cubic roots of unity are in $\mathbb{F}_{q^2} \setminus \mathbb{F}_p$. Therefore, there is an automorphism of $\mathbb{F}_{q^2}$ sending each third root of unity to its inverse. Then by part (g) of [GLS98, Theorem 2.5.12], these groups are not Bing.

For the group $^2E_6(2)$, we have $M_c = \mathbb{Z}_3$, $M_e = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the action of $\mathrm{Out} = S_3$ on M_e is faithful. The order 3 automorphisms of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ have trace -1 (it is not necessary to understand the action on M_c).

$E_7(q)$

The Schur multiplier is trivial or $\mathbb{Z}_2$.

$E_8(q)$

These groups have trivial Schur multiplier.

1.5.4 Sporadic groups

By [GLS98, Definition 6.1.1] the sporadic groups have $M_c = 1$. The action of Out on M_e for each of these groups is described in [GLS98, Theorem 6.3.1 and Table 3.6.1].

M_{11}

This group has trivial Schur multiplier.

M_{12}

The Schur multiplier has order 2.

M_{22}

The Schur multiplier is $\mathbb{Z}_{12}$. There is an outer automorphism that acts as multiplication by -1.

M_{23}

This group has trivial Schur multiplier.

M_{24}

This group has trivial Schur multiplier.

J_1

This group has trivial Schur multiplier.

J_2

The Schur multiplier has order 2.

J_3

The Schur multiplier is $\mathbb{Z}_3$. There is an outer automorphism that acts as multiplication by -1.

J_4

This group has trivial Schur multiplier.

HS

The Schur multiplier has order 2.

He

This group has trivial Schur multiplier.

Mc

The Schur multiplier is $\mathbb{Z}_3$. There is an outer automorphism that acts as multiplication by -1.

Suz

The Schur multiplier is $\mathbb{Z}_6$. There is an outer automorphism that acts as multiplication by -1.

Ly

This group has trivial Schur multiplier.

Ru

The Schur multiplier has order 2.

$O'N$

The Schur multiplier is $\mathbb{Z}_3$. There is an outer automorphism that acts as multiplication by -1.

Co_1

The Schur multiplier has order 2.

Co_2

This group has trivial Schur multiplier.

Co_3

This group has trivial Schur multiplier.

Fi_{22}

The Schur multiplier is $\mathbb{Z}_6$. There is an outer automorphism that acts as multiplication by -1.

Fi_{23}

This group has trivial Schur multiplier.

Fi'_{24}

The Schur multiplier is $\mathbb{Z}_3$. There is an outer automorphism that acts as multiplication by -1.

F_5

This group has trivial Schur multiplier.

F_3

This group has trivial Schur multiplier.

F_2

The Schur multiplier has order 2.

F_1

This group has trivial Schur multiplier.

1.6 GAP code for Bing Groups

These GAP functions are similar to those used to find the examples in Section 1.4. We remark that the results of Section 1.4 do not depend on the correctness of the code in this section. As before, we use the packages HAP and SONATA.

```
IsBingGroupWithNontrivialMultiplier:=function(G)
  # true if G is Bing and H2(G) is nontrivial
  # false otherwise
  local R,d,endG,H2,A,phi,psi,RemIntPositive,BingTrace,ExponentImage,
f;
  G := Image(IsomorphismPermGroup(G));
  R := ResolutionFiniteGroup(G,3);
  d := Homology(TensorWithIntegers(R),2);
  if d=[] then # H2(G)=0
    return false;
  fi;
  if RemInt(Size(d),d[1]) = d[1]-1 then # tr(id)=-1
    return false;
  fi;
  endG:=Endomorphisms(G);
```

```
  H2:= f -> Homology(TensorWithIntegers(EquivariantChainMap(R,R,f
)),2);
  A:=AbelianGroup(IsFpGroup,d);
  phi:=IsomorphismGroups(Source(H2(endG[1])),A);
  psi:=InverseGeneralMapping(phi);
  RemIntPositive := function(a,n)
    return RemInt(RemInt(a,n)+n,n);
  end;
  ExponentImage := function(f,g) # exponent of generator g in f(g)
    return ExponentSumWord(UnderlyingElement(Image(f,g)),
                           UnderlyingElement(g));
  end;
  BingTrace := function(f)
    local gens,f_star;
    gens:=GeneratorsOfGroup(A);
    f_star:=CompositionMapping(phi,H2(f),psi);
    return RemIntPositive(Sum(List(gens,
                                  g -> ExponentImage(f_star,g))),
                          d[1]);
  end;
  for f in endG do
    if BingTrace(f) = d[1]-1 then
      return false;
    fi;
  od;
  return true;
end;;
```

We can use the following functions to find the Bing groups with nontrivial Schur multiplier of order at most m.

```
SmallGroups:= function(m)
  # The groups of order at most m
  return Concatenation(List([1..m],
    o -> List([1..NumberSmallGroups(o)], i-> [o,i]))
  );
end;;

SmallBingGroups:= function(m)
```

```
  # Bing groups of order at most m with
  # nontrivial Schur multiplier
  return Filtered(
    SmallGroups(m),
    p-> (not IsAbelian(SmallGroup(p)))
         and
         IsBingGroupWithNontrivialMultiplier(SmallGroup(p))
  );
end;;
```

As an example, we find the Bing groups with nontrivial Schur multiplier of order at most 50.

```
gap> l:=SmallBingGroups(50);
[ [ 16, 3 ], [ 32, 5 ], [ 32, 6 ], [ 32, 9 ], [ 32, 24 ],
  [ 32, 29 ], [ 32, 30 ], [ 32, 31 ], [ 32, 35 ], [ 32, 41 ],
  [ 32, 42 ], [ 32, 44 ], [ 48, 14 ], [ 48, 19 ], [ 48, 21 ] ]
```

In some cases the following function can be used to find an efficient presentation of a finite group. We shall mention that the function may return `fail` even if the group admits an efficient presentation.

```
EfficientPresentation:=function(G)
    # G finite
    local P,d,R,H2;
    G:=Image(IsomorphismFpGroup(G));
    G:=SimplifiedFpGroup(G);
    R:=ResolutionFiniteGroup(G,3);
    H2:=Homology(TensorWithIntegers(R),2);
    d:=Size(RelatorsOfFpGroup(G))-Size(GeneratorsOfGroup(G));
    if d=Size(H2) then
        return PresentationFpGroup(G);
    fi;
    P:=PresentationViaCosetTable(G);
    G:=FpGroupPresentation(P);
    d:=Size(RelatorsOfFpGroup(G))-Size(GeneratorsOfGroup(G));
    if d=Size(H2) then
        return P;
    fi;
    return fail;
end;;
```

Using `EfficientPresentation` we find some efficient Bing groups with nontrivial Schur multiplier.

```
gap> Filtered(l, p -> EfficientPresentation(SmallGroup(p)) <> fail);
[ [ 16, 3 ], [ 32, 5 ], [ 32, 6 ],
  [ 32, 9 ], [ 32, 29 ], [ 32, 35 ] ]
```

1.7 A way to prove a group is Bing

In this section we obtain a condition in terms of the automorphisms of retracts of G which we prove is equivalent to G being a Bing group. We use this equivalent definition to give alternative proofs of Proposition 1.4.5 and Proposition 1.4.10 that do not use GAP. Although these proofs are lengthy, they may be preferred by readers who are not comfortable with computer assisted proofs. We use a number of tools and ideas that may be interesting in their own sake. For example we exhibit a stem extension of H to give a lower bound on the size of $H_2(H)$ and we use the Lyndon–Hochschild–Serre spectral sequence to obtain an upper bound on its size. From this we conclude $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The proofs presented in this section are human–verifiable, however we shall mention that some of the intermediate steps were hinted by GAP. We also show how these ideas can be used to prove the group $H \times H$ is not Bing.

The results of this section did not appear in the Licentiate Thesis [SC15] and are not published elsewhere.

1.7.1 Finite Bing groups

Definition 1.7.1. Let A be a finite and nontrivial abelian group and let d be its first invariant factor. We say that A *satisfies property P* if for every endomorphism $f\colon A \to A$ such that $\operatorname{tr}(f \otimes \mathbb{1}_{\mathbb{Z}_d}) = -1$ there is an arbitrarily large integer k such that $\operatorname{tr}(f^k \otimes \mathbb{1}_{\mathbb{Z}_d}) = -1$.

Theorem 1.7.2. *Let G be a finite group such that $H_2(G) \neq 0$ satisfies property P. Let d be the first invariant factor of $H_2(G)$. The following are equivalent*
(i) G is a Bing group.
(ii) For every split epimorphism $G \to K$ and every $\phi \in \operatorname{Aut}(K)$ we have $\operatorname{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_d}) \neq -1$ in $\mathbb{Z}_d$.

Proof. We first prove (ii) implies (i). Suppose that (ii) holds and G is not Bing to obtain a contradiction. Let $f\colon G \to G$ be an endomorphism with $\operatorname{tr}(H_2(f) \otimes \mathbb{1}_{\mathbb{Z}_d}) = -1$. Since G is finite, the descending chain

$$G \supset f(G) \supset f^2(G) \supset \dots$$

eventually becomes equal to certain subgroup $K = f^{k'}(G)$. Since $H_2(G)$ satisfies property P, there is an integer $k > k'$ such that $\operatorname{tr}(H_2(f^k) \otimes \mathbb{1}_{\mathbb{Z}_d}) = -1$. Then $f^k\colon G \to K$ is an epimorphism and $\phi = f^k|_K \in \mathrm{Aut}(K)$. Now let $i\colon K \hookrightarrow G$ be the inclusion and let $p = \phi^{-1}f^k\colon G \to K$. We have $p \circ i = \mathbb{1}_K$. Thus, G can be decomposed as a semidirect product $G = N \rtimes K$, where $N = \ker(p)$. Therefore

$$\begin{aligned}\mathrm{tr}(H_2(f^k) \otimes \mathbb{1}_{\mathbb{Z}_d}) &= \mathrm{tr}(H_2(i\phi p) \otimes \mathbb{1}_{\mathbb{Z}_d})\\ &= \mathrm{tr}(H_2(\phi p i) \otimes \mathbb{1}_{\mathbb{Z}_d})\\ &= \mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_d})\\ &\neq -1,\end{aligned}$$

a contradiction. Now we prove (i) implies (ii). Suppose $p\colon G \to K$ and $i\colon K \to G$ satisfy $pi = \mathbb{1}_K$. Then for any $\phi \in \mathrm{Aut}(K)$ we have

$$\begin{aligned}\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_d}) &= \mathrm{tr}(H_2(pi\phi) \otimes \mathbb{1}_{\mathbb{Z}_d})\\ &= \mathrm{tr}(H_2(i\phi p) \otimes \mathbb{1}_{\mathbb{Z}_d})\\ &\neq -1,\end{aligned}$$

a contradiction. □

Groups with property P

Obviously $\mathbb{Z}_d$ has property P for every d. In this section we prove that if the first invariant factor d of A is squarefree, A has property P.

Definition 1.7.3. Let n,d be natural numbers. We say $P(n,d)$ holds if for every matrix $M \in M_n(\mathbb{Z}_d)$ with $\mathrm{tr}(M) = -1$ we can find an arbitrarily large $k \in \mathbb{N}$ such that $\mathrm{tr}(M^k) = -1$.

If $P(n,d)$ holds and A is an abelian group with n invariant factors and first invariant factor d, then A obviously satisfies property P.

Proposition 1.7.4. *If d is squarefree then $P(n,d)$ holds for all $n \in \mathbb{N}$.*

Proof. Suppose $M \in M_n(\mathbb{Z}_d)$ has trace -1. For each prime factor $p \mid d$ we consider the class $\overline{M}$ of M in $M_n(\mathbb{F}_p)$ and we consider the Jordan form of $\overline{M}$ in an algebraic closure of $\mathbb{F}_p$. Let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of M. There is a number n_p such that $\lambda_i \in \mathbb{F}_{p^{n_p}}$ for every i. Since the elements of $\mathbb{F}_{p^{n_p}}$ are the roots of $x^{p^{n_p}} - x$, if $p^{n_p} - 1 \mid k-1$ we have $\lambda_i^k = \lambda_i$. Therefore

$$\mathrm{tr}(\overline{M}^k) = \mathrm{tr}(J^k) = \sum \lambda_i^k = \sum \lambda_i = \mathrm{tr}(J) = \mathrm{tr}(A) = -1$$

in $\mathbb{F}_p$. Finally, if $k \equiv 1 \pmod{\prod_{p|d}(p^{n_p}-1)}$ we have $\mathrm{tr}(A^k) = -1$ in $\mathbb{Z}_d$. □

Corollary 1.7.5. *Let A be a finite and nontrivial abelian group and let d be its first invariant factor. If d is squarefree then A has property P.*

The following lemma will be needed later.

Lemma 1.7.6. *Let p be a prime divisor of $n \in \mathbb{N}$ and $A \in GL(n, \mathbb{F}_p)$. Then* $\mathrm{tr}(A) = 0$ *whenever the order of A is a power of p.*

Proof. Let p^k be the order of A. Then the minimal polynomial of A divides $x^{p^k} - 1 = (x-1)^{p^k}$. Thus 1 is the unique eigenvalue of A and we have $\mathrm{tr}(A) = n = 0$ in $\mathbb{F}_p$. $\square$

1.7.2 Presenting semidirect products

Lemma 1.7.7 ([HAMS93, Chapter V, Section 3.3])**.** *Consider a semidirect product $G = N \rtimes_\phi H$ given by a group homomorphism $\phi : H \to \mathrm{Aut}(N)$. Let $\langle \mathcal{S}_N \mid \mathcal{R}_N \rangle$ and $\langle \mathcal{S}_H \mid \mathcal{R}_H \rangle$ be group presentations for N and H respectively. For every $n \in \mathcal{S}_N$ and $h \in \mathcal{S}_H$ let $\omega(n,h)$ be a word in the free group generated by $\mathcal{S}_N$ such that $\omega(n,h) = \phi_h(n)^{-1}$ in N. Then we have a presentation $\langle \mathcal{S}_G \mid \mathcal{R}_G \rangle$ of G with generating set $\mathcal{S}_G = \mathcal{S}_N \cup \mathcal{S}_H$ and relators*

$$\mathcal{R}_G = \mathcal{R}_N \cup \mathcal{R}_H \cup \{hnh^{-1}\omega(n,h) \,:\, n \in \mathcal{S}_N, h \in \mathcal{S}_H\}.$$

1.7.3 Stem extensions

Definition 1.7.8. An extension

$$1 \to A \to E \to G \to 1$$

is *central* if $A \subseteq Z(E)$. An extension is *stem* if it is central and $A \subseteq [E,E]$. If G is a finite group, a *Schur covering group* of G is a stem extension

$$1 \to H_2(G) \to E \to G \to 1.$$

The Schur covering group is not always unique up to isomorphism but we have the following result:

Theorem 1.7.9 ([Rot09, Section 9.4, p. 553])**.** *Any stem extension is a homomorphic image of a Schur covering group of G. In particular, the order of any stem extension of G is at most $|G| \cdot |H_2(G)|$.*

1.7.4 Another proof of Proposition 1.4.10

The following proof reduces Proposition 1.4.10 to a series of verifications that are carried out later in the section.

Proposition 1.4.10. *The group H presented by $\mathcal{Q} = \langle x, y \mid x^4, y^4, (xy)^2, (x^{-1}y)^2 \rangle$ is a finite group of order 2^4. We have $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $\mathcal{Q}$ is efficient. Moreover H is a Bing group.*

Proof. By Proposition 1.7.10 H is isomorphic to the group presented by

$$\mathcal{Q}' = \langle a,b,c \mid a^2, b^4, c^2, [a,b], [a,c], cbc^{-1} = ab \rangle$$

By Lemma 1.7.7 $H = (\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ thus $|H| = 16$. By Proposition 1.7.15 we have $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By Corollary 1.7.5, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ satisfies property P and we can apply Theorem 1.7.2. By Proposition 1.7.12 if we write $H = N \rtimes K$ then K must be either 1, $\mathbb{Z}_2$, $\mathbb{Z}_4$ or H. We only need to address the case $K = H$, since the remaining groups have trivial Schur multiplier and thus the trace of any automorphism is 0. Thus it is enough to show $\operatorname{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_2}) \neq -1$ in $\mathbb{Z}_2$ for every $\phi \in \operatorname{Aut}(H)$. By Proposition 1.7.13 we have $|\operatorname{Aut}(H)| = 32$ and we use Lemma 1.7.6 with $p = n = 2$. □

For the rest of the section H denotes the group presented by $\mathcal{Q}'$. We frequently use the fact that every element of H can be written uniquely in the *standard form* $a^i b^j c^k$ with $0 \leq i < 2$, $0 \leq j < 4$ and $0 \leq k < 2$. We will also use the fact that we know how to take any word to standard form.

Alternative presentation

Proposition 1.7.10. *The groups presented by*

$$\mathcal{Q} = \langle x,y \mid x^4, y^4, (xy)^2, (x^{-1}y)^2 \rangle$$

and

$$\mathcal{Q}' = \langle a,b,c \mid a^2, b^4, c^2, [a,b], [a,c], cbc^{-1} = ab \rangle$$

are isomorphic.

Proof. We exhibit mutually inverse maps. The map $\mathcal{Q} \to \mathcal{Q}'$ is given by $x \mapsto b$, $y \mapsto bc$. Checking this is indeed a morphism is easy using the normal form for elements in the group presented by $\mathcal{Q}'$. The other map $\mathcal{Q}' \to \mathcal{Q}$ is given by $a \mapsto x^{-1}y^2x^{-1}$, $b \mapsto x$, $c \mapsto x^{-1}y$. The computations needed to check that this defines a morphism are the following.

$$\begin{aligned} a^2 \mapsto (x^{-1}y^2x^{-1})(x^{-1}y^2x^{-1}) &= x^{-1}y^2x^2y^2x^{-1} \\ &= (x^{-1}yx^{-1})(xyx)(xyx)(x^{-1}yx^{-1}) \\ &= y^{-1}y^{-1}y^{-1}y^{-1} \\ &= 1 \end{aligned}$$

$$b^4 \mapsto x^4 = 1$$

$$c^2 \mapsto (x^{-1}y)(x^{-1}y) = 1$$

$$\begin{aligned}[a,b] \mapsto (x^{-1}y^2x^{-1})x(xy^{-2}x)x^{-1} &= x^{-1}y^2xy^{-2}\\ &= (x^{-1}yx^{-1})(xyx)y^{-2}\\ &= y^{-1}y^{-1}y^{-2}\\ &= 1\end{aligned}$$

$$\begin{aligned}[a,c] \mapsto (x^{-1}y^2x^{-1})(x^{-1}y)(xy^{-2}x)(y^{-1}x) &= x^{-1}y^2x^2yxy^{-2}xy^{-1}x\\ &= (x^{-1}yx^{-1})(xyx)(xyx)(y^{-2}xy^{-1}x)\\ &= y^{-1}y^{-1}y^{-1}(y^{-2}xyy^{-1}x)\\ &= y^{-1}xyy^{-1}x\\ &= (x^{-1}yx^{-1}y)^{-1}\\ &= 1\end{aligned}$$

$$\begin{aligned}[b,c]a^{-1} \mapsto x(x^{-1}y)x^{-1}(y^{-1}x)(xy^{-2}x) &= yx^{-1}y^{-1}x^2y^{-2}x\\ &= yx^{-1}y^{-1}x^{-2}y^{-2}x\\ &= y(x^{-1}y^{-1}x^{-1})x^{-1}y^{-2}x\\ &= yyx^{-1}y^{-2}x\\ &= y^2(x^{-1}y^{-1}x^{-1})(xy^{-1}x)\\ &= y^2yy\\ &= 1\end{aligned}$$

It is easy to see these maps are inverses. □

Conjugacy classes

Proposition 1.7.11. *The conjugacy classes of H are:*

- $\{1\}$, $\{a\}$, $\{b^2\}$ *y* $\{ab^2\}$. *The center is given by these classes.*
- $\{b,ab\}$, $\{b^3,ab^3\}$. *If g is in one of these classes g has order 4 and we have $g^2 = b^2$.*
- $\{bc,abc\}$, $\{b^3c,ab^3c\}$. *If g is in one of these classes, g has order 4 and we have $g^2 = ab^2$.*
- $\{c,ac\}$, $\{b^2c,ab^2c\}$. *If g is in one of these classes, g has order 2.*

Possible descriptions as a semidirect product

Proposition 1.7.12. *If $H = N \rtimes K$ then K is either 1, $\mathbb{Z}_2$, $\mathbb{Z}_4$ or H.*

Proof. We consider two cases.

- If K has order 4 we need to eliminate the possibility of $K = \mathbb{Z}_2 \times \mathbb{Z}_2$. If H/N has no elements of order 4, the square of every element of order 4 of H is an element of N. In particular b^2 and $ab^2 = (bc)^2$ are elements of N. Then $N = \{1, a, b^2, ab^2\}$. Clearly $K = H/N = \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by $\overline{b}$ and $\overline{c}$. To split the map $H \to K$, we need an element of order 2 in bN, but all these elements have order 4. Thus it is not possible to have $H = N \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

- If K has order 8 we have $N = \mathbb{Z}_2$ and N has to be central. We have three cases.

  - $N = \{1, a\}$. The quotient is $K = \mathbb{Z}_2 \times \mathbb{Z}_4$, generated by $\overline{b}$ and $\overline{c}$. To split the projection we would need to choose i, j such that $a^i b$ and $a^j c$ commute, but this is impossible.
  - $N = \{1, b^2\}$. Then $\overline{b}$ has order 2 in K. If there is a split of $H \to K$ then either b or b^3 has order 2, but both has order 4.
  - $N = \{1, ab^2\}$. If there is a section $s\colon K \to H$ we would have $s(\overline{b}) \in \{b, ab^3\}$ and $s(\overline{c}) \in \{c, ab^2c\}$. In each of the four possible cases $s(\overline{b})$ and $s(\overline{c})$ generate H.Thus there can be no such section s.

□

Automorphisms

Proposition 1.7.13. *We have* $|\mathrm{Aut}(H)| = 32$ *and* $|\mathrm{Out}(H)| = 8$.

Proof. Note that a is the only element in $Z(H)$ which is not a square. Thus a is fixed by any automorphism of H. Any automorphism maps b to an element of order 4. There are 8 elements of order 4, which have the form $a^i b^{2j+1} c^k$. Any automorphism maps c to a noncentral element of order 2. There are 4 such elements which have the form $a^l b^{2m} c$. Thus there are at most $8 \cdot 4$ automorphisms. Using the normal form for words in H, it is easy to see that no matter how we choose i, j, k, l we obtain a homomorphism. It is easy to see that $\{a, a^i b^{2j+1} c^k, a^l b^{2m} c\}$ generates H for any choice of i, j, l and m. Thus these homomorphisms are bijective and we have $|\mathrm{Aut}(H)| = 32$. Since $|Z(H)| = 4$ we have $|\mathrm{Inn}(H)| = |H|/|Z(H)| = 16/4 = 4$. Therefore $|\mathrm{Out}(H)| = |\mathrm{Aut}(H)|/|\mathrm{Inn}(H)| = 32/4 = 8$. □

Schur multiplier

Proposition 1.7.14. *There is a stem extension*

$$1 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to E \to H \to 1.$$

Proof. Let E be the group presented by

$$\begin{aligned}\langle p, q, a, b, c \mid p^2,\ & q^2,\ [p,q], \\ & [a,p],\ [b,p],\ [c,p],\ [a,q],\ [b,q],\ [c,q], \\ & a^2p,\ b^4,\ c^2,\ [a,b]q^{-1},\ [a,c]p^{-1},\ [c,b]a^{-1}\rangle.\end{aligned}$$

Let A be the subgroup of E generated by p and q, which is central since $[p,q]=[a,p]=[b,p]=[c,p]=[a,q]=[b,q]=[c,q]=1$. Moreover $p=[a,c]$ and $q=[a,b]$. Note that $H=E/A$, since taking the quotient we recover the presentation $\mathcal{Q}'$ of H. We thus have a stem extension

$$1 \to A \to E \to H \to 1.$$

We now describe E as a semidirect product, by using Lemma 1.7.7 repeatedly. Let E_0 be the group presented by

$$\langle p, a \mid p^2, [a,p], a^2p\rangle.$$

The group E_0 is isomorphic to $\mathbb{Z}_4$, it is generated by a and we have $p=a^2$. Now the group E_1 presented by

$$\langle p,q,a \mid p^2,q^2,[p,q],[a,p],[a,q],a^2p\rangle$$

is $\mathbb{Z}_4\times\mathbb{Z}_2$. The subgroup $\mathbb{Z}_4$ is generated by a, $\mathbb{Z}_2$ is generated by q and we have $p=a^2$. It is easy to verify that there is an order 2 automorphism of E_1 defined by $p\mapsto p$, $q\mapsto q$ and $a\mapsto aq$. Thus by Lemma 1.7.7, the group E_2 presented by

$$\langle p,q,a,b \mid p^2,q^2,[p,q],[a,p],[b,p],[a,q],[b,q],a^2p,b^4,[a,b]q^{-1}\rangle$$

is a semidirect product $(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4$. We have an order 2 automorphism of E_2 defined by $p\mapsto p$, $q\mapsto q$, $a\mapsto pa$ and $b\mapsto ab$. By Lemma 1.7.7, the group E has the description $((\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_4)\rtimes\mathbb{Z}_2$ and we obtain $|E|=64$. Thus $|A|=4$ and since both p,q have order at most 2, we conclude $A=\mathbb{Z}_2\oplus\mathbb{Z}_2$. □

Proposition 1.7.15. *The Schur multiplier of H is $H_2(H)=\mathbb{Z}_2\oplus\mathbb{Z}_2$*

Proof. Let $N=\langle a,b\rangle$ and $K=H/N$. Thus we have $H=N\rtimes K$ and we can apply the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{p,q}=H_p(K;H_q(N))\Rightarrow H_{p+q}(H).$$

By [Bro94, Chapter III, Section 1] we have

$$\begin{aligned}E^2_{0,2}&=H_0(\mathbb{Z}_2;H_2(\mathbb{Z}_2\times\mathbb{Z}_4))=H_2(\mathbb{Z}_2\times\mathbb{Z}_4)_{\mathbb{Z}_2}=\mathbb{Z}_2\\ E^2_{1,1}&=H_1(\mathbb{Z}_2;H_1(\mathbb{Z}_2\times\mathbb{Z}_4))=\operatorname{coker}(\overline{N}\colon H_1(\mathbb{Z}_2\times\mathbb{Z}_4)_{\mathbb{Z}_2}\to H_1(\mathbb{Z}_2\times\mathbb{Z}_4)^{\mathbb{Z}_2})=\mathbb{Z}_2\\ E^2_{2,0}&=H_2(\mathbb{Z}_2;H_0(\mathbb{Z}_2\times\mathbb{Z}_4))=H_2(\mathbb{Z}_2)=0\end{aligned}$$

Thus $|\bigoplus_{p+q=2}E^\infty_{p,q}|\le|\bigoplus_{p+q=2}E^2_{p,q}|=4$, giving the bound $|H_2(H)|\le 4$. From Proposition 1.7.14 and Theorem 1.7.9 we have $|H_2(H)|\ge 4$. Thus the stem extension in Proposition 1.7.14 is a Schur covering group of H and we have $H_2(H)=\mathbb{Z}_2\oplus\mathbb{Z}_2$. □

1.7.5 Another proof of Proposition 1.4.5

Following the same scheme of the previous section, we reduce Proposition 1.4.5 to a series of verifications.

Proposition 1.4.5. *The group G presented by*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

is a finite group of order 3^5. We have $H_2(G) = \mathbb{Z}_3$, so $\mathcal{P}$ is efficient. Moreover G is a Bing group.

Proof. By Proposition 1.7.16, G is isomorphic to the group presented by

$$\mathcal{P}' = \langle a, b, c \mid a^9, b^9, c^3, [a,b], cac^{-1}b^{-2}, cbc^{-1}a^{-1}b^{-5} \rangle.$$

By Lemma 1.7.7, $G = (\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3$ thus G has order 243. By Proposition 1.7.33 we have $H_2(G) = \mathbb{Z}_3$. By Corollary 1.7.5, $\mathbb{Z}_3$ satisfies property P and we can apply Theorem 1.7.2. By Proposition 1.7.23 if $G = N \rtimes K$ then K is either 1, $\mathbb{Z}_3$ or G. We only need to address the case $K = G$, since the remaining groups have trivial Schur multiplier and thus the trace of any automorphism is 0. Thus it is enough to show $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_3}) \neq -1$ in $\mathbb{Z}_3$ for each automorphism $\phi \in \mathrm{Aut}(G)$. By Proposition 1.7.30 we have $|\mathrm{Aut}(G)| = 2 \cdot 3^6$. Proposition 1.7.34 tells us $f_* \colon H_2(G) \to H_2(G)$ is the identity, where f is the order 2 automorphism defined in Proposition 1.7.31 . Thus $\langle\langle f \rangle\rangle \lhd \mathrm{Aut}(G)$ acts trivially on $H_2(G)$, since $\mathrm{Aut}(H_2(G)) = \mathrm{Aut}(\mathbb{Z}_3) = \mathbb{Z}_2$ is abelian. Therefore, it is enough to see that the action of $\mathrm{Aut}(G)/\langle\langle f \rangle\rangle$ on $H_2(G)$ is trivial. But this quotient is a 3-group and $\mathrm{Aut}(H_2(G)) = \mathbb{Z}_2$, thus the action is trivial. This concludes the proof that G is Bing. □

For the rest of the section G denotes the group presented by $\mathcal{P}'$. We frequently use the fact that every element of G can be written uniquely in the *standard form* $a^i b^j c^k$ with $0 \leq i, j < 9$, $0 \leq k < 3$. We will also use the fact that we know how to take any word to standard form. We have a description $G = (\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3$ and the conjugation action of $\mathbb{Z}_3$ on $\mathbb{Z}_9 \times \mathbb{Z}_9$ is given by $a^c = b^2$ and $b^c = ab^5$. Thus $(a^i b^j)^c = a^j b^{2i+5j}$.

Alternative presentation

Proposition 1.7.16. *The groups presented by*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

and

$$\mathcal{P}' = \langle a, b, c \mid a^9, b^9, c^3, [a,b], cac^{-1}b^{-2}, cbc^{-1}a^{-1}b^{-5} \rangle$$

are isomorphic.

Proof. We will prove that $x \mapsto c$, $y \mapsto a$ and $a \mapsto y$, $b \mapsto xy^{-4}x^{-1}$, $c \mapsto x$ give inverse isomorphisms. Once we show that these are homomorphisms, it is clear that these are inverses. For the first map, we can use the normal form for words in the group presented by $\mathcal{P}'$. For the second, we need to show the following words are trivial in the group presented by $\mathcal{P}$:

- y^9
- $(xy^{-4}x^{-1})^9 = x(y^9)^{-4}x^{-1}$
- $x^3 = 1$
- $[y, xy^{-4}x^{-1}]$
- $xyx^{-1}(xy^{-4}x^{-1})^{-2} = xy^9x^{-1}$
- $x(xy^{-4}x^{-1})x^{-1}y^{-1}(xy^{-4}x^{-1})^{-5} = x^2y^{-4}x^{-2}y^{-1}xy^{20}x^{-1}$

Thus it is enough to see the following words are trivial in the group presented by $\mathcal{P}$.

- y^9
- $[y, xy^{-4}x^{-1}]$
- $x^2y^{-4}x^{-2}y^{-1}xy^{20}x^{-1}$

The following relations equivalent to $xyx^{-1}yxy^{-1}x^{-1}y^{-1} = 1$ will be useful.

$$\begin{aligned}
x^{-1}yxy^{-1} &= y^{-1}x^{-1}yx \\
yxy^{-1}x^{-1} &= xy^{-1}x^{-1}y \\
yx^{-1}y^{-1}x &= x^{-1}y^{-1}xy \\
xyx^{-1}y^{-1} &= y^{-1}xyx^{-1}
\end{aligned}$$

Using these relations and $x^3 = 1$, $y^{-4}x^{-1}y^2x^{-1}y^{-1}x^{-1} = 1$, we deduce $y^9 = 1$.

$$\begin{aligned}
y^9 &= y^5x^{-1}y^2x^{-1}y^{-1}x^{-1} \\
&= yx^{-1}y^2x^{-1}y^{-1}x^{-1}x^{-1}y^2x^{-1}y^{-1}x^{-1} \\
&= yx^{-1}y(yx^{-1}y^{-1}x)y^2x^{-1}y^{-1}x^{-1} \\
&= yx^{-1}y(x^{-1}y^{-1}xy)y^2x^{-1}y^{-1}x^{-1} \\
&= yx^{-1}(yx^{-1}y^{-1}x)y^3x^{-1}y^{-1}x^{-1} \\
&= yx^{-1}(x^{-1}y^{-1}xy)y^3x^{-1}y^{-1}x^{-1} \\
&= yx^{-1}x^{-1}y^{-1}xy^4x^{-1}y^{-1}x^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= yx^{-1}x^{-1}y^{-1}x(x^{-1}y^2x^{-1}y^{-1}x^{-1})x^{-1}y^{-1}x^{-1} \\
&= y(xyx^{-1}y^{-1})xy^{-1}x^{-1} \\
&= y(y^{-1}xyx^{-1})xy^{-1}x^{-1} \\
&= 1
\end{aligned}
$$

Now we prove $[y, xy^{-4}x^{-1}] = 1$.

$$
\begin{aligned}
[y, xy^{-4}x^{-1}] &= yxy^{-4}x^{-1}y^{-1}x(y^4)x^{-1} \\
&= yx(xyxy^{-2}x)x^{-1}y^{-1}x(x^{-1}y^2x^{-1}y^{-1}x^{-1})x^{-1} \\
&= yx^{-1}(yxy^{-1}x^{-1})y^{-1}x \\
&= yx^{-1}(xy^{-1}x^{-1}y)y^{-1}x \\
&= 1
\end{aligned}
$$

To conclude, we prove $x^2y^{-4}x^{-2}y^{-1}xy^{20}x^{-1} = 1$.

$$
\begin{aligned}
x^2y^{-4}x^{-2}y^{-1}xy^{20}x^{-1} &= x^2(y^{-4})x^{-2}y^{-1}xy^2x^{-1} \\
&= x^2(xyxy^{-2}x)x^{-2}y^{-1}xy^2x^{-1} \\
&= yxy^{-2}x^{-1}y^{-1}xy^2x^{-1} \\
&= yxy^{-2}(x^{-1}y^{-1}xy)yx^{-1} \\
&= yxy^{-2}(yx^{-1}y^{-1}x)yx^{-1} \\
&= (yxy^{-1}x^{-1})y^{-1}xyx^{-1} \\
&= (xy^{-1}x^{-1}y)y^{-1}xyx^{-1} \\
&= 1
\end{aligned}
$$

□

Conjugacy classes

The proof of the following lemma will be omitted.

Lemma 1.7.17. *We have the following description of the conjugacy classes of elements in G.*

- $Z(G) = \{1, a^3b^3, a^6b^6\}$.
- *If a^ib^j is not central, its conjugacy class has size* 3.
- *If $3 \nmid k$, the conjugacy class of $a^ib^jc^k$ has size* 27. *There are* 6 *such classes:*
  - $\{c\}^G$ *and* $\{c^2\}^G$ *are classes of elements of order* 3.
  - $\{ac\}^G$, $\{a^2c\}^G$, $\{ac^2\}^G$ *and* $\{a^2c^2\}^G$ *are classes of elements of order* 9.

Possible descriptions as a semidirect product

Lemma 1.7.18. *We have* $\langle\langle a^i b^j \rangle\rangle^G = \langle a, b\rangle$ *whenever* $i \neq j \pmod 3$.

Proof. Recall that $(a^i b^j)^c = a^j b^{2i+5j}$. We claim $\langle a^i b^j, a^j b^{2i+5j}\rangle = \langle a, b\rangle$. This is equivalent to $\langle (i, j), (j, 2i+5j)\rangle = \mathbb{Z}_9 \times \mathbb{Z}_9$. We only need to show

$$M = \begin{pmatrix} i & j \\ j & 2i+5j \end{pmatrix}$$

is in $\mathrm{GL}_2(\mathbb{Z}_9)$, which follows from $\det(M) = i(2i+5j) - j^2 = 2i^2 + 5ij - j^2 \equiv 2(i-j)^2 \not\equiv 0 \pmod 3$. $\square$

Lemma 1.7.19. $\langle\langle a^i b^j \rangle\rangle^G = \langle a^3, b^3, ab\rangle$ *provided* $i \equiv j \not\equiv 0 \pmod 3$.

Proof. Since the other inclusion is obvious, we only prove $\langle a^3, b^3, ab\rangle \subset \langle\langle a^i b^j \rangle\rangle^G$. Let $K = \langle\langle a^i b^j \rangle\rangle^G$. Without loss of generality we may assume $j = 1$. Then $i = 3k+1$. We have $(a^{3k+1} b^1)^c = ab^{2+6k+5} = ab^{6k+7} \in K$. Moreover $(a^1 b^{6k+7})^{3k+1} = a^{3k+1} b^{(6k+7)(3k+1)} = a^{3k+1} b^7 \in K$. Then $b^6 \in K$, and we have $b^3 \in K$ so $a^{3k+1} b^{3k+1} \in K$. This implies $ab \in K$ and finally since $a^3 b^3 \in K$ we obtain $a^3 \in K$. $\square$

Lemma 1.7.20. *If* $i \equiv j \equiv 0 \pmod 3$ *and* $i \not\equiv j \pmod 9$ *we have* $\langle\langle a^i b^j \rangle\rangle^G = \langle a^3, b^3\rangle$.

Proof. Write $i = 3m$, $j = 3n$. We have $m \not\equiv n \pmod 3$. Then $(a^i b^j)^c = a^j b^{2i+5j} = a^{3n} b^{6m+6n}$. We claim $\langle a^{3m} b^{3n}, a^{3n} b^{6m+6n}\rangle = \langle a^3, b^3\rangle$, or equivalently $\langle (m, n), (n, 2m+2n)\rangle = \mathbb{Z}_3 \times \mathbb{Z}_3$. To prove this it is enough to show

$$M = \begin{pmatrix} m & n \\ n & 2m+2n \end{pmatrix}$$

is in $\mathrm{GL}_2(\mathbb{Z}_3)$. But this follows from $\det(M) = m(2m+2n) - n^2 = 2m^2 + 2mn - n^2 \equiv 2(m-n)^2 \not\equiv 0 \pmod 3$. $\square$

Proposition 1.7.21. *If* $N \lhd G$ *is a normal subgroup then either* $N \leq \langle a, b\rangle$ *or* $|G : N| \leq 3$.

Proof. The conjugacy class of $a^i b^j c^k$ has 27 elements if $k \neq 0$. Then a normal subgroup which is not a subgroup of $\langle a, b\rangle$ has at least $27+1$ elements, and thus has index at most 3 in G. $\square$

Proposition 1.7.22. *If* $N \lhd G$ *is a normal subgroup and* $N \leq \langle a, b\rangle$ *then* N *is either* 1, $\langle a^3 b^3\rangle$, $\langle a^3, b^3\rangle$, $\langle a^3, b^3, ab\rangle$ *or* $\langle a, b\rangle$.

Proof. By the previous lemmas, any normal subgroup of G strictly included in $\langle a, b\rangle$ is contained in $\langle a^3, b^3, ab\rangle$. Then we only need to determine which subgroups of $\langle a^3, b^3, ab\rangle$ are normal in G. By Lemma 1.7.19 this boils down to find the normal subgroups of G strictly contained in $\langle a^3, b^3\rangle$. Finally by Lemma 1.7.20 these are 1 and $\langle a^3 b^3\rangle$. $\square$

Proposition 1.7.23. *If we have a decomposition $G = N \rtimes K$ then K is either 1, $\mathbb{Z}_3$ or G.*

Proof. We want to show that it is not possible to have a decomposition $G = N \rtimes K$ with $|N|$ is equal to 3, 9 or 27. By Proposition 1.7.21 and Proposition 1.7.22, the normal subgroups of these orders are $\langle a^3b^3\rangle$, $\langle a^3, b^3\rangle$ and $\langle a^3, b^3, ab\rangle$. If $N = \langle a^3b^3\rangle$, the projection $G \to G/N$ is not split because we would need to find $\alpha = a(a^3b^3)^i$ and $\beta = b(a^3b^3)^j$ such that $\alpha^3\beta^3 = 1$ while we actually have $\alpha^3\beta^3 = a^3b^3 \neq 1$ for any choice of i and j. If $N = \langle a^3, b^3\rangle$, the projection $G \to G/N$ is not split because a has order 3 in G/N and any preimage $\alpha = aa^{3k}b^{3l}$ has order 9. If $N = \langle a^3, b^3, ab\rangle$ the projection is not split because a has order 3 in G/N and any preimage $\alpha = aa^{3k}b^{3l}(ab)^j$ has order 9. $\square$

Automorphisms

Lemma 1.7.24. *We have*

$$(a^ib^jc^k)^3 = \begin{cases} a^{3i}b^{3j} & \text{if } 3 \mid k \\ a^{3i+6j}b^{3i+6j} & \text{if } 3 \nmid k \end{cases}$$

Proof. This follows from the following computation.

$$\begin{aligned}
(a^ib^jc^k)^3 &= (a^ib^jc^k)(a^ib^jc^k)(a^ib^jc^k) \\
&= a^ib^j(a^ib^j)^{c^k}(a^ib^j)^{c^{2k}} \\
&= \begin{cases} a^{3i}b^{3j} & \text{if } 3 \mid k \\ a^ib^j(a^ib^j)^c(a^ib^j)^{c^2} & \text{if } 3 \nmid k \end{cases} \\
&= \begin{cases} a^{3i}b^{3j} & \text{if } 3 \mid k \\ a^{3i+6j}b^{3i+6j} & \text{if } 3 \nmid k \end{cases}
\end{aligned}$$

$\square$

Lemma 1.7.25. *The following sets are invariant by any $\phi \in \mathrm{Aut}(G)$.*
(i) $W = \{a^ib^j : i \not\equiv j \pmod 3\} = \{g \in G : g^3 \notin Z(G)\}$.
(ii) $N = \langle a, b\rangle$.

Proof. By Lemma 1.7.24 we have the equality claimed in (i) and the right side is clearly invariant by any automorphism. And (ii) follows from $N = \langle W\rangle$. $\square$

We will use the following basic result on the automorphisms of a semidirect product.

Lemma 1.7.26. *Let $G = N \rtimes H$ be a semidirect product. If $\phi_N \in \mathrm{Aut}(N)$ and $\phi_H \in \mathrm{Aut}(H)$ satisfy $\phi_N(n^h) = \phi_N(n)^{\phi_H(h)}$ for all $n \in N$ and $h \in H$, then there exists a unique automorphism $\phi \in \mathrm{Aut}(G)$ such that $\phi|_N = \phi_N$ and $\phi|_H = \phi_H$.*

If $\phi \in \mathrm{Aut}(G)$ *satisfies* $\phi(N) = N$ *and* $\phi(H) = H$ *then* ϕ *is obtained in this way from* $\phi_N = \phi|_N$ *and* $\phi_H = \phi|_H$.

Lemma 1.7.27. *There is no matrix* $A \in \mathrm{GL}_2(\mathbb{Z}_9)$ *such that* $C^2A = AC$ *where* C *is the matrix*

$$\begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix}.$$

Proof. Suppose such a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ exists. Then

$$\begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 2a+5c & 2b+5d \\ a & b \end{pmatrix} = \begin{pmatrix} 2b & a+5b \\ 2d & c+5d \end{pmatrix}$$

We obtain the following system

$$\begin{aligned} 2a+5c &= 2b \\ 2b+5d &= a+5b \\ a &= 2d \\ b &= c+5d \end{aligned}$$

and eliminating a and b we have:

$$\begin{aligned} 3d &= 6c \\ 3c &= 6d \end{aligned}$$

Then $c \equiv 2d \pmod 3$. Write $c = 2d+3k$. Using $a = 2d$ and $b = 7d+3k$ we obtain

$$A = \begin{pmatrix} 2d & 7d+3k \\ 2d+3k & d \end{pmatrix}$$

Since $A \in \mathrm{GL}_2(\mathbb{Z}_9)$, the determinant $\det(A)$ must be a unit in $\mathbb{Z}_9$. But

$$\det(A) = 2d^2 - (7d+3k)(2d+3k) = -12d^2 \notin \mathbb{Z}_9^*,$$

contradiction. □

Lemma 1.7.28. *The centralizer of* C *in* $GL_2(\mathbb{Z}_9)$ *has order* 54.

Proof. Let A be the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and suppose A commutes with C. Then

$$\begin{pmatrix} c & d \\ 2a+5c & 2b+5d \end{pmatrix} = \begin{pmatrix} 2b & a+5b \\ 2d & c+5d \end{pmatrix}$$

This is the same as

$$\begin{aligned} c &= 2b \\ d &= a+5b \\ 2a+5c &= 2d \\ 2b+5d &= c+5d \end{aligned}$$

The last two equations are consequences of the first two equations. Then A has the form

$$A = \begin{pmatrix} a & b \\ 2b & a+5b \end{pmatrix}$$

We need to count how many of these $9 \cdot 9$ matrices are in $\mathrm{GL}_2(\mathbb{Z}_9)$. We must have $\det(A) = a(a+5b) - b(2b) = a^2 + 5ab - 2b^2 \not\equiv 0 \pmod 3$. If $3 \mid b$ then $3 \nmid \det(A)$ if and only if $3 \nmid a$. Thus there are $6 \cdot 3$ solutions with $3 \mid b$. If $3 \nmid b$ we write $\det(A)b^{-2} = (ab^{-1})^2 + 5(ab^{-1}) - 2$. Now $t^2 + 5t - 2$ is divisible by 3 if and only if $t \equiv 1 \pmod 3$. Thus A is invertible if and only if $ab^{-1} \not\equiv 1 \pmod 3$. Equivalently, whenever $a \not\equiv b \pmod 3$. We have 6 possibilities for b since we assuming b is a unit. And we have 6 possibilities for a. This gives $6 \cdot 6$ solutions. In total we obtained $18 + 36 = 54$ solutions. □

Lemma 1.7.29. *There is no automorphism $\phi \in \mathrm{Aut}(G)$ such that $\phi(c) = c^2$.*

Proof. This follows from Lemma 1.7.27 and Lemma 1.7.26. □

Proposition 1.7.30. *We have $|\mathrm{Aut}(G)| = 2 \cdot 3^6$ and $|\mathrm{Out}(G)| = 2 \cdot 3^2$.*

Proof. In Lemma 1.7.25 we have seen that $\phi(N) = N$ for every automorphism ϕ of G. Thus $\phi(c)$ cannot be an element of N. Since c has order 3 we deduce that $\phi(c)$ belongs to either $\{c\}^G$ or $\{c^2\}^G$. By Lemma 1.7.29, there is no automorphism mapping c to c^2. Thus the orbit of c by $\mathrm{Aut}(G)$ is $\{c\}^G$, in particular has size 27. Now by Lemma 1.7.26, the stabilizer of c is the centralizer of C in $\mathrm{GL}_2(\mathbb{Z}_9)$ which has order 54 by Lemma 1.7.28. Thus $|\mathrm{Aut}(G)| = 27 \cdot 54 = 2 \cdot 3^6$. Since $|Z(G)| = 3$, $|\mathrm{Inn}(G)| = |G : Z(G)| = 3^4$. Thus $|\mathrm{Out}(G)| = |\mathrm{Aut}(G) : \mathrm{Inn}(G)| = 2 \cdot 3^2$. □

As a consequence of Lemma 1.7.26 we have.

Proposition 1.7.31. *There is an order 2 automorphism $f\colon G \to G$ defined by*

$$f(a^p b^q c^r) = a^{-p} b^{-q} c^r.$$

Schur multiplier

To compute $H_2(G)$ we use the following result.

Theorem 1.7.32 (Evens, [Eve72, Theorem 2.1])**.** *Let $G = A \rtimes K$ be a p-group. Then, if A abelian and p is odd we have*

$$H_2(G) = H_2(K) \oplus H_1(K; A) \oplus H_2(A)_K$$

and the inclusion of $H_2(A)_K$ as a direct summand of $H_2(G)$ is given by passing to the quotient the map $i_*\colon H_2(A) \to H_2(G)$ induced by the inclusion $i\colon A \hookrightarrow G$.

Proposition 1.7.33. *The Schur multiplier of G is $H_2(G) = \mathbb{Z}_3$.*

Proof. Recall that $G = (\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes \mathbb{Z}_3$. By Theorem 1.7.32 we have

$$H_2(G) = H_2(\mathbb{Z}_3) \oplus H_1(\mathbb{Z}_3; \mathbb{Z}_9 \times \mathbb{Z}_9) \oplus H_2(\mathbb{Z}_9 \times \mathbb{Z}_9)_{\mathbb{Z}_3}$$

We have $H_2(\mathbb{Z}_3) = 0$. By [Bro94, Chapter III, Section 1, Example 2] we compute

$$H_1(\mathbb{Z}_3; \mathbb{Z}_9 \times \mathbb{Z}_9) = \operatorname{coker}(\overline{N}\colon (\mathbb{Z}_9 \times \mathbb{Z}_9)_{\mathbb{Z}_3} \to (\mathbb{Z}_9 \times \mathbb{Z}_9)^{\mathbb{Z}_3}) = 0,$$

Therefore we have $H_2(G) = H_2(\mathbb{Z}_9 \times \mathbb{Z}_9)_{\mathbb{Z}_3}$. To compute $H_2(\mathbb{Z}_9 \times \mathbb{Z}_9)_{\mathbb{Z}_3}$ we need to understand the action of $\mathbb{Z}_3$ on $H_2(\mathbb{Z}_9 \times \mathbb{Z}_9)$. We do this topologically. Consider the complex X associated to the following presentation of $\mathbb{Z}_9 \times \mathbb{Z}_9$

$$\langle a, b | a^9, b^9, [a, b] \rangle.$$

The action of $\mathbb{Z}_3 = \langle c \rangle$ on $\mathbb{Z}_9 \times \mathbb{Z}_9 = \langle a, b \rangle$ is given by $a^c = b^2$ y $b^c = ab^5$. We need to define a cellular map $c\colon X \to X$ inducing conjugation by c on $\pi_1(X)$. We define c on $X^{(1)}$ so that the 1-cell $a \mapsto b^2$ and $b \mapsto ab^5$. Then we extend this to the 2-skeleton of X. The 2-cell a^9 wraps twice over b^9. The 2-cell b^9 wraps once over the 2-cell a^9, five times the 2-cell b^9 and a certain number of times the 2-cell $[a, b]$, as many as needed to turn $(ab^5)^9$ into $a^9 b^{45}$. Finally note that the boundary of the 2-cell $[a, b]$ is mapped to $b^2 ab^5 b^{-2} b^{-5} a^{-1} = b^2 ab^{-2} a^{-1}$. The definition of c on this 2-cell is indicated in Figure 1.2.

Then the 2-cell $[a, b]$ wraps twice over the 2-cell $[a, b]$ with the opposite orientation. Now $H_2(X) = \mathbb{Z}$ and $c_*\colon H_2(X) \to H_2(X)$ is multiplication by -2. We have the following commutative diagram

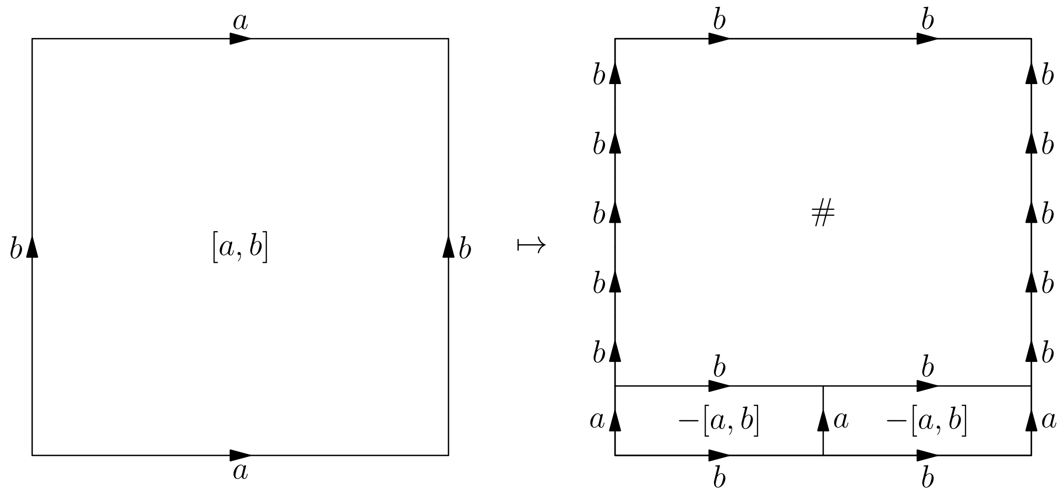

Figure 1.2: The definition of $c\colon X \to X$ on the 2-cell $[a,b]$

$$\begin{tikzcd} H_2(X) \arrow[r, two heads] \arrow[d, "c_*"'] & H_2(\mathbb{Z}_9\times\mathbb{Z}_9) \arrow[d, "c_*"] \\ H_2(X) \arrow[r, two heads] & H_2(\mathbb{Z}_9\times\mathbb{Z}_9) \end{tikzcd}$$

Then since $H_2(\mathbb{Z}_9\times\mathbb{Z}_9)=\mathbb{Z}_9$ is generated by the 2-cell $[a,b]$, the action of c on this group is multiplication by -2 and we have $H_2(\mathbb{Z}_9\times\mathbb{Z}_9)_{\mathbb{Z}_3}=\mathbb{Z}_3$. □

Proposition 1.7.34. *The automorphism f given by Proposition 1.7.31 induces the identity morphism of $H_2(G)$.*

Proof. Consider the following diagram where j is obtained passing to the quotient.

$$\begin{tikzcd} H_2(\mathbb{Z}_9\times\mathbb{Z}_9) \arrow[r, two heads] \arrow[dr, "i_*"'] & H_2(\mathbb{Z}_9\times\mathbb{Z}_9)_{\mathbb{Z}_3} \arrow[d, "j"] \\ & H_2(G) \end{tikzcd}$$

By Theorem 1.7.32, j is the inclusion of $H_2(\mathbb{Z}_9\times\mathbb{Z}_9)_{\mathbb{Z}_3}$ as a direct summand of $H_2(G)$, which in our case is an isomorphism. Then $i_*\colon H_2(\mathbb{Z}_9\times\mathbb{Z}_9)\to H_2(G)$ is an epimorphism. Now considering the following diagram

$$\begin{tikzcd} H_2(\mathbb{Z}_9\times\mathbb{Z}_9) \arrow[r, two heads, "i_*"] \arrow[d, "f|_*"'] & H_2(G) \arrow[d, "f_*"] \\ H_2(\mathbb{Z}_9\times\mathbb{Z}_9) \arrow[r, two heads, "i_*"] & H_2(G) \end{tikzcd}$$

we see it is enough to prove that the restriction of f to $\mathbb{Z}_9 \times \mathbb{Z}_9$ (which is the same as multiplication by -1) induces the identity of $H_2(\mathbb{Z}_9 \times \mathbb{Z}_9)$. We prove this topologically. Consider the standard complex X of the presentation

$$\langle a,b|a^9,b^9,[a,b]\rangle.$$

Consider the map $g\colon X \to X$ defined on the 1-skeleton of X by $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ and extended to X in the obvious way. It is clear that the map induced by g in $H_2(X)$ is the identity. Commutativity of the following diagram concludes the proof:

$$\begin{tikzcd}
H_2(X) \arrow[r, two heads] \arrow[d, "g_*"'] & H_2(\mathbb{Z}_9 \times \mathbb{Z}_9) \arrow[d, "f|_*"] \\
H_2(X) \arrow[r, two heads] & H_2(\mathbb{Z}_9 \times \mathbb{Z}_9)
\end{tikzcd}$$

□

1.7.6 Another application

Proposition 1.7.35. *let H be the Bing of order 16 of Proposition 1.4.10. Then the direct product $H \times H$ is not a Bing group.*

Proof. By the Künneth formula we have $H_2(H\times H) = (\mathbb{Z}_2)^7 \oplus \mathbb{Z}_4$. Since $H = (\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ we have

$$H \times H = ((\mathbb{Z}_2 \times \mathbb{Z}_4) \times (\mathbb{Z}_2 \times \mathbb{Z}_4)) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

The group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ admits an automorphism ϕ with $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_2}) = -1$. Then by Theorem 1.7.2 we conclude $H \times H$ is not Bing. □

Remark 1.7.36. Using GAP directly to check if $H \times H$ satisfies Definition 1.4.1 may be complicated. The group $\mathrm{Aut}(H)$ has order $524288 = 2^{19}$ and can be computed using GAP, however computing $\mathrm{End}(H \times H)$ seems unfeasible. Even if we are able to compute the automorphisms of $H_2(H \times H)$ induced by automorphisms of $H \times H$, by Lemma 1.7.6 we would only obtain trace 0 automorphisms. Of course, we could still use the ideas in this section to write a more sophisticated method to check if a given group is Bing.

Resumen del Capítulo 1: La propiedad del punto fijo para 2-complejos

Un espacio X tiene la propiedad del punto fijo si toda función continua $f\colon X \to X$ tiene un punto fijo. En este capítulo se estudian las siguientes dos preguntas que permanecían abiertas.

Pregunta 1 de Bing. *¿Existe un poliedro compacto de dimensión* 2 *con la propiedad del punto fijo y característica de Euler par?*

Pregunta 8 de Bing. *¿Cuál es el menor entero positivo n tal que la propiedad del punto fijo no es un invariante homotópico para los poliedros de dimensión a lo sumo n?*

Se presentan resultados que dan respuesta a estas preguntas. Estos resultados aparecieron previamente en la Tesis de Licenciatura [SC15] y en los artículos [BSC17] y [SC17b]. En la Sección 1.7 se dan demostraciones alternativas, no son asistidas por computadora e inéditas de los principales resultados del capítulo. Además se presentan algunas demostraciones que sólo estaban disponibles en castellano, como por ejemplo una demostración alternativa, más accesible del siguiente resultado de Borsuk.

Teorema 1.1.17 Borsuk. *Un complejo simplicial X tiene a S^1 como retracto si y solamente si $\mathbb{Z}$ es un sumando directo de $H_1(X)$.*

Se exponen los resultados obtenidos por Lopez que motivaron las preguntas 1 y 8 de Bing.

Los resultados del capítulo utilizan fuertemente la correspondencia entre presentaciones de grupo y tipos homotópicos de 2-complejos. Utilizando teoría de Nielsen y la clasificación de tipos homotópicos de 2-complejos compactos con grupo fundamental abeliano se prueba el siguiente resultado.

Teorema 1.3.21 (Barmak–Sadofschi Costa)**.** *Un poliedro compacto de dimensión* 2 *con característica de Euler distinta de* 1 *y la propiedad del punto fijo no puede tener grupo fundamental abeliano.*

Con ideas similares se prueba que el grupo fundamental de un tal espacio no puede ser un subgrupo finito de $\mathrm{SO}(3)$.

Teorema 1.3.22 (Barmak–Sadofschi Costa)**.** *Un poliedro compacto de dimensión* 2 *con característica de Euler distinta de* 1 *y la propiedad del punto fijo no puede tener grupo fundamental A_4, S_4, A_5 o D_n.*

A continuación se construyen poliedros compactos de dimensión 2 con la propiedad del punto fijo y característica de Euler igual a un entero positivo arbitrario n. De este modo se obtiene una respuesta afirmativa a la Pregunta 1 de Bing. Si $n \leq 0$, del Teorema 1.1.17 se sigue que no puede existir un tal ejemplo. Con este fin, se introduce la noción de *grupo de Bing*.

Definición 1.4.1. Sea G un grupo finitamente presentable y sean $d_1 \mid \ldots \mid d_k$ los factores invariantes de $H_2(G)$. Se dice que G es un *grupo de Bing* si $H_1(G)$ es finito y para todo endomorfismo $\phi\colon G \to G$ se tiene $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ en $\mathbb{Z}_{d_1}$.

La definición anterior tiene sentido si $H_2(G) \neq 0$. Si G es un grupo finitamente presentable tal que $H_1(G)$ es finito y $H_2(G) = 0$, se toma como convención que G es de Bing.

El siguiente resultado permite obtener 2-complejos con la propiedad del punto fijo.

Teorema 1.4.2. *Si $\mathcal{P}$ es una presentación eficiente de un grupo de Bing G entonces $X_{\mathcal{P}}$ tiene la propiedad del punto fijo.*

El resultado anterior permite reducir el problema a hallar ejemplos de grupos de Bing eficientes con multiplicador de Schur no trivial. Si G es un grupo finito simple, todo endomorfismo de G es trivial o un automorfismo y entonces resulta particularmente simple decidir si es de Bing. Utilizando la clasificación de los grupos finitos simples se prueba el siguiente resultado.

Teorema 1.5.1. *Los únicos grupos finitos simples de Bing con multiplicador de Schur no trivial son los grupos $D_{2m}(q)$ con q impar y $m > 2$.*

Si alguno de estos grupos fuera eficiente daría un poliedro de dimensión 2 con la propiedad del punto fijo y característica de Euler 3. Con el fin de responder la Pregunta 1 de Bing se utiliza el software GAP para hallar un grupo de Bing de una naturaleza distinta:

Proposición 1.4.5. *El grupo G presentado por*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

es un grupo finito de orden 3^5. Se tiene $H_2(G) = \mathbb{Z}_3$, y por lo tanto $\mathcal{P}$ es eficiente. Más aún G es un grupo de Bing.

Inmediatamente se obtiene:

Corolario 1.4.8. *El complejo $X_{\mathcal{P}}$ asociado a la presentación*

$$\mathcal{P} = \langle x, y \mid x^3, xyx^{-1}yxy^{-1}x^{-1}y^{-1}, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

tiene la propiedad del punto fijo y característica de Euler 2.

La propiedad del punto fijo es un invariante homotópico de los grafos (poliedros de dimensión 1) y por lo tanto la respuesta a la Pregunta 8 de Bing es por lo menos 2. Utilizando el Teorema 1.4.2 se prueba que la respuesta es 2.

Teorema 1.4.12. *Existe un poliedro Y compacto y de dimensión 2 sin la propiedad del punto fijo y tal que el poliedro X obtenido de Y mediante un colapso elemental de dimensión 2 tiene la propiedad del punto fijo.*

Chapter 2

The Casacuberta-Dicks conjecture

A famous result of Jean-Pierre Serre [Ser80] states that every action of a finite group on a contractible 1-complex (i.e. a tree) has a fixed point. By Smith theory, every action of a p-group on the disk D^n has a fixed point. The group A_5 acts simplicially and fixed point freely on the barycentric subdivision X of the 2-skeleton of the Poincaré homology sphere which is an acyclic 2-complex. By considering $X * A_5$, Edwin E. Floyd and Roger W. Richardson [FR59] proved that A_5 acts simplicially and fixed point freely on a contractible 3-complex. Moreover, by embedding $X * A_5$ in $\mathbb{R}^{81}$ and taking a regular neighborhood they proved that A_5 acts simplicially and fixed point freely on a triangulation of the disk D^{81}. This was the only example known of this kind until Bob Oliver obtained a complete classification of the groups that act fixed point freely on a disk D^n [Oli75]. Floyd and Richardson's example makes clear that Serre's result does not hold in dimension 3, but does it hold for 2-complexes? Carles Casacuberta and Warren Dicks [CD92] made the following conjecture.

Conjecture 2.0.1 (Casacuberta–Dicks)**.** *Let G be a group. If X is a 2-dimensional finite contractible G-complex then $X^G \neq \varnothing$.*

Throughout the chapter, by G-complex we mean a G-CW complex. That is, a CW complex with a continuous G-action that is *admissible* (i.e. the action permutes the open cells of X, and maps a cell to itself only via the identity). For more details see [OS02, Appendix A].

We mention that in the original formulation by Casacuberta and Dicks X is not required to be finite. In the finite case, the same question was raised independently by Aschbacher and Segev [AS93a, Question 3]. In this chapter we study the Casacuberta–Dicks conjecture in the finite case, as stated above.

In [CD92] the conjecture is proved for solvable groups. Independently, Yoav Segev studied the question of which groups act fixed point freely on an acyclic 2-complex and proved Conjecture 2.0.1 for solvable groups and the alternating groups A_n for $n \geq 6$ [Seg93]. In [Seg94], Segev proved the conjecture for collapsible 2-complexes. Using the classification of the finite simple groups, Michael Aschbacher and Yoav Segev proved that for many groups any action on a finite 2-dimensional acyclic complex has a fixed point and then Oliver and Segev [OS02] gave a complete classification of the groups that act fixed point freely on a finite acyclic 2-complex. Before [OS02], A_5 was the only group known to act fixed point freely on an acyclic 2-complex. An excellent survey on this topic is A. Adem's exposition at the Séminaire Bourbaki [Ade03]. In [Cor01], J.M. Corson proved that the Casacuberta–Dicks conjecture holds for diagrammatically reducible complexes (in particular it holds for collapsible complexes).

In this chapter of the Thesis we will study the Casacuberta–Dicks conjecture from different points of view. We will establish connections between this problem and other well-known open questions in topology and geometric group theory. One such connection is with the Quillen conjecture on the poset of p-subgroups of a group. We will show that modulo an open conjecture on equations over groups, if there is a counterexample of Conjecture 2.0.1, then there is a counterexample of a particular form. We will study the fundamental group of acyclic complexes which are potential counterexamples to the conjecture and prove that some perfect groups cannot appear as the fundamental group of one such space. We will find algebraic reformulations of the conjecture involving the group of outer automorphisms of a free group and we will use ideas of Bass-Serre theory to restate the conjecture in terms of presentations. This is connected with the relation gap problem and results on equations over groups. Some of the relevant examples which appear in this chapter were obtained with the software GAP, and the code is included here. Although the answer to the question raised by Casacuberta, Dicks, Aschbacher and Segev is not given in this work, we hope that the ideas developed here could motivate future work on this problem and eventually lead to the proof of the conjecture or the proof of existence of a counterexample.

In Section 2.1 we recall the results obtained by Oliver and Segev [OS02] that we use heavily in Section 2.2 and Section 2.3.

The main result of Section 2.2 is Theorem 2.2.11 which roughly says that, assuming a special case of the Kervaire-Laudenbach-Howie conjecture, if the Casacuberta–Dicks conjecture is false, there is a counterexample of a very special form.

In Section 2.3 we use tools from [OS02] to prove the p-rank 3 case of Quillen's conjecture on the poset of p-subgroups of a finite group. The results of this section will appear in joint work with K. Piterman and A. Viruel [PSCV18]. This result can be seen as a special case of the Casacuberta–Dicks conjecture.

In Section 2.4 we describe some examples of fixed point free 2-dimensional acyclic G-complexes. This gives some evidence supporting the Casacuberta–Dicks conjecture. This examples are studied using GAP and the code appears in Section 2.4.2.

In Section 2.5 we obtain some reformulations of the Casacuberta–Dicks conjecture involving finite subgroups of $\mathrm{Out}(F_m)$. The results of this section give some motivation for Chapter 3.

In Section 2.6, we translate the A_5-case of the Casacuberta–Dicks conjecture into a nice looking problem in combinatorial group theory. Using a result of Klyachko [Kly93] on equations over groups we show that some particular cases of this restatement hold. In Section 2.7 we explain the relation between these restatements and the Relation Gap Problem.

Finally, in Section 2.8 we prove that certain groups do not arise as the fundamental group of an acyclic 2-dimensional fixed point free G-complex of the type constructed by Oliver and Segev [OS02].

Throughout the chapter we will frequently assume that the 2-cells in a G-complex are attached along closed edge paths, this will make no difference for the questions that we study. A *graph* is a 1-dimensional CW complex. By *G-graph* we always mean a 1-dimensional G-complex.

2.1 Fixed point free actions on acyclic 2-complexes

In this section we review the results obtained by Bob Oliver and Yoav Segev in their article [OS02] that we will need later in the chapter. For another detailed exposition of the results of Oliver and Segev see [Ade03].

Definition 2.1.1 ([OS02])**.** A G-space X is *essential* if there is no normal subgroup $1 \neq N \lhd G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \to X^H$ induces an isomorphism on integral homology.

The main results of [OS02] are the following two theorems.

Theorem 2.1.2 ([OS02, Theorem A])**.** *For any finite group G, there is an essential fixed point free 2-dimensional (finite) acyclic G-complex if and only if G is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G-complex are all solvable.*

Theorem 2.1.3 ([OS02, Theorem B])**.** *Let G be any finite group, and let X be any 2-dimensional acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \lhd G$ such that $X^{N'} \neq \varnothing$. Then X^N is acyclic; X is essential if and only if $N = 1$; and the action of G/N on X^N is essential.*

The following fundamental result of Segev [Seg93] will be used frequently (sometimes without giving a reference to it).

Theorem 2.1.4 ([OS02, Theorem 4.1])**.** *Let X be any 2-dimensional acyclic G-complex (not necessarily finite). Then X^G is acyclic or empty, and is acyclic if G is solvable.*

We denote the set of subgroups of G by $\mathcal{S}(G)$.

Definition 2.1.5 ([OS02]). By a *family* of subgroups of G we mean any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. A nonempty family is said to be *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$ then $H' \in \mathcal{F}$; (c) for any $H \triangleleft K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For any family $\mathcal{F}$ of subgroups of G, a $(G,\mathcal{F})$-*complex* will mean a G-complex all of whose isotropy subgroups lie in $\mathcal{F}$. A $(G,\mathcal{F})$-complex is *universal* (resp. H-*universal*) if the fixed point set of each $H \in \mathcal{F}$ is contractible (resp. acyclic).

If G is not solvable, the separating family of solvable subgroups of G is denoted by $\mathcal{SLV}$. If G is perfect, then the family of proper subgroups of G is denoted by $\mathcal{MAX}$.

Lemma 2.1.6 ([OS02, Lemma 1.2]). *Let X be any 2-dimensional acyclic G-complex without fixed points. Let $\mathcal{F}$ be the set of subgroups $H \subseteq G$ such that $X^H \neq \varnothing$. Then $\mathcal{F}$ is a separating family of subgroups of G, and X is an* H-*universal $(G,\mathcal{F})$-complex.*

Proposition 2.1.7 ([OS02, Proposition 6.4]). *Assume that L is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subseteq \mathrm{Aut}(L)$ be any subgroup containing L, and let $\mathcal{F}$ be a separating family for G. Then there is a 2-dimensional acyclic $(G,\mathcal{F})$-complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3 \pmod 8$.*

If X is a poset, $\mathcal{K}(X)$ denotes the *order complex* of X, that is the simplicial complex with simplices the finite nonempty totally ordered subsets of X (the complex $\mathcal{K}(X)$ is also known as the *nerve* of X).

Definition 2.1.8 ([OS02, Definition 2.1]). For any family $\mathcal{F}$ of subgroups of G define

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H):H]}(1-\chi(\mathcal{K}(\mathcal{F}_{>H}))).$$

Recall that if $G \curvearrowright X$, the orbit $G \cdot x$ is said to be *of type* G/H if the stabilizer G_x is conjugate to H in G. In other words, if the action of G on $G \cdot x$ is the same as the action of G on G/H. In [OS02] the following result is only needed for separating families and so, it is stated accordingly. We will need the following more general version and it can be checked that the original proof works in this case with no modifications.

Lemma 2.1.9 ([OS02, Lemma 2.3]). *Fix a family $\mathcal{F}$, a finite* H-*universal $(G,\mathcal{F})$-complex X, and let $H \subseteq G$. For each n, let $c_n(H)$ denote the number of orbits of n-cells of type G/H in X. Then $i_{\mathcal{F}}(H) = \sum_{n \geq 0}(-1)^n c_n(H)$.*

Proposition 2.1.10 ([OS02, Tables 2,3,4]). *Let G be one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Then $i_{\mathcal{SLV}}(1) = 1$.*

2.1.1 The three families of acyclic examples

Proposition 2.1.11 ([OS02, Example 3.4]). *Set $G = \mathrm{PSL}_2(q)$, where $q = 2^k$ and $k \geq 2$. Then there is a 2-dimensional acyclic fixed point free G-complex X, all of whose isotropy subgroups are solvable. More precisely X can be constructed to have three orbits of vertices with isotropy subgroups isomorphic to $B = \mathbb{F}_q \rtimes C_{q-1}$, $D_{2(q-1)}$, and $D_{2(q+1)}$; three orbits of edges with isotropy subgroups isomorphic to C_{q-1}, C_2 and C_2; and one free orbit of 2-cells.*

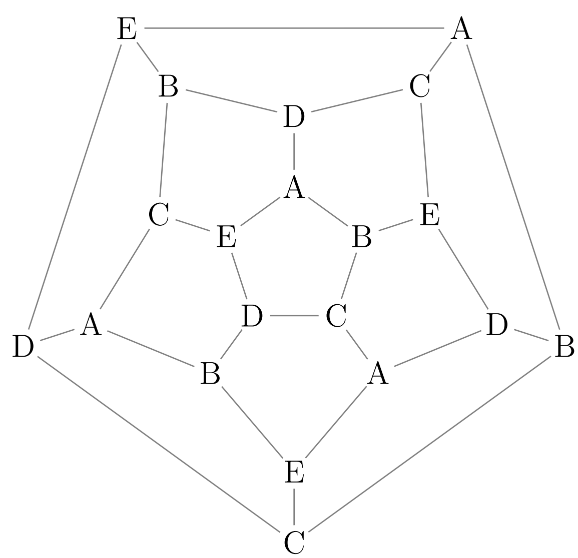

Figure 2.1: A representation of the 2-skeleton of the Poincaré dodecahedral space. Opposite faces are rotated clockwise by $\frac{2\pi}{10}$ and identified. Thus the resulting space has five 0-cells A, B, C, D, E; ten 1-cells AB, AC, AD, AE, BC, BD, BE, CD, CE, DE and five pentagonal 2-cells $ABCDE$, $ABDEC$, $ABECD$, $ACBED$, $ADBCE$.

We have $A_5 = \mathrm{PSL}_2(2^2)$. The barycentric subdivision of the 2-skeleton of the Poincaré dodecahedral space is an A_5-complex of the type given in Proposition 2.1.11 with fundamental group the binary icosahedral group $A_5^* \simeq \mathrm{SL}(2,5)$ which has order 120. In Figure 2.1 we see the usual way to describe this space. In Figure 2.3 we see it from the point of view of Proposition 2.1.11. The Poincaré dodecahedral space appears in many other natural ways, for more information see [KS79].

Proposition 2.1.12 ([OS02, Example 3.5]). *Assume that $G = \mathrm{PSL}_2(q)$, where $q = p^k \geq 5$ and $q \equiv \pm 3 \bmod 8$. Then there is a 2-dimensional acyclic fixed point free G-complex X, all of whose isotropy subgroups are solvable. More precisely, X can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to $B = \mathbb{F}_q \rtimes C_{(q-1)/2}$, D_{q-1}, D_{q+1}, and A_4; four orbits of edges with isotropy subgroups isomorphic to $C_{(q-1)/2}$, C_2^2, C_3 and C_2; and* *one free orbit of* 2*-cells.*

Proposition 2.1.13 ([OS02, Example 3.7])**.** *Set $q = 2^{2k+1}$ for any $k \geq 1$. Then there is a 2-dimensional acyclic fixed point free $\mathrm{Sz}(q)$-complex X, all of whose isotropy subgroups are solvable. More precisely, X can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to $M(q,\theta)$, $D_{2(q-1)}$, $C_{q+\sqrt{2q}+1} \rtimes C_4$, $C_{q-\sqrt{2q}+1} \rtimes C_4$; four orbits of edges with isotropy subgroups isomorphic to C_{q-1}, C_4, C_4 and C_2; and one free orbit of* 2*-cells.*

We have $A_5 = \mathrm{PSL}_2(2^2) = \mathrm{PSL}_2(5)$, so this group is addressed in both Proposition 2.1.11 and Proposition 2.1.12. There is no other such exception.

If G is one of the groups in Theorem 2.1.2, $\Gamma_{OS}(G)$ denotes the 1-skeleton of any 2-dimensional fixed point free acyclic G-complex without free orbits of 1-cells of the type constructed in Propositions 2.1.11 to 2.1.13 (here, we regard A_5 as $\mathrm{PSL}_2(2^2)$ rather than $\mathrm{PSL}_2(5)$). For example, $\Gamma_{OS}(A_5)$ can be taken to be the 1-skeleton Γ_P of the barycentric subdivision of the 2-skeleton of the Poincaré dodecahedral space (see Figure 2.3) which is a simplicial complex with $21 = 5+10+6$ vertices, $80 = 20+30+30$ edges and 60 faces. We now fix some notation regarding Γ_P. The representatives for the orbits of vertices are $v_1 = H_1 = A$, $v_2 = H_2 = AB$, $v_3 = H_3 = ABCDE$. The representatives for the orbits of edges are $e_1 = (v_1 \to v_2)$, $e_2 = (v_1 \to v_3)$ and $e_3 = (v_2 \to v_3)$.

Generally, there is more than one possible choice for the G-graph $\Gamma_{OS}(G)$. Even for $G = A_5$, thought of as $\mathrm{PSL}_2(2^2)$, the quotient graph $\Gamma_{OS}(G)/G$ is not unique as we will see in Example 1 of Section 2.4. However in Section 2.2 we will see why this choice is not relevant at all.

Recall that the *coset complex* of a tuple of subgroups $(H_1,\dots,H_k)$ of a group G is the simplicial complex with vertex set $G/H_1 \coprod G/H_2 \coprod \cdots \coprod G/H_k$ having a simplex for every subset of vertices with nonempty intersection. In [OS02, p. 21] (see also [Ade03, Section 5]) it is explained that, for $G = \mathrm{PSL}_2(2^k)$, the graph $\Gamma_{OS}(G)$ can be taken as the 1-skeleton of the coset complex of $(B, D_{2(q-1)}, D_{2(q+1)})$. Nevertheless, the coset complex itself is in general not acyclic (see [AS93a]). In Figure 2.2 we see a picture of $\Gamma_{OS}(G)/G$ for this particular choice.

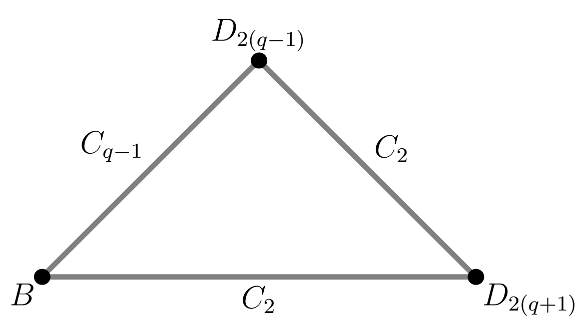

Figure 2.2: A picture of $\Gamma_{OS}(G)/G$ for the particular choice of $\Gamma_{OS}(G)$ as a coset complex in the case $G = \mathrm{PSL}_2(2^k)$.

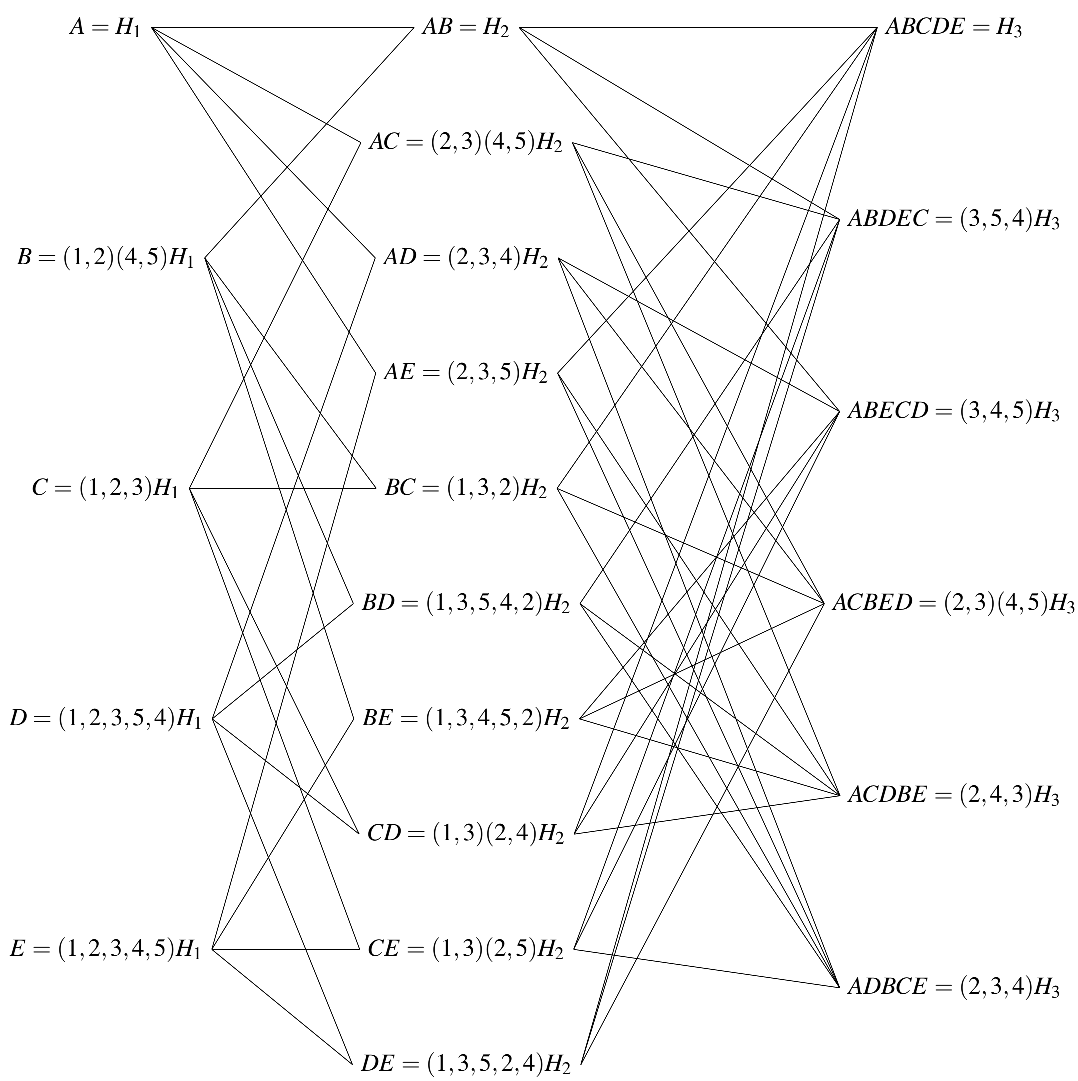

Figure 2.3: A picture of Γ_P the 1-skeleton of the barycentric subdivision of the 2-skeleton of the Poincaré dodecahedral space. In addition to the edges in the figure, every vertex in the left column is connected to every vertex in the right column by an edge that is not shown. The isotropy subgroups for the orbits of vertices are $H_1 = \langle (3,4,5),(2,4)(3,5)\rangle = A_4$, $H_2 = \langle (1,2)(4,5),(3,4,5)\rangle = D_6$ and $H_3 = \langle (1,4)(2,3),(1,3)(4,5)\rangle = D_{10}$.

A key property of the G-graph $\Gamma_{OS}(G)$ is that $H_1(\Gamma_{OS}(G))$ is a free $\mathbb{Z}[G]$-module of rank 1. From [OS02, Proposition 1.7] we deduce

Proposition 2.1.14. *Let G be one of the groups in Theorem 2.1.2. A G-graph Γ is a suitable* *choice for* $\Gamma_{OS}(G)$ *if and only if the following conditions hold*

(i) The orbits of Γ *have the types prescribed by Propositions 2.1.11 to 2.1.13.*

(ii) Γ *is connected.*

(iii) For each $1 \neq H \leq G$, Γ^H *is acyclic or empty and is acyclic if* H *has prime power order.*

2.2 A reduction modulo Kervaire-Laudenbach-Howie

In this section, using the results from [OS02] we prove Theorem 2.2.11 which roughly says that, assuming a special case of the Kervaire-Laudenbach-Howie conjecture (see Section A.6), if the Casacuberta-Dicks conjecture is false, then there is a counterexample of the type constructed in [OS02]. The special case we need is the following.

Conjecture 2.2.1. *Let* X *be a finite contractible* 2*-complex. If* $A \subset X$ *is an acyclic subcomplex, then* A *is contractible.*

By the work of Gerstenhaber-Rothaus [GR62], we know that Conjecture 2.2.1 holds under the hypothesis that $\pi_1(A)$ is locally residually finite. If the fundamental group of A is hyperlinear then Conjecture 2.2.1 is known to hold (see [NT18, Theorem 1.2], see also [Tho12], [Pes08, Section 10]). Thus the following implies Conjecture 2.2.1. In [Seg94, (4.3)] the conjecture is proved when X is collapsible.

Conjecture 2.2.2 (Connes' embedding conjecture for groups [Pes08])**.** *Every group is hyperlinear.*

2.2.1 Some equivariant modifications

Definition 2.2.3. If X,Y are G-spaces, a *G-homotopy* is an equivariant map $H\colon X \times I \to Y$. We say that $f_0(x) = H(x,0)$ and $f_1(x) = H(x,1)$ are *G-homotopic* and we denote this by $f_0 \simeq_G f_1$. An equivariant map $f\colon X \to Y$ is a *G-homotopy equivalence* if there is a map $g\colon Y \to X$ such that $fg \simeq_G 1_Y$ and $gf \simeq_G 1_X$. A G-invariant subspace A of X is a *strong G-deformation retract of* X if there is a retraction $r\colon X \to A$ such that there is a G-homotopy $H\colon ir \simeq 1_X$ relative to A, where $i\colon A \to X$ is the inclusion.

Remark 2.2.4. An equivariant map $f\colon X \to Y$ is a G-homotopy equivalence if and only if $f^H\colon X^H \to Y^H$ is a homotopy equivalence for each subgroup $H \leq G$ (see [tD08, (2.7) Proposition]). Thus, if $f\colon X \to Y$ is a G-homotopy equivalence, the action $G \curvearrowright X$ is fixed point free (resp. essential) if and only if the action $G \curvearrowright Y$ is fixed point free (resp. essential).

From the equivariant homotopy extension property for pairs of G-complexes (see [Bre67, Chapter I, Section 1]) we deduce the following.

Theorem 2.2.5. *If A is a G-subcomplex of a G-complex X and the inclusion $A \hookrightarrow X$ is a G-homotopy equivalence, then A is a strong G-deformation retract of X.*

Lemma 2.2.6. *Let X be an acyclic 2-dimensional G-complex. Let $H \leq G$ and $x_0, x_1 \in X^{(0)} \cap X^H$.Then there is a G-complex $Y \supset X$, such that X is a strong G-deformation retract of Y and Y is obtained from X by attaching an orbit of 1-cells of type G/H with endpoints $\{x_0, x_1\}$ and an orbit of 2-cells of type G/H.*

Proof. We attach an orbit of 1-cells of type G/H to X using the attaching map $\varphi\colon G/H \times S^0 \to X^{(0)}$ defined by $(gH, 1) \mapsto g \cdot x_0$, $(gH, -1) \mapsto g \cdot x_1$. Let e be the 1-cell of this new orbit corresponding to the coset H. Since X is acyclic, by Theorem 2.1.4 X^H is also acyclic. Let γ be an edge path in X^H starting at x_1 and ending at x_0. Then we attach an orbit of 2-cells of type G/H in such a way that the 2-cell corresponding to the coset H is attached along the closed edge path given by e and γ. It is clear that X is a strong G-deformation retract of Y. □

Remark 2.2.7. In the situation of Lemma 2.2.6, we say that Y is obtained from X by an *equivariant elementary expansion of dimension* 2 *and type* G/H or that X is obtained from Y by an *equivariant elementary collapse of dimension* 2 *and type* G/H.

The following definitions appear in [KLV01, Section 2].

Definition 2.2.8. A *forest* is a graph with trivial first homology. If a subcomplex Γ of a CW complex X is a forest, there is a CW complex Y obtained from X by shrinking each connected component of Γ to a point. The quotient map $q\colon X \to Y$ is a homotopy equivalence and we say Y is obtained from X by a *forest collapse*.

If X is a G-complex and $\Gamma \subset X$ is a forest which is G-invariant, the quotient map q is a G-homotopy equivalence and we say the G-complex Y is obtained from X by a *G-forest collapse*.

We say that a G-graph is *reduced* if it has no edge e such that $G \cdot e$ is a forest.

Lemma 2.2.9. *Let X be a 2-dimensional acyclic G-complex. If $X^{(1)}$ is a reduced G-graph then stabilizers of different vertices are not comparable.*

Proof. Suppose $X^{(1)}$ is a reduced G-graph. Let $\mathcal{F} = \{G_x : x \in X^{(0)}\}$. Let $M = \{v \in X^{(0)} : G_v$ is maximal in $\mathcal{F}\}$. We first prove, by contradiction, that $X^{(0)} = M$. Consider $v \in X^{(0)} - M$ such that G_v is maximal in $\{G_x : x \in X^{(0)} - M\}$. Then since X^{G_v} contains v, by Theorem 2.1.4 it must be acyclic. Since $v \notin M$, there is a vertex $w \in X^{G_v} \cap M$. By connectivity there is an edge $e \in X^{G_v}$ whose endpoints v' and w' satisfy $v' \notin M$ and $w' \in M$. Since $G_{v'} \geq G_v$ and $v' \notin M$, by our choice of v we have $G_v = G_{v'}$. Since $e \in X^{G_v}$ we have $G_v \leq G_e$ and since v' is an endpoint of e we have $G_e \leq G_{v'}$. Thus $G_e = G_{v'}$ and then the degree of v' in the graph $G \cdot e$ (which has vertex set $G \cdot w' \coprod G \cdot v'$) is 1. Thus $G \cdot e$ is a forest, contradiction. Therefore we must have $M = X^{(0)}$. To conclude we have to prove that different vertices $u, v \in M$ have different stabilizers. Suppose $G_u = G_v$ to get a contradiction. Since u, v are vertices of X^{G_u}

which is connected, there is an edge $e \in X^{G_u}$ and by maximality we must have $G_e = G_u$. If u', v' are the endpoints of e, we have $G_{u'} = G_{v'}$. We have two cases and in any case we obtain a contradiction. If $G \cdot u' \neq G \cdot v'$ then $G \cdot e$ is a forest consisting of $|G/G_e|$ parallel edges, contradiction. Otherwise, there is a nontrivial element $g \in G$ such that $g \cdot u' = v'$ and we have $G_{u'} = G_{v'} = gG_{u'}g^{-1}$. Thus $g \in N_G(G_{u'})$. Consider the action of $\langle g \rangle$ on $X^{G_{u'}}$, which is acyclic and thus has a fixed point by the Lefschetz fixed point theorem. But this cannot happen, since this would imply that $\langle G_{u'}, g \rangle \gneq G_{u'}$ fixes a point of X, which is a contradiction since $u' \in M$. □

Corollary 2.2.10. *If X is a 2-dimensional acyclic G-complex and X^G is nonempty then there is a G-invariant maximal tree.*

Proof. We define a sequence of G-complexes $X_0, \dots, X_k$ such that $X_i^G \neq \varnothing$. Let $X_0 = X$. If X_i is defined and $X_i^{(0)} \neq *$ then by Lemma 2.2.9 there is an edge e_{i+1} of X_i such that $G \cdot e_{i+1}$ is a forest. Then X_{i+1} is obtained from X_i by collapsing the G-forest $G \cdot e_{i+1}$. Then $G \cdot \{e_1, \dots, e_k\}$ is a G-invariant spanning tree for X. □

2.2.2 The reduction

Now we prove the main result of the section: assuming Conjecture 2.2.1, if the Casacuberta-Dicks conjecture fails, there is a counterexample of a particular form.

Theorem 2.2.11. *Assume Conjecture 2.2.1 holds. If the Casacuberta–Dicks conjecture 2.0.1 is false, then there is a 2-dimensional essential, fixed point free and contractible G-complex X where G is one of the following groups:*

(i) $\mathrm{PSL}_2(2^p)$ *for p prime.*

(ii) $\mathrm{PSL}_2(3^p)$ *for an odd prime p.*

(iii) $\mathrm{PSL}_2(q)$ *for a prime $q > 3$ such that* $q \equiv \pm 3 \bmod 5$ *and* $q \equiv \pm 3 \bmod 8$.

(iv) $\mathrm{Sz}(2^p)$ *for p an odd prime.*

Moreover, for any choice of $\Gamma_{OS}(G)$, it is possible to take X so that it is obtained from $\Gamma_{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells and $k+1$ free orbits of 2-cells.

Proof. Suppose X is a counterexample for the Casacuberta–Dicks conjecture. We may assume that $|G|$ is minimal. Since we are assuming Conjecture 2.2.1, by Theorem 2.1.3 we have that X is essential. Then G must be one of the groups in Theorem 2.1.2. By minimality of $|G|$, we have that $X^H \neq \varnothing$ for every $H \lneq G$. Then by Lemma 2.1.6 X is an H-universal $(G, \mathcal{MAX})$-complex. By Proposition 2.1.7, we must have $\mathcal{MAX} = \mathcal{SLV}$. Then every proper subgroup of G is solvable. By [OS02, Proposition 3.3], if every proper subgroup of a group $\mathrm{PSL}_2(2^k)$

($k \geq 2$) is solvable then k is a prime (note that when $k = 2$ the group is A_5). Also by [OS02, Proposition 3.3], if every proper subgroup of a group $\mathrm{PSL}_2(q)$ (with $q \equiv \pm 3 \bmod 8$, $q > 5$) is solvable then either $q = 3^p$ for p an odd prime or q is prime and $q \equiv \pm 3 \bmod 5$ (since otherwise A_5 is a subgroup). Finally by [OS02, Proposition 3.6], if every proper subgroup of a group $\mathrm{Sz}(2^k)$ is solvable then k is an odd prime. Thus G is one of the groups in the statement of Theorem 2.2.11.

Now we prove the second part of the theorem. By doing enough G-forest collapses we can assume that $X^{(1)}$ is a reduced G-graph. The stabilizers of the vertices of $\Gamma_{OS}(G)$ are precisely the maximal subgroups of G. Therefore, since every proper subgroup of G fixes a point of X, by Lemma 2.2.9, we have $X^{(0)} = \Gamma_{OS}(G)^{(0)}$. Applying Lemma 2.2.6 enough times to modify X, we may further assume $\Gamma_{OS}(G)$ is a subcomplex of X.

Finally we will prove that X can be taken so that for every subgroup $1 \neq H \lneq G$, we have $X^H = \Gamma_{OS}(G)^H$. We prove this by reverse induction on $|H|$. Assume that we have X such that it holds for every subgroup K with $H \lneq K \lneq G$. Since $\Gamma_{OS}(G)^H$ is a tree (it is acyclic and 1-dimensional) and X^H is contractible by Conjecture 2.2.1, the inclusion $\Gamma_{OS}(G)^H \hookrightarrow X^H$ is a $N_G(H)$-homotopy equivalence and by Theorem 2.2.5, $\Gamma_{OS}(G)^H$ is a strong $N_G(H)$-deformation retraction of X^H. Thus we can take a $N_G(H)$-retraction $r_H\colon X^H \to \Gamma_{OS}(G)^H$ which is also a $N_G(H)$-homotopy equivalence. Moreover, the stabilizer of the cells in $X^H - \Gamma_{OS}(G)^H$ is H (the stabilizer cannot be bigger by the induction hypothesis). We define retractions $r_{H^g}\colon X^{H^g} \to \Gamma_{OS}(G)^{H^g}$ by $r_{H^g}(gx) = g \cdot r_H(x)$ which glue to give a strong G-deformation retraction

$$r\colon \Gamma_{OS}(G) \bigcup_{g \in G} X^{H^g} \to \Gamma_{OS}(G).$$

We may replace X by the pushout of the diagram

$$\begin{array}{ccc} \Gamma_{OS}(G) \bigcup_{g\in G} X^{H^g} & \xrightarrow{r} & \Gamma_{OS}(G) \\ \big\downarrow & & \\ X & & \end{array}$$

which is G-homotopy equivalent to X. This procedure removes the excessive orbits of cells of type G/H. Thus by induction we may assume that $X^{(1)}$ coincides with $\Gamma_{OS}(G)$ up to free orbits of 1-cells. By Lemma 2.1.9 we conclude that every orbit of 2-cells of X is free and that there are exactly $k+1$ orbits of 2-cells. $\square$

In particular we have the following:

Corollary 2.2.12. *Assuming Conjecture 2.2.1, if the Casacuberta–Dicks conjecture is false, then there is a counterexample where every orbit of 2-cells is free.*

The attaching maps for the free orbits of 2-cells of two acyclic 2-dimensional G-complexes with 1-skeleton $\Gamma_{OS}(G)$ determine elements in $H_1(\Gamma_{OS}(G)) \simeq \mathbb{Z}[G]$ which differ by a unit of $\mathbb{Z}[G]$. Two attaching maps based at x_0 which give the same element of $H_1(\Gamma_{OS}(G))$ give elements of $\pi_1(\Gamma_{OS}(G), x_0)$ which differ by an element in the commutator subgroup of $\pi_1(\Gamma_{OS}(G), x_0)$.

The following explains why our particular choice of $\Gamma_{OS}(G)$ and the way the free orbits of 1-cells are attached is not relevant.

Proposition 2.2.13. *Any two choices for $\Gamma_{OS}(G)$ are G-homotopy equivalent. Moreover, attaching $k \geq 0$ free orbits of 1-cells to any two choices for $\Gamma_{OS}(G)$ produces G-homotopy equivalent graphs.*

Proof. Since any choice of $\Gamma_{OS}(G)$ is a universal $(G, \mathcal{SLV} - \{1\})$-complex, the first part follows from [OS02, Proposition A.6]. The second part follows easily from the first and Theorem A.7.1. □

Remark 2.2.14. Another way to prove the first part of Proposition 2.2.13 is using the arguments contained in the proof of Theorem 2.2.11. Concretely, if Γ and Γ' are different choices for $\Gamma_{OS}(G)$, we can modify Γ using Lemma 2.2.6 to obtain a G-complex X having both Γ and Γ' as G-subcomplexes. Moreover $\Gamma \hookrightarrow X$ is a G-homotopy equivalence. Now we repeat the process of replacing X by a pushout G-homotopy equivalent to X in the same way as in the proof of Theorem 2.2.11, until we reach Γ'. Note that in this argument we do not use Conjecture 2.2.1 to obtain the retractions; instead we use the fact that each of our modifications is a G-homotopy equivalence.

Corollary 2.2.15. *Let Γ be a graph obtained from $\Gamma_{OS}(G)$ by attaching $k \geq 0$ free orbits of 1-cells. The set of G-homotopy equivalence classes of 2-dimensional acyclic fixed point free G-complexes with 1-skeleton Γ does not depend on the particular choice of $\Gamma_{OS}(G)$ or the way the k free orbits of 1-cells are attached. In particular, the set of isomorphism classes of groups that occur as the fundamental group of such spaces does not depend on such choices.*

Proof. Again, this is an easy application of Theorem A.7.1. □

2.3 Relationship with Quillen's conjecture

In this section we prove the p-rank 3 case of Quillen's conjecture on the poset of p-subgroups of a finite group G. This is joint work with K. Piterman and A. Viruel that will appear in [PSCV18].This result can also be seen as a special case of the Casacuberta–Dicks conjecture.

Quillen's conjecture concerns the poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups of G, which was introduced by K.S. Brown in [Bro75], where he proved that the Euler characteristic $\chi(\mathcal{K}(\mathcal{S}_p(G)))$ of its order complex is 1 modulo the greatest power of p dividing the order of G. Some years later, Quillen [Qui78] studied some homotopy properties of its order complex $\mathcal{K}(\mathcal{S}_p(G))$ and proved that $\mathcal{K}(\mathcal{A}_p(G)) \simeq \mathcal{K}(\mathcal{S}_p(G))$ [Qui78, Proposition 2.1]. Here, $\mathcal{A}_p(G)$ denotes the subposet of nontrivial elementary abelian p-subgroups of G. Recall that a p-group is *elementary abelian* if it is isomorphic to a product $(\mathbb{Z}_p)^n$ for some n.

Quillen also proved that if $O_p(G)$, the greatest normal p-subgroup of G, is nontrivial then $\mathcal{K}(\mathcal{A}_p(G)) \simeq *$. [Qui78, Proposition 2.4] and conjectured that the converse should hold. In this section we consider the following stronger version of Quillen's conjecture, stated by Aschbacher and Smith [AS93b].

Conjecture 2.3.1 (Quillen's conjecture)**.** *If $O_p(G) = 1$ then $\widetilde{H}_*(\mathcal{A}_p(G)) \neq 0$.*

Quillen proved some cases of this conjecture. For example, he proved it for solvable groups [Qui78, Theorem 12.1] and observed that it holds for groups of p-rank 2. Recall that the *p-rank* of G is the maximum possible rank of an elementary abelian p-subgroup of G and equals $\dim \mathcal{K}(\mathcal{A}_p(G)) + 1$. Thus the p-rank 2 case follows from Serre's result that an action of finite group acting on a tree ha a fixed point and the p-rank 3 case may be seen as a special case of the Casacuberta–Dicks conjecture. In [AS93b], M. Aschbacher and S.D. Smith made a huge progress on the study of this conjecture. By using the classification of finite simple groups, they proved that Quillen's conjecture holds if $p > 5$ and G does not contain certain unitary components. Previously, Aschbacher and Kleidman [AK90] had proved Quillen's conjecture for almost simple groups (i.e. finite groups G such that $L \leq G \leq \mathrm{Aut}(L)$ for some simple group L).

We will use the results of Oliver and Segev [OS02] to prove Quillen's conjecture for groups of p-rank 3. If X is a poset, we define $H_*(X)$ as $H_*(\mathcal{K}(X))$. Note that the order complex of a G-poset is always a G-complex.

Theorem 2.3.2 (Piterman – Sadofschi Costa – Viruel [PSCV18])**.** *Let G be a finite group of p-rank 3. If $\widetilde{H}_*(\mathcal{A}_p(G)) = 0$ then $O_p(G) \neq 1$.*

Proof. Suppose the statement is false and consider a counterexample G. Then $X = \mathcal{K}(\mathcal{A}_p(G))$ is a 2-dimensional acyclic complex. Equipped with the conjugation action of G, X is a G-complex. Since we are assuming $O_p(G) = 1$, the action is fixed point free. Consider the subgroup N generated by the subgroups $N' \triangleleft G$ such that $X^{N'} \neq \varnothing$. Clearly N is normal in G. By Theorem 2.1.3 $Y = X^N$ is acyclic (in particular it is nonempty) and the action of G/N on Y is essential and fixed point free. By Lemma 2.1.6 $\mathcal{F} = \{H \leq G/N : Y^H \neq \varnothing\}$ is a separating family and Y is an H-universal $(G/N, \mathcal{F})$-complex. Thus, Theorem 2.1.2 asserts that G/N must be one of the groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. In any case, by Proposition 2.1.7 we must have $\mathcal{F} = \mathcal{SLV}$. By Proposition 2.1.10, $i_{\mathcal{SLV}}(1) = 1$. Finally by Lemma 2.1.9, Y must have at least one free G/N-orbit. Therefore X has a G-orbit of type G/N. Let $\sigma = (A_0 < \ldots < A_j)$ be a simplex of X with stabilizer N. Since

$A_0 \triangleleft N$, we have that $O_p(N)$ is nontrivial. Since $N \triangleleft G$ and $O_p(N)\,\mathrm{char}\,N$ we have $O_p(N) \triangleleft G$ and therefore $O_p(N) \leq O_p(G)$. So $O_p(G)$ is nontrivial, a contradiction. $\square$

With the same argument we can prove the following generalization of Theorem 2.3.2.

Theorem 2.3.3 (Piterman - Sadofschi Costa - Viruel [PSCV18])**.** *If $\mathcal{K}(\mathcal{S}_p(G))$ has an acyclic and G-invariant 2-dimensional subcomplex, then $O_p(G) \neq 1$.*

By Theorem 2.3.3, Quillen's conjecture also holds when $\mathcal{K}(\mathcal{B}_p(G))$ is 2-dimensional. Recall that the subposet $\mathcal{B}_p(G) = \{Q \in \mathcal{S}_p(G) : Q = O_p(N_G(Q))\}$ is homotopy equivalent to $\mathcal{S}_p(G)$. See [Smi11] for an account of the relations between the different p-group complexes.

Finally we mention that a possible approach to prove Conjecture 2.3.1 is to find an acyclic and G-invariant 2-dimensional subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$. If Quillen's conjecture were true, then this would be possible. Therefore, by Theorem 2.3.3 we have the following equivalent version of the conjecture.

Conjecture 2.3.4 (Restatement of Quillen's conjecture)**.** *Assume $\mathcal{K}(\mathcal{S}_p(G))$ is acyclic. Then there exists a G-invariant acyclic subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$ of dimension at most 2.*

2.4 Experimental results

We developed a GAP package, G2Comp [SC18a] to study 2-dimensional G-complexes. The original aim of this was to find a counterexample to the Casacuberta–Dicks conjecture. The plan was to construct acyclic examples of the type described by Oliver and Segev. To do this we can take a random closed edge path in $\Gamma_{OS}(A_5)$ and check if the resulting space is acyclic. If so, we can try to prove (again using GAP) that the fundamental group is trivial. Of course there is no algorithm to decide if a finite presentation $\mathcal{P}$ presents the trivial group. But hopefully, there is a particular example where the methods implemented in GAP are successful.

As predicted by the Casacuberta–Dicks conjecture, we could not find any contractible examples. For many of the acyclic examples we have a satisfying description of the fundamental group or at least a proof that the group is not trivial. Our experimentation lead us to make the following conjecture.

Conjecture 2.4.1. *Let X be a fixed point free 2-dimensional finite and acyclic A_5-complex. If $\pi_1(X)$ is finite then $\pi_1(X) = A_5^*$.*

In [EHT01], a similar phenomenon is described for perfect nontrivial *cyclically presented* groups (i.e. the fundamental groups of $\mathbb{Z}_n$-complexes with one 0-cell, one free orbit of 1-cells and one free orbit of 2-cells). We will return briefly to cyclically presented groups in Section 2.6.2. Kervaire proved that if the fundamental group of a homology 3-sphere is finite, then it is either the trivial group or A_5^* [Ker69, Theorem 2]. We state some more questions regarding these examples.

Question 2.4.2. *Is it true that any acyclic example with* 1*-skeleton* $\Gamma_{OS}(A_5)$ *is homotopy equivalent to the spine of a homology* 3*-sphere?*

Question 2.4.3. *Is it true that the fundamental group of an acyclic example with* 1*-skeleton* $\Gamma_{OS}(G)$ *is a* 3*-manifold group?*

Question 2.4.4. *Is it true that the fundamental group of an acyclic example with* 1*-skeleton* $\Gamma_{OS}(G)$ *is hyperbolic?*

In Section 2.4.1 we describe some interesting examples of 2-dimensional fixed point free acyclic G-complexes, highlighting some properties. The complete description of each of the examples and the relevant verifications appear in Section 2.4.2. Though it is not essential, we recommend to go back and forth between the code and the description of each example. In most of the examples the group acting is A_5 and the 1-skeleton is the graph Γ_P described in Figure 2.3. In some of the examples we use our package SmallCancellation [SC18b] to check if some presentations of the fundamental group satisfy conditions $C(p)$ and $C'(\lambda)$. For the basic definitions and results of small cancellation theory see Section A.5.

2.4.1 The examples

Example 1: $G = A_5$, fundamental group A_5^*

a. The Poincaré dodecahedral space

In this example we construct the barycentric subdivision of the 2-skeleton of the Poincaré homology sphere to demonstrate the usage of G2Comp. We first construct Γ_P. Then we attach a free orbit of 2-cells using the closed edge path (e_1, e_3, e_2^{-1}).

Note that if we modify the attaching map by adding a commutator we can produce different acyclic examples. We were not able to prove that all of these examples have nontrivial fundamental group. Another interesting family of paths to consider are the paths

$$\alpha = (e_1, e_2, e_3^{-1}, g_1 e_1, g_2 e_1^{-1}, \dots, g_i e_1^{i+1}, \dots, g_{2k} e_1^{-1}).$$

In this case we can easily prove that the fundamental group has a presentation with 6 generators and 6 relations. Examples 4, 5 and 6 arose in this way.

b. A different 1-skeleton

We now describe an example which makes clear that, even for $G = A_5$, considered as $\mathrm{PSL}_2(4)$, there are different choices for $\Gamma_{OS}(G)$. From Theorem 2.1.4 and Proposition 2.1.14 we conclude that $X^{(1)}$ is a suitable choice for $\Gamma_{OS}(A_5)$. The quotient graph $X^{(1)}/A_5$ is pictured in Figure 2.4 and as we can see, is different from Γ_P/A_5. The fundamental group of this example is also A_5^*.

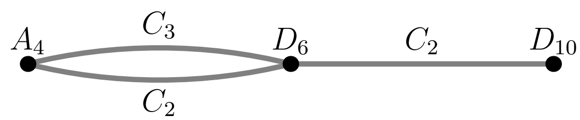

Figure 2.4: A picture of $X^{(1)}/A_5$ for Example 1 b.

Example 2: $G = A_5$, free product of 6 copies of A_5^*

a. An example with a free orbit of 1-cells

The 1-skeleton of this space is obtained from Γ_P by attaching a free orbit of 1-cells. The representative e_4 of this free orbit has both endpoints in v_1. Both orbits of 2-cells are attached along paths of length 5. The computations show this space has fundamental group $\mathop{\ast}\limits_{i=1}^{6} A_5^*$. We can explain why we obtain this fundamental group. Let f_1 and f_2 be the closed edge paths used to attach the free orbits of 2-cells We also denote the corresponding 2-cells in these orbits by f_1 and f_2. There is an acyclic subcomplex X_1 given by v_1, $H_1 \cdot e_4$ and $H_1 \cdot f_2$ and we have $\pi_1(X_1, v_1) = A_5^*$. At this point we already know that the fundamental group is nontrivial, since Conjecture 2.2.1 holds for finite groups. At each vertex $g \cdot v_1$ (there are $[G : H_1] = 5$ such vertices) we have a translate of this subcomplex. The closed edge path $((2,5)(3,4)e_4, e_4)$ gives the trivial element of $\pi_1(X_1, v_1)$ and thus is trivial in $\pi_1(X, v_1)$. Thus attaching the free orbits of 2-cells along f_1 and f_2 gives the same fundamental group as attaching the free orbits of 2-cells along $f_1' = ((3,4,5)e_2, (2,5,4)e_3^{-1}, (2,3,4)e_1^{-1})$ and f_2. And the complex Y given by the orbits of $v_1, v_2, v_3, e_1, e_2, e_3$ and f_1' has fundamental group A_5^* and gives the sixth copy of A_5^*.

This example can be modified easily to obtain a space with fundamental group

$$A_5^* * \mathop{\ast}\limits_{i=1}^{60} \pi$$

for any deficiency 0 perfect group π. This is why we focus mainly in understanding the fundamental group of examples without free orbits of 1-cells.

b. An example without free orbits of 1-cells

In this example we attach a free orbit of 2-cells to Γ_P along a path of length 7. The fundamental group of the resulting space is $\mathop{\ast}\limits_{i=1}^{6} A_5^*$. In this case we have no conceptual proof of this fact.

Example 3: $G = A_5$, free product of 7 copies of A_5^*

This example shows that we can attach a free orbit of 2-cells to Γ_P along a path of length 7 in such a way that we obtain an acyclic space with fundamental group $\mathop{\ast}\limits_{i=1}^{7} A_5^*$. We could not find any other group with a neat description which is the fundamental group of a 2-dimensional acyclic fixed point free A_5-complex with 1-skeleton $\Gamma_{OS}(A_5)$. In particular, we do not know if $\mathop{\ast}\limits_{i=1}^{4} A_5^*$ appears in this way.

Example 4: $G = A_5$**, a** $C'(1/6)$ **group with an epimorphism to** A_5

In this example, by attaching a free orbit of 2-cells to Γ_P along a path of length 9, we obtain a space such that the fundamental group admits a presentation with 6 generators and 6 relators satisfying condition $C'(1/6)$. Thus the fundamental group is nontrivial and torsion free (in particular it is not a free product of copies of A_5^*). We may wonder if this fundamental group is related to A_5^* or A_5 in some way. Indeed, this group (as each of the fundamental groups of the previous examples) has an epimorphism to A_5.

Example 5: $G = A_5$**, a** $C(8)$ **group that is not** $C'(1/6)$

In this example with fundamental group Γ_P, the attaching map has length 11. The fundamental group π has a presentation with 6 generators and 6 relators that satisfies the small cancellation condition $C(8)$ but does not satisfies $C'(1/6)$. Thus π is nontrivial and torsion free. There is an epimorphism $\pi \to A_5$.

Example 6: $G = A_5$**, a group without an epimorphism to** A_5

At some point it seemed plausible to conjecture that each of the examples has an epimorphism to A_5. After some time, we found a space with 1-skeleton Γ_P where the fundamental group π does not have A_5 as a quotient. The attaching map has length 7. The group π has a $C(7)$ presentation. Thus π is torsion free and nontrivial by Proposition A.5.1.

Example 7: $G = A_5$**, TzGoGo produces a long presentation**

This example, constructed from a length 17 attaching map, shows that computing the fundamental group is not always as easy as in the previous examples. Calling `TzGoGo` on the presentation of the fundamental group produces (after some minutes) a balanced presentation with 20 generators and total length 493056. We cannot say much about this group.

Example 8: An acyclic example for $G = \mathrm{PSL}_2(8)$

Since the space in this example is acyclic, by Theorem 2.1.4 and Proposition 2.1.14 the 1-skeleton of this example is a suitable choice for $\Gamma_{OS}(\mathrm{PSL}_2(8))$. The attaching map has length 7. In this case, calling `TzGoGo` produces a long presentation. We could not find a neat description for the fundamental group of any acyclic 2-dimensional G-complex with 1-skeleton $\Gamma_{OS}(\mathrm{PSL}_2(8))$.

Example 9: A suitable choice for $\Gamma_{OS}(\mathrm{PSL}_2(13))$

In this example, we construct a $\mathrm{PSL}_2(13)$-graph with the prescribed orbit types and we use the function `IsSuitableChoiceForGammaOS` which implements Proposition 2.1.14 to verify that it is a suitable choice for $\Gamma_{OS}(\mathrm{PSL}_2(13))$. For this group, we could not produce a single example of a closed edge path which gives an acyclic example. There are really many ways to choose a path of a fixed length ℓ, even for small values of ℓ. It may be the case that the least length of a closed edge path producing an acyclic example is beyond what we tried, but most probably we could not find an example because we did not try for enough time.

Example 10: A suitable choice for $\Gamma_{OS}(\mathrm{Sz}(2^3))$

We construct a $\mathrm{Sz}(2^3)$-graph with the prescribed orbit types and we verify that it is a suitable choice for $\Gamma_{OS}(\mathrm{Sz}(2^3))$. In this case, checking if a single closed edge path produces an acyclic example is already impractical for we would have to compute the Smith normal form of a square matrix of side 29120.

2.4.2 Code for the examples

Code for example 1 a.

We use G2Comp to construct the barycentric subdivision of the 2-skeleton of the Poincaré dodecahedral space and we compute its fundamental group which is isomorphic to A_5^*.

```
gap> LoadPackage("G2Comp");
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:= [ [ e1, 1 ], [ e3, 1 ], [ e2, -1 ] ];;
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
gap> pi:=Pi1(K);;
```

```
gap> IdGroup(pi)=IdGroup(SL(2,5));
true
```

Code for example 1 b.

```
gap> LoadPackage("G2Comp");
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges (K,Intersection(H1,H2),v1,v2,"D");;
gap> e2:=AddOrbitOfEdges (K,Intersection(H1,H3),v1,v3,"E");;
gap> e3:=AddOrbitOfEdges (K,Intersection(H1,H2^(1,2,3)),ActionVertex
((1,2,3)^2,v2),v1,"F");;
gap> f:= [ ActionOrientedEdge((1,3,5), [e2,1]),
           ActionOrientedEdge((1,2,3,5,4), [e2,-1]),
           ActionOrientedEdge((1,2,3,5,4), [e1,1]),
           ActionOrientedEdge((1,2,4,3,5), [e3,1]) ];
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
gap> pi:=Pi1(K);;
gap> IdGroup(pi)=IdGroup(SL(2,5));
true
```

Code for example 2 a.

```
gap> LoadPackage("G2Comp");
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
```

```
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> e4:=AddOrbitOfEdges(K, Group(()), v1, v1, "G");;
gap> f1:= [ ActionOrientedEdge((), [e4,1]),
>           ActionOrientedEdge((3,4,5), [e2,1]),
>           ActionOrientedEdge((2,5,4), [e3,-1]),
>           ActionOrientedEdge((2,3,4), [e1,-1]),
>           ActionOrientedEdge((2,5)(3,4), [e4,1])];;
gap> f2:= [ ActionOrientedEdge((), [e4,1]),
>           ActionOrientedEdge((2,3,5), [e4,-1]),
>           ActionOrientedEdge((2,3,4), [e4,-1]),
>           ActionOrientedEdge((3,5,4), [e4,1]),
>           ActionOrientedEdge((2,5,4), [e4,1])];;
gap> AddOrbitOfTwoCells(K, Group(()),f1,"f1");;
gap> AddOrbitOfTwoCells(K, Group(()),f2,"f2");;
gap> IsAcyclic(K);
true
gap> Pi:=Pi1(K);;
gap> P:=PresentationFpGroup(Pi);;
gap> TzGoGo(P);;
#I  there are 12 generators and 12 relators of total length 106
#I  there are 12 generators and 12 relators of total length 92
gap> TzSubstitute(P);;
#I  substituting new generator _x141 defined by f109*f134
#I  eliminating _x141 = f109*f134
gap> TzGoGo(P);;
#I  there are 12 generators and 12 relators of total length 78
gap> GeneratorsOfPresentation(P);
[ f19, f25, f35, f63, f73, f76, f77, f93, f109, f111, f132, f134 ]
gap> TzPrintRelators(P);
#I  1. f77*f93*f77^-1*f93*f77*f93^-1
#I  2. f109^-1*f134*f109*f134^-1*f109*f134
#I  3. f76^-1*f111^-1*f76^-1*f111*f76*f111
#I  4. f19*f25*f19^-1*f25^-1*f19^-1*f25
#I  5. f35^-1*f63^-1*f35*f63^-1*f35^-1*f63
#I  6. f73*f132*f73^-1*f132*f73*f132^-1
```

```
#I  7. f109^-1*f134^-3*f109^-1*f134^2
#I  8. f63^-1*f35*f63^3*f35*f63^-1
#I  9. f19^3*f25^-1*f19^-2*f25^-1
#I  10. f111*f76^-1*f111^-1*f76*f111^-1*f76^-1*f111
#I  11. f132^-2*f73^-1*f132*f73*f132*f73^-1
#I  12. f93*f77^-1*f93^-2*f77^-1*f93*f77
```

We can see that the presentation is an union of 6 subpresentations each one having 2 generators and 2 relators. Using GAP it is easy to see that each of these presents A_5^*.

Code for example 2 b.

In the following example we specify a fundamental group which makes the presentation of the fundamental group particularly simple.

```
gap> LoadPackage("G2Comp");
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:= [ ActionOrientedEdge((2,4,3), [e2, -1]),
>          ActionOrientedEdge((2,3,5), [e1, 1]),
>          ActionOrientedEdge((2,4,5), [e3, 1]),
>          ActionOrientedEdge((2,3,4), [e3, -1]),
>          ActionOrientedEdge((2,3,4), [e1, -1]),
>          ActionOrientedEdge((2,3)(4,5), [e1, 1]),
>          ActionOrientedEdge((2,4,3), [e3, 1])];
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
# We use the following spanning tree to
# compute the fundamental group of K
gap> T:=[ ActionEdge((1,4,5,2,3), e3),
>         ActionEdge((1,3,5), e1),
```

```
>         ActionEdge((1,4,2), e3),
>         ActionEdge((3,5,4), e2),
>         ActionEdge((1,2,5,4,3), e2),
>         ActionEdge((1,4,2,5,3), e3),
>         ActionEdge((1,3,2,4,5), e2),
>         ActionEdge((1,5,4,2,3), e3),
>         ActionEdge((1,3)(4,5), e1),
>         ActionEdge((1,2,4,3,5), e2),
>         ActionEdge((1,2,3,4,5), e1),
>         ActionEdge((1,4)(2,3), e2),
>         ActionEdge((1,2)(4,5), e2),
>         ActionEdge((1,3,5), e2),
>         ActionEdge((1,5,3,4,2), e3),
>         ActionEdge((1,3)(2,4), e1),
>         ActionEdge((2,3,4), e3),
>         e3,
>         ActionEdge((1,3,5,2,4), e1),
>         ActionEdge((1,2,3), e1) ];;
gap> IsSpanningTreeOfComplex(K,T);
true
gap> pi:=Pi1(K,T);;
gap> P:=PresentationFpGroup(pi);;
gap> TzGoGo(P);;
gap> for i in [1..9] do
>      TzSubstitute(P,1);
>    od;
#I  there are 12 generators and 12 relators of total length 75
gap> TzPrintRelators(P);
#I  1. _x87^-1*_x89^-1*_x87^-1*_x89^2
#I  2. f44*_x84^2*f44*_x84^-1
#I  3. f55*_x82^-1*f55*_x82^2
#I  4. f7^-1*_x86*f7*_x86*f7^-1*_x86^-1
#I  5. f41^-1*f10^-1*f41*f10*f41*f10^-1
#I  6. f58^-1*f61*f58*f61*f58^-1*f61^-1
#I  7. _x87^-1*_x89^-1*_x87^4*_x89^-1
#I  8. f7^-1*_x86*f7^2*_x86*f7^-2
#I  9. f44^-4*_x84^-1*f44*_x84^-1
#I  10. f55^-1*_x82*f55^4*_x82
```

```
#I  11. f41*f10^-1*f41^-1*f10^2*f41^-1*f10^-1
#I  12. f58^-1*f61*f58^-1*f61^-1*f58^2*f61^-1
```

Thus the group is a free product of 6 copies of A_5^*.

Code for example 3

```
gap> LoadPackage("G2Comp");
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:=[ [e1,1],
>         [e3,1],
>         ActionOrientedEdge((1,2)(3,5),[e2,-1]),
>         ActionOrientedEdge((1,2)(4,5),[e1,1]),
>         [e3,1],
>         ActionOrientedEdge((2,5)(3,4),[e3,-1]),
>         ActionOrientedEdge((2,3,5),[e1,-1]) ];;
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
gap> pi:=Pi1(K);;
gap> P:=PresentationFpGroup(pi);;
gap> TzGoGo(P);;
#I  there are 14 generators and 14 relators of total length 201
#I  there are 14 generators and 14 relators of total length 98
gap> TzSubstitute(P,1);;
#I  substituting new generator _x81 defined by f43*f50
#I  eliminating f43 = _x81*f50^-1
#I  there are 14 generators and 14 relators of total length 95
gap> TzSubstitute(P,1);;
```

```
#I  substituting new generator _x82 defined by f50^-1*f66^-1
#I  eliminating f66 = _x82^-1*f50^-1
#I  there are 14 generators and 14 relators of total length 90
gap> TzPrintRelators(P);
#I  1. _x81*_x82^2*_x81*_x82^-1
#I  2. f22^-1*f3^-1*f22^-1*f3*f22*f3
#I  3. f19*f10*f19^-1*f10^-1*f19^-1*f10
#I  4. f5^-1*f21*f5*f21*f5^-1*f21^-1
#I  5. f13^-1*f18^-1*f13*f18*f13*f18^-1
#I  6. f50^-1*f48*f50*f48*f50^-1*f48^-1
#I  7. f20*f7^-1*f20^-1*f7^-1*f20*f7
#I  8. _x81^-2*_x82^-1*_x81*_x82^-1*_x81^-2
#I  9. f50^-1*f48^-1*f50^2*f48^-1*f50^-1*f48
#I  10. f21^-1*f5*f21*f5^-2*f21*f5
#I  11. f10^-1*f19*f10*f19^-2*f10*f19
#I  12. f13^-2*f18*f13*f18^-1*f13*f18
#I  13. f22^-1*f3*f22*f3^-2*f22*f3
#I  14. f20^-1*f7*f20*f7^-2*f20*f7
```

We can see that the presentation is an union of 7 subpresentations each one having 2 generators and 2 relators. Using GAP it is easy to see that each of these presents A_5^*.

Code for example 4

```
gap> LoadPackage("G2Comp");;
gap> LoadPackage("SmallCancellation");;
gap> LoadPackage("G2Comp");;
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:=[ [e3,1],
>         [e2,-1],
```

```
>          ActionOrientedEdge((2,3,4), [e1,1]),
>          ActionOrientedEdge((1,2,3,5,4), [e1,-1]),
>          ActionOrientedEdge((1,3,5,2,4), [e1,1]),
>          ActionOrientedEdge((1,3,4,2,5), [e1,-1]),
>          ActionOrientedEdge((1,3,5), [e1,1]),
>          ActionOrientedEdge((1,3,4,5,2), [e1,-1]),
>          ActionOrientedEdge((1,2)(4,5), [e1,1]) ];;
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
gap> pi:=Pi1(K);;
gap> P:=PresentationFpGroup(pi);;
gap> TzGoGo(P);
#I  there are 6 generators and 6 relators of total length 126
gap> PresentationSatisfiesCPrime(P,1/6);
true
gap> pi:=FpGroupPresentation(P);;
gap> GroupHomomorphismByImages(
>       pi,
>       AlternatingGroup(5),
>       GeneratorsOfGroup(pi),
>       [ (), (), (), (1,2,3,4,5), (1,2,4,5,3), (1,2,5,3,4) ]
>     );
[ f46, f48, f50, f59, f61, f62 ] -> [ (), (), (), (1,2,3,4,5),
(1,2,4,5,3), (1,2,5,3,4) ]
```

Code for example 5

```
gap> LoadPackage("G2Comp");;
gap> LoadPackage("SmallCancellation");
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
```

```
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:=[ [e1,1],
>        [e3,1],
>        [e2,-1],
>        [e1,1],
>        ActionOrientedEdge((1,2)(4,5), [e1,-1]),
>        ActionOrientedEdge((1,3,4,5,2), [e1,1]),
>        ActionOrientedEdge((1,3,5), [e1,-1]),
>        ActionOrientedEdge((1,3,4,2,5), [e1,1]),
>        ActionOrientedEdge((1,3,5,2,4), [e1,-1]),
>        ActionOrientedEdge((1,2,3,5,4), [e1,1]),
>        ActionOrientedEdge((2,3,4), [e1,-1]) ];;
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
gap> pi:=Pi1(K);;
gap> P:=PresentationFpGroup(pi);;
gap> while Size(GeneratorsOfPresentation(P))>6 do
>     TzEliminate(P);;
>    od;;
gap> pi:=FpGroupPresentation(P);;
gap> GroupHomomorphismByImages(
>     pi,
>     AlternatingGroup(5),
>     GeneratorsOfGroup(pi),
>     [ (), (3,4,5), (2,3,4), (1,3,4,2,5), (1,2,5,4,3), (1,2,5) ]
>    );
[ f46, f48, f50, f59, f61, f62 ] -> [ (), (3,4,5), (2,3,4),
(1,3,4,2,5),
  (1,2,5,4,3), (1,2,5) ]
gap> PresentationSatisfiesC(P,8);
true
gap> PresentationSatisfiesCPrime(P,1/6);
false
```

Code for example 6

```
gap> LoadPackage("G2Comp");
```

```
gap> LoadPackage("SmallCancellation");
gap> G:=AlternatingGroup(5);;
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:=[ [e3, 1],
>          [e2, -1],
>          ActionOrientedEdge((2,3,4), [e1, 1]),
>          ActionOrientedEdge((1,2,3,5,4), [e1, -1]),
>          ActionOrientedEdge((1,3,4), [e1, 1]),
>          ActionOrientedEdge((1,3,5,4,2), [e1, -1]),
>          ActionOrientedEdge((1,2)(4,5), [e1, 1]) ];;
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
gap> pi:=Pi1(K);;
gap> P:=PresentationFpGroup(pi);;
gap> for i in [1..36] do
>      TzEliminate(P);;
>    od;;
#I  there are 6 generators and 6 relators of total length 90
gap> PresentationSatisfiesC(P,7);
true
gap> TzGoGo(P);;
#I  there are 5 generators and 5 relators of total length 217
gap> Epimorphism(FpGroupPresentation(P),G);
false
```

Code for example 7

```
gap> LoadPackage("G2Comp");;
gap> G:=AlternatingGroup(5);;
```

```
gap> H1:=Group([ (3,4,5), (2,4)(3,5) ]);;
gap> H2:=Group([ (1,2)(4,5), (3,4,5) ]);;
gap> H3:=Group([ (1,4)(2,3), (1,3)(4,5) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:= [ [e1, 1],
>          ActionOrientedEdge((1,2)(4,5), [e1, -1]),
>          ActionOrientedEdge((1,2)(4,5), [e1, 1]),
>          ActionOrientedEdge((1,2)(4,5), [e1, -1]),
>          ActionOrientedEdge((1,2)(3,4), [e2, 1]),
>          ActionOrientedEdge((1,4,5,2,3), [e3, -1]),
>          ActionOrientedEdge((1,3)(2,5), [e3, 1]),
>          ActionOrientedEdge((1,2,4,3,5), [e2, -1]),
>          ActionOrientedEdge((1,3,2,4,5), [e1, 1]),
>          ActionOrientedEdge((1,4,5,2,3), [e3, 1]),
>          ActionOrientedEdge((1,2)(3,4), [e2, -1]),
>          ActionOrientedEdge((1,3,4,5,2), [e1, 1]),
>          ActionOrientedEdge((1,3,5), [e1, -1]),
>          ActionOrientedEdge((1,2,3,4,5), [e1, 1]),
>          ActionOrientedEdge((2,3,5), [e1, -1]),
>          ActionOrientedEdge((2,3,4), [e1, 1]),
>          ActionOrientedEdge((2,3,4), [e1, -1]) ];;
gap> AddOrbitOfTwoCells(K, Group(()), f, "f");;
gap> IsAcyclic(K);
true
gap> pi:=Pi1(K);;
gap> P:=PresentationFpGroup(pi);;
gap> TzGoGo(P);
#I  there are 20 generators and 20 relators of total length 493056
```

Code for example 8

```
gap> LoadPackage("G2comp");;
gap> G := SmallGroup(IdGroup(PSL(2,8)));;
```

```
gap> H1:=Group([ (1,7)(2,9)(3,5)(4,6), (1,3,4,6,7,9,5) ]);;
gap> H2:=Group([ (1,7)(3,4)(5,8)(6,9), (2,3)(4,6)(5,8)(7,9) ]);;
gap> H3:=Group([ (1,8)(2,6)(3,7)(4,5), (1,3)(2,9)(5,8)(6,7) ]);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"A");;
gap> v2:=AddOrbitOfVertices(K,H2,"B");;
gap> v3:=AddOrbitOfVertices(K,H3,"C");;
gap> e1:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "D");;
gap> e2:=AddOrbitOfEdges(K, Intersection(H1,H3), v1, v3, "E");;
gap> e3:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "F");;
gap> f:=[ [ ActionEdge( (2,3,4,5,7,6,9),     e1), 1],
>         [ ActionEdge( (3,6,7,4,8,5,9),     e1),-1],
>         [ ActionEdge( (1,3,5,2,4,8,9,6,7), e2), 1],
>         [ ActionEdge( (1,5,3,2,7,9,4),     e2),-1],
>         [ ActionEdge( (2,9,6,7,5,4,3),     e1), 1],
>         [ ActionEdge( (2,9,6,7,5,4,3),     e3), 1],
>         [ ActionEdge( (2,9,6,7,5,4,3),     e2),-1] ];;
gap> AddOrbitOfTwoCells(K,Group(Identity(G)),f,"f");;
gap> IsAcyclic(K);
true
```

Code for example 9

The following function implements Proposition 2.1.14.

```
IsSuitableChoiceForGammaOS:=function(K)
  # checks if a G-graph K having the correct orbit types
  # is a suitable choice for Gamma_OS( G )
  local G,H,KH,conj;
  G:=GroupOfComplex(K);
  if not IsConnected(K) then
    Print("# The complex is not connected.\n");
    return false;
  fi;
  conj:= List(ConjugacyClassesSubgroups(G),Representative);
  for H in conj do
    if Order(H)<> 1 then
      KH:=FixedSubcomplex(K,H);
      if (not (IsEmptyComplex(KH) or IsAcyclic(KH) ))
        or
```

```
          (Size(Set(Factors(Order(H))))=1 and not IsAcyclic(KH))
        then
          Print("# Fails for ", H, "\n");
          return false;
        fi;
      fi;
    od;
    return true;
  end;;
```

Using this function we can verify the following is a suitable choice for $\Gamma_{OS}(\mathrm{PSL}_2(13))$.

```
gap> LoadPackage("G2Comp");;
gap> G:=PSL(2,13);;
gap> Borel := Group([
>   (1,13)(2,14)(3,10)(4,7)(5,12)(8,11),
>   (1,8,4,2,10,14)(5,11,6,13,7,12) ]);;
gap> D14    := Group([
>   (2,13)(3,11)(4,6)(5,8)(7,14)(9,10),
>   (1,2)(4,14)(5,13)(6,12)(7,11)(8,10) ]);;
gap> D12    := Group([
>   (3,9)(4,10)(5,11)(6,12)(7,13)(8,14),
>   (1,11)(2,13)(3,4)(5,7)(6,12)(8,9)]);;
gap> A4     := Group([
>   (1,3,6)(2,9,12)(4,14,10)(5,13,11),
>   (1,2)(3,9)(4,8)(5,7)(10,14)(11,13) ]);;
gap> g1     := (2,8,6)(3,4,10)(5,7,14)(9,13,12);;
gap> g2     :=(2,10,11,14,12,7,3,9,13,6,8,5,4);;
gap> C6     := Intersection(Borel,D12^g1);;
gap> C2xC2 := Intersection(D12,A4);;
gap> C3     := Intersection(A4,D12^g2);;
gap> C2     := Intersection(D14,Borel);;
gap> K:=NewEquivariantTwoComplex(G);;
gap> v_B:=AddOrbitOfVertices(K,Borel,"A");;
gap> v_D14:=AddOrbitOfVertices(K,D14,"B");;
gap> v_D12:=AddOrbitOfVertices(K,D12,"C");;
gap> v_A4:=AddOrbitOfVertices(K,A4,"D");;
gap> AddOrbitOfEdges(K,C6, v_B,ActionVertex(g1^-1,v_D12),"E");;
gap> AddOrbitOfEdges(K,C2xC2, v_D12,v_A4,"F");;
```

```
gap> AddOrbitOfEdges(K,C3,v_A4,ActionVertex(g2^-1,v_D12),"G");;
gap> AddOrbitOfEdges(K,C2,v_D14,v_B,"H");;
gap> IsSuitableChoiceForGammaOS(K);
true
```

Code for example 10

We give a suitable choice for $\Gamma_{OS}(\mathrm{Sz}(2^3))$.

```
gap> LoadPackage("G2Comp");;
gap> G:=Group([
>  (2,3,4,6)(5,8,11,16)(7,10,14,19)(9,13,18,24)(12,17,22,30)
>  (15,20,27,34)(21,29,37,46)(23,32,41,25)(26,33,42,52)
>  (28,36,44,54)(31,40,50,59)(35,38,48,57)(39,49,58,51)
>  (43,53,61,65)(45,55,63,64)(47,56,60,62),
>  (1,2)(3,5)(4,7)(6,9)(8,12)(10,15)(11,13)(16,21)(17,23)
>  (18,25)(19,26)(20,28)(22,31)(24,32)(27,35)(29,38)(30,39)
>  (33,42)(34,43)(36,45)(37,47)(40,51)(41,49)(44,54)(46,55)
>  (48,52)(50,60)(53,62)(56,64)(57,65)(58,59)(61,63) ]);;
gap> IsomorphismGroups(G,Sz(8))<>fail;
true
gap> H1:= Group([
>  (1,35,31,37,39,4,50)(2,13,64,20,40,55,29)(3,18,34,30,52,38,32)
>  (5,56,47,61,17,26,16)(6,15,58,10,33,42,59)(7,19,44,11,25,49,57)
>  (8,43,63,46,62,54,27)(12,41,51,23,60,24,36)(21,45,22,28,65,48,53),
>  (1,10,54,33)(2,15,44,42)(3,52,11,55)(4,27,23,36)(5,48,13,46)
>  (6,21,58,24)(7,35,17,45)(8,41,18,20)(9,16,59,32)(12,49,25,28)
>  (19,65,62,30)(22,34,64,37)(26,57,53,39)(29,63,50,40)
>  (31,43,56,47)(38,61,60,51) ]);; # Borel
gap> H2:=Group([
>  (1,48)(2,41)(3,54)(4,5)(6,26)(7,16)(8,9)(10,46)(11,60)(12,24)
>  (13,44)(14,49)(15,23)(17,25)(18,20)(19,33)(21,56)(22,52)(27,29)
>  (28,58)(30,62)(31,39)(32,51)(34,40)(35,45)(36,37)(38,65)(42,59)
>  (43,55)(47,50)(57,61)(63,64),
>  (1,45)(2,37)(3,25)(4,32)(6,33)(7,64)(8,51)(9,53)(10,11)(12,56)
>  (13,22)(14,49)(15,27)(16,35)(17,62)(18,41)(19,63)(20,50)(21,29)
>  (23,61)(24,60)(26,39)(28,40)(30,42)(31,48)(34,55)(36,43)(38,52)
>  (44,54)(46,57)(47,58)(59,65) ]);; # D14
gap> H3:=Group([
```

```
>  (1,49)(2,11)(3,35)(4,6)(5,55)(7,29)(8,41)(9,20)(10,30)(12,62)
>  (13,46)(14,60)(15,28)(16,51)(17,25)(18,21)(19,27)(22,64)(24,26)
>  (31,54)(32,63)(33,59)(34,39)(36,57)(37,50)(38,58)(40,47)(42,52)
>  (43,44)(45,61)(48,56)(53,65),
>  (1,40,11,32)(2,36,60,41)(3,44,5,17)(4,42,55,50)(6,58,65,43)
>  (7,22,18,61)(8,49,48,15)(9,52,35,38)(10,59,39,64)(12,29,33,13)
>  (14,56,19,47)(16,51,23,26)(20,37,53,25)(21,31,30,62)
>  (27,63,28,57)(34,45,46,54) ]);; # C5 \rtimes C4
gap> H4:=Group([
>  (1,10)(2,11)(3,32)(4,36)(5,28)(6,65)(7,54)(8,21)(9,38)(12,34)
>  (13,55)(14,29)(15,61)(16,60)(17,30)(18,19)(20,37)(22,24)(23,53)
>  (25,42)(26,45)(27,31)(33,58)(35,56)(39,50)(40,46)(41,51)(43,62)
>  (44,59)(47,64)(48,52)(49,63),
>  (1,56,28,41)(2,63,49,47)(3,58,20,42)(4,38,53,44)(5,37,6,52)
>  (7,59,46,62)(8,11,57,14)(9,50,65,17)(10,12,18,54)(13,31,39,61)
>  (15,36,19,32)(16,23,24,51)(21,45,29,64)(22,30,33,34)
>  (25,35,43,55)(27,40,60,48) ]);; # C13 \rtimes C4
gap> H12:=Intersection(H1,H2);; # C7
gap> H23:=Intersection(H2,H3);; # C2
gap> H24:=Intersection(H2,H4);; # C2
gap> H14:=Intersection(H1,H4);; # C4
gap> H34:=Intersection(H3,H4);; # C4
gap> K:=NewEquivariantTwoComplex(G);;
gap> v1:=AddOrbitOfVertices(K,H1,"H1");;
gap> v2:=AddOrbitOfVertices(K,H2,"H2");;
gap> v3:=AddOrbitOfVertices(K,H3,"H3");;
gap> v4:=AddOrbitOfVertices(K,H4,"H4");;
gap> e12:=AddOrbitOfEdges(K, Intersection(H1,H2), v1, v2, "H12");;
gap> e23:=AddOrbitOfEdges(K, Intersection(H2,H3), v2, v3, "H23");;
gap> e14:=AddOrbitOfEdges(K, Intersection(H1,H4), v1, v4, "H14");;
gap> e34:=AddOrbitOfEdges(K, Intersection(H3,H4), v3, v4, "H34");;
gap> IsSuitableChoiceForGammaOS(K);
true
```

2.5 Some reformulations involving $\mathrm{Out}(F_m)$

We already know that attaching a free orbit of 2-cells to $\Gamma_{OS}(G)$ along a curve representing a generator of $H_1(\Gamma_{OS}(G)) = \mathbb{Z}[G]$ gives an acyclic space. If we want to study the fundamental group of these examples we need an algebraic object that captures more information. We do not have an action of G on $\pi_1(\Gamma_{OS}(G))$ but we have an action by outer automorphisms: there is an injective map $G \to \mathrm{Out}(\pi_1(\Gamma_{OS}(G)))$. In this section we give some restatements of the conjecture in terms of the action of a finite subgroup of $\mathrm{Out}(F_m)$ on the conjugacy classes of elements of F_m. Since the group $\mathrm{Out}(F_m)$ is deeply studied and much is known about it [Vog02], we expected to use the results of this area to study our problem. The restatements presented here come from the naïve idea that attaching an orbit of 2-cells should be the same as considering the orbit of a conjugacy class of a free group, but there are some technical difficulties. At the end of this section, we explain how these restatements were our initial motivation for Chapter 3.

2.5.1 Finite subgroups of $\mathrm{Out}(F_m)$

The *rank* of a graph Γ is the dimension $\dim H_1(\Gamma)$. Let Γ be a connected G-graph of rank $m > 1$ without edges of valence 1 and let $x_0 \in \Gamma$. Assume the action of G on Γ is faithful. Let $g \in G$, and let γ be a path from x_0 to gx_0. Then $[\omega] \mapsto [\gamma * (g \cdot \omega) * \gamma^{-1}]$ defines an automorphism of $\pi_1(\Gamma, x_0)$ whose class $\alpha(g) \in \mathrm{Out}(\pi_1(\Gamma, x_0))$ does not depend on the particular choice of γ. Then $\alpha \colon G \to \mathrm{Out}(\pi_1(\Gamma, x_0))$ is an injective homomorphism (see [Zim96, Lemma 1]). If we choose an isomorphism h to identify $\pi_1(\Gamma, x_0)$ with F_m, we obtain an injective homomorphism $\alpha \colon G \to \mathrm{Out}(F_m)$. We say that $\alpha \colon G \to \mathrm{Out}(F_m)$ is *realized* by (Γ, h). The conjugacy class of the subgroup $\alpha(G) \leq \mathrm{Out}(F_m)$ does not depend on h.

Theorem 2.5.1 (Realization Theorem [Zim96, Theorem 1])**.** *Any finite subgroup $G \leq \mathrm{Out}(F_m)$ is realized by a G-graph Γ.*

By doing G-forest collapses, we may assume the G-graph Γ in the previous theorem is reduced.

Abelianization induces a map $\mathrm{Aut}(F_m) \to \mathrm{GL}(m, \mathbb{Z})$ which passes to the quotient by $\mathrm{Inn}(F_m)$. Thus we obtain a map $\mathrm{Out}(F_m) \to \mathrm{GL}(m, \mathbb{Z})$.

Theorem 2.5.2 ([Zim96, Corollary 1])**.** *The canonical projection $\mathrm{Out}(F_m) \to \mathrm{GL}(m, \mathbb{Z})$ is injective on finite subgroups.*

If $G \leq \mathrm{Out}(F_m)$ is realized by Γ, the subgroup $G \leq \mathrm{GL}(m, \mathbb{Z})$ obtained in this way is the same as the $\mathbb{Z}[G]$-module $H_1(\Gamma)$. Conjugate finite subgroups of $\mathrm{Out}(F_m)$ give conjugate subgroups of $\mathrm{GL}(m, \mathbb{Z})$. For example, we have an injection $A_5 \hookrightarrow \mathrm{Out}(F_{60})$ given by the graph $\Gamma_{OS}(A_5)$. One may wonder if this subgroup is conjugate to the subgroup $A_5 \hookrightarrow \mathrm{Out}(F_{60})$ realized by the A_5-graph with one 0-cell and one free orbit of 1-cells (note that these subgroups give conjugate subgroups of $\mathrm{GL}(60, \mathbb{Z})$). As we will see, these subgroups are not conjugate in $\mathrm{Out}(F_{60})$.

The map $p\colon \mathrm{Aut}(F_m) \to \mathrm{Out}(F_m)$ is injective on finite subgroups, since $\ker(p) = \mathrm{Inn}(F_m) = F_m$ is torsion free. We say that a finite subgroup $G \leq \mathrm{Out}(F_m)$ *lifts* to a subgroup of $\mathrm{Aut}(F_m)$ if there is a finite subgroup $\widetilde{G} \leq \mathrm{Aut}(F_m)$ such that $G = p(\widetilde{G})$. In this case $p(\widetilde{G})$ is isomorphic to G. It turns out that not every finite subgroup of $\mathrm{Out}(F_m)$ lifts to $\mathrm{Aut}(F_m)$. Consider the short exact sequence

$$1 \to F_m \to \mathrm{Aut}(F_m) \to \mathrm{Out}(F_m) \to 1.$$

Taking the preimage of $G \leq \mathrm{Out}(F_m)$ we get a short exact sequence

$$1 \to F_m \to \widetilde{G} \to G \to 1.$$

Any short exact sequence $1 \to N \to G \to H \to 1$ gives a map $H \to \mathrm{Out}(N)$. This allows to recover the subgroup $G \leq \mathrm{Out}(F_m)$ from the short exact sequence above. We will return to the study of this exact sequence in Section 2.6 using results closely related to those stated here (both approaches are based on Bass-Serre theory).

Proposition 2.5.3 ([Zim96, Theorem 1])**.** *Let $G \leq \mathrm{Out}(F_m)$ be finite. The following are equivalent:*

(i) G lifts to a subgroup of $\mathrm{Aut}(F_m)$.

(ii) The short exact sequence $1 \to F_m \to \widetilde{G} \to G \to 1$ splits.

(iii) For every G-graph Γ realizing G the action of G on Γ has a fixed point.

(iv) There is a G-graph Γ realizing G such that the action of G on Γ has a fixed point.

Now it is clear that the two subgroups of $\mathrm{Out}(F_{60})$ considered above, each one isomorphic to A_5, are not conjugate. In general, it is possibly to decide algorithmically if two finite subgroups of $\mathrm{Out}(F_m)$ are conjugate: there is a correspondence between conjugacy classes of finite subgroups of $\mathrm{Out}(F_m)$ and reduced G-graphs up to *equivariant Whitehead moves* (see [KLV01]). Another reference for the results in this section is [Krs89].

2.5.2 Free orbits of conjugacy classes

Note that if $G \leq \mathrm{Out}(F_m)$ then G acts on the conjugacy classes of F_m. The normal subgroup $\langle\langle S \rangle\rangle$ generated by a set of conjugacy classes S is well defined.

Conjecture 2.5.4. *Let $k \geq 1$ and G be a finite subgroup of $\mathrm{Out}(F_{k|G|})$. If $C_1, \ldots, C_k$ are conjugacy classes of $F_{k|G|}$ such that $\langle\langle G \cdot C_1, \ldots, G \cdot C_k \rangle\rangle = F_{k|G|}$, then G lifts to a subgroup of $\mathrm{Aut}(F_{k|G|})$.*

Conjecture 2.5.5. *Let $k \geq 1$ and G be a finite subgroup of $\mathrm{Out}(F_{k|G|})$. Suppose $C_1, \ldots, C_k$ are conjugacy classes of $F_{k|G|}$ such that $\langle\langle G \cdot C_1, \ldots, G \cdot C_k \rangle\rangle = F_{k|G|}$. Then G lifts to a subgroup of $\mathrm{Aut}(F_{k|G|})$ and there are elements $c_1, \ldots, c_k \in F_{k|G|}$ such that $\{g \cdot c_i\}_{g \in G, 1 \leq i \leq k}$ is a basis of $F_{k|G|}$.*

Obviously Conjecture 2.5.5 implies Conjecture 2.5.4.

Proposition 2.5.6. *The Casacuberta–Dicks conjecture implies Conjecture 2.5.5. Moreover, if we assume Conjecture 2.2.1, Conjecture 2.5.4 implies the Casacuberta-Dicks conjecture.*

Proof. Assume the Casacuberta–Dicks conjecture holds and let Γ be a reduced G-graph realizing the subgroup $G \leq \mathrm{Out}(F_m)$. Let X be obtained from Γ attaching free orbits of 2-cells along curves representing the conjugacy classes $C_1, \ldots, C_k$. Then X is contractible, thus $X^G \neq \varnothing$. Therefore, G must fix a point of Γ and by Lemma 2.2.9, the graph Γ must have a unique vertex. Suppose Γ has a 1-cell with nontrivial stabilizer H. By Theorem 2.1.4, X^H is acyclic. But $H_1(X^H) \neq 0$. Thus every orbit of 1-cells is free and we are done.

For the other assertion, assuming Conjecture 2.2.1, if the Casacuberta-Dicks conjecture is false, by Corollary 2.2.12 there is a counterexample X where every orbit of 2-cells is free. Considering the subgroup $G \leq \mathrm{Out}(\pi_1(\Gamma, x_0))$ realized by the action of G on $\Gamma = X^{(1)}$ we obtain a counterexample to Conjecture 2.5.4. □

Conjecture 2.5.5 can be “factored” as the product of the following two conjectures.

Conjecture 2.5.7. *Let $k \geq 1$ and G be a finite subgroup of $\mathrm{Out}(F_{k|G|})$. Suppose $C_1, \ldots, C_k$ are conjugacy classes of $F_{k|G|}$ such that $\langle\langle G \cdot C_1, \ldots, G \cdot C_k \rangle\rangle = F_{k|G|}$. Then there is a basis $\{c_{g,i}\}_{g \in G, 1 \leq i \leq k}$ of $F_{k|G|}$ such that $g \cdot [\![c_{h,i}]\!] = [\![c_{gh,i}]\!]$.*

Conjecture 2.5.8. *Let $k \geq 1$ and G be a finite subgroup of $\mathrm{Out}(F_{k|G|})$. Suppose there is a basis $\{c_{g,i}\}_{g \in G, 1 \leq i \leq k}$ of $F_{k|G|}$ such that $g \cdot [\![c_{h,i}]\!] = [\![c_{gh,i}]\!]$. Then G lifts to a subgroup of $\mathrm{Aut}(F_{k|G|})$ and there are elements $c_1, \ldots, c_k \in F_{k|G|}$ such that $\{g \cdot c_i\}_{g \in G, 1 \leq i \leq k}$ is a basis of $F_{k|G|}$.*

2.5.3 Permutational subgroups of $\mathrm{Out}(F_m)$

In this section, instead of considering a free action, we consider a fixed point free action on a set of conjugacy classes. Employing some techniques from [OS02], we obtain another restatement of the Casacuberta–Dicks conjecture.

Definition 2.5.9 ([Kal92])**.** A subgroup $G \leq \mathrm{Aut}(F_m)$ is *permutational* if G acts on a basis of F_m.

A permutational subgroup of $\mathrm{Aut}(F_m)$ is realized by a G-graph with one 0-cell and m loops.

Definition 2.5.10. A set $\{C_1, \ldots, C_m\}$ of conjugacy classes of F_m is a *normal-basis* if $F_m = \langle\langle C_1, \ldots, C_m \rangle\rangle$. A set $\{C_1, \ldots, C_m\}$ of conjugacy classes of F_m is an H-*basis* if $F_m / \langle\langle C_1, \ldots, C_m \rangle\rangle$ is perfect.

Obviously every basis is a normal-basis and every normal basis is an H-basis.

Example 2.5.11. The action $G \curvearrowright \Gamma_{OS}(G)$ gives a subgroup $G \leq \mathrm{Out}(F_m)$ where $m = |G|$. We know it is possible to obtain an acyclic space by attaching a free orbit of 2-cells to $\Gamma_{OS}(G)$. Let C be a conjugacy class arising from such an attaching map. Then $G \cdot C$ is an H-basis.

Definition 2.5.12. A subgroup $G \leq \mathrm{Out}(F_m)$ is *permutational* if G acts on a a normal-basis of F_m.

A permutational subgroup $G \leq \mathrm{Aut}(F_m)$ gives a permutational subgroup $G \leq \mathrm{Out}(F_m)$. As we can see in Figure 2.5, not every permutational subgroup of $\mathrm{Out}(F_m)$ lifts to a permutational subgroup of $\mathrm{Aut}(F_m)$.

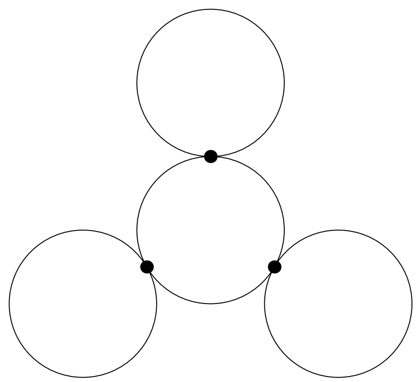

Figure 2.5: The generator of $\mathbb{Z}_3$ acts as rotation by $\frac{2\pi}{3}$ on this graph. The subgroup $\mathbb{Z}_3 \leq \mathrm{Out}(F_4)$ realized by this action is permutational but does not lift to $\mathrm{Aut}(F_4)$.

We will prove the Casacuberta–Dicks conjecture is equivalent to the following

Conjecture 2.5.13. *Any permutational subgroup of* $\mathrm{Out}(F_m)$ *that acts fixed point freely on a normal-basis of F_m lifts to a permutational subgroup of* $\mathrm{Aut}(F_m)$.

The following technical proposition explains how a conjugacy class C in the fundamental group of a G-graph Γ can be used to attach an orbit of 2-cells of type G/G_C to Γ. We assume familiarity with some notions on edge paths which are explained in Section A.2.

Proposition 2.5.14. *Let Γ be a G-graph. Let $C \neq \{1\}$ be a conjugacy class of $\pi_1(\Gamma, x_0)$. Then*

(a) Up to cyclic permutation, there is a unique cyclically reduced closed edge path ω in Γ representing the conjugacy class C.

(b) The stabilizer of ω is contained in the stabilizer of C. Moreover $G_\omega \triangleleft G_C$ and G_C/G_ω is cyclic.

(c) Let T be a set of representatives for the left cosets of G_C in G. Consider the CW-complex X obtained from Γ by attaching 2-cells along each of the closed edge paths in $T \cdot \omega$. If C is not a proper power, we can extend the action of G on Γ to an action on X in such a way that the action on $X - \Gamma = G/G_C \times (D^2)^\circ$ is by isometries.

Moreover, the centers of the disks form an orbit of type G/G_C, the action preserves the orientation of each open disk (it maps a disk to itself only through a rotation) and the stabilizer of a point inside the 2-cell attached along ω other than the center of the disk is G_ω.

(d) Suppose the conjugacy classes $C_1, \dots, C_k$ are not proper powers and consider the complex X obtained from Γ attaching orbits as in (c). Then the barycentric subdivision X' is a G-complex and for any $H \leq G$ we have

$$\sum_{n\geq 0}(-1)^n c_n(H,X) = \sum_{n\geq 0}(-1)^n c_n(H,X')$$

where $c_n(H,X)$ denotes the number of orbits of n-cells of X of type G/H.

Proof. (a) is explained in Section A.2. The first part of (b) is clear, for the second part consider the action of G_C on the set of reduced closed edge paths representing C. For part (c), we view $D^2 \subset \mathbb{C}$. We consider the loop $f_\omega \colon S^1 \to \Gamma$ given by traveling along the closed edge path ω at constant speed. We have a map $f : G/G_C \times S^1 \to \Gamma$ defined by $f(tG_C, s) = t f_\omega(s)$. We use this map to attach the 2-cells. Since C is not a proper power, for every $h \in G_C$ there is a unique $z_h \in S^1 \subset \mathbb{C}$ such that $h f_\omega(s) = f_\omega(z_h s)$ for each $s \in S^1$ and the map $G_C \to S^1$ given by $h \mapsto z_h$ is a homomorphism with kernel G_ω. Now we define an action $G \curvearrowright G/G_C \times D^2$. If $g \in G$, $t \in T$ and $s \in D^2$, we set $g \cdot (tG_C, s) = (t'G_C, z_h s)$ where $t' \in T$, $h \in G_C$ are given by $gt = t'h$. We define $g \cdot f(tG_C, s) = f(g \cdot (tG_C, s))$. This determines a well defined action $G \curvearrowright X$. It is clear from the construction that the second part of (c) holds. Finally, (d) follows from (c) and the fact that an open 2-cell with stabilizer G_C is subdivided into a 0-cell with stabilizer G_C and an equal number of 1-cells and 2-cells with stabilizer G_ω. $\square$

By an *almost separating family* we mean a family $\mathcal{F}$ that only satisfies conditions (b) and (c) of Definition 2.1.5.

Proposition 2.5.15. *Let $G \leq \mathrm{Out}(F_m)$ be permutational. Then any two* H*-bases of F_m stable by G are isomorphic as G-sets.*

Proof. Let Γ be a G-graph inducing the subgroup $G \leq \mathrm{Out}(F_m)$. Let $\mathcal{F}$ be the smallest almost separating family containing the family $\{H \in \mathcal{S}(G) : \Gamma^H \neq \varnothing\}$. Let B be an H-basis of F_m. Note that an element of an H-basis cannot be a proper power. Using part (c) of Proposition 2.5.14 we attach G-orbits of 2-cells to Γ along representatives of B/G to obtain an acyclic complex X with an action of G. We will prove the number of orbits of 2-cells of type G/H does not depend on the particular choice of the H-basis B. It is enough to show that $\sum_{n\geq 0}(-1)^n c_n(H,X)$ does not depend on B. By part (d) we have

$$\sum_{n\geq 0}(-1)^n c_n(H,X) = \sum_{n\geq 0}(-1)^n c_n(H,X')$$

and we know that X' is a G-complex. Moreover from part (b) of Proposition 2.5.14 and Segev's result Theorem 2.1.4 it follows that X' is an H-universal $(G,\mathcal{F})$-complex. Now by Lemma 2.1.9 we have

$$\sum_{n\geq 0}(-1)^n c_n(H,X) = i_{\mathcal{F}}(H)$$

which does not depend on B.

□

Proposition 2.5.16. *The Casacuberta–Dicks conjecture is equivalent to Conjecture 2.5.13.*

Proof. First suppose the Casacuberta-Dicks conjecture holds. Let $G \leq \mathrm{Out}(F_m)$ be a permutational subgroup and consider a G-graph Γ realizing G. Let B be a normal-basis where G acts without fixed points. As before, no element of B is a proper power and using part (c) of Proposition 2.5.14, we can attach orbits of 2-cells along representatives of B/G to obtain a 2-complex X with an action of G. Since B is a normal basis, X is contractible and thus by the Casacuberta–Dicks conjecture, G must fix a point of X. Since the action on B is fixed point free, we must have $\Gamma^G \neq \varnothing$ and thus G lifts to $\mathrm{Aut}(F_m)$. By Corollary 2.2.10 there is a G-invariant spanning tree. The edges in the complement of this tree show $G \leq \mathrm{Aut}(F_m)$ give a basis permuted by G (here we use the admissibility of the action $G \curvearrowright \Gamma$).

Now assume Conjecture 2.5.13 holds. Let X be a 2-dimensional contractible G-complex. As always, we may assume the action is admissible. Let $\Gamma = X^{(1)}$ and let m be the rank of $\pi_1(\Gamma)$. The attaching maps of the 2-cells determine m different conjugacy classes of $\pi_1(\Gamma)$ which form a normal-basis B. If the orientations of these attaching maps are chosen carefully, G acts on B. If $(X-\Gamma)^G = \varnothing$ then the action of G on B is fixed point free. Thus by Conjecture 2.5.13, G lifts to $\mathrm{Aut}(\pi_1(\Gamma))$ giving a fixed point for the action $G \curvearrowright \Gamma$. □

Now we explain the motivation for Chapter 3. A *partial basis* of F_m is a subset of a basis of F_m. There is a simplicial complex $\mathrm{PB}(F_m)$ whose simplices are the nonempty partial bases of F_m. This complex has a natural action of $\mathrm{Aut}(F_m)$ and thus any subgroup $G \leq \mathrm{Aut}(F_m)$ acts on this complex. A related simplicial complex is $\mathcal{B}(F_m)$, whose simplices are the sets of conjugacy classes of elements in a partial basis and which has an action of $\mathrm{Out}(F_m)$. If G is one of the groups in Theorem 2.2.11, we have $G \leq \mathrm{Out}(\pi_1(\Gamma_{OS}(G)))$ and thus we would like to know if there is a maximal simplex σ of $\mathcal{B}(\pi_1(\Gamma_{OS}(G)))$ fixed by G. If there is such σ, by Proposition 2.5.15 the action of G on σ would be free. Thus we would obtain a counterexample to the Casacuberta–Dicks conjecture. The homotopy type of the complexes $\mathrm{PB}(F_m)$ and $\mathcal{B}(F_m)$ is the theme studied in Chapter 3.

2.6 Brown's short exact sequence

Using Bass-Serre theory, K.S. Brown gave a method to produce a presentation for a group G acting on a simply connected complex X [Bro84, Theorem 1]. Brown also describes a presentation for an extension $\widetilde{G}_X$ of G by $\pi_1(X)$, when X is not simply connected [Bro84, Theorem 2]. The group $\widetilde{G}_X$ has a description as a quotient of the fundamental group of a graph of groups. A similar result in the simply connected case was given by Corson [Cor92, Theorem 5.1] in terms of *complexes of groups* (higher dimensional analogues of graphs of groups).

Using Brown's result we translate the A_5 case of the Casacuberta–Dicks conjecture into a nice looking problem in combinatorial group theory. If we are looking for examples without free orbits of 1-cells the equivalent problem is particularly simple. This translation can be done in general, but to obtain similar results for the rest of the groups G that appear in Theorem 2.2.11 we need a choice of $\Gamma_{OS}(G)$ and presentations for the stabilizers of its vertices.

In Brown's original formulation, the result deals with actions that need not to be admissible (Brown uses the term $G-CW$-complex in a different way than us). Since the actions we are interested in are admissible, we state Brown's result only in that case.

Let X be a connected G-complex. By admissibility of the action, the group G acts on the set of oriented edges (see Section A.2 for our conventions on oriented edges, closed edge paths and the definition of the edge path group). The group $\widetilde{G}_X$ depends on a number of choices that we now specify. For each 1-cell of X we choose a preferred orientation in such a way that these orientations are preserved by G. This determines a set P of oriented edges. We choose a *tree of representatives* for X/G. That is, a tree $T \subset X$ such that the vertex set V of T is a set of representatives of $X^{(0)}/G$. Such tree always exists and the 1-cells of T are inequivalent modulo G. We give an orientation to the 1-cells of T so that they are elements of P. We also choose a set of representatives E of P/G in such a way that $s(e) \in V$ for every $e \in E$ and such that each oriented edge of T is in E. If e is an oriented edge, the unique element of V that is equivalent to $t(e)$ modulo G will be denoted by $w(e)$. For every $e \in E$ we choose an element $g_e \in G$ such that $t(e) = g_e \cdot w(e)$. If $e \in T$, we specifically choose $g_e = 1$. For each orbit of 2-cells we choose a closed edge path τ based at a vertex of T and representing the attaching map for this orbit of 2-cells. Let F be the set given by these closed edge paths.

The group $\widetilde{G}_X$ is defined as a quotient of

$$\mathop{\ast}_{v\in V} G_v \ast \mathop{\ast}_{e\in E} \mathbb{Z}$$

by certain relations. In order to define these relations we introduce some notation. If $v \in V$ and $g \in G_v$ we denote the copy of g in the free factor G_v by g_v. The generator of the copy of $\mathbb{Z}$ that corresponds to e is denoted by x_e. The relations are the following:

(i) $x_e = 1$ if $e \in T$.

(ii) $x_e^{-1} g_{s(e)} x_e = (g_e^{-1} g g_e)_{w(e)}$ for every $e \in E$ and $g \in G_e$.

(iii) $r_\tau = 1$ for every $\tau \in F$.

We state Brown's theorem before giving the definition of the element r_ω associated to a closed edge path ω.

Theorem 2.6.1 (Brown, [Bro84, Theorems 1 and 2])**.** *The group*

$$\widetilde{G}_X = \frac{\mathop{\ast}_{v\in V} G_v \ast \mathop{\ast}_{e\in E} \mathbb{Z}}{\langle\langle R \rangle\rangle}$$

where R consists of relations (i)-(iii) is an extension

$$1 \to \pi_1(X,x_0) \xrightarrow{i} \widetilde{G}_X \xrightarrow{\overline{\phi}} G \to 1.$$

The map $\overline{\phi}$ is defined passing to the quotient the coproduct ϕ of the inclusions $G_v \to G$ and the mappings $\mathbb{Z} \to G$ given by $x_e \mapsto g_e$. The map i sends a closed edge path ω based at $x_0 \in V$ to r_ω.

The group $\widetilde{G}_X$ can be described as the quotient of the fundamental group of certain graph of groups by relations of type (iii). Now we explain how to obtain the elements r_τ. If α is an oriented edge, we define

$$\varepsilon(\alpha) = \begin{cases} \;\;1 & \alpha \in P \\ -1 & \text{if } \alpha \notin P \end{cases}$$

and we can always take $e \in E$ and $g \in G$ such that $\alpha = ge^{\varepsilon(\alpha)}$. Note that e is unique but g is not. Moreover, if α starts at $v \in V$, we can write

$$\alpha = \begin{cases} he & \text{with } h \in G_{s(e)}, \text{ if } \alpha \in P \\ hg_e^{-1}e^{-1} & \text{with } h \in G_{w(e)}, \text{ if } \alpha \notin P \end{cases}$$

Again, h is not unique.

Now if $\tau = (\alpha_1, \dots, \alpha_n)$ is a closed edge path starting at a vertex $v_0 \in V$ we define an element $r_\tau \in \mathop{\ast}\limits_{v\in V} G_v * \mathop{\ast}\limits_{e\in E} \mathbb{Z}$. Recursively, we define some sequences. Since the oriented edge α_1 starts at $v_0 \in V$, we can obtain an oriented edge e_1 and an element $h_1 \in G_{v_0}$ as above. We set $\varepsilon_1 = \varepsilon(\alpha_1)$ and $g_1 = h_1 g_{e_1}^{\varepsilon_1}$. Set $v_1 = w(e_1)$ if $\alpha_1 \in P$ and otherwise $v_1 = s(e_1)$. Now suppose we have defined $e_1, \dots, e_k$, $h_1, \dots, h_k$, $\varepsilon_1, \dots, \varepsilon_k$, $g_1, \dots, g_k$ and $v_1, \dots, v_k$. The oriented edge $(g_1 g_2 \cdots g_k)^{-1}\alpha_{k+1}$ starts at $v_k \in V$, so we can obtain an oriented edge e_{k+1} and an element $h_{k+1} \in G_{v_k}$ as before. We set $\varepsilon_{k+1} = \varepsilon(\alpha_{k+1})$ and $g_{k+1} = h_{k+1} g_{e_{k+1}}^{\varepsilon_{k+1}}$. Set $v_{k+1} = w(e_{k+1})$ if $\alpha_{k+1} \in P$ and otherwise $v_{k+1} = s(e_{k+1})$. When we conclude, we have an element $g_1 g_2 \cdots g_n \in G_{v_0}$. Finally the relation associated to τ is given by

$$r_\tau = (h_1)_{v_0} x_{e_1}^{\varepsilon_1} \, (h_2)_{v_1} x_{e_2}^{\varepsilon_2} \, \cdots \, (h_n)_{v_{n-1}} x_{e_n}^{\varepsilon_n} \, (g_1 g_2 \cdots g_n)_{v_0}^{-1}.$$

The description of the inclusion i along with the exactness at the middle in Brown's short exact sequence say that for any word in $w \in \ker(\phi)$ we can find a closed edge path ω for a 2-cell such that $w = r_\omega$. We give a hands-on proof of this fact.

Proposition 2.6.2. *Let Γ be a G-graph and let $w \in \widetilde{G}_\Gamma$. If $\overline{\phi}(w) = 1$, then there is a closed edge path ω such that $w = r_\omega$.*

Proof. Consider a word in $\mathop{\ast}_{v\in V} G_v * \mathop{\ast}_{e\in E} \mathbb{Z}$ representing w. If we insert letters x_e with $e\in T$ and 1_{G_v} with $v\in V$ this word still represents w. Using these two moves we can assume the word has the form

$$(h_1)_{v_0} x_{e_1}^{\varepsilon_1} (h_2)_{v_1} x_{e_2}^{\varepsilon_2} \cdots (h_n)_{v_{n-1}} x_{e_n}^{\varepsilon_n}$$

and that we have $v_i = t(e_i^{\varepsilon_i}) = s(e_{i+1}^{\varepsilon_{i+1}}) \mod G$ for $i = 1,\dots,n-1$ and $v_0 = t(e_n^{\varepsilon_n}) = s(e_1^{\varepsilon_1}) \mod G$. Let $g_i = h_i g_{e_i}^{\varepsilon_i}$. Then setting

$$\alpha_i = \begin{cases} g_1\cdots g_{i-1} h_i e_i & \text{if } \varepsilon_i = 1 \\ g_1\cdots g_{i-1} h_i g_{e_i}^{-1} e_i^{-1} & \text{if } \varepsilon_i = -1 \end{cases}$$

we have that $\omega = (\alpha_1,\dots,\alpha_n)$ is a closed edge path. Moreover, we have $r_\omega = w$. □

A closed edge path ω in X determines a conjugacy class $[\![\omega]\!]$ of $\pi_1(X)$. The following describes the conjugation action of $\widetilde{G}_X$ on $\pi_1(X)$.

Proposition 2.6.3 ([Bro84, Proposition 1])**.** *Let ω be a closed edge path in X and $g\in G$. Then the conjugacy classes $[\![\omega]\!]$ and $[\![g\omega]\!]$ of $\pi_1(X)$ are contained in the same $\widetilde{G}_X$-conjugacy class. Moreover for any element $\widetilde{g}\in\overline{\phi}^{-1}(g)$ we have $[\![\omega]\!]^{\widetilde{g}} = [\![g\omega]\!]$.*

The following proposition summarizes many ideas of this section.

Proposition 2.6.4. *Let Γ be a G-graph and let $w_1,\dots,w_k \in \ker(\overline{\phi}\colon \widetilde{G}_\Gamma \to G)$. Then there is a 2-complex X obtained by attaching orbits of 2-cells to Γ along closed edge paths $\omega_1,\dots,\omega_k$ such that $r_{\omega_i} = w_i$ and we have the following diagram with exact rows and columns.*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \langle\langle \omega_i \rangle\rangle^{\widetilde{G}_\Gamma} \arrow[r, equal] \arrow[d] & \langle\langle w_i \rangle\rangle^{\widetilde{G}_\Gamma} \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\Gamma) \arrow[r, "i"] \arrow[d, "i_*"] & \widetilde{G}_\Gamma \arrow[r, "\overline{\phi}"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X) \arrow[r, "i"] \arrow[d] & \widetilde{G}_X \arrow[r, "\overline{\phi}"] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Remark 2.6.5. If Γ is a G-graph, the extension $1\to\pi_1(\Gamma)\to\widetilde{G}_\Gamma\to G\to 1$ is the same as the extension in Proposition 2.5.3 (this is the fundamental theorem of Bass-Serre theory, see [Zim96]).

Remark 2.6.6. If X is a connected G-complex, the group $\widetilde{G}_X$ is isomorphic to the group formed by the lifts $\widetilde{g}$ of elements $g\colon X \to X$ to the universal cover $\widetilde{X}$ of X (see [Bro84]). Suppose Y is another G-complex and $h\colon X \to Y$ is equivariant and a homotopy equivalence. Let $\widetilde{h}\colon \widetilde{X} \to \widetilde{Y}$ be a lift of h to the universal covers. Then if $g \in G$, for each lift $\widetilde{g}_X : \widetilde{X} \to \widetilde{X}$ of $g\colon X \to X$ there is a unique lift $\widetilde{g}_Y : \widetilde{Y} \to \widetilde{Y}$ of $g\colon Y \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{h}"] \arrow[d, "\widetilde{g}_X"'] & \widetilde{Y} \arrow[d, "\widetilde{g}_Y"] \\ \widetilde{X} \arrow[r, "\widetilde{h}"'] & \widetilde{Y} \end{tikzcd}$$

Then it is easy to check that there is an isomorphism $\widetilde{G}_X \to \widetilde{G}_Y$ given by $\widetilde{g}_X \mapsto \widetilde{g}_Y$. In particular, the isomorphism type of $\widetilde{G}_{\Gamma_{OS}(G)}$ does not depend on any choice.

2.6.1 Applying Brown's result

Consider the following subgroups of A_5.

$$\begin{aligned} H_1 &= A_4 = \langle (2,5)(3,4), (3,5,4) \rangle \\ H_2 &= D_6 = \langle (3,5,4), (1,2)(3,5) \rangle \\ H_3 &= D_{10} = \langle (1,2)(3,5), (2,5)(3,4) \rangle. \end{aligned}$$

In this section we take $\Gamma_{OS}(A_5)$ to be the 1-skeleton of the coset complex of (H_1, H_2, H_3), which is the same as the graph depicted in Figure 2.3. Suppose that we have an acyclic 2-complex X obtained from $\Gamma_{OS}(A_5)$ by attaching a free A_5-orbit of 2-cells. We want to apply Brown's method to obtain a presentation for the extension $\widetilde{G}_X$. We consider the vertices $v_1 = H_1$, $v_2 = H_2$ and $v_3 = H_3$ of $\Gamma_{OS}(A_5)$. Then the stabilizers of the oriented edges $e_{12} = (v_1 \to v_2)$, $e_{23} = (v_2 \to v_3)$, $e_{13} = (v_1 \to v_3)$ are

$$\begin{aligned} H_{12} &= H_1 \cap H_2 = \langle (3,5,4) \rangle \\ H_{23} &= H_2 \cap H_3 = \langle (1,2)(3,5) \rangle \\ H_{13} &= H_1 \cap H_3 = \langle (2,5)(3,4) \rangle. \end{aligned}$$

We take $T = \{e_{12}, e_{23}\}$. Thus $V = \{v_1, v_2, v_3\}$. We take $E = \{e_{12}, e_{23}, e_{13}\}$. Note that we have $w(e) = t(e)$ for every $e \in E$. We can take $g_e = 1$ for every $e \in E$.

Then Brown's result gives

$$\widetilde{G}_X = \frac{(H_1 *_{H_{12}} H_2 *_{H_{23}} H_3) *_{H_{13}}}{\langle\langle w \rangle\rangle}$$

We explain this. First we amalgamate the groups H_1, H_2, H_3 identifying the copy of H_{12} in H_1 with the copy of H_{12} in H_2 and the copy of H_{23} in H_2 with the copy of H_{23} in H_3. This comes from the relations of type (ii) for $e \in T$. Then we form an HNN extension with stable letter $x = x_{e_{13}}$ that corresponds to the relation of type (ii) coming from e_{13}. The associated subgroups of this HNN extension are the copies of H_{13} in H_1 and H_3. The quotient by the word w comes from the only relation of type (iii).

Now we obtain an explicit presentation for $\widetilde{G}_X$. We have $A_4 = \langle a,b \mid a^2, b^3, (ab)^3\rangle$ and the isomorphism maps $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$. We have $D_6 = \langle b,c \mid b^3, c^2, (bc)^2\rangle$ and the isomorphism maps $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$. Finally $D_{10} = \langle c,d \mid c^2, d^2, (cd)^5\rangle$ and the isomorphism maps $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$. Thus we have a presentation

$$\widetilde{G}_X = \langle a,b,c,d,x \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x^{-1}ax = d, w\rangle$$

where the word w depends on the attaching map. The mapping $\overline{\phi}\colon \widetilde{G}_X \to A_5$ is given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x \mapsto 1$.

The following conjecture is equivalent to the A_5 without free orbits of 1-cells case of the Casacuberta-Dicks conjecture.

Conjecture 2.6.7. *There is no word $w \in \ker(\phi)$ such that*

$$\langle a,b,c,d,x \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x^{-1}ax = d, w\rangle$$

is a presentation of A_5.

If a word w comes from an attaching map of an acyclic 2-complex X with 1-skeleton $\Gamma_{OS}(A_5)$ the total exponent of x in w must be ± 1. We can see this algebraically by abelianizing or geometrically by noting this is the same as X/G being acyclic. If we take into account additional free orbits of 1 and 2 cells we obtain the following conjecture.

Conjecture 2.6.8. *There is no presentation of A_5 of the form*

$$\langle a,b,c,d,x_0,\dots,x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0^{-1}ax_0 = d, w_0,\dots,w_k\rangle$$

with $w_0,\dots,w_k \in \ker(\phi)$, where $\phi\colon F(a,b,c,d,x_0,\dots,x_k) \to A_5$ is given by $a \mapsto (2,5)(3,4)$, $b \mapsto (3,5,4)$, $c \mapsto (1,2)(3,5)$, $d \mapsto (2,5)(3,4)$ and $x_i \mapsto 1$.

Note that this reformulation gives another proof of the fact that the way in which the free orbits of 1-cells are attached is irrelevant.

Theorem 2.6.9. *Conjecture 2.6.8 is a special case of the Casacuberta–Dicks conjecture for A_5. Moreover, assuming Conjecture 2.2.1, Conjecture 2.6.8 is equivalent to the Casacuberta–Dicks conjecture for A_5.*

Proof. This follows from Theorem 2.2.11, Theorem 2.6.1 and Proposition 2.6.2. ∎

Remark 2.6.10. For each of the groups in Theorem 2.2.11 we could obtain a similar restatement. Computer experimentation suggests that Conjecture 2.6.8 also holds without the restriction that the words lie in $\ker(\phi)$.

There is an extensive literature on one-relator quotients of an HNN extension. From small cancellation theory for one relator quotients of HNN extensions [SS74] it follows that in *most* cases the group in Conjecture 2.6.7 is infinite and thus cannot be A_5. The theory of equations over groups (see Section A.6) helps to establish some special cases of Conjecture 2.6.7. We state here a classical result of Klyachko [Kly93, Lemma 2] on equations over groups.

Theorem 2.6.11 (Klyachko, [FR96, Theorem 4.1])**.** *Let H,H' be two isomorphic subgroups of Λ and let $\varphi\colon H \to H'$ be an isomorphism. Suppose that for each i, a_i, b_i are elements of Λ such that a_i is free relative to H and b_i is free relative to H'. Let c be an arbitrary element of Λ. Then the system of equations*

$$(b_0t^{-1}a_0t)(b_1t^{-1}a_1t)\cdots(b_rt^{-1}a_rt)ct = 1$$
$$\varphi(h) = t^{-1}ht,\, h \in H$$

has a solution in an overgroup of Λ.

We recall that an element $g \in \Lambda$ is *free relative* to $H \leq \Lambda$ if the map $\mathbb{Z} * H \to \langle g, H\rangle$ given by $1 \mapsto g$ and $h \mapsto h$ for $h \in H$ is an isomorphism.

Theorem 2.6.12. *For any choice of a word $w_0 \in F(a,b,c,d)$ and $k,l > 0$, Conjecture 2.6.7 is satisfied for the word $w = b(db)^k x^{-1} cd(acd)^l x w_0 x$.*

Proof. Let $\Lambda = \langle a,b,c,d \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5\rangle$. Recall that Λ has a description as an amalgamated product. From the normal form for amalgamated products (see Theorem A.4.1) we conclude that $b(db)^k$ is free relative to $H' = \langle d\rangle$ and $cd(acd)^k$ is free relative to $H = \langle a\rangle$. Thus by Theorem 2.6.11, the infinite group Λ injects into $\Lambda * \mathbb{Z}/\langle\langle x^{-1}ax = d, w\rangle\rangle$ and we are done. □

In Example 2.7.6 we will show that infinitely many of the words considered in the previous theorem give 2-dimensional acyclic A_5-complexes. Thus it will follow that infinitely many of the acyclic potential counterexamples to the Casacuberta–Dicks conjecture for A_5 have non-trivial fundamental group.

2.6.2 Cyclic presentations

A *cyclic presentation* is a presentation arising from a $\mathbb{Z}_n$-complex with one 0-cell, a free orbit of 1-cells and a free orbit of 2-cells. If a group G is given by a cyclic presentation, there is a split extension

$$1 \to G \to H \to \mathbb{Z}_n \to 1.$$

where H is given by a presentation

$$H = \langle x, t \mid t^n, w(x,t) \rangle$$

This particular case of Brown's result follows easily from the fact that $\mathbb{Z}_n$ fixes the 0-cell. Cyclic presentations are studied in the articles [EHT01], [Edj03], [HR03], [ES14] using diverse techniques including computer experimentation and curvature tests. In [ES14] it is conjectured that if the length l of the word $w(x,t)$ is at most $2n$ then the cyclically presented group G is nontrivial and this conjecture is verified for $n \leq 100$ and $l \leq 17$. A deeper understanding of cyclic presentations could lead to advances in the Casacuberta–Dicks conjecture.

2.7 The relation gap problem

In this section we explain the connection between the Casacuberta–Dicks conjecture and the relation gap problem. We give here a brief introduction to this subject, for a detailed account see [Har18, Har15, Har00]. Suppose we have a group extension

$$1 \to N \to F \to G \to 1$$

where F is not necessarily free. The conjugation action of F on N induces a G-module structure on $N/[N,N]$. We will refer to this module as the *relation module* of the extension, though this terminology is usually reserved for the case that F is free. If N is a G-group, that is a group with an action of G, then $d_G(N)$ is the minimum number of generators for N as a G-group. That is, the least number k such that there exist elements $n_1, \dots, n_k \in N$ with $N = \langle G \cdot n_1, \dots, G \cdot n_k \rangle$. Thus $d_G(N/[N,N])$ is the minimum number of generators for $N/[N,N]$ as a $\mathbb{Z}[G]$-module and $d_F(N)$ is the minimum number of normal generators for N in F. Since a set of normal generators gives a set of generators of the relation module, we have an inequality $d_F(N) \geq d_G(N/[N,N])$. The difference $d_F(N) - d_G(N/[N,N])$ is called the *relation gap*. Now we may state the relation gap problem.

Question 2.7.1 (The relation gap problem)**.** *Is there an extension*

$$1 \to N \to F \to G \to 1$$

with F free of finite rank, G finitely presentable and a positive relation gap?

We remark that it is believed that such a presentation must exist. We can rephrase Conjecture 2.6.7 or more generally, Conjecture 2.6.8 in terms of a relation gap.

Conjecture 2.7.2. *Let* $E = \langle a, b, c, d, x_0, \dots, x_k \mid a^2, b^3, c^2, d^2, (ab)^3, (bc)^2, (cd)^5, x_0^{-1} a x_0 = d \rangle$ *and consider the morphism* $\overline{\phi} \colon E \to A_5$ *given by* $a \mapsto (2,5)(3,4)$*,* $b \mapsto (3,5,4)$*,* $c \mapsto (1,2)(3,5)$*,* $d \mapsto (2,5)(3,4)$ *and* $x_i \mapsto 1$*. Then if* $N = \ker(\overline{\phi})$ *the extension*

$$1 \to N \to E \to A_5 \to 1$$

has a relation gap.

Remark 2.7.3. The group N in Conjecture 2.7.2 is free (by Brown's result), though the group E is obviously not free.

We may rephrase some earlier results in terms of the relation module.

Proposition 2.7.4. *In the setting of Proposition 2.6.4, we have that $\pi_1(X)$ is perfect if and only if the classes of $w_1,\dots,w_k$ generate the relation module of*

$$1 \to \pi_1(\Gamma) \to \widetilde{G}_\Gamma \to G \to 1,$$

which is isomorphic to $H_1(\Gamma)$.

Proposition 2.7.5. *Conjecture 2.7.2 is equivalent to Conjecture 2.6.8*

Proof. The relevant extension comes from applying Brown's result to the action of A_5 on a graph Γ obtained from $\Gamma_{OS}(A_5)$ by attaching k free orbits of 1-cells. Thus the relation module is isomorphic to $\mathbb{Z}[A_5]^{k+1}$. This proves the equivalence. □

Example 2.7.6. In Theorem 2.6.12 we proved that words of a particular form are not counterexamples to Conjecture 2.6.7. Now we show that if $k,l \geq 0$ the word

$$w = b(db)^{3k+1}x^{-1}cd(acd)^{2l+1}x(b(db)^{3k+1}cd(acd)^{2l+1})^{-1}x$$

gives an acyclic fixed point free A_5-complex (which as we have seen, has nontrivial fundamental group). First note that $\phi(w) = 1$. Let $G = A_5$. We only have to show that the class $[w]$ is a generator of the relation module $H_1(\Gamma_{OS}(G)) \simeq \mathbb{Z}[A_5]$ of

$$1 \to \pi_1(\Gamma_{OS}(G)) \to \widetilde{G}_{\Gamma_{OS}(G)} \xrightarrow{\phi} G \to 1.$$

We can write

$$w = (b(db)^{3k+1})x^{-1}(b(db)^{3k+1})^{-1} \cdot (b(db)^{3k+1}cd(acd)^{2l+1})x(b(db)^{3k+1}cd(acd)^{2l+1})^{-1} \cdot x$$

therefore we have

$$\begin{aligned}
[w] &= -\phi(b(db)^{3k+1}) \cdot [x] + \phi(b(db)^{3k+1}cd(acd)^{2l+1}) \cdot [x] + [x] \\
&= (1 - (2,4,3) + (2,3,5))[x]
\end{aligned}$$

Since $[x]$ is a generator of the relation module and $u = 1 - (2,4,3) + (2,3,5)$ is a unit in $\mathbb{Z}[A_5]$ (the inverse is $v = 1 - (3,5,4) + (2,5,3) - (2,4,5) + (2,4,3) - (2,3,5) + (2,3,4)$) we conclude that $[w]$ is a generator.

Bridson and Tweedale prove the following result which looks strikingly similar to Conjecture 2.6.7.

Theorem 2.7.7 ([BT07, Proposition 3.6])**.** *Let* $\rho_n(x,t) = (txt^{-1})x(txt^{-1})^{-1}x^{-n-1}$*. Let* $Q_n = \langle x,t \mid \rho_n(x,t), x^n \rangle$*. Then the group* $\Gamma_{m,n} = Q_m * Q_n$ *does not admit a presentation of the form*

$$\langle x,t,y,s \mid \rho_m(x,t), \rho_n(y,s), r \rangle$$

for any word $r \in F(x,t,y,s)$.

Their proof relies on a result of Howie on one relator products of locally indicable groups. It seems that this method will not work to attack Conjecture 2.6.7. We mention some related articles containing interesting ideas and methods that we unsuccessfully tried to apply to our problem. Rhemtulla proved that a group extension with a relation gap must satisfy certain condition [Rhe81]. Mannan proposed a method to reduce this type of problems to questions in commutative algebra [Man13]. Osin and Thom [OT13, Conjecture 6.1] conjectured that for a torsion free group G there is an inequality $b_1^{(2)}(G/\langle\langle g_1, \dots, g_n \rangle\rangle) \geq b_1^{(2)}(G) - n$ of ℓ^2-Betti numbers. Another approach that we considered is to represent the groups $\widetilde{G}_X$ by mimicking Howie's proof of the Scott-Wiegold conjecture [How02] or the proof of the Gerstenhaber-Rothaus theorem on equations over groups [GR62].

2.8 Groups that are not fundamental groups of certain acyclic spaces

In Section 2.4 we have seen some of the possible fundamental groups of a fixed point free 2-dimensional G-complex with 1-skeleton $\Gamma_{OS}(G)$. In this section we prove that certain super-perfect groups do not arise in this way. This gives some evidence for Conjecture 2.4.1. Note that by Corollary 2.2.15, the results of this section do not depend on the particular choice for $\Gamma_{OS}(G)$.

2.8.1 Classification of group extensions

We review some classical results on classification of group extensions that can be found in [Bro94, Chapter IV, Section 6]. Given two groups N and Q we want to classify group extensions

$$1 \to N \to G \to Q \to 1$$

of Q by N up to equivalence. Recall that two extensions G, G' of Q by N are *equivalent* if there is an isomorphism $G \to G'$ such that the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & Q & \longrightarrow & 1. \end{array}$$

Given such an extension, there is an induced homomorphism $\alpha\colon Q \to \mathrm{Out}(N)$. We also have (for any group N) a homomorphism $\rho\colon \mathrm{Out}(N) \to \mathrm{Aut}(Z(N))$. Then $Z(N)$ is a Q-module with the structure given by $\rho\alpha$. Thus we can consider the group cohomology $H^2(Q;Z(N))$. The following result shows that group extensions inducing α are parametrized by this homology group.

Theorem 2.8.1 ([Bro94, Chapter IV, Theorem 6.6])**.** *Let $\alpha\colon Q \to \mathrm{Out}(N)$. Then either*

(i) There is no group extension of Q by N inducing α.

(ii) The elements of $H^2(Q;Z(N))$ are in bijection with the group extensions of Q by N inducing α.

Moreover, to every $\alpha\colon Q \to \mathrm{Out}(N)$ we can associate an element $o(\alpha) \in H^3(Q;Z(N))$ that is nonzero if and only if there is no group extension inducing α. We shall not need this subtler result. For the precise definition of the obstruction $o(\alpha)$ see [Bro94, Chapter IV, Theorem 6.7]. As a consequence we obtain the following result.

Theorem 2.8.2. *Let G be one of the groups in Theorem 2.2.11. Let π be a group. Suppose that the following conditions hold:*

(i) G is not a subgroup of $\mathrm{Out}(\pi)$.

(ii) $H^2(G;Z(\pi)) = 0$ for the trivial action of G on $Z(\pi)$.

(iii) π is not a quotient of the group $\widetilde{G}_{\Gamma_{OS}(G)}$ given by Theorem 2.6.1.

Then π cannot be the fundamental group of a G-complex X with $X^{(1)} = \Gamma_{OS}(G)$.

Proof. Since G is simple, by (i), the only morphism $G \to \mathrm{Out}(\pi)$ is the trivial one. By Theorem 2.8.1 the only extension of G by π is the direct product $\pi \times G$. Thus if such a space exists, by Theorem 2.6.1 there is an epimorphism $\widetilde{G}_{\Gamma_{OS}(G)} \to \pi \times G$. Thus, π is a quotient of $\widetilde{G}_{\Gamma_{OS}(G)}$, contradicting (iii). □

Remark 2.8.3. From the proof of the previous result it is obvious that we can replace the hypothesis (iii) by the following: $\pi \times G$ is not a quotient of $\widetilde{G}_{\Gamma_{OS}(G)}$. However, we will only use the result in the weaker form given above.

Theorem 2.8.4. *An acyclic, 2-dimensional A_5-complex X with $X^{(1)} = \Gamma_{OS}(A_5)$ cannot have fundamental group $\mathrm{PSL}_2(2^3)$.*

Proof. Let $G = A_5$. We want to apply Theorem 2.8.2. We have $\mathrm{Out}(\mathrm{PSL}_2(2^3)) = \mathbb{Z}_3$. Moreover since $Z(\mathrm{PSL}_2(2^3)) = 1$ condition (ii) is satisfied. Now we show there is no epimorphism

$$\widetilde{G}_{\Gamma_{OS}(G)} = \langle a,b,c,d,x \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x^{-1}ax = d\rangle \to \mathrm{PSL}_2(2^3).$$

This can be checked algorithmically, but to speed up the computation we first do some reductions. Since $|\mathrm{PSL}_2(2^3)| = 504$ is not divisible by 5, any epimorphism must map $cd \mapsto 1$. Thus if there is such a morphism, it must factor through

$$G_1 = \langle a,b,c,d,x \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,cd,x^{-1}ax = d\rangle.$$

We will see that there is no epimorphism $G_1 \to \mathrm{PSL}_2(2^3)$. We ask GAP to compute a simpler presentation of the group G_1.

```
gap> F:=FreeGroup(["a","b","c","d","x"]);;
gap> AssignGeneratorVariables(F);;
#I  Assigned the global variables [ a, b, c, d, x ]
gap> R:=[a^2,b^3,c^2,d^2,(a*b)^3,(b*c)^2, c*d, x^-1*a*x*d^-1];;
gap> G1:=SimplifiedFpGroup(F/R);;
<fp group on the generators [ a, b, x ]>
gap> RelatorsOfFpGroup(G1);
[ a^2, b^3, (a*b)^3, (b*x^-1*a*x)^2 ]
```

Thus G_1 has the presentation $\langle a,b,x \mid a^2,b^3,(ab)^3,(bx^{-1}ax)^2\rangle$. We use the following function.

```
EpimorphismFromG1:=function(pi)
  local elements, elementsOfOrder1or2,elementsOfOrder1or3,a,b,x;
  elements:=Set(pi);
  elementsOfOrder1or2:=Filtered(elements, x-> Order(x) in [1,2]);
  elementsOfOrder1or3:=Filtered(elements, x-> Order(x) in [1,3]);
  for a in elementsOfOrder1or2 do
    for b in elementsOfOrder1or3 do
      if (a*b)^3=Identity(pi) then
        for x in elements do
          if (b*x^-1*a*x)^2=Identity(pi) then
            if Order(Group([a,b,x]))=Order(pi) then
              return [a,b,x];
            fi;
          fi;
        od;
      fi;
    od;
  od;
  return fail;
end;;
```

The following computation

```
gap> EpimorphismFromG1( PSL(2,2^3) );
fail
```

shows there is no epimorphism, concluding the proof. $\square$

Corollary 2.8.5. *An acyclic, 2-dimensional A_5-complex X with $X^{(1)} = \Gamma_{OS}(A_5)$ cannot have fundamental group* $\mathrm{PSL}_2(2^5)$.

Proof. We want to apply Theorem 2.8.2. We have $\mathrm{Out}(\mathrm{PSL}_2(2^5)) = \mathbb{Z}_5$ and $Z(\mathrm{PSL}_2(2^5)) = 1$. To conclude we need to prove that $\mathrm{PSL}_2(2^5)$ is not a quotient of $\widetilde{G}_{\Gamma_{OS}(A_5)}$. To do this we first note that $|\mathrm{PSL}_2(2^5)| = 32736$ is not divisible by 5. By the same argument in the proof of Theorem 2.8.4 (using the same function `EpimorphismFromG1`) the following computation

```
gap> EpimorphismFromG1( PSL(2,2^5) );
fail
```

concludes the proof. $\square$

2.8.2 Outer automorphisms of a free product

In this section we prove that in some cases the fundamental group of an acyclic example cannot be a free product of copies of A_5^*. In order to do this we must get some understanding of the outer automorphisms of a free product of groups.

Proposition 2.8.6. *Let $G_1, \dots, G_n$ be finite groups. Let $p > n$ be a prime such that G_i and* $\mathrm{Aut}(G_i)$ *are p-torsion-free for every i. Then* $\mathrm{Out}\left(\ast_{i=1}^n G_i\right)$ *is p-torsion-free.*

Proof. We assume $G = \mathop{\ast}\limits_{i=1}^{n} G_i$ has p-torsion to get a contradiction. We maintain the notation from [MM96]. Since no free factor of G is infinite cyclic, we have $\mathrm{Out}(G) = \Sigma\mathrm{Out}(G)$ and by [MM96, Theorem 5.6] there is an exact sequence

$$1 \to \overline{L}(G) \to \Sigma\mathrm{Out}(G) \to \overline{P} \to 1.$$

By [MM96, Theorem 6.1], $\overline{L}(G)$ is torsion free. Thus $\overline{P}$ must have p-torsion. Also by [MM96, Theorem 5.6] there is an exact sequence

$$1 \to \prod_{i=1}^{n} G_i \to P \to \overline{P} \to 1,$$

where $P = \left(\prod_{i=1}^n G_i^{n-1} \circ \mathrm{Aut}(G_i)\right) \circ \Omega$. Since $\Omega \leq S_n$ it follows that $p \nmid |P|$. Therefore $p \nmid |\overline{P}|$, contradiction. $\square$

Corollary 2.8.7. *If $1 \leq n \leq 6$, the group* $\mathrm{PSL}_2(2^3)$ *is not a subgroup of* $\mathrm{Out}(\mathop{\ast}\limits_{i=1}^{n} A_5^*)$.

Proof. We have $|PSL_2(2^3)| = 2^3 \cdot 3^2 \cdot 7$ and $|A_5^*| = |\mathrm{Aut}(A_5^*)| = 2^3 \cdot 3 \cdot 5$. Thus we can take $p = 7$ and use Proposition 2.8.6. $\square$

Theorem 2.8.8. *An acyclic, 2-dimensional* $\mathrm{PSL}_2(2^3)$*-complex* X *with* $X^{(1)} = \Gamma_{OS}(\mathrm{PSL}_2(2^3))$ *cannot have fundamental group a free product of* $1 \leq n \leq 6$ *copies of* A_5^*.

Proof. Let $G = \mathrm{PSL}_2(2^3)$ and take $\Gamma_{OS}(G)$ arising from a coset complex, so $\Gamma_{OS}(G)/G$ is as in Figure 2.2. We want to apply Theorem 2.8.2. If $n = 1$ we have $Z(A_5^*) = \mathbb{Z}_2$ and by the universal coefficient theorem we have $H^2(\mathrm{PSL}_2(2^3);\mathbb{Z}_2) = 0$. If $n > 1$, we have $Z(\ast_{i=1}^n A_5^*) = 1$. Thus condition (ii) is satisfied. By Corollary 2.8.7 we have condition (i). Finally we verify condition (iii). We apply Brown's result to obtain a group $\widetilde{G}_{\Gamma_{OS}(G)} = (B \ast_{C_2} D_{18} \ast_{C_2} D_{14}) \ast_{C_3}$. An epimorphism $\widetilde{G}_{\Gamma_{OS}(G)} \to A_5$ kills 7-torsion, thus must factor through the quotient of $\widetilde{G}_{\Gamma_{OS}(G)}$ by the normal closure of the 7-torsion. We now prove this quotient is isomorphic to $\mathbb{Z}$. The 7-torsion in the Borel subgroup $B = \mathbb{F}_{2^3} \rtimes C_7$ normally generates B and the groups D_{14} and D_{18} are normally generated by any element of order 2. Therefore from the description of $\widetilde{G}_{\Gamma_{OS}(G)}$ we conclude that this quotient is $\mathbb{Z}$. But there is no epimorphism from $\mathbb{Z}$ to a free product of $n \geq 1$ copies of A_5^*. □

Resumen del Capítulo 2: La conjetura de Casacuberta–Dicks

En este capítulo se utilizan diferentes enfoques para estudiar la siguiente conjetura formulada por Carles Casacuberta y Warren Dicks en [CD92].

Conjetura 2.0.1 (Casacuberta–Dicks). *Toda acción de un grupo finito G en un 2-complejo contráctil y finito tiene un punto fijo.*

Utilizando los resultados obtenidos por Oliver y Segev [OS02] y asumiendo el siguiente caso particular de la conjetura de Kervaire–Laudenbach–Howie se obtiene una descripción de los posibles contraejemplos a la conjetura de Casacuberta–Dicks.

Conjetura 2.2.1. *Sea X un complejo de dimensión 2 finito y contráctil. Si $A \subset X$ es un subcomplejo acíclico, entonces A es contráctil.*

Teorema 2.2.11. *Asumiendo la Conjetura 2.2.1, si la conjetura de Casacuberta–Dicks resulta falsa, debe existir un G-complejo X contráctil de dimensión 2, esencial y sin puntos fijos donde G es alguno de los siguientes grupos:*

(i) $\mathrm{PSL}_2(2^p)$ *con p primo.*

(ii) $\mathrm{PSL}_2(3^p)$ *con p un primo impar.*

(iii) $\mathrm{PSL}_2(q)$ *con $q > 3$ primo tal que $q \equiv \pm 3 \text{ mód } 5$ y $q \equiv \pm 3 \text{ mód } 8$.*

(iv) $\mathrm{Sz}(2^p)$ *con p un primo impar.*

Más aún, es posible tomar X obtenido a partir del grafo $\Gamma_{OS}(G)$ adjuntando $k \geq 0$ órbitas libres de 1-celdas y $k+1$ órbitas libres de 2-celdas.

El grafo $\Gamma_{OS}(G)$ que aparece en el Teorema 2.2.11 es cualquier elección particular del 1-esqueleto de un 2-complejo como los que describen Oliver y Segev en [OS02, Examples 3.4, 3.5, 3.7].

Una conjetura relacionada con la de Casacuberta–Dicks es la conjetura de Quillen sobre el poset $\mathcal{S}_p(G)$ de p-subgrupos no triviales de un grupo finito G [Qui78]. Dicha conjetura dice que la acción por conjugación de G en $\mathcal{S}_p(G)$ tiene un punto fijo siempre que el complejo simplicial $\mathcal{K}(\mathcal{S}_p(G))$ es contráctil. Utilizando los resultados de [OS02] se prueba el siguiente resultado, que aparecerá en [PSCV18].

Teorema 2.3.2 (Piterman – Sadofschi Costa – Viruel)**.** *La conjetura de Quillen vale para grupos de p-rango 3.*

Este resultado es el caso 2-dimensional de la conjetura de Quillen (en su versión enunciada a partir del poset de p-subgrupos elementales abelianos) y a la vez puede ser visto como un caso particular de la conjetura de Casacuberta–Dicks.

El objeto que resulta clave para construir los ejemplos de 2-complejos acíclicos con una acción sin puntos fijos de G es el $\mathbb{Z}[G]$-módulo libre $H_1(\Gamma_{OS}(G))$. El G-grafo $\Gamma_{OS}(G)$ induce un subgrupo $G \leq \mathrm{Out}(F_{|G|})$ que resulta ser el objeto análogo para estudiar si alguno de estos 2-complejos es contráctil. Con estas ideas, se obtienen diversas reformulaciones algebraicas de la Conjetura de Casacuberta–Dicks, como la siguiente:

Conjetura 2.5.13. *Todo subgrupo permutacional de* $\mathrm{Out}(F_m)$ *que actúa sin puntos fijos en una base normal de* F_m *se levanta a un subgrupo permutacional de* $\mathrm{Aut}(F_m)$.

La conjetura de Casacuberta–Dicks permanece abierta incluso en el caso en que el grupo es $A_5 = \mathrm{PSL}_2(2^2)$. Utilizando la teoría de Bass-Serre, K.S. Brown [Bro84] probó un resultado que permite traducir casos particulares de la conjetura de Casacuberta–Dicks en preguntas de teoría geométrica de grupos.

Conjetura 2.6.7. *No existe una palabra* $w \in \ker(\phi)$ *tal que*

$$\langle a,b,c,d,x \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x^{-1}ax = d,w\rangle$$

es una presentación de A_5*, donde* $\phi : F(a,b,c,d,x) \to A_5$ *es el morfismo definido por* $a \mapsto (2,5)(3,4)$*,* $b \mapsto (3,5,4)$*,* $c \mapsto (1,2)(3,5)$*,* $d \mapsto (2,5)(3,4)$ *y* $x \mapsto 1$.

Del mismo modo se obtiene la siguiente conjetura que (asumiendo la Conjetura 2.2.1) resulta equivalente al caso A_5 de la conjetura de Casacuberta–Dicks.

Conjetura 2.6.8. *No existe una presentación de* A_5 *de la forma*

$$\langle a,b,c,d,x_0,\dots,x_k \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x_0^{-1}ax_0 = d,w_0,\dots,w_k\rangle$$

con $w_0,\dots,w_k \in \ker(\phi)$*, donde* $\phi\colon F(a,b,c,d,x_0,\dots,x_k) \to A_5$ *es el morfismo dado por* $a \mapsto (2,5)(3,4)$*,* $b \mapsto (3,5,4)$*,* $c \mapsto (1,2)(3,5)$*,* $d \mapsto (2,5)(3,4)$ *y* $x_i \mapsto 1$ *para todo i.*

Las Conjeturas 2.6.7 y 2.6.8 están profundamente relacionadas con otro famoso problema abierto, el *relation gap problem* (ver [Har18, Har15]). Se puede reformular la Conjetura 2.6.8 en términos de un *relation gap*.

Conjetura 2.7.2. *Sea E el grupo presentado por*

$$\langle a,b,c,d,x_0,\dots,x_k \mid a^2,b^3,c^2,d^2,(ab)^3,(bc)^2,(cd)^5,x_0^{-1}ax_0 = d\rangle$$

y sea $\overline{\phi}\colon E \to A_5$ *el morfismo dado por* $a \mapsto (2,5)(3,4)$*,* $b \mapsto (3,5,4)$*,* $c \mapsto (1,2)(3,5)$*,* $d \mapsto (2,5)(3,4)$ *y* $x_i \mapsto 1$*. Entonces, si* $N = \ker(\overline{\phi})$ *la extensión*

$$1 \to N \to E \to A_5 \to 1$$

tiene un relation gap.

Usando un resultado de ecuaciones sobre grupos probado por Klyachko [Kly93] se prueba un caso particular de la Conjetura 2.6.7.

Teorema 2.6.12. *Una palabra $w = b(db)^k x^{-1} cd(acd)^l x w_0 x$, con $w_0 \in F(a,b,c,d)$ y $k,l > 0$, no puede ser un contraejemplo de la Conjetura 2.6.7.*

Este resultado implica que infinitos de los posibles contraejemplos acíclicos de la conjetura de Casacuberta–Dicks de hecho no son contráctiles.

También se estudia la conjetura de Casacuberta–Dicks a través de experimentos con el software GAP [GAP18]. Con este fin, se desarrollaron dos paquetes, G2Comp [SC18a] que permite trabajar computacionalmente con G-complejos de dimensión 2 y SmallCancellation [SC18b] que es una implementación de las condiciones clásicas de *small cancellation* [LS77]. Se exhiben ejemplos de A_5-complejos acíclicos de dimensión 2 con 1-esqueleto $\Gamma_{OS}(A_5)$ y tales que el grupo fundamental: es isomorfo a A_5^*; es isomorfo a $\mathop{\ast}\limits_{i=1}^{6} A_5^*$; es isomorfo a $\mathop{\ast}\limits_{i=1}^{7} A_5^*$; tiene un epimorfismo a A_5 y cumple la condición $C'(1/6)$ de small cancellation; no tiene un epimorfismo a A_5, pero cumple la condición $C(7)$ y por lo tanto es no trivial. A partir de estos resultados experimentales se formula la siguiente conjetura:

Conjetura 2.4.1. *Sea X un 2-complejo acíclico y finito con una acción sin puntos fijos de A_5. Si $\pi_1(X)$ es finito, entonces $\pi_1(X) = A_5^*$.*

Los resultados experimentales y una sugerencia de Bob Oliver motivaron los siguientes resultados.

Teorema 2.8.4. *Un 2-complejo acíclico y finito X con una acción sin puntos fijos de A_5 tal que $X^{(1)} = \Gamma_{OS}(A_5)$ no puede tener grupo fundamental* $\mathrm{PSL}_2(2^3)$.

Teorema 2.8.5. *Un 2-complejo acíclico y finito X con una acción sin puntos fijos de A_5 tal que $X^{(1)} = \Gamma_{OS}(A_5)$ no puede tener grupo fundamental* $\mathrm{PSL}_2(2^5)$.

Teorema 2.8.8. *Un 2-complejo acíclico y finito X con una acción sin puntos fijos de* $\mathrm{PSL}_2(2^3)$ *tal que $X^{(1)} = \Gamma_{OS}(\mathrm{PSL}_2(2^3))$ no puede tener grupo fundamental isomorfo a un producto libre de $1 \leq n \leq 6$ copias de A_5^*.*

Chapter 3

The complex of partial bases

In this chapter we study the simplicial complex $\mathrm{PB}(F_n)$ with simplices the nonempty partial bases of F_n. The main result of the chapter, which appeared in our article [SC17a], is Theorem 3.4.5, which states that $\mathrm{PB}(F_n)$ is Cohen-Macaulay. Neil Fullarton and Andrew Putman independently obtained a proof of this result (personal communication with A. Putman). As explained in Section 2.5, our original motivation for the study of this object came from the Casacuberta–Dicks conjecture, but besides this relation these results are of independent interest. We now explain the context for these results.

The curve complex $\mathcal{C}(S_g)$ of an oriented surface S_g of genus g was introduced by Harvey [Har81] as an analogue of Tits buildings for the mapping class group $\mathrm{Mod}(S_g)$. Harer proved that $\mathcal{C}(S_g)$ is homotopy equivalent to a wedge of $(g-1)$-spheres [Har85]. Masur and Minsky proved that $\mathcal{C}(S_g)$ is hyperbolic [MM99]. The curve complex became a fundamental object in the study of $\mathrm{Mod}(S_g)$. Since there is an analogy between $\mathrm{Aut}(F_n)$ and $\mathrm{Mod}(S_g)$, it is natural to seek for an analogue of $\mathcal{C}(S_g)$ in this context. There are many candidates that share properties with the curve complex.

One of these analogues is the poset $\mathrm{FC}(F_n)$ of proper free factors of F_n. Hatcher and Vogtmann [HV98] proved that its order complex $\mathcal{K}(\mathrm{FC}(F_n))$ is Cohen-Macaulay (in particular, that it is homotopy equivalent to a wedge of $(n-2)$-spheres). Bestvina and Feighn [BF14] proved that $\mathcal{K}(\mathrm{FC}(F_n))$ is hyperbolic. Subsequently, different proofs of this fact appeared in [KR14] and [HH17].

Other natural analogues are defined in terms of partial bases. A *partial basis* of a free group F is a subset of a basis of F. Day and Putman [DP13] defined the complex $\mathcal{B}(F_n)$ whose simplices are sets $\{C_0,\dots,C_k\}$ of conjugacy classes of F_n such that there exists a partial basis $\{v_0,\dots,v_k\}$ with $C_i = [\![v_i]\!]$ for $0 \le i \le k$. They proved that $\mathcal{B}(F_n)$ is 0-connected for $n \ge 2$ and 1-connected for $n \ge 3$ [DP13, Theorem A], that a certain quotient is $(n-2)$-connected [DP13, Theorem B] and they conjectured that $\mathcal{B}(F_n)$ is $(n-2)$-connected [DP13, Conjecture 1.1]. As an application, they used $\mathcal{B}(F_n)$ to prove that the Torelli subgroup is finitely generated.

In Section 3.1, we obtain a presentation for $\mathrm{SAut}(F_n, \{v_1, \dots, v_l\})$, analogous to Gersten's presentation of $\mathrm{SAut}(F_n)$ which was used by Day and Putman in [DP13]. First we use McCool's method to present $\mathrm{Aut}(F_n, \{v_1, \dots, v_l\})$ and then we apply the Reidemeister-Schreier method.

In Section 3.2 we prove that the link $\mathrm{lk}(B, \mathrm{PB}(F_n))$ of a partial basis B is 0-connected if $n - |B| \geq 2$ and 1-connected if $n - |B| \geq 3$. The proof is the proof of [DP13, Theorem A] with only minor modifications. Instead of Gersten's presentation, we use the presentation obtained in Section 3.1.

In Section 3.3 we prove Theorem 3.3.7, which is a version of a result due to Quillen [Qui78, Theorem 9.1]. Our version produces an explicit basis of the top homology group of X. The proof is based on Piterman's proof of Quillen's Theorem [Pit16, Teorema 2.1.28], which uses Barmak-Minian's non-Hausdorff mapping cylinder argument [BM08].

In Section 3.4 we prove Theorem 3.4.5. The key idea is to compare the link $\mathrm{lk}(B, \mathrm{PB}(F_n))$ (which is $(n - |B| - 1)$-dimensional) with $\mathrm{FC}(F_n)_{>\langle B\rangle}$ (which is $(n - |B| - 2)$-dimensional). In order do this, we have to consider the $(n - |B| - 2)$-skeleton of $\mathrm{lk}(B, \mathrm{PB}(F_n))$. Finally, using the basis given by Theorem 3.3.7, we can understand what happens when we pass from $\mathrm{lk}(B, \mathrm{PB}(F_n)^{(n-2)})$ to $\mathrm{lk}(B, \mathrm{PB}(F_n))$. We proceed by induction on $n - |B|$ and to start we need the result proved in Section 3.2.

3.1 A presentation for $\mathrm{SAut}(F_n, \{v_1, \dots, v_l\})$

The main result of this section is Theorem 3.1.10, which gives a finite presentation for the group $\mathrm{SAut}(F_n, \{v_1, \dots, v_l\})$. When $l = 0$, this presentation reduces to the presentation of $\mathrm{SAut}(F_n)$ given by Gersten in [Ger84] and used by Day and Putman in [DP13]. To obtain this presentation we first obtain a presentation for $\mathrm{Aut}(F_n, \{v_1, \dots, v_l\})$ using McCool's method [McC75] and then we get to the desired presentation using the Reidemeister-Schreier method.

3.1.1 Definitions and Notations

Throughout the chapter automorphisms act on the left and compose from right to left as usual. Let F_n be a free group with basis $\{v_1, \dots, v_n\}$. Recall that $\mathrm{SAut}(F_n)$, *the special automorphism group of* F_n, is the subgroup of $\mathrm{Aut}(F_n)$ consisting of automorphisms whose images in $\mathrm{Aut}(\mathbb{Z}^n)$ have determinant 1. If A is a subset of F_n we define $\mathrm{Aut}(F_n, A) = \{\phi \in \mathrm{Aut}(F_n) : \phi|_A = 1_A\}$ and $\mathrm{SAut}(F_n, A) = \{\phi \in \mathrm{SAut}(F_n) : \phi|_A = 1_A\}$.

Let $L = \{v_1, \dots, v_n, v_1^{-1}, \dots, v_n^{-1}\} \subseteq F_n$ be the set of letters. We fix a number l, $1 \leq l \leq n$ and we define $L' = L - \{v_1, \dots, v_l, v_1^{-1}, \dots, v_l^{-1}\}$. We consider the subgroup $\Omega(F_n)$ of $\mathrm{Aut}(F_n)$ given by the automorphisms that permute L. The order of $\Omega(F_n)$ is $2^n n!$. If $A \subseteq L$, $a \in A$ and

$a^{-1} \notin A$, there is an automorphism $(A;a)$ of F_n defined on L by

$$(A;a)(x) = \begin{cases} x & \text{if } x \in \{a, a^{-1}\} \\ a^{-1}xa & \text{if } x, x^{-1} \in A \\ xa & \text{if } x \in A, x^{-1} \in L - A \\ a^{-1}x & \text{if } x^{-1} \in A, x \in L - A \\ x & \text{if } x, x^{-1} \in L - A \end{cases}$$

The set of these automorphisms will be denoted by $\Lambda(F_n)$. We consider the set of Whitehead automorphisms $\mathcal{W}(F_n) = \Lambda(F_n) \cup \Omega(F_n)$.

If $a, b \in L$ with $a \neq b^{\pm 1}$ we denote by $E_{a,b}$ the automorphism that maps a to ab and fixes $L - \{a, a^{-1}\}$ and by $M_{a,b}$ the automorphism that maps a to ba and fixes $L - \{a, a^{-1}\}$. We have $E_{a,b} = (\{a,b\};b)$, $M_{a,b} = (\{a^{-1}, b^{-1}\}; b^{-1})$ and $E_{a,b} = M_{a^{-1},b^{-1}}$. If $a, b \in L$ and $a \neq b^{\pm 1}$ we also consider the automorphism $w_{a,b}$ that takes a to b^{-1}, b to a and fixes $L - \{a, a^{-1}, b, b^{-1}\}$.

3.1.2 McCool's method

We recall the classical presentation of $\mathrm{Aut}(F_n)$ obtained by McCool.

Theorem 3.1.1 ([McC74]). *There is a presentation of* $\mathrm{Aut}(F_n)$ *with generators* $\mathcal{W}(F_n)$ *and relators R1-R7 below.*

$$(A;a)^{-1} = (A - \{a\} + \{a\}^{-1}; a^{-1}) \tag{R1}$$

$$(A;a)(B;a) = (A \cup B; a) \qquad \text{if } A \cap B = \{a\} \tag{R2}$$

$$[(A;a),(B;b)] = 1 \qquad \text{if } A \cap B = \varnothing,\ a^{-1} \notin B,\ b^{-1} \notin A \tag{R3}$$

$$(B;b)(A;a) = (A \cup B - \{b\}; a)(B;b) \qquad \text{if } A \cap B = \varnothing,\ a^{-1} \notin B,\ b^{-1} \in A \tag{R4}$$

$$(A - \{a\} \cup \{a\}^{-1}; b)(A;a) = (A - \{b\} \cup \{b^{-1}\}; a) w_{a,b} \qquad \text{if } b \in A,\ b^{-1} \notin A,\ a \neq b \tag{R5}$$

$$T(A;a)T^{-1} = (T(A); T(a)) \qquad \text{if } T \in \Omega(F_n) \tag{R6}$$

$$\text{Multiplication table of } \Omega(F_n) \tag{R7}$$

Additionally the following relations hold

$$(A;a) = (L - A; a^{-1})(L - \{a^{-1}\}; a) = (L - \{a^{-1}\}; a)(L - A; a^{-1}) \tag{R8}$$

$$(L - \{b^{-1}\}; b)(A;a)(L - \{b\}; b^{-1}) = (A;a) \qquad \text{if } b, b^{-1} \in L - A \tag{R9}$$

$$(L - \{b^{-1}\}; b)(A;a)(L - \{b\}; b^{-1}) = (L - A; a^{-1}) \qquad \text{if } b \neq a,\ b \in A,\ b^{-1} \in L - A \tag{R10}$$

The *length* of an element $u \in F_n$, denoted $|u|$, is the number of letters of the unique reduced word in L that represents u. The *total length* of an m-tuple $(u_1, \ldots, u_m)$ of elements of F_n

is $|u_1| + \ldots + |u_m|$. In [McC75], McCool considered the action of $\mathrm{Aut}(F_n)$ on the set of cyclic words and described a method to obtain a finite presentation for the stabilizer of a tuple of cyclic words. Jensen and Wahl [JW04, Theorem 7.1] used McCool's method to give a presentation of the group $\mathrm{Aut}(F_n, \{[\![v_1]\!], \ldots, [\![v_k]\!]\})$ of automorphisms of F_n that fix the conjugacy classes of the elements of a partial basis $\{v_1, \ldots, v_k\}$. In [McC75, Section 4. (1)-(2)] McCool explains that the method also works if we replace cyclic words by ordinary words. The version for ordinary words is the one we use in the proof of Theorem 3.1.4.

Theorem 3.1.2 (McCool's method)**.** *Let F_n be the free group with basis $\{v_1, \ldots, v_n\}$. The group $\mathcal{A} = \mathrm{Aut}(F_n)$ acts on $(F_n)^m$ by $\phi \cdot (u_1, \ldots, u_m) = (\phi(u_1), \ldots, \phi(u_m))$. Then if $U = (u_1, \ldots, u_m)$ is an m-tuple of words in F_n, the stabilizer $\mathcal{A}_U$ of U is finitely presented.*

Moreover, we can construct a finite 2-complex K with fundamental group $\mathcal{A}_U$ as follows. Let $K^{(0)}$ be the set of tuples in the orbit of U which have minimum total length. We may assume $U \in K^{(0)}$. Each triple (V, V', ϕ) such that $V, V' \in K^{(0)}$, $\phi \in \mathcal{W}(F_n)$ and $\phi(V) = V'$ represents an oriented edge from V to V' labelled ϕ. The oriented edges (V, V', ϕ) and (V', V, ϕ^{-1}) together determine a single 1-cell in $K^{(1)}$. Finally a 2-cell is attached following each closed edge path in $K^{(1)}$ such that the word obtained reading the labels is a relator corresponding to a relation of type R1-R10. Then $\mathcal{A}_U = \pi_1(K, U)$.

Remark 3.1.3. Let $\mathcal{P}$ be the presentation of $\mathrm{Aut}(F_n)$ from Theorem 3.1.1 and let $X_{\mathcal{P}}$ be the associated 2-complex (see Section A.3). The complex K in Theorem 3.1.2 is a subcomplex of the covering space of $X_{\mathcal{P}}$ that corresponds to the subgroup $\mathcal{A}_U \leq \pi_1(X_{\mathcal{P}}, x_0)$.

Theorem 3.1.4. *There is a presentation of* $\mathrm{Aut}(F_n, \{v_1, \ldots, v_l\})$ *with generators*

$$\mathcal{W}(F_n) \cap \mathrm{Aut}(F_n, \{v_1, \ldots, v_l\})$$

and relators R1-R7 that involve only those generators.

Proof. Let $U = (v_1, \ldots, v_l)$. We use Theorem 3.1.2 to construct a 2-complex K with fundamental group $\mathrm{Aut}(F_n, \{v_1, \ldots, v_l\}) = \mathcal{A}_U$. We note that U has minimum total length and the 0-skeleton $K^{(0)}$ is the set of tuples $V = (v_{\sigma(1)}^{s_1}, \ldots, v_{\sigma(l)}^{s_l})$ with $\sigma \in S_n$ and $s_i \in \{1, -1\}$.

Now we can obtain a presentation of $\pi_1(K, U)$. The presentation has a generator for each edge and a relation for each 2-cell of K and also a relation for each edge in a fixed spanning tree of K^1. The spanning tree we choose consists of an edge (V, U, σ_V) for each $V \in K^0 - \{U\}$, for some fixed $\sigma_V \in \Omega(F_n)$. We note that if $x \in L$ and $(A;a)(x) \in L$ then $x = (A;a)(x)$. Therefore an edge labeled $(A;a)$ is necessarily a loop. Hence the relations of types R1-R5 and R8-R10 are products of loops on a same vertex V. If we consider such a relation

$$(V, V, (A_1; a_1)) \cdots (V, V, (A_k; a_k))$$

with $V \neq U$, using relations R6 we can replace it by

$$(V,U,\sigma)^{-1}(U,U,(\sigma(A_1);\sigma(a_1)))\cdots(U,U,(\sigma(A_k);\sigma(a_k)))(V,U,\sigma)$$

for some $\sigma \in \Omega(F_n)$, which in turn is equivalent to

$$(U,U,(\sigma(A_1);\sigma(a_1)))\cdots(U,U,(\sigma(A_k);\sigma(a_k)))$$

But this relation already appears in the presentation. For this reason we can discard every relation of types R1-R5 and R8-R10 which is based at a vertex different from U. Now the generators $(V,V,(A;a))$ with $V \neq U$ appear only in the relations of type R6. We will show that we can remove almost all relations of type R6, so that the only ones left are those based at U and those given by

$$(V,V,(A;a)) = (V,U,\sigma_V)^{-1}(U,U,(\sigma_V(A);\sigma_V(a)))(V,U,\sigma_V)$$

with $V \neq U$. Then we will eliminate the generators $(V,V,(A;a))$ along with these relations, so that the only relations of type R6 left are based at U.

First we show that relations

$$(V,V,(A;a)) = (V,V',\tau)^{-1}(V',V',(\tau(A);\tau(a)))(V,V',\tau)$$

with $V,V' \neq U$ and $\tau(V) = V'$ are redundant. To do this we take $\sigma \in \Omega(F_n)$ such that $\sigma(U) = V$. Conjugating by (U,V,σ) and using a relation of type R7 we can replace our relation by

$$(U,V,\sigma)^{-1}(V,V,(A;a))(U,V,\sigma) = (U,V',\tau\sigma)^{-1}(V',V',(\tau(A);\tau(a)))(U,V',\tau\sigma)$$

Now using $(U,U,(\sigma^{-1}(A);\sigma^{-1}(a))) = (U,V,\sigma)^{-1}(V,V,(A;a))(U,V,\sigma)$ we see that this is equivalent to

$$(U,U,(\sigma^{-1}(A);\sigma^{-1}(a))) = (U,V',\tau\sigma)^{-1}(V',V',(\tau(A);\tau(a)))(U,V',\tau\sigma)$$

which is repeated.

Now we show that if $\tau \neq \sigma_V$ satisfies $\tau(V) = U$, the relation

$$(V,V,(A;a)) = (V,U,\tau)^{-1}(U,U,(\tau(A);\tau(a)))(V,U,\tau)$$

is redundant, since it can be replaced by

$$(V,U,\sigma_V)^{-1}(U,U,(\sigma_V(A);\sigma_V(a)))(V,U,\sigma_V) = (V,U,\tau)^{-1}(U,U,(\tau(A);\tau(a)))(V,U,\tau)$$

that can be rewritten (using R7) as

$$(U,U,(\sigma_V(A);\sigma_V(a))) = (U,U,\tau\sigma_V^{-1})^{-1}(U,U,(\tau(A);\tau(a)))(U,U,\tau\sigma_V^{-1})$$

that is a relation of type R6 based at U. Relations

$$(U,U,(A;a)) = (U,V,\tau)^{-1}(V,V,(\tau(A);\tau(a)))(U,V,\tau)$$

are obviously equivalent to those considered before.

In this way we can eliminate the generators corresponding to the edges $(V,V,(A;a))$ and the only relations of type R6 left are those given by loops in U. At this point the only generators left are those labelled with an element of $\Omega(F_n)$ or based at U and the only relators left are those of type R7 or of types R1-R10 based at U. In a similar way we eliminate the generators that are not based at U along with the relations of type R7 not based at U.

To finish we must eliminate the relators of type R8-R10 using that they follow from R1-R7 (see [McC74, 3.]). If $l = 0$ we already know that these relations are redundant. If $l > 1$, the only letters fixed by $(L - \{a^{-1}\};a)$ are a and a^{-1} so in this case there are no relators of type R8-R10. To deal with the case $l = 1$ we must check that the every relation of type R1-R7 used in [McC74, 3.] to check the relation of type R8-R10 we intend to eliminate fixes v_1. For example, if we are eliminating a relation of type R9, we have $b = v_1^{\pm 1}$ so any generator $(X;b)$ or $(X;b^{-1})$ fixes v_1. In addition $(A;a)$ fixes v_1. So every generator in the intermediate steps fixes v_1 and we are done. □

3.1.3 The Reidemeister-Schreier method

To simplify the presentation we change the generating set following Gersten [Ger84].

Theorem 3.1.5. *The group* $\mathrm{Aut}(F_n,\{v_1,\dots,v_l\})$ *has a presentation with generators*

$$\{M_{a,b} : a \in L', b \in L, a \neq b^{\pm 1}\} \cup (\Omega(F_n) \cap \mathrm{Aut}(F_n,\{v_1,\dots,v_l\}))$$

subject to the following relations:

S0. Multiplication table of $\Omega(F_n) \cap \mathrm{Aut}(F_n,\{v_1,\dots,v_l\})$.

S1. $M_{a,b}M_{a,b^{-1}} = 1$.

S2. $[M_{a,b},M_{c,d}] = 1$ *if* $b \neq c^{\pm 1}$, $a \neq d^{\pm 1}$ *and* $a \neq c$.

S3. $[M_{b,a^{-1}},M_{c,b^{-1}}] = M_{c,a}$.

S4. $w_{a,b} = M_{b^{-1},a^{-1}}M_{a^{-1},b}M_{b,a}$.

S5. $\sigma M_{a,b}\sigma^{-1} = M_{\sigma(a),\sigma(b)}$ *if* $\sigma \in \Omega(F_n) \cap \mathrm{Aut}(F_n,\{v_1,\dots,v_l\})$.

Proof. The generators of this presentation are elements of $\mathrm{Aut}(F_n,\{v_1,\dots,v_l\})$ that verify relations S0-S5. We have

$$(A;a) = \prod_{b\in A, b\neq a} E_{b,a} = \prod_{b\in A, b\neq a} M_{b^{-1},a^{-1}}$$

(S2 guarantees the product is well-defined). Therefore, by Theorem 3.1.4 we have a generating set of $\mathrm{Aut}(F_n,\{v_1,\dots,v_l\})$. It suffices to check that R1-R7 can be deduced from S0-S5. The proof is the same as in [Ger84, Theorem 1.2]. □

Theorem 3.1.6 (c.f. [Ger84, Theorem 1.4])**.** *The group* $\mathrm{SAut}(F_n,\{v_1,\dots,v_l\})$ *has a presentation with generators*

$$\{M_{a,b} : a\in L', b\in L, a\neq b^{\pm 1}\}\cup(\Omega(F_n)\cap\mathrm{SAut}(F_n,\{v_1,\dots,v_l\}))$$

subject to the following relations:

S0. Multiplication table of $\Omega(F_n)\cap\mathrm{SAut}(F_n,\{v_1,\dots,v_l\})$.

S1. $M_{a,b}M_{a,b^{-1}}=1$.

S2. $[M_{a,b},M_{c,d}]=1$ *if* $b\neq c^{\pm 1}$, $a\neq d^{\pm 1}$ *and* $a\neq c$.

S3. $[M_{b,a^{-1}},M_{c,b^{-1}}]=M_{c,a}$.

S4. $w_{a,b}=M_{b^{-1},a^{-1}}M_{a^{-1},b}M_{b,a}$.

S5. $\sigma M_{a,b}\sigma^{-1}=M_{\sigma(a),\sigma(b)}$ *if* $\sigma\in\Omega(F_n)\cap\mathrm{SAut}(F_n,\{v_1,\dots,v_l\})$.

Proof. The presentation is obtained applying the Reidemeister-Schreier method to the presentation $\mathcal{P}$ of Theorem 3.1.5. That is, we consider the associated 2-complex $X_{\mathcal{P}}$ and we construct the covering space corresponding to the subgroup $\mathrm{SAut}(F_n,\{v_1,\dots,v_l\})\leq\mathrm{Aut}(F_n,\{v_1,\dots,v_l\})$ which has index 2. This covering has two cells over each cell of $X_{\mathcal{P}}$. In the same way as in the proof of Theorem 3.1.1, we obtain a presentation of its fundamental group and then we eliminate generators and relators until we get the desired presentation. □

Lemma 3.1.7 ([Ger84, Lemma 1.3])**.** *The group* $\Omega(F_n)\cap\mathrm{SAut}(F_n)$ *has a presentation with generators* $\{w_{a,b} : a,b\in L, a\neq b^{\pm 1}\}$ *and relations:*

- $w_{a,b^{-1}}=w_{a,b}^{-1}$
- $w_{a,b}w_{c,d}w_{a,b}^{-1}=w_{w_{a,b}(c),w_{a,b}(d)}$
- $w_{a,b}^4=1$

Corollary 3.1.8. *The group* $\Omega(F_n)\cap\mathrm{SAut}(F_n,\{v_1,\dots,v_l\})$ *has a presentation with generators* $\{w_{a,b} : a,b\in L', a\neq b^{\pm 1}\}$ *and the following relations:*

- $w_{a,b^{-1}}=w_{a,b}^{-1}$
- $w_{a,b}w_{c,d}w_{a,b}^{-1}=w_{w_{a,b}(c),w_{a,b}(d)}$

- $w_{a,b}^4 = 1$

Proof. This follows immediately from the previous lemma using $\Omega(F_n) \cap \mathrm{SAut}(F_n, \{v_1, \ldots, v_l\}) \simeq \Omega(F_{n-l}) \cap \mathrm{SAut}(F_{n-l})$. □

Theorem 3.1.9. *The group* $\mathrm{SAut}(F_n, \{v_1, \ldots, v_l\})$ *has a presentation with generators*

$$\{M_{a,b} : a \in L', b \in L, a \neq b^{\pm 1}\} \cup \{w_{a,b} : a, b \in L', a \neq b^{\pm 1}\}$$

subject to the following relations

(1) $M_{a,b} M_{a,b^{-1}} = 1$.

(2) $[M_{a,b}, M_{c,d}] = 1$ *if* $b \neq c^{\pm 1}$, $a \neq d^{\pm 1}$ *and* $a \neq c$.

(3) $[M_{b,a^{-1}}, M_{c,b^{-1}}] = M_{c,a}$.

(4) $w_{a,b} = M_{b^{-1},a^{-1}} M_{a^{-1},b} M_{b,a}$.

(5') $w_{a,b} M_{c,d} w_{a,b}^{-1} = M_{w_{a,b}(c), w_{a,b}(d)}$.

(6) $w_{a,b}^4 = 1$.

Proof. This presentation can be obtained from Theorem 3.1.6 using Corollary 3.1.8. □

Now we state and prove the main result of this section.

Theorem 3.1.10. *If* $n - l \geq 3$, *the group* $\mathrm{SAut}(F_n, \{v_1, \ldots, v_l\})$ *has a presentation with generators*

$$\{M_{a,b} : a \in L', b \in L, a \neq b^{\pm 1}\} \cup \{w_{a,b} : a, b \in L', a \neq b^{\pm 1}\}$$

subject to the following relations:

1. $M_{a,b} M_{a,b^{-1}} = 1$.
2. $[M_{a,b}, M_{c,d}] = 1$ *if* $b \neq c^{\pm 1}$, $a \neq d^{\pm 1}$ *and* $a \neq c$.
3. $[M_{b,a^{-1}}, M_{c,b^{-1}}] = M_{c,a}$.
4. $w_{a,b} = M_{b^{-1},a^{-1}} M_{a^{-1},b} M_{b,a}$.
5. $w_{a,b} = w_{a^{-1},b^{-1}}$.
6. $w_{a,b}^4 = 1$.

Proof. We only need to show that relation (5') follows from relations (1)-(6). To do this we do the same computations as in [Ger84, Proof of Theorem 2.7]. We separate in cases. In each case different letters represent elements which belong to different orbits of the action of $\mathbb{Z}_2$ on L given by $x \mapsto x^{-1}$.

- Cases with four orbits.
  - $w_{a,b}M_{c,d}w_{a,b}^{-1} = M_{w_{a,b}(c),w_{a,b}(d)}$

$$\begin{aligned}
w_{a,b}M_{c,d}w_{a,b}^{-1} &= M_{b^{-1},a^{-1}}M_{a^{-1},b}M_{b,a}M_{c,d}w_{a,b}^{-1}\\
\text{(Using 2)} &= M_{b^{-1},a^{-1}}M_{a^{-1},b}M_{c,d}M_{b,a}w_{a,b}^{-1}\\
\text{(Using 2)} &= M_{b^{-1},a^{-1}}M_{c,d}M_{a^{-1},b}M_{b,a}w_{a,b}^{-1}\\
\text{(Using 2)} &= M_{c,d}M_{b^{-1},a^{-1}}M_{a^{-1},b}M_{b,a}w_{a,b}^{-1}\\
&= M_{c,d}w_{a,b}w_{a,b}^{-1}\\
&= M_{c,d}\\
&= M_{w_{a,b}(c),w_{a,b}(d)}
\end{aligned}$$

- Cases with three orbits.

  In every case we use $w_{a,b} = w_{a^{-1},b^{-1}} = M_{b,a}M_{a,b^{-1}}M_{b^{-1},a^{-1}}$.
  - $w_{a,b}M_{a,d}w_{a,b}^{-1} = M_{w_{a,b}(a),w_{a,b}(d)}$

$$\begin{aligned}
w_{a,b}M_{a,d}w_{a,b}^{-1} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},a^{-1}}M_{a,d}M_{b^{-1},a}M_{a,b}M_{b,a^{-1}}\\
\text{(Using 3)} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},d}M_{a,d}M_{b^{-1},a^{-1}}M_{b^{-1},a}M_{a,b}M_{b,a^{-1}}\\
\text{(Using 1)} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},d}M_{a,d}M_{a,b}M_{b,a^{-1}}\\
\text{(Using 2)} &= M_{b,a}M_{a,b^{-1}}M_{a,d}M_{b^{-1},d}M_{a,b}M_{b,a^{-1}}\\
\text{(Using 3)} &= M_{b,a}M_{a,b^{-1}}M_{a,d}M_{a,d^{-1}}M_{a,b}M_{b^{-1},d}M_{b,a^{-1}}\\
\text{(Using 1)} &= M_{b,a}M_{a,b^{-1}}M_{a,b}M_{b^{-1},d}M_{b,a^{-1}}\\
\text{(Using 1)} &= M_{b,a}M_{b^{-1},d}M_{b,a^{-1}}\\
\text{(Using 2)} &= M_{b,a}M_{b^{-1},d}M_{b,a^{-1}}\\
\text{(Using 2)} &= M_{b^{-1},d}M_{b,a}M_{b,a^{-1}}\\
\text{(Using 1)} &= M_{b^{-1},d}\\
&= M_{w_{a,b}(a),w_{a,b}(d)}
\end{aligned}$$

  - $w_{a,b}M_{a^{-1},d}w_{a,b}^{-1} = M_{w_{a,b}(a^{-1}),w_{a,b}(d)}$
    By (5) this case follows from the previous one.
  - $w_{a,b}M_{c,a}w_{a,b}^{-1} = M_{w_{a,b}(c),w_{a,b}(a)}$
    By (5) this case follows from the next one.

- $w_{a,b}M_{c,a^{-1}}w_{a,b}^{-1} = M_{w_{a,b}(c),w_{a,b}(a^{-1})}$

$$\begin{aligned}
w_{a,b}M_{c,a^{-1}}w_{a,b}^{-1} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},a^{-1}}M_{c,a^{-1}}M_{b^{-1},a}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 2)} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},a^{-1}}M_{b^{-1},a}M_{c,a^{-1}}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 1)} &= M_{b,a}M_{a,b^{-1}}M_{c,a^{-1}}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 3)} &= M_{b,a}M_{c,b}M_{c,a^{-1}}M_{a,b^{-1}}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 1)} &= M_{b,a}M_{c,b}M_{c,a^{-1}}M_{b,a^{-1}} \\
\text{(Using 2)} &= M_{b,a}M_{c,b}M_{b,a^{-1}}M_{c,a^{-1}} \\
\text{(Using 3)} &= M_{c,b}M_{b,a}M_{c,a}M_{b,a^{-1}}M_{c,a^{-1}} \\
\text{(Using 2)} &= M_{c,b} \\
&= M_{w_{a,b}(c),w_{a,b}(a^{-1})}
\end{aligned}$$

- $w_{a,b}M_{b,d}w_{a,b}^{-1} = M_{w_{a,b}(b),w_{a,b}(d)}$

$$\begin{aligned}
w_{a,b}M_{b,d}w_{a,b}^{-1} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},a^{-1}}M_{b,d}M_{b^{-1},a}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 2)} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},a^{-1}}M_{b^{-1},a}M_{b,d}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 1)} &= M_{b,a}M_{a,b^{-1}}M_{b,d}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 3)} &= M_{b,a}M_{a,b^{-1}}M_{a,b}M_{b,d}M_{a,d}M_{b,a^{-1}} \\
\text{(Using 1)} &= M_{b,a}M_{b,d}M_{a,d}M_{b,a^{-1}} \\
\text{(Using 3)} &= M_{b,a}M_{b,d}M_{b,d^{-1}}M_{b,a^{-1}}M_{a,d} \\
\text{(Using 1)} &= M_{b,a}M_{b,a^{-1}}M_{a,d} \\
\text{(Using 1)} &= M_{a,d} \\
&= M_{w_{a,b}(a),w_{a,b}(d)}
\end{aligned}$$

- $w_{a,b}M_{b^{-1},d}w_{a,b}^{-1} = M_{w_{a,b}(b^{-1}),w_{a,b}(d)}$
  By (5) this case follows from the previous one.
- $w_{a,b}M_{c,b}w_{a,b}^{-1} = M_{w_{a,b}(c),w_{a,b}(b)}$

$$\begin{aligned}
w_{a,b}M_{c,b}w_{a,b}^{-1} &= M_{b,a}M_{a,b^{-1}}M_{b^{-1},a^{-1}}M_{c,b}M_{b^{-1},a}M_{a,b}M_{b,a^{-1}} \\
\text{(Using 3)} &= M_{b,a}M_{a,b^{-1}}M_{c,a}M_{c,b}M_{b^{-1},a^{-1}}M_{b^{-1},a}M_{a,b}M_{b,a^{-1}}
\end{aligned}$$

$$
\begin{aligned}
(\text{Using } 1) &= M_{b,a}M_{a,b^{-1}}M_{c,a}M_{c,b}M_{a,b}M_{b,a^{-1}} \\
(\text{Using } 2) &= M_{b,a}M_{a,b^{-1}}M_{c,a}M_{a,b}M_{c,b}M_{b,a^{-1}} \\
(\text{Using } 3) &= M_{b,a}M_{c,a}M_{a,b^{-1}}M_{c,b^{-1}}M_{a,b}M_{c,b}M_{b,a^{-1}} \\
(\text{Using } 2) &= M_{b,a}M_{c,a}M_{a,b^{-1}}M_{a,b}M_{c,b^{-1}}M_{c,b}M_{b,a^{-1}} \\
(\text{Using } 1) &= M_{b,a}M_{c,a}M_{c,b^{-1}}M_{c,b}M_{b,a^{-1}} \\
(\text{Using } 1) &= M_{b,a}M_{c,a}M_{b,a^{-1}} \\
(\text{Using } 2) &= M_{c,a}M_{b,a}M_{b,a^{-1}} \\
(\text{Using } 1) &= M_{c,a} \\
&= M_{w_{a,b}(c),w_{a,b}(b)}
\end{aligned}
$$

- $w_{a,b}M_{c,b^{-1}}w_{a,b}^{-1} = M_{w_{a,b}(c),w_{a,b}(b^{-1})}$
  By (5) this case follows from the previous one.

• Cases with two orbits

- $w_{a,b}M_{a,b}w_{a,b}^{-1} = M_{w_{a,b}(a),w_{a,b}(b)}$
  We consider $c \in L'$ such that the orbit of c is different from the orbits of a and b. Using $M_{a,b} = [M_{c,b^{-1}}, M_{a,c^{-1}}]$ we have

$$
\begin{aligned}
w_{a,b}M_{a,b}w_{a,b}^{-1} &= w_{a,b}[M_{c,b^{-1}}, M_{a,c^{-1}}]w_{a,b}^{-1} \\
&= [w_{a,b}M_{c,b^{-1}}w_{a,b}^{-1}, w_{a,b}M_{a,c^{-1}}w_{a,b}^{-1}] \\
(\text{Case of 3 orbits already proved}) &= [M_{w_{a,b}(c),w_{a,b}(b^{-1})}, M_{w_{a,b}(a),w_{a,b}(c^{-1})}] \\
(\text{Using } 3) &= M_{w_{a,b}(a),w_{a,b}(b)}
\end{aligned}
$$

- $w_{a,b}M_{a,b^{-1}}w_{a,b}^{-1} = M_{w_{a,b}(a),w_{a,b}(b^{-1})}$
  This follows from the previous case taking inverse at both sides.
- $w_{a,b}M_{a^{-1},b}w_{a,b}^{-1} = M_{w_{a,b}(a^{-1}),w_{a,b}(b)}$
  Using (5) this is equivalent to $w_{a^{-1},b^{-1}}M_{a^{-1},b}w_{a^{-1},b^{-1}}^{-1} = M_{w_{a^{-1},b^{-1}}(a^{-1}),w_{a^{-1},b^{-1}}(b)}$. But this is the previous case applied to a^{-1} and b^{-1}.
- $w_{a,b}M_{a^{-1},b^{-1}}w_{a,b}^{-1} = M_{w_{a,b}(a^{-1}),w_{a,b}(b^{-1})}$
  This follows from the previous case taking inverse at both sides.
- $w_{a,b}M_{b,a}w_{a,b}^{-1} = M_{w_{a,b}(b),w_{a,b}(a)}$
  We consider $c \in L'$ such that the orbit of c is different from the orbits of a and b. Using $M_{b,a} = [M_{c,a^{-1}}, M_{b,c^{-1}}]$ we have

$$
w_{a,b}M_{b,a}w_{a,b}^{-1} = w_{a,b}[M_{c,a^{-1}}, M_{b,c^{-1}}]w_{a,b}^{-1}
$$

$$
\begin{aligned}
&= [w_{a,b}M_{c,a^{-1}}w_{a,b}^{-1}, w_{a,b}M_{b,c^{-1}}w_{a,b}^{-1}] \\
\text{(Case of 3 orbits already proved)} &= [M_{w_{a,b}(c),w_{a,b}(a^{-1})}, M_{w_{a,b}(b),w_{a,b}(c^{-1})}] \\
&= M_{w_{a,b}(b),w_{a,b}(a)}
\end{aligned}
$$

- $w_{a,b}M_{b,a^{-1}}w_{a,b}^{-1} = M_{w_{a,b}(b),w_{a,b}(a^{-1})}$
  This follows from the previous case taking inverse at both sides.
- $w_{a,b}M_{b^{-1},a}w_{a,b}^{-1} = M_{w_{a,b}(b^{-1}),w_{a,b}(a)}$
  Using (5) this is equivalent to $w_{a^{-1},b^{-1}}M_{b^{-1},a}w_{a^{-1},b^{-1}}^{-1} = M_{w_{a^{-1},b^{-1}}(b^{-1}),w_{a^{-1},b^{-1}}(a)}$. But this is the previous case applied to a^{-1} and b^{-1}.
- $w_{a,b}M_{b^{-1},a^{-1}}w_{a,b}^{-1} = M_{w_{a,b}(b^{-1}),w_{a,b}(a^{-1})}$
  This follows from the previous case taking inverse at both sides.

□

3.2 The links are 1-connected

Recall that if G is a group and $\mathcal{S} \subseteq G$ is a generating set, the *Cayley graph* $\mathrm{Cay}(G,\mathcal{S})$ is the graph with vertex set G and an edge $\{g,g'\}$ whenever there exists $s \in \mathcal{S}$ such that $g' = gs$. If $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a presentation of G, then $\mathrm{Cay}(G,\mathcal{S})$ can be identified with the 1-skeleton of the universal covering space of the presentation complex $X_{\mathcal{P}}$.

Theorem 3.2.1 (c.f. [DP13, Theorem A])**.** *Let B be a partial basis of F_n. The complex $\mathrm{lk}(B,\mathrm{PB}(F_n))$ is connected for $n-|B| \geq 2$ and 1-connected for $n-|B| \geq 3$.*

Proof. The same proof of [DP13, Theorem A] works with subtle changes. We extend $B = \{v_1,\dots,v_l\}$ to a basis $\{v_1,\dots,v_n\}$ of F_n. Let $\mathcal{P} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be the presentation from Theorem 3.1.10. We know that $\mathcal{S}$ is a generating set for $\mathrm{SAut}(F_n,B)$, even if $n-|B| = 2$ (for example, by Theorem 3.1.9).

We define a cellular $\mathrm{SAut}(F_n,B)$-equivariant map $\Phi\colon \mathrm{Cay}(\mathrm{SAut}(F_n,B),\mathcal{S}) \to \mathrm{lk}(B,\mathrm{PB}(F_n))$. If f is a 0-cell, $\Phi(f) = f(v_{l+1})$. Now we have to define Φ on the 1-cells. If $f - fs$ is a 1-cell there are three cases:

- $v_{l+1} = s(v_{l+1})$. In this case Φ maps the entire 1-cell to $f(v_{l+1})$.
- $B \cup \{v_{l+1}, s(v_{l+1})\}$ is a partial basis. In this case Φ maps the 1-cell homeomorphically to the edge $\{f(v_{l+1}), fs(v_{l+1})\}$.
- $s = M^{e''}_{v^{e}_{l+1},v^{e'}_{i}}$ for some $1 \leq i \leq l$ and $e,e',e'' \in \{1,-1\}$. In this case $v_{l+2} = s(v_{l+2})$ and the image of the 1-cell is the edge-path $f(v_{l+1}) - f(v_{l+2}) - fs(v_{l+1})$.

Note that this is well-defined, since the definitions for $f-fs$ and $fs-f$ agree.

If $n-|B|\geq 2$, the action of $\mathrm{SAut}(F_n,B)$ on the vertex set of $\mathrm{lk}(B,\mathrm{PB}(F_n))$ is transitive and the image of Φ contains every vertex of $\mathrm{lk}(B,\mathrm{PB}(F_n))$. Since $\mathrm{Cay}(\mathrm{SAut}(F_n,B),\mathcal{S})$ is connected we conclude that $\mathrm{lk}(B,\mathrm{PB}(F_n))$ is connected when $n-|B|\geq 2$.

Now suppose $n-|B|\geq 3$. To prove that $\mathrm{lk}(B,\mathrm{PB}(F_n))$ is 1-connected, we will show that

$$\Phi_*\colon \pi_1(\mathrm{Cay}(\mathrm{SAut}(F_n,B),\mathcal{S}),1)\to\pi_1(\mathrm{lk}(B,\mathrm{PB}(F_n)),v_{l+1})$$

is surjective and has trivial image.

Claim 1. *The map* $\Phi_*\colon \pi_1(\mathrm{Cay}(\mathrm{SAut}(F_n,B),\mathcal{S}),1)\to\pi_1(\mathrm{lk}(B,\mathrm{PB}(F_n)),v_{l+1})$ *has trivial image.*

We will show that Φ extends to the universal cover $\widetilde{X}_{\mathcal{P}}$ of the presentation 2-complex $X_{\mathcal{P}}$ of $\mathcal{P}$. To prove this it is enough to prove that for every relation in $\mathcal{R}$, the image by Φ of every attaching map (in $\widetilde{X}_{\mathcal{P}}$) associated to that relation is null-homotopic. Since Φ is equivariant, it is enough to prove this for the lifts at 1.

Let $s_1\cdots s_k=1$ be a relation in $\mathcal{R}$. The associated closed edge path is

$$1-s_1-s_1s_2-\cdots-s_1s_2\cdots s_k=1$$

and its image by Φ_* is the concatenation of the paths

$$\Phi_*(s_1\cdots s_{i-1}-s_1\cdots s_{i-1}s_i)=s_1\cdots s_{i-1}\Phi_*(1-s_i)$$

for $i=1,\dots,k$.

Inspecting the relations in $\mathcal{R}$ we see that there exists $x\in\{v_{l+1},\dots,v_n\}$ such that $s_i(x)=x$ for $1\leq i\leq k$ (here we use $n-|B|\geq 3$). Therefore we have that either v_{l+1} equals x or these vertices are joined by an edge. Hence, if $1\leq i\leq k$, $s_1\cdots s_i(v_{l+1})$ and $s_1\cdots s_i(x)=x$ are either equal or joined by an edge. Therefore it suffices to show that the closed edge path

$$x-s_1\cdots s_{i-1}\Phi_*(1-s_i)-x$$

is trivial for every i. Using the action of $\mathrm{SAut}(F_n,B)$, it is enough to show the loops

$$x-\Phi_*(1-s_i)-x$$

are trivial. We separate in cases:

- If $v_{l+1}=s_i(v_{l+1})$ it is immediate.
- If $B\cup\{v_{l+1},s(v_{l+1})\}$ is a partial basis we have two cases.
  - If $B\cup\{v_{l+1},s_i(v_{l+1}),x\}$ is a partial basis it is immediate.

- If $B \cup \{v_{l+1}, s_i(v_{l+1}), x\}$ is not a partial basis, inspecting $\mathcal{S}$, we see that $s_i = M^{e''}_{v^{e}_{l+1}, x^{e'}}$ for certain $e, e', e'' \in \{1, -1\}$. Considering $y \in \{v_{l+1}, \ldots, v_n\}$ distinct from x and v_{l+1} we conclude that the loop $x - v_{l+1} - s_i(v_{l+1}) - x$ contracts to y.

- If $s_i = M^{e''}_{v^{e}_{l+1}, v^{e'}_i}$ for some $1 \leq i \leq l$ and $e, e', e'' \in \{1, -1\}$, we have to show that the loop $x - v_{l+1} - v_{l+2} - s_i(v_{l+1}) - x$ is null-homotopic. Again we have two cases.
  - If $x = v_{l+2}$ it is immediate.
  - If $x \neq v_{l+2}$, then $\{x, v_{l+1}, v_{l+2}\}$ is a 2-simplex. Additionally $s_i(v_{l+1}) = (v^{e'}_i v^{e}_{l+1})^{ee''}$ therefore $\{x, v_{l+2}, s_i(v_{l+1})\}$ is also a 2-simplex and we are done.

Claim 2. *The map* $\Phi_*\colon \pi_1(\mathrm{Cay}(\mathrm{SAut}(F_n, B), \mathcal{S}), 1) \to \pi_1(\mathrm{lk}(B, \mathrm{PB}(F_n)), v_{l+1})$ *is surjective.*

Let $u_0 - u_1 - \ldots - u_k$ be a closed edge path in $\mathrm{lk}(B, \mathrm{PB}(F_n))$, with $u_0 = u_k = v_{l+1}$. We will show that it is in the image of Φ_*. For $0 \leq i < k$, we have that $B \cup \{u_i, u_{i+1}\}$ is a partial basis of F_n.

Next we inductively define elements $\phi_1, \ldots, \phi_k \in \mathrm{SAut}(F_n, B \cup \{v_{l+1}\})$. Since $B \cup \{v_{l+1}, u_1\}$ is a partial basis and $n - l \geq 3$, there is $\phi_1 \in \mathrm{SAut}(F_n, B \cup \{v_{l+1}\})$ such that $\phi_1(v_{l+2}) = u_1$. Hence $u_1 = \phi_1 w_{v_{l+2}, v_{l+1}}(v_{l+1})$.

Now suppose we have defined ϕ_i so that $u_i = (\phi_1 w_{v_{l+2}, v_{l+1}}) \cdots (\phi_i w_{v_{l+2}, v_{l+1}})(v_{l+1})$. Since $B \cup \{u_i, u_{i+1}\}$ is a partial basis, applying the inverse of $(\phi_1 w_{v_{l+2}, v_{l+1}}) \cdots (\phi_i w_{v_{l+2}, v_{l+1}})$ we see that

$$B \cup \{v_{l+1}, (\phi_i w_{v_{l+2}, v_{l+1}})^{-1} \cdots (\phi_1 w_{v_{l+2}, v_{l+1}})^{-1}(u_{i+1})\}$$

is a partial basis and hence there is $\phi_{i+1} \in \mathrm{SAut}(F_n, B \cup \{v_{l+1}\})$ such that

$$\phi_{i+1}(v_{l+2}) = (\phi_i w_{v_{l+2}, v_{l+1}})^{-1} \cdots (\phi_1 w_{v_{l+2}, v_{l+1}})^{-1}(u_{i+1}).$$

Equivalently, $u_{i+1} = (\phi_1 w_{v_{l+2}, v_{l+1}}) \cdots (\phi_{i+1} w_{v_{l+2}, v_{l+1}})(v_{l+1})$. We define

$$\phi_{k+1} = ((\phi_1 w_{v_{l+2}, v_{l+1}}) \cdots (\phi_k w_{v_{l+2}, v_{l+1}}))^{-1}.$$

Since $(\phi_1 w_{v_{l+2}, v_{l+1}}) \cdots (\phi_k w_{v_{l+2}, v_{l+1}})(v_{l+1}) = u_k = v_{l+1}$, we have $\phi_{k+1} \in \mathrm{SAut}(F_n, B \cup \{v_{l+1}\})$.

For every $1 \leq i \leq k+1$, we can find $s^i_1, \ldots, s^i_{m_i} \in \mathcal{S}^{\pm 1}$ that additionally fix v_{l+1} and such that

$$\phi_i = s^i_1 \cdots s^i_{m_i}.$$

We have $(s^1_1 \cdots s^1_{m_1}) w_{v_{l+2}, v_{l+1}} \cdots w_{v_{l+2}, v_{l+1}} (s^{k+1}_1 \cdots s^{k+1}_{m_{k+1}}) = 1$. Therefore there is a closed edge path in $\mathrm{Cay}(\mathrm{SAut}(F_n), \mathcal{S})$ whose image by Φ_* is

$$v_{l+1} - s^1_1(v_{l+1}) - s^1_1 s^1_2(v_{l+1}) - \cdots - s^1_1 \cdots s^1_{m_1}(v_{l+1}) - s^1_1 \cdots s^1_{m_1} w_{v_{l+2}, v_{l+1}}(v_{l+1}) - \cdots$$

Since $s^i_j(v_{l+1}) = v_{l+1}$, for every $1 \leq i \leq k+1$ and $1 \leq j \leq m_i$, after deleting repeated vertices this path equals

$$v_{l+1} - (s^1_1 s^1_2 \cdots s^1_{m_1} w_{v_{l+2}, v_{l+1}})(v_{l+1}) - (s^1_1 s^1_2 \cdots s^1_{m_1} w_{v_{l+2}, v_{l+1}})(s^2_1 s^2_2 \cdots s^2_{m_2} w_{v_{l+2}, v_{l+1}})(v_{l+1}) - \cdots$$

$$-(s_1^1 s_2^1 \cdots s_{m_1}^1 w_{v_{l+2},v_{l+1}})(s_1^k s_2^k \cdots s_{m_k}^k w_{v_{l+2},v_{l+1}})(v_{l+1}) - v_{l+1}$$

which is precisely $u_0 - u_1 - \cdots - u_k$. $\square$

3.3 A variant of Quillen's result

Recall that if X is a poset, $\mathcal{K}(X)$ denotes the order complex of X, that is the simplicial complex with simplices the chains of X. Both the face poset and the order complex are functorial. If K is a simplicial complex, $\mathcal{X}(K)$ denotes the face poset of K, that is the poset of simplices of K ordered by inclusion. The complex $\mathcal{K}(\mathcal{X}(K))$ is the barycentric subdivision K' of K. Throughout the chapter we consider homology with integer coefficients and $\widetilde{C}_\bullet(K)$ is the augmented simplicial chain complex. Let $\lambda : \widetilde{C}_\bullet(K) \to \widetilde{C}_\bullet(K')$ be the subdivision operator $\alpha \mapsto \alpha'$. If X is a poset X^{op} denotes the poset X with the opposite order and we write $\widetilde{H}_\bullet(X)$ for the homology $\widetilde{H}_\bullet(\mathcal{K}(X))$. We thus have $\widetilde{H}_\bullet(X) = \widetilde{H}_\bullet(X^{\mathrm{op}})$. Recall that if X is a poset and $x \in X$ the *height* of x denoted $h(x)$ is the dimension of $\mathcal{K}(X_{\leq x})$. If K is a simplicial complex we can identify $\mathcal{X}(\mathrm{lk}(\sigma, K)) = \mathcal{X}(K)_{>\sigma}$ by the map $\tau \mapsto \sigma \cup \tau$. If K_1, K_2 are simplicial complexes we have $\widetilde{C}_\bullet(K_1 * K_2) = \widetilde{C}_\bullet(K_1) * \widetilde{C}_\bullet(K_2)$ (here $*$ denotes the join of chain complexes, defined as the suspension of the tensor product). Recall that the join of two posets X_1, X_2 is the disjoint union of X_1 and X_2 keeping the given ordering within X_1 and X_2 and setting $x_1 \leq x_2$ for every $x_1 \in X_1$ and $x_2 \in X_2$ [Bar11, Definition 2.7.1]. We have $\mathcal{K}(X_1 * X_2) = \mathcal{K}(X_1) * \mathcal{K}(X_2)$. If X is a poset and $x \in X$, then $\mathrm{lk}(x, X) = X_{<x} * X_{>x}$ is the subposet of X consisting of elements that can be compared with x. We have $\mathrm{lk}(x, \mathcal{K}(X)) = \mathcal{K}(\mathrm{lk}(x, X))$.

Definition 3.3.1. Let $f : X \to Y$ be an order preserving map. The *non-Hausdorff mapping cylinder* $M(f)$ is the poset given by the following order on the disjoint union of X and Y. We keep the given ordering within X and Y and for $x \in X$, $y \in Y$ we set $x \leq y$ in $M(f)$ if $f(x) \leq y$ in Y.

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & M(f) \\ & Y \arrow[u, hook, "i"'] \end{tikzcd}$$

If $j : X \to M(f)$, $i : Y \to M(f)$ are the inclusions, then $\mathcal{K}(i)$ is a homotopy equivalence. Since $j \leq if$ we also have $\mathcal{K}(j) \simeq \mathcal{K}(if)$. For more details on this construction see [Bar11, 2.8].

Definition 3.3.2. A simplicial complex K is said to be *n-spherical* if $\dim(K) = n$ and K is $(n-1)$-connected. We say that K is *homologically n-spherical* if $\dim(K) = n$ and $\widetilde{H}_i(K) = 0$ for every $i < n$. Recall that K is *Cohen-Macaulay* if K is n-spherical and the link $\mathrm{lk}(\sigma, K)$ is $(n - \dim(\sigma) - 1)$-spherical for every simplex $\sigma \in K$. A poset X is *(homologically) n-spherical* if $\mathcal{K}(X)$ is (homologically) n-spherical.

Recall that if $f\colon X \to Y$ is a map of posets, the *fiber of f under y* is the subposet $f/y = \{x : f(x) \leq y\} \subseteq X$.

Definition 3.3.3. An order preserving map $f\colon X \to Y$ is *(homologically) n-spherical*, if $Y_{>y}$ is (homologically) $(n-h(y)-1)$-spherical and f/y is (homologically) $h(y)$-spherical for all $y \in Y$.

Proposition 3.3.4. *Let $f\colon X \to Y$ be homologically n-spherical. Then for every $x \in X$ we have $h(f(x)) \geq h(x)$.*

Proof. Let $y = f(x)$. Since $x \in f/y$ and f/y is homologically $h(y)$-spherical we have $h(x) \leq \dim(f/y) = h(y)$. □

Proposition 3.3.5. *A homologically n-spherical map $f\colon X \to Y$ is surjective.*

Proof. Let $y \in Y$ and let $r = h(y)$. Since f/y is homologically r-spherical, $\dim(f/y) = r$. So there is $x \in f/y$ with $h(x) = r$. Let $\widetilde{y} = f(x)$. We obviously have $\widetilde{y} \leq y$. By Proposition 3.3.4 we have $h(\widetilde{y}) \geq h(x) = r$. Therefore we have $\widetilde{y} = y$. □

From the definition of spherical map we also have the following:

Proposition 3.3.6. *If $f\colon X \to Y$ is homologically n-spherical then* $\dim(X) = \dim(Y) = n$*.*

The first part of the following result is due to Quillen [Qui78, Theorem 9.1]. To prove the second part we build on the proof of the first part given by Piterman [Pit16, Teorema 2.1.28]. The idea of considering the non-Hausdorff mapping cylinder of $f\colon X \to Y$ and removing the points of Y from bottom to top is originally due to Barmak and Minian [BM08].

Theorem 3.3.7. *Let $f\colon X \to Y$ be a homologically n-spherical map between posets such that Y is homologically n-spherical. Then X is homologically n-spherical, $f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ is an epimorphism and*

$$\widetilde{H}_n(X) \simeq \widetilde{H}_n(Y) \bigoplus_{y \in Y} \widetilde{H}_{h(y)}(f/y) \otimes \widetilde{H}_{n-h(y)-1}(Y_{>y}).$$

Moreover suppose that $X = \mathcal{X}(K)$ for certain simplicial complex K and
(i) If $f(\sigma_1) \leq f(\sigma_2)$ then $\mathrm{lk}(\sigma_2, K) \subseteq \mathrm{lk}(\sigma_1, K)$*.*
(ii) If $f(\sigma_1) \leq f(\sigma_2)$ and $f(\tau_1) \leq f(\tau_2)$ then $f(\sigma_1 \cup \tau_1) \leq f(\sigma_2 \cup \tau_2)$, whenever $\sigma_1 \cup \tau_1, \sigma_2 \cup \tau_2 \in K$.
(iii) For every $y \in Y$ and every $\sigma \in f^{-1}(y)$, the map $f_*\colon \widetilde{H}_{n-h(y)-1}(X_{>\sigma}) \to \widetilde{H}_{n-h(y)-1}(Y_{>y})$ is an epimorphism.

Then we can produce a basis of $\widetilde{H}_n(K)$ as follows. Since f_* is an epimorphism, we can take $\{\gamma_i\}_{i \in I} \subseteq \widetilde{H}_n(K)$ such that $\{f_*(\gamma_i')\}_{i \in I}$ is a basis of $\widetilde{H}_n(Y)$. In addition, for every $y \in Y$ we *choose $x \in f^{-1}(y)$ and we consider the subcomplexes $K_y = \{\sigma : f(\sigma) \leq y\}$ and $K^y = \mathrm{lk}(x,K)$. By (i), K^y does not depend on the choice of x. Also by (i), $K_y * K^y$ is a subcomplex of K. Let $\widetilde{f}\colon \mathcal{X}(K^y) \to Y_{>y}$ be defined by $\widetilde{f}(\tau) = f(x \cup \tau)$. By (ii), $\widetilde{f}$ does not depend on the choice of x. We take a basis $\{\alpha_i\}_{i\in I_y}$ of $\widetilde{H}_{h(y)}(K_y)$ and using (iii) we take $\{\beta_j\}_{j\in J_y} \subseteq \widetilde{H}_{n-h(y)-1}(K^y)$ such that $\{\widetilde{f}_*(\beta_j')\}_{j\in J_y}$ is a basis of $\widetilde{H}_{n-h(y)-1}(Y_{>y})$. Then*

$$\{\gamma_i : i \in I\} \cup \{\alpha_i * \beta_j : y \in Y, i \in I_y, j \in J_y\}$$

is a basis of $\widetilde{H}_n(K)$.

Proof. Let $M = M(f)$ be the non-Hausdorff mapping cylinder of f and let $j\colon X \to M$, $i\colon Y \to M$ be the inclusions. We have $j_* = i_* f_*$. Since f is n-spherical we have $\dim(M) = n+1$.

Let $Y_r = \{y \in Y : h(y) \geq r\}$. For each r we consider the subspace $M_r = X \cup Y_r$ of M. We have $M_{n+1} = X$ and $M_0 = M$. Let

$$L_r = \coprod_{h(y)=r} \mathrm{lk}(y, M_r) = \coprod_{h(y)=r} f/y * Y_{>y}.$$

For each r we consider the Mayer-Vietoris sequence for the open covering $\{U,V\}$ of $\mathcal{K}(M_{r-1})$ given by

$$\begin{aligned} U &= \mathcal{K}(M_{r-1}) - \{y \in Y : h(y) = r-1\} \\ V &= \bigcup_{h(y)=r-1} \mathrm{st}(y, \mathcal{K}(M_{r-1})) \end{aligned}$$

where $\mathrm{st}(v,K)$ denotes the open star of v in K. We have homotopy equivalences $U \simeq \mathcal{K}(M_r)$ and $U \cap V \simeq \mathcal{K}(L_r)$. Since f is a homologically n-spherical map $\mathrm{lk}(y, M_{r-1})$ is homologically n-spherical, so the homology of L_r is concentrated in degrees 0 and n. The tail of the sequence is $0 \to \widetilde{H}_{n+1}(M_r) \to \widetilde{H}_{n+1}(M_{r-1})$ and since $\widetilde{H}_{n+1}(M_0) = \widetilde{H}_{n+1}(Y) = 0$ we have $\widetilde{H}_{n+1}(M_r) = 0$ for every r. We also have isomorphisms $\widetilde{H}_i(M_r) \to \widetilde{H}_i(M_{r-1})$ if $0 \leq i \leq n-1$ (since L_r may not be connected, we have to take some care when $i = 0,1$). From this we conclude that X is homologically n-spherical and we also have short exact sequences

$$0 \to \widetilde{H}_n(L_n) \xrightarrow{i_{n+1}} \widetilde{H}_n(X) \xrightarrow{p_{n+1}} \widetilde{H}_n(M_n) \to 0$$

$$\dots$$

$$0 \to \widetilde{H}_n(L_{r-1}) \xrightarrow{i_r} \widetilde{H}_n(M_r) \xrightarrow{p_r} \widetilde{H}_n(M_{r-1}) \to 0$$

$$\dots$$

$$0 \to \widetilde{H}_n(L_0) \xrightarrow{i_1} \widetilde{H}_n(M_1) \xrightarrow{p_1} \widetilde{H}_n(M) \to 0.$$

Here the map i_r is the map induced by the map $L_{r-1} \to M_r$ given by the coproduct of the inclusions $\mathrm{lk}(y, M_{r-1}) \to M_r$ and the map p_r is induced by the inclusion $M_r \to M_{r-1}$. By induction on r, it follows that these sequences are split and that $\widetilde{H}_n(M_r)$ is free for every r. We have

$$\widetilde{H}_n(L_r) = \bigoplus_{h(y)=r} \widetilde{H}_r(f/y) \otimes \widetilde{H}_{n-r-1}(Y_{>y})$$

and therefore using the isomorphism $i_* \colon \widetilde{H}_n(Y) \to \widetilde{H}_n(M)$ we obtain

$$\widetilde{H}_n(X) = \widetilde{H}_n(Y) \bigoplus_{y \in Y} \widetilde{H}_{h(y)}(f/y) \otimes \widetilde{H}_{n-h(y)-1}(Y_{>y}).$$

Now $\widetilde{H}_n(j) = p_1 \cdots p_n$ is an epimorphism so $f_* \colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ is also an epimorphism. We will need the following claim which is proved at the end of the proof.

Claim. Let $y \in Y$, $r = h(y)$. Then for every $\alpha \in Z_r(K_y)$, $\beta \in Z_{n-r-1}(K^y)$ we have $[(\alpha * \beta)'] = [\alpha' * \widetilde{f}_*(\beta')]$ in $\widetilde{H}_n(M_{r+1})$.

Let $j_r \colon X \to M_r$ be the inclusion. We have $j_{r*} = p_{r+1} \circ \ldots \circ p_{n+1}$. Now by induction on r we prove that for $0 \leq r \leq n+1$

$$\{j_{r*}(\gamma_i') \,:\, i \in I\} \cup \{j_{r*}((\alpha_i * \beta_j)') \,:\, y \in Y, i \in I_y, j \in J_y, h(y) < r\}$$

is a basis of $\widetilde{H}_n(M_r)$. Since $j_0 = j$ and $j_* = i_* f_*$ it holds when $r = 0$. Now, assuming it holds for r, we prove it also holds for $r+1$. By the split exact sequence obtained above, it suffices to check that

$$\{j_{r+1}((\alpha_i * \beta_j)') \,:\, i \in I_y, j \in J_y\}$$

is a basis of $\widetilde{H}_n(\mathrm{lk}(y, M_r))$ for every $y \in Y$ of height r. Now in $\widetilde{H}_n(M_{r+1})$ we have

$$j_{r+1}((\alpha_i * \beta_j)') = (\alpha_i * \beta_j)' = \alpha_i' * \widetilde{f}_*(\beta_j')$$

and the induction is complete, for $\{\alpha_i' * \widetilde{f}_*(\beta_j')\}_{i \in I_y, j \in J_y}$ is a basis of $\widetilde{H}_n(\mathrm{lk}(y, M_r))$. We have $j_{n+1} = 1_X$ and taking $r = n+1$ we get the desired basis of $\widetilde{H}_n(K)$.

Proof (claim). We consider chain maps $\phi_1, \phi_2 \colon \widetilde{C}_\bullet(K_y * K^y) \to \widetilde{C}_\bullet(\mathcal{K}(M_{r+1}))$ defined by

$$\phi_1 \colon \widetilde{C}_\bullet(K_y * K^y) \hookrightarrow \widetilde{C}_\bullet(K) \xrightarrow{\lambda} \widetilde{C}_\bullet(\mathcal{K}(X)) \hookrightarrow \widetilde{C}_\bullet(\mathcal{K}(M_{r+1}))$$

and

$$\begin{aligned} \phi_2 \colon \widetilde{C}_\bullet(K_y * K^y) = \widetilde{C}_\bullet(K_y) * \widetilde{C}_\bullet(K^y) &\xrightarrow{\lambda * \lambda} \widetilde{C}_\bullet(\mathcal{K}(f/y)) * \widetilde{C}_\bullet(\mathcal{K}(\mathcal{X}(K^y))) \xrightarrow{1 * \widetilde{f}_*} \\ \widetilde{C}_\bullet(\mathcal{K}(f/y)) * \widetilde{C}_\bullet(\mathcal{K}(Y_{>y})) &= \widetilde{C}_\bullet(\mathcal{K}(f/y * Y_{>y})) \hookrightarrow \widetilde{C}_\bullet(\mathcal{K}(M_{r+1})). \end{aligned}$$

Note that $\phi_1(\alpha*\beta) = (\alpha*\beta)'$ and $\phi_2(\alpha*\beta) = \alpha' * \widetilde{f}_*(\beta')$. Using the Acyclic Carrier Theorem (see Section A.8) we will prove that ϕ_1 and ϕ_2 are chain homotopic. We define an acyclic carrier $\Phi\colon K_y * K^y \to \mathcal{K}(M_{r+1})$. If $\sigma\cup\tau$ is a simplex in $K_y * K^y$, with $\sigma \in K_y$ and $\tau \in K^y$, we define

$$\Phi(\sigma\cup\tau) = \begin{cases} \mathcal{K}\left(M_{r+1_{\leq \widetilde{f}(\tau)}}\right) & \text{if } \tau \neq \varnothing. \\ \mathcal{K}\left(M_{r+1_{\leq\sigma}}\right) & \text{if } \tau = \varnothing. \end{cases}$$

If $\sigma_1\cup\tau_1 \subseteq \sigma_2\cup\tau_2$ are simplices of $K_y * K^y$ where $\sigma_i \in K_y$ and $\tau_i \in K^y$ are possibly empty, we have $\sigma_1 \subseteq \sigma_2$ and $\tau_1 \subseteq \tau_2$. In M we have $\sigma_1 \leq \sigma_2 \leq y \leq \widetilde{f}(\tau_1) \leq \widetilde{f}(\tau_2)$ so in any case $\Phi(\sigma_1\cup\tau_1) \subseteq \Phi(\sigma_2\cup\tau_2)$. It is clear that $\Phi(\sigma\cup\tau)$ is acyclic, therefore Φ is an acyclic carrier.

Now we prove that ϕ_1 and ϕ_2 are carried by Φ. To show that ϕ_1 is carried by Φ we need to show that $\phi_1(\sigma\cup\tau) = (\sigma\cup\tau)'$ is supported on $\Phi(\sigma\cup\tau)$. If τ is empty it is clear. If τ is nonempty, we consider $x \in f^{-1}(y)$. In M, by (ii) we have $\sigma\cup\tau \leq f(\sigma\cup\tau) \leq f(x\cup\tau) = \widetilde{f}(\tau)$. Therefore $(\sigma\cup\tau)'$ is supported on $\Phi(\sigma\cup\tau) = \mathcal{K}\left(M_{r+1_{\leq \widetilde{f}(\tau)}}\right)$. It is easy to see that ϕ_2 is also carried by Φ.

Finally by the Acyclic Carrier Theorem we have

$$[(\alpha*\beta)'] = [\phi_1(\alpha*\beta)] = [\phi_2(\alpha*\beta)] = [\alpha' * \widetilde{f}_*(\beta')]$$

and we are done. □

□

Remark 3.3.8. We can consider $\varphi\colon X \to M_{r+1}$ given by

$$\varphi(x) = \begin{cases} x & \text{if } h(x) < r+1 \\ f(x) & \text{if } h(x) \geq r+1 \end{cases}.$$

Then $j_{r+1} \leq \varphi$. Therefore $j_{r+1*} \simeq \mathcal{K}(\varphi)$ and $j_* = \varphi_*$. In the previous proof we actually have $\varphi_*((\alpha*\beta)') = \alpha' * \widetilde{f}_*(\beta')$ in $Z_n(M_{r+1})$.

3.4 PB(F_n) is Cohen-Macaulay

To prove that PB(F_n) is Cohen-Macaulay we need to consider other related spaces. The *free factor poset* FC(F) of a free group F is the poset of proper free factors of F ordered by inclusion. This poset was studied by Hatcher and Vogtmann [HV98]. If H is a free factor of F_n and B is a basis of H then B is a partial basis of F_n. If B is a partial basis of F_n then $H = \langle B\rangle$ is a free factor of F_n. There is an order preserving map

$$g\colon \mathcal{X}\left(\mathrm{PB}(F_n)^{(n-2)}\right) \to \mathrm{FC}(F_n)$$
$$\sigma \mapsto \langle\sigma\rangle$$

and if B_0 is a partial basis we have the restriction $g\colon \mathcal{X}\left(\mathrm{PB}(F_n)^{(n-2)}\right)_{>B_0} \to \mathrm{FC}(F_n)_{>\langle B_0\rangle}$.

Proposition 3.4.1 ([MKS76, p. 117]). *Suppose H is a free factor of F_n and $K \leq H$. Then K is a free factor of H if and only if K is a free factor of F_n.*

Theorem 3.4.2 (Hatcher-Vogtmann, [HV98, §4]). *If $H \leq F_n$ is a free factor,* $\mathrm{FC}(F_n)_{>H}$ *is* $(n-\mathrm{rk}(H)-2)$*-spherical.*

We will consider the following simplicial complex Y with vertices the free factors of F_n that have rank $n-1$. A simplex of Y is a set of free factors $\{H_1,\dots,H_k\}$ such that there is a basis $\{w_1,\dots,w_n\}$ of F_n such that for $1 \leq i \leq k$ we have $H_i = \langle w_1,\dots,w_{i-1},w_{i+1},\dots,w_n\rangle$. If $H \leq F_n$ is a free factor, we consider the full subcomplex Y_H of Y spanned by the vertices which are free factors containing H. There is another equivalent definition for Y and Y_H in terms of sphere systems, see [HV98, Remark after Corollary 3.4].

Theorem 3.4.3 (Hatcher-Vogtmann,[HV98, Theorem 2.4]). *Let H be a free factor of F_n. Then Y_H is* $(n-\mathrm{rk}(H)-1)$*-spherical.*

There is a spherical map $f\colon \mathcal{X}(Y_H^{(n-\mathrm{rk}(H)-2)}) \to (\mathrm{FC}(F_n)_{>H})^{\mathrm{op}}$ that maps a simplex $\sigma = \{H_1,\dots,H_k\}$ to $H_1 \cap \cdots \cap H_k$. Hatcher and Vogtmann used the map f to prove Theorem 3.4.2. We also consider the map $\widetilde{g}\colon \mathcal{X}\left(\mathrm{lk}\left(B,\mathrm{PB}(F_n)^{(n-2)}\right)\right) \to \mathrm{FC}(F_n)_{>\langle B\rangle}$ given by $\sigma \mapsto \langle B \cup \sigma\rangle$ which can be identified with $g\colon \mathcal{X}\left(\mathrm{PB}(F_n)^{(n-2)}\right)_{>B} \to \mathrm{FC}(F_n)_{>\langle B\rangle}$. The following technical lemma will be needed later.

Lemma 3.4.4. *Let B be a partial basis of F_n, $|B| = l$. Let $\overline{\gamma} \in \widetilde{H}_{n-l-2}(\mathrm{FC}(F_n)_{>\langle B\rangle})$. There exists $\gamma \in B_{n-l-2}(\mathrm{lk}(B,\mathrm{PB}(F_n)))$ such that $\widetilde{g}_*(\gamma') = \overline{\gamma}$.*

Proof. We define a map $\phi\colon C_{n-l-1}(Y_{\langle B\rangle}) \to C_{n-l-1}(\mathrm{lk}(B,\mathrm{PB}(F_n)))$ as follows. For each $(n-l-1)$-simplex $\sigma = \{H_{l+1},\dots,H_n\}$ of $Y_{\langle B\rangle}$ we choose a basis $\{w_1,\dots,w_n\}$ of F_n such that

$$H_i = \langle w_1,\dots,w_{i-1},w_{i+1},\dots,w_n\rangle$$

for $l+1 \leq i \leq n$. Then by Proposition 3.4.1 we have

$$\langle B\rangle = \bigcap_{i=l+1}^{n} H_i = \langle w_1,\dots w_l\rangle$$

so $B \cup \{w_{l+1},\dots,w_n\}$ is a basis of F_n. Therefore $\widetilde{\sigma} = \{w_{l+1},\dots,w_n\}$ is an $(n-l-1)$-simplex of $\mathrm{lk}(B,\mathrm{PB}(F_n))$. Then we define the map ϕ on σ by $\phi(\sigma) = \widetilde{\sigma}$.

Now since $f\colon \mathcal{X}(Y_{\langle B\rangle}^{(n-l-2)}) \to (\mathrm{FC}(F_n)_{>\langle B\rangle})^{\mathrm{op}}$ is $(n-l-2)$-spherical [HV98, §4], by Theorem 3.3.7 we have an epimorphism $f_*\colon \widetilde{H}_{n-l-2}(\mathcal{X}(Y_{\langle B\rangle}^{(n-l-2)})) \to \widetilde{H}_{n-l-2}(\mathrm{FC}(F_n)_{>\langle B\rangle})$ and since $Y_{\langle B\rangle}$ is $(n-l-2)$-connected, there is $c \in C_{n-l-1}(Y_{\langle B\rangle})$ such that $f_*(d(c)') = \overline{\gamma}$. We define $\gamma = d\phi(c)$. We immediately have $\gamma \in B_{n-l-2}(\mathrm{lk}(B,\mathrm{PB}(F_n)))$. It is easy to verify that $\widetilde{g}_*(d\phi(\sigma)') = f_*(d\sigma')$ and from this it follows that $\widetilde{g}_*(\gamma') = \widetilde{g}_*(d\phi(c)') = f_*(dc') = \overline{\gamma}$. $\square$

Theorem 3.4.5. *The complex* $\mathrm{PB}(F_n)$ *is Cohen-Macaulay of dimension* $n-1$.

Proof. We prove that $\mathrm{lk}(B_0,\mathrm{PB}(F_n))$ is $(n-|B_0|-1)$-spherical for any partial basis B_0 of F_n by induction on $k=n-|B_0|$. If $k\leq 3$ it follows from Theorem 3.2.1. Now if $k\geq 4$ we want to apply Theorem 3.3.7 to the map $g\colon \mathcal{X}\left(\mathrm{PB}(F_n)^{(n-2)}\right)_{>B_0}\to \mathrm{FC}(F_n)_{>\langle B_0\rangle}$.

By Theorem 3.4.2, $\mathrm{FC}(F_n)_{>\langle B_0\rangle}$ is $(n-|B_0|-2)$-spherical. In addition g is spherical, since $\mathrm{FC}(F_n)_{>H}$ is $(n-\mathrm{rk}(H)-2)$-spherical if $H\in \mathrm{FC}(F_n)_{>\langle B_0\rangle}$ and by the induction hypothesis $g/H=\mathcal{X}(\mathrm{PB}(H))_{>B_0}=\mathcal{X}(\mathrm{lk}(B_0,\mathrm{PB}(H)))$ is $(\mathrm{rk}(H)-|B_0|-1)$-spherical. Then by Theorem 3.3.7, $\mathcal{X}\left(\mathrm{PB}(F_n)^{(n-2)}\right)_{>B_0}$ is homologically $(n-|B_0|-2)$-spherical.

We identify $\mathcal{X}\left(\mathrm{PB}(F_n)^{(n-2)}\right)_{>B_0}=\mathcal{X}(\mathrm{lk}(B_0,\mathrm{PB}(F_n)^{(n-2)}))$. Now we check the hypotheses (i), (ii) and (iii) of Theorem 3.3.7. If $\widetilde{g}(B_1)\subseteq\widetilde{g}(B_2)$ it is easy to see that

$$\mathrm{lk}(B_2,\mathrm{lk}(B_0,\mathrm{PB}(F_n)^{(n-2)}))\subseteq \mathrm{lk}(B_1,\mathrm{lk}(B_0,\mathrm{PB}(F_n)^{(n-2)}))$$

so (i) holds. Obviously (ii) holds. And by the induction hypothesis (iii) holds. Thus, the second part of Theorem 3.3.7 gives a basis of $\widetilde{H}_{n-|B_0|-2}(\mathrm{lk}(B_0,\mathrm{PB}(F_n)^{(n-2)}))$. By Lemma 3.4.4 we can choose the γ_i to be borders. We need to prove that the remaining elements of this basis are trivial in $\widetilde{H}_{n-|B_0|-2}(\mathrm{PB}(F_n)_{>B_0})$. We only have to show that for all $H\in\mathrm{FC}(F_n)_{>\langle B_0\rangle}$, $i\in I_H$, $j\in J_H$

$$\alpha_i*\beta_j\in B_{n-|B_0|-2}(\mathrm{lk}(B_0,\mathrm{PB}(F_n))).$$

We take a basis B of H. By the induction hypothesis we have $\widetilde{H}_{n-|B|-2}(\mathrm{lk}(B,\mathrm{PB}(F_n)))=0$. So there is $\omega\in C_{n-|B|-1}(\mathrm{lk}(B,\mathrm{PB}(F_n)))$ such that $d(\omega)=(-1)^{|\alpha_i|}\beta_j$. Therefore

$$d(\alpha_i*\omega)=d(\alpha_i)*\omega+(-1)^{|\alpha_i|}\alpha_i*d(\omega)=\alpha_i*\beta_j.$$

Therefore $\mathrm{lk}(B_0,\mathrm{PB}(F_n))$ is homologically $(n-|B_0|-1)$-spherical and by Theorem 3.2.1 it is $(n-|B_0|-1)$-spherical. □

Remark 3.4.6. Theorem 3.3.7 also holds without the word *homologically* (see [Qui78, Theorem 9.1]). Thus, we may easily modify the previous proof so that Theorem 3.2.1 is only used as the base case $k\leq 3$.

Resumen del Capítulo 3: El complejo de bases parciales

El complejo de curvas $\mathcal{C}(S_g)$ de una superficie S_g de género g fue introducido por Harvey [Har81] como un análogo del Tits building para el mapping class group $\mathrm{Mod}(S_g)$. Harer probó que $\mathcal{C}(S_g)$ es homotópicamente equivalente a un wedge de $(g-1)$-esferas [Har85]. Masur y Minsky probaron que $\mathcal{C}(S_g)$ es hiperbólico [MM99]. Desde entonces, el complejo de curvas se volvió un objeto fundamental en el estudio de $\mathrm{Mod}(S_g)$. Dado que hay una analogía entre $\mathrm{Aut}(\mathbb{F}_n)$ y $\mathrm{Mod}(S_g)$, es natural buscar en este contexto un objeto análogo a $\mathcal{C}(S_g)$. Hay varios candidatos que tienen propiedades similares a las del complejo de curvas.

Uno de estos análogos es el poset $\mathrm{FC}(\mathbb{F}_n)$ de factores libres propios de $\mathbb{F}_n$. Hatcher y Vogtmann [HV98] probaron que su order complex $\mathcal{K}(\mathrm{FC}(\mathbb{F}_n))$ es Cohen-Macaulay (en particular, que es homotópicamente equivalente a un wedge de $(n-2)$-esferas). Bestvina y Feighn [BF14] probaron que $\mathcal{K}(\mathrm{FC}(\mathbb{F}_n))$ es hiperbólico. Posteriormente, distintas demostraciones de este resultado aparecieron en [KR14] y [HH17].

Otros análogos naturales se construyen a partir de las bases parciales. Una *base parcial* de un grupo libre $\mathbb{F}$ es un subconjunto de una base de $\mathbb{F}$. Day y Putman [DP13] definieron el complejo $\mathcal{B}(\mathbb{F}_n)$ que tiene como símplices a los conjuntos$\{C_0,\dots,C_k\}$ de clases de conjugación de $\mathbb{F}_n$ tales que existe una base parcial $\{v_0,\dots,v_k\}$ con $C_i = [\![v_i]\!]$ para $0 \leq i \leq k$. En dicho artículo, Day y Putman probaron que $\mathcal{B}(\mathbb{F}_n)$ es 0-conexo si $n \geq 2$ y 1-conexo si $n \geq 3$ [DP13, Theorem A], y que cierto cociente resulta $(n-2)$-conexo [DP13, Theorem B]. Además conjeturaron que $\mathcal{B}(\mathbb{F}_n)$ es $(n-2)$-conexo [DP13, Conjecture 1.1]. Como aplicación, usaron $\mathcal{B}(\mathbb{F}_n)$ para probar que el subgrupo de Torelli es finitamente generado.

En este capítulo se estudia el complejo simplicial $\mathrm{PB}(\mathbb{F}_n)$ que tiene como símplices las bases parciales no vacías de $\mathbb{F}_n$. El siguiente teorema es resultado principal del capítulo y se prueba en cuatro pasos.

Teorema 3.4.5. *El complejo de bases parciales* $\mathrm{PB}(\mathbb{F}_n)$ *es Cohen-Macaulay.*

El primer paso consiste en obtener una presentación de $\mathrm{SAut}(\mathbb{F}_n,\{v_1,\dots,v_l\})$, análoga a la presentación de $\mathrm{SAut}(\mathbb{F}_n)$ obtenida por Gersten. Con este fin, se utiliza el método de McCool para presentar $\mathrm{Aut}(\mathbb{F}_n,\{v_1,\dots,v_l\})$ y luego se aplica el método de Reidemeister-Schreier para presentar el subgrupo $\mathrm{SAut}(\mathbb{F}_n,\{v_1,\dots,v_l\})$ que tiene índice 2.

En un segundo paso, realizando algunas modificaciones menores a la demostración de [DP13, Theorem A], se prueba que el link $\mathrm{lk}(B,\mathrm{PB}(\mathbb{F}_n))$ de una base parcial B es 0-conexo si $n-|B| \geq 2$ y 1-conexo si $n-|B| \geq 3$. En vez de la presentación de Gersten del grupo $\mathrm{SAut}(\mathbb{F}_n)$, utilizada por Day y Putman, se utiliza la presentación de $\mathrm{SAut}(\mathbb{F}_n,\{v_1,\dots,v_l\})$ obtenida en el primer paso.

En el tercer paso se prueba una versión de un resultado de Quillen [Qui78, Theorem 9.1] que produce una base explícita del grupo de homología de grado máximo de X. La demostración se basa en la demostración del Teorema de Quillen dada por Piterman [Pit16, Teorema 2.1.28], que utiliza el argumento del cilindro no Hausdorff de Barmak y Minian [BM08].

Finalmente en el cuarto paso se prueba el Corolario 3.4.5. La idea clave es comparar el link $\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n))$ (que tiene dimensión $(n-|B|-1)$) con $\mathrm{FC}(\mathbb{F}_n)_{>\langle B\rangle}$ (que tiene dimensión $(n-|B|-2)$). Para poder hacer esto, se considera el $(n-|B|-2)$-esqueleto de $\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n))$. La base provista por el resultado obtenido en el tercer paso permite comprender qué ocurre al pasar de $\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n)^{(n-2)})$ a $\mathrm{lk}(B, \mathrm{PB}(\mathbb{F}_n))$. Se procede por inducción en $n-|B|$ y como caso base se usa el resultado probado en el segundo paso.

Appendix

A.1 Words and cyclically reduced words in free groups

Let F be a free group with basis $x_1,\dots,x_m$. Every element of F is given by a word

$$w = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_k}^{\varepsilon_k}$$

for some $k \geq 0$, $i_1,\dots,i_k \in \{1,2,\dots,m\}$ and $\varepsilon_1,\dots,\varepsilon_k \in \{1,-1\}$. Such a word w is said to be *reduced* if $i_j = i_{j+1}$ implies $\varepsilon_j = \varepsilon_{j+1}$ for $1 \leq j \leq k-1$. If in addition $i_k = i_1$ implies $\varepsilon_k = \varepsilon_1$ we say the word w is *cyclically reduced*.

Every element of F is represented by a unique reduced word. Every conjugacy class of elements of F is represented, up to cyclic permutation of the letters, by a unique cyclically reduced word.

A.2 Edge paths and the edge path group

Let X be a CW-complex. If e is an oriented edge, the *source* of e is denoted by $s(e)$ and the *target* of e is denoted by $t(e)$. If e is an oriented edge, the opposite edge is denoted by e^{-1}. An *edge path* in X is a tuple $\alpha = (e_1,\dots,e_k)$ of oriented edges such that $t(e_j) = s(e_{j+1})$ for $1 \leq j \leq k-1$. If in addition $t(e_k) = s(e_1)$ we say that α is a *closed edge path*. An edge path $\alpha = (e_1,\dots,e_k)$ is said to be *reduced* if $e_{i+1} \neq e_i^{-1}$ for $1 \leq i \leq k-1$. If in addition, α is closed and $e_1 \neq e_k^{-1}$ we say that it is *cyclically reduced*. If $\alpha = (e_1,\dots,e_k)$ is a closed edge path and $x_0 = s(e_1)$ we say that α is *based* at x_0.

A closed edge path based at x_0 determines an element of $\pi_1(X,x_0)$ and every element of the fundamental group may be represented in this way. The edge path group $\mathcal{E}(X,x_0)$ provides a convenient description of the fundamental group of a CW complex. Once we fix a preferred orientation for each 1-cell, any tuple of edges α determines a word $w(\alpha)$ in the free group with basis the 1-cells of X. We consider the subgroup F of this group given by words coming from the closed edge paths based at x_0. For each 2-cell e_i^2 in X, we consider a closed edge path $\alpha_{e_i^2}$ based at x_0 and giving a loop in X freely homotopic to the attaching map of e_i^2. Let N be the normal closure of the words $w(\alpha_{e_i^2})$. The group N does not depend on the choice of the paths $\alpha_{e_i^2}$. The *edge path group* is defined by $\mathcal{E}(X,x_0) = F/N$. The canonical mapping $\mathcal{E}(X,x_0) \to \pi_1(X,x_0)$ is an isomorphism.

For a graph Γ, the edge path group $\mathcal{E}(\Gamma,x_0)$ coincides with the free group F. By Section A.1, every element of $\pi_1(\Gamma,x_0)$ is represented by a unique reduced closed edge path based at x_0; and every conjugacy class of $\pi_1(\Gamma,x_0)$ is represented up to cyclic permutation by a unique cyclically reduced closed edge path.

A.3 Presentation complexes

Now we recall the relation between presentations of a group G and 2-complexes with fundamental group G.

Definition A.3.1. Recall that a presentation $\mathcal{P} = \langle x_1,\dots,x_n \mid r_1,\dots,r_k\rangle$ of a group G has an associated CW-complex $X_{\mathcal{P}}$, the *presentation complex* (also called *Cayley complex* or *standard complex*) of $\mathcal{P}$, with one 0-cell, a 1-cell for each generator x_i and a 2-cell for every relator r_j, attached along the closed edge path given by the word r_j.

The fundamental group of $X_{\mathcal{P}}$ is isomorphic to G. The second barycentric subdivision of this complex is a triangulation of $X_{\mathcal{P}}$, so any presentation complex is in fact a polyhedron. Conversely, every compact connected 2-complex X is homotopy equivalent to the presentation complex of a presentation $\mathcal{P}$ of $\pi_1(X)$. To prove this, we consider a maximal tree T in $X^{(1)}$. The CW-complex $Y = X/T$ has a unique 0-cell. The fundamental group of $Y^{(1)}$ is free with basis the 1-cells of Y and the attaching maps for the 2-cells of Y determine words in this free group. In this way we obtain a presentation and the complex associated to this presentation is homotopy equivalent to X.

Example A.3.2. If $\mathcal{P} = \langle x_1,\dots,x_n \mid\rangle$, then $X_{\mathcal{P}} = \bigvee_{i=1}^n S^1$. If $\mathcal{P} = \langle x \mid 1\rangle$, then $X_{\mathcal{P}} = S^1 \vee S^2$. If $\mathcal{P} = \langle x \mid x^2\rangle$, then $X_{\mathcal{P}} = \mathbb{RP}^2$. If $\mathcal{P} = \langle x,y \mid [x,y]\rangle$, then $X_{\mathcal{P}} = S^1 \times S^1$. If $\mathcal{P} = \langle x,y \mid xyx^{-1}y\rangle$, then $X_{\mathcal{P}}$ is the Klein bottle.

If $\mathcal{P} = \langle X \mid R\rangle$ is a presentation of G, the *Cayley graph* $\mathrm{Cay}(G,X)$ can be defined as the 1-skeleton of the universal covering of the presentation complex $X_{\mathcal{P}}$.

A.4 Free products, amalgamated products and HNN extensions

In this section, following [LS77], we review the basics on amalgamated free products and HNN extensions.

Let A, B be two groups $C \leq A$ and $C' \leq B$ two subgroups and let $\phi : C \to C'$ be an isomorphism. Then we can form the *amalgamated product* defined by

$$A *_C B = A * B/\langle\langle \phi(c)\cdot c^{-1} \,:\, c \in C\rangle\rangle.$$

A sequence $g_1, \dots, g_n$, $n \geq 0$ of elements of $A * B$ is called *reduced* if

(1) Each g_i is in one of the factors A, B.

(2) Consecutive elements g_i, g_{i+1} come from different factors.

(3) If $n > 1$, no g_i is in C or C'.

(4) If $n = 1$, $g_1 \neq 1$.

It is easy to see that every element of $A *_C B$ is equal to the product of the elements in a reduced sequence. We have the following normal form for amalgamated products:

Theorem A.4.1 ([LS77, Chapter IV, Theorem 2.6])**.** *If $g_1, \dots, g_n$ is a reduced sequence and $n \geq 1$, then $g_1 g_2 \cdots g_n \neq 1$ in $A *_C B$. In particular A and B are embedded in $A *_C B$.*

If $\phi : C \to C'$ is an isomorphism between subgroups of a group A we can form the *HNN extension*

$$A *_C = A * F(t) / \langle\langle t^{-1} c t = \phi(c) \, : \, c \in C \rangle\rangle.$$

The canonical map $A \hookrightarrow A *_C$ is injective. There is also a normal form for HNN extensions, see [LS77, Chapter IV, Theorem 2.1].

A.5 Small cancellation theory

We review here some basic definitions and results from [LS77, Chapter V].

A subset R of a free group F is called *symmetrized* if all elements of R are cyclically reduced and for each r in R all cyclically reduced conjugates of r and r^{-1} are also in R. If R is a set of cyclically reduced words but is not symmetrized we may work instead with the *symmetrization* R^* of R (the smallest symmetrized set containing R).

A *piece* of R is a word that is a common prefix of two different words in the symmetrized set R^*. We say that a presentation $\mathcal{P} = \langle X \mid R \rangle$ *satisfies the small cancellation condition* $C(p)$ if no relator $r \in R^*$ is a product of fewer than p pieces. We say that $\mathcal{P}$ *satisfies the small cancellation condition* $C'(\lambda)$ if for all $r \in R^*$, if $r = bc$ and b is a piece then $|b| < \lambda |r|$. We say that $\mathcal{P}$ *satisfies the small cancellation condition* $T(q)$ if for all h such that $3 \leq h < q$ and for all elements $r_1, \dots, r_h$ in R^*, if no consecutive elements r_i, r_{i+1} are inverses, then at least one of the products $r_1 r_2, \dots r_{h-1} r_h, r_h r_1$ is reduced without cancellation.

These conditions can be understood geometrically in terms of diagrams (for the precise definition of diagram, see [LS77, Chapter V. Section 1]). Condition $C(p)$ implies that every face in the interior of a reduced diagram has at least p sides. Condition $T(q)$ implies that every vertex in the interior of a reduced diagram has degree at least q.

If a group $G = \langle X \mid R \rangle$ satisfies $C'(1/6)$ then it is hyperbolic and Dehn's algorithm (see [LS77, Chapter V, Section 4] solves the word problem for G. If a group $G = \langle X \mid R \rangle$ satisfies $C(6)$ or $C(4) - T(4)$ or $C(3) - T(6)$ then G has solvable word problem and solvable conjugacy problem (see [LS77, Chapter V Sections 5 and 6]). If a presentation $\mathcal{P} = \langle X \mid R \rangle$ satisfies $C(6)$ and no relator of $\mathcal{P}$ is a proper power then the presentation complex of $\mathcal{P}$ is aspherical.

The following result is probably well known but we could not find a suitable reference. We include a proof here.

Proposition A.5.1. *Let $\mathcal{P} = \langle X \mid R \rangle$ be a $C(6)$ presentation and suppose there is a generator $x \in X$ which is a piece. Then x is nontrivial in the group presented by $\mathcal{P}$.*

Proof. Suppose x is trivial and consider a reduced simply connected diagram D with boundary x. Let V, E, F be the number of vertices, edges and faces of D. Note that the boundary of D has just one vertex and one edge (thus there is just one face having an edge on the boundary). We will arrive at a contradiction by an Euler characteristic argument. Since each interior edge is shared by exactly two faces we have $2(E-1)+1 = \sum_f d(F) \geq 6(F-1)$. We rewrite this as $F \leq \frac{2E+5}{6}$. Every vertex has degree at least 3 except possibly the vertex in the boundary. But this vertex also has degree at least 3, since otherwise the diagram would have just one vertex, one edge and one face, which is impossible since x is a piece and $\mathcal{P}$ satisfies condition $C(6)$. Now since every vertex has degree at least 3 we have $2E = \sum_v d(v) \geq 3V$. Now $1 = \chi(D) = V - E + F \leq \frac{2}{3}E - E + \frac{2E+5}{6} = \frac{5}{6}$, a contradiction. $\square$

A.6 Equations over groups

Let G be a group. An *equation* over G in the variables $x_1, \dots, x_n$ is an element $w \in G * F(x_1, \dots, x_n)$. We say that a system of equations

$$\begin{gathered} w_1(x_1, \dots, x_n) = 1 \\ w_2(x_1, \dots, x_n) = 1 \\ \cdots \\ w_m(x_1, \dots, x_n) = 1 \end{gathered}$$

has a solution in an overgroup of G if the map $G \to G * F(x_1, \dots, x_m)/\langle\langle w_1, \dots, w_m \rangle\rangle$ is injective. Such a system of equations determines an $(m \times n)$-matrix M where $M_{i,j}$ is given by the total exponents of the letter x_j in the word w_i. A system is said to be *independent* if the rank of M is m.

One of the most important open problems in the theory of equations over groups is the Kervaire–Laudenbach–Howie conjecture [How81, Conjecture].

Conjecture A.6.1 (Kervaire–Laudenbach–Howie)**.** *An independent system of equations over G has a solution in an overgroup of G.*

Now we explain why Conjecture 2.2.1 follows from the Kervaire–Laudenbach–Howie conjecture for perfect groups which admit a balanced presentation. Let A be an acyclic subcomplex of a contractible 2-complex X. Take a maximal tree T for A and consider a maximal tree $\overline{T}$ of X containing T. Then $A/T \simeq A$ is an acyclic subcomplex of the contractible 2-complex $X/\overline{T}$. Then the group $G = \pi_1(A/T)$ is perfect. As usual, from A/T we can read a presentation for G which is balanced since A/T is acyclic. Now we consider a variable x_i for each 1-cell of $X/\overline{T}$ which is not in A/T and we read words from the attaching maps for the 2-cells of $X/\overline{T}$ which are not part of A/T. In this way we obtain equations in these variables with coefficients in A. Since $X/\overline{T}$ is acyclic, there is an equal number of variables and equations and the determinant of the exponent matrix is 1. Thus if the Kervaire–Laudenbach–Howie conjecture holds for perfect groups which admit a balanced presentation, $\pi_1(A)$ injects into $\pi_1(X)$ and if X is contractible then A is contractible too.

A.7 The Gluing Theorem

Theorem A.7.1 ([Bro06, 7.5.7])**.** *Let*

$$\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dd, "i'"] & & Y' \arrow[dd, "\bar{i}'"] \\
A \arrow[ur, "\phi_A"] \arrow[rr, "f"] \arrow[dd, "i"'] & & Y \arrow[ur, "\phi_Y"] \arrow[dd, "\bar{i}"] & \\
& X' \arrow[rr, "\bar{f}'"] & & X' \cup_f Y' \\
X \arrow[ur, "\phi_X"] \arrow[rr, "\bar{f}"] & & X \cup_f Y \arrow[ur, "\phi"'] &
\end{tikzcd}$$

be a commutative diagram such that i, i' are closed cofibrations, the front and back faces are pushouts and ϕ_A, ϕ_X, ϕ_Y are homotopy equivalences. Then ϕ is a homotopy equivalence.

A.8 The Acyclic Carrier Theorem

Definition A.8.1. Let K and L be simplicial complexes. An *acyclic carrier* from K to L is a function Φ that assigns to each simplex σ of K, a subcomplex $\Phi(\sigma)$ of L such that:
(i) $\Phi(\sigma)$ is acyclic.
(ii) If τ is a face of σ, then $\Phi(\tau) \subseteq \Phi(\sigma)$.

If $f\colon \widetilde{C}_p(K) \to \widetilde{C}_q(L)$ is a homomorphism, we say that f is *carried by* Φ if, for each oriented p-simplex σ of K, we have $f(\sigma) \in \widetilde{C}_q(\Phi(\sigma))$.

Theorem A.8.2 ([Mun84, Theorem 13.3])**.** *Let $\Phi\colon K \to L$ be an acyclic carrier.*

(a) If $\phi, \psi\colon \widetilde{C}_*(K) \to \widetilde{C}_*(L)$ *are two augmentation-preserving chain maps carried by* Φ*, there exists a chain homotopy* D *from* ϕ *to* ψ *that is also carried by* Φ*.*
(b) There exists an augmentation-preserving chain map from $\widetilde{C}_*(K) \to \widetilde{C}_*(L)$ *that is carried by* Φ*.*

List of Symbols

Groups and actions

F_n	the free group of rank n
C_n	the cyclic group of order n
D_{2n}	the dihedral group of order $2n$
A_n	the alternating group on $\{1,\dots,n\}$
A_5^*	the binary icosahedral group
$\mathbb{F}_q$	the finite field of order q
$\mathrm{PSL}_n(q)$	the projective special linear group over $\mathbb{F}_q$
$\mathrm{Sz}(q)$	for $q = 2^{2n+1}$ denotes the Suzuki group over $\mathbb{F}_q$
$\mathrm{Aut}(G)$	the automorphism group of G
$\mathrm{Out}(G)$	the outer automorphism group of G
$\mathrm{SAut}(G)$	the special automorphism group of G
$G \curvearrowright X$	a group action of G on X
G_x	the stabilizer of $x \in X$
X^H	the fixed point set of H
$N \triangleleft G$	a normal subgroup of G
$N \operatorname{char} G$	a characteristic subgroup of G
$N_G(H)$	the normalizer of H in G
$N \rtimes H$	a semidirect product
$\langle S \rangle$	is the subgroup generated by $S \subset G$
$\langle\langle S \rangle\rangle^G$	is the normal subgroup of G generated by $S \subset G$
$[\![g]\!]$	the conjugacy class of g
h^g	$= ghg^{-1}$
$[g,h]$	the commutator $ghg^{-1}h^{-1}$
$O_p(G)$	is the intersection of the Sylow p-subgroups of G
$A *_C B$	an amalgamated product of groups
$A *_C$	an HNN extension

Topological spaces

D^n	the unit disk in $\mathbb{R}^n$
S^n	the unit sphere in $\mathbb{R}^{n+1}$
$\mathbb{CP}^n$	the complex projective space of dimension n
$\mathbb{HP}^n$	the quaternionic projective space of dimension n
$X \simeq Y$	means X and Y are homotopy equivalent
$X^{(n)}$	the n-skeleton of a CW-complex X
$X_{\mathcal{P}}$	the presentation complex (or standard complex) of $\mathcal{P}$
$\mathrm{lk}(v,K)$	the link of a vertex v in K
$\mathrm{st}(v,K)$	the open star of a vertex v in K

Posets and families of groups

$\mathcal{K}(X)$	the order complex of a poset X
$\mathcal{X}(K)$	the face poset of a simplicial complex K
$X_{>x}$	$=\{x' \in X : x' > x\}$
$X_{\geq x}$	$=\{x' \in X : x' \geq x\}$
f/y	$=\{x \in X : f(x) \leq y\}$ is the fiber of $f\colon X \to Y$ under y
X^{op}	the poset X with the opposite order
$\mathcal{S}(G)$	the family of all subgroups of G
$\mathcal{A}_p(G)$	the poset of nontrivial elementary abelian p-subgroups of G
$\mathcal{SLV}$	the family of solvable subgroups of G

Specific notation

$\mathrm{tr}_R(f)$	the trace of an endomorphism of R-modules f, see Definition 1.1.19
$\widetilde{G}_X$	the group extension given by Brown's theorem, see Section 2.6
$\Gamma_{OS}(G)$	any G-graph satisfying certain properties, see Section 2.1.1
$d_G(N)$	denotes the minimum number of elements needed to generate a G-group N
$\mathrm{PB}(\mathbb{F}_n)$	the complex of partial bases of $\mathbb{F}_n$
$\mathcal{B}(\mathbb{F}_n)$	the complex of conjugacy classes of partial bases of $\mathbb{F}_n$
$\mathrm{FC}(\mathbb{F}_n)$	the poset of free factors of $\mathbb{F}_n$
$\mathrm{Cay}(G,X)$	the Cayley graph of G for the generating set $X \subset G$

Bibliography

[ABE$^+$12] Erhard Aichinger, Franz Binder, Jürgen Ecker, Peter Mayr, and Christof Nöbauer, *SONATA, system of nearrings and their applications, Version 2.6*, http://www.algebra.uni-linz.ac.at/Sonata/, Nov 2012, Refereed GAP package.

[AC65] James J. Andrews and Morton L. Curtis, *Free groups and handlebodies*, Proc. Amer. Math. Soc. **16** (1965), 192–195. MR 0173241

[Ade03] Alejandro Adem, *Finite group actions on acyclic 2-complexes*, Astérisque (2003), no. 290, Exp. No. 894, vii, 1–17, Séminaire Bourbaki. Vol. 2001/2002. MR 2074048

[AK90] Michael Aschbacher and Peter B. Kleidman, *On a conjecture of Quillen and a lemma of Robinson*, Arch. Math. (Basel) **55** (1990), no. 3, 209–217. MR 1075043

[AS93a] Michael Aschbacher and Yoav Segev, *A fixed point theorem for groups acting on finite* 2*-dimensional acyclic simplicial complexes*, Proc. London Math. Soc. (3) **67** (1993), no. 2, 329–354. MR 1226605

[AS93b] Michael Aschbacher and Stephen D. Smith, *On Quillen's conjecture for the p-groups complex*, Ann. of Math. (2) **137** (1993), no. 3, 473–529. MR 1217346

[Bar11] Jonathan A. Barmak, *Algebraic topology of finite topological spaces and applications*, Lecture Notes in Mathematics, vol. 2032, Springer, Heidelberg, 2011. MR 3024764

[BF14] Mladen Bestvina and Mark Feighn, *Hyperbolicity of the complex of free factors*, Adv. Math. **256** (2014), 104–155. MR 3177291

[Bin69] R H. Bing, *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119–132. MR 0236908 (38 #5201)

[BM08] Jonathan Ariel Barmak and Elias Gabriel Minian, *Simple homotopy types and finite spaces*, Adv. Math. **218** (2008), no. 1, 87–104. MR 2409409

[Bor31] Karol Borsuk, *Quelques théorèmes sur les ensembles unicohérents*, Fund. Math. **17** (1931), no. 1, 171–209.

[Bor33] ______, *Über die Abbildungen der metrischen kompakten Räume auf die Kreislinie*, Fund. Math. **20** (1933), no. 1, 224–231.

[Bre67] Glen E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Mathematics, No. 34, Springer-Verlag, Berlin-New York, 1967. MR 0214062

[Bre71] ______, *Some examples for the fixed point property*, Pacific J. Math. **38** (1971), 571–575. MR 0310864 (46 #9962)

[Bro75] Kenneth S. Brown, *Euler characteristics of groups: the p-fractional part*, Invent. Math. **29** (1975), no. 1, 1–5. MR 0385008

[Bro82] Robert F. Brown, *The fixed point property and Cartesian products*, Amer. Math. Monthly **89** (1982), no. 9, 654–678. MR 678810 (84e:54047)

[Bro84] Kenneth S. Brown, *Presentations for groups acting on simply-connected complexes*, J. Pure Appl. Algebra **32** (1984), no. 1, 1–10. MR 739633

[Bro94] ______, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. MR 1324339 (96a:20072)

[Bro06] Ronald Brown, *Topology and groupoids*, BookSurge, LLC, Charleston, SC, 2006, Third edition of ıt Elements of modern topology [McGraw-Hill, New York, 1968; MR0227979], With 1 CD-ROM (Windows, Macintosh and UNIX). MR 2273730

[BSC17] Jonathan Ariel Barmak and Iván Sadofschi Costa, *On a question of R. H. Bing concerning the fixed point property for two-dimensional polyhedra*, Adv. Math. **305** (2017), 339–350. MR 3570138

[BT07] Martin R. Bridson and Michael Tweedale, *Deficiency and abelianized deficiency of some virtually free groups*, Math. Proc. Cambridge Philos. Soc. **143** (2007), no. 2, 257–264. MR 2364648

[CD92] Carles Casacuberta and Warren Dicks, *On finite groups acting on acyclic complexes of dimension two*, Publicacions Matemàtiques (1992), 463–466.

[CHRR07] Colin M. Campbell, George Havas, Colin Ramsay, and Edmund F. Robertson, *On the efficiency of the simple groups of order less than a million and their covers*, Experiment. Math. **16** (2007), no. 3, 347–358. MR 2367323 (2009a:20051)

[CHRR14] ———, *All simple groups with order from 1 million to 5 million are efficient*, Int. J. Group Theory **3** (2014), no. 1, 17–30. MR 3081126

[Coh67] Marshall M. Cohen, *Simplicial structures and transverse cellularity*, Ann. of Math. (2) **85** (1967), 218–245. MR 0210143 (35 #1037)

[Cor92] Jon M. Corson, *Complexes of groups*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 199–224. MR 1162493

[Cor01] ———, *On finite groups acting on contractible complexes of dimension two*, Geom. Dedicata **87** (2001), no. 1-3, 161–166. MR 1866846

[Dol95] Albrecht Dold, *Lectures on algebraic topology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1972 edition. MR 1335915 (96c:55001)

[DP13] Matthew Day and Andrew Putman, *The complex of partial bases for F_n and finite generation of the Torelli subgroup of* $\mathrm{Aut}(F_n)$, Geom. Dedicata **164** (2013), 139–153. MR 3054621

[Edj03] Martin Edjvet, *On irreducible cyclic presentations*, J. Group Theory **6** (2003), no. 2, 261–270. MR 1961572

[EG57] Samuel Eilenberg and Tudor Ganea, *On the Lusternik-Schnirelmann category of abstract groups*, Ann. of Math. (2) **65** (1957), 517–518. MR 0085510

[EHT01] Martin Edjvet, Paul Hammond, and Nathan Thomas, *Cyclic presentations of the trivial group*, Experiment. Math. **10** (2001), no. 2, 303–306. MR 1837678

[Ell13] Graham Ellis, *HAP, Homological Algebra Programming, Version 1.10.15*, http://hamilton.nuigalway.ie/Hap/www, Dec 2013, Refereed GAP package.

[ES14] Martin Edjvet and Jerry Swan, *On irreducible cyclic presentations of the trivial group*, Exp. Math. **23** (2014), no. 2, 181–189. MR 3223773

[Eve72] Leonard Evens, *The Schur multiplier of a semi-direct product*, Illinois J. Math. **16** (1972), 166–181. MR 0417310

[Fad70] Edward Fadell, *Recent results in the fixed point theory of continuous maps*, Bull. Amer. Math. Soc. **76** (1970), 10–29. MR 0271935 (42 #6816)

[FP90] Rudolf Fritsch and Renzo A. Piccinini, *Cellular structures in topology*, Cambridge Studies in Advanced Mathematics, vol. 19, Cambridge University Press, Cambridge, 1990. MR 1074175 (92d:55001)

[FR59] Edwin E. Floyd and Roger W. Richardson, *An action of a finite group on an n-cell without stationary points*, Bull. Amer. Math. Soc. **65** (1959), 73–76. MR 0100848

[FR96] Roger Fenn and Colin Rourke, *Klyachko's methods and the solution of equations over torsion-free groups*, Enseign. Math. (2) **42** (1996), no. 1-2, 49–74. MR 1395041

[GAP18] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3*, 2018.

[Ger84] Steve M. Gersten, *A presentation for the special automorphism group of a free group*, J. Pure Appl. Algebra **33** (1984), no. 3, 269–279. MR 761633

[GKKL11] Robert M. Guralnick, William M. Kantor, Martin Kassabov, and Alexander Lubotzky, *Presentations of finite simple groups: a computational approach*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 2, 391–458. MR 2746771 (2011m:20035)

[GL91] Mauricio Gutierrez and M. Paul Latiolais, *Partial homotopy type of finite two-complexes*, Math. Z. **207** (1991), no. 3, 359–378. MR 1115169 (92h:55007)

[GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple *K*-groups. MR 1490581 (98j:20011)

[GLS00] ———, *The classification of the finite simple groups. Number 1. Part I. Chapter 1*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 2000.

[GR62] Murray Gerstenhaber and Oscar S. Rothaus, *The solution of sets of equations in groups*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1531–1533. MR 0166296

[Hag07] Charles L. Hagopian, *An update on the elusive fixed-point property*, Open Problems in Topology. II (E. Pearl, ed.), Elsevier B. V., 2007, pp. 263–277.

[HAMS93] Cynthia Hog-Angeloni, Wolfgang Metzler, and Allan J. Sieradski (eds.), *Two-dimensional homotopy and combinatorial group theory*, London Mathematical Society Lecture Note Series, vol. 197, Cambridge University Press, Cambridge, 1993. MR 1279174 (95g:57006)

[Har81] William J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 245–251. MR 624817

[Har85] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) **121** (1985), no. 2, 215–249. MR 786348

[Har00] Jens Harlander, *Some aspects of efficiency*, Groups—Korea '98 (Pusan), de Gruyter, Berlin, 2000, pp. 165–180. MR 1751092

[Har15] ———, *On the relation gap and relation lifting problem*, Groups St Andrews 2013, London Math. Soc. Lecture Note Ser., vol. 422, Cambridge Univ. Press, Cambridge, 2015, pp. 278–285. MR 3495661

[Har18] ———, *The relation gap problem*, Advances in two-dimensional homotopy and combinatorial group theory, London Math. Soc. Lecture Note Ser., vol. 446, Cambridge Univ. Press, Cambridge, 2018, pp. 128–148. MR 3752470

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[HH17] Arnaud Hilion and Camille Horbez, *The hyperbolicity of the sphere complex via surgery paths*, J. Reine Angew. Math. **730** (2017), 135–161. MR 3692016

[HK93] Ian Hambleton and Matthias Kreck, *Cancellation of lattices and finite two-complexes*, J. Reine Angew. Math. **442** (1993), 91–109. MR 1234837 (94i:57036)

[How81] James Howie, *On pairs of* 2*-complexes and systems of equations over groups*, J. Reine Angew. Math. **324** (1981), 165–174. MR 614523

[How02] ———, *A proof of the Scott-Wiegold conjecture on free products of cyclic groups*, J. Pure Appl. Algebra **173** (2002), no. 2, 167–176. MR 1915093

[HR03] George Havas and Edmund F. Robertson, *Irreducible cyclic presentations of the trivial group*, Experiment. Math. **12** (2003), no. 4, 487–490. MR 2043998

[Hus77] Sufian Y. Husseini, *The products of manifolds with the f.p.p. need not have the f.p.p*, Amer. J. Math. **99** (1977), no. 5, 919–931. MR 0454961 (56 #13203)

[HV98] Allen Hatcher and Karen Vogtmann, *The complex of free factors of a free group*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 196, 459–468. MR 1660045

[Jia80] Boju Jiang, *On the least number of fixed points*, Amer. J. Math. **102** (1980), no. 4, 749–763. MR 584467 (82k:55004)

[Jia83] ———, *Lectures on Nielsen fixed point theory*, Contemporary Mathematics, vol. 14, American Mathematical Society, Providence, R.I., 1983. MR 685755 (84f:55002)

[JM06] Jerzy Jezierski and Waclaw Marzantowicz, *Homotopy methods in topological fixed and periodic points theory*, Topological Fixed Point Theory and Its Applications, vol. 3, Springer, Dordrecht, 2006. MR 2189944 (2006i:55003)

[JW04] Craig A. Jensen and Nathalie Wahl, *Automorphisms of free groups with boundaries*, Algebr. Geom. Topol. **4** (2004), 543–569. MR 2077676

[Kal92] Sašo Kalajdževski, *Automorphism group of a free group: centralizers and stabilizers*, J. Algebra **150** (1992), no. 2, 435–502. MR 1176906

[Ker69] Michel A. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. **144** (1969), 67–72. MR 0253347

[Kin53] Shin'ichi Kinoshita, *On some contractible continua without fixed point property*, Fund. Math. **40** (1953), 96–98. MR 0060225 (15,642b)

[KLV01] Sava Krstić, Martin Lustig, and Karen Vogtmann, *An equivariant Whitehead algorithm and conjugacy for roots of Dehn twist automorphisms*, Proc. Edinb. Math. Soc. (2) **44** (2001), no. 1, 117–141. MR 1879214

[Kly93] Anton A. Klyachko, *A funny property of sphere and equations over groups*, Comm. Algebra **21** (1993), no. 7, 2555–2575. MR 1218513

[KR14] Ilya Kapovich and Kasra Rafi, *On hyperbolicity of free splitting and free factor complexes*, Groups Geom. Dyn. **8** (2014), no. 2, 391–414. MR 3231221

[Krs89] Sava Krstić, *Actions of finite groups on graphs and related automorphisms of free groups*, J. Algebra **124** (1989), no. 1, 119–138. MR 1005698

[KS79] Robion C. Kirby and Martin G. Scharlemann, *Eight faces of the Poincaré homology* 3*-sphere*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977) (1979), 113–146. MR 537730

[KS17] Slawomir Kwasik and Fang Sun, *Manifolds with the fixed point property and their squares*, Topology Appl. **216** (2017), 129–136. MR 3584128

[Kur30] Kazimierz Kuratowski, *Problem 49*, Fund. Math. **15** (1930), 356.

[Kur68] ———, *Topology vol. ii*, Academic Press, 1968.

[Lop67] William Lopez, *An example in the fixed point theory of polyhedra*, Bulletin of the American Mathematical Society **73** (1967), no. 6, 922–924.

[LS77] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin-New York, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. MR 0577064

[Man13] Wajid H. Mannan, *A commutative version of the group ring*, J. Algebra **379** (2013), 113–143. MR 3019248

[McC74] James McCool, *A presentation for the automorphism group of a free group of finite rank*, J. London Math. Soc. (2) **8** (1974), 259–266. MR 0340421

[McC75] ______, *Some finitely presented subgroups of the automorphism group of a free group*, J. Algebra **35** (1975), 205–213. MR 0396764

[MKS76] Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory*, revised ed., Dover Publications, Inc., New York, 1976, Presentations of groups in terms of generators and relations. MR 0422434

[MM96] Darryl McCullough and Andy Miller, *Symmetric automorphisms of free products*, Mem. Amer. Math. Soc. **122** (1996), no. 582, viii+97. MR 1329943

[MM99] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149. MR 1714338

[Mun84] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984. MR 755006 (85m:55001)

[Neu56] Bernhard H. Neumann, *On some finite groups with trivial multiplicator*, Publ. Math. Debrecen **4** (1956), 190–194. MR 0078997 (18,12e)

[NT18] Martin Nitsche and Andreas Thom, *Universal solvability of group equations*, https://arxiv.org/abs/1811.07737, 2018.

[Oli75] Robert Oliver, *Fixed-point sets of group actions on finite acyclic complexes*, Comment. Math. Helv. **50** (1975), 155–177. MR 0375361

[OS02] Bob Oliver and Yoav Segev, *Fixed point free actions on* $\mathbf{Z}$*-acyclic 2-complexes*, Acta Math. **189** (2002), no. 2, 203–285. MR 1961198

[OT13] Denis Osin and Andreas Thom, *Normal generation and* ℓ^2*-Betti numbers of groups*, Math. Ann. **355** (2013), no. 4, 1331–1347. MR 3037017

[Pes08] Vladimir G. Pestov, *Hyperlinear and sofic groups: a brief guide*, Bulletin of Symbolic Logic **14** (2008), no. 4, 449–480.

[Pit16] Kevin I. Piterman, *El tipo homotópico de los posets de p-subgrupos*, Tesis de Licenciatura, Departamento de Matemática, FCEyN, UBA. Available at http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2016/Kevin_Piterman.pdf, 2016.

[PSCV18] Kevin I. Piterman, Iván Sadofschi Costa, and Antonio Viruel, *Quillen's conjecture for groups of p-rank* 3, In preparation (2018).

[Qui78] Daniel Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Adv. in Math. **28** (1978), no. 2, 101–128. MR 493916

[Rhe81] Akbar H. Rhemtulla, *Groups of finite weight*, Proc. Amer. Math. Soc. **81** (1981), no. 2, 191–192. MR 593454

[Ros94] Jonathan M. Rosenberg, *Algebraic K-theory and its applications*, Graduate Texts in Mathematics, vol. 147, Springer-Verlag, New York, 1994. MR 1282290 (95e:19001)

[Rot09] Joseph J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, New York, 2009. MR 2455920

[SC15] Iván Sadofschi Costa, *La propiedad del punto fijo para poliedros de dimensión dos*, Tesis de Licenciatura, Departamento de Matemática, FCEyN, UBA. Available at http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2015/Ivan_Sadofschi_Costa.pdf, 2015.

[SC17a] ———, *The complex of partial bases of a free group*, https://arxiv.org/abs/1711.09954, 2017.

[SC17b] ———, *Presentation complexes with the fixed point property*, Geom. Topol. **21** (2017), no. 2, 1275–1283. MR 3626602

[SC18a] ———, *G2Comp - Equivariant Two Complexes, Version 1.0.2*, GAP package, https://github.com/isadofschi/g2comp, 2018.

[SC18b] ———, *SmallCancellation - Metric and nonmetric small cancellation conditions, Version 1.0.2*, GAP package, https://github.com/isadofschi/smallcancellation, 2018.

[Seg93] Yoav Segev, *Group actions on finite acyclic simplicial complexes*, Israel J. Math. **82** (1993), no. 1-3, 381–394. MR 1239057

[Seg94] ———, *Some remarks on finite* 1*-acyclic and collapsible complexes*, J. Combin. Theory Ser. A **65** (1994), no. 1, 137–150. MR 1255267

[Ser80] Jean-Pierre Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980, Translated from the French by John Stillwell. MR 607504

[Shi66] Gen-hua Shi, *On least number of fixed points and nielsen numbers*, Acta Math. Sinica **16** (1966), 223–232.

[Smi11] Stephen D. Smith, *Subgroup complexes*, Mathematical Surveys and Monographs, vol. 179, American Mathematical Society, Providence, RI, 2011. MR 2850680

[Spa66] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1966, Corrected reprint of the 1966 original. MR 1325242 (96a:55001)

[SS74] George S. Sacerdote and Paul E. Schupp, *SQ-universality in HNN groups and one relator groups*, J. London Math. Soc. (2) **7** (1974), 733–740. MR 0364464

[Swa65] Richard G. Swan, *Minimal resolutions for finite groups*, Topology **4** (1965), 193–208. MR 0179234 (31 #3482)

[tD08] Tammo tom Dieck, *Algebraic topology*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR 2456045

[Tho12] Andreas Thom, *Are acyclic subcomplexes of finite contractible 2-complexes contractible?*, MathOverflow, 2012, https://mathoverflow.net/q/87601 (version: 2012-02-05).

[Vog02] Karen Vogtmann, *Automorphisms of free groups and outer space*, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000), vol. 94, 2002, pp. 1–31. MR 1950871

[Wag72] Roger Waggoner, *A fixed point theorem for* $(n-2)$*-connected n-polyhedra*, Proc. Amer. Math. Soc. **33** (1972), 143–145. MR 0293622 (45 #2699)

[Wag75] ———, *A method of combining fixed points*, Proc. Amer. Math. Soc. **51** (1975), 191–197. MR 0402713 (53 #6527)

[Wec42] Franz Wecken, *Fixpunktklassen. III. Mindestzahlen von Fixpunkten*, Math. Ann. **118** (1942), 544–577. MR 0010281 (5,275b)

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)

[Whi41] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2) **42** (1941), 409–428. MR 0004123

[Zee64] E. Christopher Zeeman, *On the dunce hat*, Topology **2** (1964), 341–358. MR 0156351

[Zim96] Bruno Zimmermann, *Finite groups of outer automorphisms of free groups*, Glasgow Math. J. **38** (1996), no. 3, 275–282. MR 1417356