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Cohomología de Hochschild de álgebras de operadores diferenciales asociadas a arreglos de hiperplanos

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Cohomología de Hochschild de álgebras de operadores diferenciales asociadas a arreglos de hiperplanos

Dado un arreglo de hiperplanos $\mathcal{A}$ en un espacio vectorial V sobre un cuerpo de característica cero, estudiamos el álgebra $\mathsf{Diff}(\mathcal{A})$ de operadores diferenciales en V tangentes a los hiperplanos de $\mathcal{A}$ desde el punto de vista del álgebra homológica.

Hacemos un estudio detallado de este álgebra para el caso de un arreglo central de rectas en un espacio vectorial de dimensión 2. Entre otras cosas, determinamos la cohomología de Hochschild $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ como álgebra de Gerstenhaber, establecemos un vínculo entre ésta y la cohomología de de Rham del complemento $M(\mathcal{A})$ del arreglo, determinamos el grupo de isomorfismos de $\mathsf{Diff}(\mathcal{A})$, clasificamos las álgebras de esta forma a menos de isomorfismo y estudiamos las deformaciones formales de $\mathsf{Diff}(\mathcal{A})$.

Mostramos que en el contexto general de un arreglo de hiperplanos de dimensión arbitraria el álgebra $\mathsf{Diff}(\mathcal{A})$ es isomorfa al álgebra envolvente del par de Lie–Rinehart formado por el álgebra de funciones coordenadas del espacio vectorial y el álgebra de Lie de derivaciones tangentes al arreglo. El cálculo de la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ puede ser ubicado entonces en el contexto del cálculo de la del álgebra envolvente U de un par de Lie–Rinehart (S, L): damos un método para hacer esto en el caso en que L es un S-módulo proyectivo. Concretamente, presentamos una sucesión espectral que converge a $HH^\bullet(U)$ cuya segunda página involucra la cohomología de Lie–Rinehart del par (S, L) y la cohomología de Hochschild de S a valores en U.

Palabras clave: Arreglos de hiperplanos, Cohomología de Hochschild, Álgebras de operadores diferenciales, Pares de Lie–Rinehart, Teoría de deformaciones.

Hochschild cohomology of algebras of differential operators associated with hyperplane arrangements

Given a free hyperplane arrangement $\mathcal{A}$ in a vector space V over a field of characteristic zero, we study the algebra $\mathsf{Diff}(\mathcal{A})$ of differential operators on V which are tangent to the hyperplanes of $\mathcal{A}$ from the point of view of homological algebra.

We make a thorough study of this algebra for the case of a central arrangement of lines in a vector space of dimension 2. Among other things, we determine the Hochschild cohomology $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ as a Gerstenhaber algebra, establish a connection between that cohomology and the de Rham cohomology of the complement $M(\mathcal{A})$ of the arrangement, determine the isomorphism group of $\mathsf{Diff}(\mathcal{A})$, classify the algebras of that form up to isomorphism and study the formal deformations of $\mathsf{Diff}(\mathcal{A})$.

We show that in the general setting of a free arrangement of hyperplanes of arbitrary dimension the algebra $\mathsf{Diff}(\mathcal{A})$ is isomorphic to the enveloping algebra of the Lie–Rinehart pair formed by the algebra of coordinates functions on the vector space and the Lie algebra of derivations tangent to the arrangement. The computation of the Hochschild cohomology of $\mathsf{Diff}(\mathcal{A})$ can be then put in the context of computing that of the enveloping algebra U of a Lie–Rinehart pair (S, L): we provide a method to do this if L is S-projective. Concretely, we present a spectral sequence which converges to $HH^\bullet(U)$ and whose second page involves the Lie—Rinehart cohomology of the pair and the Hochschild cohomology of S with values on U.

Keywords: Hyperplane arrangements, Hochschild cohomology, Algebras of differential operators, Lie–Rinehart pairs, Deformation theory.

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Contents

Introduction

Let us fix a ground field $\Bbbk$ of characteristic zero, a vector space V of finite dimension and a central arrangement of hyperplanes $\mathcal{A}$ in V, that is, a finite set $\{H_1, \ldots, H_l\}$ of subspaces of V of codimension 1. For each $i \in \{1, \ldots, l\}$, let $\alpha_i : V \to \Bbbk$ be a linear form with kernel H_i. We let S be the algebra of polynomial functions of V, fix a defining polynomial $Q = \alpha_1 \cdots \alpha_l \in S$ for $\mathcal{A}$, and consider, following K. Saito in [Sai80], the Lie algebra

$$\mathsf{Der}\,\mathcal{A} = \{\delta \in \mathsf{Der}(S) : \delta(Q) \in QS\}$$

of derivations of S logarithmic with respect to $\mathcal{A}$, which is, geometrically speaking, the Lie algebra of vector fields on V which are tangent to the hyperplanes of $\mathcal{A}$. This Lie algebra is a very interesting invariant of the arrangement and has been the subject of a lot of work — we refer to the book of P. Orlik and H. Terao [OT92] and the one by A. Dimca [Dim17] for surveys on this subject. In particular, using this Lie algebra we can define an important class of arrangements: we say that an arrangement $\mathcal{A}$ is *free* if $\mathsf{Der}\,\mathcal{A}$ is free as a left S-module. For example, central arrangements of lines in the plane are free, as are, according to a beautiful result of Terao [Ter80a], the arrangements of reflecting hyperplanes of a finite group generated by pseudo-reflections.

Now, along with $\mathsf{Der}\,\mathcal{A}$ we can consider also the associative algebra $\mathsf{Diff}(\mathcal{A})$ of differential operators on S which preserve the ideal QS of S and all its powers: we call it the *algebra of differential operators tangent* to the arrangement $\mathcal{A}$. As shown by F. J. Calderón-Moreno in [CM99] or by M. Suárez-Álvarez in [SÁ18], when $\mathcal{A}$ is free $\mathsf{Diff}(\mathcal{A})$ coincides with the subalgebra of the algebra $\mathsf{End}_\Bbbk(S)$ of linear endomorphisms of the vector space S generated by $\mathsf{Der}\,\mathcal{A}$ and the set of maps given by left multiplication by elements of S. The algebraic structure of $\mathsf{Diff}(\mathcal{A})$ is determined by both the S-module structure of $\mathsf{Der}\,\mathcal{A}$ and its Lie structure, so it is a very natural object to study. The main goal of this thesis is precisely to do this from the point of view of homological algebra and deformation theory in the special situation in which the arrangement $\mathcal{A}$ is free.

Our first step is to find a description of the algebra $\mathsf{Diff}(\mathcal{A})$ that is convenient for performing explicit calculations. The language of Lie–Rinehart pairs provides the required formalism to do this: indeed, the pair $(S, \mathsf{Der}\,\mathcal{A})$ determined by the polynomial algebra S and the Lie algebra of derivations tangent to $\mathcal{A}$ is a Lie–Rinehart pair, as those studied by G. Rinehart in [Rin63] and by J. Huebschmann in [Hue90], and the algebra $\mathsf{Diff}(\mathcal{A})$ can be identified to the universal enveloping algebra $U(S, \mathsf{Der}\,\mathcal{A})$ of this pair. This is the content of our Theorem 2.19.

Theorem. *Let $\mathcal{A}$ be a free hyperplane arrangement on a vector space V and let S be the algebra of coordinate functions on V. There is a canonical isomorphism of algebras*

$$U(S, \mathsf{Der}\,\mathcal{A}) \to \mathsf{Diff}(\mathcal{A}).$$

That there is such a morphism and that it is surjective is an interpretation of the results in [CM99] and in [SÁ18]. To prove that it is injective we use the calculation of the Gelfand-Kirillov dimension of the enveloping algebra of a Lie–Rinehart pair done by J. Matczuk in [Mat88] and the fact that $\mathsf{Diff}(\mathcal{A})$ and the algebra of differential operators on S become isomorphic after localization at the single element Q. With this theorem at hand we are able to give in Proposition 2.20 a presentation of $\mathsf{Diff}(\mathcal{A})$ by generators and relations and, using the results by Th. Lambre and P. Le Meur in [LLM18], to prove in **2.25** that $\mathsf{Diff}(\mathcal{A})$ has the twisted Calabi–Yau property.

We then focus on central arrangement of lines $\mathcal{A}$ in a 2-dimensional vector space V with at least five lines, which are the simplest free arrangements. The bulk of Chapter 3 is a lengthy calculation that culminates in Proposition 3.15, where we give a description of the Hochschild cohomology of $\mathsf{Diff}(\mathcal{A})$ in a completely explicit fashion. Let us just state here the following result, which follows from the proposition, and omit the details.

Proposition. *If $\mathcal{A}$ is a central line arrangement of l lines with $l \geq 5$, the Hilbert series of $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ is*

$$h_{HH^\bullet(U)}(t) = 1 + lt + (2l-1)t^2 + lt^3.$$

When the arrangement consists of less than five lines, the conclusion of the proposition does not hold: we deal with this special case using different techniques by the end of the thesis.

The next step is to describe the algebra structure of $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ and its Gerstenhaber structure: it is in order to do this that we need such an explicit description. The results appear in Proposition 3.19 and 3.22 and are, again, too technical to reproduce here. In any case, these structures provide a better understanding of our computations and allow us to relate $\mathsf{Diff}(\mathcal{A})$ with a well-known invariant of the arrangement, the Orlik–Solomon algebra. This algebra, studied by P. Orlik and L. Solomon in [OS80], is a combinatorial analogue of the algebra obtained as the de Rham cohomology of the complement of $\mathcal{A}$ for the case in which $\Bbbk = \mathbb{C}$, which was found by E. Brieskorn in [Bri73] and, previously, by V. I. Arnold in [Arn69] for the family of braid arrangements. This algebra appears in our situation in Proposition 3.20:

Proposition. *The subalgebra $\mathcal{H}$ of $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ generated by $HH^1(\mathsf{Diff}(\mathcal{A}))$ is isomorphic to the Orlik–Solomon algebra of $\mathcal{A}$.*

Along with these results, we are also able to obtain the Hochschild homology, the cyclic homology, the periodic cyclic homology, the K-theory of $\mathsf{Diff}(\mathcal{A})$ and a *direct* proof of the twisted Calabi–Yau property for the special case of line arrangements: these are the contents of Propositions 3.23 and 3.25.

We can extract consequences of our computation of cohomology. Indeed, applying the methods developed by J. Alev and M. Chamarie in [AC92], we are able to describe the automorphism group $\mathsf{Diff}(\mathcal{A})$ in Theorem 4.7:

Theorem. *The group* $\mathsf{Aut}(\mathsf{Diff}(\mathcal{A}))$ *is the semidirect product* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A})) \ltimes \mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *of the subgroups* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *of automorphisms of* $\mathsf{Diff}(\mathcal{A})$ *that preserve the grading and* $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *of exponentials of locally nilpotent inner derivations of* $\mathsf{Diff}(\mathcal{A})$. *The action of* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *on* $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *is given by*

$$\theta_0 \cdot \exp \mathsf{ad}(f) = \exp \mathsf{ad}(\theta^{-1}(f))$$

for all $\theta_0 \in \mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *and* $f \in S$.

Along with this theorem, we give in Chapter 4 a complete description of the groups $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A})))$ and $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$. We show that the first one is a finite dimensional algebraic group which "sees" the symmetries of the arrangement and the second one is an infinite dimensional group whose structure does not depend on the arrangement at all. This description of the automorphism group, in turn, allows us to give a complete solution to the problem of determining which pairs of arrangements of lines $\mathcal{A}$ and $\mathcal{A}'$ have isomorphic algebras $\mathsf{Diff}(\mathcal{A})$ and $\mathsf{Diff}(\mathcal{A}')$.

Proposition. *Two central arrangements of lines have isomorphic algebras of differential operators if and only if they are themselves isomorphic.*

The explicitness of our calculation of the Hochschild cohomology of $\mathsf{Diff}(\mathcal{A})$ continues to be useful: in Section 5.2 we put to use our findings on $HH^2(\mathsf{Diff}(\mathcal{A}))$ to study the formal deformation theory of the algebra $\mathsf{Diff}(\mathcal{A})$ in the sense of M. Gerstenhaber [Ger64]. With the help of the Diamond Lemma of G. Bergman [Ber78] we show, on one hand, that many of the infinitesimal deformations of the algebra can be integrated to formal deformations and, on the other, exhibit obstructed infinitesimal deformations.

Let us go back to the general case of free arrangements of hyperplanes of arbitrary dimension. As we mentioned above, the pair $(S, \mathsf{Der}\,\mathcal{A})$ determined by the polynomial algebra S and the Lie algebra $\mathsf{Der}\,\mathcal{A}$ is a Lie–Rinehart pair and its enveloping algebra is isomorphic to $\mathsf{Diff}(\mathcal{A})$. In view of this observation, the problem of determining the Hochschild cohomology of $\mathsf{Diff}(\mathcal{A})$ has a rather natural generalization: given a Lie–Rinehart pair (S, L) with universal enveloping algebra $U = U(S, L)$, to determine the Hochschild cohomology $HH^\bullet(U)$.

Following the ideas of Th. Lambre and P. Le Meur in [LLM18], we construct a spectral sequence that reduces that problem to the computation of the Hochschild cohomology of the commutative algebra S with values in U and the Lie–Rinehart cohomology of the pair (S, L). Explicitly, we obtain the following result in Corollary 6.8.

Theorem. *For each* U*-bimodule* M *there is a first-quadrant spectral sequence* $E_\bullet$ *converging to* $HH^\bullet(U, M)$ *such that*

$$E_2^{p,q} \cong H^p(L|S, H^q(S, M)).$$

We give several concrete examples in which this spectral sequence makes it possible to completely determine $HH^\bullet(U)$ and we show how this method is applied to the special case of the computation of the cohomology of the algebra $\mathsf{Diff}(\mathcal{A})$ associated to a line arrangement. In particular, we extend our computation of the Hochschild cohomology of $\mathsf{Diff}(\mathcal{A})$ as a graded vector space to arrangements with 3 or 4 lines, which were excluded before. This result appears in the text as Proposition 6.50 for the case of 3 lines. In order to perform this computation, it is important to have a concrete description of the action of U on the Hochschild cohomology $H^\bullet(S, U)$ as computed from a projective resolution of S: we are able to obtain it in Theorem 6.18 following [SÁ17]. Finally, to enrich the description of our spectral sequence, we provide in Theorem 6.30 an interpretation of the differentials of its page E_2 in terms of appropriate *cup* products that is obtained emulating what is done in [SÁ07].

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Let us end this introduction with a brief summary of the contents of the thesis.

In Chapter 1, we provide definitions, examples and results from the theory of hyperplane arrangements that will be useful throughout the thesis. We first focus on the general setting of hyperplane arrangements, the module of derivations and the complex of logarithmic forms. With these notions at hand, we present some of the results that started to raise interest in the area and that relate the cohomology of the complement space of a complex arrangement with other constructions, such as those by V. I. Arnold in [Arn69], by E. Brieskorn's in [Bri73] and by P. Orlik and L. Solomon in [OS80].

In Chapter 2 we present the algebra $\mathsf{Diff}(\mathcal{A})$ of differential operators tangent to a hyperplane arrangement $\mathcal{A}$ and give a useful description of this algebra for the case of a free arrangement building on the results of [SÁ18]. We then turn on to the case of central line arrangements, providing a presentation and showing that in this case the algebra is isomorphic to an iterated Ore extension. After that, we recall the notions of Lie–Rinehart pairs and of their enveloping algebras, due to G. Rinehart in [Rin63]. These concepts are proven vital for us because the algebra $\mathsf{Diff}(\mathcal{A})$ is isomorphic to the enveloping algebra of the pair $(S, \mathsf{Der}\,\mathcal{A})$ whenever $\mathcal{A}$ is free. Using this, we find a presentation for $\mathsf{Diff}(\mathcal{A})$ and show it has the twisted Calabi–Yau property using a result from [LLM18].

From Chapters 3 to 5 we study the case of a central arrangement of lines $\mathcal{A}$ in a 2-dimensional vector space over a field of characteristic zero. We determine the Hochschild cohomology of $\mathsf{Diff}(\mathcal{A})$ as a Gerstenhaber algebra, establish a connection between that cohomology and the de Rham cohomology of the complement $M(\mathcal{A})$ of the arrangement, determine the isomorphism group of $\mathsf{Diff}(\mathcal{A})$, classify the algebras of that form up to isomorphism and study their deformation theory.

In our final Chapter 6, we construct a spectral sequence converging to the Hochschild cohomology of the enveloping algebra of a Lie–Rinehart pair, we show that the differentials on its second page are given by cup products and we end the thesis by using the spectral sequence to extend our results on the Hochschild cohomology of $\mathsf{Diff}(\mathcal{A})$ to the case in which $\mathcal{A}$ has few lines — this was in fact our motivation for the construction of the sequence.

The contents of Chapters 3 and 4 form part of the article [KSÁ18], which has been accepted for publication in *Documenta Mathematica*. On the other hand, the preprint [Kor18] contains most of Chapter 6 and has been submitted.

Introducción

Sean $\Bbbk$ un cuerpo de característica cero, V un espacio vectorial de dimensión finita y $\mathcal{A}$ un arreglo de hiperplanos de V, esto es, un conjunto finito $\{H_1, \dots, H_n\}$ de subespacios de V de codimensión 1. Para cada $i \in \{1, \dots, l\}$, sea $\alpha_i : V \to \Bbbk$ una forma lineal con núcleo H_i. Llamemos S al álgebra de funciones coordenadas en V, fijemos el polinomio $Q = \alpha_1 \cdots \alpha_l \in S$, que *define* $\mathcal{A}$, y consideremos, siguiendo a K. Saito in [Sai80], el álgebra de Lie

$$\operatorname{Der} \mathcal{A} = \{\delta \in \operatorname{Der}(S) : \delta(Q) \in QS\}$$

de derivaciones de S logarítmicas con respecto a $\mathcal{A}$, que es, en términos geométricos, el álgebra de Lie de campos vectoriales en V que son tangentes a los hiperplanos de $\mathcal{A}$. Este álgebra de Lie es un invariante interesante del arreglo y ha sido objeto de estudio de varios trabajos: el libro de P. Orlik y H. Terao [OT92] y el de A. Dimca [Dim17] son útiles como referencias generales. Sirviéndonos del álgebra de Lie de derivaciones, podemos definir una clase importante de arreglos: decimos que un arreglo $\mathcal{A}$ es *libre* si $\operatorname{Der} \mathcal{A}$ es un S-módulo libre. Por ejemplo, un arreglo central de rectas en el plano es libre y son libres también, de acuerdo a un resultado de H. Terao en [Ter80a], los arreglos de hiperplanos de reflexión de un grupo finito generado por pseudo-reflexiones.

Junto con el álgebra de Lie $\operatorname{Der} \mathcal{A}$ podemos considerar el álgebra asociativa $\operatorname{Diff}(\mathcal{A})$ de operadores diferenciales en S que presevan el ideal QS de S y todas sus potencias: la llamamos el *álgebra de operadores diferenciales tangentes* al arreglo $\mathcal{A}$. Es un resultado de F. J. Calderón-Moreno en [CM99] y de M. Suárez-Álvarez en [SÁ18] que cuando $\mathcal{A}$ es libre, $\operatorname{Diff}(\mathcal{A})$ coincide con la subálgebra de $\operatorname{End}(S)$, el álgebra de endomorfismos lineales del espacio vectorial S, generada por $\operatorname{Der} \mathcal{A}$ y el conjunto de funciones dadas por la multiplicación a izquierda por elementos de S.

La estructura algebraica de $\operatorname{Diff}(\mathcal{A})$ está determinada por la estructura de S-módulo de $\operatorname{Der} \mathcal{A}$ y por su estructura de Lie, de manera que es un objeto muy natural de estudiar. El objetivo principal de esta tesis es precisamente hacer esto desde el punto de vista del álgebra homológica y la teoría de deformaciones en la situación especial en que el arreglo $\mathcal{A}$ es libre.

Nuestro primer paso es encontrar una descripción del álgebra $\operatorname{Diff}(\mathcal{A})$ que sea conveniente para realizar cálculos explícitos. El lenguaje de los pares de Lie–Rinehart nos provee del formalismo necesario: el par $(S, \operatorname{Der} \mathcal{A})$ determinado por el álgebra de polinomios S y el álgebra de Lie de derivaciones tangentes a $\mathcal{A}$ es un par de Lie–Rinehart, como los estudiados por G. Rinehart en [Rin63] y por J. Huebschmann en [Hue90], y el álgebra $\operatorname{Diff}(\mathcal{A})$ puede identificarse con el álgebra envolvente universal $U(S, \operatorname{Der} \mathcal{A})$ de este par. Este es el contenido de nuestro Teorema 2.19.

Teorema. *Sea $\mathcal{A}$ un arreglo de hiperplanos libre en un espacio vectorial V y sea S el álgebra de funciones coordenadas en V. Hay un isomorfismo canónico de álgebras*

$$U(S, \mathsf{Der}\,\mathcal{A}) \to \mathsf{Diff}(\mathcal{A}).$$

La existencia de este morfismo sigue inmediatamente de los resultados en [CM99] y en [SÁ18]. Para probar que es inyectivo, utilizamos el cálculo de la dimensión de Gelfand-Kirillov del álgebra envolvente de un par de Lie–Rinehart hecho por J. Matczuk en [Mat88] y el hecho de que $\mathsf{Diff}(\mathcal{A})$ y el álgebra de operadores diferenciales en S se tornan isomorfas al localizar en el elemento Q. Con este teorema a mano, damos en la Proposición 2.20 una presentation de $\mathsf{Diff}(\mathcal{A})$ por generadores y relaciones y, usando los resultados de Th. Lambre y P. Le Meur en [LLM18], probamos en **2.25** que $\mathsf{Diff}(\mathcal{A})$ tiene la propiedad de Calabi–Yau torcida.

Nos enfocamos después en los arreglos centrales de rectas $\mathcal{A}$ en un espacio vectorial V de dimensión 2 con al menos cinco rectas, que son los arreglos libres más simples. Una buena parte del Capítulo 3 es un cálculo extenso que culmina en la Proposición 3.15, en la que damos una descripción de la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ de manera completamente explícita. Sin entrar en detalles, la proposición nos da la siguiente información.

Proposición. *Si $\mathcal{A}$ es un arreglo central de rectas de l rectas con $l \geq 5$, la serie de Hilbert de $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ es*

$$h_{HH^\bullet(U)}(t) = 1 + lt + (2l-1)t^2 + lt^3.$$

Cuando el arreglo tiene menos de cinco rectas, la conclusión de la proposición no sigue siendo cierta: lidiamos con esta situación especial utilizando técnicas diferentes sobre el final de la tesis.

El siguiente paso es describir la estructura de álgebra de $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ y su estructura de Gerstenhaber: es para esto que necesitamos una descripción tan explícita. Los resultados aparecen en las Proposiciones 3.19 y 3.22 y son, una vez más, demasiado técnicos para reproducir aquí. De cualquier manera, estas estructuras nos dan un mejor entendimiento de nuestros cálculos previos y nos permiten relacionar $\mathsf{Diff}(\mathcal{A})$ con un invariante conocido del arreglo, el álgebra de Orlik–Solomon. Este álgebra, estudiada por P. Orlik y L. Solomon en [OS80], es un análogo combinatorio del álgebra encontrada por E. Brieskorn en [Bri73] y, previamente, por V. I. Arnold en [Arn69] para la familia de arreglos de trenzas, como la cohomología de de Rham del complemento de $\mathcal{A}$ para el caso en que $\Bbbk = \mathbb{C}$. Aparece en nuestra situación en la Proposición 3.20:

Proposición. *La subálgebra $\mathcal{H}$ de $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ generada por $HH^1(\mathsf{Diff}(\mathcal{A}))$ es isomorfa al álgebra de Orlik–Solomon de $\mathcal{A}$.*

Además de estos resultados, calculamos la homología de Hochschild, la homología cíclica, la homología cíclica periódica y la K-teoría de $\mathsf{Diff}(\mathcal{A})$, y obtenemos una prueba *directa* de la propiedad de Calabi–Yau para el caso especial de arreglos centrales de rectas: estos son los contenidos de las Proposiciones 3.23 y 3.25.

Procedemos a continuación a extraer consecuencias de nuestro cálculo de la cohomología. Utlizando los métodos desarrollados por J. Alev y M. Chamarie en [AC92], describimos el grupo de automorfismos de $\mathsf{Diff}(\mathcal{A})$ en el Teorema 4.7:

Teorema. *El grupo* $\mathsf{Aut}(\mathsf{Diff}(\mathcal{A}))$ *es el producto semidirecto* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A})) \ltimes \mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *de los subgrupos* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *de automorfismos de* $\mathsf{Diff}(\mathcal{A})$ *que preservan la graduación y* $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *de las exponenciales de derivaciones internas localmente nilpotentes de* $\mathsf{Diff}(\mathcal{A})$. *La acción de* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *en* $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *está dada por*

$$\theta_0 \cdot \exp \mathsf{ad}(f) = \exp \mathsf{ad}(\theta^{-1}(f))$$

para cada $\theta_0 \in \mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *y* $f \in S$.

Junto con este teorema, damos en el Capítulo 4 una descripción completa de los grupos $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ y $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$. Mostramos que el primero es un grupo algebraico de dimensión finital que "ve" las simetrías del arreglo y que el segundo es un grupo de dimensión infinita cuya estructura es independiente del arreglo. Esta descripción del grupo de automorfismos, a su vez, nos permite dar una solución completa al problema de determinar cuáles pares de arreglos de rectas $\mathcal{A}$ y $\mathcal{A}'$ tienen álgebras $\mathsf{Diff}(\mathcal{A})$ y $\mathsf{Diff}(\mathcal{A}')$ isomorfas.

Proposición. *Dos arreglos de rectas tienen álgebras de operadores diferenciales isomorfas si y solo si son isomorfos.*

Lo explícito de nuestros resultados sobre la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ continúa siendo útil: en la Sección 5.2 utilizamos nuestra descripción de $HH^2(\mathsf{Diff}(\mathcal{A}))$ para estudiar la teoría de deformaciones formales del álgebra $\mathsf{Diff}(\mathcal{A})$ en el sentido de M. Gerstenhaber [Ger64]. Con la ayuda del Lema del diamante de G. Bergman [Ber78] mostramos, por un lado, que muchas de las deformaciones infinitesimales del álgebra pueden ser integradas a deformaciones y, por otro, exhibimos deformaciones infinitesimales obstruidas.

Volvamos ahora a la situación general de un arreglo libre de hiperplanos de dimensión arbitraria. Como mencionamos arriba, el par $(S, \mathsf{Der}\,\mathcal{A})$ determinado por el álgebra de polinomios S y el álgebra de Lie $\mathsf{Der}\,\mathcal{A}$ es un par de Lie–Rinehart y su álgebra envolvente es isomorfa a $\mathsf{Diff}(\mathcal{A})$. En vista de esta observación, el problema de determinar la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ tiene una generalización natural más bien clara: dado un par de Lie–Rinehart (S, L) con álgebra envolvente $U = U(S, L)$, determinar la cohomología de Hochschild $HH^\bullet(U)$.

Siguiendo las ideas de Th. Lambre y P. Le Meur en [LLM18], construimos una sucesión espectral que reduce el problema al del cálculo de la cohomología de Hochschild del álgebra conmutativa S a valores en U y al de la cohomología de Lie–Rinehart del par (S, L). Explícitamente, obtenemos el siguiente resultado en el Corolario 6.8.

Teorema. *Para cada U-bimódulo M hay una sucesión espectral $E_\bullet$ en el primer cuadante que converge a $HH^\bullet(U, M)$ tal que*

$$E_2^{p,q} \cong H^p(L|S, H^q(S, M)).$$

Después de este resultado, damos varios ejemplos en los que la sucesión espectral hace posible determinar completamente $HH^\bullet(U)$ y mostramos cómo este método se aplica al caso especial del cálculo de la cohomología del álgebra $\mathsf{Diff}(\mathcal{A})$ asociada a un arreglo de rectas. En particular, extendemos nuestros resultados del Capítulo 3 sobre la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ como espacio vectorial graduado a arreglos con 3 o 4 rectas, que habíamos excluído anteriormente. Este resultado aparece en el texto como la Proposición 6.50 para el caso de 3 rectas. Para realizar este cálculo, es importante tener una descripción concreta de la acción de U sobre la cohomología de Hochschild $H^\bullet(S, U)$ en la situación en que esta última es calculada mediante una resolución proyectiva de S: obtenemos tal descripción en el Teorema 6.18, siguiendo [SÁ17]. Finalmente, para enriquecer la descripción de nuestra sucesión espectral, damos en el Teorema 6.30 una interpretación de los diferenciales de la página E_2 en términos de productos *cup* apropiados, que se obtiene emulando lo hecho en [SÁ07].

∗ ∗ ∗

Terminamos esta introducción con un breve sumario de los contenidos de la tesis.

En el Capítulo 1, damos definiciones, ejemplos y resultados de la teoría de arreglos de hiperplanos que serán útiles a través de la tesis. Primero nos enfocamos en las nociones generales sobre los arreglos de hiperplanos, el módulo de derivaciones y el complejo de formas logarítmicas. Después, presentamos algunos de los resultados que empezaron a generar interés en el área y que relacionan la cohomología de de Rham del complemento de un arreglo complejo con otras construcciones, tales como las de V. I. Arnold en [Arn69], de E. Brieskorn en [Bri73] y de P. Orlik y L. Solomon en [OS80].

En el Capítulo 2 presentamos el álgebra $\mathsf{Diff}(\mathcal{A})$ de operadores diferenciales tangentes a un arreglo de hiperplanos $\mathcal{A}$ y damos una descripción conveniente de este álgebra para el caso de arreglos libres a partir de los resultados de [SÁ18]. A continuación, nos centramos en el caso de los arreglos centrales de rectas, dando una presentación y mostrando que en este caso el álgebra es isomorfa a una extensión de Ore iterada. Volviendo a la situación general, presentamos los pares de Lie–Rinehart y sus álgebras envolventes, que se deben a G. Rinehart en [Rin63]. Estos conceptos son vitales para nosotros puesto que el álgebra $\mathsf{Diff}(\mathcal{A})$ es isomorfa al álgebra envolvente del par $(S, \mathsf{Der}\,\mathcal{A})$ si $\mathcal{A}$ es libre. Usando esto, encontramos una presentación para $\mathsf{Diff}(\mathcal{A})$ y mostramos que tiene la propiedad de Calabi–Yau torcida usando un resultado de [LLM18].

Entre los Capítulos 3 y 5 estudiamos el caso de un arreglo central de rectas $\mathcal{A}$ en un espacio vectorial de dimensión 2 sobre un cuerpo de característica cero. Determinamos la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ como álgebra de Gerstenhaber, establecemos una conexión entre esta cohomología y la de de Rham del complemento $M(\mathcal{A})$ del arreglo, determinamos el grupo de isomorfismo de $\mathsf{Diff}(\mathcal{A})$, clasificamos las álgebras de esa forma a menos de isomorfismo y estudiamos su teoría de deformaciones.

En nuestro capítulo final, el Capítulo 6, construimos una sucesión espectral que converge a la cohomología de Hochschild del álgebra envolvente de un par de Lie–Rinehart, mostramos que los diferenciales de la segunda página están dados por productos *cup* y terminamos la tesis utilizando la sucesión espectral para extender nuestros resultados sobre la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ en el caso en que $\mathcal{A}$ tiene pocas rectas: ésta fue, de hecho, nuestra motivación para la construcción de la sucesión.

Los contenidos de los Capítulos 3 y 4 forman parte del artículo [KSÁ18], que ha sido aceptado para publicación en *Documenta Mathematica*. Por otro lado, el preprint [Kor18] contiene la mayoría del Capítulor 6 y ha sido submitido.

– 1 –
Hyperplane arrangements

In this chapter we define and illustrate the objects with which we deal throughout the thesis. The first definitions cover combinatorial aspects of hyperplane arrangements, the Lie module of derivations tangent to an arrangement, and the complex of logarithmic forms. Afterwards, we deal with the cohomology of the complement of a complex arrangement and its relation with our previous constructions. Finally, we present the Orlik–Solomon algebra, which is a combinatorial analogue of the algebra obtained as the cohomology of the complement in the general situation where the ground field is not $\mathbb{C}$.

1.1 Basic definitions

1.1. Let us first introduce some notation that we will keep throughout this thesis. We let $\Bbbk$ be a ground field and assume that all vector spaces and algebras are implicitly defined over $\Bbbk$. We will also take unadorned $\otimes$ and $\hom$ with respect to $\Bbbk$ and, sometimes, we will write $|$ instead of $\otimes$.

1.2. A *hyperplane arrangement* or an *arrangement of hyperplanes* $\mathcal{A}$ over $\Bbbk$ is a finite collection of affine hyperplanes $\{H_1, \dots, H_l\}$ of a $\Bbbk$-vector space V of finite dimension. Most of the times, we shall omit the reference to $\Bbbk$. An arrangement is *central* if its hyperplanes are actually subspaces.

We will denote the dimension of V by n and call it the dimension of $\mathcal{A}$. Choosing a basis, we may identify the algebra of coordinate functions on V with the polynomial algebra $S = \Bbbk[x_1, \dots, x_n]$. Let us denote, if l is any positive integer, the set $\{1, \dots, l\}$ by $[\![l]\!]$. For each $i \in [\![l]\!]$, let $\alpha_i : V \to \Bbbk$ be an affine function such that H_i is the zero locus of α_i. The *defining polynomial* of $\mathcal{A}$ is

$$Q(\mathcal{A}) = \alpha_1 \alpha_2 \cdots \alpha_l$$

and is usually denoted simply by Q. As different choices of linear forms give rise to the same arrangement, Q is defined up to a scalar multiple. If $\mathcal{A}$ is a central arrangement, Q is an homogeneous polynomial of degree l.

Unless we claim otherwise, we will keep in the general situation and with the notation explained in this paragraph.

1.3. Let $\mathcal{A}$ be a hyperplane arrangement. The *rank of* $\mathcal{A}$ is the dimension of the space $\mathcal{A}^\perp$ generated by the normals of its hyperplanes. We call $\mathcal{A}$ *essential* if its rank equals its dimension. The *complement* of $\mathcal{A}$ is the set $M(\mathcal{A}) = V \setminus \bigcup_{H\in\mathcal{A}} H$. If $\Bbbk = \mathbb{R}$, a connected component of $M(\mathcal{A})$ is called a *chamber*; the set of chambers of $\mathcal{A}$ is denoted by $C(\mathcal{A})$.

1.4. *Example.* The boolean arrangement Bool_n in $V = \Bbbk^n$ is defined by $Q = x_1x_2\cdots x_n$. It is an essential central arrangement. To determine the chamber to which a point belongs it is enough to give the signs of its coordinates: the number of chambers is thus 2^n.

1.5. *Example.* The braid arrangement $\mathcal{B}_n$ in $\Bbbk^n$ has hyperplanes

$$H_{ij} = \{(x_1,\dots,x_n) \in \Bbbk^n : x_i - x_j = 0\} \qquad \text{for } 1 \le i < j \le n,$$

so it has $\binom{n}{2}$ hyperplanes. This central arrangement is not essential, for the normal of each hyperplane satisfies the equation $x_1 + \dots + x_n = 0$: in fact, the rank of $\mathcal{B}_n$ is $n-1$. Let us now assume that $\Bbbk = \mathbb{R}$ and let $p = (p_1,\dots,p_n) \in \mathbb{R}^n$. Given i and j such that $1 \le i < j \le n$, we observe that p lies on one or another side of the hyperplane $x_i - x_j$ if and only if p_i is greater or smaller than p_j. As a consequence of this, a connected component of $M(\mathcal{A})$ is determined by a total order on $[\![n]\!]$, or, in other words, to a permutation of that set. The number of chambers of $M(\mathcal{A})$ is then $n!$.

1.6. There are many ways to construct arrangements; let us review two of the most important ones. If $\mathcal{A}_1$ is an arrangement in V_1 and $\mathcal{A}_2$ is an arrangement in V_2, their *product* $\mathcal{A}_1\times\mathcal{A}_2$ is the arrangement in $V_1\oplus V_2$ with hyperplanes $H\oplus V_2$ for $H \in \mathcal{A}_1$ and $V_1\oplus H$ for $H \in \mathcal{A}_2$. For example, the nth boolean arrangement Bool_n can be viewed as the n-fold product of the arrangement in $\Bbbk$ whose only hyperplane is the point at the origin. Furthermore, every arrangement can be viewed as the product of an empty arrangement and an essential arrangement, the *essentialization of* $\mathcal{A}$, whose hyperplanes are the intersections of those of $\mathcal{A}$ with the subspace $\mathcal{A}^\perp$. Notice that this construction produces an arrangement not in V but in $\mathcal{A}^\perp$. We call an arrangement reducible if it is, after a change of coordinates, the product of two arrangements on nonzero vector spaces.

Another basic construction is that of *coning*, whose point is to relate affine —that is, not necessarily central— and central arrangements: given an affine arrangement $\mathcal{A}$ in V, the cone $c\mathcal{A}$ is a central arrangement in $\Bbbk \oplus V$ such that $\mathcal{A}$ is "embedded" in $c\mathcal{A}$. Let us denote the algebra of coordinates on $\Bbbk\oplus V$ by $S' = \Bbbk[x_0,\dots,x_n]$: if $Q \in S$ is the defining polynomial of $\mathcal{A}$, we let $Q' \in S'$ be the homogenization of Q and $c\mathcal{A}$ be the arrangement determined by x_0Q'. For instance, if $Q = (x_1+1)(x_2-2)$ then $c\mathcal{A}$ has defining polynomial $x_0(x_1-x_0)(x_2-2x_0)$. There is one more hyperplane in $c\mathcal{A}$ than in $\mathcal{A}$: the one defined by $x_0 = 0$.

1.7. We now describe an important family of arrangements that is, in fact, a big motivation of the theory. We call an automorphism $s \in \mathsf{GL}(V)$ of V a *pseudo-reflection* if it has finite order and its fixed point set is a hyperplane, which we call the *reflecting hyperplane of* s, and we call it a *reflection* if its order is two. A finite subgroup G of $\mathsf{GL}(V)$ is a *(pseudo-) reflection group* if it is generated by (pseudo-) reflections; if $\Bbbk = \mathbb{R}$ it is called a Coxeter group. The set of reflecting hyperplanes $\mathcal{A}(G)$ of a reflection group G is the *reflection arrangement* of G.

A root system is a finite set R of nonzero vectors in $\mathbb{R}^n$, the *roots*, that satisfy certain combinatorial properties that can be found, for instance, in [Bou68, Chapitre VI]. The set $\mathcal{A}(R)$ of hyperplanes in $\mathbb{R}^n$ orthogonal to the roots of R is an arrangement, and one can show that every such arrangement is, in fact, a reflection arrangement: the corresponding group is the one generated by the reflections with respect to its hyperplanes. We call these Coxeter arrangements. For instance, the arrangement associated to the class of root systems

$$A_{n-1} = \left\{e_i - e_j : 1 \leq i < j \leq n\right\} \subset \mathbb{R}^n$$

is the real nth braid arrangement $\mathcal{B}_n = \mathcal{A}(A_{n-1})$ that we saw in Example 1.5. Identifying the reflection with respect to the plane $x_i - x_j = 0$ with the permutation $(ij) \in \mathbb{S}_n$, we see that the corresponding reflection group is $\mathbb{S}_n$.

There is a complete description of the family of pseudo-reflection arrangements in the complex case due to G. Shephard and J. Todd, who have classified irreducible finite complex pseudo-reflection groups in [ST54].

1.2 Combinatorics

1.8. Let $\mathcal{A}$ be an arrangement in V. The *intersection poset* $\mathcal{L}(A)$ is the set of all nonempty intersections of hyperplanes in $\mathcal{A}$ —including V, the intersection of the empty set— with order given by reverse inclusion, that is, $X \leq Y$ if and only if $Y \subseteq X$. When $\mathcal{A}$ is central, $\mathcal{L}(\mathcal{A})$ is a lattice.

1.9. *Example.* Let us consider the boolean arrangement Bool_n of Example 1.4, defined by $x_1 \ldots x_n = 0$. As every subset of hyperplanes in Bool_n has a different nonempty intersection, $\mathcal{L}(\mathcal{A})$ is isomorphic to the poset of all subsets of $[\![n]\!]$ ordered by inclusion. As a matter of fact, this is the reason for the name of the arrangement.

Let us show that, on the other hand, the intersection poset $\mathcal{L}(\mathcal{B}_n)$ of the braid arrangement $\mathcal{B}_n = \{H_{ij} : 1 \leq i < j \leq n\}$, introduced in Example 1.5, is isomorphic to the lattice $\mathbb{P}$ of partitions of the set $[\![n]\!]$ ordered by refinement. Indeed, if $X \in \mathcal{L}(\mathcal{B}_n)$, there is an equivalence relation $\sim_X$ on $[\![n]\!]$ such that

$$i \sim_X j \iff X \subset H_{ij},$$

with the convention that H_{ij} denotes V if $i = j$, and we may therefore consider the partition Λ_X of $[\![n]\!]$ into its corresponding equivalence classes. The map

$$\varphi : \mathcal{L}(\mathcal{B}_n) \ni X \mapsto \Lambda_X \in \mathbb{P}$$

is the desired isomorphism of lattices: it preserves order, it is injective, for we can write X as the intersection of the hyperplanes H_{ij} such that $i \sim_X j$, and it is also surjective: given a partition Λ of $[\![n]\!]$ that determines a relation $\sim$, we may define X as the intersection of all hyperplanes H_{ij} such that $i \sim j$.

1.10. From an arrangement $\mathcal{A}$ and a choice of $X \in \mathcal{L}(\mathcal{A})$ we can construct two arrangements: the first one is

$$\mathcal{A}_X := \{H \in \mathcal{A} : X \subseteq H\},$$

which is a subarrangement of $\mathcal{A}$, and the second one is the arrangement

$$\mathcal{A}^X := \{X \cap H : X \nsubseteq H \text{ and } X \cap H \neq \varnothing\}$$

in X, which we call the *restriction* of $\mathcal{A}$ to X. If H_0 is a hyperplane of $\mathcal{A}$, we let $\mathcal{A}' = A \setminus \{H_0\}$ and $\mathcal{A}'' = A^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a *triple with distinguished hyperplane* H_0. This construction is useful to perform inductive arguments, as we show in the next example, which will be relevant in the proof of Zaslavsky's Theorem 1.18.

1.11. *Example.* Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of real arrangements with distinguished hyperplane H. We claim that the number of chambers of each arrangement satisfy

$$|C(\mathcal{A})| = |C(\mathcal{A}'')| + |C(\mathcal{A}')|. \tag{1.1}$$

Let us, in order to prove this equality, denote by P the set of chambers of $\mathcal{A}'$ that intersect H and by Q the set of those that do not. Of course, every chamber in P gives rise to two chambers in $\mathcal{A}$ when it is split by H; on the other hand, each chamber in Q is a chamber of $\mathcal{A}$. We see in this way that

$$|C(\mathcal{A})| = 2|P| + |Q| = |P| + |C(\mathcal{A}')|,$$

and this, together with the observation that the map $P \ni c \mapsto c \cap H \in C(\mathcal{A}'')$ is a bijection, finishes the proof.

1.12. Let L be a finite poset. The Möbius function $\mu : L \times L \to \mathbb{Z}$ is defined recursively by

- $\mu(x, x) = 1$;
- $\sum_{x \leq z \leq y} \mu(x, z) = 0$ if $x < y$;
- $\mu(x, y) = 0$ if $x > y$.

This function plays a key role in the Möbius inversion formulas, which we now state for the special case of in which L is the lattice of intersections $\mathcal{L}(\mathcal{A})$ of a central hyperplane arrangement $\mathcal{A}$.

1.13. Proposition. *Let f and g be functions on $\mathcal{L}(\mathcal{A})$ with values on an abelian group. The following two equivalencies hold:*

$$g(Y) = \sum_{X \in \mathcal{L}(\mathcal{A}_Y)} f(X) \iff f(Y) = \sum_{X \in \mathcal{L}(\mathcal{A}_Y)} \mu(X, Y) g(X),$$
$$g(X) = \sum_{Y \in \mathcal{L}(\mathcal{A}^X)} f(Y) \iff f(X) = \sum_{Y \in \mathcal{L}(\mathcal{A}^X)} \mu(X, Y) g(Y).$$

1.14. Let $\mathcal{A}$ be a central hyperplane arrangement and let us write $\mu(X) = \mu(V, X)$ for each $X \in \mathcal{L}(\mathcal{A})$; notice that V is the unique minimal element of the lattice $\mathcal{L}(\mathcal{A})$. The *characteristic polynomial* $\chi_{\mathcal{A}} \in \mathbb{Z}[t]$ of the arrangement $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(t) = \sum_{X \in \mathcal{L}(\mathcal{A})} \mu(X) t^{\dim(X)}.$$

This polynomial collects important combinatorial information of the arrangement —the number of hyperplanes, for instance: it is readly seen, using the recurrence that defines μ, that $\mu(X) = -1$ if X is a hyperplane, so that $\chi_{\mathcal{A}}(t) = t^n - |\mathcal{A}|t^{n-1} + \dots$.

1.15. *Example.* The characteristic polynomial of the boolean arrangement is $(t-1)^n$. To see this, let us first show by induction with respect to the codimension $r(Y)$, with $Y \in \mathcal{L}(\mathcal{A})$, that $\mu(Y) = (-1)^{r(Y)}$. This is immediate when $Y = V$, so let us suppose that $k := r(Y)$ is positive. If $0 \le i \le k$, the number of subspaces $X \in \mathcal{L}(\mathcal{A})$ such that $X \le Y$ and $r(X) = i$ is $\binom{k}{i}$. Using this, the second property in the definition of μ and the inductive hypothesis we see that

$$0 = \sum_{X \le Y} \mu(X) = \sum_{0 \le i < k} \binom{k}{i} (-1)^i + \mu(Y).$$

As $0 = (1-1)^k = \sum_{i=0}^{k} \binom{k}{i}(-1)^i$, we conclude that $\mu(Y) = (-1)^k$ and, finally, that

$$\chi_{\mathcal{A}}(t) = \sum_{X \in \mathcal{L}(\mathcal{A})} (-1)^{r(X)} t^{\dim(X)} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i t^{n-i} = (t-1)^n.$$

1.16. *Example.* Let $\Bbbk = \mathbb{F}_q$ be the finite field of q elements and $\mathcal{A}$ the arrangement in $V = \Bbbk^n$ of all hyperplanes that pass through the origin.

If W is a $\mathbb{F}_q$-vector space of finite dimension and w denotes its cardinality then, evidently, $|\mathsf{hom}(W, V)| = w^n$. Let us define, for $X \in \mathcal{L}(\mathcal{A})$, P_X to be the subset of $\mathsf{hom}(W, V)$ of maps with image equal to X and Q_X that of maps with image contained in X but not necessarily equal to it. Of course, we have $Q_X = \bigcup_{Y \ge X} P_Y$ and, with the help of the Möbius inversion formulas in 1.12, we see that $|P_Y| = \sum_{X \ge y} \mu(Y, X)|Q_X|$. In particular, for $Y = V$, this means that

$$|P_V| = \sum_{X \ge Y} \mu(X) w^{\dim X}. \tag{1.2}$$

As a linear map $W \to V$ is surjective if and only if its transpose $V^* \to W^*$ is injective, $|P_V|$ is the number of injective maps in $\mathsf{hom}(V^*, W^*)$. Let us now fix a basis $\{x_1, \dots, x_n\}$ of V^* and suppose that ϕ is a monomorphism from V^* to W^*. There are $w^n - 1$ possibilities for $\phi(x_1)$; once we have chosen one, we remove all its multiples to see that there are $w^n - q$ elements where x_2 can be sent: an inductive argument following these lines shows that

$$|P_V| = (w^n - 1)(w^n - q) \cdots (w^n - q^{n-1}).$$

Since equation (1.2) means that $|P_V| = \chi_{\mathcal{A}}(w)$, we may conclude that

$$\chi_{\mathcal{A}}(t) = (t^n - 1)(t^n - q) \cdots (t^n - q^{n-1}),$$

for this result holds for an infinite number of integer values of t.

1.17. *Example.* We follow the ideas in the previous example, now to compute the characteristic polynomial of a braid arrangement. Let $n \in \mathbb{N}$ and let $\mathcal{A}$ be the nth braid arrangement. Recall from Example 1.9 that $\mathcal{L}(\mathcal{A})$ is isomorphic to the lattice of partitions of $I := [\![n]\!]$; the partition associated to $X \in \mathcal{L}(\mathcal{A})$ is denoted by Λ_X.

Let W be a set of cardinality w. If $\phi : I \to W$, we write Λ_ϕ the partition $\{\phi^{-1}(w) : w \in W\}$ of I and for each $X \in \mathcal{L}(\mathcal{A})$ we put

$$P_X = \{\phi : I \to W : \Lambda_\phi = \Lambda_X\}, \qquad Q_X = \{\phi : I \to W : \Lambda_\phi \geq \Lambda_X\}.$$

As the disjoint union of P_Y with $Y \geq X$ is Q_X, we can apply the Möbius inversion formula to see that

$$|P_Y| = \sum_{X \geq Y} \mu(Y, X)|Q_X|. \tag{1.3}$$

Let us now compute $|Q_X|$. As maps $\phi \in Q_X$ are constant on each of the classes determined by the equivalence relation $\sim_X$ that we saw in Example 1.9, there is a bijection between Q_X and W^{Λ_X}. Moreover, $|\Lambda_X| = \dim X$, for the subset of V formed by the vectors $v^\lambda = \sum_{i \in \lambda} e_i$ with $\lambda \in \Lambda_X$ is a basis of X. If we now set $Y = V$ in (1.3) we obtain that

$$|P_V| = \sum_{X \in \mathcal{L}(\mathcal{A})} \mu(X) w^{\dim X}.$$

On the other hand, $|P_V|$ is the number of injective maps $I \to W$ and therefore equals to $w(w-1)\cdots(w-(n-1))$. We conclude that

$$\chi_{\mathcal{A}}(t) = t(t-1)\cdots(t-(n-1)),$$

for we have seen, once again, that the equality holds for an infinite number of instances of t.

1.18. The Poincaré polynomial of $\mathcal{A}$,

$$\pi(\mathcal{A}, t) = \sum_{X \in \mathcal{L}(\mathcal{A})} \mu(X)(-t)^{r(X)},$$

has the same information as the characteristic polynomial, since we have

$$\chi_{\mathcal{A}}(t) = t^n \pi(\mathcal{A}, -t^{-1}).$$

For instance, we may deduce from Example 1.17 that the Poincaré polynomial of the braid arrangement $\mathcal{B}_n$ is given by

$$\pi(M(\mathcal{B}_n), t) = (1+t)(1+2t)\cdots(1+(n-1)t). \tag{1.4}$$

This polynomial can be sometimes more convenient than the characteristic polynomial to work with. The following two results should help to illustrate this point. The first assertion is known as the Deletion–Restriction Theorem and the second one, due to T. Zaslavsky, is a consequence of the comparison of first one with (1.1) of Example 1.11.

Theorem. *If* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *is a triple then*

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t).$$

Theorem. *If* $\mathcal{A}$ *is a real arrangement then the number of chambers of* $\mathcal{A}$ *is* $\pi(\mathcal{A}, 1)$.

Proof. This appears in I.2.A of [Zas75]. □

1.3 Derivations

1.19. From now on and unless we say otherwise all our arrangements will be central.

1.20. We denote by $\mathsf{Der}(S)$ the set of derivations of S, that is, the linear maps $\theta : S \to S$ such that the Leibniz rule

$$\theta(fg) = \theta(f)g + f\theta(g)$$

holds for every f and g in S. It is straightforward to see that $\mathsf{Der}(S)$ is an S-submodule and a sub-Lie algebra of the algebra $\mathsf{End}(S)$ of endomorphisms of S. We view S as a graded algebra as usual, with each variable x_i of degree 1 for $i \in [\![n]\!]$ and for each $p \geq 0$ we write S_p the homogeneous component of S of degree p. The Lie algebra $\mathsf{Der}(S)$ is a left graded S-module, and it is freely generated by the set of partial derivatives $\{\partial_i : S \to S : i \in [\![n]\!]\}$, which are homogeneous elements of $\mathsf{Der}(S)$ of degree -1.

The Lie algebra of *derivations of the arrangement* $\mathcal{A}$ is the Lie subalgebra

$$\mathsf{Der}(\mathcal{A}) := \{\theta \in \mathsf{Der}(S) : \theta(Q) \in QS\}$$

of $\mathsf{Der}(S)$, which happens to be also a graded S-submodule of $\mathsf{Der}(S)$. This invariant of $\mathcal{A}$ was first considered by K. Saito in the more general context of the study of differential forms with logarithmic singularities along a divisor of a complex manifold in [Sai80] and, in particular, its Lie algebra and S-module structures subtly codify geometric, arithmetic and combinatorial properties of the arrangement. In geometrical terms, $\mathsf{Der}(\mathcal{A})$ has a rather clear description: it is the algebra of polynomial vector fields tangent to each of the hyperplanes of $\mathcal{A}$.

1.21. A derivation θ belongs to $\mathsf{Der}\,\mathcal{A}$ if and only if α divides $\theta(\alpha)$ for every linear form α such that $\ker\alpha$ belongs to $\mathcal{A}$. Indeed, if $Q = \alpha_1 \cdots \alpha_l$ for coprime linear forms $\alpha_1, \ldots, \alpha_l$ then the claim follows from the equality

$$\theta(Q) = \theta(\alpha_1)\alpha_2 \cdots \alpha_l + \alpha_1\theta(\alpha_2 \cdots \alpha_l).$$

As an immediate consequence of this observation we see that if $\mathcal{A}$ is a central arrangement then the eulerian derivation $E := x_1\partial_1 + \ldots + x_n\partial_n$ is a derivation of $\mathcal{A}$, for $E(\alpha) = \alpha$ if α is a homogeneous linear form.

1.22. An arrangement $\mathcal{A}$ is *free* if $\operatorname{Der}\mathcal{A}$ is a free S-module. The notion of freeness was introduced in [Sai80] as well; as we shall illustrate in Example 1.31, freeness is not a generic property, but this condition is nevertheless satisfied in many important examples. Indeed, it is a theorem by H. Terao in [Ter80a] that reflection arrangements (see **1.7**) over $\mathbb{C}$ are free —we recommend the exposition of this subject in [OT92, §6.3]. In [Ter80b], H. Terao states the yet unsettled conjecture that the freeness of an arrangement is a combinatorial property, that is, that it depends only on the intersection poset $\mathcal{L}(\mathcal{A})$. We do know, as a consequence of Theorem 1.52 in Section 1.5 below, that the cohomology of the complement of a complex arrangement is a combinatorial property; in contrast, Rybnikov in [Ryb11] has constructed two complex arrangements with the same combinatorics but whose complements have nonisomorphic fundamental groups.

1.23. As a first example of a free arrangement, we may see by hand that the module of derivations of the boolean arrangement Bool_n of Example 1.4 admits $\{x_i\partial_i : 1 \leq i \leq n\}$ as a basis. Let us show that this is actually a consequence of the fact that we saw in Example 1.6 that Bool_n is the n-fold product of Bool_1, which is evidently a free arrangement. Indeed, let us consider the general situation in which $\mathcal{A}_1$ and $\mathcal{A}_2$ are arrangements in V_1 and V_2 and let $\mathcal{A}_1 \times \mathcal{A}_2$ be their product, as we defined in Example 1.6, which is an arrangement in $V = V_1 \oplus V_2$. Let S, S_1 and S_2 be the algebras of coordinate functions on V, V_1 and V_2, respectively, and let us identify as usual S with the tensor product algebra $S_1 \otimes S_2$ and view S_1 and S_2 as subalgebras of S. If $\operatorname{Der}(\mathcal{A}_1)$ and $\operatorname{Der}(\mathcal{A}_2)$ are the S_1- and S_2-modules of derivations of $\mathcal{A}_1$ and $\mathcal{A}_2$, then it is easy to see that there is an isomorphism of S-modules

$$\operatorname{Der}(\mathcal{A}) \cong S \otimes_{S_1} \operatorname{Der}(\mathcal{A}_1) \oplus S \otimes_{S_2} \operatorname{Der}(\mathcal{A}_2).$$

In particular, the product $\mathcal{A}_1 \times \mathcal{A}_2$ is free if and only if the factors $\mathcal{A}_1$ and $\mathcal{A}_2$ are free. The sufficiency of the condition is obvious, and the necessity follows from the facts that *(i)* projective finitely generated graded S-modules are free and *(ii)* a finitely generated S_1-module M is projective if the S-module $S \otimes_{S_1} M$ is projective, since the inclusion $S_1 \to S$ is faithfully flat; see [TSpa19, Proposition 058S].

1.24. To provide more examples efficiently, we need to make a few preliminary observations. The first one is that if an arrangement is free then the rank of the module of derivations is n and that, moreover, there is a basis of $\operatorname{Der}\mathcal{A}$ formed by homogeneous derivations. As a consequence of this last fact, each time that we pick a basis of $\operatorname{Der}\mathcal{A}$ we will be able to assume without loss of generality that it is one of that form.

Proposition. *If an arrangement $\mathcal{A}$ in a vector space of dimension n is free, then its S-module of derivations $\operatorname{Der}\mathcal{A}$ has a basis of n homogeneous elements.*

Proof. Let r be the rank of the S-module $\operatorname{Der}\mathcal{A}$. As the set of derivations $\{\partial_i : 1 \leq i \leq n\}$ is an S-basis of $\operatorname{Der}(S)$, the set $\{Q\partial_i : 1 \leq i \leq n\}$ is one of $Q\operatorname{Der}(S)$ and therefore, looking at the ranks of each of the S-modules in the chain $Q\operatorname{Der}(S) \subset \operatorname{Der}\mathcal{A} \subset \operatorname{Der}(S)$, we conclude $n \leq r \leq n$. Let us now consider all the homogeneous components of the members of a basis of $\operatorname{Der}\mathcal{A}$: they form a set of generators and, then, choosing a minimal set of generators among them we find a homogeneous basis of $\operatorname{Der}\mathcal{A}$. □

1.25. If $\theta_1, \dots, \theta_n \in \operatorname{Der}\mathcal{A}$ are derivations of $\mathcal{A}$, the *Saito matrix* is

$$M(\theta_1, \dots, \theta_n) \coloneqq \begin{pmatrix} \theta_1(x_1) & \cdots & \theta_1(x_n) \\ \vdots & \ddots & \vdots \\ \theta_n(x_1) & \cdots & \theta_n(x_n) \end{pmatrix}.$$

Lemma. *The defining polynomial Q divides $\det M(\theta_1, \dots, \theta_n)$ in S.*

Proof. Let α be a linear form defining a hyperplane in $\mathcal{A}$. Without losing generality, we may write $\alpha = x_1 + c_2x_2 + \cdots + c_nx_n$, for scalars $c_2, \dots, c_n$ —we may, if needed, rename the variables and multiply α by an appropriate scalar. If θ is a derivation then

$$\theta(x_1) = \theta(\alpha) - c_2\theta(x_2) - \cdots - c_n\theta(x_n),$$

and therefore $\det M(\theta_1, \dots, \theta_n)$ is equal to

$$\det \begin{pmatrix} \theta_1(\alpha) & \cdots & \theta_1(x_n) \\ \vdots & & \vdots \\ \theta_n(\alpha) & \cdots & \theta_n(x_n) \end{pmatrix} = \alpha \det \begin{pmatrix} \theta_1(\alpha)/\alpha & \cdots & \theta_1(x_n) \\ \vdots & & \vdots \\ \theta_n(\alpha)/\alpha & \cdots & \theta_n(x_n) \end{pmatrix} \in \alpha S.$$

Since α was arbitrary, it follows from is that $\det M(\theta_1, \dots, \theta_n) \in QS$. □

1.26. Theorem (Saito's criterion, [Sai80, Theorem 1.8.ii])**.** *A set of n derivations $\{\theta_1, \dots, \theta_n\}$ in $\operatorname{Der}\mathcal{A}$ is an S-basis if and only if the determinant of the matrix $M(\theta_1, \dots, \theta_n)$ is a nonzero scalar multiple of Q.*

Proof. Suppose, to begin with, that the condition on the determinant holds. The derivations $\theta_1, \dots, \theta_n$ are linearly independent over S: indeed, if θ_1 were equal to an S-linear combination of $\theta_2, \dots, \theta_n$ then the evaluation of the determinant of the matrix $M(\theta_1, \dots, \theta_n)$ at any point p of V would be equal to zero and, since the field is infinite, the determinant itself would be zero, contradicting the hypothesis.

We may assume that $\det M(\theta_1, \dots, \theta_n) = Q$; else, we simply replace θ_1 with an scalar multiple. For each $i \in [\![n]\!]$, we have $\theta_i = \sum \theta_i(x_j)\partial_j$, so, applying essentialy Cramer's rule, we get that

$$Q\partial_j = \det \begin{pmatrix} \theta_1(x_1) & \cdots & \theta_1 & \cdots & \theta_1(x_n) \\ \vdots & & \vdots & & \vdots \\ \theta_n(x_1) & \cdots & \theta_n & \cdots & \theta_n(x_n) \end{pmatrix}$$

and therefore

$$Q\partial_j \in S\theta_1 + \ldots + S\theta_n. \tag{1.5}$$

Let now $\eta \in \operatorname{Der}\mathcal{A}$ and $i \in [\![n]\!]$. We see from (1.5) that there exist $f_1, \ldots, f_n \in S$ such that $Q\eta = \sum_{j=1}^n f_j\theta_j$. As Q divides $\det M(\theta_1, \ldots, \theta_{i-1}, \eta, \theta_{i+1}, \ldots, \theta_n)$ in virtue of Lemma 1.25 we have that

$$\begin{aligned}
&Q \det M(\theta_1, \ldots, \theta_{i-1}, \eta, \theta_{i+1}, \ldots, \theta_n) \\
&\qquad = \det M(\theta_1, \ldots, \theta_{i-1}, Q\eta, \theta_{i+1}, \ldots, \theta_n) \\
&\qquad = \det M(\theta_1, \ldots, \theta_{i-1}, f_i\theta_i, \theta_{i+1}, \ldots, \theta_n) \\
&\qquad = f_i \det M(\theta_1, \ldots, \theta_n) \\
&\qquad = f_i Q,
\end{aligned}$$

so that f_iQ belongs to Q^2S and then Q divides f_i. This shows that $\eta = \sum_{j=1}^n \frac{f_j}{Q}\theta_j$ and, therefore, that $\{\theta_1, \ldots, \theta_n\}$ spans the S-module $\operatorname{Der}\mathcal{A}$, which is what it remained to see.

Let us suppose now that $\{\theta_1, \ldots, \theta_n\}$ is a basis of $\operatorname{Der}\mathcal{A}$. Using again **1.25**, we know there exists $f \in S \setminus \{0\}$ such that $fQ = \det M(\theta_1, \ldots, \theta_n)$. Given a hyperplane H in $\mathcal{A}$, which we may suppose to be defined by the linear form x_1, the arrangement $\mathcal{A} \setminus \{H\}$ is defined by $Q_H = Q/x_1$. Consider now the derivations

$$\eta_1 = Q\partial_1, \qquad\qquad \eta_i = Q_H\partial_i \qquad \text{for } 2 \le i \le n.$$

These derivations belong to $\operatorname{Der}\mathcal{A}$, so, in view of our hypothesis, can be written as an S-linear combination of $\theta_1, \ldots, \theta_n$: this implies that there exists an square matrix N with entries in S such that $M(\eta_1, \ldots, \eta_n) = M(\theta_1, \ldots, \theta_n)N$. As

$$QQ_H^{l-1} = \det M(\eta_1, \ldots, \eta_n) = \det M(\theta_1, \ldots, \theta_n) \det N = fQ \det N,$$

we see that f divides Q_H^{l-1}. As this is true for every $H \in \mathcal{A}$ and $\gcd_{H\in\mathcal{A}} Q_H^{l-1} = 1$, we conclude that $f \in \Bbbk$. □

1.27. Recall from **1.24** that if an arrangement is free we may take a basis $\{\theta_1, \ldots, \theta_n\}$ consisting of homogeneous derivations. Looking at the degrees in the equality $\det M(\theta_1, \ldots, \theta_n) = fQ$ of Saito's criterion we arrive at the following result.

Corollary. *The set $B = \{\theta_1, \ldots, \theta_n\}$ of homogeneous and linearly independent derivations in $\operatorname{Der}\mathcal{A}$ is a basis of $\operatorname{Der}\mathcal{A}$ if and only if $\sum_{i=1}^n |\theta_i| = l - n$.* □

1.28. *Example.* Let us consider a central arrangement $\mathcal{A}$ of lines in the plane $\Bbbk^2$, and let us denote x and y the coordinates of $\Bbbk^2$. Up to a change of coordinates, we may assume that the line with equation $x = 0$ is one of the lines in $\mathcal{A}$, so that the defining polynomial Q of the arrangement is of the form xF for some square-free homogeneous polynomial $F \in S$ which does not have x as a factor. Saito's criterion allows us to show that the two derivations

$$E = x\partial_x + y\partial_y, \qquad D = F\partial_y$$

form an S-basis of $\operatorname{Der}\mathcal{A}$. Indeed, we have

$$\det M(E, D) = \det\begin{pmatrix} x & y \\ 0 & F \end{pmatrix} = Q.$$

1.29. *Example.* As we said in **1.22**, reflection arrangements are free. In particular, the braid arrangement $\mathcal{B}_n$, defined in Example 1.5, admits the set $\{\delta_{-1}, \dots, \delta_{n-2}\}$ with

$$\delta_i = \sum_{j=1}^{n} x_j^{i+1}\partial_j$$

as a basis of $\operatorname{Der}\mathcal{B}_n$, as we check using Saito's criterion once again. The matrix $M(\delta_{-1}, \dots, \delta_{n-2})$ is the Vandermonde matrix and its determinant $Q = \prod_{1 \le i < j \le n}(x_i - x_j)$, the *discriminant* of $x_1, \dots, x_n$, is the defining polynomial of $\mathcal{B}_n$.

1.30. Let $\mathcal{A}$ be a free arrangement and let $\{\theta_1, \dots, \theta_l\}$ be an S-basis of $\operatorname{Der}\mathcal{A}$. The multiset of *exponents* of $\mathcal{A}$ is

$$\exp\mathcal{A} = \{|\theta_1| + 1, \dots |\theta_l| + 1\}.$$

For instance, we deduce from Example 1.17 that for the braid arrangement $\mathcal{B}_n$ we have

$$\exp\mathcal{B}_n = \{0, 1, 2, \dots, n-1\}.$$

The following result is a part of [OT92, Proposition 4.29] and will be helpful to show an example of a non-free arrangement.

Proposition. *Let $\mathcal{A}$ be a free arrangement and let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be irreducible arrangements such that $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$. The multiplicity of 0 in $\exp\mathcal{A}$ is the difference between the dimension of $\mathcal{A}$ and its rank, and the multiplicity of 1 is k.* □

If G is a reflection group, the exponents of the corresponding reflection arrangement $\mathcal{A}(G)$ have an interpretation in terms of invariant theory and this allows their determination using the character table of G. This is done in [OT92, Appendix B] for each of the groups appearing in the Shephard–Todd classification.

1.31. *Example.* Let us present an example of the fact, noted above, that freeness is not a generic property. The arrangement in $V = \Bbbk^3$ defined by

$$Q = xyz(ax + by + cz)$$

is free if and only if $abc = 0$. On one hand, if, for instance, $c = 0$ then Saito's criterion allows us to see that the derivations $x\partial_x + y\partial_y$, $y(ax + by)\partial_y$ and $z\partial_z$ form an S-basis of $\operatorname{Der}\mathcal{A}$.

Let us show that, on the other hand, if $abc \neq 0$ then $\mathcal{A}$ is not free. We can extract two consequences from Proposition 1.30: as the arrangement is essential, 0 is not an exponent of $\mathcal{A}$ and, as it is irreducible, the multiplicity of 1 as an exponent is 1. According to Proposition 1.24, if $\mathcal{A}$ were free then the number of exponents would be equal to its rank, which is 3, and two of the exponents would be at least 2. As the number of planes is 4, Corollary 1.27 implies that $\operatorname{Der}\mathcal{A}$ is not free.

Even though in this case $\operatorname{Der}\mathcal{A}$ is not free, it not too far from being free: the S-module $\operatorname{Der}\mathcal{A}$ has projective dimension 1. In order to simplify our calculations, performing a change of coordinates we may assume that $a = b = c = 1$. We claim that the derivations

$$\begin{aligned} E &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3, \\ D_{ij} &= x_ix_j(x_j - x_i) \qquad \text{for } (i,j) \in \{(1,2),(2,3),(3,1)\} \end{aligned}$$

generate the S-module $\operatorname{Der}\mathcal{A}$. Let, in order to see this, θ be a derivation in $\operatorname{Der}\mathcal{A}$. In view of **1.21**, we have that

$$x_i \mid \theta(x_i) \qquad \text{for } 1 \leq i \leq 3, \text{ and} \qquad\qquad x_1 + x_2 + x_3 \mid \theta(x_1 + x_2 + x_3).$$

We thus see that there exist polynomials a_1, a_2, a_3 and f in S such that $\theta(x_i) = x_ia_i$ and $(x_1 + x_2 + x_3)f = x_1a_1 + x_2a_2 + x_3a_3$. This last equation amounts to the equality

$$0 = x_1(a_1 - f) + x_2(a_2 - f) + x_3(a_3 - f). \tag{1.6}$$

Let now $K_\bullet$ be the Koszul complex described in [Wei94, §4.5] associated to the ring S and the regular sequence (x_1, x_2, x_3). Denoting by W the $\Bbbk$-vector space with basis $\{x_1, x_2, x_3\}$, we have $K_\bullet = S \otimes \Lambda^\bullet W$ and the differential $d_1 : K_1 \to K_0$ is given by

$$d_1(b_1 \otimes x_1 + b_2 \otimes x_2 + b_3 \otimes x_3) = b_1x_1 + b_2x_2 + b_3x_3.$$

In particular, equation (1.6) tells us that the 1-cochain $\omega = \sum_{i=1}^3 (a_i - f) \otimes x_i$ in K_1 is a cocycle, and, since the complex is exact, a coboundary: there exist therefore c_1, c_2 and c_3 in S such that

$$\begin{aligned} \omega &= d_2(c_3 \otimes x_1 \wedge x_2 - c_2 \otimes x_1 \wedge x_3 + c_1 \otimes x_2 \wedge x_3) \\ &= (c_2x_3 - c_3x_2) \otimes x_1 + (c_3x_1 - c_1x_3) \otimes x_2 + (c_1x_2 - c_2x_1) \otimes x_3. \end{aligned}$$

This equation implies at once that $\theta = fE + c_1D_{23} + c_2D_{31} + c_3D_{12}$, as we wanted. One way to restate this fact is that the morphism of S-modules $\pi : S^{\oplus 4} \to S$ such that

$$(f, c_1, c_2, c_3) \mapsto fE + c_1D_{23} + c_2D_{31} + c_3D_{12}$$

is surjective and therefore the first step towards a projective resolution of $\operatorname{Der}\mathcal{A}$. In order to complete the resolution, we observe that $\operatorname{Der}\mathcal{A}$ is a submodule of $\operatorname{Der} S$, which is free of rank 3, and that the composition $S^{\oplus 4} \to \operatorname{Der}\mathcal{A} \hookrightarrow \operatorname{Der} S \cong S^{\oplus 3}$ has matrix

$$\begin{pmatrix} x_1 & 0 & x_1x_3 & -x_1x_2 \\ x_2 & -x_2x_3 & 0 & x_1x_2 \\ x_3 & x_2x_3 & -x_1x_3 & 0 \end{pmatrix}$$

Using this as input for Macaulay command `resolution`, we find that the kernel of π is free of rank one generated by $(0, x_1, x_2, x_3)$. We have thus found an S-projective resolution of $\operatorname{Der}\mathcal{A}$ of length 1

$$0 \longleftarrow \operatorname{Der}\mathcal{A} \xleftarrow{\pi} S^{\otimes 4} \longleftarrow S \longleftarrow 0$$

Since this resolution is in fact minimal, we see that the projective dimension of $\operatorname{Der}\mathcal{A}$ is 1.

1.4 FORMS

THE COMPLEX OF LOGARITHMIC FORMS

In this subsection we present a few basic facts and constructions regarding the complex of differential forms and that of logarithmic forms on a hyperplane arrangement. We refer to [OT92, §4.4] for the missing proofs.

1.32. Let S, as before, be the algebra of coordinates on V and denote by F the field of fractions of S. We identify S and F with the algebra of polynomials $\Bbbk[x_1, \dots, x_n]$ and the field of rational functions $\Bbbk(x_1, \dots, x_n)$. Let $\Omega^1(V)$ be the F-vector space $F \otimes_\Bbbk V^*$ and denote by $\Omega^\bullet(V)$ the exterior algebra of $\Omega^1(V)$ over F. This is a graded algebra and the elements of its pth homogeneous component, which we write $\Omega^p(V)$, are called *rational differential p-forms.* We denote, as usual, the product of $\Omega^\bullet(V)$ by the symbol $\wedge$ and often simply omit it.

There is an unique $\Bbbk$-linear map $d : F \to \Omega^1(V)$ such that $d(fg) = d(f)g + fd(g)$ for f and g in F and $d(x_i) = 1 \otimes x_i \in F \otimes V^*$ for each $i \in [\![n]\!]$, which is given by the formula

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

whenever $f \in F$. This map extends naturally to $\Omega^\bullet(V)$ as described in the next proposition.

Proposition. *There exists an unique $\Bbbk$-linear map $d : \Omega^\bullet(V) \to \Omega^\bullet(V)$ such that*

(*i*) *the restriction of d to $\Omega^0(V) = F$ coincides with the map $d : F \to \Omega^1(V)$ defined above;*

(*ii*) *the map d is a differential, so that $d^2 = 0$;*

(*iii*) *the graded Leibniz identity holds, that is,*

$$d(\omega\eta) = (d\omega)\eta + (-1)^p \omega(d\eta) \qquad \text{if } \omega \in \Omega^p(V) \text{ and } \eta \in \Omega^q(V).$$

We have, in fact, that

$$d(f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

if $1 \le p \le n$, $1 \le i_1 < \cdots < i_p \le n$ and $f \in F$. □

The graded vector space $\Omega^\bullet(V)$, along with the differential d, is the *complex of rational differential forms on* V. We define the *complex of regular differential forms on* V to be the subcomplex $\Omega^\bullet[V]$ of $\Omega^\bullet(V)$ such that $\Omega^0[V] = S$ and, for $p \geq 0$,

$$\Omega^p[V] = \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} S dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

1.33. Let now $\mathcal{A}$ be a hyperplane arrangement in V with defining polynomial Q. For each $p \geq 0$, we define the *module of logarithmic p-forms with poles along $\mathcal{A}$* —or, for short, of *logarithmic p-forms on $\mathcal{A}$*— to be

$$\Omega^p(\mathcal{A}) = \left\{ \omega \in \Omega^p(V) \text{ such that } Q\omega \in \Omega^p[V] \text{ and } Qd\omega \in \Omega^{p+1}[V] \right\}.$$

For instance, we have that $\Omega^n(\mathcal{A}) = (1/Q)\Omega^n[V]$. One can check that the module of logarithmic p-forms is an S-submodule and that, moreover,

$$\Omega^\bullet(\mathcal{A}) := \bigoplus_{p \geq 0} \Omega^p(\mathcal{A})$$

is an S-subcomplex of $\Omega^\bullet(V)$ which is closed under exterior product. We can find some useful examples of forms in $\Omega^1(\mathcal{A})$ with the help of the next proposition.

Proposition. (*i*) *The* 1*-form* dQ/Q *belongs to* $\Omega^1(\mathcal{A})$.

(*ii*) *If* $\alpha \in V^*$*, then* $d\alpha/\alpha \in \Omega^1(\mathcal{A})$ *if and only if* $\ker \alpha \in \mathcal{A}$.

(*iii*) *A rational p-form ω belongs to $\Omega^p(\mathcal{A})$ if and only if the forms $Q\omega$ and $dQ \wedge \omega$ are regular.*

Proof. The first assertion is immediate. For the second one, we observe that for any linear form α we have $d(d\alpha/\alpha) = 0$ and therefore the condition $d\alpha/\alpha \in \Omega^1(\mathcal{A})$ reduces to $Qd\alpha/\alpha = \frac{Q}{\alpha}d\alpha \in \Omega^1[V]$, which is easily seen to be equivalent to the condition that α divides Q. The third statement can be found in [OT92, Proposition 4.69]. □

1.34. The module of logarithmic 1-forms on $\mathcal{A}$ is closely related to the Lie algebra of derivations of $\mathcal{A}$: they are S-dual to each other. We now make explicit the pairing that induces this duality. Given $1 \leq p \leq n$, the *interior product*

$$\langle \cdot, \cdot \rangle : \mathsf{Der}_{\Bbbk}(S) \times \Omega^p(V) \to \Omega^{p-1}(V)$$

is the S-bilinear map defined by

$$\langle \theta, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \rangle = \sum_{k=1}^{p} (-1)^{k-1} \theta(x_{i_k}) dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_k}} \wedge \cdots \wedge dx_{i_p},$$

for $\theta \in \mathsf{Der}_{\Bbbk}(S)$ and $1 \leq i_1 < \cdots < i_p \leq n$. A somewhat tedious calculation shows that the interior product restricts to a pairing

$$\langle \cdot, \cdot \rangle : \mathsf{Der}_{\Bbbk}(\mathcal{A}) \times \Omega^p(\mathcal{A}) \to \Omega^{p-1}(\mathcal{A}),$$

which gives our desired duality.

Proposition. *The morphisms of S-modules*

$$\alpha : \mathsf{Der}_{\Bbbk}(\mathcal{A}) \ni \theta \mapsto \langle \theta, - \rangle \in \mathsf{hom}_S(\Omega^1(\mathcal{A}), S)$$

and

$$\beta : \Omega^1(\mathcal{A}) \ni \omega \mapsto \langle -, \omega \rangle \in \mathsf{hom}_S(\mathsf{Der}_{\Bbbk}(\mathcal{A}), S)$$

are isomorphisms.

Proof. We first show that α is a monomorphism: if $\theta \in \ker \alpha$, then for all f in S we have $0 = \langle \theta, df \rangle = \theta(f)$, and therefore θ is the zero derivation. In order to see that it is also an epimorphism, let $\eta \in \mathsf{hom}(\Omega^1(\mathcal{A}), S)$. The map $\theta : S \ni f \mapsto \eta(df) \in S$ is a derivation of S. We may evaluate η at the form $dQ/Q \in \Omega^1(\mathcal{A})$ to obtain $\eta(dQ/Q)$. As this is an element of S, we see that $\theta(Q) = Q\eta(dQ/Q) \in QS$, so that θ is a derivation of $\mathcal{A}$.

In order to see that β is a monomorphism, let $\omega = \sum_{i=1}^n \omega_i dx_i \in \Omega^1(\mathcal{A})$ and assume that $\beta(\omega) = 0$. Since $Q\omega$ is regular, there are $f_1, \dots, f_n \in S$ such that $\omega_i = Qf_i$; evaluating, we see that $f_i = \beta(\omega)(Q\partial_i) = 0$ and therefore that $\omega = 0$.

Finally, let $\xi \in \mathsf{hom}_S(\mathsf{Der}_{\Bbbk}(\mathcal{A}), S)$. We put $\omega_i \coloneqq (1/Q)\xi(Q\partial_i)$ for each $1 \in [\![n]\!]$ and claim that the form $\omega \coloneqq \sum_{i=1}^n \omega_i dx_i$ belongs $\Omega^1(\mathcal{A})$. It is clear that $Q\omega$ is regular; to prove the claim it is enough, in view of Proposition 1.33, to see that $dQ \wedge \omega$ is also regular: this follows from the fact that the coefficient of $dx_i \wedge dx_j$ in $dQ \wedge \omega$ is

$$\frac{1}{Q}\left(\partial_i Q\omega(Q\partial_j) - \partial_j Q\omega(Q\partial_i)\right) = \omega(\partial_i Q\partial_j - \partial_j Q\partial_i),$$

which is an element in S. Now, as ω is regular, for every $\theta \in \mathsf{Der}\,\mathcal{A}$ we have

$$\begin{aligned}\beta(\omega)(\theta) = \omega(\theta) &= \sum \omega_i\theta(x_i) = \sum (1/Q)\xi(Q\partial_i)\theta(x_i) \\ &= \xi\left(\sum (1/Q)Q\theta(x_i)\partial_i\right) = \xi(\theta),\end{aligned}$$

from which we conclude that $\beta(\omega) = \xi$, as we wanted. □

1.35. A consequence of this proposition is that both S-modules $\mathsf{Der}\,\mathcal{A}$ and $\Omega^1(\mathcal{A})$ are reflexive. In particular, if $\dim V = 2$ then $\mathsf{Der}\,\mathcal{A}$ is free: this follows from a result that states that a reflexive module over a finitely generated ring of dimension 2 is free. This gives us another, independent, proof of the freeness of arrangements of lines that we established in Example 1.28.

1.36. We finish this subsection by stating a result that relates the complex of logarithmic forms with the freeness of the arrangement.

Proposition. *(i) The S-module $\Omega^1(\mathcal{A})$ is free if and only if $\mathcal{A}$ is free.*

(ii) If $\Omega^1(\mathcal{A})$ is S-free with basis $(\omega_i : 1 \leq i \leq l)$ then for every $p \in [\![n]\!]$ the S-module $\Omega^p(\mathcal{A})$ is free with basis

$$\left\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\right\}.$$

Proof. This can be found in Corollary 4.76 and Proposition 4.81 of [OT92]. □

The algebra $R^\bullet(\mathcal{A})$

1.37. The algebra $R^\bullet(\mathcal{A})$ was first considered by Arnold for braid arrangements in [Arn69], which is one of the works that started to raise interest on hyperplane arrangements, and appears several times throughout this thesis. We will see that when $\Bbbk = \mathbb{C}$ it is isomorphic to the cohomology of the complement $M(\mathcal{A})$ of $\mathcal{A}$ in $\mathbb{C}^n$.

For each hyperplane H in $\mathcal{A}$ we choose a linear form $\alpha_H : V \to \Bbbk$ such that $\ker \alpha_H = H$ and consider the 1-form $\omega_H \coloneqq \frac{d\alpha_H}{\alpha_H} \in \Omega^1(V)$. Observe that ω_H does not depend on the choice of the linear form α_H but only on the hyperplane H.

We define the graded associative algebra $R^\bullet(\mathcal{A})$ to be the subalgebra of $\Omega^\bullet(V)$ generated by the set $\{\omega_H : H \in \mathcal{A}\}$. It follows from Proposition 1.33 that $R^1(\mathcal{A})$ is a subspace of $\Omega^1(\mathcal{A})$; as $\Omega^\bullet(\mathcal{A})$ is closed under exterior product, $R^\bullet(\mathcal{A})$ is in fact a subalgebra of $\Omega^\bullet(\mathcal{A})$. The $\mathbb{Z}$-grading on $R^\bullet(\mathcal{A})$ is induced by that of $\Omega^\bullet(V)$, so that $R^p(\mathcal{A}) = R^\bullet(\mathcal{A}) \cap \Omega^p(V)$ for each $p \in \mathbb{Z}$. Since $d\omega_H = 0$ for every hyperplane H in $\mathcal{A}$, the restriction of the differential d of $\Omega^\bullet(V)$ to $R^\bullet(\mathcal{A})$ is zero. We observe as well that $R^0(\mathcal{A}) = \Bbbk$ and that $R^p(\mathcal{A}) = 0$ if $p > n$.

1.38. *Example.* We return to the situation of Example 1.28, where we consider a central arrangement $\mathcal{A}$ of lines $H_1, \dots, H_l$. For each $1 \le i \le l$ we let $\alpha_i : V \to \Bbbk$ be a linear form with kernel H_i and we put $\omega_i = \frac{d\alpha_i}{\alpha_i}$. The 1-forms $\omega_1, \dots \omega_l$ span $R^1(\mathcal{A})$ and they are in fact linearly independent. For each $i \in [\![l]\!]$ there are scalars a_i and b_i such that $\alpha_i = a_i x + b_i y$; let us suppose that $\lambda_1, \dots, \lambda_l$ in $\Bbbk$ are such that

$$0 = \sum \lambda_i \omega_i = \sum \frac{\lambda_i a_i}{\alpha_i} dx + \frac{\lambda_i b_i}{\alpha_i} dy.$$

The coefficients of dx and of dy must be zero, so that, for instance, $0 = \sum \frac{\lambda_i a_i}{\alpha_i}$ in F or, equivalently, $0 = \sum \lambda_i a_i \frac{Q}{\alpha_i}$ in S . Now, if $1 \le j \le l$, we see that $\lambda_j a_j \frac{Q}{\alpha_j} \equiv 0$ modulo α_j and, since Q is square-free, this implies that actually $\lambda_j a_j = 0$. The same argument with the coefficients of dy allows us to conclude that $\lambda_j = 0$ for every j, as we claimed.

Since $\omega_i^2 = 0$ and $\omega_i\omega_j = -\omega_j\omega_i$, the set $\{\omega_i\omega_j : i < j\}$ spans $R^2(\mathcal{A})$. We immediately see that $\omega_i\omega_j = (a_i b_j - b_i a_j) dx dy$ and therefore for any i, j and k we have

$$\alpha_k d\alpha_i d\alpha_j + \alpha_i d\alpha_j d\alpha_k + \alpha_j d\alpha_k d\alpha_i = \det \begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ \alpha_i & \alpha_j & \alpha_k \end{pmatrix} dx dy = 0.$$

Multiplying by $\frac{1}{\alpha_i\alpha_j\alpha_k}$, we obtain the relation

$$0 = \omega_i\omega_j + \omega_j\omega_k + \omega_k\omega_i \qquad \text{for any } i, j, k \text{ in } [\![l]\!].$$

This relation allows to write any $\omega_i\omega_j$ as $\omega_i\omega_l - \omega_j\omega_l$ and, as a consequence of this, the set $\{\omega_i\omega_l : 1 \le i < l\}$ spans $R^2(\mathcal{A})$: we claim that this set is linearly independent. Let $\mu_1, \dots, \mu_{l-1}$ be in $\Bbbk$ and suppose that

$$\sum \mu_i \omega_i \omega_l = 0. \tag{1.7}$$

There is an F-linear map $\partial : \Omega^2(V) \to \Omega^1(V)$ such that $f dx dy \mapsto f x dy - f y dx$ and one can see that $\partial(\omega_i \omega_j) = \omega_j - \omega_i$. Now, applying ∂ to the linear combination in the left hand side of (1.7) we obtain $\sum \mu_i(\omega_l - \omega_i)$ and, as $\{\omega_1, \dots \omega_l\}$ is $\Bbbk$-linearly independent, we get that $\mu_1 = \cdots = \mu_{l-1} = 0$.

Let $\mathcal{F} = \bigoplus_{k\geq 0} \mathcal{F}_k$ be the free graded-commutative algebra generated by l generators $w_1, \dots, w_l$ of degree 1 subject to the relations $w_i w_j + w_j w_k + w_k w_i = 0$, one for each choice of $i, j, k \in [\![l]\!]$. We have $\mathcal{F}_k = 0$ if $k \geq 3$: indeed, if $i, j, k \in [\![l]\!]$ then

$$w_i w_j w_k = (w_i w_j + w_j w_k + w_k w_i) w_k = 0,$$

because of the graded-commutativity. Since the generators w_i satisfy the same relations as the forms ω_i, we may proceed as before to find that the set of monomials $\{w_1 w_i : 1 \leq i \leq l\}$ spans $\mathcal{F}_2$ and, therefore, that the dimension of $\mathcal{F}_2$ is at most $l-1$. There is clearly a surjective morphism of graded algebras $f : \mathcal{F} \to R^\bullet(\mathcal{A})$ such that $f(w_i) = \partial_{\alpha_i}$ for all $i \in [\![l]\!]$. This map is also injective because the dimension of $R^2(\mathcal{A})$ is $l-1$, so that there is an isomorphism of graded algebras $\mathcal{F} \cong R^\bullet(\mathcal{A})$.

1.39. *Example.* For the braid arrangement $\mathcal{B}_n$ defined in Example 1.5 we have that the 1-forms given by $\omega_{ij} = \frac{dx_i - dx_j}{x_i x_j}$, with $i, j \in [\![n]\!]$, generate the algebra $R^\bullet(\mathcal{B}_n)$. Let us show by induction that $\{\omega_{ij} : 1 \leq i < j \leq n\}$ is a basis of $R^1(\mathcal{B}_n)$: we immediately see that the claim holds for $n = 2$. For the inductive step, let c_{ij} be scalars such that $0 = \sum_{1\leq i<j\leq n} \mu_{ij}\omega_{ij}$. The component in dx_n of equation

$$0 = \sum_{1\leq i<j\leq n-1} \mu_{ij}\omega_{ij} + \sum_{i=1}^{n-1} \mu_{in}\omega_{in} \tag{1.8}$$

is $0 = -\sum_{i=1}^{n-1} \frac{c_{in}}{x_i - x_n}$ in F or, equivalently, $0 = -\sum_{i=1}^{n-1} \prod_{j\neq i}(x_j - x_n) c_{in}$ in S. If $i \in [\![n-1]\!]$, we see that this equation means that $\prod_{j\neq i}(x_j - x_n)c_{in} = 0$ and therefore that $c_{in} = 0$. We conclude now from (1.8) and the inductive hypothesis that $c_{ij} = 0$ for every possible i and j.

It is straightforward to check that the relation $0 = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij}$ with i, j and k in $[\![n]\!]$ holds in $R^2(\mathcal{B}_n)$. Choosing $n = 3$, we see that $R^3(\mathcal{A})$ is generated by

$$\omega_{12}\omega_{23}\omega_{13} = -(\omega_{23}\omega_{31} + \omega_{31}\omega_{12})\omega_{13} = 0$$

and therefore $R^p(\mathcal{B}_3) = 0$ if $p \geq 3$. We now claim that the dimension of $R^2(\mathcal{B}_3)$ is 2: as the forms $\omega_{12}\omega_{13}$ and $\omega_{12}\omega_{23}$ are generators, we need only see that they are linearly independent. This is easily achieved following the idea in our previous example, for the F-linear map $\partial : \Omega^2(V) \to \Omega^1(V)$ such that $\partial(dx_i dx_j) = x_j dx_i - x_j dx_i$ if $i, j \in [\![3]\!]$ can be seen to satisfy $\partial(\omega_{ij}\omega_{jk}) = \omega_{jk} - \omega_{ij}$ whenever $i, j, k \in [\![3]\!]$.

Let now $A^\bullet = \bigoplus_{i\geq 0} A^i$ be the free graded-commutative algebra generated by the three symbols w_{12}, w_{13} and w_{23} of degree 1 subject to the relations

$$0 = w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij}, \qquad \text{if } 1 \leq i, j, k \leq 3. \tag{1.9}$$

The surjective morphism of graded algebras $\varphi : A^\bullet \to R^\bullet(\mathcal{A})$ such that $\varphi(w_{ij}) = \omega_{ij}$ evidently restricts to an isomorphism in degrees zero and one. Proceeding, again, as in our previous example, we see that relations (1.9) imply that $w_{12}w_{13}$ and $w_{12}w_{23}$ span A^2, and then the fact that the restriction of φ to degree two is surjective implies that it is an isomorphism.

1.5 THE COHOMOLOGY OF $M(\mathcal{A})$ AND THE ORLIK–SOLOMON ALGEBRA

THE COHOMOLOGY OF $M(\mathcal{A})$

In this subsection, our base field is $\mathbb{C}$ and all cohomology groups have complex values. We will go over the seminal paper of E. Brieskorn [Bri73] in which he deals with the cohomology of the complement $M(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ as a topological space.

1.40. Given a hyperplane H with defining linear form $\alpha_H : V \to \mathbb{C}$ we denote M_H its complement $V \setminus H$. The restriction $\alpha_H : M_H \to \mathbb{C}^\times$ induces a morphism of groups in cohomology

$$\alpha_H^* : H^\bullet(\mathbb{C}^\times) \to H^\bullet(M_H).$$

The class of the rational form $\eta = \frac{1}{2\pi i}\frac{dz}{z}$ on $\mathbb{C}^\times$ is a generator of $H^1(\mathbb{C}^\times)$ and the rational 1-form on V

$$\eta_H = \frac{1}{2\pi i}\frac{d\alpha_H}{\alpha_H}. \tag{1.10}$$

has $\alpha_H^*([\eta]) = [\eta_H]$, where the brackets denote taking cohomology class. Restricting along the inclusion $M \hookrightarrow M_H$, the 1-form η_H pulls back to a form on M which we will denote also by η_H.

With this notation in place we are ready to state the main result on the cohomology of $M(\mathcal{A})$.

Theorem (E. Brieskorn)**.** *The cohomology classes of the forms η_H corresponding to the hyperplanes H of $\mathcal{A}$ generate the algebra $H^\bullet(M(\mathcal{A}))$. Moreover, there is an isomorphism of graded algebras $R^\bullet(\mathcal{A}) \cong H^\bullet(M(\mathcal{A}))$ which maps ω_H to $[\eta_H]$.*

Proof. See [Bri73, Lemme 5]. □

1.41. The normalization in (1.10) is chosen so that the class $[\eta_H]$ is integral. Indeed, E. Brieskorn proves the corresponding result of Theorem 1.40 with integral coefficients.

1.42. *Example.* The cohomology ring of the complement of a braid arrangement $\mathcal{B}_n$ was described by V. Arnold in [Arn69] some years before the general result of Brieskorn while studying the cohomology of braid groups. As we saw in 1.5, the hyperplanes of $\mathcal{B}_n$ are defined by equations $x_i - x_j = 0$ for $1 \le i < j \le n$. Arnold showed that there is an isomorphism of graded rings between $H^\bullet(M(\mathcal{B}_n))$ and the quotient of the exterior algebra of the vector space with basis $\{\omega_{ij} : 1 \le i < j \le l\}$ by the ideal generated by the relations

$$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij}.$$

This isomorphism is induced by the identification of the class of ω_{ij} with the class of $\frac{1}{2\pi i}\frac{dz_i - dz_j}{z_i - z_j}$ for each $1 \le i < j \le n$. Brieskorn's result is in fact a generalization of this statement.

1.43. In addition to the precedent remarkable theorem, E. Brieskorn gives a description of the cohomology ring of the complement of a reflection arrangement with reflection group G in terms of its exponents.

The action of G on V induces another action of G on S; let S^G be its subalgebra of invariants polynomials. A result from C. Chevalley in [Che55, 1.(A)] states that there exists algebraically independent homogeneous polynomials $f_1, \dots, f_n$ in S^G such that $S^G = \Bbbk[f_1, \dots, f_n]$. These polynomials are not unique, but their degrees are. The integers $\deg f_i - 1$ with $1 \le i \le n$ are the *exponents* of the group G.

Theorem (E. Brieskorn). *For each $p \ge 0$, the dimension of $H^p(M(\mathcal{A}(G)))$ is the number of words in G of length p, where the length of a word is the minimal number of reflections required to factorize it. If G is a Coxeter group then the Poincaré polynomial of $M(\mathcal{A}(G))$ is*

$$\prod_{i=1}^{n}(1 + m_i t),$$

where $m_1, \dots, m_n$ are the exponents of G, which coincide with those of the arrangement $\mathcal{A}(G)$.

If the reflection group G is not a Coxeter group, Brieskorn shows that there is a similar formula for the Poincaré polynomial but in terms of the *coexponents* of G; this numbers coincide with the exponents in the Coxeter case.

Proof. See [Bri73, Théorème 6]. □

1.44. *Example.* Recall that the braid arrangement $\mathcal{B}_n$ is the reflection arrangement corresponding to the symmetric group $G = \mathbb{S}_n$ acting on $\Bbbk^n$ by permuting its coordinates. The Fundamental Theorem of Symmetric Functions tells us that the algebra of invariants S^G is freely generated by the elementary symmetric polynomials

$$s_k = \sum_{1 \le j_1 < \ldots < j_k \le n} x_{j_1} \cdots x_{j_k} \qquad \text{for } 1 \le k \le n.$$

The exponents of G are therefore $0, \dots, n-1$ and an immediate application of Brieskorn's theorem yields

$$\pi(M(\mathcal{B}_n), t) = (1 + t)(1 + 2t) \cdots (1 + (n-1)t).$$

THE ORLIK-SOLOMON ALGEBRA

We now return to the situation in which $\Bbbk$ need not be $\mathbb{C}$. The Orlik-Solomon algebra $A^\bullet(\mathcal{A})$, presented by P. Orlik and L. Solomon in [OS80], gathers important combinatorial information of $\mathcal{A}$ and, if $\Bbbk$ is $\mathbb{C}$, it is also isomorphic to the cohomology of the complement $M(\mathcal{A})$.

1.45. Let E^1 be the vector space freely generated by symbols e_H, one for each $H \in \mathcal{A}$, let $E^\bullet(\mathcal{A}) = \Lambda^\bullet E^1$ be the exterior algebra of E^1 and write $uv := u \wedge v$ if $u, v \in E^\bullet(\mathcal{A})$. The pth homogeneous component of $E^\bullet(\mathcal{A})$ is spanned as a vector space by the monomials $e_{H_1}e_{H_2}\ldots e_{H_p}$ with each H_i in $\mathcal{A}$. For each $H \in \mathcal{A}$, there is a unique linear graded derivation $\partial : E^\bullet(\mathcal{A}) \to E^\bullet(\mathcal{A})$ of degree -1 such that $\partial(e_H) = 1$. This map satisfies

$$\partial(e_{H_1}\cdots e_{H_p}) = \sum_{k=1}^{k-1} e_{H_1}\cdots \hat{e}_{H_k}\cdots e_{H_p}$$

for $p \geq 2$ and $H_1, \ldots, H_p \in \mathcal{A}$ and $\partial^2 = 0$, as can be seen by a direct computation.

Let I be the ideal of $E^\bullet(\mathcal{A})$ generated by all elements of the form $\partial\left(e_{H_1}\cdots e_{H_p}\right)$ such that the hyperplanes $H_1, \ldots, H_p$ are not in general position, that is, that the corresponding linear forms are linearly dependent. As I is generated by homogeneous elements, it is a graded ideal: its pth homogeneous component is $I_p = I \cap E^p(\mathcal{A})$.

The *Orlik–Solomon algebra* $A^\bullet(\mathcal{A})$ is the quotient of $E^\bullet(\mathcal{A})$ by I. It is a graded commutative algebra and, since $I_0 = 0$, connected. Denoting the class of e_H in $A^\bullet(\mathcal{A})$ by a_H for each $H \in \mathcal{A}$, we observe that $\{a_H : H \in \mathcal{A}\}$ is a basis of $A^1(\mathcal{A})$. If $p \geq 1$ and $H_1, \ldots, H_p$ are hyperplanes not in general position then

$$e_{H_1}\cdots e_{H_p} = e_{H_1}\partial(e_{H_1}\cdots e_{H_p}) \in I,$$

so that $a_{H_1}\cdots a_{H_p} = 0$ in $A^\bullet(\mathcal{A})$. In particular, $A^p(\mathcal{A}) = 0$ if $p > n$.

1.46. Let us write, if $S = \{H_1, \ldots, H_p\}$ is a subset of $\mathcal{A}$, $e_S := e_{H_1}\cdots e_{H_p}$. Given $S, T \subset \mathcal{A}$, we have $\partial(e_T\partial e_S) = \partial e_T \partial e_S$ and therefore we see that $\partial(I) \subset I$. As a consequence of this, $\partial : E^\bullet(\mathcal{A}) \to E^\bullet(\mathcal{A})$ descends to $A^\bullet(\mathcal{A})$, inducing a graded derivation $\partial : A^\bullet(\mathcal{A}) \to A^\bullet(\mathcal{A})$ that satisfies $\partial^2 = 0$.

Proposition. *The complex* $(A^\bullet(\mathcal{A}), \partial)$ *is acyclic.*

Proof. Let us choose $H \in \mathcal{A}$. As $\partial a_H = 1$, for every $b \in A^\bullet(\mathcal{A})$ we have $b = \partial(ba_H) + a_H\partial b$. It follows that if b is a cocycle then it is a coboundary. □

1.47. There is a standard basis for $A^\bullet(\mathcal{A})$, the *broken circuit basis.* We do not give here an explicit construction of this basis —it can be found, for instance, in [OT92, §3.1]—, but we remark that it depends only on the poset of intersections $\mathcal{L}(\mathcal{A})$ and an arbitrary total order on $\mathcal{A}$; the idea is essentially that of the Gröbner bases. The existence of this basis emphasizes the fact that $A^\bullet(\mathcal{A})$ depends only on the combinatorics of $\mathcal{A}$.

We begin now to state a series of propositions that lead to the main result of this section, Theorem 1.52, which asserts that $A^\bullet(\mathcal{A})$ is isomorphic to $H^\bullet(M(\mathcal{A}))$ when $\Bbbk = \mathbb{C}$. In view of our previous remark, this result implies that the cohomology of $M(\mathcal{A})$ depends only on the combinatorics of $\mathcal{A}$, as we promised in **1.22**. To prove that the algebra $A^\bullet(\mathcal{A})$ is isomorphic to $H^\bullet(M(\mathcal{A}))$ or, equivalently, in view of Theorem 1.40, to $R^\bullet(\mathcal{A})$, we will first construct an epimorphism $A^\bullet(\mathcal{A}) \to R^\bullet(\mathcal{A})$ and then, comparing dimensions, show that it is an isomorphism.

1.48. We start with a particular case of a result of P. Orlik and L. Solomon, who consider, more generally, geometric lattices and not only those that come from hyperplane arrangements. Both the statement and its proof are purely combinatorial.

Theorem. *The Hilbert series of $A^\bullet(\mathcal{A})$ is the Poincaré polynomial of the arrangement $\pi(\mathcal{A}, t)$.* □

Proof. This is Theorem 2.6 of [OS80]. □

1.49. The next step in our plan is the construction of an algebra morphism $A^\bullet(\mathcal{A}) \to R^\bullet(\mathcal{A})$.

Proposition (P. Orlik and L. Solomon, [OS80])**.** *There exists a surjective morphism of graded algebras $A^\bullet(\mathcal{A}) \to R^\bullet(\mathcal{A})$ such that $a_H \mapsto \omega_H$ for every $H \in \mathcal{A}$.*

Proof. Let $\gamma : E^\bullet(\mathcal{A}) \to R^\bullet(\mathcal{A})$ be the morphism of algebras such that $\gamma(e_H) = \omega_H$ for each $H \in \mathcal{A}$. Evidently, γ is surjective: we will prove that $\gamma(I) = 0$, so that γ factors through $A^\bullet(\mathcal{A})$.

Let $S = \{H_1, \dots, H_p\}$ be a subset of $\mathcal{A}$ in general position and, for each $i \in [\![p]\!]$, let α_i be a linear form with kernel H_i, so that, in particular, the set $\hat{S} \coloneqq \{\alpha_, \dots, \alpha_p\}$ is linearly dependent. We need to show that $\gamma(\partial e_S) = 0$.

We may assume without loss of generality that no proper subset of S is in general position. Indeed, if, for example, the subset $\left\{\alpha_2, \dots, \alpha_p\right\}$ of $\hat{S}$ is linearly dependent then

$$\partial e_S = e_{H_2} \cdots e_{H_p} + e_{H_1} \partial(e_{H_2} \cdots e_{H_p})$$

and therefore, as $e_{H_2} \cdots e_{H_p} = 0$ because of linear dependence, we see that $\gamma(\partial e_S)$ is zero if and only if $\gamma(\partial(e_{H_2} \cdots e_{H_p})) = 0$.

Now, our assumption is that there is a linear combination $\sum_{i=1}^p c_i \alpha_i = 0$ with every c_i nonzero; up to rescaling the elements of $\hat{S}$, we may in fact take $c_i = 1$ for every i. We then have that $\sum_{i=1}^p d\alpha_i = 0$ and therefore, for each $j \in [\![p-1]\!]$, that

$$0 = \left(\sum_{i=1}^p d\alpha_i\right) d\alpha_1 \cdots \widehat{d\alpha_j}\widehat{d\alpha}_{j+1} \cdots d\alpha_p = d\alpha_1 \cdots \widehat{d\alpha_j} \cdots d\alpha_p + d\alpha_1 \cdots \widehat{d\alpha}_{j+1} \cdots d\alpha_p.$$

For each $j \in [\![p]\!]$ we define the rational form η_j

$$\eta_j = \frac{(-1)^{j-1}}{\alpha_j} \omega_1 \cdots \hat{\omega}_j \cdots \omega_p.$$

Assuming that $j < p$, we multiply this equation by $\alpha_1 \cdots \alpha_p$ and obtain that

$$\begin{aligned} \alpha_1 \cdots \alpha_p \eta_j &= (-1)^{j-1} d\alpha_1 \cdots \widehat{d\alpha_j} \cdots d\alpha_p \\ &= (-1)^j d\alpha_1 \cdots \widehat{d\alpha}_{j+1} \cdots d\alpha_p \\ &= \alpha_1 \cdots \alpha_p \eta_{j+1}, \end{aligned}$$

so that in fact $\eta_1 = \cdots = \eta_p$. With all this in hand, we conclude that

$$\gamma(\partial e_S) = \sum_{j=1}^{p}(-1)^{j-1}\omega_1 \cdots \hat{\omega}_j \cdots \omega_p = \left(\sum_{j=1}^{p} \alpha_j\right)\eta_1 = 0,$$

as we wanted. □

1.50. The following result is known as Brieskorn's Lemma and is useful when performing inductive arguments.

Lemma (E. Brieskorn)**.** *Let $X \in \mathcal{L}(\mathcal{A})$ and recall that $\mathcal{A}_x$ is the set of hyperplanes of $\mathcal{A}$ that contain X. Let k be an integer such that $0 \le k \le n$. The inclusion maps $i_X : M(\mathcal{A}) \to M(\mathcal{A}_X)$ induce isomorphisms*

$$\bigoplus_{\operatorname{rank}(X)=k} H^k(M(\mathcal{A}_X)) \cong H^k(M(\mathcal{A})).$$

Proof. See [Bri73, Lemme 3]. □

1.51. The following theorem is of vital importance in the theory of hyperplane arrangements.

1.52. Theorem (P. Orlik and L. Solomon, [OS80, Theorem 5.2])**.** *Let $\mathcal{A}$ be a* complex *hyperplane arrangement. There exists an isomorphism of graded algebras $A^\bullet(\mathcal{A}) \cong H^\bullet(M(\mathcal{A}))$ such that $\alpha_H \mapsto [\eta_H]$.*

Proof. Recall from Theorem 1.40 that the map

$$R^\bullet(\mathcal{A}) \ni \omega_H \mapsto [\eta_H] \in H^\bullet(M(\mathcal{A}))$$

is an isomorphism of graded algebras. As the morphism $A^\bullet(\mathcal{A}) \to R^\bullet(\mathcal{A})$ from Proposition 1.49 is surjective, it will be enough to see that $\dim A^\bullet(\mathcal{A}) = \dim H^\bullet(M(\mathcal{A}))$.

The dimension of $A^\bullet(\mathcal{A})$ is the value at $t = 1$ of the Hilbert series of $A^\bullet(\mathcal{A})$ that we gave in Theorem **1.48**: in this way we see that

$$\dim A^\bullet(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} (-1)^{r(X)}\mu(X).$$

It suffices to show, then, that this number equals $\dim H^\bullet(M(\mathcal{A}))$ and, in order to prove this equality, we let $Z := \bigcap_{H\in\mathcal{A}} H$ and check that if q is the codimension of Z then

$$\dim H^q(M(\mathcal{A})) = (-1)^q\mu(Z).$$

This is is immediate if $q = 0$: both sides are equal to one. We proceed by induction supposing that q is a positive integer and that $X \in \mathcal{L}(\mathcal{A})$ is such that $r(X) < q$. Applying the inductive hypothesis to the complement $M(\mathcal{A}_X)$ and using the fact that $X = \bigcap_{H\in\mathcal{A}_X} H$, we see that the dimension of $H^{r(X)}(M(\mathcal{A}_X))$ is $(-1)^{r(X)}\mu(X)$. As the Euler characteristic of $M(\mathcal{A}_X)$ is 0 because $r(X) > 0$ we can use the isomorphism in Brieskorn's Lemma 1.50 to we see that

$$\begin{aligned} 0 &= \sum_{p=0}^{q}(-1)^p \dim H^p(M) = \sum_{p=0}^{q-1}\sum_{r(X)=p} \dim H^p(M(\mathcal{A}_X)) + (-1)^q H(M) \\ &= \sum_{r(X)<q} (-1)^{r(X)}\mu(X) + (-1)^q \dim H^q(M). \end{aligned}$$

The second defining property of the Möbius function in **1.12** now tells us that

$$0 = -\mu(\bigcap_{H\in\mathcal{A}} H) + (-1)^q \dim H^q(M)$$

and this completes the inductive step. With this at hand, we use one more time Brieskorn's Lemma 1.50 to finally obtain

$$\begin{aligned} \dim H^\bullet(M) &= \sum_{p=0}^{q} \dim H^p(M) = \sum_{p=0}^{q}\sum_{r(X)=p} \dim H^p(M(\mathcal{A}_X)) \\ &= \sum_{X\in\mathcal{L}(\mathcal{A})} (-1)^{r(X)}\mu(X), \end{aligned}$$

which is what we wanted to prove. □

1.53. Combining Theorem 1.52 with Theorem 1.48 we obtain at once our next remark.

Corollary. *The Poincaré polynomial of $M(\mathcal{A})$ is equal to the Poincaré polynomial of the arrangement $\pi(\mathcal{A}, t)$.*

This statement generalizes the fact that the Poincaré polynomial of the braid arrangement $\mathcal{B}_n$ that we computed in (1.4) agrees with the Poincaré polynomial of $M(\mathcal{B}_n)$ that we found in Example 1.44.

A remarkable consequence of this corollary is the following. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ and $\mathcal{A}'$ be the arrangement in $\mathbb{C}^n$ whose hyperplanes are defined by the same equations as those of $\mathcal{A}$. This corollary and Theorem 1.18 imply that the dimension of the total cohomology $H^\bullet(M(\mathcal{A}'))$ is equal to the number of chambers of the arrangement $\mathcal{A}$ in $\mathbb{R}^n$.

1.54. Some years after the proof of Theorem 1.52 , P. Orlik, L. Solomon and H. Terao were able to generalize the statement to the case in which the ground field $\Bbbk$ is not $\mathbb{C}$.

Theorem (P. Orlik, L. Solomon and H. Terao, [OST84])**.** *The surjective morphism of algebras $A^\bullet(\mathcal{A}) \to R^\bullet(\mathcal{A})$ of Proposition 1.49 is an isomorphism.* □

The exists of this isomorphism, in particular, shows the non-evident combinatorial nature of $R^\bullet(\mathcal{A})$. An improved version of the proof can be found in [OT92, §3.5]. The argument is different from the one we used above: their key idea is that if $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple of arrangements as in **1.10** then there are exact sequences of algebras

$$0 \longrightarrow A^\bullet(\mathcal{A}') \longrightarrow A^\bullet(\mathcal{A}) \longrightarrow A^\bullet(\mathcal{A}'') \longrightarrow 0$$

and

$$0 \longrightarrow R^\bullet(\mathcal{A}') \longrightarrow R^\bullet(\mathcal{A}) \longrightarrow R^\bullet(\mathcal{A}'') \longrightarrow 0$$

which, along with the epimorphism $A^\bullet(\mathcal{A}) \to R^\bullet(\mathcal{A})$, are used to set up an inductive argument.

1.55. We end this chapter by stating an important result that describes the cohomology of the complex of logarithmic forms on $\mathcal{A}$ for a large family of arrangements.

A central arrangement $\mathcal{A}$ is *tame* if for every $p \geq 0$ the projective dimension of the S-module $\Omega^p(\mathcal{A})$ is at most p. This condition is satisfied in many important situations. First, as a consequence of **1.36**, for a free central arrangement each $\Omega^p(\mathcal{A})$ is free and thus its projective dimension is 0: it follows that free arrangements are tame. Another big family of examples is that of *generic* arrangements, that is, those arrangements $\mathcal{A}$ with at least n hyperplanes, any n of which are in general position. For example, the arrangement defined by $x_1 \cdots x_n(x_1 + \ldots + x_n) = 0$ that we worked with in Example 1.31 is generic. L. Rose and H. Terao have found in [RT91] a projective resolution of $\Omega^p(\mathcal{A})$ of length p for each $p \in [\![n]\!]$, so that, in particular, generic arrangements are tame —more information on this class of arrangements can be found in [OT92, §5.1]. As a final example, one can show that all arrangements in $\mathbb{k}^3$ are tame. Not all arrangements are tame, though: the smallest example of a non-tame arrangement is the set of fifteen hyperplanes in $\mathbb{k}^4$ with equations $\sum_{i=0}^4 a_i x_i = 0$ for $a_i \in \{0, 1\}$. These last two facts are explained by J. Wiens and S. Yuzvinsky in [WY97, Section 2].

We are interested in tame arrangements because of the following result, which is known sometimes as the Logarithmic Comparison Theorem.

Theorem (J. Wiens and S. Yuzvinsky)**.** *Let $\mathcal{A}$ be a tame arrangement. The natural embedding of $R^\bullet(\mathcal{A})$ into $\Omega^\bullet(\mathcal{A})$ is a quasi-isomorphism.*

Proof. See [WY97, Corollary 2.3]. □

1.6 RESUMEN

El capítulo empieza con las definiciones básicas sobre arreglos de hiperplanos, estableciendo notación y presentando los ejemplos y construcciones con los que lidiamos a lo largo de la tesis. Concretamente, fijado un cuerpo $\mathbb{k}$, un *arreglo de hiperplanos* $\mathcal{A}$ es un conjunto finito de hiperplanos afines $\{H_1, \ldots, H_l\}$ en un espacio vectorial V de dimensión finita y es *central* si todos sus hiperplanos son en verdad subespacios. Llamamos S al álgebra de funciones coordenadas de V y la identificamos con $\mathbb{k}[x_1, \ldots, x_n]$. Para cada $i \in \{1, \ldots, l\}$, sea $\alpha_i : V \to \mathbb{k}$ una forma lineal con núcleo H_i: decimos que el polinomio $Q = \alpha_1 \cdots \alpha_l \in S$ *define* al arreglo $\mathcal{A}$.

Una vez establecido esto, estudiamos en **1.10** el poset de intersecciones $\mathcal{L}(\mathcal{A})$ del arreglo. A continuación, vemos los polinomios característico y de Poincaré del arreglo, que son importantes invariantes combinatorios, y los calculamos en los casos concretos del arreglo booleano, el arreglo de todos los subespacios de codimensión 1 en un espacio vectorial sobre un cuerpo finito y el arreglo de trenzas. Este es el contenido de los Ejemplos 1.15, 1.16 y 1.17.

Desde la Sección 1.3 hasta el final de la tesis suponemos que los arreglos son centrales. En esta sección nos ocupamos del álgebra de *derivaciones tangentes al arreglo* $\mathcal{A}$, definida por

$$\mathsf{Der}(\mathcal{A}) := \{\theta \in \mathsf{Der}(S) : \theta(Q) \in QS\}$$

y que es una subálgebra de Lie y un S-submódulo de $\mathsf{Der}(S)$, el módulo de las derivaciones de S. Este álgebra de Lie es un invariante interesante del arreglo y ha sido objeto de estudio de varios trabajos: el libro de P. Orlik y H. Terao [OT92] y el de A. Dimca [Dim17] sirven como referencias generales. Sirviendonos del álgebra de Lie de derivaciones, podemos definir una clase importante de arreglos: decimos que un arreglo $\mathcal{A}$ es *libre* si $\mathsf{Der}\,\mathcal{A}$ es un S-módulo libre. Por ejemplo, un arreglo central de rectas en el plano es libre; también son libres, de acuerdo a un resultado de H. Terao en [Ter80a], los arreglos de hiperplanos de reflexión de un grupo finito generado por pseudo-reflexiones.

En la Sección 1.4 estudiamos el subcomplejo de $\Omega^\bullet(\mathcal{A})$ del complejo formas racionales $\Omega^\bullet(V)$ llamado de *formas logarítmicas en* $\mathcal{A}$, dado por

$$\Omega^p(\mathcal{A}) = \left\{\omega \in \Omega^p(V) \text{ tal que } Q\omega \in \Omega^p[V] \text{ y } Qd\omega \in \Omega^{p+1}[V]\right\}$$

si $p \geq 0$. En particular, vemos en la Propisición 1.34 que el S-módulo $\Omega^1(\mathcal{A})$ es S-dual al álgebra $\mathsf{Der}\,\mathcal{A}$. Definimos a continuación el álgebra $R^\bullet(\mathcal{A})$ como la subálgebra de $\Omega^\bullet(\mathcal{A})$ generada por el conjunto $\{\omega_H : H \in \mathcal{A}\}$ y encontramos una presentación por generadores y relaciones para los casos de arreglos centrales de rectas y el arreglos de trenzas $\mathcal{B}_3$ en los Ejemplos 1.38 y 1.39.

Terminamos el capítulo con la Sección 1.5, en que damos algunos resultados de la teoría que son importantes para la tesis. El primero, que se se debe a E. Brieskorn en [Bri73], fue encontrado para el caso especial de arreglos de trenzas por V. I. Arnold en [Arn69] y aparece en la Sección 1.5 como Teorema 1.40.

Teorema. *Sea $\mathcal{A}$ un arreglo de hiperplanos en un espacio vectorial complejo. Hay un isomorfismo de álgebras entre $R(\mathcal{A})$ y la cohomología de de Rham $H^\bullet(M(\mathcal{A}))$ del espacio complementario al arreglo $M(\mathcal{A})$.*

El otro resultado importante muestra que si $\mathcal{A}$ es un arreglo en un espacio vectorial complejo entonces la cohomología del espacio complementario a $\mathcal{A}$ depende solamente de la combinatoria del arreglo. Para ver esto estudiamos el álgebra $A^\bullet(\mathcal{A})$, definida para un arreglo sobre un cuerpo cualquiera de característica cero por P. Orlik y L. Solomon en [OS80] en términos combinatorios, y probamos en el Teorema 1.52 que, efectivamente, este álgebra es un análogo combinatorio del álgebra de cohomología de $H^\bullet(M(\mathcal{A}))$:

Teorema. *Si $\mathcal{A}$ un arreglo de hiperplanos en un espacio vectorial complejo, hay un isomorfismo de álgebras graduadas $A^\bullet(\mathcal{A}) \cong H^\bullet(M(\mathcal{A}))$.*

– 2 –
The algebra of differential operators tangent to a hyperplane arrangement

In this chapter we introduce the associative algebra $\mathsf{Diff}(\mathcal{A})$ of differential operators tangent to a hyperplane arrangement. We show that this algebra is the subalgebra generated by S and $\mathsf{Der}\,\mathcal{A}$ inside $\mathsf{Diff}(S)$ if the arrangement is free and, moreover, that it is isomorphic to the enveloping algebra of a suitable Lie–Rinehart pair. With these results at hand, we are able to give a precise description of $\mathsf{Diff}(\mathcal{A})$ in the case of a central line arrangement and also to study the twisted Calabi–Yau property for $\mathsf{Diff}(\mathcal{A})$ in the general situation.

2.1 Algebras of differential operators

2.1. We assume from now on that the characteristic of the ground field $\Bbbk$ is zero. Let B be a commutative algebra and write $\mathsf{End}\,B$ the algebra of $\Bbbk$-linear endomorphisms of B as a vector space. We inductively define subspaces $\mathsf{Diff}(B)_p$ of $\mathsf{End}\,B$, one for each $p \geq -1$, setting $\mathsf{Diff}(B)_{-1} = 0$ and

$$\mathsf{Diff}(B)_p = \{f : B \to B : fb - bf \in \mathsf{Diff}(B)_{p-1} \text{ for all } b \in B\} \qquad \text{if } p \geq 0.$$

In [MR01, §15.5] we can find the following result.

Lemma. *The union* $\mathsf{Diff}(B) := \bigcup_{p\geq -1} \mathsf{Diff}(B_p)$ *is a subalgebra of* $\mathsf{End}(B)$ *and* $\{\mathsf{Diff}(B_p)\}_{p\geq -1}$ *is an exhaustive and increasing filtration of* $\mathsf{Diff}(B)$ *which is compatible with its multiplicative structure and such that the associated graded algebra* $\mathsf{gr}\,\mathsf{Diff}(B)$ *is commutative.* □

The algebra $\mathsf{Diff}(B)$ is called the *algebra of differential operators on* B. We say that $f \in \mathsf{Diff}(B)$ has *order* p if it belongs to $\mathsf{Diff}(B)_p$ and not to $\mathsf{Diff}(B)_{p-1}$. There is an injective morphism of algebras $\phi : B \to \mathsf{Diff}(B)$ such that $\phi(b)(x) = xb$ for all $b, x \in B$ which we will view as an identification. On the other hand, a non-zero derivation θ in $\mathsf{Der}\,B$ is a differential operator of order 1, so that $\mathsf{Der}\,B$ is a subspace of $\mathsf{Diff}(B)$ which is easily seen to be a Lie subalgebra.

The following theorem of Grothendieck gives us generators of $\mathsf{Diff}(B)$ in an important case.

2.2. Theorem. *Let B be a regular commutative finitely generated algebra which is a domain. The algebra* $\mathsf{Diff}(B)$ *is generated as a subalgebra of* $\mathsf{End}(B)$ *by B and* $\mathsf{Der}\,B$. □

Y. Nakai in [Nak70] has conjectured that, conversely, B has to be regular if $\mathsf{Diff}\,B$ is generated by B and $\mathsf{Der}\,B$. This conjecture is open except in very special cases.

2.3. *Example.* Let $S = \Bbbk[x_1, \dots, x_n]$ be the polynomial algebra on n variables and let us denote the usual partial derivatives on S by $\partial_1, \dots, \partial_n$. Theorem 2.2 allows us to find a presentation of $\mathsf{Diff}(S)$ as follows.

As $\{x_1, \dots, x_n\}$ and $\{\partial_1, \dots, \partial_n\}$ generate S as an algebra and $\mathsf{Der}\, S$ as an S-module, the set $\{x_1, \dots, x_n, \partial_1, \dots, \partial_n\}$ generate $\mathsf{Diff}\, S$ as an algebra. A straightforward calculation shows that

$$[\partial_i, \partial_j] = 0, \qquad [x_i, x_j] = 0, \qquad [\partial_i, x_j] = \delta_{i,j}$$

for each i and j in $[\![n]\!]$. On the other hand, recall the nth Weyl algebra A_n is the quotient of the free algebra with $2n$ generators $q_1, \dots, q_n, p_1, \dots, p_n$ by the two-sided ideal generated by the elements

$$[q_i, q_j], \qquad [p_i, p_j], \qquad [q_i, p_j] - \delta_{i,j}$$

for every $1 \leq i, j \leq n$. There is then a unique morphism of algebras $\phi : A_n \to \mathsf{Diff}(B)$ such that $x_i \mapsto q_i$ and $p_i \mapsto \partial_i$ for $i \in [\![n]\!]$ and it is clearly surjective. Since A_n is, as it is well-known, a simple algebra, this morphism is also injective and therefore an isomorphism.

This example generalizes, with a substantial amount of work, in the following way: V. V. Bavula in [Bav10] gives an explicit construction of a finite set of algebra generators and a finite set of defining relations for the ring of differential operators on a regular algebra in terms of a presentation of the algebra.

2.4. Let $\mathcal{A}$ be a hyperplane arrangement on V and let us keep the usual notation; in particular, let Q be the defining polynomial of $\mathcal{A}$. We would like to construct a version of the algebra of differential operators on V relative to $\mathcal{A}$. For this we need the following notion: if R is an algebra and $I \subset R$ is a right ideal, the largest subalgebra $\mathbb{I}_R(I)$ of R that contains I as an ideal can be seen to be $\{r \in R : rI \subset I\}$ and it is called the *idealizer* of I in R. The *algebra of differential operators tangent to the arrangement* $\mathcal{A}$ is

$$\mathsf{Diff}(\mathcal{A}) = \bigcap_{t \geq 1} \mathbb{I}_{\mathsf{Diff}(S)}(Q^t\, \mathsf{Diff}(S)).$$

We have a variant of Theorem 2.2 for this situation.

Theorem. *If $\mathcal{A}$ is a free hyperplane arrangement then the algebra $\mathsf{Diff}(\mathcal{A})$ is generated by $S \cup \mathsf{Der}(\mathcal{A})$.* □

This theorem is proved by F. J. Calderón Moreno in [CM99] and by M. Schulze in [Sch07] using techniques from analytic geometry for the case $\Bbbk = \mathbb{C}$ and by M. Suárez-Álvarez in [SÁ18] for any field of characteristic zero by "extending to differential operators of arbitrary order" Saito's criterion **1.26**.

The algebra $\mathsf{Diff}(\mathcal{A})$ may be generated by $S \cup \mathsf{Der}\,\mathcal{A}$ even if the arrangement is not free. Indeed, this is the case of the arrangement in $\Bbbk^3$ with defining polynomial $xyz(x + y + z)$ that we studied in Example 1.31: this was shown by [Sch07, §5]. There are no known necessary and sufficient conditions for the conclusion of the theorem to hold.

2.5. *Example.* The nth Boolean arrangement $\mathcal{B}_n$ can be viewed as the product $\mathcal{B}_1^{\times n}$ of n copies of the 1-dimension non-empty central arrangement $\mathcal{B}_1$. In view of **1.23**, we have an isomorphism of algebras $\mathsf{Diff}(\mathcal{B}_n) \cong \mathsf{Diff}(\mathcal{B}_1)^{\otimes n}$. Now the arrangement $\mathcal{B}_1$, defined in $V = \Bbbk$ by $Q = x$, is free, with $\mathsf{Der}(\mathcal{B}_1)$ freely generated by the derivation $\theta = x\partial_x$. It follows that the algebra $\mathsf{Diff}(\mathcal{B}_1)$ is generated by x and θ. Computing, we find that $[\theta, x] = x$. If we let D be the quotient algebra

$$D = \frac{\Bbbk\langle y, t\rangle}{(ty - yt - y)},$$

there there is a surjective map of algebras $\pi : D \to \mathsf{Diff}(\mathcal{B}_1)$ such that $\pi(y) = x$ and $\pi(t) = \theta$. The algebra D is manifestly the enveloping algebra of the non-abelian Lie algebra of dimension 2 spanned by t and y with $[t, y] = t$. In particular, the set $\{y^i t^j : i, j \geq 0\}$ is a basis of D. Using this it is easy to check that the map π is injective so, putting everything together, an isomorphism.

The conclusion of this is that the algebra $\mathsf{Diff}(\mathcal{B}_n)$ is isomorphic to the algebra freely generated by letters $y_1, \dots, y_n, t_1, \dots, t_n$, subject to the relations

$$[y_i, y_j] = [t_i, t_j] = 0, \qquad [t_i, y_j] = \delta_{i,j} y_j, \qquad \text{with } 1 \leq i, j \leq n.$$

2.2 The algebra of differential operators tangent to a central arrangement of lines

2.6. We fix a ground field $\Bbbk$ of characteristic zero and put $S = \Bbbk[x, y]$. We view S as a graded algebra as usual, with both x and y of degree 1, and for each $p \geq 0$ we write S_p the homogeneous component of S of degree p. The Lie algebra $\mathsf{Der}(S)$ of derivations of S, which is a free left graded S-module, is freely generated by the usual partial derivatives ∂_x, $\partial_y : S \to S$, which are homogeneous elements of $\mathsf{Der}(S)$ of degree -1.

Recall, as in **2.1**, that $\mathsf{Diff}(S)$ is the associative algebra of regular differential operators on S, that we may view S as a subalgebra of $\mathsf{Diff}(S)$ and, from Example 2.3, that $\mathsf{Diff}(S)$ is generated as a subalgebra of $\mathsf{End}(S)$ by S and $\mathsf{Der}(S)$. The algebra $\mathsf{Diff}(S)$ is generated by x, y, ∂_x and ∂_y, and in fact these elements generate it freely subject to the relations

$$[x, y] = [\partial_x, y] = [\partial_y, x] = [\partial_x, \partial_y] = 0, \qquad [\partial_x, x] = [\partial_y, y] = 1.$$

It follows easily from this that $\mathsf{Diff}(S)$ has a $\mathbb{Z}$-grading with x and y in degree 1 and ∂_x and ∂_y in degree -1, and that with respect to this grading, S is a graded $\mathsf{Diff}(S)$-module.

2.7. We fix an integer $r \geq -1$ and consider a central arrangement $\mathcal{A}$ of $r + 2$ lines in the plane $\Bbbk^2$. Up to a change of coordinates, we may assume that the line with equation $x = 0$ is one of the lines in $\mathcal{A}$, so that the defining polynomial Q of the arrangement is of the form xF for some square-free homogeneous polynomial $F \in S$ of degree $r + 1$ which does not have x as a factor. Up to multiplying by a scalar, which does not change anything substantial, we may assume that $F = x\bar{F} + y^{r+1}$ for some $\bar{F} \in S_r$.

The Lie algebra of derivations of S that preserve the arrangement, defined in **1.20**, is a graded Lie subalgebra of $\mathsf{Der}(S)$. The two derivations

$$E = x\partial_x + y\partial_y, \qquad D = F\partial_y$$

are elements of $\mathsf{Der}(\mathcal{A})$ of degrees 0 and r, and it follows immediately from Saito's criterion **1.26** that the set $\{E, D\}$ is a basis of $\mathsf{Der}(\mathcal{A})$ as a graded S-module: this is the content of Example 1.28.

2.8. The *algebra of differential operators tangent to the arrangement* $\mathcal{A}$ is the subalgebra $\mathsf{Diff}(\mathcal{A})$ of $\mathsf{Diff}(S)$ generated by S and $\mathsf{Der}(\mathcal{A})$, as we saw in 2.4. It follows immediately from the remarks above that $\mathsf{Diff}(\mathcal{A})$ is generated by x, y, E and D, and a computation shows that the following commutation relations hold in $\mathsf{Diff}(\mathcal{A})$:

$$\begin{aligned} &[y, x] = 0, \\ &[D, x] = 0, && [D, y] = F, \\ &[E, x] = x, && [E, y] = y, && [E, D] = rD. \end{aligned} \tag{2.1}$$

Since these generators are homogeneous elements in $\mathsf{Diff}(S)$ —with E of degree 0, x and y of degree 1 and D of degree r— we see that the algebra $\mathsf{Diff}(\mathcal{A})$ is a graded subalgebra of $\mathsf{Diff}(S)$ and, by restricting the structure from $\mathsf{Diff}(S)$, that S is a graded $\mathsf{Diff}(\mathcal{A})$-module.

The set of commutation relations given above is in fact a presentation of the algebra $\mathsf{Diff}(\mathcal{A})$. More precisely, we have the following lemma.

Lemma. *The algebra* $\mathsf{Diff}(\mathcal{A})$ *is isomorphic to the iterated Ore extension* $S[D][E]$. *It is a noetherian domain and the set* $\{x^i y^j D^k E^l : i, j, k, l \geq 0\}$ *is a* $\Bbbk$*-basis for* $\mathsf{Diff}(\mathcal{A})$.

Here we view D as a derivation of S, so that we way construct the Ore extension $S[D]$, and view E as a derivation of this last algebra, so as to be able extend once more to obtain $S[D][E]$.

Proof. It is clear at this point that the obvious map $\pi : S[D][E] \to \mathsf{Diff}(\mathcal{A})$ is a surjective morphism of algebras, so we need only prove that it is injective. To do that, let us suppose that there exists a non-zero element L in $S[D][E]$ whose image under the map π is zero, and suppose that $L = \sum_{i,j\geq 0} f_{i,j} D^i E^j$, with coefficients $f_{i,j} \in S$ for all $i, j \geq 0$, almost all of which are zero. As L is non-zero, we may consider the number $m = \max\{i + j : f_{i,j} \neq 0\}$.

Let us now fix a point $p = (a, b) \in \mathbb{A}^2$ which is not on any line of the arrangement $\mathcal{A}$, so that $aF(a, b) \neq 0$, and let O_p be the completion of S at the ideal $(x - a, y - b)$ or, more concretely, the algebra of formal series in $x - a$ and $y - b$. We view O_p as a left module over $\mathsf{Diff}(S)$ in the tautological way and, by restriction, as a left $\mathsf{Diff}(\mathcal{A})$-module. There exist formal series ϕ and ψ in O_p such that

$$E \cdot \phi = 1, \qquad D \cdot \phi = 0, \qquad E \cdot \psi = 0, \qquad D \cdot \psi = x^r.$$

Indeed, we may choose $\phi = \ln x$ to satisfy the first two conditions, and the last two ones are equivalent to the equations

$$\partial_x \psi = -\frac{x^{r-1} y}{F}, \qquad \partial_y \psi = \frac{x^r}{F},$$

which can be solved for ψ, as the usual well-known sufficient integrability condition from elementary calculus holds. If now $s, t \in \mathbb{N}_0$ are such that $s + t = m$, a straightforward computation shows that $L \cdot \phi^s \psi^t = s!t!x^{rt} f_{s,t}$ in $\mathcal{O}_p$, and this implies that $f_{s,t} = 0$. This contradicts the choice of m and this contradiction proves what we want. □

2.3 Lie–Rinehart pairs

2.9. In Section 2.2 we were able to give a very concrete description of the algebra of differential operators tangent to an arrangement of lines. As soon as one tries to do the same by hand in larger examples the task becomes prohibitively laborious. The language of Lie–Rinehart pairs provides a formalism that allows us to handle this complexity. Originally, this pairs were defined by G. Rinehart in [Rin63], in order to generalize the algebraic structure of vector fields and smooth functions on a manifold to commutative algebras and Lie algebras.

A *Lie–Rinehart pair* (S, L) consists of a commutative $\Bbbk$-algebra S and a $\Bbbk$-Lie algebra L such that L acts on S by $\Bbbk$-linear derivations, L is an S-module and

$$(s\alpha)(t) = s(\alpha(t)), \qquad [\alpha, s\beta] = s[\alpha, \beta] + \alpha(s)\beta$$

for s, t in S and α and β in L. Given such a pair, a *Lie–Rinehart module* —or (S, L)-module— is a vector space M that is at the same time an S-module and an L-Lie module in such a way that

$$(s\alpha)(m) = s(\alpha(m)), \qquad \alpha(sm) = s\alpha(m) + \alpha(s)m \tag{2.2}$$

for $s \in S$, $\alpha \in L$ and $m \in M$. The first important example of a module is given by $M = S$, with the obvious actions of S and of L.

2.10. *Example.* A Lie–Rinehart pair (S, L) in which the action of L on S is trivial can be simply described as an S-Lie algebra and the corresponding (S, L)-modules are just Lie L-modules. We encounter this situation often with $S = \Bbbk$: this is Lie theory.

2.11. *Example.* If S is a commutative algebra and L is a subalgebra of the Lie algebra of derivations $\operatorname{Der} S$ that is at the same time an S-submodule, then (S, L) is a Lie–Rinehart pair.

A particular case of this is obtained by taking $S = \Bbbk[x_1, \dots, x_n]$ and $L = \operatorname{Der} S$, the full algebra of derivations, which is freely generated as an S-module by the derivations $\partial_1, \dots, \partial_n$. It is easy to construct Lie–Rinehart modules for the pair (S, L). One need only notice that the Weyl algebra A_n of Example 2.3 is an (S, L)-module with actions induced by left multiplications and then use the fact that any A_n-module can be viewed as an (S, L)-module in a similar way.

A similar example but of a different category can be constructed as follows. If M is a finite dimensional smooth manifold, we put $S = C^\infty(M)$, the algebra of smooth functions, and $L = \mathfrak{X}(M)$, the Lie-algebra of vector fields on M. Notice that L is the Lie algebra of derivations of S; this is the content of Lemma 3.3 in [KMS93]. We can construct Lie–Rinehart modules for the pair (S, L) from geometric data as follows. Let $E \to M$ be a smooth vector bundle on M and let $\Gamma(E)$ be the space of smooth sections of E: an (S, L)-module structure on $\Gamma(E)$ compatible with the usual S-module structure turns out to be the same a linear connection on $E \to M$ with zero curvature.

2.12. *Example.* Another instance of the Example 2.11 that interests us particularly arises from hyperplane arrangements. If $\mathcal{A}$ is a hyperplane arrangement in a vector space V, it is straightforward to check that the algebra of coordinate functions $S = \Bbbk[x_1, \dots, x_l]$ of V and $L = \operatorname{Der} \mathcal{A}$ form a Lie–Rinehart pair.

2.13. Let (S, L) be a Lie–Rinehart pair. J. Huebschmann shows in [Hue90] that there is an associative algebra $U = U(S, L)$, the *universal enveloping algebra of the pair*, endowed with a morphism of algebras $i : S \to U$ and a morphism of Lie algebras $j : L \to U$ that satisfy, for $s \in S$ and $\alpha \in L$,

$$i(s)j(\alpha) = j(s\alpha), \qquad j(\alpha)i(s) - i(s)j(\alpha) = i(\alpha(s))$$

and universal with these properties. Let us briefly describe the construction of $U(S, L)$ presented in [Hue90, §1]; there is an alternative, less conceptual, description in [Rin63]. We start by considering the usual enveloping algebra $U(L)$ of L as a Lie $\Bbbk$-algebra. As S is L-Lie module, we can view S as a left $U(L)$-module and, using this structure, we can turn the vector space $S \otimes U(L)$ into an associative algebra in such a way that the obvious maps $S \to S \otimes U(L)$ and $U(L) \to S \otimes U(L)$ are multiplicative and

$$(1 \otimes \alpha) \cdot (s \otimes 1) = s \otimes \alpha + \alpha(s) \otimes 1$$

whenever $s \in S$ and $\alpha \in L$. The enveloping algebra of the pair (S, L) is the quotient of $S \otimes U(L)$ by the right ideal generated by the elements $st \otimes \alpha - s \otimes t\alpha$ for s and t in S and α in L, which turn out to be a bilateral ideal.

One of the points of this construction is that the category of U-modules is isomorphic to that of (S, L)-modules. As a particular example, since S is an (S, L)-module, as we saw, it is also an U-module.

2.14. *Example.* If $\mathfrak{g}$ is Lie algebra, the universal enveloping algebra of the pair $(\Bbbk, \mathfrak{g})$ is simply the usual enveloping algebra of $\mathfrak{g}$. Indeed, this is clear from the construction we have just described.

2.15. *Example.* If $S = \Bbbk[x_1, \dots, x_n]$, then full Lie algebra of derivations $L = \operatorname{Der} S$ is freely generated as an S-module by the n derivations $y_i = \frac{\partial}{\partial x_i} : S \to S$ with $1 \le i \le n$. The construction sketched above now shows us that the enveloping algebra of the pair (S, L) admits the presentation

$$\frac{\Bbbk\langle x_i, y_i : 1 \le i \le n\rangle}{\left(y_i x_j - x_j y_i - \delta_{ij}\right)},$$

so it isomorphic to the algebra of differential operators $\operatorname{Diff}(S) = A_n$, the Weyl algebra.

2.16. *Example.* In the situation of Example 2.11, the enveloping algebra of the Lie–Rinehart pair $(C^\infty(M), \mathfrak{X}(M))$ can be seen to be isomorphic to the algebra of globally defined differential operators on the manifold —we refer for this to the first section of [Hue90].

2.17. A key result about the enveloping algebra is the following generalization of the Poincaré–Birkhoff–Witt Theorem.

Theorem. *Let (S, L) be a Lie–Rinehart pair such that L is a free S-module of finite rank and let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis. There is an increasing algebra filtration $F_\bullet$ on $U(S, L)$ with*

$$F_0 = S, \qquad F_1 = S + L, \qquad F_p = (F_1)^p \text{ for each } p \geq 2,$$

and a canonical isomorphism of algebras from the symmetric algebra $\mathsf{Symm}_S(L)$ to the associated graded algebra $\mathsf{gr}\, U(S, L)$. Moreover, the set of monomials

$$\alpha_1^{k_1} \ldots \alpha_n^{k_n} \qquad \text{with } k_1, \ldots, k_n \geq 0$$

is a basis of $U(S, L)$ as a left S-module.

Proof. See [Rin63, §3]. □

We deduce immediately from this PBW theorem the following.

Corollary. *If (S, L) is a Lie–Rinehart pair such that L is a free S-module of finite rank then the algebra $U(S, L)$ is a noetherian domain.*

2.18. A less trivial consequence of Theorem 2.17 is the following result. In order to state, we need the notion of Gelfand–Kirillov dimension, GKdim, for which we refer to the book [KL00] by G. R. Krause and T. H.Lanagan or to [MR01, Chapter 8].

Corollary. *Let (S, L) be a Lie–Rinehart pair such that L is a free S-module of finite rank. If S is a finitely generated algebra, then*

$$\mathsf{GKdim}\, U(S, L) = \mathsf{GKdim}\, S + \mathsf{rank}_S\, L.$$

Proof. This follows from Theorem A in J. Matczuk's article [Mat88]. □

2.19. The reason we are interested in these last results is that they allow us to describe the algebra of differential operators $\mathsf{Diff}(\mathcal{A})$ in the case of a free arrangement.

Theorem. *Let $\mathcal{A}$ be a free hyperplane arrangement on a vector space V of dimension n and let S be the algebra of coordinate functions on V. There is a canonical isomorphism of algebras*

$$U(S, \mathsf{Der}\, \mathcal{A}) \to \mathsf{Diff}(\mathcal{A}).$$

In particular, $\mathsf{Diff}(\mathcal{A})$ is a noetherian domain of Gelfand-Kirillov dimension $2n$.

Proof. In view of Theorem 2.4 there is an obvious surjective morphism of algebras

$$\phi : U(S, \mathsf{Der}\,\mathcal{A}) \to \mathsf{Diff}(\mathcal{A}).$$

Let I be its kernel and suppose, to reach a contradiction, that $I \neq 0$. As $U(S, L)$ is a domain and contains non-zero regular elements, Proposition 3.15 in [KL00] tells us that

$$2n = \mathsf{GKdim}\,U(S, L) \geq \mathsf{GKdim}\,U(S, L)i/I + 1 = \mathsf{GKdim}\,\mathsf{Diff}(\mathcal{A}) + 1.$$

Let $\Omega = \{Q^i : i \geq 0\}$. This is a multiplicatively closed subset of $\mathsf{Diff}(S)$, its elements are regular and commute, and the corresponding linear derivations are locally nilpotent: Theorem 4.9 of [KL00] tells us that Ω is an Ore set in $\mathsf{Diff}(S)$ and that

$$\mathsf{GKdim}\,\mathsf{Diff}(S)\Omega^{-1} = \mathsf{GKdim}\,\mathsf{Diff}(S).$$

This last number is $2n$, as can be deduced from Corollary 2.18 in view that $\mathsf{Diff}(S)$ is the enveloping algebra of the Lie–Rinehart pair $(S, \mathsf{Der}\,S)$. On the other hand, Ω is contained in $\mathsf{Diff}(\mathcal{A})$ and has the same properties as in $\mathsf{Diff}(S)$, so that the same theorem now tells us that

$$\mathsf{GKdim}\,\mathsf{Diff}(\mathcal{A})\Omega^{-1} = \mathsf{GKdim}\,\mathsf{Diff}(\mathcal{A}).$$

To find the contradiction we want, it is therefore enough to show that $\mathsf{Diff}(S)\Omega^{-1} = \mathsf{Diff}(\mathcal{A})\Omega^{-1}$. As $\mathsf{Diff}(\mathcal{A})$ is contained in $\mathsf{Diff}(S)$, to see this we need only show that for each $u \in \mathsf{Diff}(S)$ there exists $i \geq 0$ such that $Q^i u \in \mathsf{Diff}(\mathcal{A})$ and, according to Proposition 8 in [SÁ18], we may take $i = \binom{p+1}{2}$ with p the order of u. □

2.20. The result in last theorem can be made completely explicit.

Proposition. *Let $\mathcal{A}$ be a free hyperplane arrangement in a vector space V with coordinate algebra S and let $\mathcal{B} = \{\theta_1, \dots, \theta_n\}$ be a basis of* $\mathsf{Der}\,\mathcal{A}$. *Let $c_{ij}^k \in S$ be the structure coefficients of* $\mathsf{Der}\,\mathcal{A}$ *with respect to $\mathcal{B}$, so that*

$$[\theta_i, \theta_j] = \sum_{k=1}^{n} c_{ij}^k \theta_k.$$

The algebra $\mathsf{Diff}(\mathcal{A})$ *is isomorphic to the free algebra generated by letters $x_1, \dots, x_n, \theta_1, \dots, \theta_n$ subject to the relations*

$$[x_i, x_j] = 0, \qquad [\theta_i, x_j] = \theta_i(x_j), \qquad [\theta_i, \theta_j] = \sum_{k=1}^{n} c_{ij}^k \theta_k$$

for every i, j and k in $[\![n]\!]$. □

2.21. In Section 2.2 we found that when $\mathcal{A}$ is an arrangement of lines, $\mathsf{Diff}(\mathcal{A})$ is an iterated Ore extension of S: this is not the case in the general situation. Indeed, if $\mathcal{A} = \mathcal{B}_3$, the third braid arrangement, then $\mathsf{Der}\,\mathcal{A} \cong S \otimes \mathfrak{sl}_2$ as a Lie algebra, and this can be used to show that $\mathsf{Diff}(\mathcal{A})$ is not an iterated Ore extension.

2.22. A simple and final observation that we can make at this point, and that we have actually proved at the end of the proof of Theorem 2.19, is that our algebra $\mathsf{Diff}(\mathscr{A})$ and the full algebra $\mathsf{Diff}(S)$ of regular differentials operators of S are birational, that is, that they have the same skew-fields of quotients. In fact, the two algebras become isomorphic already after localization at the single element Q.

Proposition. *The inclusion* $\mathsf{Diff}(\mathscr{A}) \to \mathsf{Diff}(S)$ *induces after localization at* Q *an isomorphism* $\mathsf{Diff}(\mathscr{A})[\frac{1}{Q}] \to \mathsf{Diff}(S)[\frac{1}{Q}]$ *and, in particular,* $\mathsf{Diff}(\mathscr{A})$ *and* $\mathsf{Diff}(S)$ *have isomorphic fields of fractions.*

2.4 Twisted Calabi–Yau algebras

2.23. Let us recall the notion of twisted Calabi–Yau algebras from the article [Gin06] by V. Ginzburg. We will see that when a hyperplane arrangement is free, its algebra of differential operators is twisted Calabi–Yau.

Let $n \geq 0$. An algebra A has Van den Bergh duality of dimension n if A has a resolution of finite length by finitely generated projective A-bimodules and there exists an invertible A-bimodule D such that there is an isomorphism of A-bimodules

$$\mathsf{Ext}^i_{A^e}(A, A \otimes A) = \begin{cases} 0 & \text{if } i \neq n; \\ D & \text{if } i = n. \end{cases}$$

An algebra A is *twisted Calabi–Yau* or has the *twisted Calabi–Yau* property of dimension n if, moreover, there exists an automorphism σ of A, the *Nakayama automorphism*, such that the dualizing bimodule D can be taken to be A_σ, the A-bimodule obtained from A by twisting its right action using the automorphism σ, so that $a \triangleright x \triangleleft b = ax\sigma(b)$ for all $a, b \in A$ and all $x \in A_\sigma$. If the automorphism σ is the identity of A, we simply say that A is *Calabi–Yau*.

The Van den Bergh duality property for an algebra A is important because, as can be seen in [vdB98], it relates the Hochschild cohomology of A with its homology in a way analogue to Poincaré duality. Explicitly, for each A-bimodule M there is a canonical isomorphism

$$H^i(A, M) \to H_{n-i}(A, D \otimes_A M).$$

In the case that A is twisted Calabi–Yau, so that there exists an automorphism σ of A such that we may take $D = A_\sigma$, we observe that if σ is not the identity of A, the bimodule $A_\sigma \otimes_A M$ is not generally isomorphic to M.

2.24. Let (S, L) be a Lie–Rinehart pair. The following result by Th. Lambre and P. Le Meur gives a sufficient condition for the enveloping algebra of the pair to have the twisted Calabi–Yau property that is satisfied in important examples. Notice that the following theorem includes the hypothesis that L be finitely generated projective of constant rank —that is, that the localizations of L are all free of the same rank— implies that L is finitely generated as a consequence of Proposition 1.3 of [Vas69].

Theorem. *Let (S, L) be a Lie–Rinehart pair. If S is twisted Calabi–Yau of dimension n, L is finitely generated and projective of constant rank d and $\Lambda_S^d L$ is free then the enveloping algebra $U(S, L)$ is twisted Calabi–Yau of dimension $n + d$.*

In addition to the precedent theorem, the authors give concrete simple formulas for the Nakayama automorphism of $U(S, L)$.

Proof. This is Theorem 2 in [LLM18]. One can streamline their argument using the spectral sequence we construct in Chapter 6 of this thesis to compute $\mathsf{Ext}_{U^e}(U, U \otimes U)$. □

2.25. Let $\mathcal{A}$ be a free hyperplane arrangement in V and let, as usual, n be the dimension of V. As we have seen in **1.24** and in Theorem 2.19, the free module $\mathsf{Der}\,\mathcal{A}$ has rank n and the algebra of differential operators on $\mathcal{A}$ is isomorphic to the enveloping algebra of the Lie–Rinehart pair $(S, \mathsf{Der}\,\mathcal{A})$, and therefore Theorem 2.24 tells us that $\mathsf{Diff}(\mathcal{A})$ is a twisted Calabi–Yau algebra of dimension $2n$. We will give an direct proof of this fact for the case of an arrangement of lines as in Section 2.2 and describe explicitly the Nakayama automorphism.

2.5 Resumen

En este capítulo presentamos el álgebra de operadores diferenciales $\mathsf{Diff}(\mathcal{A})$ tangentes a los hiperplanos de un arreglo $\mathcal{A}$, que es el principal objeto de estudio de la tesis. Primero, vemos en **2.4** que $\mathsf{Diff}(\mathcal{A})$ admite un sistema de generadores manejable en el caso en que $\mathcal{A}$ es un arreglo libre.

Teorema. *Si $\mathcal{A}$ es un arreglo de hiperplanos libre entonces el álgebra* $\mathsf{Diff}(\mathcal{A})$ *está generada por* $S \cup \mathsf{Der}(\mathcal{A})$.

Este resultado fue demostrado por F. J. Calderón Moreno en [CM99] y M. Schulze en [Sch07] para el caso de arreglos complejos usando técnicas de geometría analítica, y por M. Suárez-Álvarez en [SÁ18] para el caso en que $\Bbbk$ es un cuerpo cualquiera de característica cero.

A continuación, nos detenemos a analizar el caso de un arreglo central $\mathcal{A}$ de rectas, que es el que más nos interesa en esta tesis, para encontrar en **2.8** una presentación de $\mathsf{Diff}(\mathcal{A})$ que no reproducimos aquí. Inmediatamente después obtenemos la siguiente descripción:

Proposición. *El álgebra* $\mathsf{Diff}(\mathcal{A})$ *es isomorfa a una extensión de Ore iterada.*

Volvemos luego al caso general de un arreglo libre de hiperplanos de dimensión arbitraria. Introducimos en la Sección 2.3 la noción de pares de Lie–Rinehart. Un par de Lie–Rinehart (S, L) consta de un álgebra conmutativa S y un álgebra de Lie L que es un S-módulo y actúa en S por derivaciones de manera que

$$(s\alpha)(t) = s(\alpha(t)), \qquad [\alpha, s\beta] = s[\alpha, \beta] + \alpha(s)\beta$$

si s, t pertenecen a S y α y β a L. En **2.13** damos una construcción del álgebra envolvente $U = U(S, L)$ de un par de Lie–Rinehart (S, L), que es la “menor” álgebra asociativa que contiene a S y a L. Esta construcción es central para nosotros: probamos en el Teorema 2.19 que el álgebra $\mathsf{Diff}(\mathcal{A})$ puede identificarse con el álgebra envolvente del par dado por el álgebra de funciones coordenadas de V y el álgebra de Lie de derivaciones de $\mathcal{A}$.

Teorema. *Sea $\mathcal{A}$ un arreglo de hiperplanos libre en un espacio vectorial V y sea S el álgebra de funciones coordenadas en V. Hay un isomorfismo canónico de álgebras*

$$U(S, \mathsf{Der}\,\mathcal{A}) \to \mathsf{Diff}(\mathcal{A}).$$

La existencia de este morfismo sigue inmediatamente de los resultados de [CM99] y [SÁ18] que recién mencionamos. Para probar que es inyectivo, utilizamos el cálculo de la dimensión de Gelfand-Kirillov del álgebra envolvente de un par de Lie–Rinehart hecho por J. Matczuk en [Mat88] y el hecho de que $\mathsf{Diff}(\mathcal{A})$ y el álgebra de operadores diferenciales en S se tornan isomorfas al localizar en el elemento Q. La sección termina dando en la Proposición 2.20 una presentación de $\mathsf{Diff}(\mathcal{A})$ por generadores y relaciones.

Finalmente, en la Sección 2.4 nos dedicamos a estudiar la dualidad de Van den Bergh y la propiedad de Calabi–Yau torcida para un álgebra. Usando los resultados de Th. Lambre y P. Le Meur en [LLM18], obtenemos en **2.25** lo siguiente:

Proposición. *Si $\mathcal{A}$ es un arreglo libre de hiperplanos, el álgebra* $\mathsf{Diff}(\mathcal{A})$ *tiene la propiedad de Calabi–Yau torcida.*

– 3 –
The Hochschild cohomology of the algebra of differential operators tangent to a line arrangement

In this chapter we study the Hochschild cohomology of the algebra of differential operators tangent to a central arrangement of lines as a Gerstenhaber algebra. We start by constructing a useful projective resolution for the algebra, which we then use to compute explicitly Hochschild cohomology, the cup product and the Gerstenhaber bracket. We devote the last two sections to the much simpler calculation of Hochschild and cyclic homology, K-theory and to a direct proof of the twisted Calabi–Yau property.

The results we obtain are pivotal to the study of the automorphisms and the deformations of A that we develop further ahead.

3.1. As in Section 2.2, we let $\mathcal{A}$ be a central line arrangement in $\Bbbk^2$ and denote by A the associative algebra $\mathsf{Diff}(\mathcal{A})$ defined in **2.8**. We let S be the algebra of coordinate functions on $\Bbbk^2$ and identify it, as usual, with $\Bbbk[x, y]$; if $p \geq 0$, we denote by S_p the homogeneous component of S of degree p. Recall that we have written the defining polynomial Q of $\mathcal{A}$ as $Q = xF$ for a square free homogeneous polynomial $F \in S$ of degree $r + 1$ such that $x \nmid F$. After multiplying by an scalar if necessary, we may in fact write $F = y^{r+1} + x\bar{F}$, for $\bar{F} \in S_r$.

We will use frequently the following non-standard notation from now on: if M is a vector space we write $\boxed{M}$ for an element of M about which we do not need to be specific.

3.1 A projective resolution

3.2. Our immediate objective is to construct a projective resolution of A as an A-bimodule, and we do this by looking at A as a deformation of a commutative polynomial algebra, which suggests that it should have a resolution resembling the usual Koszul complex.

3.3. Let us introduce some more notation that will be useful throughout our calculations. If U is a vector space and $u \in U$, there are derivations $\nabla_x^u, \nabla_y^u : S \to S \otimes U \otimes S$ of S into the S-bimodule $S \otimes U \otimes S$ uniquely determined by the condition that

$$\nabla_x^u(x) = 1 \otimes u \otimes 1, \qquad \nabla_x^u(y) = 0, \qquad \nabla_y^u(x) = 0, \qquad \nabla_y^u(y) = 1 \otimes u \otimes 1,$$

and, in fact, for every $i, j \geq 0$ we have that

$$\nabla_x^u(x^i y^j) = \sum_{s+t+1=i} x^s \otimes u \otimes x^t y^j, \qquad \nabla_y^u(x^i y^j) = \sum_{s+t+1=j} x^i y^s \otimes u \otimes y^s.$$

We consider the derivation $\nabla = \nabla_x^x + \nabla_y^y : S \to S \otimes S_1 \otimes S$; it is the unique derivation such that $\nabla(\alpha) = 1 \otimes \alpha \otimes 1$ for all $\alpha \in S_1$. There is, on the other hand, a unique morphism of S-bimodules $d : S \otimes S_1 \otimes S \to S \otimes S$ such that $d(1 \otimes \alpha \otimes 1) = \alpha \otimes 1 - 1 \otimes \alpha$ for all $\alpha \in S_1$, and we have

$$d(\nabla(f)) = f \otimes 1 - 1 \otimes f$$

for all $f \in S$. To check this last equality, it is enough to notice that $d \circ \nabla : S \to S \otimes S$ is a derivation and, since S_1 generates S as an algebra, that the equality holds when $f \in S_1$.

3.4. Let V be the subspace of A spanned by x, y, D and E. This is a homogeneous subspace and its grading induces on the exterior algebra $\Lambda^\bullet(V)$ an internal grading. If ω is an element of an exterior power $\Lambda^p(V)$ of V, we write $(-) \wedge \omega$ for the map of A-bimodules

$$A \otimes S_1 \otimes A \to A \otimes \Lambda^{p+1} V \otimes A$$

such that $(1 \otimes \alpha \otimes 1) \wedge \omega = 1 \otimes \alpha \wedge \omega \otimes 1$ for all $\alpha \in S_1$.

3.5. There is a chain complex $\mathbf{P}$ of free graded A-bimodules of the form

$$A|\Lambda^4 V|A \xrightarrow{d_4} A|\Lambda^3 V|A \xrightarrow{d_3} A|\Lambda^2 V|A \xrightarrow{d_2} A|V|A \xrightarrow{d_1} A|A \tag{3.1}$$

where, we recall from **1.1**, the symbol $|$ stands for tensor product over $\Bbbk$, and with A^e-linear maps homogeneous of degree zero and such that

$$d_1(1|v|1) = [v, 1|1], \qquad \forall v \in V;$$

$$\begin{aligned}
d_2(1|x \wedge y|1) &= [x, 1|y|1] - [y, 1|x|1]; \\
d_2(1|x \wedge E|1) &= [x, 1|E|1] - [E, 1|x|1] + 1|x|1; \\
d_2(1|y \wedge E|1) &= [y, 1|E|1] - [E, 1|y|1] + 1|y|1; \\
d_2(1|x \wedge D|1) &= [x, 1|D|1] - [D, 1|x|1]; \\
d_2(1|y \wedge D|1) &= [y, 1|D|1] - [D, 1|y|1] + \nabla(F); \\
d_2(1|D \wedge E|1) &= [D, 1|E|1] - [E, 1|D|1] + r|D|1;
\end{aligned}$$

$$\begin{aligned}
d_3(1|x \wedge y \wedge D|1) &= [x, 1|y \wedge D|1] - [y, 1|x \wedge D|1] + [D, 1|x \wedge y|1] + \nabla(F) \wedge x; \\
d_3(1|x \wedge y \wedge E|1) &= [x, 1|y \wedge E|1] - [y, 1|x \wedge E|1] + [E, 1|x \wedge y|1] - 2|x \wedge y|1; \\
d_3(1|x \wedge D \wedge E|1) &= [x, 1|D \wedge E|1] - [D, 1|x \wedge E|1] + [E, 1|x \wedge D|1] \\
&\qquad - (r+1)|x \wedge D|1; \\
d_3(1|y \wedge D \wedge E|1) &= [y, 1|D \wedge E|1] - [D, 1|y \wedge E|1] + [E, 1|y \wedge D|1] \\
&\qquad + \nabla(F) \wedge E - (r+1)|y \wedge D|1;
\end{aligned}$$

$$
\begin{aligned}
d_4(1|x \wedge y \wedge D \wedge E|1) = [x, 1|y \wedge D \wedge E|1] - [y, 1|x \wedge D \wedge E|1] \\
+ [D, 1|x \wedge y \wedge E|1] - [E, 1|x \wedge y \wedge D|1] \\
+ \nabla(F) \wedge x \wedge E + (r+2)|x \wedge y \wedge D|1.
\end{aligned}
$$

That $\mathbf{P}$ is indeed a complex follows from a direct calculation. More interestingly, it is exact:

Lemma. *The complex* $\mathbf{P}$ *is a projective resolution of* A *as an* A*-bimodule, with augmentation* $d_0 : A|A \to A$ *such that* $d_0(1|1) = 1$.

Proof. For each $p \in \mathbb{N}_0$ we consider the subspace $\bar{F}_p A = \langle x^i y^j D^k E^l : k + l \leq p \rangle$ of A. As a consequence of Lemma 2.8, we see that $\bar{F}A = (\bar{F}_p A)_{p \geq 0}$ is an exhaustive and increasing algebra filtration on A and that the corresponding associated graded algebra $\mathsf{gr}(A)$ is isomorphic to the usual commutative polynomial ring $\Bbbk[x, y, D, E]$. Since V is a subspace of A, we can restrict the filtration of A to one on V, and the latter induces as usual a filtration on each exterior power $\Lambda^p V$. In this way we obtain a filtration on each component of the complex $\mathbf{P}$, which turns out to be compatible with its differentials, as can be checked by inspection. The complex $\mathsf{gr}(\mathbf{P})$ obtained from $\mathbf{P}$ by passing to associated graded objects in each degree is isomorphic to the Koszul resolution of $\mathsf{gr}(A)$ as a $\mathsf{gr}(A)$-bimodule and it is therefore acyclic over $\mathsf{gr}(A)$. A standard argument using the filtration of $\mathbf{P}$ concludes from this that the complex $\mathbf{P}$ is itself acyclic over A. As its components are manifestly free A-bimodules, this proves the lemma. ☐

3.6. One almost immediate application of having a bimodule projective resolution for our algebra is in computing its global dimension.

Proposition. *The global dimension of* A *is equal to* 4.

Of course, as A is noetherian, there is no need to distinguish between the left and the right global dimensions.

Proof. If $\lambda \in \Bbbk$ we let M_λ be the left A-module which as a vector space is freely spanned by an element u_λ and on which the action of A is such that $x \cdot u_\lambda = y \cdot u_\lambda = D \cdot u_\lambda = 0$ and $E \cdot u_\lambda = \lambda u_\lambda$. It is easy to see that all 1-dimensional A-modules are of this form and that $M_\lambda \cong M_\mu$ iff $\lambda = \mu$, but we will not need this.

The complex $\mathbf{P} \otimes_A M_\lambda$ is a projective resolution of M_λ as a left A-module, and therefore the cohomology of $\mathsf{hom}_A(\mathbf{P} \otimes_A M_\lambda, M_\mu)$ is canonically isomorphic to $\mathsf{Ext}^\bullet_A(M_\lambda, M_\mu)$. Identifying as usual $\mathsf{hom}_A(\mathbf{P} \otimes_A M_\lambda, M_\mu)$ to $M_\mu \otimes \Lambda^\bullet V^* \otimes M_\lambda^*$, we compute that the complex is

$$
\begin{aligned}
M_\mu \otimes M_\lambda^* \xrightarrow{\delta^0} M_\mu \otimes V^* \otimes M_\lambda^* \xrightarrow{\delta^1} M_\mu \otimes \Lambda^2 V^* \otimes M_\lambda^* \xrightarrow{\delta^2} \\
\longrightarrow M_\mu \otimes \Lambda^3 V^* \otimes M_\lambda^* \xrightarrow{\delta^3} M_\mu \otimes \Lambda^4 V^* \otimes M_\lambda^*
\end{aligned}
$$

with differentials given by

$$
\delta^0(1) = (\mu - \lambda) \otimes \hat{E},
$$

$$\begin{aligned}
&\delta^1(a \otimes \hat{x} + b \otimes \hat{y} + c \otimes \hat{D} + d \otimes \hat{D}) \\
&\qquad = (\lambda + 1 - \mu) a \otimes \hat{x} \wedge \hat{E} + (\lambda + 1 - \mu) b \otimes \hat{y} \wedge \hat{E} + (\lambda + r - \mu) c \otimes \hat{D} \wedge \hat{E}, \\
&\delta^2(a \otimes \hat{x} \wedge \hat{y} + b \otimes \hat{x} \wedge \hat{E} + c \otimes \hat{y} \wedge \hat{E} + d \otimes \hat{x} \wedge \hat{D} + e\hat{y} \wedge \hat{D} + f\hat{D} \wedge \hat{E}) \\
&\qquad = (\mu - \lambda - 2) a \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + (\mu - \lambda - r - 1) d \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \\
&\qquad\qquad + (\mu - \lambda - r - 1) e \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta^3(a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + b \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + c \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + d \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) \\
&\qquad = (\lambda + r + 2 - \mu) a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}.
\end{aligned}$$

An easy computation shows that

$$\dim \operatorname{Ext}_A^p(M_\lambda, M_{\lambda+r+2}) = \begin{cases} 1, & \text{if } p = 3 \text{ or } p = 4; \\ 0, & \text{in any other case.} \end{cases}$$

In particular, $\operatorname{Ext}_A^4(M_\lambda, M_{\lambda+r+2}) \neq 0$ and therefore $\operatorname{gldim} A \geq 4$. On the other hand, we have constructed a projective resolution of A as an A-bimodule of length 4, so that the projective dimension of A as a bimodule is $\operatorname{pdim}_{A^e} A \leq 4$. The proposition now follows from this and that $\operatorname{gldim} A \leq \operatorname{pdim}_{A^e} A$. □

3.7. We will use the following two simple lemmas a few times; the conclusion of its statement is false if $r < 2$.

Lemma. *Suppose that $r \geq 2$. If $\alpha, \beta \in S_1$ are such that $\alpha F_x + \beta F_y = 0$, then $\alpha = \beta = 0$.*

Proof. Suppose that F_1, F_2 and F_3 are three distinct linear factors of F (here is where we need the hypothesis that r is at least 2) so that $F = F_1 F_2 F_3 F'$ for some $F' \in S_{r-2}$; as F has degree at least 3, this is possible. We have $F_x \equiv F_{1x} F_2 F_3 F'$ and $F_y \equiv F_{1y} F_2 F_3 F'$ modulo F_1, so that $(\alpha F_{1x} + \beta F_{1y}) F_2 F_3 F' \equiv 0 \mod F_1$. Since F is square free, this tells us that F_1 divides $\alpha F_{1x} + \beta F_{1y}$ and, since both polynomials have the same degree and $F_1 \neq 0$, that there exists a scalar λ such that $\alpha F_{1x} + \beta F_{1y} = \lambda F_1$. Of course, we can do the same with the other two factors F_2 and F_3. We can state this by saying that the matrix $\left(\begin{smallmatrix} \alpha_x & \beta_x \\ \alpha_y & \beta_y \end{smallmatrix}\right)$ has the three vectors $\left(\begin{smallmatrix} F_{1x} \\ F_{1y} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} F_{2x} \\ F_{2y} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} F_{3x} \\ F_{3y} \end{smallmatrix}\right)$ as eigenvectors. Since no two of these are linearly dependent, because F is square-free, this implies that the matrix is in fact a scalar multiple of the identity, and there is a $\mu \in \Bbbk$ such that $\alpha = \mu x$ and $\beta = \mu y$. The hypothesis is then that $\mu(r+1)F = \mu(xF_x + yF_y) = 0$, so that $\mu = 0$. This proves the claim. □

3.8. Lemma. *If $\alpha_1, \ldots, \alpha_{r+1} \in S_1$ are such that $F = \prod_{i=1}^{r+1} \alpha_i$, the set of quotients $\{\frac{F}{\alpha_1}, \ldots, \frac{F}{\alpha_{r+1}}\}$ is a basis for S_r.*

Proof. Suppose $c_1, \ldots, c_{r+1} \in \Bbbk$ are scalars such that $\sum_{i=1}^{r+1} c_i \frac{F}{\alpha_i} = 0$. If $j \in \{1, \ldots, r+1\}$, we then have $c_j \frac{F}{\alpha_j} \equiv 0$ modulo α_j and, since F is square-free, this implies that in fact $c_j = 0$. The set $\{\frac{F}{\alpha_1}, \ldots, \frac{F}{\alpha_{r+1}}\}$ is therefore linearly independent. Since $\dim S_r = r + 1$, this completes the proof. □

3.2 THE HOCHSCHILD COHOMOLOGY OF $\mathsf{Diff}(\mathcal{A})$

3.9. We want to compute the Hochschild cohomology of the algebra $A = \mathsf{Diff}(\mathcal{A})$. Applying the functor $\mathsf{hom}_{A^e}(-, A)$ to the resolution $\mathbf{P}$ of 3.5 we get, after standard identifications, the cochain complex

$$\begin{tikzcd} A \arrow[r, shift left, "d^0"] & A \otimes V^* \arrow[l, dashed, shift left, "s^1"] \arrow[r, shift left, "d^1"] & A \otimes \Lambda^2 V^* \arrow[l, dashed, shift left, "s^2"] \arrow[r, shift left, "d^1"] & A \otimes \Lambda^3 V^* \arrow[l, dashed, shift left, "s^3"] \arrow[r, shift left, "d^2"] & A \otimes \Lambda^4 V^* \arrow[l, dashed, shift left, "s^4"] \arrow[r] & 0 \end{tikzcd}$$

which we denote simply by $A \otimes \Lambda V^*$, with differentials such that

$$d^0(a) = [x, a] \otimes \hat{x} + [y, a] \otimes \hat{y} + [D, a] \otimes \hat{D} + [E, a] \otimes \hat{E};$$

$$\begin{aligned} d^1(a \otimes \hat{x}) &= -[y, a] \otimes \hat{x} \wedge \hat{y} + (a - [E, a]) \otimes \hat{x} \wedge \hat{E} - [D, a] \otimes \hat{x} \wedge \hat{D} \\ &\qquad + \nabla_x^a(F) \otimes \hat{y} \wedge \hat{D}; \\ d^1(a \otimes \hat{y}) &= [x, a] \otimes \hat{x} \wedge \hat{y} + (a - [E, a]) \otimes \hat{y} \wedge \hat{E} + (\nabla_y^a(F) - [D, a]) \otimes \hat{y} \wedge \hat{D}; \\ d^1(a \otimes \hat{D}) &= [x, a] \otimes \hat{x} \wedge \hat{D} + [y, a] \otimes \hat{y} \wedge \hat{D} + (ra - [E, a]) \otimes \hat{D} \wedge \hat{E}; \\ d^1(a \otimes \hat{E}) &= [x, a] \otimes \hat{x} \wedge \hat{E} + [y, a] \otimes \hat{y} \wedge \hat{E} + [D, a] \otimes \hat{D} \wedge \hat{E}; \end{aligned}$$

$$\begin{aligned} d^2(a \otimes \hat{x} \wedge \hat{y}) &= ([D, a] - \nabla_y^a(F)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + ([E, a] - 2a) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E}; \\ d^2(a \otimes \hat{x} \wedge \hat{E}) &= -[y, a] \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} - [D, a] \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \nabla_x^a(F) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^2(a \otimes \hat{y} \wedge \hat{E}) &= [x, a] \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + (\nabla_y^a(F) - [D, a]) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^2(a \otimes \hat{x} \wedge \hat{D}) &= -[y, a] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + ([E, a] - (r + 1)a) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E}; \\ d^2(a \otimes \hat{y} \wedge \hat{D}) &= [x, a] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + ([E, a] - (r + 1)a) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^2(a \otimes \hat{D} \wedge \hat{E}) &= [x, a] \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + [y, a] \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \end{aligned}$$

$$\begin{aligned} d^3(a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}) &= (-[E, a] + (r + 2)a) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^3(a \otimes \hat{x} \wedge \hat{y} \wedge \hat{E}) &= ([D, a] - \nabla_y^a(F)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^3(a \otimes \hat{x} \wedge \hat{D} \wedge \hat{E}) &= -[y, a] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^3(a \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) &= [x, a] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}. \end{aligned}$$

These differentials are homogeneous with respect to the natural internal grading on the complex $A \otimes \Lambda V^*$ coming from the grading of A. We denote $\gamma : A \otimes \Lambda V^* \to A \otimes \Lambda V^*$ the $\Bbbk$-linear map whose restriction to each homogeneous component of $A \otimes \Lambda V^*$ is simply multiplication by the degree. There is a homotopy, drawn in the diagram (3.1) with dashed arrows, with

$$\begin{aligned} &s^1(a \otimes \hat{x} + b \otimes \hat{y} + c \otimes \hat{D} + d \otimes \hat{E}) = d, \\ &s^2(a \otimes \hat{x} \wedge \hat{y} + b \otimes \hat{x} \wedge \hat{E} + c \otimes \hat{y} \wedge \hat{E} + d \otimes \hat{x} \wedge \hat{D} + e \otimes \hat{y} \wedge \hat{D} + f \otimes \hat{D} \wedge \hat{E}) \\ &\qquad\qquad = -b \otimes \hat{x} - c \otimes \hat{y} - f \otimes \hat{D}, \\ &s^3(a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + b \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + c \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + d \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) \\ &\qquad\qquad = b \otimes \hat{x} \wedge \hat{y} + c \otimes \hat{x} \wedge \hat{D} + d \otimes \hat{y} \wedge \hat{D}, \end{aligned}$$

$$s^4(a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}) = -a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}$$

and such that $d \circ s + s \circ d = \gamma$: this tells us that γ induces the zero map on cohomology. Since our ground field $\Bbbk$ has characteristic zero, this implies that the inclusion $(A \otimes \Lambda V^*)_0 \to A \otimes \Lambda V^*$ of the component of degree zero of our complex $A \otimes \Lambda V^*$ is a quasi-isomorphism.

3.10. From now on and until the end of this section, we will assume that $r \geq 3$**.** Let us write the complex $(A \otimes \Lambda V^*)_0$ simply $\mathfrak{X}$ and let us put $T = \Bbbk[E]$, which coincides with A_0. The complex $\mathfrak{X}$ has components

$$\begin{aligned}
\mathfrak{X}^0 &= A_0, \\
\mathfrak{X}^1 &= A_1 \otimes (\Bbbk\hat{x} \oplus \Bbbk\hat{y}) \oplus A_r \otimes \Bbbk\hat{D} \oplus A_0 \otimes \Bbbk\hat{E}, \\
\mathfrak{X}^2 &= A_2 \otimes \Bbbk\hat{x} \wedge \hat{y} \oplus A_1 \otimes (\Bbbk\hat{x} \wedge \hat{E} \oplus \Bbbk\hat{y} \wedge \hat{E}) \oplus A_r \otimes \Bbbk\hat{D} \wedge \hat{E} \\
&\qquad \oplus A_{r+1} \otimes (\Bbbk\hat{x} \wedge \hat{D} \oplus \Bbbk\hat{y} \wedge \hat{D}), \\
\mathfrak{X}^3 &= A_2 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \oplus A_{r+1} \otimes (\Bbbk\hat{x} \wedge \hat{D} \wedge \hat{E} \oplus \Bbbk\hat{y} \wedge \hat{D} \wedge \hat{E}) \\
&\qquad \oplus A_{r+2} \otimes \Bbbk\hat{x} \wedge \hat{y} \wedge \hat{D}, \\
\mathfrak{X}^4 &= A_{r+2} \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}
\end{aligned}$$

and, since $r > 2$, we have

$$\begin{aligned}
A_0 &= T, & A_1 &= S_1 T, & A_2 &= S_2 T, \\
A_r &= (S_r \oplus \Bbbk D)T, & A_{r+1} &= (S_{r+1} \oplus S_1 D)T, & A_{r+2} &= (S_{r+2} \oplus S_2 D)T.
\end{aligned}$$

In fact, this is where our assumption that $r \geq 3$ intervenes: if $r \leq 2$, these subspaces of A have a different description.

The differentials in $\mathfrak{X}$ can be computed to be given by

$$\delta^0(a) = x\tau_1(a) \otimes \hat{x} + y\tau_1(a) \otimes \hat{y} + D\tau_r(a) \otimes \hat{D},$$

$$\begin{aligned}
\delta^1(\phi a \otimes \hat{x}) &= -\phi y \tau_1(a) \otimes \hat{x} \wedge \hat{y} - (F\phi_y a + \phi D\tau_r(a)) \otimes \hat{x} \wedge \hat{D} + \nabla_x^{\phi a}(F) \otimes \hat{y} \wedge \hat{D}, \\
\delta^1(\phi a \otimes \hat{y}) &= \phi x \tau_1(a) \otimes \hat{x} \wedge \hat{y} + (\nabla_y^{\phi a}(F) - F\phi_y a - \phi D\tau_r(a)) \otimes \hat{y} \wedge \hat{D}, \\
\delta^1((\phi + \lambda D)a \otimes \hat{D}) &= (\phi x \tau_1(a) + \lambda x D\tau_1(a)) \otimes \hat{x} \wedge \hat{D} \\
&\qquad + (\phi y \tau_1(a) + \lambda F(\tau_1(a) - a) + \lambda y D\tau_1(a)) \otimes \hat{y} \wedge \hat{D}, \\
\delta^1(a \otimes \hat{E}) &= x\tau_1(a) \otimes \hat{x} \wedge \hat{E} + y\tau_1(a) \otimes \hat{y} \wedge \hat{E} + D\tau_r(a) \otimes \hat{D} \wedge \hat{E},
\end{aligned}$$

$$\begin{aligned}
\delta^2(\phi a \otimes \hat{x} \wedge \hat{y}) &= (F\phi_y a + \phi D\tau_r(a) - \nabla_y^{\phi a}(F)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}, \\
\delta^2(\phi a \otimes \hat{x} \wedge \hat{E}) &= -\phi y \tau_1(a) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} - (F\phi_y a + \phi D\tau_r(a)) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \\
&\qquad + \nabla_x^{\phi a}(F) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
\delta^2(\phi a \otimes \hat{y} \wedge \hat{E}) &= \phi x \tau_1(a) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \\
&\qquad + (\nabla_y^{\phi a}(F) - F\phi_y a - \phi D\tau_r(a)) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
\delta^2((\phi + \psi D)a \otimes \hat{x} \wedge \hat{D}) &= (-\phi y \tau_1(a) - \psi F(\tau_1(a) - a) - \psi y D\tau_1(a)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D},
\end{aligned}$$

$$\delta^2((\phi+\psi D)a\otimes\hat{y}\wedge\hat{D}) = (\phi x\tau_1(a)+\psi xD\tau_1(a))\otimes\hat{x}\wedge\hat{y}\wedge\hat{D},$$
$$\begin{aligned}\delta^2((\phi+\lambda D)a\otimes\hat{D}\wedge\hat{E}) = {}&(\phi x\tau_1(a)+\lambda xD\tau_1(a))\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}\\ &+(\phi y\tau_1(a)+\lambda yD\tau_1(a)+\lambda F(\tau_1(a)-a))\otimes\hat{y}\wedge\hat{D}\wedge\hat{E},\end{aligned}$$

$$\delta^3((\phi+\psi D)a\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}) = 0,$$
$$\delta^3(\phi a\otimes\hat{x}\wedge\hat{y}\wedge\hat{E}) = (F\phi_y a+\phi D\tau_r(a)-\nabla_y^{\phi a}(F))\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E},$$
$$\begin{aligned}&\delta^3((\phi+\psi D)a\otimes\hat{x}\wedge\hat{D}\wedge\hat{E})\\ &\qquad= -(\phi y\tau_1(a)+\psi yD\tau_1(a)+\psi F(\tau_1(a)-a))\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E},\end{aligned}$$
$$\delta^3((\phi+\psi D)a\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}) = (\phi x\tau_1(a)+\psi xD\tau_1(a))\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}.$$

Here and below $\tau_t : T\to T$ is the $\Bbbk$-linear map such that $\tau_t(E^n) = E^n-(E+t)^n$ for all $n\in\mathbb{N}_0$, and ϕ and ψ denote homogeneous elements of S of appropriate degrees and λ a scalar.

3.11. We proceed to compute the cohomology of the complex $\mathfrak{X}$, starting with degrees zero and four, for which the computation is almost immediate. Indeed, since the kernel of τ_1 and of τ_r is $\Bbbk\subseteq T$, it is clear that $H^0(\mathfrak{X}) = \ker\delta^0 = \Bbbk$. On the other hand, if $\psi\in S_2$ and $a\in T$, we can write $\psi = \psi_1x+\psi_2y$ for some $\psi_1,\psi_2\in S_1$ and there is a $b\in T$ such that $\tau_1(b) = a$, so that

$$\delta^3(-\psi_2Db\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}+\psi_1Db\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}) = (\psi Da+\boxed{S_{r+2}T})\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}.$$

Similarly, we have $\delta^3(S_{r+1}T\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}+S_{r+1}T\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}) = S_{r+2}T\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}$. These two facts imply that the map δ^3 is surjective, so that $H^4(\mathfrak{X}) = 0$.

3.12. Let $\omega\in\mathfrak{X}^1$ be a 1-cocycle in $\mathfrak{X}$. There are then $a,b,c,d,e,f\in T$, $k\in\mathbb{N}_0$ and $\phi_0,\dots,\phi_k\in S_r$ such that either $k=0$ or $\phi_k\neq 0$, and

$$\omega = (xa+yb)\otimes\hat{x}+(xc+yd)\otimes\hat{y}+\left(\sum_{i=0}^{k}\phi_iE^i+De\right)\otimes\hat{D}+f\otimes\hat{E}.$$

If $\bar{e}\in T$ is such that $\tau_r(\bar{e}) = e$, then by replacing ω by $\omega-\delta^0(\bar{e})$, which does not change the cohomology class of ω, we can assume that $e=0$. The formula for δ^0 then shows that ω is a coboundary iff it is equal to zero. The coefficient of $\hat{x}\wedge\hat{y}$ in $\delta^1(\omega)$ is

$$x^2\tau_1(c)+xy(\tau_1(d)-\tau_1(a))-y^2\tau_1(b) = 0.$$

We therefore have b, c, $d-a\in\Bbbk$. The coefficient of $\hat{D}\wedge\hat{E}$, on the other hand, is $D\tau_r(f) = 0$, so that also $f\in\Bbbk$; exactly the same information comes from the vanishing of the coefficients of $\hat{x}\wedge\hat{E}$ and of $\hat{y}\wedge\hat{E}$. Since $b\in\Bbbk$, the coefficient of $\hat{x}\wedge\hat{D}$ is

$$-Fb-xD\tau_r(a)+\sum_{i=0}^{k}\phi_ix\tau_1(E^i) = 0.$$

We thus see that $\tau_r(a) = 0$, so that $a\in\Bbbk$, and that $\sum_{i=0}^{k}\phi_ix\tau_1(E^i) = Fb$. This implies that $k\leq 1$, that $-\phi_1x = Fb$ and therefore, since x is not a factor of F by hypothesis, that $\phi_1 = 0$ and $b = 0$.

Finally, using all the information we have so far, we can see that the vanishing of the coefficient of $\hat{y} \wedge \hat{D}$ in $\delta^1(\omega)$ implies that $F_x xa + F_y(xc + yd) = Fd$. Together with Euler's relation $F_x x + F_y y = (r + 1)F$ this tells us that

$$(cx + (d - a)y)F_y = (d - (r + 1)a)F. \tag{3.2}$$

As F is square-free, it follows[1] from this equality that the polynomial $cx + (d - a)y$ is zero, so that $c = 0$ and $d = a$, and, finally, that $a = 0$. We conclude in this way that the set of 1-cocycles

$$\phi \otimes \hat{D} + f \otimes \hat{E}, \qquad \phi \in S_r, f \in \Bbbk$$

is a complete, irredundant set of representatives for the elements of $H^1(\mathfrak{X})$.

3.13. Let now $\omega \in \mathfrak{X}^3$ be a 3-cocycle, so that

$$\omega = a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + b \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + c \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + d \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}$$

for some $a \in (S_{r+2} \oplus S_2 D)T$, $b \in S_2 T$, c, $d \in (S_{r+1} \oplus S_1 D)T$ and $\delta^3(\omega) = 0$. For all $\phi \in S_1$ and $e \in T$ we have

$$\delta^2(\phi e \otimes \hat{x} \wedge \hat{E}) = -\phi y \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}$$

and

$$\delta^2(\phi e \otimes \hat{y} \wedge \hat{E}) = \phi x \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$

so that by adding to ω an element of $\delta^2(S_1 T \otimes \hat{x} \wedge \hat{E} + S_1 T \otimes \hat{y} \wedge \hat{E})$, which does not change the cohomology class of ω, we can suppose that $b = 0$. Similarly, for all $\phi \in S_2$ and all $e \in T$ we have that

$$\delta^2(\phi e \otimes \hat{x} \wedge \hat{y}) = (\boxed{S_{r+2}T} + \phi D \tau_r(e)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D},$$

and, for all $\phi \in S_{r+1}$ and all $e \in T$, that

$$\delta^2(\phi e \otimes \hat{x} \wedge \hat{D}) = -\phi y \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}$$

and

$$\delta^2(\phi e \otimes \hat{y} \wedge \hat{D}) = \phi x \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}.$$

Using this we see that, up to changing ω by adding to it a 3-coboundary, we can suppose that $a = 0$. Finally, for each $\phi \in S_r$ and all $e \in T$ we have

$$\delta^2(\phi e \otimes \hat{D} \wedge \hat{E}) = \phi x \tau_1(e) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$
$$\delta^2(De \otimes \hat{D} \wedge \hat{E}) = xD\tau_1(e) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}$$

[1] Suppose that $u = cx + (d - a)y$ is not zero. Differentiating in (3.2) with respect to y, we find that $-raF_y = uF_{yy}$. Since x does not divide F, we have $F_{yy} \neq 0$, and then $a \neq 0$ and u divides F_y: from (3.2) it follows then that u^2 divides F, since the left hand side of that equality is non-zero, and this is absurd because F is square-free.

and

$$\delta^2(-y \otimes \hat{x} \wedge \hat{E} + \bar{F}E \otimes \hat{D} \wedge \hat{E}) = y^{r+1} \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$

so we can also suppose that $c \in y^{r+1}ET + yDT$.

There are $l \geq 0$, $\lambda_1, \ldots, \lambda_l$, $\mu_0, \ldots, \mu_l \in \Bbbk$, $\phi_0, \ldots, \phi_l \in S_{r+1}$, $\psi_0, \ldots, \psi_l \in S_1$, $\zeta_0, \ldots, \zeta_l \in S_1$ such that $c = \sum_{i=1}^{l} \lambda_i y^{r+1} E^i + \sum_{i=0}^{l} \mu_i y D E^i$ and $d = \sum_{i=0}^{l} (\phi_i + \psi_i D) E^i$. The vanishing of $\delta^3(\omega)$ means precisely that

$$\sum_{i=1}^{l} \lambda_i y^{r+2} \tau_1(E^i) + \sum_{i=0}^{l} \Big(\mu_i y^2 D \tau_1(E^i) - \mu_i y F (E+1)^i - \phi_i x \tau_1(E^i) - \psi_i x D \tau_1(E^i) \Big) = 0.$$

The left hand side of this equation is an element of $S_{r+2}T \oplus S_2DT$. The component in S_2DT is $\sum_{i=0}^{l} (\mu_i y^2 - \psi_i x) D \tau_1(E^i) = 0$ and therefore $\mu_i = \psi_i = 0$ for all $i \in \{1, \ldots, l\}$. On the other hand, the component in $S_{r+2}T$ is

$$\sum_{i=1}^{l} \lambda_i y^{r+2} \tau_1(E^i) - \mu_0 y F - \sum_{i=0}^{l} \phi_i x \tau_1(E^i) = 0.$$

This implies that $\lambda_i y^{r+2} - \phi_i x = 0$ if $i \in \{2, \ldots, l\}$, so that $\lambda_i = \phi_i = 0$ for such i, and then the equation reduces to $\lambda_1 y^{r+2} + \mu_0 y F - \phi_1 x = 0$. Recalling from **3.1** that $F = y^{r+1} + x\bar{F}$, we deduce from this that $\lambda_1 = -\mu_0$ and $\phi_1 = \mu_0 y \bar{F}$ and, putting everything together, that every 3-cocycle is cohomologous to one of the form

$$(\mu_0 y D - \mu_0 y^{r+1} E) \hat{x} \wedge \hat{D} \wedge \hat{E} + (\phi_0 + \psi_0 D + \mu_0 y \bar{F} E) \hat{y} \wedge \hat{D} \wedge \hat{E} \tag{3.3}$$

with $\mu_0 \in \Bbbk$, $\phi_0 \in S_{r+1}$ and $\psi_0 \in S_1$. A direct computation shows that moreover every 3-cochain of this form is a 3-cocycle.

Let now η be a 2-cochain η in $\mathfrak{X}$, so that

$$\eta = \boxed{A_2 \otimes \Bbbk\hat{x} \wedge \hat{y} \oplus A_{r+1} \otimes (\Bbbk\hat{x} \wedge \hat{D} \oplus \Bbbk\hat{y} \wedge \hat{D})} + u \otimes \hat{x} \wedge \hat{E} + v \otimes \hat{y} \wedge \hat{E} + w \otimes \hat{D} \wedge \hat{E}$$

with $u, v \in A_1$ and $w \in A_r$, and let us suppose that $\delta^2(\eta)$ is equal to the 3-cocycle (3.3). There are $l \geq 0$, $\alpha_0, \ldots, \alpha_l$, $\beta_0, \ldots, \beta_l \in S_1$, $\gamma_0, \ldots, \gamma_l \in S_r$ and $\xi_0, \ldots, \xi_l \in \Bbbk$ such that $u = \sum_{i=0}^{l} \alpha_i E^i$, $v = \sum_{i=0}^{l} \beta_i E^i$ and $w = \sum_{i=0}^{l} (\gamma_i + \xi_i D) E^i$. The coefficient of $\hat{x} \wedge \hat{y} \wedge \hat{E}$ in $\delta^2(\eta)$ must be equal to zero, so that

$$\sum_{i=0}^{l} (-\alpha_i y + \beta_i x) \tau_1(E^i) = 0,$$

and this implies that there are scalars $\rho_1, \ldots, \rho_l \in \Bbbk$ such that $\alpha_i = \rho_i x$ and $\beta_i = \rho_i y$ for all $i \in \{1, \ldots, l\}$. Looking now at the coefficient of $\hat{x} \wedge \hat{D} \wedge \hat{E}$ in $\delta^2(\eta)$ and comparing with (3.3) we find that

$$\sum_{i=0}^{l} \Big(-F \alpha_{iy} E^i - \alpha_i D \tau_r(E^i) + \gamma_i x \tau_1(E^i) + \xi_i x D \tau_1(E^i) \Big) = \mu_0 y D - \mu_0 y^{r+1} E. \tag{3.4}$$

This is an equality of two elements of $ST \oplus SDT$. Considering the components in DT, we find that $xD\sum_{i=1}^{l}(-\rho_i\tau_r(E^i) + \xi_i\tau_1(E^i)) = \mu_0 yD$, and this tells us that $\mu_0 = 0$ and that

$$\sum_{i=1}^{l}\left(-\rho_i\tau_r(E^i) + \xi_i\tau_1(E^i)\right) = 0. \tag{3.5}$$

On the other hand, as the components in ST of the two sides of (3.4) are equal, we have

$$-F\alpha_{0y} + \sum_{i=0}^{l}\gamma_i x\tau_1(E^i) = 0,$$

so that $\gamma_i = 0$ for all $i \in \{2, \dots, l\}$ and $F\alpha_{0y} + \gamma_1 x = 0$. As x does not divide F, we must have $\alpha_{0y} = 0$ and $\gamma_1 = 0$: in particular, there is $\rho_0 \in \Bbbk$ such that $\alpha_0 = \rho_0 x$.

Finally, considering the coefficient of $\hat{y} \wedge \hat{D} \wedge \hat{E}$ of $\delta^2(\eta)$ and of (3.3) we see that

$$\begin{aligned}\sum_{i=0}^{l}\Big(\nabla_x^{\alpha_i E^i}(F) + \nabla_y^{\beta_i E^i}(F) - F\beta_{iy}E^i - \beta_i D\tau_r(E^i)\\ + \gamma_i y\tau_1(E^i) + \xi_i yD\tau_1(E^i) - \xi_i F(E+1)^i\Big) = \phi_0 + \psi_0 D,\end{aligned}$$

which at this point we can rewrite, using in the process the equality (3.5) above and the fact that $\nabla_x^{xE^i}(F) + \nabla_x^{yE^i}(F) = F\sum_{t=0}^{r}(E+t)^i$, as

$$\rho_0 xF_x + \beta_0 F_y - F\left(\beta_{0y} + \xi_0 - \sum_{i=1}^{l}\left(\rho_i\sum_{t=1}^{r}(E+t)^i - \xi_i(E+1)^i\right)\right) = \phi_0 + \psi_0 D.$$

It follows at once that $\psi_0 = 0$ and that, in fact,

$$\rho_0 xF_x + \beta_0 F_y - F\left(\beta_{0y} + \xi_0 - \sum_{i=1}^{l}\left(\rho_i\sum_{t=1}^{r}t^i - \xi_i\right)\right) = \phi_0.$$

The polynomial ϕ_0 is then in the linear span of xF_x, xF_y, yF_y and F inside S_{r+1}. Euler's relation implies that already the first three polynomials span this subspace, and we have

$$\begin{aligned}\delta(x \otimes \hat{x} \wedge \hat{E}) &= xF_x \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},\\ \delta(x \otimes \hat{y} \wedge \hat{E}) &= xF_y \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},\\ \delta(y \otimes \hat{y} \wedge \hat{E} - D \otimes \hat{D}\hat{E}) &= yF_y \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}.\end{aligned} \tag{3.6}$$

We conclude in this way that the only 3-coboundaries among the cocycles of the form (3.3) are the linear combinations of the right hand sides of the equalities (3.6); these three cocycles are, moreover, linearly independent. This means that there is an isomorphism

$$H^3(\mathfrak{X}) \cong \Bbbk\omega_3 \oplus S_1 D \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \oplus \frac{S_{r+1}}{\langle xF_x, xF_y, yF_y\rangle} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \tag{3.7}$$

with

$$\omega_3 = (yD - y^{r+1}E) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + y\bar{F}E \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}.$$

In particular, we have that $\dim H^3(\mathfrak{X}) = r + 2$, since the denominator appearing in the right hand side of the isomorphism (3.7) is a 3-dimensional vector space —this follows at once from Lemma **3.7**.

3.14. It only remains for us to compute the second cohomology space $HH^2(A)$. We consider a 2-cocycle $\omega \in \mathfrak{X}^2$ and $a \in S_2T$, $b, c \in S_1T$, $d, e \in S_{r+1}T \oplus S_1DT$ and $f \in S_rT \oplus DT$ such that

$$\omega = a \otimes \hat{x} \wedge \hat{y} + b \otimes \hat{x} \wedge \hat{E} + c \otimes \hat{y} \wedge \hat{E} + d \otimes \hat{x} \wedge \hat{D} + e \otimes \hat{y} \wedge \hat{D} + f \otimes \hat{E} \wedge \hat{D}.$$

Adding to ω an element of $\delta^1(T \otimes \hat{E})$, we can assume that $f \in S_rT$; adding an element of $\delta^1(S_1T\otimes\hat{x}\oplus S_1T\otimes\hat{y})$, we can suppose that $a = 0$; finally, adding an element of $\delta^1((S_rT\oplus DT)\otimes\hat{D})$ we can suppose that $d \in y^{r+1}T \oplus yDT$. In this situation, there are an integer $l \geq 0$, $\alpha_0, \dots, \alpha_l$, $\beta_0, \dots, \beta_l \in S_1$, $\lambda_0, \dots, \lambda_l$, $\mu_0, \dots, \mu_l \in \mathbb{k}$, $\phi_0, \dots, \phi_l \in S_{r+1}$, $\psi_0, \dots, \psi_l \in S_1$ and $\xi_0, \dots, \xi_l \in S_r$ such that

$$b = \sum_{i=0}^{l} \alpha_i E^i, \qquad c = \sum_{i=0}^{l} \beta_i E^i, \qquad d = \sum_{i=0}^{l} (\lambda_i y^{r+1} + \mu_i yD)E^i, \qquad e = \sum_{i=0}^{l} (\phi_i + \psi_i D)E^i,$$

and

$$f = \sum_{i=0}^{l} \xi_i E^i.$$

As

$$\delta^1(-y \otimes \hat{x} + \bar{F}E \otimes \hat{D}) = y^{r+1} \otimes \hat{x} \wedge \hat{D} + \boxed{S_{r+1}} \otimes \hat{y} \wedge \hat{D},$$

we can assume that $\lambda_0 = 0$.

The coefficient of $\hat{x} \wedge \hat{y} \wedge \hat{E}$ in $\delta^2(\omega)$ is $\sum_{i=0}^{l}(-\alpha_i y + \beta_i x)\tau_1(E^i) = 0$, and this implies that there are scalars $\rho_1, \dots, \rho_l \in \mathbb{k}$ such that $\alpha_i = \rho_i x$ and $\beta_i = \rho_i y$ for each $i \in \{1, \dots, l\}$. The coefficient of $\hat{x} \wedge \hat{D} \wedge \hat{E}$ in $\delta^2(\omega)$ is

$$\sum_{i=0}^{l} \Big(-F\alpha_{iy}E^i - \alpha_i D\tau_r(E^i) + \xi_i x\tau_1(E^i)\Big) = 0. \tag{3.8}$$

It follows that $\sum_{i=0}^{l} \alpha_i D\tau_r(E^i) = 0$, so that $\alpha_1 = \cdots = \alpha_l = 0$; as a consequence of this, we have that $\rho_1 = \cdots = \rho_l = 0$ and $\beta_1 = \cdots = \beta_l = 0$. The equality (3.8) also tells us that $-F\alpha_{0y} + \sum_{i=0}^{l} \xi_i x\tau_1(E^i) = 0$, and from this we see that $\xi_2 = \cdots = \xi_l = 0$ and $-F\alpha_{0y} - \xi_1 x = 0$, so that $\alpha_{0y} = 0$ and $\xi_1 = 0$, since x does not divide F. In particular, there is a $\rho_0 \in \mathbb{k}$ such that $\alpha_0 = \rho_0 x$.

The coefficient of $\hat{y} \wedge \hat{D} \wedge \hat{E}$ in $\delta^2(\omega)$ is

$$\begin{aligned}\sum_{i=0}^{l}\Big(\nabla_x^{\alpha_i E^i}(F) + \nabla_y^{\beta_i E^i}(F) - F\beta_{iy}E^i - \beta_i D\tau_r(E^i) + \xi_i y\tau_1(E^i)\Big)&\\ = \rho_0 x F_x + \beta_0 F_y - \beta_{0y}F&\\ = \Big(\rho_0 - (r+1)^{-1}\beta_{0y}\Big)xF_x + \Big(\beta_{0x}x + (1-(1+r)^{-1})\beta_{0y}y\Big)F_y = 0,&\end{aligned}$$

and our Lemma 3.7 implies then that $\beta_0 = 0$ and $\rho_0 = 0$. Finally, we consider the coefficient of $\hat{x} \wedge \hat{y} \wedge \hat{D}$:

$$\sum_{i=0}^{l}\Big(-\lambda_i y^{r+2}\tau_1(E^i) + \mu_i yF(E+1)^i - \mu_i y^2 D\tau_1(E^i) + \phi_i x\tau_1(E^i) + \psi_i xD\tau_1(E^i)\Big) = 0.$$

Looking at the terms involving D in this equation, we see that

$$\sum_{i=0}^{l}(-\mu_i y^2 + \psi_i x)D\tau_1(E^i) = 0,$$

so $\mu_1 = \cdots = \mu_l = 0$ and $\psi_1 = \cdots = \psi_l = 0$. The terms not involving D add up to

$$\mu_0 yF + \sum_{i=0}^{l}(-\lambda_i y^{r+2} + \phi_i x)\tau_1(E^i) = 0,$$

so that $\lambda_2 = \cdots = \lambda_l = 0$, $\phi_2 = \cdots = \phi_l = 0$ and $\mu_0 yF + \lambda_1 y^{r+1} - \phi_1 x = 0$, which implies that $\lambda_1 = -\mu_0$ and $\phi_1 = \mu_0 y\bar{F}$.

After all this, we see that every 2-cocycle in our complex is cohomologous to one of the form

$$(\mu_0 yD - \mu_0 y^{r+1}E)\hat{x} \wedge \hat{D} + (\phi_0 + \psi_0 D + \mu_0 y\bar{F}E)\hat{y} \wedge \hat{D} + \xi_0 \hat{D} \wedge \hat{E} \tag{3.9}$$

with $\mu_0 \in \Bbbk$, $\phi_0 \in S_{r+1}$, $\psi_0 \in S_1$ and $\xi_0 \in S_r$. Thanks to a direct computation we find that all elements of this form are 2-cocycles.

Let us now suppose that the cocycle (3.9), which we call again ω, is a coboundary, so that there exist $k \geq 0$, $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_k \in S_1$, $\sigma_1, \ldots, \sigma_k \in S_r$, $\zeta_0, \ldots, \zeta_k \in \Bbbk$ and $u \in T$ such that if

$$\eta = \sum_{i=0}^{k}\alpha_i E^i \hat{x} + \sum_{i=0}^{k}\beta_i E^i \hat{y} + \sum_{i=0}^{k}(\sigma_i + \zeta_i D)E^i \hat{D} + u\hat{E},$$

we have $\delta^1(\eta) = \omega$. The coefficient of $\hat{D} \wedge \hat{E}$ in $\delta^1(\eta)$ is $D\tau_r(u)$ so, comparing with (3.9), we see that we must have $\xi_0 = 0$ and $u \in \Bbbk$; it follows from this that the coefficients of $\hat{E} \wedge \hat{E}$ and of $\hat{y} \wedge \hat{E}$ in $\delta^1(\eta)$ vanish. On the other hand, the coefficient of $\hat{x} \wedge \hat{y}$ in $\delta^1(\eta)$ is $\sum_{i=0}^{k}(-\alpha_i y + \beta_i x)\tau_1(E^i)$: as this has to be zero, we see that there exist $\rho_1, \ldots, \rho_k \in \Bbbk$ such that $\alpha_i = \rho_i x$ and $\beta_i = \rho_i y$ for each $i \in \{1, \ldots, k\}$.

Next, the coefficient of $\hat{x} \wedge \hat{D}$ in $\delta^1(\eta)$ is

$$\sum_{i=0}^{k}\Big(-F\alpha_{iy}E^i - \alpha_i D\tau_r(E^i) + \sigma_i x\tau_1(E^i) + \zeta_i xD\tau_1(E^i)\Big) = \mu_0 yD - \mu_0 y^{r+1}E. \tag{3.10}$$

This means, first, that $\sum_{i=1}^{k}\Big(-\rho_i xD\tau_r(E^i) + \zeta_i xD\tau_1(E^i)\Big) = \mu_0 yD$, and this is only possible if $\mu_0 = 0$ and

$$\sum_{i=1}^{k}\Big(-\rho_i\tau_r(E^i) + \zeta_i\tau_1(E^i)\Big) = 0. \tag{3.11}$$

Second, the equality (3.10) implies that

$$\sum_{i=0}^{k}\Big(-F\alpha_{iy}E^i + \sigma_i x\tau_1(E^i)\Big) = -F\alpha_{0y} + \sum_{i=1}^{k}\sigma_i x\tau_1(E^i) = 0,$$

so that $\sigma_2 = \cdots = \sigma_k = 0$ and $F\alpha_{0y} + \sigma_1 x = 0$, which tells us that $\sigma_1 = 0$ and $\alpha_{0y} = 0$: there is then a $\rho_0 \in \Bbbk$ such that $\alpha_0 = \rho_0 x$.

Finally, the coefficient of $\hat{y} \wedge \hat{D}$ in $\delta^1(\eta)$ is

$$\sum_{i=0}^{k}\Big(\nabla_x^{\alpha_i E^i}(F) + \nabla_y^{\beta_i E^i}(F) - F\beta_{iy}E^i - \beta_i D\tau_r(E^i) + \sigma_i y\tau_1(E^i) - \zeta_i F(E+1)^i + \zeta_i yD\tau_1(E^i)\Big)$$
$$= \phi_0 + \psi_0 D.$$

Looking only at the terms which are in S_1DT, we see that

$$yD\sum_{i=1}^{k}(-\rho_i\tau_r(E^i) + \zeta_i\tau(E^i)) = \psi_0 D$$

and, in view of (3.11), it follows from this that $\psi_0 = 0$. The terms in $S_{r+1}T$, on the other hand, are

$$\rho_0 xF_x + \beta_0 F_y + F\left(-\beta_{0y} - \zeta_0 + \sum_{i=0}^{k}\Big(\rho_i\sum_{t=1}^{r}(E+t)^i - \zeta_i(E+1)^i\Big)\right) = \phi_0,$$

and proceeding as before we see that ϕ_0 is in the linear span of xF_x, xF_y and yF_y. Computing, we find that

$$\delta^1(x \otimes \hat{x}) = xF_x \otimes \hat{y} \wedge \hat{D},$$
$$\delta^1(x \otimes \hat{y}) = xF_y \otimes \hat{y} \wedge \hat{D},$$
$$\delta^1(y \otimes \hat{y} - D\hat{D}) = yF_y \otimes \hat{y} \wedge \hat{D}.$$

We thus conclude that there is an isomorphism

$$H^2(\mathfrak{X}) \cong \Bbbk\omega_2 \oplus \frac{S_{r+1}}{\langle xF_x, xF_y, yF_y\rangle} \otimes \hat{y} \wedge \hat{D} \oplus S_1 D \otimes \hat{y} \wedge \hat{D} \oplus S_r \otimes \hat{D} \wedge \hat{E},$$

with $\omega_2 = (yD - y^{r+1}E) \otimes \hat{x} \wedge \hat{D} + y\bar{F}E \otimes \hat{y} \wedge \hat{D}$, and that, in particular, the dimension of $H^2(\mathfrak{X})$ is $2r + 3$.

3.15. We can summarize our findings as follows:

Proposition. *Suppose that $r \geq 3$. For all $p \geq 4$ we have $HH^p(A) = 0$. There are isomorphisms*

$$\begin{aligned}
&HH^0(A) \cong \Bbbk, \\
&HH^1(A) \cong S_r \otimes \hat{D} \oplus \Bbbk \otimes \hat{E}, \\
&HH^2(A) \cong \Bbbk\omega_2 \oplus \frac{S_{r+1}}{\langle xF_x, xF_y, yF_y\rangle} \otimes \hat{y} \wedge \hat{D} \oplus S_1 D \otimes \hat{y} \wedge \hat{D} \oplus S_r \otimes \hat{D} \wedge \hat{E}, \\
&HH^3(A) \cong \Bbbk\omega_3 \oplus \frac{S_{r+1}}{\langle xF_x, xF_y, yF_y\rangle} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \oplus S_1 D \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},
\end{aligned}$$

with

$$\begin{aligned}
\omega_2 &= (yD - y^{r+1}E) \otimes \hat{x} \wedge \hat{D} + y\bar{F}E \otimes \hat{y} \wedge \hat{D}, \\
\omega_3 &= (yD - y^{r+1}E) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + y\bar{F}E \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}.
\end{aligned}$$

The Hilbert series of the Hochschild cohomology of A is

$$\begin{aligned}
h_{HH^\bullet(A)}(t) &= 1 + (r+2)t + (2r+3)t^2 + (r+2)t^3 \\
&= (1+t)(1+(r+1)t+(r+2)t^2).
\end{aligned}$$

□

In fact, in each of the isomorphisms appearing in the statement of the proposition we have given a set of representing cocycles. This will be important in what follows, when we compute the Gerstenhaber algebra structure on the cohomology of A.

We have chosen a system of coordinates in the vector space containing the arrangement $\mathcal{A}$ in such a way that one of the lines is given by the equation $x = 0$. This was useful in picking a basis for the S-module of derivations $\mathsf{Der}(\mathcal{A})$ and, as a consequence, obtaining a presentation of the algebra A amenable to the computations we wanted to carry out, but the unnaturality of our choice is reflected in the rather unpleasant form of the representatives that we have found for cohomology classes. In the next section we will be able to obtain a more natural description.

3.16. In Proposition **3.15** we considered only line arrangements with $r \geq 3$, that is, with at least 5 lines. As we explained in **3.10**, without this restriction the method of calculation that we followed has to be modified, and it turns out that this is not only a technical difference: the actual results are different. Let us describe what happens, starting with the factorizable cases:

- If there are no lines, so that $r = -2$, the arrangement is empty and $\mathsf{Diff}(\mathcal{A})$ is the second Weyl algebra $A_2 = \Bbbk\langle x, y, \partial_x, \partial_y\rangle$.
- If there is one line, then $\mathsf{Diff}(\mathcal{A})$ is $\Bbbk\langle x, y, x\partial_x, \partial_y\rangle$ and this is isomorphic to $U(\mathfrak{s}) \otimes A_1$, with $U(\mathfrak{s})$ the enveloping algebra of the non-abelian 2-dimensional Lie algebra $\mathfrak{s}$ and $A_1 = \Bbbk\langle y, \partial_y\rangle$, the first Weyl algebra.
- If there are two lines, so that $r = 0$, then $\mathsf{Diff}(\mathcal{A})$ is $\Bbbk\langle x, y, x\partial_x, y\partial_y\rangle$, which is isomorphic to $U(\mathfrak{s}) \otimes U(\mathfrak{s})$.

The Hochschild cohomology of the Weyl algebras is well-known as is that of $U(s)$ —see, for example [Sri61]. Using this and Künneth's formula we find that when $-2 \leq r \leq 0$ we have for all $i \geq 0$ that

$$\dim HH^i(\mathsf{Diff}(\mathcal{A})) = \binom{r+2}{i}.$$

Finally, we have the cases of three and four lines. Up to isomorphism of arrangements, we can assume that the defining polynomials are, respectively, $Q = xy(x-y)$ and $Q = xy(x-y)(x-\lambda y)$ for some $\lambda \in \Bbbk \setminus \{0, 1\}$. One can compute the cohomology of $\mathsf{Diff}(\mathcal{A})$ in these cases along the lines of what we did above, but the computation is surprisingly much more involved. We have done the computation using an alternative, much more efficient approach —using a spectral sequence that computes in general the Hochschild cohomology of the enveloping algebra of a Lie–Rinehart pair— with which we deal in Chapter 6. Let us for now simply summarize the result: when r is 2 or 3, the Hilbert series of $HH^\bullet(A)$ is

$$h_{HH^\bullet(A)}(t) = 1 + (r+2)t + (2r+4)t^2 + (r+3)t^3.$$

This differs from the general case of Proposition **3.15** in the coefficients of t^2 and t^3.

3.3 THE GERSTENHABER ALGEBRA STRUCTURE ON $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$

3.17. Let $\mathcal{B}A$ be the usual bar resolution for A as an A-bimodule. There is a morphism of complexes $\phi : \mathbf{P} \to \mathcal{B}A$ over the identity map of A such that $\phi = \phi_K + \phi_N$ with $\phi_K, \phi_N : \mathbf{P} \to \mathcal{B}A$ maps of A-bimodules such that

$$\phi_K(1|v_1 \wedge \cdots \wedge v_p|1) = \sum_{\pi \in S_p} (-1)^{\varepsilon(\pi)} 1|v_{\pi(1)}|\cdots|v_{\pi(p)}|1,$$

whenever $p \geq 0$ and $v_1, \ldots, v_p \in V$, with the sum running over all permutations of degree p, and

$$\phi_N(1|1) = 0;$$

$$\phi_N(1|v|1) = 0, \quad \forall v \in V;$$

$$\begin{aligned}\phi_N(1|x \wedge y|1) = \phi_N(1|x \wedge E|1) = \phi_N(1|y \wedge E|1) = \phi_N(1|x \wedge D|1)& \\ = \phi_N(1|D \wedge E|1) = 0;&\end{aligned}$$

$$\phi_N(1|y \wedge D|1) = q_{(1)}|\bar{q}_{(2)}|q_{(3)}|1 - F|1|1|1;$$

$$\phi_N(1|x \wedge y \wedge E|1) = \phi_N(1|x \wedge D \wedge E|1) = 0;$$

$$\begin{aligned}\phi_N(1|x \wedge y \wedge D|1) = q_{(1)}|\bar{q}_{(2)}|q_{(3)}|x|1 - q_{(1)}|\bar{q}_{(2)}|x|q_{(3)}|1 + q_{(1)}|x|\bar{q}_{(2)}|q_{(3)}|1& \\ - F|x|1|1|1 - F|1|1|x|1;&\end{aligned}$$

$$\begin{aligned}\phi_N(1|y \wedge D \wedge E|1) = q_{(1)}|\bar{q}_{(2)}|q_{(3)}|E|1 - q_{(1)}|\bar{q}_{(2)}|E|q_{(3)}|1 + q_{(1)}|E|\bar{q}_{(2)}|q_{(3)}|1& \\ - F|E|1|1|1 - F|1|1|E|1.&\end{aligned}$$

Here $q_{(1)}|\bar{q}_{(2)}|q_{(3)}$ denotes the element $\nabla(F) \in S \otimes S_1 \otimes S$, with an omitted sum.

On the other hand, there is a morphism of complexes of A-bimodules $\psi : \mathcal{B}A \to \mathbf{P}$ over the identity map of A such that

$$
\begin{aligned}
&\psi_0(1|1) = 1|1, \\
&\psi_1(1|w|1) = w_{(1)}|w_{(2)}|w_{(3)}, && \text{for all standard monomials } w; \\
&\psi_2(1|yD|y|1) = -y|y \wedge D|1 - q_{(1)}|q_{(2)} \wedge y|q_{(3)}; \\
&\psi_2(1|y^{r+1}E|y|1) = -y^{r+1}|y \wedge E|1; \\
&\psi_2(1|E|w|1) = -w_{(1)}|w_{(2)} \wedge E|w_{(3)} && \text{for all standard monomials } w; \\
&\psi_2(1|v|w|1) = -1|w \wedge v|1, && \text{if } v, w \in \{x, y, D, E\} \text{ and } vw \text{ is not standard}; \\
&\psi_2(1|w|x|1) = -w_{(1)}|x \wedge w_{(2)}|w_{(3)} && \text{for all standard monomials } w;
\end{aligned}
$$

and

$$\psi_2(1|u|v|1) = 0$$

whenever u and v are standard monomials of A such that the concatenation uv is also a standard monomial. We omit the description of ψ_3 and ψ_4 because we do not need them. This morphism ψ can and will be taken normalized, so that it vanishes on elementary tensors of $\mathcal{B}A$ with a scalar factor.

3.18. We need the comparison morphisms that we have just described in order to compute the Gerstenhaber bracket on $HH^\bullet(A)$, but we start with a more immediate application: obtaining a natural basis of the first cohomology space $HH^1(A)$.

Proposition. (*i*) *If α is a non-zero element of S_1 that divides Q, so that $\ker\alpha$ is one of the lines in $\mathcal{A}$, then there exists a unique derivation $\partial_\alpha : A \to A$ such that $\partial_\alpha(f) = 0$ for all $f \in S$ and*

$$\partial_\alpha(\delta) = \frac{\delta(\alpha)}{\alpha}$$

for all $\delta \in \mathrm{Der}(\mathcal{A})$.

(*ii*) *If $Q = \alpha_0 \dots \alpha_{r+1}$ is a factorization of Q as a product of elements of S_1, then the cohomology classes of the $r + 2$ derivations $\partial_{\alpha_0}, \dots, \partial_{\alpha_{r+1}}$ of A freely span the vector space $HH^1(A)$.*

Here we are viewing $HH^1(A)$ as the vector space of outer derivations of A, as usual. It should be noticed that the derivation ∂_α associated to a linear factor of Q does not change if we replace α by one of its non-zero scalar multiples: this means that the basis of $HH^1(A)$ is really indexed by the lines of the arrangement $\mathcal{A}$.

Proof. (*i*) Let us fix a non-zero element α in S_1 dividing Q. There is at most one derivation $\partial_\alpha : A \to A$ as in the statement of the proposition simply because the algebra A is generated by the set $S \cup \mathrm{Der}(\mathcal{A})$. In order to prove that there is such a derivation, we need only recall from **1.21** that $\delta(\alpha) \in \alpha S$ for all $\delta \in \mathsf{Der}(\mathcal{A})$ and check that the candidate derivation respects the relations (2.1) of **3.1** that present the algebra A.

(*ii*) We need to pass from the description of $HH^1(A)$ as the space of outer derivations to its description in terms of the complex $\mathfrak{X}$ that was used to compute it: we do this with the comparison morphism $\phi : \mathbf{P} \to \mathcal{B}A$ over the identity map that we described in **3.17**. If $\delta : A \to A$ is a derivation of A and $\tilde{\delta} : A \otimes A \otimes A \to A$ is the map such that $\tilde{\delta}(a \otimes b \otimes c) = a\delta(b)c$ for all $a, b, c \in A$, which is a 1-cocycle on $\mathcal{B}A$ then the composition $\bar{\delta} \circ \phi_1 : A \otimes V \otimes A \to A$ is a 1-cocycle in the complex $\hom_{A^e}(\mathbf{P}, A)$ whose cohomology class corresponds to δ in the usual description of $HH^1(A)$ as the space of outer derivations of A. In the notation that we used in **3.9**, this cohomology class is that of

$$\delta(x) \otimes \hat{x} + \delta(y) \otimes \hat{y} + \delta(D) \otimes \hat{D} + \delta(E) \otimes \hat{E} \in A \otimes \hat{V}.$$

Using this, we can now prove the second part of the proposition. We can suppose without loss of generality that $\alpha_0 = x$, and then the class of δ_{α_0} in $HH^1(A)$ is that of $1 \otimes \hat{E}$. On the other hand, for each $i \in \{1, \dots, r+1\}$, a direct computation shows that the class of ∂_{α_i} is

$$\alpha_{iy} \frac{F}{\alpha_i} \otimes \hat{D} + 1 \otimes \hat{E}.$$

It follows easily from the second part of Lemma **3.7** that these $r + 2$ classes span $HH^1(A)$ and, since the dimension of this space is exactly $r + 2$, do so freely. □

THE CUP PRODUCT

3.19. We describe the associative algebra structure on $HH^\bullet(A)$ given by the cup product.

Proposition. *The cup product on $HH^\bullet(A)$ is such that*

$$\begin{aligned}
&S_r \otimes \hat{D} \smile S_r \otimes \hat{D} = 0; && \\
&\phi\hat{D} \smile \hat{E} = \phi\hat{D} \wedge \hat{E}, && \forall \phi \in S_r; \\
&S_r \otimes \hat{D} \smile HH^2(A) = 0; && \\
&1 \otimes \hat{E} \smile \omega_2 = \omega_3; && \\
&1 \otimes \hat{E} \smile \kappa \otimes \hat{y} \wedge \hat{D} = \kappa \otimes \hat{y} \wedge \hat{E} \wedge \hat{D}, && \forall \kappa \in S_{r+1}/\langle xF_x, xF_y, yF_y \rangle; \\
&1 \otimes \hat{E} \smile \psi D \otimes \hat{y} \wedge \hat{D} = \psi D \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, && \forall \psi \in S_1; \\
&1 \otimes \hat{E} \smile S_r \otimes \hat{D} \wedge \hat{E} = 0. &&
\end{aligned}$$

These equalities completely describe the multiplicative structure on $HH^\bullet(A)$.

Proof. There is a morphism of complexes of A-bimodules $\Delta : \mathbf{P} \to \mathbf{P} \otimes_A \mathbf{P}$ that lifts the canonical isomorphism $A \to A \otimes_A A$ such that $\Delta = \Delta_K + \Delta_N$, with

- $\Delta_K : \mathbf{P} \to \mathbf{P} \otimes_A \mathbf{P}$ the map of A-bimodules such that for whenever $p \geq 0$ and $v_1, \dots, v_p \in V$ we have

$$\Delta_K(1|v_1 \wedge \cdots \wedge v_p|1) = \sum (-1)^\varepsilon 1|v_{i_1} \wedge \cdots \wedge v_{i_r}|1 \otimes 1|v_{j_1} \wedge \cdots \wedge v_{j_s}|1,$$

  with the sum taken over all decompositions $r + s = p$ with $r, s \geq 0$, and all permutations $(i_1, \dots, i_r, j_1, \dots, j_s)$ of $(1, \dots, p)$ such that $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_s$, and where ε is the signature of the permutations,
- and $\Delta_N : \mathbf{P} \to \mathbf{P} \otimes_A \mathbf{P}$ the map of A-bimodules such that

$$\begin{aligned}
&\Delta_N(1|1) = 0;\\
&\Delta_N(1|v|1) = 0, \qquad \forall v \in V;\\
&\Delta_N(1|v \wedge w|1) = 0, \qquad \text{if } v, w \in \{x, y, D, E\},\ v \neq w \text{ and } \{v, w\} \neq \{y, D\};\\
&\Delta_N(1|y \wedge D|1) = f_{(1)}|f_{(2)}|f_{(3)} \otimes 1|f_{(4)}|f_{(5)};\\
&\Delta_N(1|x \wedge y \wedge D|1) = \Delta_N(1|x \wedge y \wedge E|1) = \Delta_N(1|x \wedge D \wedge E|1) = 0;\\
&\Delta_N(1|y \wedge D \wedge E|1) = -f_{(1)}|f_{(2)} \wedge E|f_{(3)} \otimes 1|f_{(4)}|f_{(5)}\\
&\qquad\qquad + f_{(1)}|f_{(2)}|f_{(3)} \otimes 1|f_{(4)} \wedge E|f_{(5)}.
\end{aligned}$$

  Here we have written $f_{(1)}|f_{(2)}|f_{(3)}|f_{(4)}|f_{(5)}$ for the image of F under the composition

$$S \xrightarrow{\nabla} S \otimes S_1 \otimes S \xrightarrow{\mathrm{id}_S \otimes \mathrm{id}_{S_1} \otimes \nabla} S \otimes S_1 \otimes S \otimes S_1 \otimes S,$$

  with an omitted sum, *à la* Sweedler.

We leave the verification that this does define a morphism of complexes to the reader.

One can compute the cup product on $HH^\bullet(A)$ using this diagonal morphism Δ. Indeed, we view $HH^\bullet(A)$ as the cohomology of the complex $\mathsf{hom}_{A^e}(\mathbf{P}, A)$, and if ϕ and ψ are a p- and a q-cocycle in that complex, the cup product of their cohomology classes is represented by the composition

$$P_{p+q} \xrightarrow{\Delta_{p,q}} P_p \otimes_A P_q \xrightarrow{\phi \otimes \psi} A \otimes_A A \cong A,$$

with $\Delta_{p,q}$ the component $P_{p+q} \to P_p \otimes P_q$ of the morphism Δ. The multiplication table given in the statement of the composition can be computed in this way, item by item. □

3.20. Using our description of the cup product we may understand a part of the cohomology in geometrical terms.

Proposition. (*i*) *For all $i, j, k \in \{0, \dots, r+1\}$ we have*

$$\partial_{\alpha_i} \smile \partial_{\alpha_j} + \partial_{\alpha_j} \smile \partial_{\alpha_k} + \partial_{\alpha_k} \smile \partial_{\alpha_i} = 0 \tag{3.12}$$

and $HH^1(A) \smile HH^1(A) = S_r \otimes \hat{D} \wedge \hat{E}$.

(*ii*) *The subalgebra $\mathcal{H}$ of $HH^\bullet(A)$ generated by $HH^1(A)$ is the graded-commutative algebra freely generated by its elements $\partial_{\alpha_0}, \dots, \partial_{\alpha_{r+1}}$ of degree 1 subject to the $\binom{r+2}{3}$ relations (3.12).*

This subalgebra $\mathcal{H}$ is isomorphic to the algebra $R^\bullet(\mathcal{A})$ of Example 1.38 and, as in Theorem 1.54, to the Orlik–Solomon algebra of the arrangement. Of course, when the base field is $\mathbb{C}$, there is therefore an isomorphism of algebras between $\mathcal{H}$ and the cohomology of the complement of the arrangement, as we saw in Theorem 1.52.

Proof. Using Proposition **3.19** and the description given in the proof of Proposition **3.18** for the derivations ∂_{α_i} we compute immediately that

$$\partial_{\alpha_i} \smile \partial_{\alpha_j} = -\begin{vmatrix} \alpha_{ix} & \alpha_{jx} \\ \alpha_{iy} & \alpha_{jy} \end{vmatrix} \frac{Q}{\alpha_i \alpha_j}$$

for all $i, j \in \{0, \dots, r+1\}$. Using this, we see that for all $i, j, k \in \{0, \dots, r+1\}$ we have

$$\partial_{\alpha_i} \smile \partial_{\alpha_j} + \partial_{\alpha_j} \smile \partial_{\alpha_k} + \partial_{\alpha_k} \smile \partial_{\alpha_i} = -\begin{vmatrix} \alpha_i & \alpha_j & \alpha_k \\ \alpha_{ix} & \alpha_{jx} & \alpha_{kx} \\ \alpha_{iy} & \alpha_{jy} & \alpha_{ky} \end{vmatrix} \frac{Q}{\alpha_i \alpha_j \alpha_k} = 0,$$

as the determinant vanishes. This proves the first claim of (*i*). The second one follows immediately from the description of the cup product of Proposition **3.19**.

(*ii*) Let, as in Example 1.38, $\mathcal{F} = \bigoplus_{n \geq 0} \mathcal{F}_n$ be the free graded-commutative algebra generated by $r+2$ generators $w_0, \dots, w_{r+1}$ of degree 1 subject to the relations $w_i w_j + w_j w_k + w_k w_i = 0$, one for each choice of $i, j, k \in \{0, \dots, r+1\}$. Recall that we have $\mathcal{F}_n = 0$ and $\dim \mathcal{F}_2 = r+1$. The first part of the proposition implies that there is a surjective morphism of graded algebras $f : \mathcal{F} \to \mathcal{H}$ such that $f(w_i) = \partial_{\alpha_i}$ for all $i \in \{0, \dots, r+1\}$, and this map is also injective because the dimension of the component of degree 2 of $\mathcal{H}$, which is $S_r \otimes \hat{D} \wedge \hat{E}$, is $r+1$. □

3.21. Proposition **3.20** describes meaningfully a part of the associative algebra $HH^\bullet(A)$, the subalgebra $\mathcal{H}$ generated by $HH^1(A)$, in terms of the geometry of the arrangement $\mathcal{A}$. It is not clear how to make sense of the complete algebra. We can make the following observation, though. Let us write

$$HH^2(A)' = \Bbbk\omega_2 \oplus (S_{r+1}/\langle xF_x, xF_y, yF_y\rangle \oplus S_1 D) \otimes \hat{y} \wedge \hat{D},$$

which is a complement of $\mathcal{H}^2$ in $HH^2(A)$, and let $Q = \alpha_0 \dots \alpha_{r+1}$ be a factorization of Q as a product of linear factors. If $\delta : A \to A$ is derivation of A, then our description of $HH^1(A)$ implies that there exist scalars $\delta_0, \dots, \delta_{r+1} \in \Bbbk$ and an element $u \in A$ such that $\delta = \sum_{i=0}^{r+1} \delta_i \partial_{\alpha_u} + \mathsf{ad}(u)$, and it follows easily from Proposition **3.19** that the map

$$\zeta \in HH^2(A)' \mapsto \delta \smile \zeta \in HH^3(A)$$

is either zero or an isomorphism, provided $\sum_{i=0}^{r+1} \delta_i$ is zero or not.

THE GERSTENHABER BRACKET

3.22. Using the comparison morphisms of 3.17, we can now compute the Gerstenhaber bracket. As usual, this is very laborious.

Proposition. *In $HH^\bullet(A)$ we have*

$$
\begin{array}{ll}
[0,\bullet] & \left\{ [HH^0(A), HH^\bullet(A)] = 0, \right. \\
[1,1] & \left\{ [HH^1(A), HH^1(A)] = 0, \right. \\
[1,2] & \left\{ \begin{array}{l} [HH^1(A), S_r \otimes \hat{D} \wedge \hat{E}] = 0, \\ [u \otimes \hat{D} + \lambda \otimes \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D}] = uw \otimes \hat{y} \wedge \hat{D}, \\ [u \otimes \hat{D} + \lambda \otimes \hat{E}, \omega_2] = ((\mu - \lambda)yF_x + \mu y\bar{F} - y^2\bar{u}) \otimes \hat{y} \wedge \hat{D}, \end{array} \right. \\
[1,3] & \left\{ \begin{array}{l} [u \otimes \hat{D} + \lambda \otimes \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}] = uw \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\ [u \otimes \hat{D} + \lambda \otimes \hat{E}, \omega_3] = ((\mu - \lambda)yF_x + \mu y\bar{F} - y^2\bar{u}) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \end{array} \right. \\
[2,2] & \left\{ \begin{array}{l} [S_r \otimes \hat{D} \wedge \hat{E}, S_r \otimes \hat{D} \wedge \hat{E}] = 0, \\ [u \otimes \hat{D} \wedge \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D}] = uw \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\ [u \otimes \hat{D} \wedge \hat{E}, \omega_2] = (\mu yFx + \mu y\bar{F} - y^2\bar{u}) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\ [(S_{r+1} + S_1 D) \otimes \hat{y} \wedge \hat{D}, (S_{r+1} + S_1 D) \otimes \hat{y} \wedge \hat{D}] = 0, \\ [(S_{r+1} + S_1 D) \otimes \hat{y} \wedge \hat{D}, \omega_2] = 0, \\ [\omega_2, \omega_2] = 0. \end{array} \right.
\end{array}
$$

Here $u \in S_r$, $\lambda \in \Bbbk$, $v \in S_{r+1}$, $w \in S_1$ and $\mu \in \Bbbk$ and $\bar{u} \in S_{r-1}$ are such that $u = \lambda y^r + x\bar{u}$.

Proof. Let us first recall from [Ger64] how one can compute the Gerstenhaber bracket in the standard complex $\mathsf{hom}_{A^e}(\mathcal{B}A, A)$. If $f : A^{\otimes q} \to A$ is a q-cochain in the standard complex $\mathsf{hom}_{A^e}(\mathcal{B}A, A)$, which we identify as usual with $\mathsf{hom}(A^{\otimes \bullet}, A)$, and $p \geq q$, we denote $\mathsf{w}_p(f) : A^{\otimes p} \to A^{p-q+1}$ the p-cochain in the same complex such that

$$
\begin{aligned}
&\mathsf{w}_p(f)(a_1 \otimes \cdots \otimes a_p) \\
&\qquad = \sum_{i=1}^{p-q+1} (-1)^{(q-1)(i-1)} a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+q-1}) \otimes a_{i+q} \otimes \cdots \otimes a_p.
\end{aligned}
$$

If now α and β are a p- and a q-cocycle in the standard complex, the Gerstenhaber composition $\diamond$ (which is usually written simply $\circ$) of α and β is the $(p+q-1)$-cochain

$$\alpha \diamond \beta = \alpha \circ \mathsf{w}_{p+q-1}(\beta)$$

and the Gerstenhaber bracket is the graded commutator for this composition, so that

$$[\alpha, \beta] = \alpha \diamond \beta - (-1)^{(p-1)(q-1)} \beta \diamond \alpha.$$

Next, if α and β are now a p- and a q-cochain in the complex $\hom_{A^e}(\mathbf{P}, A)$, we can lift them to a p-cochain $\tilde\alpha = \alpha \circ \psi_p$ and a q-cochain $\tilde\beta = \beta \circ \psi_q$ in the standard complex $\hom_{A^e}(\mathcal{B}A, A)$, and the Gerstenhaber bracket of the classes of α and β is then represented by the $(p+q-1)$-cochain $[\tilde\alpha, \tilde\beta] \circ \phi_{p+q-1}$. This is the computation we have to do in order to compute brackets in $HH^\bullet(A)$, except that in some favorable circumstances we can take advantage of the compatibility of the bracket with the product to cut down the work. We do this in several steps.

- Since the morphism ψ is normalized and $HH^0(A)$ is spanned by $1 \in \Bbbk$, it follows immediately that

$$[HH^0(A), HH^\bullet(A)] = 0.$$

- The Gerstenhaber bracket on $HH^1(A)$ is induced by the commutator of derivations. From Proposition **3.18** we have a basis of $HH^1(A)$ whose elements are classes of certain derivations, and it is immediate to check that those derivations commute, so that

$$[HH^1(A), HH^1(A)] = 0. \tag{3.13}$$

- We know that the subspace $S_r \otimes \hat D \wedge \hat E$ of $HH^2(A)$ is $HH^1(A) \smile HH^1(A)$. Since $HH^\bullet(A)$ is a Gerstenhaber algebra and we now that (3.13) holds, it follows that

$$[HH^1(A), S_r \otimes \hat D \wedge \hat E] = 0.$$

  For exactly the same reasons we also have that

$$[S_r \otimes \hat D \wedge \hat E, S_r \otimes \hat D \wedge \hat E] = 0.$$

- Let $\alpha = u \otimes \hat D + \lambda \otimes \hat E$, with $u \in S_r$ and $\lambda \in \Bbbk$. If $\beta = (v + wD) \otimes \hat y \wedge \hat D$, with $v \in S_{r+1}$ and $w \in S_1$, one can compute that $(\tilde\alpha \diamond \tilde\beta) \circ \phi = uw \otimes \hat y \wedge \hat D$ and that $(\tilde\beta \diamond \tilde\alpha) \circ \phi = 0$: it follows from this that

$$[\alpha, (v + wD) \otimes \hat y \wedge \hat D] = uw \otimes \hat y \wedge \hat D.$$

  On the other hand, we have $(\tilde\omega_2 \diamond \tilde\alpha) \circ \phi = 0$ and

$$\begin{aligned}[][\tilde\alpha, \tilde\omega_2] \circ \phi = (\tilde\alpha \diamond \tilde\omega_2) \circ \phi &= (yu - \lambda y^{r+1}) \otimes \hat x \wedge \hat D + \lambda y \bar F \otimes \hat y \wedge \hat D \\ &= \Big((\mu - \lambda) y F_x + \mu y \bar F - y^2 \bar u\Big) \otimes \hat y \wedge \hat D \\ &\qquad - \delta^1\Big(((\mu - \lambda)\bar F - y\bar u)E \otimes \hat D + (\lambda - \mu) y \otimes \hat x\Big)\end{aligned}$$

  with $\bar u \in S_{r-1}$ and $\mu \in \Bbbk$ chosen so that $u = \mu y^r + x\bar u$.

  Finally, if $v \in S_{r+1}$ and $w \in S_1$, using the compatibility of the bracket and the product and what we know so far we see that

$$\begin{aligned}[][\alpha, (v + wD) \otimes \hat y \wedge \hat D \wedge \hat E] &= [\alpha, 1 \otimes E \smile (v + wD) \otimes \hat y \wedge \hat D] \\ &= 1 \otimes E \smile [\alpha, (v + wD) \otimes \hat y \wedge \hat D]\end{aligned}$$

$$\begin{aligned} &= 1 \otimes E \smile uw \otimes \hat{y} \wedge \hat{D} \\ &= uw \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \end{aligned}$$

and, similarly, that

$$\begin{aligned} [\alpha, \omega_3] &= [\alpha, \omega_2 \smile 1 \otimes \hat{E}] = [\alpha, \omega_2] \smile 1 \otimes \hat{E} + \omega_2 \smile [\alpha, 1 \otimes \hat{E}] \\ &= \Big((\mu - \lambda)yF_x + \mu y\bar{F} - y^2\bar{u}\Big) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}. \end{aligned}$$

- Let $u \in S_r$. If $v \in S_{r+1}$ and $w \in S_1$, we have

$$\begin{aligned} &[u \otimes \hat{D} \wedge \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D}] = [u \otimes \hat{D} \smile 1 \otimes \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D}] \\ &\quad = [u \otimes \hat{D}, (v + wD) \otimes \hat{y} \wedge \hat{D}] \smile 1 \otimes \hat{E} \\ &\qquad\qquad + u \otimes \hat{D} \smile [1 \otimes \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D}] \\ &\quad = uw \otimes \hat{y} \wedge \hat{D} \smile 1 \otimes \hat{E} = uw \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}. \end{aligned}$$

  Similarly,

$$\begin{aligned} [u \otimes \hat{D} \wedge \hat{E}, \omega_2] &= [u \otimes \hat{D} \smile 1 \otimes \hat{E}, \omega_2] \\ &= [u \otimes \hat{D}, \omega_2] \smile 1 \otimes \hat{E} + u \otimes \hat{D} \smile [1 \otimes \hat{E}, \omega_2] \\ &= \Big(\mu y F_x + \mu y\bar{F} - y^2\bar{u}\Big) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}. \end{aligned}$$

  if $u = \mu y^r + x\bar{u}$ with $\mu \in \Bbbk$ and $\bar{u} \in S_{r-1}$.
- Let now $\alpha = (v + wD) \otimes \hat{y} \wedge \hat{D}$ and $\beta = (s + tD) \otimes \hat{y} \wedge \hat{D}$, with $v, s \in S_{r+1}$ and $w, t \in S_1$. We claim that $(\tilde{\alpha} \diamond \tilde{\beta}) \circ \phi = 0$, so that, by symmetry, we have $[\tilde{\alpha}, \tilde{\beta}] \circ \phi = 0$. To verify our claim, we compute:

$$\begin{aligned} &1|x \wedge y \wedge E|1 \overset{\phi}{\longmapsto} \boxed{\Bbbk[x, y, E]^{\otimes 5}} \xmapsto{\mathsf{w}_3(\tilde{\beta})} 0; \\ &1|x \wedge D \wedge E|1 \overset{\phi}{\longmapsto} \boxed{\Bbbk[x, D, E]^{\otimes 5}} \xmapsto{\mathsf{w}_3(\tilde{\beta})} 0; \\ &1|x \wedge y \wedge D|1 \overset{\phi}{\longmapsto} 1|x|y|D|1 - 1|x|D|y|1 + 1|D|x|y|1 \\ &\qquad\qquad - 1|D|y|x|1 + 1|y|D|x|1 - 1|y|x|D|1 + \boxed{S^{\otimes 5}} \\ &\qquad \xmapsto{\mathsf{w}_3(\tilde{\beta})} 1|(s + tD)|x|1 - 1|x|(s + tD)|1 \\ &\qquad \overset{\psi}{\longmapsto} -s_{(1)}|x \wedge s_{(2)}|s_{(3)} - t_{(1)}|x \wedge t_{(2)}|t_{(3)}D - t|x \wedge D|1 \\ &\qquad \overset{\alpha}{\longmapsto} 0; \\ &1|y \wedge D \wedge E|1 \overset{\phi}{\longmapsto} 1|y|D|E|1 - 1|y|E|D|1 + 1|E|y|D|1 \\ &\qquad\qquad - 1|E|D|y|1 + 1|D|E|y|1 - 1|D|y|E|1 + \boxed{\Bbbk[x, y, E]^{\otimes 5}} \\ &\qquad \xmapsto{\mathsf{w}_3(\tilde{\beta})} 1|(s + tD)|E|1 - 1|E|(s + tD)|1 \end{aligned}$$

$$\begin{aligned}
&\overset{\psi}{\longmapsto} s_{(1)}|s_{(2)} \wedge E|s_{(3)} + t_{(1)}|t_{(2)} \wedge E|t_{(3)}D + t|D \wedge E|1 \\
&\overset{\alpha}{\longmapsto} 0.
\end{aligned}$$

- Let again $\alpha = (v + wD) \otimes \hat{y} \wedge \hat{D}$, with $v \in S_{r+1}$ and $w \in S_1$, and let us compute that $(\tilde{\omega}_2 \diamond \tilde{\alpha}) \circ \phi_3 = -w(yD - y^{r+1}E) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}$.

$$\begin{aligned}
1|x \wedge y \wedge z|1 &\overset{\phi_3}{\longmapsto} \boxed{\Bbbk[x, y, E]^{\otimes 5}} \xmapsto{\mathsf{w}_2(\tilde{\alpha})} 0 \\
1|x \wedge D \wedge E|1 &\overset{\phi_3}{\longmapsto} \boxed{\Bbbk[x, D, E]^{\otimes 5}} \xmapsto{\mathsf{w}_3(\tilde{\alpha})} 0 \\
1|x \wedge y \wedge D|1 &\overset{\phi_3}{\longmapsto} 1|x|y|D|1 - 1|x|D|y|1 + 1|D|x|y|1 \\
&\qquad\qquad - 1|D|y|x|1 + 1|y|D|x|1 - 1|y|x|D|1 + \boxed{S^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\tilde{\alpha})} 1|(v + wD)|x|1 + 1|x|(v + wD)|1 \\
&\overset{\psi_2}{\longmapsto} -v_{(1)}|x \wedge v_{(2)}|v_{(3)} - w_{(1)}|x \wedge w_{(2)}|w_{(3)}D - w|x \wedge D|1 \\
&\overset{\omega_2}{\longmapsto} -w(yD - y^{r+1}E) \\
1|y \wedge D \wedge E|1 &\overset{\phi_3}{\longmapsto} 1|y|D|E|1 - 1|y|E|D|1 + 1|E|y|D|1 \\
&\qquad - 1|E|D|y|1 + 1|D|E|y|1 - 1|D|y|E|1 + \boxed{\Bbbk[x, y, E]^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\tilde{\alpha})} 1|(v + wD)|E|1 - 1|E|(v + wD)|1 \\
&\overset{\psi_2}{\longmapsto} v_{(1)}|v_{(2)} \wedge E|v_{(3)} + w_{(1)}|w_{(2)} \wedge E|w_{(3)}D + w|D \wedge E|1 \\
&\overset{\omega_2}{\longmapsto} 0.
\end{aligned}$$

Similarly, we have that $(\tilde{\alpha} \diamond \tilde{\omega}_2) \circ \phi_3 = y(v + wD) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}$:

$$\begin{aligned}
1|x \wedge y \wedge z|1 &\overset{\phi_3}{\longmapsto} \boxed{\Bbbk[x, y, E]^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_2(\tilde{\omega}_2)} 0 \\
1|x \wedge D \wedge E|1 &\overset{\phi_3}{\longmapsto} 1|x|D|E|1 - 1|x|E|D|1 + 1|E|x|D|1 \\
&\qquad\qquad - 1|E|D|x|1 + 1|D|E|x|1 - 1|D|x|E|1 \\
&\xmapsto{\mathsf{w}_3(\tilde{\omega}_2)} -1|E|(yD - y^{r+1}E)|1 + 1|(yD - y^{r+1})|E|1 \\
&\overset{\psi_2}{\longmapsto} -1|y \wedge E|D - y|D \wedge E|1 + \sum_{i=0}^{r} y^i|y \wedge E|y^{r-i} \\
&\overset{\alpha}{\longmapsto} 0 \\
1|x \wedge y \wedge D|1 &\overset{\phi_3}{\longmapsto} 1|x|y|D|1 - 1|x|D|y|1 + 1|D|x|y|1 \\
&\qquad - 1|D|y|x|1 + 1|y|D|x|1 - 1|y|x|D|1 + \boxed{S^{\otimes 5}}
\end{aligned}$$

$$
\begin{aligned}
&\xmapsto{\mathsf{w}_3(\tilde\omega_2)} -1|x|y\bar F E|x|1 - 1|(yD - y^{r+1}E)|y|1 \\
&\qquad\qquad + 1|y\bar F E|x|1 + 1|y|(yD - y^{r+1}E)|1 \\
&\xmapsto{\psi_2} y|y \wedge D|1 - y^{r+1}|y \wedge E|1 \\
&\qquad\qquad - (y\bar F E)_{(1)}|x \wedge (y\bar F E)_{(2)}|(y\bar F E)_{(3)} \\
&\xmapsto{\alpha} -y(v + wD)
\end{aligned}
$$

$$
\begin{aligned}
1|y \wedge D \wedge E|1 &\xmapsto{\phi_3} 1|y|D|E|1 - 1|y|E|D|1 + 1|E|y|D|1 \\
&\qquad - 1|E|D|y|1 + 1|D|E|y|1 - 1|D|y|E|1 + \boxed{\Bbbk[x, y, E]^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\tilde\omega_2)} -1|E|y\bar F E|1 + 1|y\bar F E|E|1 \\
&\xmapsto{\psi_2} (y\bar F E)_{(1)}|(y\bar F E)_{(2)} \wedge E|(y\bar F E)_{(3)} \\
&\xmapsto{\alpha} 0.
\end{aligned}
$$

It follows from this that

$$
\begin{aligned}
[\tilde\omega_2, \tilde\alpha] \circ \phi_3 &= -w(yD - y^{r+1}E) \otimes \hat x \wedge \hat y \wedge \hat D + y(v + wD) \otimes \hat x \wedge \hat y \wedge \hat D \\
&= (yv + y^{r+1}E) \otimes \hat x \wedge \hat y \wedge \hat D
\end{aligned}
$$

and, as we say in **3.13**, this is a coboundary.

- The one computation that remains is that of the bracket of ω_2 with itself, which is represented by the 3-cocycle

$$
[\tilde\omega_2, \tilde\omega_2] \circ \phi_3 = 2(\tilde\omega_2 \diamond \tilde\omega_2) \circ \phi_3 = 2y^2 \bar F E \otimes \hat x \wedge \hat y \wedge \hat D, \tag{3.14}
$$

as can be seen from the following calculation:

$$
\begin{aligned}
1|x \wedge y \wedge z|1 &\xmapsto{\phi_3} \boxed{\Bbbk[x, y, E]^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_2(\tilde\omega_2)} 0
\end{aligned}
$$

$$
\begin{aligned}
1|x \wedge D \wedge E|1 &\xmapsto{\phi_3} 1|x|D|E|1 - 1|x|E|D|1 + 1|E|x|D|1 \\
&\qquad\qquad - 1|E|D|x|1 + 1|D|E|x|1 - 1|D|x|E|1 \\
&\xmapsto{\mathsf{w}_3(\tilde\omega_2)} -1|E|(yD - y^{r+1}E)|1 + 1|(yD - y^{r+1})|E|1 \\
&\xmapsto{\psi_2} -1|y \wedge E|D - y|D \wedge E|1 + \sum_{i=0}^{r} y^i|y \wedge E|y^{r-i} \\
&\xmapsto{\omega_2} 0
\end{aligned}
$$

$$
\begin{aligned}
1|x \wedge y \wedge D|1 &\xmapsto{\phi_3} 1|x|y|D|1 - 1|x|D|y|1 + 1|D|x|y|1 \\
&\qquad - 1|D|y|x|1 + 1|y|D|x|1 - 1|y|x|D|1 + \boxed{S^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\tilde\omega_2)} -1|x|y\bar F E|x|1 - 1|(yD - y^{r+1}E)|y|1 \\
&\qquad\qquad + 1|y\bar F E|x|1 + 1|y|(yD - y^{r+1}E)|1
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\psi_2}{\longmapsto} y|y\wedge D|1 - y^{r+1}|y\wedge E|1 \\
&\qquad\qquad\qquad\qquad - (y\bar{F}E)_{(1)}|x\wedge(y\bar{F}E)_{(2)}|(y\bar{F}E)_{(3)} \\
&\overset{\omega_2}{\longmapsto} -y^2\bar{F}E \\
1|y\wedge D\wedge E|1 &\overset{\phi_3}{\longmapsto} 1|y|D|E|1 - 1|y|E|D|1 + 1|E|y|D|1 \\
&\qquad - 1|E|D|y|1 + 1|D|E|y|1 - 1|D|y|E|1 + \boxed{\Bbbk[x,y,E]^{\otimes 5}} \\
&\xrightarrow{\mathsf{w}_3(\tilde{\omega}_2)} -1|E|y\bar{F}E|1 + 1|y\bar{F}E|E|1 \\
&\overset{\psi_2}{\longmapsto} (y\bar{F}E)_{(1)}|(y\bar{F}E)_{(2)}\wedge E|(y\bar{F}E)_{(3)} \\
&\overset{\omega_2}{\longmapsto} 0.
\end{aligned}
$$

Now the 3-cocycle (3.14) is a coboundary, again by what we saw in **3.13**, so that the class of ω_2 has bracket-square zero.

This completes the proof of the proposition. □

3.4 Hochschild homology, cyclic homology and K-theory

3.23. For completeness, we determine the rest of the 'usual' homological invariants of our algebra A. Recall that our ground field $\Bbbk$ is of characteristic zero.

Proposition. *The inclusion $T = \Bbbk[E] \to A$ induces an isomorphism in Hochschild homology and in cyclic homology. In particular, there are isomorphisms of vector spaces*

$$
HH_i(A) \cong \begin{cases} T, & \text{if } i = 0 \text{ or } i = 1; \\ 0, & \text{if } i \geq 2; \end{cases}
\qquad\qquad
HC_i(A) \cong \begin{cases} T, & \text{if } i = 0; \\ HC_i(\Bbbk), & \text{if } i > 0. \end{cases}
$$

On the other hand, the inclusion $\Bbbk \to A$ induces an isomorphism in periodic cyclic homology and in higher K-theory.

Proof. As we know, the algebra A is $\mathbb{N}_0$-graded and for each $n \in \mathbb{N}_0$ its homogeneous component A_n of degree n is the eigenspace corresponding to the eigenvalue n of the derivation $\mathsf{ad}(E) : A \to A$. On one hand, this grading of A induces as usual an $\mathbb{N}_0$-grading on the Hochschild homology $HH_\bullet(A)$ of A; on the other, the derivation $\mathsf{ad}(E)$ induces a linear map $L_{\mathsf{ad}(E)} : HH_\bullet(A) \to HH_\bullet(A)$ as in [Lod92, §4.1.4] and, in fact, for all $n \in \mathbb{N}_0$ the homogeneous component $HH_\bullet(A)_n$ of degree n for that grading coincides with the eigenspace corresponding to the eigenvalue n of $L_{\mathsf{ad}(E)}$. As the derivation $\mathsf{ad}(E)$ is inner, it follows from [Lod92, Proposition 4.1.5] that the map $L_{\mathsf{ad}(E)}$ is actually the zero map and this tells us in our situation that $HH_\bullet(A)_n = 0$ for all $n \neq 0$. Of course, this means that $HH_\bullet(A) = HH_\bullet(A)_0$ and, since A is non-negatively graded, it is immediate that the 0th homogeneous component $HH_\bullet(A)_0$ coincides with the Hochschild homology $HH_\bullet(A_0)$ of A_0 and that the map $HH_\bullet(A_0) \to HH_\bullet(A)$ induced by the inclusion $A_0 \hookrightarrow A$ is an isomorphism. Now, in the notation of [Lod92, Theorem

4.1.13], this tells us that $\overset{\approx}{HH}_\bullet(A) = 0$ so that, by that theorem, we also have $\overset{\approx}{HC}_\bullet(A) = 0$: this means precisely that the inclusion $A_0 \hookrightarrow A$ induces an isomorphism $HC_\bullet(A_0) \to HC_\bullet(A)$ in cyclic homology. Together with the well-known computation of the Hochschild homology of a polynomial ring and that of the cyclic homology of symmetric algebras [Lod92, Theorem 3.2.5], this proves the first claim of the statement.

In the proof of the lemma of **3.5** we constructed an increasing filtration F on the algebra A with $F_{-1}A = 0$ and such that the corresponding graded algebra is the commutative polynomial algebra $\operatorname{gr} A = \Bbbk[x, y, D, E]$ with generators x and y in degree 0 and D and E in degree 1. In particular, both $\operatorname{gr} A$ and its subalgebra $\operatorname{gr}_0 A$ of degree 0 have finite global dimension. It follows from a theorem of D. Quillen [Qui73, p. 117, Theorem 7] that the inclusion $\Bbbk[x, y] = F_0A \to A$ induces an isomorphism $K_i(\Bbbk[x, y]) \to K_i(A)$ in K-theory for all $i \geq 0$. Similarly, the theorem of J. Block [Blo87, Theorem 3.4] tells us that that inclusion induces an isomorphism $HP_\bullet(\Bbbk[x, y]) \to HP_\bullet(A)$ in periodic cyclic homology. As the inclusion $\Bbbk \to \Bbbk[x, y]$ induces an isomorphism in K-theory and in periodic cyclic homology, we see that the second claim of the proposition holds. □

3.5 THE TWISTED CALABI–YAU PROPERTY

3.24. The enveloping algebra A^e of A is a bimodule over itself, with left and right actions $\triangleright$ and $\triangleleft$ given by 'outer' and 'inner' multiplication, respectively, so that if $a \otimes b$, $c \otimes d$ and $e \otimes f$ are elementary tensors in A^e, we have

$$a \otimes b \triangleright c \otimes d \triangleleft e \otimes f = ace \otimes fdb.$$

From this bimodule structure we obtain a duality functor

$$\mathsf{hom}_{A^e}(-, A^e) : {}_{A^e}\mathsf{Mod} \to \mathsf{Mod}_{A^e}.$$

On the other hand, using the anti-automorphism $\tau : A^e \to A^e$ such that $\tau(a \otimes b) = b \otimes a$ for all $a, b \in A$, we can turn a right A^e-module M into a left A^e-module, with action $u \triangleright m = m \triangleleft \tau(u)$ for $u \in A^e$ and $m \in M$. In this way, we obtain an isomorphism of categories $\tau^* : \mathsf{Mod}_{A^e} \to {}_{A^e}\mathsf{Mod}$. We denote $(-)^\vee : {}_{A^e}\mathsf{Mod} \to {}_{A^e}\mathsf{Mod}$ the composition $\tau^* \circ \mathsf{hom}_{A^e}(-, A^e)$.

Let now W be a finite dimensional vector space, let W^* be the vector space dual to W, and view $A \otimes W \otimes A$ and $A \otimes W^* \otimes A$ as left A^e-modules using the usual 'exterior' action. There is a unique $\Bbbk$-linear map

$$\Phi : A \otimes W^* \otimes A \to (A \otimes W \otimes A)^\vee$$

such that $\Phi(a \otimes \phi \otimes b)(1 \otimes w \otimes 1) = \phi(w)b \otimes a$ and it is an isomorphism of left A^e-modules: we will view it in all that follows as an identification.

Notice that we have already proved in **2.25** that for any free hyperplane arrangement the algebra of differential operators is twisted Calabi–Yau. Since in the case of a line arrangement the algebra $\mathsf{Diff}(\mathcal{A})$ is an iterated Ore extension —as we have shown in Lemma 2.8—, we can also deduce this fact using the results by L. Liu, S. Wang and Q. Wu in [LWW14]. We prefer to give a straightforward, computational proof, as the isomorphism of complexes that intervenes in it is useful when one tries to explicit the duality between homology and cohomology described in **2.23**.

3.25. Proposition. *The algebra A is twisted Calabi-Yau of dimension 4 with modular automorphism $\sigma : A \to A$ such that*

$$\sigma(x) = x, \qquad \sigma(y) = y, \qquad \sigma(D) = D + F_y, \qquad \sigma(E) = E + r + 2.$$

Let us recall from Section 2.4 that this means that A has a resolution of finite length by finitely generated projective A-bimodules, that $\mathsf{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq 4$ and that $\mathsf{Ext}^4_{A^e}(A, A^e) \cong A_\sigma$, the A-bimodule obtained from A by twisting its right action using the automorphism σ, so that $a \triangleright x \triangleleft b = ax\sigma(b)$ for all $a, b \in A$ and all $x \in A_\sigma$.

Proof. A direct computation shows that there is indeed an automorphism σ of A as in the statement of the proposition. We already know that A has a resolution $\mathbf{P}$ of length 4 by finitely generated free A-bimodules, so we need only compute $\mathsf{Ext}^\bullet_{A^e}(A, A^e)$, and this is the cohomology of the complex $\mathbf{P}^\vee$ obtained by applying the functor described in **3.24** to $\mathbf{P}$. Using the identifications introduced there, this complex $\mathbf{P}^\vee$ is

$$A \otimes A \xrightarrow{d_1^\vee} A \otimes V^* \otimes A \xrightarrow{d_2^\vee} A \otimes \Lambda^2 V^* \otimes A \xrightarrow{d_3^\vee} A \otimes \Lambda^3 V^* \otimes A \xrightarrow{d_4^\vee} A \otimes \Lambda^4 V^* \otimes A$$

with left A^e-linear differentials such that

$$\begin{aligned}
d_1^\vee(1 \otimes 1) &= -[x, 1 \otimes \hat{x} \otimes 1] - [y, 1 \otimes \hat{y} \otimes 1] - [D, 1 \otimes \hat{D} \otimes 1] - [E, 1 \otimes \hat{E} \otimes 1];\\
d_2^\vee(1 \otimes \hat{x} \otimes 1) &= [y, 1 \otimes \hat{x} \wedge \hat{y} \otimes 1] + [D, 1 \otimes \hat{x} \wedge \hat{D} \otimes 1] + [E, 1 \otimes \hat{x} \wedge \hat{E} \otimes 1]\\
&\qquad + 1 \otimes \hat{x} \wedge \hat{E} \otimes 1 + \tilde{\nabla}_x^{\hat{y} \wedge \hat{D}}(F);\\
d_2^\vee(1 \otimes \hat{y} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{y} \otimes 1] + [D, 1 \otimes \hat{y} \wedge \hat{D} \otimes 1] + [E, 1 \otimes \hat{y} \wedge \hat{E} \otimes 1]\\
&\qquad + 1 \otimes \hat{y} \wedge \hat{E} \otimes 1 + \tilde{\nabla}_y^{\hat{y} \wedge \hat{D}}(F);\\
d_2^\vee(1 \otimes \hat{D} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{D} \otimes 1] - [y, 1 \otimes \hat{y} \wedge \hat{D} \otimes 1] + [E, 1 \otimes \hat{D} \wedge \hat{E} \otimes 1]\\
&\qquad + r \otimes \hat{D} \wedge \hat{E} \otimes 1;\\
d_2^\vee(1 \otimes \hat{E} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{E} \otimes 1] - [y, 1 \otimes \hat{y} \wedge \hat{E} \otimes 1] - [D, 1 \otimes \hat{D} \wedge \hat{E} \otimes 1];\\
d_3^\vee(1 \otimes \hat{x} \wedge \hat{y} \otimes 1) &= -[D, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \otimes 1] - \tilde{\nabla}_y^{\hat{x} \wedge \hat{y} \wedge \hat{D}}(F)\\
&\qquad - [E, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \otimes 1] - 2 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \otimes 1;\\
d_3^\vee(1 \otimes \hat{x} \wedge \hat{D} \otimes 1) &= [y, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \otimes 1] - [E, 1 \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \otimes 1]\\
&\qquad - (r+1) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \otimes 1;\\
d_3^\vee(1 \otimes \hat{x} \wedge \hat{E} \otimes 1) &= [y, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \otimes 1] + [D, 1 \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \otimes 1]\\
&\qquad + \tilde{\nabla}_x^{\hat{y} \wedge \hat{D} \wedge \hat{E}}(F);
\end{aligned}$$

$$
\begin{aligned}
d_3^\vee(1\otimes \hat{y}\wedge \hat{D}\otimes 1) &= -[x, 1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\otimes 1] - [E, 1\otimes \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1] \\
&\qquad\qquad - (r+1)\otimes \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1; \\
d_3^\vee(1\otimes \hat{y}\wedge \hat{E}\otimes 1) &= -[x, 1\otimes \hat{x}\wedge \hat{y}\wedge \hat{E}\otimes 1] + [D, 1\otimes \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1] + \tilde{\nabla}_y^{\hat{y}\wedge\hat{D}\wedge\hat{E}}(F); \\
d_3^\vee(1\otimes \hat{D}\wedge \hat{E}\otimes 1) &= -[x, 1\otimes \hat{x}\wedge \hat{D}\wedge \hat{E}\otimes 1] - [y, 1\otimes \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1]; \\
d_4^\vee(1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\otimes 1) &= [E, 1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1] \\
&\qquad\qquad + (r+2)\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1; \\
d_4^\vee(1\otimes \hat{x}\wedge \hat{y}\wedge \hat{E}\otimes 1) &= -[D, 1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1] - \tilde{\nabla}_y^{\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}}(F); \\
d_4^\vee(1\otimes \hat{x}\wedge \hat{D}\wedge \hat{E}\otimes 1) &= [y, 1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1]; \\
d_4^\vee(1\otimes \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1) &= -[x, 1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1],
\end{aligned}
$$

where each $\tilde{\nabla}_x^u$ is the image of ∇_x^u under the map $a\otimes u\otimes b \mapsto b\otimes u\otimes a$, and the same with each $\tilde{\nabla}_y^u$.

Let us now identify $\mathbf{P}\otimes_A A_\sigma$ with $\mathbf{P}$ as vector spaces, remembering that the bimodule structure on $\mathbf{P}$ with this identification is given by $a \triangleright x \triangleleft b = ax\sigma(b)$ for all $a, b \in A$ and all $x \in \mathbf{P}$. There is a morphism of complexes of A-bimodules $\psi : \mathbf{P}^\vee \to \mathbf{P}\otimes_A A_\sigma$ such that

$$
\begin{aligned}
\psi(1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1) &= 1\otimes 1; \\
\psi(1\otimes \hat{y}\wedge \hat{D}\wedge \hat{E}\otimes 1) &= -1\otimes x\otimes 1; \\
\psi(1\otimes \hat{x}\wedge \hat{D}\wedge \hat{E}\otimes 1) &= 1\otimes y\otimes 1; \\
\psi(1\otimes \hat{x}\wedge \hat{y}\wedge \hat{E}\otimes 1) &= -1\otimes D\otimes 1 - \xi; \\
\psi(1\otimes \hat{x}\wedge \hat{y}\wedge \hat{D}\otimes 1) &= 1\otimes E\otimes 1; \\
\psi(1\otimes \hat{D}\wedge \hat{E}\otimes 1) &= -1\otimes x\wedge y\otimes 1; \\
\psi(1\otimes \hat{x}\wedge \hat{D}\otimes 1) &= 1\otimes y\wedge E\otimes 1; \\
\psi(1\otimes \hat{y}\wedge \hat{D}\otimes 1) &= -1\otimes x\wedge E\otimes 1; \\
\psi(1\otimes \hat{y}\wedge \hat{E}\otimes 1) &= 1\otimes x\wedge D\otimes 1 + x\wedge\xi; \\
\psi(1\otimes \hat{x}\wedge \hat{E}\otimes 1) &= -1\otimes y\wedge D\otimes 1 + \zeta; \\
\psi(1\otimes \hat{x}\wedge \hat{y}\otimes 1) &= -1\otimes D\wedge E\otimes 1 - \xi\wedge E; \\
\psi(1\otimes \hat{E}\otimes 1) &= 1\otimes x\wedge y\wedge D\otimes 1; \\
\psi(1\otimes \hat{D}\otimes 1) &= -1\otimes x\wedge y\wedge E\otimes 1; \\
\psi(1\otimes \hat{y}\otimes 1) &= 1\otimes x\wedge D\wedge E\otimes 1 + x\wedge\xi\wedge E; \\
\psi(1\otimes \hat{x}\otimes 1) &= -1\otimes y\wedge D\wedge E\otimes 1 + \zeta\wedge E; \\
\psi(1\otimes 1) &= 1\otimes x\wedge y\wedge D\wedge E\otimes 1,
\end{aligned}
$$

where $\xi \in A\otimes V\otimes A$ and $\zeta \in A\otimes \Lambda^2 V\otimes A$ are chosen so that

$$
d_1(\xi) = \tilde{\nabla}_y(F) - 1|F_y, \qquad\qquad d_2(\zeta) = \xi y - y\xi - 1|y|F_y - \tilde{\nabla}_x^x(F) + \nabla(F).
$$

That there are elements which satisfy these two conditions follows immediately from the exactness of the Koszul resolution of S as an S-bimodule —indeed, the right hand sides of the two conditions are cycles in that complex— but we can exhibit a specific choice: if we write $F = \sum_{a+b=r+1} c_a x^a y^b$, with $c_0, \dots, c_{r-1} \in \Bbbk$, then we can pick

$$\xi = \sum_{\substack{a+b=r+1 \\ s+t+1=b-1}} (t+1) c_a y^s |y| x^a y^t, \qquad \zeta = \sum_{\substack{a+b=r+1 \\ s+t+1=b \\ s'+t'+1=a}} c_a x^{s'} y^s | x \wedge y | x^{t'} y^t.$$

That these formulas for ψ do indeed define a morphism of complexes follows from a direct computation and it is easy to see that it is in fact an isomorphism, as for an appropriate ordering of the bases of the bimodules involved the matrices for the components of ψ are upper triangular. Of course, it therefore induces an isomorphism in cohomology and, since A_σ is A-projective on the left, we conclude that there are isomorphisms of A-bimodules

$$H^i(\mathbf{P}^\vee) \cong H^i(\mathbf{P} \otimes_A A_\sigma) \cong \begin{cases} A_\sigma & \text{if } i = 4; \\ 0 & \text{if } i > 0. \end{cases}$$

This completes the proof. □

3.6 Resumen

En este capítulo nos enfocamos en el estudio del álgebra $\mathsf{Diff}(\mathcal{A})$ para el caso de un arreglo $\mathcal{A}$ en un espacio vectorial V de dimensión 2 con al menos cinco rectas: así son los arreglos libres más simples. Recordemos del Capítulo 2 que ya disponemos de una presentación para $\mathsf{Diff}(\mathcal{A})$ en **2.8**: escribiendo al polinomio que define al arreglo como $Q = xF$, vemos que $\mathsf{Diff}(\mathcal{A})$ es el álgebra generada por las letras x, y, E y D de manera que se satisfacen las relaciones de conmutación

$$\begin{aligned} &[y, x] = 0, && && \\ &[D, x] = 0, && [D, y] = F, && \\ &[E, x] = x, && [E, y] = y, && [E, D] = (l-2)D, \end{aligned}$$

en donde l es la cantidad de rectas de $\mathcal{A}$. Encontramos en la Sección 3.1 una resolución proyectiva de $\mathsf{Diff}(\mathcal{A})$ como bimódulo sobre si mismo para, después de un cálculo extenso, dar en la Proposición 3.15 una descripción de la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ de manera completamente explícita. Sin entrar en detalles, la proposición nos da la siguiente información.

Proposición. *Si $\mathcal{A}$ es un arreglo central de rectas de l rectas con $l \geq 5$, la serie de Hilbert de $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ es*

$$h_{HH^\bullet(U)}(t) = 1 + lt + (2l-1)t^2 + lt^3.$$

Cuando el arreglo tiene menos de cinco rectas, la conclusión de la proposición no sigue siendo cierta: lidiamos con esta situación especial utilizando técnicas diferentes sobre el final de la tesis.

El siguiente paso es describir la estructura de álgebra de $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ y su estructura de Gerstenhaber: es para esto que necesitamos una descripción tan explícita. Los resultados aparecen en las Proposiciones 3.19 y 3.22 y son, una vez más, demasiado técnicos para reproducir aquí. De cualquier manera, estas estructuras nos dan un mejor entendimiento de nuestros cálculos previos y nos permiten relacionar $\mathsf{Diff}(\mathcal{A})$ con un invariante conocido del arreglo, el álgebra de Orlik–Solomon. Ésta aparece en nuestra situación en la Proposición 3.20:

Proposición. *La subálgebra $\mathcal{H}$ de $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ generada por $HH^1(\mathsf{Diff}(\mathcal{A}))$ es isomorfa al álgebra de Orlik–Solomon de $\mathcal{A}$.*

Además de estos resultados, calculamos en la Proposición 3.23 la homología de Hochschild, la homología cíclica y la homología cíclica periódica y la K-teoría de $\mathsf{Diff}(\mathcal{A})$.

Proposición. *La inclusión $T = \Bbbk[E] \to \mathsf{Diff}(\mathcal{A})$ induce un isomorfismo en homología de Hochschild y homología cíclica. En particular, hay isomorfismos de espacios vectoriales*

$$HH_i(\mathsf{Diff}(\mathcal{A})) \cong \begin{cases} T, & \text{si } i = 0 \text{ o } i = 1; \\ 0, & \text{si } i \geq 2; \end{cases} \qquad HC_i(\mathsf{Diff}(\mathcal{A})) \cong \begin{cases} T, & \text{si } i = 0; \\ HC_i(\Bbbk), & \text{si } i > 0. \end{cases}$$

Más aún, la inclusión $\Bbbk \to \mathsf{Diff}(\mathcal{A})$ induce un isomorfismo en homología cíclica periódica y en K-teoría superior.

Para terminar el capítulo, obtenemos en la Sección 3.5 una prueba *directa* de la propiedad de Calabi–Yau para el caso especial de arreglos centrales de rectas.

Proposición. *El álgebra $\mathsf{Diff}(\mathcal{A})$ es twisted Calabi-Yau torcida de dimensión 4 con automorfismo modular $\sigma : \mathsf{Diff}(\mathcal{A}) \to \mathsf{Diff}(\mathcal{A})$ dado por*

$$\sigma(x) = x, \qquad \sigma(y) = y, \qquad \sigma(D) = D + F_y, \qquad \sigma(E) = E + l.$$

– 4 –
Automorphisms of $\mathsf{Diff}(\mathcal{A})$ and the isomorphism problem

In this chapter we continue with the study of the algebra $A = \mathsf{Diff}(\mathcal{A})$ of differential operators tangent to a central arrangement of lines $\mathcal{A}$ that we started in Chapter 3, bearing in mind that the arrangement has $r+2$ lines and that r is at least 3. We take advantage of the explicitness of the calculation of the first Hochschild cohomology group of A and employ the methods developed by J. Alev and M. Chamarie in [AC92] to give a description of the group of automorphisms of A: we show that $\mathsf{Aut}(A)$ is the semidirect product of the subgroup of homogeneous automorphisms of degree 0 and that of the exponentials of locally ad-nilpotent elements. With this description at hand, we solve the problem of determining which pairs of arrangements $\mathcal{A}$ and $\mathcal{A}'$ have isomorphic algebras $\mathsf{Diff}(\mathcal{A})$ and $\mathsf{Diff}(\mathcal{A}')$ and, in particular, we show that the arrangement $\mathcal{A}$ can be recovered from the algebra $\mathsf{Diff}(\mathcal{A})$.

4.1 Automorphisms

4.1. Our next objective is to compute the group of automorphisms of the algebra A. We start by describing some graded automorphisms of A. Later we will see that these are, in fact, *all* the graded automorphisms of our algebra, and that together with the exponentials of locally ad-nilpotent elements they generate the whole group $\mathsf{Aut}(A)$.

Lemma. *If* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{GL}_2(\Bbbk)$ *and* $e \in \Bbbk^\times$ *are such that*

$$\frac{1}{(ad-bc)e} Q(ax+by, cx+dy) = Q(x,y),$$

and $v \in \Bbbk$ *and* $\phi_0 \in S_r$, *then there is a homogeneous algebra automorphism* $\theta : A \to A$ *such that*

$$\theta(x) = ax+by, \qquad \theta(y) = cx+dy, \qquad \theta(E) = E+v$$

and

$$\theta(D) = \begin{cases} \phi_0 - \frac{ebF}{ax+by}E + eD, & \text{if } b \neq 0; \\ \phi_0 + eD, & \text{if not.} \end{cases} \tag{4.1}$$

Proof. This is proved by a straightforward calculation. It should be noted that the quotient appearing in the formula (4.1) is always a polynomial. □

4.2. Recall form [AC92] that a *higher derivation* of A is a sequence $d = (d_i)_{i\geq 0}$ of linear maps $A \to A$ such that $d_0 = \mathrm{id}_A$ and for all $a, b \in A$ and all $i \geq 0$ we have the *higher Leibniz identity*

$$d_i(ab) = \sum_{s+t=i} d_s(a)d_t(b).$$

It is clear that if $d = (d_i)_{i\geq 0}$ is a higher derivation and $m \geq 0$, then the sequence $d^{[m]} = (d_i^{[m]})_{i\geq 0}$ with

$$d_i^{[m]} = \begin{cases} d_{i/m}, & \text{if } i \text{ is divisible by } m; \\ 0, & \text{if not} \end{cases}$$

is also a higher derivation. On the other hand, if $d = (d_i)_{i\geq 0}$ and $d' = (d'_i)_{i\geq 0}$ are higher derivations of A, we can construct a new higher derivation $(d''_i)_{i\geq 0}$, which we denote $d \circ d'$, putting $d''_i = \sum_{s+t=i} d_s \circ d'_t$ for all $i \geq 0$. Finally, if $\delta : A \to A$ is a derivation of A, then the sequence $(\frac{1}{i!}\delta^i)_{i\geq 0}$ is a higher derivation, which we denote by $\exp(\delta)$; notice that this makes sense because our ground field $\Bbbk$ has characteristic zero.

We let $D(A)$ be the associative subalgebra of $\mathsf{End}_{\Bbbk}(A)$ generated by $\mathsf{Der}(A)$, and say that two higher derivations $d = (d_i)_{i\geq 0}$ and $d' = (d'_i)_{i\geq 0}$ of A are equivalent, and write $d \sim d'$, if for all $i \geq 0$ the map $d_i - d'_i$ is in the subalgebra of $\mathsf{End}_{\Bbbk}(A)$ generated by $D(A)$ and $d_0, \dots, d_{i-1}$; one can check that this is indeed an equivalence relation on the set of higher derivations.

4.3. We recall the following very useful lemma from [AC92]:

Lemma. *If $d = (d_i)_{i\geq 0}$ is a higher derivation of A, then $d_i \in D(A)$ for all $i \geq 0$.*

Proof. The result is an easy consequence of the fact that

> *if d is a higher derivation of A and $j \geq 1$, then there exists a higher derivation $d' = (d'_i)_{i\geq 0}$ such that $d' \sim d$, $d'_i = 0$ if $1 < i < j$, and d'_j is an element of $\mathsf{Der}(A)$.* (4.2)

To prove that this holds, let $d = (d_i)_{i\geq 0}$ and suppose there is an $j \geq 1$ such that that $d_i = 0$ if $1 < i < j$. The higher Leibniz identity implies that d_j is an element of $\mathsf{Der}(A)$, and then we can consider the higher derivation $\exp(-d_j)^{[j]}$. We let $d' = (d'_i)_{i\geq 0}$ be the composition $\exp(-d_j)^{[j]} \circ d$. It is immediate that $d \sim d'$ and a simple computation shows that $d'_i = 0$ if $1 < i < j+1$. The claim (4.2) follows inductively from this. □

4.4. Lemma. *An element of A commutes with x and with y if and only if it belongs to S.*

Proof. The sufficiency of the condition is clear. To prove the necessity, let $e \in A$ be such that $[x, e] = [y, e] = 0$. There are an integer $m \geq 0$ and elements $\phi_0, \dots, \phi_m$ in the subalgebra generated by x, y and D in A such that $e = \sum_{i=0}^{m} \phi_i E^i$, and we have $0 = [x, e_l] = \sum_{i=0}^{m} \phi_i \tau_1(E^i)$: this tells us that $\phi_i = 0$ if $i > 0$, and that $e = \phi_0$. In particular, there are an integer $n \geq 0$

and elements $\psi_0, \ldots, \psi_n$ in S such that $e = \sum_{i=0}^{n} \psi_i D^i$. If $i \geq 0$ we have $[D^i, y] \equiv iFD^{i-1} \mod \bigoplus_{j=0}^{i-2} SD^j$, so that

$$0 = [e, y] = \sum_{i=0}^{n} \psi_i [D^i, y] \equiv n\psi_n FD^{n-1} \mod \bigoplus_{j=0}^{n-2} SD^i.$$

Proceeding by descending induction we see from this that $\psi_i = 0$ if $i > 0$, so that $e = \psi_0 \in S$. □

4.5. Proposition. *If $\theta : A \to A$ is an automorphism of A such that for all $i \geq 0$ and all $a \in A_i$ we have $\theta(a) \in a + \bigoplus_{j>i} A_j$, then here exists an $f \in S$, uniquely determined up to the addition of a constant, such that*

$$\theta(x) = x, \qquad \theta(y) = y, \qquad \theta(D) = D - Ff_y, \qquad \theta(E) = E - [E, f].$$

Conversely, every $f \in S$ determines in this way an automorphism of A satisfying that condition.

Proof. Let $\theta : A \to A$ be an automorphism of A as in the statement. For each $j \geq 0$ there is a unique linear map $\theta_j : A \to A$ of degree j such that for each $i \geq 0$ and each $a \in A_i$ the element $\theta_j(a)$ is the $(i+j)$th homogeneous component of $\theta(a)$. We have that for all $a \in A$ we have $\theta_j(a) = 0$ for $j \geq 0$ and $\theta(a) = \sum_{j\geq 0} \theta_i(a)$ and, moreover, the sequence $(\theta_j)_{j\geq 0}$ is a higher derivation of A. In particular, it follows from Lemma 4.3 that

$$\theta_i \in D(A) \text{ for all } i \geq 0. \tag{4.3}$$

We know from Proposition 3.15 that $\mathsf{Der}(A) = S_r\hat{D} \oplus \Bbbk\hat{E} \oplus \mathsf{InnDer}(A)$. If u is an irreducible factor of xF, then $(\phi\hat{D})(uA)$, $\hat{E}(uA)$ and $[a, uA]$ are all contained in uA for all $\phi \in S_r$ and all $a \in A$, and therefore (4.3) implies that that $\theta(uA) \subseteq uA$. As our argument also applies to the inverse automorphism θ^{-1}, we have $\theta^{-1}(uA) \subseteq uA$ and, therefore, $\theta(uA) = uA$. Since all units of A are in $\Bbbk$, we see that $\theta(u) = u$. Since of xF has two linearly independent linear factors, we can conclude that $\theta(x) = x$ and $\theta(y) = y$.

Let $\theta(E) = E + e_1 + \cdots + e_l$ with $e_i \in A_i$ for each $i \in \{1, \ldots, l\}$. We have

$$x = \theta(x) = [\theta(E), \theta(x)] = [E, x] + [e_1, x] + \cdots + [e_l, x]$$

and, by looking at homogeneous components, we see that $[e_i, x] = 0$ for all $i \in \{1, \ldots, l\}$. Similarly, $[e_i, y] = 0$ for such i, and therefore Lemma 4.4 tells us that $e_1, \ldots, e_l \in S$.

Suppose now that $\theta(D) = D + d_{r+1} + \cdots + d_l$ with $d_j \in A_j$ for each $j \in \{r+1, \ldots, l\}$. Considering the equality $[\theta(E), \theta(D)] = r\theta(D)$ we see that $d_{r+i} = \frac{1}{i}Fe_{iy}$ for each $i \in \{1, \ldots, l\}$. Putting $f = -\sum_{i=1}^{l} \frac{1}{i} e_i$, we obtain the first part of the lemma. The second part follows from a direct verification. □

4.6. The automorphisms described in Proposition **4.5** are precisely the exponentials of the inner derivations corresponding to locally ad-nilpotent elements of A. This is a consequence of the following result:

Proposition. *An element of A is locally* ad*-nilpotent if and only if it belongs to S. If $f \in S$, then the automorphism* $\exp \mathsf{ad}(f)$ *maps x, y, D and E to x, y, $D - F f_y$ and $E - [E, f]$, respectively.*

Proof. Suppose that $e \in A$ is a locally ad-nilpotent element. The kernel $\ker \mathsf{ad}(e)$ is a factorially closed subalgebra of A, so that whenever $a, b \in A$ and $\mathsf{ad}(e)(ab) = 0$ we have $\mathsf{ad}(e)(a) = 0$ or $\mathsf{ad}(e)(b) = 0$; see [Fre06]for the proof of this in the commutative case, which adapts to ours.

Since $[x^i y^j D^k E^l, x] = -x^{i+1} y^j D^k \tau_1(E^l)$ for all $i, j, k, l \geq 0$, we have $[A, x] \subseteq xA$ and from this we see immediately that $[A, xA] \subseteq xA$. This implies that there is a sequence $(u_k)_{k\geq 0}$ in A such that $\mathsf{ad}(e)^k(x) = xu_k$ for all $k \geq 0$. Since e is locally ad-nilpotent, we can consider the integer $k_0 = \max\{k \in \mathbb{N}_0 : \mathsf{ad}(e)^k(x) \neq 0\}$, and then we have $0 \neq xu_{k_0} \in \ker \mathsf{ad}(e)$. As $\ker \mathsf{ad}(e)$ is factorially closed, we see that $\mathsf{ad}(e)(x) = 0$. In other words, the element e commutes with x.

There are an integer $m \geq 0$ and elements $\phi_0, \dots, \phi_m$ in the subalgebra generated by x, y and D in A such that $e = \sum_{i=0}^m \phi_i E^i$, and we have $0 = [x, e] = \sum_{i=0}^m \phi_i \tau_1(E^i)$: this tells us that $\phi_i = 0$ if $i > 0$, and that $e = \phi_0$. In particular, there are an integer $n \geq 0$ and elements $\psi_0, \dots, \psi_n$ in S such that $e = \sum_{i=0}^n \psi_i D^i$.

An induction shows that $[D^i, F] \in FA$ for all $i \geq 0$, and using this we can easily see that $[e, F] = \sum_{i=0}^n \psi_i [D^i, F] \in FA$, from which it follows that in fact $[e, FA] \subseteq FA$. There is therefore a sequence $(v_i)_{i\geq 0}$ of elements of A such that $\mathsf{ad}(e)^i(F) = Fv_i$ for all $i \geq 0$. The local nilpotence of the map $\mathsf{ad}(e)$ allows us to consider the integer

$$i_0 = \max\{i \in \mathbb{N}_0 : \mathsf{ad}(e)^i(F) \neq 0\},$$

and then $0 \neq Fv_{i_0} \in \ker \mathsf{ad}(e)$. If $ax + by$ is any of the factors of F, we have $b \neq 0$ and $ax + by \in \ker \mathsf{ad}(e)$: clearly, this implies that y commutes with e.

In view of Lemma **4.4**, we see that $e \in S$: this proves the necessity of the condition for local ad-nilpotency given in the lemma. Its sufficiency is a direct consequence of the fact that the graded algebra associated to the filtration on A described in **3.1** is commutative. Finally, the truth of the last sentence of the proposition can be verified by an easy computation. □

4.7. We write $\mathsf{Aut}_0(A)$ the set all automorphisms of A described in Lemma **4.1**, and $\mathsf{Exp}(A)$ the set of all automorphisms of A described in Proposition **4.5**; they are subgroups of the full group of automorphisms $\mathsf{Aut}(A)$.

Theorem. *The group* $\mathsf{Aut}(A)$ *is the semidirect product* $\mathsf{Aut}_0(A) \ltimes \mathsf{Exp}(A)$*, corresponding to the action of* $\mathsf{Aut}_0(A)$ *on* $\mathsf{Exp}(A)$ *given by*

$$\theta_0 \cdot \exp \mathsf{ad}(f) = \exp \mathsf{ad}(\theta^{-1}(f))$$

for all $\theta_0 \in \mathsf{Aut}_0(A)$ and $f \in S$. The subgroup $\mathsf{Aut}_0(A)$ is precisely the set of automorphisms of A preserving the grading and $\mathsf{Exp}(A)$ is the set of exponentials of locally nilpotent inner derivations of A.

Notice that the action described in this statement makes sense, as $\theta_0(S) = S$ whenever θ_0 belongs to $\mathsf{Aut}_0(A)$.

Proof. Let $\theta : A \to A$ be an automorphism and let us write $\theta(E) = e_0 + \cdots + e_l$, $\theta(x) = x_0 + \cdots + x_l$, $\theta(y) = e_0 + \cdots + y_l$, $\theta(D) = d_0 + \cdots + d_l$ with $e_i, x_i, y_i, d_i \in A_i$ for each $i \in \{0, \dots, l\}$. Since θ is an automorphism, we have

$$[\theta(E), \theta(x)] = \theta(x), \qquad [\theta(E), \theta(y)] = \theta(y), \qquad [\theta(E), \theta(D)] = r\theta(D). \tag{4.4}$$

Looking at the degree zero parts of these equalities, and remembering that A_0 is a commutative ring, we see $x_0 = y_0 = d_0 = 0$. As $\theta(x) \neq 0$, we can consider the number $s = \min\{i \in \mathbb{N}_0 : x_i \neq 0\}$ and we have $s > 0$. Looking that the component of degree s of the first equality in (4.4), we see that $[e_0, x_s] = x_s$. This means that the restriction $\mathsf{ad}(e_0) : A_s \to A_s$ has a non-zero fixed vector. Now A_s as a right $\Bbbk[E]$-module is free with basis $\{x^i y^j D^k : i + j + rk = s\}$, the map $\mathsf{ad}(e_0)$ is right $\Bbbk[E]$-linear and coincides with right multiplication by $-\tau_s(e_0)$ on A_s. Clearly, the existence of non-zero fixed vector implies that $-\tau_s(e_0) = 1$, so that $e_0 = uE + v$ for some $u \in \Bbbk^\times$ and $v \in \Bbbk$ with $su = 1$. Putting now $s' = \min\{i \in \mathbb{N}_0 : y_i \neq 0\}$ and $s'' = \min\{u \in \mathbb{N}_0 : d_i \neq 0\}$ and looking at the components in the least possible degree in the second and third equations of (4.4), we find that $s'u = 1$ and $s''u = r$. In particular, $s = s'$ and $s'' = rs$.

Suppose for a moment that $s > 1$. As $\theta(x)$, $\theta(y)$ and $\theta(D)$ are in the ideal (A_s) generated by A_s, the composition $q : A \to A$ of θ with the quotient map $A \to A/(A_s)$ is a surjection such that $q(A_0) = A/(A_s)$. This is impossible, as A_0 is a commutative ring and $A/(A_s)$ is not: we therefore have $s = 1$ and, as a consequence, $u = 1$.

There exist $a, b, c, d \in \Bbbk[E]$ such that $x_1 = xa + yb$ and $y_1 = xc + yd$. The four elements $\theta(E)$, $\theta(x)$, $\theta(y)$ and $\theta(D)$ generate A and, as $\theta(D)$ is in $\bigoplus_{i \geq r} A_i$, the elements x and y are in the subalgebra generated by the first three. It follows at once that $x, y \in x_1\Bbbk[E] + y_1\Bbbk[E]$ and, therefore, that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{GL}_2(\Bbbk[E])$.

Let us write $f \in \Bbbk[E] \mapsto \vec{f} \in \Bbbk[E]$ the unique algebra morphism such that $\vec{E} = E + 1$. We have $[\theta(x), \theta(y)] = 0$ and in degree 2 this tells us that

$$x^2(a\vec{c} - \vec{a}c) + xy\boxed{T} + y^2(b\vec{d} - \vec{b}d) = 0,$$

so that

$$a\vec{c} = \vec{a}c, \qquad b\vec{d} = \vec{c}d. \tag{4.5}$$

Suppose that a is not constant. As the characteristic of $\Bbbk$ is zero (and possibly after replacing $\Bbbk$ by an algebraic extension, which does not change anything) there is then a $\xi \in \Bbbk$ such that $a(\xi) = 0$ and $\vec{a}(\xi) = a(\xi + 1) \neq 0$, and the first equality in (4.5) implies that $c(\xi) = 0$. The determinant of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is thus divisible by $E-\xi$, and this is impossible. Similarly, we find that all of b, c, d must be constant.

Since $d_r \in A_r$, there exist $k \geq 0$, $\phi_0, \dots, \phi_k \in S_r$ and $h \in \Bbbk[E]$ such that $d_r = \sum_{i=0}^{k} \phi_i E^i + Dh$. The component of degree $r+1$ of $[\theta(D), \theta(x)]$ is

$$0 = [d_r, x_1] = -\sum_{i=0}^{k} (ax+by)\phi_i \tau_1(E^i) - (ax+by)D\tau_1(h) + bF\vec{h}.$$

We thus see that h is constant, that $\phi_i = 0$ if $i \geq 2$, and that

$$(ax+by)\phi_1 + bhF = 0.$$

If $b=0$, then $\phi_1 = 0$, and if instead $b \neq 0$, then either $h \neq 0$ and we see that $ax+by$ divides F and that $\phi_1 = -bhF/(ax+by)$, or $h=0$ and $\phi_1 = 0$. In any case, we see that

$$d_r = \begin{cases} \phi_0 - \frac{hbF}{ax+by}E + hD, & \text{if } b \neq 0; \\ \phi_0 + hD, & \text{if not.} \end{cases}$$

Finally, the component of degree $r+1$ of the equality $[\theta(D), \theta(y)] = \theta(F)$ tells us that

$$F(ax+by, cx+dy) = (ad-bc)h\frac{xF}{ax+by}.$$

It follows now from Lemma 4.1 that there is a graded automorphism $\theta_0 : A \to A$ such that $\theta_0(x) = ax+by$, $\theta_0(y) = cx+dy$, $\theta_0(E) = E+v$ and $\theta_0(D) = d_r$. The composition $\theta_0^{-1} \circ \theta$ satisfies the hypothesis of Proposition 4.5, and then there exists an $f \in S$ such that $\theta = \theta_0 \circ \exp \mathsf{ad}(f)$. This shows that $\mathsf{Aut}(A) = \mathsf{Aut}_0(A) \cdot \mathsf{Exp}(A)$. Moreover, if θ is a graded automorphism, then so is $\exp \mathsf{ad}(f) = \theta_0^{-1} \circ \theta$ and, since it maps E to $E - [E, f]$, this is possible if and only if $f \in \Bbbk$, that is, if and only if $\exp \mathsf{ad}(f) = \mathrm{id}_A$; this proves the last claim of the theorem.

Finally, computing the action of both sides of the equation on the generators of A, we see that

$$\exp \mathsf{ad}(f) \circ \theta_0 = \theta_0 \circ \exp \mathsf{ad}(\theta^{-1}(f))$$

for all $f \in S$ and all $\theta_0 \in \mathsf{Aut}_0(A)$, and this tells us that $\mathsf{Aut}(A)$ is indeed a semidirect product $\mathsf{Aut}_0(A) \ltimes \mathsf{Exp}(A)$. □

4.2 THE ISOMORPHISM PROBLEM

In this section we make use of our description of the group of automorphisms of $\mathsf{Diff}(\mathcal{A})$ to give a complete solution of the problem of determining which pairs of arrangements of lines $\mathcal{A}$ and $\mathcal{A}'$ have isomorphic algebras $\mathsf{Diff}(\mathcal{A})$ and $\mathsf{Diff}(\mathcal{A}')$. In particular, we show that the arrangement $\mathcal{A}$ can be recovered from the algebra $\mathsf{Diff}(\mathcal{A})$.

4.8. As usual, we say that an element u of A is *normal* if $uA = Au$. Such an element, since it is not a zero-divisor, determines an automorphism $\theta_u : A \to A$ uniquely by the condition that $ua = \theta_u(a)u$ for all $u \in A$.

Proposition. *Let $Q = \alpha_0 \cdots \alpha_{r+1}$ be a factorization of Q as a product of linear factors. The set of non-zero normal elements of A is*

$$\mathscr{N}(A) = \{\lambda\alpha_0^{i_0} \cdots \alpha_{r+1}^{i_{r+1}} : \lambda \in \Bbbk^\times, i_0, \dots, i_{r+1} \in \mathbb{N}_0\}.$$

This set is the saturated multiplicatively closed subset generated by Q both in A or in S.

Proof. A direct computation shows that each of the factors $\alpha_0, \dots, \alpha_{r+1}$ of Q is normal in A, so the set $\mathscr{N}(A)$ is contained in the set of normal elements of A, for the latter is multiplicatively closed. The set $\mathscr{N}(A)$ is multiplicatively closed and it is saturated because S is closed under divisors in A, and it is clear that as a saturated multiplicatively closed it is generated by Q. To conclude the proof, we have to show that every non-zero normal element of A belongs to $\mathscr{N}(A)$.

Let u be a normal element in A and let $\theta_u : A \to A$ be the associated automorphism, so that $ua = \theta_u(a)u$ for all $a \in A$. There are $k, l \in \mathbb{N}_0$ with $k \leq l$ and elements $u_k, \dots u_l \in A$ such that $u = u_k + \cdots + u_l$, $u_i \in A_i$ if $k \leq i \leq l$, and $u_k \neq 0 \neq u_l$. Similarly, there are $s, t \in \mathbb{N}_0$ with $s \leq t$ and elements $e_s, \dots, e_t \in A$ such that $\theta_u(E) = e_s + \cdots + e_t$, $e_i \in A_i$ if $s \leq i \leq t$, and $e_s \neq 0 \neq e_t$. As we have

$$u_k E + \cdots + u_l E = uE = \theta_u(E)u = e_s u_k + \cdots + e_t u_l$$

with $u_k E$, $u_l E$, $e_s u_k$ and $e_t u_l$ all non-zero, looking at the homogeneous components of both sides we see that $s = t = 0$. This means that $\theta_u(E) = f(E) \in \Bbbk[E]$, and therefore the above equality is really of the form

$$u_k E + \cdots + u_l E = f(E)u_k + \cdots + f(E)u_l.$$

It follows from this that $u_i E = f(E)u_i = u_i f(E+i)$ for all $i \in \{k, \dots, l\}$ and therefore that $E = f(E+k)$ and that $E = f(E+l)$. Since our ground field has characteristic zero, this is only possible if $k = l$: the element u is thus homogeneous of degree l.

Now, since $ua = \theta_u(a)u$ for all $a \in A$, the homogeneity of u implies immediately that θ_u is a homogeneous map. There are $n \in \mathbb{N}_0$ and $\phi_0, \dots, \phi_n$ in the subalgebra of A generated by x, y and D, such that $\phi_n \neq 0$ and $u = \sum_{i=0}^n \phi_i E^i$. As $\theta_u(x)$ has degree 1, it belongs to S_1 and we have

$$\theta_u(x) \sum_{i=0}^n \phi_i E^i = \theta_u(x)u = ux = \sum_{i=0}^n \phi_i E^i x = x \sum_{i=0} \phi_i (E+1)^i.$$

Considering only the terms that have E^n as a factor we see that $\theta_u(x) = x$, and then the equality tells us that in fact $\sum_{i=0}^n \phi_i E^i = \sum_{i=0} \phi_i (E+1)^i$. Looking now at the terms which have E^{n-1} as a factor here we see that moreover $n = 0$, so that $u \in \Bbbk[x, y, D]$. There exist then $m \in \mathbb{N}_0$ and $\psi_0, \dots, \psi_m \in S$ such that $\psi_m \neq 0$ and $u = \sum_{i=0}^{m} \psi_i D^i$. As $\theta_u(y)$ has degree 1, it belongs to S_1 and we have

$$\theta_u(y) \sum_{i=0}^{m} \psi_i D^i = \theta_u(y) u = uy = \sum_{i=0}^{m} \psi_i D^i y = \sum_{i=0}^{m} y \psi_i D^i + \sum_{i=0}^{m} \psi_i [D^i, y].$$

Comparing the terms that have D^m as a factor we conclude that also $\theta_u(y) = y$.

As θ_u fixes x and y, the element u commutes with x and y, and Lemma **4.4** allows us to conclude that u is in S_l. Moreover, we know that all homogeneous automorphisms of A are those described in Lemma **4.1**, so there exist $\phi \in S_r$ and $e \in \Bbbk^\times$ such that $\theta_u(D) = \phi + eD$. We then have that

$$uD = \theta_u(D) u = (\phi + eD) u = \phi u + euD + eu_y F$$

and this implies that $e = 1$ and $\phi u + u_y F = 0$. Suppose now that α is a linear factor of u and let $k \in \mathbb{N}$ and $v \in S$ be such that $u = \alpha^k v$ and v is not divisible by α. The last equality becomes $\phi \alpha^k v + k \alpha^{k-1} \alpha_y v F + \alpha^k v_y F = 0$ and implies that α divides $\alpha_y F$: this means that α is a non-zero multiple of x or a linear factor of F. As u can be factored as a product of linear factors, we can therefore conclude that u belongs to the set described in the statement of the proposition. □

4.9. There is a close connection between normal elements, the first Hochschild cohomology space that we computed in Section 3.2 and the modular automorphisms of A.

Proposition. *Let $Q = \alpha_0 \cdots \alpha_{r+1}$ be a factorization of Q as a product of linear factors.*

(*i*) *Every linear combination of the derivations $\partial_{\alpha_0}, \dots, \partial_{\alpha_{r+1}} : A \to A$ described in Proposition **3.18** is locally nilpotent.*

(*ii*) *If $u = \lambda \alpha_0^{i_0} \cdots \alpha_{r+1}^{i_{r+1}}$, with $\lambda \in \Bbbk^\times$ and $i_0, \dots, i_{r+1} \in \mathbb{N}_0$, is a normal element of A, then the automorphism $\theta_u : A \to A$ associated to u is*

$$\theta_u = \exp\left(- \sum_{j=0}^{r+1} i_j \partial_{\alpha_j} \right).$$

This automorphism is such that $\theta_u(f) = f$ for all $f \in S$ and

$$\theta_u(\delta) = \delta + \frac{\delta(u)}{u}$$

for all $\delta \in \mathsf{Der}(\mathcal{A})$.

(*iii*) *The modular automorphism $\sigma : A \to A$ described in Proposition **3.25** coincides with the automorphism θ_Q associated to the normal element Q.* □

We omit the proof since it follows from a straightforward calculation using our previous results.

4.10. Another immediate application of the determination of the set of normal elements is the classification under isomorphisms of our algebras.

Proposition. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two central arrangements of lines in $\mathbb{k}^2$. The algebras $D(\mathcal{A})$ and $D(\mathcal{A}')$ are isomorphic if and only if the arrangements $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic.*

Proof. The sufficiency of the condition being obvious, we prove only its necessity. We will denote with primes the objects associated to the arrangement $\mathcal{A}'$, so that for example $A' = D(\mathcal{A}')$ and so on. Moreover, in view of the sufficiency of the condition we can suppose without loss of generality that both arrangements $\mathcal{A}$ and $\mathcal{A}'$ contain the line with equation $x = 0$.

Let us suppose that there is an isomorphism of algebras $\phi : A \to A'$. Since ϕ maps locally ad-nilpotent elements to locally ad-nilpotent elements, it follows from Proposition 4.6 that $\phi(S) = S'$ and therefore that ϕ restricts to an isomorphism of algebras $\phi : S \to S'$. On the other hand, ϕ also maps normal elements to normal elements, so that ϕ restricts to a monoid homomorphism $\phi : \mathcal{N}(A) \to \mathcal{N}(A')$. Let $Q = \alpha_0 \cdots \alpha_{r+1}$ and $Q' = \alpha'_0 \cdots \alpha'_{r'+1}$ be the factorizations of Q and of Q' as products of linear factors. The invertible elements of the monoid $\mathcal{N}(A)$ are the units of $\mathbb{k}$ and the quotient $\mathcal{N}(A)/\mathbb{k}^\times$ is the free abelian monoid generated by (the classes of) $\alpha_0, \ldots, \alpha_{r+1}$ and, of course, a similar statement holds for the other arrangement. Since ϕ induces an isomorphism $\mathcal{N}(A)/\mathbb{k}^\times \to \mathcal{N}(A')/\mathbb{k}^\times$ we see, first, that $r = r'$ and, second, that there are a permutation π of the set $\{0, \ldots, r+1\}$ and a function $\lambda : \{0, \ldots, r+1\} \to \mathbb{k}^\times$ such that $\phi(\alpha_i) = \lambda(i)\alpha'_{\pi(i)}$ for all $i \in \{0, \ldots, r+1\}$. As there are at least two lines in each arrangement, this implies that the restriction $\phi|_S : S \to S'$ restricts to an isomorphism of vector spaces $\phi : S_1 \to S'_1$, so that $\phi|_S$ is linear, and that $\phi(Q) = Q'$. It is clear that this implies that the arrangements $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic. □

4.3 RESUMEN

Seguimos en este capítulo estudiando el álgebra de operadores diferenciales $\mathsf{Diff}(\mathcal{A})$ tangentes a un arreglo central $\mathcal{A}$ de al menos cinco rectas. Extraemos consecuencias de nuestro cálculo de la cohomología: particularmente, del primer grupo de cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$. Utlizando los métodos desarrollados por J. Alev y M. Chamarie en [AC92], describimos el grupo de automorfismos de $\mathsf{Diff}(\mathcal{A})$ en el Teorema 4.7:

Teorema. *El grupo* $\mathsf{Aut}(\mathsf{Diff}(\mathcal{A}))$ *es el producto semidirecto* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A})) \ltimes \mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *de los subgrupos* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *de automorfismos de* $\mathsf{Diff}(\mathcal{A})$ *que preservan la graudación y* $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *de exponenciales de derivaciones internas localmente nilpotentes de* $\mathsf{Diff}(\mathcal{A})$. *Concretamente, la acción de* $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *en* $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$ *está dada por*

$$\theta_0 \cdot \exp \mathrm{ad}(f) = \exp \mathrm{ad}(\theta^{-1}(f))$$

para cada $\theta_0 \in \mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ *y* $f \in S$.

Junto con este teorema, damos en el Lema 4.1 y la Proposición 4.6 una descripción completa de los grupos $\mathsf{Aut}_0(\mathsf{Diff}(\mathcal{A}))$ y $\mathsf{Exp}(\mathsf{Diff}(\mathcal{A}))$. Mostramos que el primero es un grupo algebraico de dimensión finita que “ve” las simetrías del arreglo y que el segundo es un grupo de dimensión infinita cuya estructura es independiente del arreglo. Esta descripción de grupo de automorfismos, a su vez, nos permite dar una solución completa al problema de determinar cuáles pares de arreglos de rectas $\mathcal{A}$ y $\mathcal{A}'$ tienen álgebras $\mathsf{Diff}(\mathcal{A})$ y $\mathsf{Diff}(\mathcal{A}')$ isomorfas.

Proposición. *Dos arreglos de rectas tienen álgebras de operadores diferenciales isomorfas si y solo si son isomorfos.*

– 5 –

Deformations of the algebra of differential operators tangent to a line arrangement

We continue to extract consequences of our findings of Chapter 3. Let $\mathcal{A}$ be a central arrangement of lines and let $A = \mathsf{Diff}(\mathcal{A})$. In this chapter, we study the deformation theory of A with the help of our explicit calculation of the second space of cohomology of A. We show that many of the infinitesimal deformations of the algebra can be integrated to formal deformations and we also exhibit obstructed infinitesimal deformations.

5.1 Formal and nth order deformations

5.1. Let A be an associative $\Bbbk$-algebra with underlying vector space V. A *formal deformation of A* is a $\Bbbk[\![t]\!]$-algebra B with underlying vector space $V[\![t]\!]$ such that there exists a family $F_\bullet = (F_i)_{i\geq 0}$ of maps $F_i \in \mathsf{hom}(V \otimes V, V)$ for $i \geq 0$ such that F_0 is the product of $\mathcal{A}$ and that the product $\cdot_B$ of B is continuous for the t-adic topology and given, when v and w belong to A, by the formula

$$v \cdot_B w = F_0(v, w) + F_1(v, w)t + F_2(v, w)t^2 + \cdots .$$

When this is the case, there is an isomorphism of $\Bbbk$-algebras $\phi : B \otimes_{\Bbbk[\![t]\!]} \Bbbk \to A$. For example, the algebra of formal series $A[\![t]\!]$, which is a $\Bbbk[\![t]\!]$-algebra in the obvious way, is a formal deformation of A with $F_i = 0$ for $i \geq 1$, and the isomorphism ϕ corresponds to the evaluation at $t = 0$ of formal series: we call this the *trivial (formal) deformation* of A.

Two formal deformations B and B' are *equivalent* if there is an isomorphism of $\Bbbk[\![t]\!]$-algebras $\psi : B \to B'$ such that the diagram

$$\begin{tikzcd} B \otimes_{\Bbbk[\![t]\!]} \Bbbk \arrow[rr, "\psi\otimes 1_{\Bbbk}"] \arrow[dr, "\phi"'] & & B' \otimes_{\Bbbk[\![t]\!]} \Bbbk \arrow[dl, "\phi'"] \\ & A & \end{tikzcd} \tag{5.1}$$

commutes. A formal deformation is trivial if it is equivalent to the trivial formal deformation.

5.2. We now recall the celebrated result by M. Gerstenhaber in [Ger64] that relates formal deformations with Hochschild cohomology; in his words, the second cohomology space $HH^2(A)$ "may be interpreted as the group of infinitesimal deformations of A".

Recall that the Hochschild cohomology of A can be computed as the cohomology of the Hochschild complex $(\mathsf{hom}(A^{\otimes\bullet}, A), d)$ with differentials given by

$$d(f)(a_0|\cdots|a_i) = a_0 f(a_1|\cdots|a_i) + \sum_{j=0}^{i-1} f(a_0|\cdots|a_j a_{j+1}|\cdots|a_i) + (-1)^i f(a_0|\cdots|a_{i-1}),$$

for $i \geq 0$ and $f \in \mathsf{hom}(A^{\otimes i}, A)$.

Theorem (M. Gerstenhaber)**.** *If B is a deformation of A there exists $n \geq 1$ and a deformation B' of A equivalent to B given by a family of maps $F_\bullet$ such that $F_i = 0$ for $1 \leq i \leq n-1$ and F_n is a 2-cocycle in the Hochschild complex $(\mathsf{hom}(A^{\otimes\bullet}, A), d)$. The class of F_n in $HH^2(A)$ is non-zero if B is not a trivial deformation.*

Proof. This is Proposition 1 in [Ger64]. □

5.3. There is a finite order version of the definitions above, which is useful. Let $n \in \mathbb{N}$. A *nth-order deformation* of an algebra A is a $\Bbbk[t]/(t^{n+1})$-algebra B with underlying vector space $V[t]/(t^{n+1})$ such that there exists a family $F_\bullet = (F_i)_{i=0}^n$ of maps $F_i \in \mathsf{hom}(V \otimes V, V)$ such that F_0 is the multiplication of A and that

$$v \cdot_B w = F_0(v, w) + F_1(v, w)t + F_2(v, w)t^2 + \cdots + F_n(v, w)t^n$$

whenever $v, w \in V$. One can show that this is the same as saying that B is a $\Bbbk[t]/(t^{n+1})$-algebra free as a $\Bbbk[t]/(t^{n+1})$-module such that there is an isomorphism of $\Bbbk$-algebras $B \to \Bbbk[t]/(t^{n+1}) \otimes A$. An example of an nth-order deformation is the $\Bbbk[t]/(t^n)$-algebra $A[t]/(t^n)$, which has $F_i = 0$ for every $i \geq 1$. This is called the trivial nth-order deformation. As before, two nth-order deformations B and B' are *equivalent* if there is an isomorphism of $\Bbbk[t]/(t^{n+1})$-algebras $B \to B'$ such that the diagram that corresponds to (5.1) is commutative.

If B is an nth-order deformation of A and $m < n$, the quotient algebra $B/(t^{m+1})$ is an mth-order deformation deformation of A. Similarly, a formal deformation gives rise to mth-order deformations for every $m \in \mathbb{N}$.

5.4. The purpose of this chapter is to describe nth order deformations of the algebra of differential operators tangent to a central line arrangement. In order to do this we will make use of Bergman's Diamond Lemma, which requires a certain amount of preliminaries, which we now recall following the neat exposition by F. Martin in [Mar16, §2.1].

Let us fix a commutative ring k and a set X, and let $\langle X \rangle$ denote the free monoid on X, whose elements we call *monomials*. A *monomial order* on $\langle X \rangle$ is a partial order $\leq$ with 1 as minimal element such that whenever u, v, v' and w are monomials and $v \leq v'$ we have $uvw \leq uv'w$. For example, if X is a totally ordered set then the graded lexicographical order —or *grlex*— is a monomial order on $\langle X \rangle$: monomials are sorted first by length and then lexicographically, according to the order on X. As usual, a monomial order satisfies the *descending chain condition* if every decreasing sequence of monomials is eventually constant. For example, every grlex order on $\langle X\rangle$ satisfies this condition.

A *rewriting system on* X is a subset S of $\langle X\rangle \times k\langle X\rangle$ such that for every element $\sigma = (w_\sigma, f_\sigma)$ of S we have $w_\sigma \neq f_\sigma$. We call each such pair σ a *rewriting rule* of S and sometimes we denote it by $w_\sigma \to f_\sigma$. The rewriting system S is *compatible* with a monomial order on X if for all its rules σ and every monomial u which appears with nonzero coefficient in f_σ we have $u < w_\sigma$.

A *basic reduction* is a triple $r = (u, \sigma, v)$ with u and v monomials and σ a rewriting rule. A basic reduction r defines a linear map $r : k\langle X\rangle \to \Bbbk\langle X\rangle$ that maps the word $uw_\sigma v$ to $wf_\sigma v$ and leaves the rest of the monomials fixed. A *reduction* is an element of the submonoid of $\mathsf{End}(k\langle X\rangle)$ generated by basic reductions.

Given a reduction system S, we say that an element $x \in k\langle X\rangle$ is

- *irreducible* if $r(x) = x$ for every reduction r,
- *reduction-finite* if for every sequence of reductions (r_n) there exists n_0 such that r_n acts trivially on $r_{n_0-1} \circ \cdots \circ r_1(x)$ for every $n \geq n_0$, and
- *reduction-unique* if it is reduction-finite and there exists $x' \in k\langle X\rangle$ such that if $r(x)$ is irreducible for a given reduction r then $r(x) = x'$.

We next define the important notion of ambiguity. Let σ and τ be rules of S and let u, v and w be monomials. The 5-tuple $\alpha = (\sigma, \tau, u, v, w)$ is an *overlap ambiguity of* S if u, v and w have positive length, $\omega_\sigma = uv$ and $\omega_\tau = vw$. In this case, the ambiguity α is *solvable* if there exist reductions r and r' such that $r(f_\sigma w) = r'(uf_\tau)$ —we depict this situation with a diamond-shaped diagram in Figure 5.1. On the other hand, we say that α an *inclusion ambiguity* if $\sigma \neq \tau$, $w_\sigma = v$ and $\omega_\tau = uvw$ and that it is solvable if there exist reductions r and r' such that $r(uf_\sigma w) = r'(f_\tau)$. In both cases, we say that the ambiguity is *supported* on the monomial uvw.

$$\begin{tikzcd}
 & f_\sigma w \arrow[dr, dashed] & \\
uvw \arrow[ur] \arrow[dr] & & r(f_\sigma w) = r'(uf_\tau) \\
 & uf_\tau \arrow[ur, dashed] &
\end{tikzcd}$$

Figure 5.1. A solvable overlap ambiguity

5.5. We have now given all the necessary definitions to state the next theorem, whose name is motivated by Figure 5.1.

Theorem (The Diamond Lemma of G. Bergman)**.** *Let k be a commutative algebra, let S be a rewriting system on a set X and let $\leq$ be a monomial order on X compatible with S that satisfies the descending chain condition. Denote by I_S the two-sided ideal of $k\langle X\rangle$ generated by the set $\{f_\sigma - w_\sigma : \sigma \in S\}$. The following statements are equivalent.*

(*a*) *All ambiguities are solvable.*
(*b*) *All elements of $k\langle X\rangle$ are reduction-unique.*

(c) *The k-submodule $k\langle X\rangle_{\mathrm{irr}}$ of $k\langle X\rangle$ spanned by the irreducible monomials of $\langle X\rangle$ is an irredundant set of representatives for the elements of the algebra $k\langle X\rangle/I_S$.*

We say that S is *confluent over k* if these conditions hold and, in that case, there is an isomorphism of k-modules from $k\langle X\rangle/I_S$ to $k\langle X\rangle_{\mathrm{irr}}$.

Proof. This appears in [Ber78] as part of the Theorem 1.2. □

5.6. *Example.* Let $\mathcal{A}$ be a central line arrangement and let us preserve the notation and conventions of **3.1**. We will call $A = \mathsf{Diff}(\mathcal{A})$ from now until the end of this chapter. Let $X = \{x, y, D, E\}$ and let us take the grlex monomial order on $\langle X\rangle$ with $x < y < D < E$. We claim that the rewriting system

$$
\begin{aligned}
&yx \to xy && ED \to DE + rD \\
&Ex \to xE + x && Dx \to xD \\
&Ey \to yE + y && Dy \to yD + F
\end{aligned}
$$

is confluent over $\Bbbk$. It is clear that this system is compatible with the monomial order, so, according to Theorem 5.5, we need only check that its ambiguities are solvable. There are only four ambiguities in our rewriting system, supported on the monomials Eyx, Dyx, EDx and EDy. All of them are solvable: the calculation that shows this can be deduced from the proof of the forthcoming Proposition 5.8, by taking $t = 0$ there. The algebra $\Bbbk\langle X\rangle/I_S$ that is the subject of Bergman's Diamond Lemma is A, since it admits the presentation that we gave in **2.8**.

5.7. From now on, we establish a connection between the deformations of A in the sense of **5.3** and the second cohomology space $HH^2(A)$. This connection arises from the specific choice of resolution that we used to compute cohomology and provides information even before the computation of Proposition 3.15. Let $\rho = a\hat{x}\wedge\hat{y} + b\hat{x}\wedge\hat{E} + c\hat{y}\wedge\hat{E} + u\hat{D}\wedge\hat{E} + v\hat{x}\wedge\hat{D} + w\hat{y}\wedge\hat{D}$ be a 2-cochain in the complex $\mathfrak{X}$ of 3.10 and let us consider the rewriting system on $\{x, y, D, E\}$ over $\Bbbk[t]/(t^2)$ with rules

$$
\begin{aligned}
&yx \to xy + ta && ED \to DE + rD + tu \\
&Ex \to xE + x + tb && Dx \to xD + tv \\
&Ey \to yE + y + tc && Dy \to yD + F + tw
\end{aligned}
\tag{5.2}
$$

Proposition. *The rewriting system (5.2) is confluent modulo t^2, that is, it is confluent over $\Bbbk[t]/(t^2)$, if and only if the cochain ρ is a cocycle. When that is the case, the algebra obtained from this rewriting system as in Bergman's Diamond Lemma of* **5.4** *is a first-order deformation of A which is trivial if and only if ρ is a coboundary.*

Proof. Our rewriting system has four ambiguities, supported on the monomials Eyx, Dyx, EDx and EDy. The first assertion in our proposition follows from the fact that the solvability of each ambiguity is equivalent to the vanishing of the corresponding component of $d^2(\rho)$. We illustrate this claim by studying the monomial Eyx. Starting from the right, we get

$$\begin{aligned} Eyx &\to Exy + tEa \\ &\to xEy + xy + tby + taE + t[E,a] \\ &\to xyE + 2xy + t\left(xc + [b,y] + yb + aE + [E,a]\right) \end{aligned}$$

and from the left

$$\begin{aligned} Eyx &\to yEx + yx + tcx \\ &\to yxE + yx + tyb + xy + t[c,x] + txc \\ &\to xyE + 2xy + t\left(2a + aE + yb + [c,x] + xc\right). \end{aligned}$$

As the two expressions that we found are irreducible, this ambiguity is solvable if and only if $[b,y] + [E,a] = [c,x] + 2a$. On the other hand, inspecting the differentials in **3.9** we see that the component of $d^2(\rho)$ in $\hat{x} \wedge \hat{y} \wedge \hat{E}$ is

$$[E,a] - 2a + [b,y] - [c,x],$$

and this shows that the desired instance of our claim holds. The same situation repeats when analyzing each of the other ambiguities. This proves the necessity of the condition that ρ be a cocycle for the rewriting system to be confluent. Its sufficiency follows from essentially the same calculation done in reverse.

In order to prove the second claim of the proposition, let us assume that ρ is cocycle, so that the rewriting system is confluent and the $\Bbbk[t]/(t^2)$-algebra $B = \Bbbk\langle x, y, D, E\rangle/I_S$, as in Bergman's Diamond Lemma, is free as an $\Bbbk[t]/(t^2)$-module. The obvious morphism of $\Bbbk$-algebras $B \to A$ which maps t to 0 gives rise to a morphism $B/tB \to A$ of $\Bbbk$-algebras that maps a basis to a basis: this tells us that B is a first-order deformation of A.

Suppose now that ρ is a coboundary. Let ω be a 1-cochain such that $d^1(\omega) = -\rho$ and write $\omega = p\hat{x} + q\hat{y} + s\gamma\hat{D} + t'\hat{E}$, with $p, q, s, t' \in A$. We claim that the assignment

$$\begin{aligned} 1 \otimes x &\mapsto x + tp, & 1 \otimes D &\mapsto D + ts, \\ 1 \otimes y &\mapsto y + tq, & 1 \otimes E &\mapsto tt' \end{aligned}$$

defines a morphism of $\Bbbk[t]/(t^2)$-algebras $\phi : \Bbbk[t]/(t^2) \otimes A \to B$. To check this, one has to show that it maps the defining relations of A to zero. There are six relations, let us write down this computation for the easiest one and for the most complicated one as an illustration.

One of the relations is that x and y commute in A, so we have to show that$\phi(1 \otimes x)$ and $\phi(1 \otimes y)$ commute in B. We have

$$\begin{aligned} \phi(1 \otimes y)(1 \otimes x) - \phi(1 \otimes x)\phi(1 \otimes y) &= yx - xy + t(-[x,q] + [y,p]) \\ &= t(-a - [x,q] + [y,p]). \end{aligned}$$

The component in $\hat{x} \wedge \hat{y}$ of the equality $d^1(\omega) = -\rho$ is $[x, q] - [y, p] = -a$ and therefore the desired commutation holds in B.

Let us now examine the most complicated instance, which is that of y and D. On one hand we have

$$\begin{aligned}\phi(1 \otimes D)(1 \otimes y) - \phi(1 \otimes y)\phi(1 \otimes D) &= Dy - yD + t(-[y, s] + [D, q]) \\ &= F + t(-w - [y, s] + [D, q])\end{aligned}$$

and on the other

$$\phi(1 \otimes F) = F + t(\nabla_x^p(F) + \nabla_y^q(F)),$$

so that the relation $Dy - yD - F$ is preserved if and only if the component in $\hat{y} \wedge \hat{D}$ of $d^1(\omega)$ is equal to that of $-\rho$, as we see comparing our last two equations to the expression for the second differential that we have in 3.9.

Assume, finally, that ρ is a 2-cocycle in $\mathfrak{X}$ such that the deformation B is trivial and let $\phi : \Bbbk[t]/(t^2) \otimes A \to B$ be a $\Bbbk$-algebra isomorphism as in 5.3. Since ϕ is the identity modulo t, we may write

$$\begin{aligned}\phi(1 \otimes x) &= x + t\phi'(x), & \phi(1 \otimes D) &= D + t\phi'(D), \\ \phi(1 \otimes y) &= y + t\phi'(y), & \phi(1 \otimes E) &= E + t\phi'(E)\end{aligned}$$

and it is straightforward to see that the 1-cochain

$$\omega = \phi'(x)\hat{x} + \phi'(y)\hat{y} + \phi'(D)\hat{D} + \phi'(E)\hat{E}$$

satisfies $d^1(\omega) = -\rho$. For example, its component in $\hat{x} \wedge \hat{E}$ is $\phi'(x) - [E, \phi'(x)] + [x, \phi'(E)]$ and this is equal to $-b$ because, as ϕ is a morphism of algebras,

$$\begin{aligned}0 &= \phi(1 \otimes E)\phi(1 \otimes x) - \phi(1 \otimes x)\phi(1 \otimes E) - \phi(1 \otimes x) \\ &= (E + t\phi'(E))(x + t\phi'(x)) - (x + t\phi'(x))(E + t\phi'(E)) - (x + t\phi'(x)) \\ &= Ex - xE - x + t([x, \phi'(E)] - [E, \phi'(x)] - \phi'(x)) \\ &= t(-b + [x, \phi'(E)] - [E, \phi'(x)] - \phi'(x)).\end{aligned}$$

This concludes the proof. □

5.8. We now look for second-order deformations. As a second-order deformation gives rise to a first-order one, as we said in 5.3, we may take into account our findings of 5.7. Let then $u, v \in A_r$, $w \in A_{r+1}$ and $\rho := v\hat{x} \wedge \hat{D} + w\hat{y} \wedge \hat{D} + u\hat{D} \wedge \hat{E}$ be a 2-cochain in $\mathfrak{X}$ and let us consider the rewriting system over $\Bbbk[t]/(t^3)$ given by the rules

$$\begin{aligned}yx &\to xy + t^2\varepsilon & ED &\to DE + rD + tu + t^2\gamma \\ Ex &\to xE + x + t^2\alpha & Dx &\to xD + tv + t^2\delta\end{aligned} \tag{5.3}$$

$$Ey \to yE + y + t^2\beta \qquad\qquad Dy \to yD + F + tw + t^2\zeta$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ are such that these rules are homogeneous with respect to the grading of A. Let us also define the 2-cochain $\xi = \varepsilon\hat{x} \wedge \hat{y} + \alpha\hat{x} \wedge \hat{E} + \beta\hat{y} \wedge \hat{E} + \gamma\hat{D} \wedge \hat{E} + \delta\hat{x} \wedge \hat{D} + \zeta\hat{y} \wedge \hat{D}$ in $\mathfrak{X}$. As the rewriting system is confluent modulo t^2 if $a, b \in A$, there is a well-defined element $(b, a)_1 \in A$ such that it is a standard monomial and $ba \equiv ab + t(b, a)_1 \mod t^2$.

Proposition. *The rewriting system (5.3) is confluent modulo t^3 if and only if*

(*i*) *the cochain ρ is a cocycle and*

(*ii*) *the following equation holds:*

$$d^2(\xi) = (-(v, y)_1 + (w, x)_1)\hat{x} \wedge \hat{y} \wedge \hat{D} - (E, v)_1\hat{x} \wedge \hat{D} \wedge \hat{E} + ((u, y)_1 - (E, w)_1)\hat{y} \wedge \hat{D} \wedge \hat{E}.$$

Proof. There are four ambiguities in this rewriting system.

- We begin with the one supported by Eyx. Starting from the right, we get

$$Eyx \to Exy + t^2E\varepsilon \to xyE + xy + xy + t^2(x\beta + \alpha y + E\varepsilon)$$

and from the left

$$\begin{aligned} Eyx &\to yEx + yx + t^2\beta x \to yxE + yx + t^2y\alpha + yx + t^2\beta x \\ &\to 2xy + xyE + t^2\left(y\alpha + \beta x + \varepsilon(2 + E)\right). \end{aligned}$$

We see that this ambiguity is solvable modulo t^3 if and only if the equation

$$[\alpha, y] + [x, \beta] = 0$$

holds in A.

- We consider now the ambiguity in Dyx. We have

$$\begin{aligned} Dyx &\to Dxy + t^2D\varepsilon \to (xD + tv + t^2\delta)y + t^2D\varepsilon \\ &\to x(yD + F + tw + t^2\zeta) + tvy + t^2(\delta y + D\varepsilon) \end{aligned}$$

and

$$\begin{aligned} Dyx &\to (yD + F + tw + t^2\zeta)x \to y(xD + tv + t^2\delta) + Fx + twx + t^2\zeta x \\ &\to (xy + t^2\varepsilon)D + xF + t(yv + wx) + t^2(y\delta + \zeta x + \nabla_y^\varepsilon). \end{aligned}$$

The solvability modulo t^3 of this ambiguity is then equivalent to the conditions

$$\begin{aligned} &vy + xw - yv - wx \equiv 0 \mod t, \\ &[\delta, y] + [D, \varepsilon] + [x, \zeta] - \nabla_y^\varepsilon(F) + (v, y)_1 - (w, x)_1 \equiv 0 \mod t. \end{aligned}$$

- Consider now EDx: on one hand, we get

$$EDx \to ExD + tEv + t^2E\delta$$

$$\begin{aligned}
&\to xED + xD + t^2\alpha D + tvE + t(r+1)v + t^2\delta E + t^2(r+1)\delta \\
&\to xDE + rxD + txu + t^2 x\gamma \\
&\quad + xD + t^2\alpha D + tEv + t^2\delta E + t^2(r+1)\delta
\end{aligned}$$

and on the other

$$\begin{aligned}
EDx &\to DEx + rDx + tux + t^2\gamma x \\
&\to DxE + Dx + t^2 D\alpha + rDx + tux + t^2\gamma x \\
&\to (xD + tv + t^2\delta)(E + r + 1) + t^2 D\alpha + tux + t^2\gamma x.
\end{aligned}$$

We obtain in this case the following two equations:

$$\begin{aligned}
&xu - ux + Ev - vE - (r+1)v \equiv 0 \mod t, \\
&[x, \gamma] + [\alpha, D] + (E, v)_1 = 0.
\end{aligned}$$

- Finally, we look at EDy.

$$\begin{aligned}
EDy &\to EyD + EF + tEw + t^2 E\zeta \\
&\to yED + yD + t^2\beta D + FE + (r+1)F + t^2(EF)_2 + tEw + t^2 E\zeta \\
&\to y(DE + rD + tu + t^2\gamma) + yD + t^2\beta D + FE + (r+1)F \\
&\quad + t^2(EF)_2 + tEw + t^2 E\zeta; \\
EDy &\to DEy + rDy + tuy + t^2\gamma y \\
&\to Dy + DyE + t^2 D\beta + rDy + tuy + t^2\gamma y \\
&\to (yD + F + tw + t^2\zeta)(E + r + 1) + t^2 D\alpha + tuy + t^2\gamma y.
\end{aligned}$$

We get here

$$\begin{aligned}
&yu + Ew - w(E + r + 1) - uy \equiv 0 \mod t, \\
&[y, \gamma] + [\beta, D] + \nabla_y^\beta(F) + \nabla_x^\alpha(F) + (yu)_1 + (Ew)_1 = 0.
\end{aligned}$$

Comparing these equations with the differentials in **3.9** we arrive to the desired claim. □

5.2 DEFORMATIONS OF $\mathsf{Diff}(\mathcal{A})$

5.9. We now use our characterization of $HH^2(A)$ in Proposition 3.15. We consider the rewriting system induced by a generic 2-cocycle

$$\begin{aligned}
&yx \to xy && Dx \to xD + t\lambda yD - t\lambda y^{r+1}E \\
&Dy \to yD + F + tg + thD + t\lambda y\bar{F}E && Ex \to xE + x \\
&Ey \to yE + y && ED \to DE + rD + tf,
\end{aligned}$$

with $\lambda \in \Bbbk$, $g \in S_{r+1}$, $h \in S_1$ and $f \in S_r$. We know form that proposition that every cocycle is cohomologous to one of this form.

In order to be confluent modulo t^3, this rewriting system must satisfy the two conditions of 5.8. The first one holds by construction and, considering the 3-cochain

$$\eta = (\lambda yg + \lambda^2 y^2 \bar{F} E + \lambda h y^{r+1} E)\hat{x}\hat{y}\hat{D} + \lambda y f \hat{x}\hat{D}\hat{E} + h f \hat{y}\hat{D}\hat{E} \tag{5.4}$$

in $\mathfrak{X}$, the second condition reads $d\xi = -\eta$. It is straightforward to see that $d\eta = 0$.

5.10. *Example.* Suppose $\lambda = 0$ and $h = 0$. Then $\eta = 0$ and our reduction system is confluent modulo t^3. Our rewriting system consist of

$$Dy \to yD + F + tg \qquad\qquad ED \to DE + rD + tf, \tag{5.5}$$

along with the other rules that determine A as in Example 5.6. We claim that this system is confluent over $\Bbbk[\![t]\!]$. Indeed, the only ambiguity whose solvability is different from that in the rewriting system that defined A is EDy, and we have

$$\begin{aligned}
EDy &\to EyD + EF + Etg \\
&\to yED + yD + FE + (r+1)F + tgE + t(r+1)g \\
&\to yDE + ryD + tyf + yD + FE + (r+1)F + tgE + t(r+1)g \\
EDy &\to DEy + rDy + tfy \\
&\to DyE + (r+1)Dy + tfy \\
&\to (yD + F + tg)(E + r + 1) + tfy.
\end{aligned}$$

The algebra obtained from (5.5) as in Bergman's Diamond Lemma 5.4 is a formal deformation of A. We therefore have here as many formal deformations as there are pairs (f,g). This determines a subspace of dimension $2r$ of $HH^2(A)$ of infinitesimal deformations which can be integrated.

5.11. *Example.* When $\lambda = f = 0$, the only rule that differs from the original rewriting system of A is

$$Dy \to yD + F + tg + thD.$$

The system is easily seen to be confluent over $\Bbbk[\![t]\!]$. This determines another subspace of integrable cocycles in $HH^2(A)$, this one of dimension $r+1$.

5.12. We can translate the condition that a deformation be trivial to one in terms of cocycles.

Proposition. *Let $\phi \in S_r$ and $\lambda_0 \in \Bbbk$ be such that $x\phi + \lambda_0 y^{r+1} = \lambda y f$. The cocycle η of equation (5.4) is a coboundary if and only if*

$$[\phi + \lambda_0(F_x + \bar{F})]y - hf \in \langle xF_x, xF_y, yF_y \rangle. \tag{5.6}$$

Proof. We follow the process we carried out in **3.14** to compute cohomology. Once we have finished adding coboundaries to η, the remainder will be a coboundary if and only if it is zero. To begin with, we should find a 2-cochain that covers the component $\hat{x}\hat{y}\hat{E}$ of η; as it is zero, this step is not necessary. For the component in $\hat{x} \wedge \hat{y} \wedge \hat{D}$, it is easy to see that

$$\begin{aligned}&d^2([\lambda gE + \tfrac{1}{2}(\lambda^2\bar{F}y + \lambda hy^r)(E^2 - E)]\hat{x}\hat{D}) \\ &= (\lambda yg + \lambda^2y^2\bar{F}E + \lambda hy^{r+1}E)\hat{x}\hat{y}\hat{D}.\end{aligned}$$

Finally, we must cover $\hat{x} \wedge \hat{D} \wedge \hat{E}$. We have

$$\begin{aligned}&d^2(-\lambda_0 y\hat{x}\hat{E} + (\lambda_0\bar{F} - \phi)E\hat{D}\hat{E}) \\ &= \lambda_0 F\hat{x}\hat{D}\hat{E} - \lambda_0 yF_x\hat{y}\hat{D}\hat{E} - (\lambda_0\bar{F} - \phi)x\hat{x}\hat{D}\hat{E} - (\lambda_0\bar{F} - \phi)y\hat{y}\hat{D}\hat{E} \\ &= (\lambda_0 y^{r+1} + x\phi)\hat{x}\hat{D}\hat{E} + [\phi - \lambda_0(F_x + \bar{F})]y\hat{y}\hat{D}\hat{E}\end{aligned}$$

and this implies that η is cohomologous to $([\phi - \lambda_0(F_x + \bar{F})]y - hf)\hat{y}\hat{D}\hat{E}$. The conclusion now follows from what we did in **3.14**. □

5.13. *Example.* The deformation induced by a general cocycle with $f = 0$ satisfies the condition of the proposition to be a 3-coboundary.

5.14. *Example.* Let us give an example in which confluence is not achieved. Let us choose $h \in S_1$ and $f \in S_r$ such that $hf \notin \langle xF_x, xF_y, yF_y\rangle$. If we consider the rewriting system (5.3) for this particular choice, it follows at once from Proposition 5.12 that it is not confluent. This means that the corresponding cocycle is *obstructed.* In the language of M. Gerstenhaber, we have obtained a infinitesimal deformation that does not integrate, not even to a second order deformation.

5.15. *Example.* Consider, finally, the 2-cocycle $\omega_2 = (yD - y^{r+1}E)\hat{x}\hat{D} + y\bar{F}E\hat{y}\hat{D}$ of **3.14**. The corresponding deformation is the one in **5.9** obtained by taking every parameter apart from λ equal to zero. As it is a cocycle, we know that the system is confluent modulo t^2. The obstruction is in this case

$$\eta = \lambda^2y^2\bar{F}E\hat{x}\hat{y}\hat{D}.$$

Proceeding as in **5.12**, we take $\phi = \lambda_0 = 0$ and we see that condition (5.6) is trivially fulfilled. We follow the proof of Proposition 3.15 to construct a preimage of η: it is straightforward to check that

$$d^2\left(\tfrac{1}{2}\lambda^2\bar{F}y(E^2 - E)\hat{x} \wedge \hat{D}\right) = \lambda^2y^2\bar{F}E\hat{x} \wedge \hat{y} \wedge \hat{D}.$$

We now consider the deformation given by

$$\begin{aligned}&Dx \to xD + t\lambda(yD - y^{r+1}E) - t^2\tfrac{1}{2}\lambda^2\bar{F}y(E^2 - E), \\ &Dy \to yD + F + t\lambda y\bar{F}E.\end{aligned}$$

This rewriting system is confluent over $\Bbbk[t]/(t^3)$. We claim that it confluent over $\Bbbk[\![t]\!]$. Let us call $w = \lambda y\bar{F}E$, $v = \lambda(yD - y^{r+1}E)$ and $\delta = \frac{1}{2}\lambda^2\bar{F}y(E^2 - E)$. Let us examine the nontrivial ambiguities; we start by EDy.

$$\begin{aligned}
EDy &\to EyD + EF + tEw \\
&\quad \to yED + yD + FE + (r+1)F + twE + t(r+1)w \\
EDy &\to DEy + rDy \to Dy + DyE + rDy \\
&\quad \to (yD + F + tw)(E + r + 1)
\end{aligned}$$

This is trivially satisfied. Next ambiguity is EDx.

$$\begin{aligned}
EDx &\to ExD + tEv + t^2E\delta \\
&\quad \to xED + xD + tvE + t(r+1)v + t^2\delta E + t^2(r+1)\delta \\
&\quad \to xDE + rxD + xD + tvE + t(r+1)v + t^2\delta E + t^2(r+1)\delta;
\end{aligned}$$

on the other hand,

$$\begin{aligned}
EDx &\to DEx + rDx \to DxE + Dx + rDx \\
&\quad \to (xD + tv + t^2\delta)(E + r + 1)
\end{aligned}$$

so this is also satisfied. Let us now look at Dyx; we have

$$\begin{aligned}
Dyx &\to Dxy \\
&\quad \to xDy + t\lambda(yD - y^{r+1}E)y - \tfrac{1}{2}t^2\lambda^2\bar{F}y(E^2 - E))y \\
&\quad \to xDy + t\lambda(yDy - y^{r+2}E - y^{r+2}) \\
&\qquad - t^2\tfrac{1}{2}\lambda^2\bar{F}y\left(y(E^2 - E) + 2yE\right) \\
&\quad \to x(yD + F + t\lambda y\bar{F}E) + t\lambda y(yD + F + t\lambda y\bar{F}E) - t\lambda y^{r+2}E - t\lambda y^{r+2} \\
&\qquad - t^2\tfrac{1}{2}\lambda^2\bar{F}y\left(y(E^2 - E) + 2yE\right) \\
&\quad = xyD + xF + t\lambda(xy\bar{F}E + y^2D + yF - y^{r+2}E - y^{r+2}) \\
&\qquad - t^2\lambda^2\tfrac{1}{2}\bar{F}y^2(E^2 - E)
\end{aligned}$$

and

$$\begin{aligned}
Dyx &\to (yD + F + t\lambda y\bar{F}E)x \\
&\quad \to yDx + xF + t\lambda(xy\bar{F}E + xy\bar{F}) \\
&\quad \to yxD + t\lambda(y^2D - y^{r+2}E) - t^2\tfrac{1}{2}\lambda^2y^2\bar{F}(E^2 - E) + xF + t\lambda xy\bar{F}E + t\lambda xy\bar{F} \\
&\quad = xyD + xF + t\lambda(y^2D - y^{r+2}E + xy\bar{F}E + xy\bar{F}) - t^2\tfrac{1}{2}\lambda^2y^2\bar{F}(E^2 - E)
\end{aligned}$$

We conclude our deformation of second order is confluent.

5.3 Resumen

En este capítulo continuamos extrayendo consecuencias de nuestro cálculo de la cohomología de Hochschild de del álgebra de operadores diferenciales $\mathsf{Diff}(\mathcal{A})$ de un arreglo central de rectas $\mathcal{A}$ como la del Capítulo 3: en este capítulo, estudiamos la teoría de deformaciones formales del álgebra $\mathsf{Diff}(\mathcal{A})$ en el sentido de M. Gerstenhaber [Ger64].

Primero damos las definiciones básicas que involucran a las deformaciones formales y de orden finito y recordamos en **5.2** un resultado fundamental de Gerstenhaber que relaciona las deformaciones de un álgebra con su segundo grupo de cohomología calculado a partir del complejo de Hochschild.

Una vez que establecemos el Lema del Diamante de Bergman, en **5.4**, establecemos un resultado análogo al de Gerstenhaber que relaciona las deformaciones de nuestra álgebra $\mathsf{Diff}(\mathcal{A})$ con nuestro cálculo del segundo grupo de la cohomología de Hochschild en la Proposición 3.15 a partir de la resolución $\mathfrak{X}$ de **3.10**. Sea $\rho = a\hat{x} \wedge \hat{y} + b\hat{x} \wedge \hat{E} + c\hat{y} \wedge \hat{E} + u\hat{D} \wedge \hat{E} + v\hat{x} \wedge \hat{D} + w\hat{y} \wedge \hat{D}$ una 2-cocadena genérica en este complejo.

Proposición. *El sistema de reescritura* (5.2) *es confluente módulo t^2, esto es, es confluente sobre $\Bbbk[t]/(t^2)$, si y solo si la cocadena ρ es un cociclo. En este caso, el álgebra que se obtiene del sistema de reescritura como en el Lema del Diamante de Bergman es una deformación de primer orden de* $\mathsf{Diff}(\mathcal{A})$ *que es trivial si y solo si ρ es un coborde.*

Lo explícito de nuestros resultados sobre la cohomología de Hochschild de $\mathsf{Diff}(\mathcal{A})$ continúa siendo útil: en la Sección 5.2 consideramos cociclos que vienen de nuestra caracterización de $HH^2(\mathsf{Diff}(\mathcal{A}))$ en la Proposición 3.15. Consideramos el sistema de reescritura inducido por un 2-cociclo genérico

$$
\begin{aligned}
&yx \to xy && Dx \to xD + t\lambda yD - t\lambda y^{r+1}E \\
&Dy \to yD + F + tg + thD + t\lambda y\bar{F}E && Ex \to xE + x \\
&Ey \to yE + y && ED \to DE + rD + tf,
\end{aligned}
$$

con $\lambda \in \Bbbk$, $g \in S_{r+1}$, $h \in S_1$ y $f \in S_r$.

Mostramos, por un lado, que muchas de las deformaciones infinitesimales del álgebra pueden ser integradas a deformaciones y, por otro, exhibimos deformaciones infinitesimales obstruidas. Concretamente, vemos que si $f = 0$ el sistema de reescritura es confluente y nos provee de una deformación de primer orden de $\mathsf{Diff}(\mathcal{A})$. Lo mismo sucede si tomamos $\lambda = 0$ y $h = 0$. Estas elecciones determinan subespacios de deformaciones infinitesimales que pueden ser integradas. También, el cociclo ω_2 determina una deformación de primer orden que, siendo corregida con un término de orden dos, da lugar a una deformación formal. Por otra parte, si elegimos $h \in S_1$ y $f \in S_r$ tales que $hf \notin \langle xF_x, xF_y, yF_y \rangle$, obtenemos una deformación infinitesimal que no se integra.

– 6 –
The spectral sequence

Let (S, L) be a Lie–Rinehart pair as in **2.9** and let $U(S, L)$ be its universal enveloping algebra. In this chapter we construct a spectral sequence converging to the Hochschild cohomology of $U(S, L)$, we describe its second page in a meaningful way and give an interpretation of the differential of that page. Since for a free hyperplane arrangement $\mathcal{A}$ the enveloping algebra of the pair $(S, \operatorname{Der} \mathcal{A})$ is isomorphic to the algebra of differential operators tangent to $\mathcal{A}$ —as we saw in Theorem 2.19—, this spectral sequence gives us an alternative way to obtain our results of Proposition 3.15 on the Hochschild cohomology of $A = \operatorname{Diff} \mathcal{A}$ for a central arrangement of lines and provides a possible method for extending these results. In particular, with this method one can deal with arrangements of three and four lines, for which the approach of Chapter 3 is not practical, as observed in **3.16**.

6.1 A cohomology theory for Lie–Rinehart pairs

As we saw in Section 2.3, a Lie–Rinehart pair (S, L) consists of a commutative algebra S and a Lie algebra L with an S-module structure that acts on S by derivations, and which satisfies certain compatibility conditions that generalize those satisfied by S and $\operatorname{Der} S$. An example important to us is the pair $(S, \operatorname{Der} \mathcal{A})$ with second component the Lie algebra of derivations of an arrangement $\mathcal{A}$. We denote by $U = U(S, L)$ the universal enveloping algebra of (S, L), whose construction we dealt with in **2.13**.

Let (S, L) be a Lie–Rinehart pair. If M is a U-module, or, equivalently, a Lie–Rinehart module, the *Lie–Rinehart cohomology of the pair with values on M* was defined by G. Rinehart in [Rin63] to be

$$H^\bullet(L|S, M) := \operatorname{Ext}_U^\bullet(S, M).$$

This cohomology generalizes the usual Lie algebra cohomology of L by taking into account its interaction with S.

6.1. In many important situations, some of which will be illustrated in the examples below, L is a projective S-module, and in this case there is a well-known complex that computes the Lie–Rinehart cohomology.

Proposition. *Suppose that L is S-projective and let $\Lambda_S^\bullet L$ denote the exterior algebra of L over S.*

The complex of U-modules $U \otimes_S \Lambda_S^\bullet L$ with differentials

$$d_r(u \otimes \theta_1 \wedge \cdots \wedge \theta_r) = \sum_{i=1}^{r} (-1)^{i+1} u\theta_i \otimes \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_r \\ + \sum_{1 \le i < j \le r} (-1)^{i+j} u[\theta_i, \theta_j] \otimes \theta_1 \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \theta_r$$

whenever $\theta_1, \ldots, \theta_r \in L$, $u \in U$ and $r \ge 1$ is an U-projective resolution of S with augmentation

$$\varepsilon : U \otimes_S S \ni u \otimes s \mapsto u \cdot s \in S.$$

In particular, the complex $\hom_S(\Lambda_S^\bullet L, M)$ with Chevalley–Eilenberg differentials computes the Lie–Rinehart cohomology $H^\bullet(L|S, M)$.

Proof. This is Theorem 4.2 in [Rin63]. □

6.2. *Example.* For the pair $(\Bbbk, \mathfrak{g})$ with $\mathfrak{g}$ a Lie algebra, M is simply a $\mathfrak{g}$-Lie module and the complex $\hom_{\Bbbk}(\Lambda_{\Bbbk}^\bullet L, M)$ is the standard complex that computes the Lie algebra cohomology $H^\bullet(\mathfrak{g}, M)$, as in §9 of the article [CE48] by C. Chevalley and S. Eilenberg.

6.3. *Example.* If M is a differential manifold and $S = C^\infty(M)$, then $L = \mathfrak{X}(M)$ is finitely generated and projective over S—this is Proposition 11.32 in the book by J. Nestruev [Nes03]. The complex $\hom_S(\Lambda_S^\bullet L, S)$ is precisely the de Rham complex $\Omega^\bullet(M)$ of differential forms and therefore the cohomology $H^\bullet(L|S, S)$ coincides with the de Rham cohomology of M.

6.4. *Example.* For the pair (S, L) associated to a free hyperplane arrangement $\mathcal{A}$, the complex $\hom_S(\Lambda_S^\bullet L, S)$ is the complex of logarithmic differential forms $\Omega^\bullet(\mathcal{A})$ that we met in **1.33**, and its cohomology is isomorphic to the Orlik–Solomon algebra of $\mathcal{A}$, by the result of J. Wiens and S. Yuzvinsky that we stated in **1.55**. When $\Bbbk = \mathbb{C}$, this algebra is, in turn, isomorphic to the cohomology of the complement of the arrangement.

6.2 The spectral sequence

Let (S, L) be a Lie–Rinehart pair and let U be its enveloping algebra. In this section we construct a spectral sequence that converges to the Hochschild cohomology of U. In order to do so we follow the ideas and tools developed by Th. Lambre and P. Le Meur in [LLM18]. In particular, we recall from that paper the construction of an adjunction between the category of U-modules and that of U-bimodules.

6.5. If M is a U-bimodule, *the S-invariant subspace* of M is

$$M^S := \{m \in M : sm = ms \text{ for all } s \in S\}.$$

This is the maximal symmetric S-subbimodule of M and it is an U-module if we let each $\alpha \in L$ act so that

$$\alpha \cdot m := \alpha m - m\alpha$$

for $m \in M^S$. The map $\mathsf{hom}_{S^e}(S, M) \ni f \mapsto f(1) \in M^S$ is bijective and becomes U-linear if we let U act on its domain with

$$(\alpha \cdot \varphi)(s) = \alpha\varphi(s) - \varphi(s)\alpha - \varphi(\alpha(s)), \qquad (t \cdot \varphi)(s) = t\varphi(s) \tag{6.1}$$

when $\alpha \in L$, $\varphi \in \mathsf{hom}_{S^e}(S, M)$ and $s, t \in S$. What is more, the assignment

$$G : {}_U\mathsf{Mod}_U \ni M \mapsto \mathsf{hom}_{S^e}(S, M) \in {}_U\mathsf{Mod}$$

is functorial.

Let, on the other hand, N be a left U-module. Again, the inclusion of S in U endows U with a structure of left S-module; since S is commutative, N can also be regarded as a right S-module. It is clear then that $U \otimes_S N$ is a left U-module and a right S-module. We can turn it into a right U-module setting, for $u \in U$, $n \in N$ and $\alpha \in L$

$$(u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha(n).$$

This construction extends to morphisms and defines a functor $F : {}_U\mathsf{Mod} \to {}_U\mathsf{Mod}_U$ with $F(N) = U \otimes_S N$. With these two functors in hand, we can state the very useful Proposition 3.4.1 of [LLM18].

Proposition. *The functor F is left adjoint to G.* □

6.6. Once we have established the following lemma we will be ready to construct the spectral sequence we are after.

Lemma. *Assume that L is a projective S-module. Let M be an U^e-module and let $M \to I^\bullet$ be an injective resolution of U as an U^e-module.*

(*i*) *The cohomology of the complex $\mathsf{hom}_{S^e}(S, I^\bullet)$ is $H^\bullet(S, M)$.*

(*ii*) *The U-module structure on $\mathsf{hom}_{S^e}(S, I^\bullet)$ defined in (6.1) induces an U-module structure on $H^\bullet(S, M)$.*

Proof. The PBW-theorem in [Rin63, §3] ensures that U is a projective S-module: using Proposition IX.2.3 of the book [CE56] by H. Cartan and S. Eilenberg, we obtain that U^e is S^e-projective. Given an injective U^e-module I, the functor $\mathsf{hom}_{S^e}(-, I)$ is naturally isomorphic to $\mathsf{hom}_{U^e}(U^e \otimes_{S^e} -, I)$, which is the composition of the exact functors $\mathsf{hom}_{U^e}(-, I)$ and $U^e \otimes_{S^e} -$, and therefore I is an injective S^e-module. As a consequence of this, $M \to I^\bullet$ is actually a resolution of M by S^e-injective modules, so that the cohomology of $\mathsf{hom}_{S^e}(S, I^\bullet)$ is $\mathsf{Ext}_{S^e}(S, U)$.

In order to prove the assertion of (*ii*), it is enough to see that the differential of the complex $\mathsf{hom}_{S^e}(S, I^\bullet)$ is a morphism of U-modules, and this follows from the functoriality of $G = \mathsf{hom}_{S^e}(S, -)$. □

6.7. Theorem. *Assume L is S-projective and let N and M be a left U-module and a U^e-module. There is a first-quadrant spectral sequence $E_\bullet$ converging to $\mathsf{Ext}^\bullet_{U^e}(F(N), M)$ with second page*

$$E_2^{p,q} = \mathsf{Ext}^p_U(N, H^q(S, M)).$$

Proof. Let $Q_\bullet \to N$ be an U-projective resolution of N and let $M \to I^\bullet$ be an U^e-injective resolution. Consider the double complex

$$X^{\bullet,\bullet} = \mathsf{hom}_U(Q_\bullet, G(I^\bullet))$$

and denote its total complex by $Z^\bullet$. There are two spectral sequences for this double complex: we will use the first one to compute $H^\bullet(Z)$ and the second one will be the one we are looking for. From the filtration on $Z^\bullet$ with

$$\tilde{F}^q Z^p = \bigoplus_{\substack{r+s=p\\ s\geq q}} X^{r,s}$$

we obtain a first spectral sequence converging to $H(Z^\bullet)$. Its zeroth page $\tilde{E}_0$ has

$$\tilde{E}_0^{p,q} = \mathsf{hom}_U(Q_p, G(I^q))$$

and its differential comes from the one on $Q_\bullet$. We claim that for each $s \geq 0$, the functor $\mathsf{hom}_U(-, G(I^s))$ is exact. Indeed, by the adjunction of Proposition 6.5 it is naturally isomorphic to $\mathsf{hom}_{U^e}(F(-), I^s)$, which is the composition of the functors $F = U \otimes_S (-)$ and $\mathsf{hom}_{U^e}(-, I^s)$ and these are exact because U is left projective over S and I^s is U^e-injective. The first page $\tilde{E}_1$ of the spectral sequence is therefore given by

$$\tilde{E}_1^{p,q} = \begin{cases} \mathsf{hom}_U(N, G(I^q)) \cong \mathsf{hom}_{U^e}(F(N), I^q) & \text{if } p = 0; \\ 0 & \text{if } p \neq 0 \end{cases}$$

and its differential is induced by that of $I^\bullet$. Now, as the complex $\mathsf{hom}_{U^e}(F(N), I^\bullet)$ computes $\mathsf{Ext}^\bullet_{U^e}(F(N), M)$ using injectives, we obtain that the second page has

$$\tilde{E}_2^{p,q} = \begin{cases} \mathsf{Ext}^q_{U^e}(F(N), M) & \text{if } p = 0; \\ 0 & \text{if } p \neq 0. \end{cases}$$

This spectral sequence thus degenerates at its the second page, so that we see that $H^\bullet(Z)$ is isomorphic to $\mathsf{Ext}^\bullet_{U^e}(F(N), M)$.

The other filtration on $Z^\bullet$ is given by

$$F^p Z^q = \bigoplus_{\substack{r+s=q\\ r\geq p}} X^{r,s}$$

and determines a second spectral sequence $E_\bullet$ that also converges to $H(Z^\bullet)$. Its differential on E_0 is induced by the one on $I^\bullet$; as Q_p is U-projective for each $p \geq 0$, the cohomology of $\mathsf{hom}_U(Q_p, G(I^\bullet))$ is given in its qth place precisely by $E_1^{p,q} = \mathsf{hom}_U(Q_p, H^q(S, M))$ —recall that, according to Lemma 6.6, the cohomology of $G(I^\bullet)$ is $H^\bullet(S, M)$. Since the differentials in E_1 are induced by those of $Q_\bullet$, for each $q \geq 0$ the cohomology of the row $E_1^{\bullet,q}$ is $E_2^{p,q} = \mathsf{Ext}^p_U(N, H^q(S, M))$. The spectral sequence $E_\bullet$ is therefore the one we were looking for. $\square$

6.8. Specializing Theorem 6.7 to the case in which $N = S$ we obtain the following corollary, which is in fact the result we are mainly interested in.

Corollary. *If L is S-projective then for each U^e-module M there is a first-quadrant spectral sequence $E_\bullet$ converging to $H^\bullet(U, M)$ with second page*

$$E_2^{p,q} = H^p(L|S, H^q(S, M)).$$

6.9. The following examples illustrate what happens in the two extreme situations.

Example. Suppose first that $L = 0$. The enveloping algebra U is just S and $\Lambda_S^\bullet L = S$, so the resolution $U \otimes \Lambda_S^\bullet L$ of S is simply $Q_\bullet = U \otimes_S S$. The double complex $X^{\bullet,\bullet}$ is therefore $\mathsf{hom}_S(S, \mathsf{hom}_{S^e}(S, I^\bullet))$, which is isomorphic to $\mathsf{hom}_{S^e}(S, I^\bullet)$ and the cohomology of the complex $Z^\bullet$ in the proof is $HH^\bullet(S)$, the Hochschild cohomology of S.

Example. If $S = \Bbbk$ and $L = \mathfrak{g}$ is a Lie algebra then $H^\bullet(S, U) = \mathsf{Ext}_{\Bbbk^e}^\bullet(\Bbbk, U)$ is just U, the second page of our spectral sequence is $H^\bullet(\mathfrak{g}, U)$ and we recover from Corollary 6.8 the well-known fact that the Hochschild cohomology of the enveloping algebra of a Lie algebra equals its Lie cohomology with values on U with the adjoint action, as in [CE56, XIII.5.1].

6.10. Another specialization of Theorem 6.7 allows us to recover one of the main results of [LLM18], which we recall here. In proving it, we will use the following simple lemma a few times.

Lemma. *Let A be an algebra and T and P two A-modules such that T admits a projective resolution by finitely generated A-modules and P is flat. There is an isomorphism*

$$\mathsf{Ext}_A^\bullet(T, P) \cong \mathsf{Ext}_A^\bullet(T, A) \otimes_A P.$$

Proof. Let $Q_\bullet$ be such a resolution of T. For each $i \geq 0$, the evident map from $\mathsf{hom}_A(Q_i, A) \otimes_A P$ to $\mathsf{hom}_A(Q_i, P)$ is an isomorphism because Q_i is finitely generated and projective. As P is flat, the cohomology of the complex $\mathsf{hom}_A(Q_\bullet, A) \otimes_A P$ is isomorphic to $\mathsf{Ext}_A^\bullet(T, A) \otimes_A P$. □

6.11. Theorem (Th. Lambre and P. Le Meur, [LLM18, Theorem 1])**.** *Let (S, L) be a Lie–Rinehart pair such that S has Van den Bergh duality in dimension n and L is finitely generated and projective with constant rank d as an S-module and let $L^\vee = \mathsf{hom}_S(L, S)$. The enveloping algebra U of the pair has Van den Bergh duality in dimension $n + d$ and there is an isomorphism of U-bimodules*

$$\mathsf{Ext}_{U^e}^{n+d}(U, U^e) \cong \Lambda_S^d L^\vee \otimes_S \mathsf{Ext}_{S^e}^n(S, S^e) \otimes_S U.$$

Notice that this expression for the dualizing module of U is not the one that appears in the original paper, but an immediate application of Lemma 3.5.2 in [LLM18] yields the desired identification.

Proof. The homological smoothness of U follows from Lemma 5.1.2 of [LLM18], whose proof does not depend on this theorem.

Let us write D for the dualizing bimodule $\mathrm{Ext}^n_{S^e}(S, S\otimes S)$. We take, specializing Theorem 6.7, $N = S$ and $M = U \otimes U$ to obtain a spectral sequence $E_\bullet$ such that

$$E_2^{p,q} = H^p(L|S, H^q(S, U \otimes U)) \implies H^{p+q}(U, U \otimes U).$$

Let us first deal with $H^q(S, U \otimes U)$. As we observed in the proof of Lemma 6.6, the U^e-module $U \otimes U$ is S^e-projective and, since S has Van den Berg duality, it admits a resolution by finitely generated projective S^e-modules. We may therefore use Lemma 6.10 to see that

$$H^q(S, U \otimes U) \cong H^q(S, S^e) \otimes_{S^e} (U \otimes U),$$

which is zero if $q \neq n$ and isomorphic to $D \otimes_{S^e} (U \otimes U)$ if $q = n$. As a consequence of this, our spectral sequence $E_\bullet$ degenerates at its the second page and thus $H^{p+n}(U, U \otimes U)$ is isomorphic to $H^p(L|S, U \otimes_S D \otimes_S U)$ for each $p \in \mathbb{Z}$.

As the Chevalley–Eilenberg complex from Proposition 6.1 is an U-projective resolution of S by finitely generated modules and D is S-projective because it is invertible —see Chapter 6 in the book [AF92] by F. Anderson and K. Fuller—, another application of Lemma 6.10 yields an isomorphism

$$H^\bullet(L|S, U \otimes_S D \otimes_S U) \cong H^\bullet(L|S, U) \otimes_U (U \otimes_S D \otimes_S U).$$

Now, our hypotheses on L are such that Theorem 2.10 in [Hue99] tells us that $H^p(L|S, U)$ is zero if $p \neq d$ and is isomorphic to $\Lambda^d_S L^\vee$ if $p = d$, so that actually

$$H^i(U, U \otimes U) \cong \begin{cases} \Lambda^d_S L^\vee \otimes_U (U \otimes_S D \otimes_S U), & \text{if } i = n + d; \\ 0 & \text{otherwise.} \end{cases}$$

The dualizing bimodule of U is therefore isomorphic to $\Lambda^d L^\vee \otimes_S D \otimes_S U$. □

6.3 THE LIE MODULE STRUCTURE ON $H^\bullet(S, U)$

Let (S, L) be a Lie–Rinehart pair and let U be its enveloping algebra. As we have already seen, U can be regarded as an S^e-module with the action defined by $(s|t) \cdot u = stu$ for s and t in S and u in U. The Hochschild cohomology of S with values on U, denoted as before by $H^\bullet(S, U)$, has an U-module structure —described in Lemma 6.6— that arises when we compute this cohomology from an *injective* resolution of U as a module over U^e. The computation of this structure in concrete examples is therefore rather inconvenient: indeed, we rarely compute Hochschild cohomology using injective resolutions.

The action of U on $H^\bullet(S, U)$ is determined by actions of S and of L that satisfy the identities in (2.2). Let M be a U-bimodule. In this section we construct an L-module structure on $H^\bullet(S, M)$ using this time *an S^e-projective resolution of S* and we show that when $M = U$, it coincides with the action of L on $H^\bullet(S, U)$ that we already had. This will allow us to compute the latter in practice.

The construction of the action

6.12. Let $\varepsilon : P_\bullet \to S$ be an S^e-projective resolution. Given a U-bimodule M, we will define for each $\alpha \in L$ a linear endomorphism $\alpha_\bullet^\sharp$ of the complex $\mathsf{hom}_{S^e}(P_\bullet, M)$ which induces on its cohomology $H^\bullet(S, U)$ a Lie algebra action of L. In order to do so, we will adapt with minor changes the considerations in the article [SÁ17] by M. Suárez-Álvarez. There, there is a construction, for an algebra A, a derivation $\delta : A \to A$ and a so called δ-operator $f : N \to N$, of a canonical morphism of graded vector spaces $\nabla_f : \mathsf{Ext}_A^\bullet(N, N) \to \mathsf{Ext}_A^\bullet(N, N)$ which, suitably specialized, gives a way to compute part of the Gerstenhaber bracket in the Hochschild cohomology of an associative algebra. The adaptation of this result to our situation is not obvious. Let us take $A = S^e$. Each $\alpha \in L$ determines a derivation of A; as opposed to the situation in [SÁ17], what we need here is a graded automorphism of $\mathsf{Ext}_A^\bullet(S, M)$ and not of $\mathsf{Ext}_A^\bullet(N, N)$. The observation that allows us to solve the problem is that there is a canonical action of L on U by derivations that restricts to the action of L on S.

6.13. Let A be an algebra and let $\delta : A \to A$ a derivation. Given an A-module N, we say that a linear map $f : N \to N$ is a δ-*operator* if for every $a \in A$ and $n \in N$ we have

$$f(an) = \delta(a)n + af(n).$$

If, moreover, $\varepsilon : P_\bullet \to N$ is an A-projective resolution of N, a δ-*lifting* of f to $P_\bullet$ is a family of δ-operators $f_\bullet = (f_i : P_i \to P_i, i \geq 0)$ such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & N \arrow[d, "f"] \\ \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & N \end{tikzcd}$$

commutes. The following proposition, extracted from [SÁ17, §1.4], ensures δ-liftings exist and are in some sense unique.

Proposition. *Let N be a left A-module, let $f : N \to N$ be a δ-operator and let $\varepsilon : P_\bullet \to N$ be a projective resolution.*

(*i*) *There exists a δ-lifting $f_\bullet$ of f to $P_\bullet$.*

(*ii*) *If $f_\bullet$ and $f'_\bullet$ are two δ-liftings of f to $P_\bullet$ then $f_\bullet$ and $f'_\bullet$ are homotopic by an A-linear homotopy.* □

6.14. We return to our setting with a Lie–Rinehart pair (S, L). Let $\alpha \in L$. As L acts on S by derivations, we can regard α as a derivation of S. It is easy to verify that the unique linear map $\alpha^e : S^e \to S^e$ such that

$$\alpha^e(s|t) = \alpha(s)|t + s|\alpha(t)$$

is a derivation of S^e. Viewing, as usual, S as a left S^e-module via $(s|t) \cdot f \coloneqq sft$, the map α becomes an α^e-operator: indeed, if $s|t \in S^e$ and $f \in S$ we have

$$\alpha((s|t)f) = \alpha(s)ft + s\alpha(f)t + sf\alpha(t) = \alpha^e(s|t)f + (s|t)\alpha(f).$$

6.15. *Example.* The standard *bar* resolution $B_\bullet \to S$ is an S^e-projective resolution that has $B_i = S^{\otimes i+2}$ —we refer for this to [CE56, IX.6]. Given $\alpha \in L$, there is a canonical α^e-lifting $\alpha_\bullet$ to $B_\bullet$: if $i \geq 0$; the linear map $\alpha_i : B_i \to B_i$ such that

$$\alpha_i(s_0|s_1|\dots|s_i|s_{i+1}) = \sum_{j=1}^{r} s_0|s_1|\dots|\alpha(s_j)|\dots|s_i|s_{i+1}$$

is an α^e-operator and it is not difficult to see that $\alpha_\bullet = (\alpha_i : i \geq 0)$ is an α^e-lifting of α. This particular way of choosing liftings gives us a function $L \ni \alpha \mapsto \alpha_\bullet \in \mathsf{End}_{\Bbbk}(P_\bullet)$ which is, as a small calculation shows, a morphism of Lie algebras.

6.16. Fix $\alpha \in L$, an S^e-projective resolution $P_\bullet \to S$ and a U-bimodule M. Let us *choose one* among all α^e-liftings of $\alpha : S \to S$ to $P_\bullet$ provided by Proposition 6.13 and call it $\alpha_\bullet$. Given $i \geq 0$ and $\phi \in \mathsf{hom}_{S^e}(P_i, M)$, we define $\alpha_i^\sharp(\phi) : P_i \to M$ by

$$\alpha_i^\sharp(\phi)(p) = [\alpha, \phi(p)] - \phi(\alpha_i(p)) \qquad \text{for } p \in P_i. \tag{6.2}$$

Proposition. *For each $i \geq 0$, the rule (6.2) defines a function*

$$\alpha_i^\sharp : \mathsf{hom}_{S^e}(P_i, M) \to \mathsf{hom}_{S^e}(P_i, M).$$

The collection $\alpha_\bullet^\sharp = (\alpha_i^\sharp)_{i\geq 0}$ is an endomorphism of the complex of vector spaces $\mathsf{hom}_{S^e}(P_\bullet, M)$.

Proof. For the first claim, we show that $\alpha_i^\sharp(\phi)$ is a morphism of S^e-modules: given $p \in P_i$ and $s|t \in S^e$ we have

$$\begin{aligned}\alpha_i^\sharp(\phi)\,((s|t)p) &= [\alpha, s\phi(p)t] - \phi\,(\alpha_i\,((s|t)p)) \\ &= \alpha(s)\phi(p)t + s[\alpha, \phi(p)]t + s\phi(p)\alpha(t) - \phi\,(\alpha^e(s|t)p + (s|t)\alpha_i(p)) \\ &= s[\alpha, \phi(p)]t - (s|t)\phi\,(\alpha_i(p))\,.\end{aligned}$$

For the second one, we must see that the map $\alpha_\bullet^\sharp$ commutes with the differential of $\mathsf{hom}_{S^e}(P_\bullet, M)$. Given $i \geq 0$ and ϕ in $\mathsf{hom}_{S^e}(P_i, M)$, we have

$$d^*(\alpha_i^\sharp(\phi))(p) = \alpha_i^\sharp(\phi)(d(p)) = [\alpha, \phi(d(p))] - \phi\,(\alpha_i(d(p)))$$

and, as $\alpha_\bullet$ is a morphism of complexes, this is equal to $\alpha_{i+1}^\sharp(d^*\phi)$. □

6.17. Proposition 6.16 implies that $\alpha_\bullet^\sharp$ descends to cohomology and therefore induces a graded endomorphism $\nabla_\alpha^\bullet$ of $H^\bullet(S, U)$. In order to construct $\nabla_\alpha^\bullet$ we have chosen an α^e-lifting $\alpha_\bullet$: the next lemma shows that $\nabla_\alpha^\bullet$ is independent of that choice and, moreover, of the choice of the projective resolution $\varepsilon : P \to S$.

Lemma. *Fix $\alpha \in L$ and an U-bimodule M. Let $\varepsilon : P_\bullet \to S$ and $\varepsilon' : P'_\bullet \to S$ be two S^e-projective resolutions of S, let $\alpha_\bullet$ and $\alpha'_\bullet$ be α^e-liftings of α to $P_\bullet$ and to $P'_\bullet$ respectively and, finally, let $\alpha_\bullet^\sharp$ and $\alpha_\bullet'^\sharp$ be defined as in Proposition 6.16. If $g : P'_\bullet \to P_\bullet$ is a morphism of complexes lifting the identity of S, the diagram*

$$\begin{array}{ccc}
\mathsf{hom}_{S^e}(P_\bullet, M) & \xrightarrow{\alpha_\bullet^\sharp} & \mathsf{hom}_{S^e}(P_\bullet, M) \\
\big\downarrow{\scriptstyle g_\bullet^*} & & \big\downarrow{\scriptstyle g_\bullet^*} \\
\mathsf{hom}_{S^e}(P'_\bullet, M) & \xrightarrow{\alpha_\bullet'^\sharp} & \mathsf{hom}_{S^e}(P'_\bullet, M)
\end{array}$$

commutes up to homotopy.

Proof. The morphism of complexes of vector spaces $h_\bullet : g_\bullet\alpha'_\bullet - \alpha_\bullet g_\bullet : P'_\bullet \to P_\bullet$ is S^e-linear: indeed, if $i \geq 0$, $a \in S^e$ and $q \in P'_i$ we have

$$\begin{aligned}
h_i(aq) &= g_i(\alpha^e(a)q + a\alpha'_i(q)) - \alpha_i(ag_i(q)) \\
&= \alpha^e(a)g_i(q) + ag_i(\alpha'_i(q)) - \alpha^e(a)g_i(q) - a\alpha_i(g(q)) \\
&= ah_i(q).
\end{aligned}$$

The map $h_\bullet^* : \mathsf{hom}_{S^e}(P_\bullet, M) \to \mathsf{hom}_{S^e}(P'_\bullet, M)$ induced by $h_\bullet$ is homotopic to zero because $h_\bullet$ is a lifting of the zero map in S to the projective resolution $P_\bullet$. Let us show that $h_\bullet^*$ measures the failure in the commutativity of the diagram that appears in the statement. We have, for $i \geq 0$ and $\phi \in \mathsf{hom}_{S^e}(P_i, M)$,

$$\left(\alpha_i'^\sharp g_i^* - g_i^*\alpha_i^\sharp\right)(\phi) = \alpha_i'^\sharp(\phi \circ g_i) - g_i^*(\alpha_i^\sharp(\phi)) = \alpha_i'^\sharp(\phi \circ g_i) - (\alpha_i^\sharp(\phi)) \circ g_i,$$

and evaluating this last expression on $q \in P'_i$ we find that $\left(\alpha_i'^\sharp g_i^* - g_i^*\alpha_i^\sharp\right)(\phi)(q)$ is equal to

$$\begin{aligned}
&[\alpha, \phi(g_i(q))] - \phi(g_i(\alpha'_i(q))) - [\alpha, \phi(g_i(q))] + \phi(\alpha_i(g_i(q))) \\
&\quad = \phi(\alpha_i(g_i(q))) - \phi(g_i(\alpha'_i(q))) \\
&\quad = (g_i^*\alpha_i^* - \alpha_i'^* g_i^*)(\phi)(q).
\end{aligned}$$

We see from this that $\alpha_i'^\sharp g_i^* - g_i^*\alpha_i^\sharp = h_i^*$, which is, as we wanted, homotopic to zero. □

This lemma corresponds to the Lemma 1.6 of [SÁ17]; in our case, the key step was the cancellation that happened when we evaluated $\left(\alpha_i'^\sharp g_i^* - g_i^*\alpha_i^\sharp\right)(\phi)$ on an element of P'_i.

6.18. Now, with the help of Lemma 6.17, we see that each $\alpha \in L$ defines a *canonical* graded endomorphism of $H^\bullet(S, M)$.

Theorem. *Let M be an U-bimodule and let $\alpha \in L$. There is a morphism of graded vector spaces*

$$\nabla_\alpha^\bullet : H^\bullet(S, M) \to H^\bullet(S, M)$$

such that for each S^e-projective resolution $\varepsilon : P_\bullet \to S$ and each α^e-lifting $\alpha_\bullet$ of α to $P_\bullet$ the diagram

$$\begin{tikzcd} H(\mathsf{hom}_{S^e}(P_\bullet, M)) \arrow[r, "\nabla^\bullet_{\varepsilon,\alpha}"] \arrow[d, "\cong"] & H(\mathsf{hom}_{S^e}(P_\bullet, M)) \arrow[d, "\cong"] \\ H^\bullet(S, M) \arrow[r, "\nabla^\bullet_\alpha"] & H^\bullet(S, M) \end{tikzcd} \tag{6.3}$$

commutes.

Proof. Choosing an S^e-projective resolution $\varepsilon : P_\bullet \to S$ and an α^e-lifting of $\alpha : S \to S$ to $P_\bullet$, Proposition 6.16 gives us an endomorphism of complexes $\alpha_\bullet^\sharp$ on $\mathsf{hom}_{S^e}(P_\bullet, M)$: as the cohomology of this complex is $H^\bullet(S, M)$, this induces a graded endomorphism $\nabla^\bullet_{\varepsilon,\alpha}$ of $H^\bullet(S, M)$. The square (6.3) defines an unique graded endomorphism $\nabla^\bullet_\alpha$ of $H^\bullet(S, M)$; as an immediate consequence of Lemma 6.17, this endomorphism is independent of the choices of ε and of the α^e-lifting. □

6.19. *Example.* It is easy to describe the endomorphism ∇^0_α of $H^0(S, U)$ for any given $\alpha \in L$. Let us choose a resolution $P_\bullet$ of S with $P_0 = S^e$ and augmentation $\varepsilon : S^e \to S$ defined by $\varepsilon(s|t) = st$. As α^e is a α^e-operator and $\varepsilon \circ \alpha^e = \alpha \circ \varepsilon$, we may choose an α^e-lifting with $\alpha_0 = \alpha^e$. According to the rule (6.2) just before Proposition 6.16 we have

$$\alpha_0^\sharp(\phi)(1|1) = [\alpha, \phi(1|1)] \qquad \text{for all } \phi \in \mathsf{hom}_{S^e}(P_0, M). \tag{6.4}$$

Identifying, as usual, each $\phi \in \mathsf{hom}_{S^e}(S^e, U)$ with $\phi(1|1) \in U$, we can view $H^0(S, U)$ as a subspace of U and the equality (6.4) tells us that

$$\nabla^0_\alpha(u) = [\alpha, u] \qquad \text{for all } u \in H^0(S, U).$$

6.20. Theorem 6.18 defines an assignment $\nabla : \alpha \mapsto \nabla_\alpha$; we will now show that it actually gives rise to a Lie action of L on $H^\bullet(S, M)$, that is, that the identity $\nabla^\bullet_{[\alpha,\beta]} = [\nabla^\bullet_\alpha, \nabla^\bullet_\beta]$ holds.

Given α and β in L and $\varepsilon : P \to S$ an S^e-projective resolution, let $\alpha_\bullet$ and $\beta_\bullet$ be α^e- and β^e-liftings of α and of β to $P_\bullet$. Call $\gamma = [\alpha, \beta] \in L$: a straightforward calculation shows that $\gamma^e = \alpha^e \circ \beta^e - \beta^e \circ \alpha^e$.

Lemma. *In the setting of last paragraph, let M be an U-bimodule.*

(*i*) *The morphism of complexes $\gamma_\bullet := \alpha_\bullet \circ \beta_\bullet - \beta_\bullet \circ \alpha_\bullet$ is a γ^e-lifting of $\gamma : S \to S$.*

(*ii*) *Let $\gamma_i^\sharp$ be the endomorphism of $\mathsf{hom}_{S^e}(P_i, M)$ induced by $\gamma_\bullet$ as in Proposition 6.16. We have $\gamma_i^\sharp = \alpha_i^\sharp \circ \beta_i^\sharp - \beta_i^\sharp \circ \alpha_i^\sharp$.*

Proof. For each $i \geq 0$, the map γ_i is a γ^e-operator: given $p \in P_i$ and $a \in S^e$ we have

$$\begin{aligned}(\alpha_i \circ \beta_i)(ap) &= \alpha_i\left(\beta^e(a)p + a\beta_i(p)\right) \\ &= \alpha^e(\beta^e(a))p + \beta^e(a)\alpha_i(p) + \alpha^e(a)\beta_i(p) + a\alpha_i\beta_i(p)\end{aligned}$$

and therefore $\gamma_i(ap) = [\alpha^e, \beta^e](a)p + \gamma_i(p)$. As the a morphism of complexes $\gamma_\bullet$ lifts γ because L acts as a Lie algebra on S, we have proven the first statement.

In order to see the second one, we observe that for $\phi \in \mathsf{hom}_{S^e}(P_i, M)$ and $p \in P_i$ we have

$$\gamma_i^\sharp(\phi)(p) = [[\alpha,\beta], \phi(p)] - \phi\left(\alpha_i(\beta_i(p)) - \beta_i(\alpha_i(p))\right)$$

and, on the other hand,

$$\begin{aligned}\alpha_i^\sharp(\beta_i^\sharp(\phi))(p) &= [\alpha, (\beta_i^\sharp(\phi))(p)] - (\beta_i^\sharp(\phi))(\alpha_i(p)) \\ &= [\alpha,[\beta,\phi(p)]] - [\alpha, \phi(\beta_i(p))] - [\beta, \phi(\alpha_i(p))] + \phi(\beta_i(\alpha_i(p))).\end{aligned}$$

These two expressions, together with the Jacobi identity, allow us to conclude that

$$\alpha_i^\sharp(\beta_i^\sharp(\phi))(p) - \beta_i^\sharp(\alpha_i^\sharp(\phi))(p) = \gamma_i^\sharp(\phi)(p),$$

which is just what we wanted. □

6.21. Proposition. *The assignment*

$$\nabla : L \ni \alpha \mapsto \nabla_\alpha^\bullet \in \mathsf{End}_{\Bbbk}(H^\bullet(S,M))$$

is a morphism of Lie algebras.

Proof. Let $\alpha, \beta \in L$ and call $\gamma = [\alpha,\beta]$. Let $\alpha_\bullet$, $\beta_\bullet$ and $\gamma_\bullet$ be α^e, β^e and γ^e-liftings, respectively. Observe that it not necessarily the case that $\gamma_\bullet$ is the commutator of $\alpha_\bullet$ and $\beta_\bullet$. Let $\alpha_\bullet^\sharp$, $\beta_\bullet^\sharp$ and $\gamma_\bullet^\sharp$ be the endomorphisms of $\mathsf{hom}_{S^e}(P_\bullet, M)$ defined as in Proposition 6.16 and consider the endomorphism $\theta_\bullet$ of $\mathsf{hom}_{S^e}(P_\bullet, M)$ with

$$\theta_i(\phi)(p) = [\gamma, \phi(p)] - \phi\left(\alpha_i \circ \beta_i(p) - \beta_i \circ \alpha_i(p)\right),$$

where $i \geq 0$, $\phi \in \mathsf{hom}_{S^e}(P_i, M)$ and $p \in P_i$. As we have seen in the first part of Lemma 6.20, the commutator $[\alpha_\bullet, \beta_\bullet]$ is a γ^e-lifting of γ and therefore Lemma 6.17 tells us that the diagram

$$\begin{tikzcd} \mathsf{hom}_{S^e}(P_\bullet, M) \arrow[r, "\gamma_\bullet^\sharp"] \arrow[d, equal] & \mathsf{hom}_{S^e}(P_\bullet, M) \arrow[d, equal] \\ \mathsf{hom}_{S^e}(P_\bullet, M) \arrow[r, "\theta_\bullet"] & \mathsf{hom}_{S^e}(P_\bullet, M) \end{tikzcd}$$

commutes up to homotopy. Now, the second part of Lemma 6.20 states that $\theta_i = \alpha_i^\sharp \circ \beta_i^\sharp - \beta_i^\sharp \circ \alpha_i^\sharp$ and therefore $\theta_\bullet$ and $\gamma_\bullet^\sharp$ induce the same endomorphism on cohomology, that is,

$$\nabla_\gamma^\bullet = H([\alpha_\bullet^\sharp, \beta_\bullet^\sharp]).$$

Finally, using the linearity of the functor H we can conclude that $\nabla_\gamma^\bullet = [\nabla_\alpha^\bullet, \nabla_\beta^\bullet]$. □

COMPARING THE TWO ACTIONS OF L

6.22. In Lema 6.6 we constructed an U-module structure on $H^\bullet(S,U)$ using an U^e-injective resolution of U. As we have seen in Section 6.1, this is equivalent to having S- and L-module structures on $H^\bullet(S,U)$ that satisfy the identities in (2.2). We will now show that this L-module structure coincides with the one defined in Subsection 6.3, using an S^e-projective resolution of S.

Theorem. *Suppose L is S-projective. The L-module structure on $H^\bullet(S,U)$ defined in Lemma 6.6 using injectives is equal to the one defined in Theorem 6.18 using projectives.*

Proof. To begin with, we fix an U^e-injective resolution $\eta : U \to I^\bullet$, an S^e-projective resolution $\varepsilon : P_\bullet \to S$ and $\alpha \in L$. In Proposition 6.16, we constructed endomorphisms of complexes $\alpha_\bullet^\sharp$ of $\mathsf{hom}_{S^e}(P_\bullet, U)$ and of $\mathsf{hom}_{S^e}(P_\bullet, I^j)$ for each $j \geq 0$ —we denote them the same way— which induce the map ∇_α on their cohomologies $H^\bullet(S,U)$ and $H^\bullet(S,I^j)$. We claim that the map

$$\eta_* : \mathsf{hom}_{S^e}(P_\bullet, U) \ni \phi \longmapsto \eta \circ \phi \in \mathsf{hom}_{S^e}(P_\bullet, I^\bullet)$$

satisfies

$$\eta_*(\alpha_i^\sharp(\phi)) = \alpha_i^\sharp(\eta_*(\phi)) \tag{6.5}$$

for each $i \geq 0$ and $\phi \in \mathsf{hom}_{S^e}(P_i, U)$. Indeed, we have

$$\eta_*(\alpha_i^\sharp(\phi))(p) = \eta(\alpha_i^\sharp(\phi))(p) = \eta([\alpha, \phi(p)]) - \eta\left(\phi\left(\alpha_i(p)\right)\right)$$

and this is equal to $\alpha_i^\sharp(\eta_*(\phi))$ because η is a morphism of U-bimodules.

Let, on the other hand,

$$\varepsilon^* : \mathsf{hom}_{S^e}(S, I^\bullet) \ni \varphi \longmapsto \varphi \circ \varepsilon \in \mathsf{hom}_{S^e}(P_\bullet, I^\bullet).$$

For each $\alpha \in L$ and $\varphi \in \mathsf{hom}_{S^e}(S, I^\bullet)$ we have

$$\varepsilon^*(\alpha \cdot \varphi) = \alpha_0^\sharp(\varepsilon(\varphi)) \tag{6.6}$$

because, given $p \in P_0$,

$$\varepsilon^*(\alpha \cdot \varphi)(p) = \alpha \cdot \varphi(\varepsilon(p)) = [\alpha, \varphi(\varepsilon(p))] - \varphi(\alpha(\varepsilon(p)))$$

and, since $\alpha \circ \varepsilon = \varepsilon \circ \alpha_0$, this is $\alpha_0^\sharp(\varepsilon^*(\varphi))(p)$.

As the morphisms of complexes ε^* and η_* are quasi-isomorphisms, the fact that they are equivariant with respect to the actions of α —as shown by (6.5) and (6.6)— allows us to conclude that the two actions of L on $H^\bullet(S,U)$ coincide. □

6.23. We end this section showing how the results above work in a minimal example.

Example. We take $S = \Bbbk[x]$, we fix a nonzero $h \in S$ and we consider the Lie algebra L which, as an S-submodule of $\operatorname{Der} S$, is freely generated by $y = h\frac{d}{dx}$. The enveloping algebra U of the pair (S,L) is isomorphic to the algebra A_h with presentation

$$\frac{\Bbbk\langle x,y\rangle}{(yx - xy - h)}$$

which we will identify with U. This algebra has been thoroughly studied by G. Benkart, S. Lopes and M. Ondrus in the series of articles that start with [BLO15a]; we observe that setting $h = 1$ we obtain the Weyl algebra that already appeared in Example 2.15. The augmented Koszul complex

$$0 \longrightarrow S^e \xrightarrow{\delta_1} S^e \xrightarrow{\varepsilon} S,$$

with $\delta_1(s|t) = sx|t - s|xt$ and $\varepsilon(s|t) = st$, is an S^e-projective resolution of S and therefore the Hochschild cohomology $H^\bullet(S,U)$ is the cohomology of the complex $U \xrightarrow{\delta} U$ with differential $\delta(u) = [x,u]$. After a small calculation we see that $H^0(S,U) = \ker\delta = \Bbbk[x]$ and that $H^1(S,U) = \operatorname{coker}\delta = A/hA$. As A/hA is the quotient of the free noncommutative algebra in x and y by the relations $xy - yx = h$, and $h = 0$, we may identify $H^1(S,U)$ with $\frac{\Bbbk[x]}{(h)}[y]$.

At this point we make use of our description of the action of U on $H^\bullet(S,U)$ as in Theorem 6.18. It is enough to determine the action of y. We use Example 6.19 to see that y acts on $H^0(S,U) = S$ in the obvious way. To describe its action on $H^1(S,U)$ we need a lifting $y_\bullet$: we obtain one defining $y_0(s|t) = hs'|1 + 1|ht'$ and $y_1(s|t) = hs'|1 + 1|ht' + s\Delta(h)t$, where $\Delta : S \to S^e$ is the unique derivation of S such that $\Delta(x) = 1|1$. Since the diagram

$$\begin{array}{ccccc} S^e & \xrightarrow{\delta_1} & S^e & \xrightarrow{\varepsilon} & S \\ {\scriptstyle y_1}\uparrow & & {\scriptstyle y_0}\uparrow & & {\scriptstyle y}\uparrow \\ S^e & \xrightarrow{\delta_1} & S^e & \xrightarrow{\varepsilon} & S \end{array}$$

commutes and y_0 and y_1 are y^e-operators, the action of y on $H^1(S,U)$ can be obtained as in (6.2). We now compute $H^\bullet(L|S, H^i(S,U))$. Using the complex in Proposition 6.1 to compute Lie–Rinehart cohomology of (S,L), we see that for each $i \in \mathbb{Z}$ this is the cohomology of the complex

$$H^i(S,U) \xrightarrow{\nabla_y^i} H^i(S,U).$$

For $i = 0$, this amounts to the cohomology of $S \xrightarrow{y} S$; the kernel of this map is $\Bbbk$ and its image, hS. Consider now the case $i = 1$ and recall that we have identified $H^1(S,U)$ with $\frac{\Bbbk[x]}{(h)}[y]$; if $f \in \Bbbk[x]$, let us write $\bar f$ its class in this quotient. Given $u \in H^1(S,U)$, there are $f_0, \dots, f_r \in \Bbbk[x]$ such that $u = \sum_{i=0}^r \bar f_i y^i$ and

$$\nabla_y^1(u) = \sum_{i=0}^r \overline{h' f_i} y^i.$$

This expression is explicit enough to compute the kernel and cokernel of ∇_y^1, and this calculation, along with the help of Corollary 6.8, gives us the following description of the Hochschild cohomology of A_h:

$$HH^i(A_h) \cong \begin{cases} \Bbbk & \text{if } i = 0; \\ S/(h) \oplus \bigoplus_{i\geq 0} \dfrac{S}{\mathsf{gcd}\,(h,h')} y^i & \text{if } i = 1; \\ \bigoplus_{i\geq 0} \dfrac{S}{(h,h')} y^i & \text{if } i = 2; \\ 0 & \text{otherwise.} \end{cases}$$

This result had already been obtained by M. Valle in [Val17] and, partially, in [BLO15b]. With our approach, nevertheless, we have isolated the most complicated steps to different calculations and, as a consequence of that, this computation is significantly shorter.

6.4 THE DIFFERENTIAL OF THE SECOND PAGE

In this section we make a straightforward adaptation of the ideas in the article [SÁ07] by M. Suárez-Álvarez to give a description of the differential of the second page of our spectral sequence. This is the reason why we chose to state Theorem 6.7 in a more general setting than that of Corollary 6.8: we need the extra freedom with respect to the first argument in order to use the argument of [SÁ07].

COHOMOLOGICAL OPERATORS

6.24. Let us fix an algebra U. Until **6.29**, U can be any associative algebra and form there on we will specialize to the situation in which U is the enveloping algebra of a Lie–Rinehart pair.

Let p and q be integer numbers. We define the bifunctor $\mathsf{COp}^{p,q}$ of a pair of U-modules M and N by

$$\mathsf{COp}^{p,q}(N,M) = [\mathsf{Ext}_U^p(-,N), \mathsf{Ext}_U^q(-,M)],$$

with the brackets denoting the class of natural transformations between the two functors. Given $d \geq 0$, a *cohomological operator of degree d from N to M* is a sequence $\mathcal{O} = \left(\mathcal{O}_p\right)_{p\geq 0}$ of natural transformations $\mathcal{O}_p \in \mathsf{COp}^{p,p+d}(N,M)$. We denote by $\mathsf{COp}^d(N,M)$ the class of cohomological operators of degree d from N to M.

6.25. Let $d \geq 0$ and M and N be two U-modules. A cohomological operator $\mathcal{O}$ of degree d from N to M is *stable* if for each short exact sequence $0 \to T' \to T \to T'' \to 0$ of U-modules the diagram

$$\begin{tikzcd} \mathsf{Ext}_U^p(T',N) \arrow[r, "\partial"] \arrow[d, "O_p"] & \mathsf{Ext}_U^{p+1}(T'',N) \arrow[d, "O_p"] \\ \mathsf{Ext}_U^{p+d}(T',M) \arrow[r, "\partial"] & \mathsf{Ext}_U^{p+1+d}(T'',M) \end{tikzcd}$$

commutes for each $p \geq 0$. The class of such stable cohomological operators is denoted by $\mathsf{sCOp}^d(N,M)$.

6.26. Let $d \geq 0$ and let M and N be two U-modules. We can represent a class $\zeta \in \mathsf{Ext}_A^d(N,M)$ by a d-extension of N by M, that is, an exact sequence of U-modules of length $d+1$ of the form

$$\zeta : 0 \to M \to \cdots \to N \to 0.$$

If now $p \geq 0$ and $\varepsilon \in \mathsf{Ext}_U^p(T,N)$, there is a well-defined class $\zeta \circ \varepsilon$ in $\mathsf{Ext}_U^{p+d}(T,M)$ represented by the $(p+d)$-extension that results from the splicing of extensions representing ζ and ε. In this way we can define a natural morphism

$$Y : \mathsf{Ext}_U^d(N,M) \in \mathsf{COp}^d(N,M)$$

by

$$Y(\zeta)_p : \tau \in \mathsf{Ext}_U^p(Q,N) \mapsto (-1)^{pd}\zeta \circ \tau \in \mathsf{Ext}_U^{p+d}(Q,M)$$

for each $p \geq 0$ and each U-module Q. We claim that Y takes values in $\mathsf{sCOp}^d(N,M)$. Indeed, let ζ be a class in $\mathsf{Ext}_U^d(N,M)$ and let $0 \to T' \to T \to T'' \to 0$ be an exact sequence of U-modules and σ the corresponding class in $\mathsf{Ext}_U^1(T'',T')$. We know from [Mac67, III.9.1] that for each $p \geq 0$ the connecting homomorphism $\partial : \mathsf{Ext}_U^p(T',M) \to \mathsf{Ext}_U^{p+1}(T'',M)$ is given by $\tau \mapsto (-1)^p \tau \circ \sigma$, so the commutativity of the diagram

$$\begin{tikzcd} \mathsf{Ext}_U^p(T',N) \arrow[r, "\partial"] \arrow[d, "Y(\zeta)_p"] & \mathsf{Ext}_U^{p+1}(T'',N) \arrow[d, "Y(\zeta)_{p+1}"] \\ \mathsf{Ext}_U^{p+d}(T',M) \arrow[r, "\partial"] & \mathsf{Ext}_U^{p+1+d}(T'',M) \end{tikzcd}$$

is just an instance of the associativity of the Yoneda product of extensions.

6.27. Theorem. *The map Y defined above is an isomorphism of graded bifunctors*

$$Y : \mathsf{Ext}_U^\bullet(-,-) \to \mathsf{sCOp}^\bullet(-,-).$$

Proof. It is rather clearly a monomorphism, for $Y(\zeta)(1) = \zeta$ for any class ζ of extensions of U-modules. Let now N and M be two U-modules and let $\mathcal{O} \in \mathsf{sCOp}_U^d(N,M)$. We consider the class $\zeta = \mathcal{O}(1) \in \mathsf{Ext}_U^d(N,M)$ and the operator

$$\tilde{\mathcal{O}} = Y(\zeta) - \mathcal{O} \in \mathsf{sCOp}^d(N,M).$$

We claim that $\tilde{O} = 0$, so that Y is surjective. First, we show that $O_0 = 0$: let Q be a U-module and let $f : Q \to N \in \mathrm{Ext}^0_U(Q, N)$. If $Q = N$ and $f = 1_N$ then $\tilde{O}(1_N) = 0$ immediatly. As $\mathrm{Ext}^0_U(f, N)(1_N) = f$, the fact that $\hat{O}$ is a natural transformation in its first variable implies that

$$\tilde{O}(f) = \mathrm{Ext}^0_U(f, M)(\tilde{O}(1_N)) = 0.$$

Proceeding by induction, let us suppose that $\tilde{O}_p = 0$ for a given $p \geq 0$, let Q be a U-module and choose a short exact sequence of U-modules $0 \to Q' \to P \to Q \to 0$ with P projective. As $\mathrm{Ext}^{p+1}_U(P, -) = 0$, the stability of O implies that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^{p}_U(Q', N) \arrow[r, "\partial"] \arrow[d, "\tilde{O}_p"] & \mathrm{Ext}^{p+1}_U(Q, N) \arrow[r] \arrow[d, "\tilde{O}_{p+1}"] & 0 \\
\mathrm{Ext}^{p+d}_U(Q', M) \arrow[r, "\partial"] & \mathrm{Ext}^{p+1+d}_U(Q, M) \arrow[r] & 0
\end{tikzcd}$$

and, since we are assuming $\tilde{O}_p = 0$, we see that $\hat{O}_{p+1}$ restricted to Q is zero. □

6.28. The following result, which will be useful next subsection, can be found *mutatis mutandis* in [SÁ07, 2.2.1], up to a different choice of filtration. We include the proof for completeness.

Lemma. *Let*

$$0 \longrightarrow {}_1X^{\bullet,\bullet} \xrightarrow{j} {}_2X^{\bullet,\bullet} \xrightarrow{k} {}_3X^{\bullet,\bullet} \longrightarrow 0$$

be a short exact sequence of double complexes and denote, for each $1 \leq i \leq 3$, by ${}_iZ^\bullet$ the total complex of ${}_iX^{\bullet,\bullet}$. Let us assume that the filtrations defined by

$$F^p\ {}_iZ^q = \bigoplus_{\substack{r+s=q \\ r \geq p}} X^{r,s}$$

induce a sequence of cohomologically graded spectral sequences

$$0 \longrightarrow {}_1E_1^{\bullet,\bullet} \xrightarrow{j_1} {}_2E_1^{\bullet,\bullet} \xrightarrow{k_1} {}_3E_1^{\bullet,\bullet} \longrightarrow 0$$

which is also exact. If $\partial : {}_3E_2^{p,q} \to {}_1E_2^{p+1,q}$ is the connecting homomorphism corresponding to the differentials in this last sequence then the square

$$\begin{tikzcd}
{}_3E_2^{p,q} \arrow[r, "\partial"] \arrow[d, "d_2"] & {}_1E_2^{p+1,q} \arrow[d, "d_2"] \\
{}_3E_2^{p+2,q-1} \arrow[r, "\partial"] & {}_1E_2^{p+3,q-1}
\end{tikzcd}$$

anti-commutes.

Proof. If γ is an element of some ${}_iX^{p,q}$ such that $d_v(x) = 0$, we will denote by $[\gamma]$ the class of x in ${}_iE_1^{p,q}$.

To begin with, let us fix p and q and $\alpha \in {}_3E_2^{p,q}$. Let $a \in {}_3X^{p,q}$ be such that $d_V(a) = 0$ and $d_H[a] = 0 \in {}_3E_1^p$; let $b \in {}_3X^{p+1,q-1}$ be such that $d_V(b) = d_H(a)$, so that the class of $d_H(b)$ in the second page is $d_2\alpha$. Since k_1 is surjective, there exists $c \in {}_2X^{p,q}$ such that $d_V(c) = 0$ and $k_1[c] = [a]$ or, in other words, there exists $t \in {}_3X^{p,q-1}$ such that $d_V(t) = a - k_0(c)$. Now, as

$$k_0 d_H(c) = d_H k_0(c) = d_H(a - d_V(t)) = d_V(b + d_H(t))$$

we see that the class of $d_H(c)$ belongs to the kernel of k_1 and therefore there exists $x \in {}_1X^{p+1,q}$ such that $d_v(x) = 0$ and $j_1[x] = [d_H(c)]$. We observe that $\partial[a] = [x]$.

Let $s \in {}_2X^{p+1,q-1}$ be such that $j_0(x) = d_H(c) + d_V(s)$; since

$$j_0(d_H(x)) = d_H(j_0(x)) = d_H d_V(s) = d_V(-d_H(s))$$

we have that $j_1[d_H(x)] = 0$ and, as j_1 is a monomorphism, there exists $r \in {}_1X^{p+2,q-1}$ with $d_v(r) = d_H(x)$. This tells us that the class of $[d_H(r)]$ in ${}_1E_2^{p+3,q-1}$ is equal to $d_2\partial\alpha$.

Let now $z = j_0 r + d_H(s) \in {}_2X^{p+2,q-1}$. We have

$$d_V k_0(s) = k_0 j_0(x) - k_0 d_H(c) = -d_H k_0(c) = -d_H(a) + d_H d_V(t),$$

so $d_H(a) = -d_V(k_0(s) - d_H(t))$. On the other hand, using that $k_0 j_0 = 0$,

$$k_0(z) = k_0(d_H(s)) = d_H(k_0(s)) = d_H(k_0(s) - d_H(t))$$

and therefore the class of $[k_0(z)]$ in ${}_3E_2^{p+2,q-1}$ is $-d_2\alpha$. Finally, we observe that

$$j_0 d_H(r) = d_H j_0(r) = d_H(z)$$

and this, along with the fact that $d_V k_0(z) = 0$, as yet another small calculation shows, allows us to conclude that $d_2\partial\alpha = -\partial d_2\alpha$. □

THE DIFFERENTIALS

6.29. Using Theorem 6.27 we can give a description of the differential in the second page of the spectral sequence of Theorem 6.7.

Let (S, L) be a Lie–Rinehart pair with enveloping algebra U and let M be a U^e-module. Let $M \to I^\bullet$ be an U^e-injective resolution of M and let

$$0 \longrightarrow {}_3T \xrightarrow{j} {}_2T \xrightarrow{k} {}_1T \longrightarrow 0$$

be a short exact sequence of U-modules. Using the Horseshoe Lemma from [Wei94, Lemma 2.2.8] we can take, for $1 \le i \le 3$, U-projective resolutions ${}_iP^\bullet \to {}_iT$ and morphisms j_* and k_*

such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_3T \arrow[r, "j"] & {}_2T \arrow[r, "k"] & {}_1T \arrow[r] & 0 \\
0 \arrow[r] & {}_3P^\bullet \arrow[r, "j_*"] \arrow[u] & {}_2P^\bullet \arrow[r, "k_*"] \arrow[u] & {}_1P^\bullet \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

commutes and the rows are exact. Let us recall from **6.5** the functor G on U^e-modules and consider, for $i \in [\![3]\!]$, the double complexes

$$ {}_iX^{\bullet,\bullet} = \mathsf{hom}_U({}_iP^\bullet, G(I^\bullet)).$$

As seen in the proof of 6.7, $G(I^q)$ is an U-injective module for each q and therefore the sequence

$$0 \longrightarrow {}_1X^{\bullet,\bullet} \xrightarrow{j_*} {}_2X^{\bullet,\bullet} \xrightarrow{k_*} {}_3X^{\bullet,\bullet} \longrightarrow 0 \tag{6.7}$$

is exact. Fix now $i \in [\![3]\!]$ and denote the total complex of ${}_iX^\bullet$ by ${}_iZ^\bullet$. The filtration in ${}_iZ^\bullet$ given by

$$F^p\, {}_iZ^q = \bigoplus_{\substack{r+s=q \\ r \geq p}} {}_iX^{r,s}$$

determines a spectral sequence ${}_iE_\bullet$ whose differential on ${}_iE_0$ is induced by the one on $I^\bullet$. As the sequence $0 \to {}_3P \to {}_2P \to {}_1P \to 0$ splits, applying the functor $\mathsf{hom}_U(-, G(I^\bullet))$ we see that so does

$$0 \longrightarrow {}_1E_0^{p,q} \xrightarrow{j_*} {}_2E_0^{p,q} \xrightarrow{k_*} {}_3E_0^{p,q} \longrightarrow 0$$

and thus taking cohomology we get another exact sequence

$$0 \longrightarrow {}_1E_1^{p,q} \xrightarrow{j_*} {}_2E_1^{p,q} \xrightarrow{k_*} {}_3E_1^{p,q} \longrightarrow 0.$$

Fix $p \geq 0$ and $i \in [\![3]\!]$. Since each ${}_iP^p$ is U-projective, the cohomology of $\mathsf{hom}_U({}_iP^p, G(I^\bullet))$ is precisely

$${}_iE_1^{p,q} = \mathsf{hom}_U({}_iP^p, H^q(S, M)).$$

The differentials in ${}_iE_1$ are induced by those of ${}_iP^\bullet$, so that we have

$${}_iE_2^{p,q} = \mathsf{Ext}_U^p({}_iT, H^q(S, M)) \tag{6.8}$$

and we thus see that our exact sequence (6.7) is in the situation of Lemma 6.28. As a consequence of this, the square

$$\begin{tikzcd}
{}_3E_2^{p,q} \arrow[r, "\partial"] \arrow[d, "d_2"] & {}_1E_2^{p+1,q} \arrow[d, "d_2"] \\
{}_3E_2^{p+2,q-1} \arrow[r, "\partial"] & {}_1E_2^{p+3,q-1}
\end{tikzcd}$$

is anti-commutative. This diagram, using identification (6.8), is isomorphic to

$$\begin{tikzcd} \operatorname{Ext}_U^p({}_1T, H^q(S,M)) \arrow[r, "\partial"] \arrow[d, "d_2"] & \operatorname{Ext}_U^{p+1}({}_3T, H^q(S,M)) \arrow[d, "d_2"] \\ \operatorname{Ext}_U^{p+2}({}_1T, H^{q-1}(S,M)) \arrow[r, "\partial"] & \operatorname{Ext}_U^{p+3}({}_3T, H^{q-1}(S,M)) \end{tikzcd} \tag{6.9}$$

6.30. The following theorem is the result we are after in this section.

Theorem. *For each $q \geq 0$ there exists $\zeta_q(M) \in \operatorname{Ext}_U^2(H^q(S,M), H^{q-1}(S,M))$ such that the differential of the second page in the spectral sequence of Corollary 6.8*

$$d_2^{p,q} : H^p(L|S, H^q(S,M)) \to H^{p+2}(L|S, H^{q-1}(S,M))$$

is given by $d_2^{p,q}(\xi) = (-1)^p \zeta_q(M) \circ \xi$.

Proof. We have seen in (6.9) that for each $q \geq 0$ the cohomological operator $O = (O_p)$ of degree 2 from $H^q(S,M)$ to $H^{q-1}(S,M)$ such that

$$O_p = (-1)^p d_2^{p,q} : \operatorname{Ext}_U^p(-, H^q(S,M)) \to \operatorname{Ext}_U^{p+2}(-, H^{q-1}(S,M))$$

is stable, so that Theorem 6.27 gives us the desired class $\zeta_q(M)$. □

6.31. If M is a U^e-module, one may conjecture that the 2-extension

$$0 \longrightarrow M^S = H^0(S,M) \longrightarrow M \longrightarrow \operatorname{Der}(S,M) \longrightarrow H^1(S,M) \longrightarrow 0.$$

represents the class $\zeta_1(M) \in \operatorname{Ext}_U^2(H^1(S,M), H^0(S,M))$ in Theorem **6.30**.

6.5 Central line arrangements

In this section we use the machinery developed in this chapter to tackle the problem of computing the Hochschild cohomology of the algebra $A = \operatorname{Diff}(\mathcal{A})$ of differential operators tangent to a central arrangement of $r+2$ lines $\mathcal{A}$ of Chapter 3. For $r \geq 3$, the Hochschild cohomology of U was computed in Chapter 3 from an U^e-projective resolution of U after lengthy calculations. For $r = 1$ and $r = 2$ those calculations are even more tedious and rather inconvenient. With the method developed in this chapter we recover our previous results and, what is more, we are able to obtain $HH^\bullet(U)$ as a vector space for every $r \geq 1$. We will study the case in which $r = 1$ in detail: for $r = 2$, the calculations follow the same lines.

The key fact that makes our spectral sequence useful is that, as we have seen in Section 2.3, the algebra of coordinate functions on the vector space together with the algebra of derivations $\operatorname{Der}\mathcal{A}$ form a Lie–Rinehart pair and its universal enveloping algebra U is isomorphic to A —we will take this isomorphism as an identification.

6.32. We use the notation from 3.1. Let $r \geq 1$ and $\mathcal{A}$ be a central line arrangement in $V = \Bbbk^2$ defined by the polynomial $xF \in S$, where F is a square-free homogeneous polynomial of degree $r+1$ not divisible by x, and write $F = x\bar{F} + y^{r+1}$. Let us call $S = \Bbbk[x,y]$ and $L = \operatorname{Der}\mathcal{A}$. The S-module $\operatorname{Der}\mathcal{A}$ admits the basis given by the two derivations $E = x\partial_x + y\partial_y$ and $D = F\partial_y$ and the enveloping algebra A of the Lie–Rinehart pair (S, L) admits the presentation in 2.8. We put $T = \Bbbk[E]$ and, if $\psi \in T$, we write by $\psi' = \tau_1(\psi)$ and $\dot{\psi} = \tau_r(\psi)$, where τ_t is the linear map $T \to T$ such that $\tau_t(E^n) = E^n - (E+t)^n$ for every $n \in \mathbb{N}_0$.

6.5.1 THE COHOMOLOGY $H^\bullet(L|S, M)$

6.33. Let M be a $\mathbb{Z}$-graded left U-module such that the action of E on homogeneous elements of M satisfies $E(m) = |m|m$. We can compute $H^\bullet(L|S, M)$ as the cohomology of the complex $\operatorname{hom}_S(\Lambda_S^\bullet L, M)$ with Chevalley-Eilenberg differentials. This, in turn, is isomorphic to the complex

$$M \xrightarrow{d^0} M \otimes_\Bbbk \operatorname{hom}_\Bbbk(\Bbbk D \oplus \Bbbk E, \Bbbk) \xrightarrow{d^1} M \otimes_\Bbbk \operatorname{hom}_\Bbbk(\Bbbk D \wedge E, \Bbbk)$$

with differentials

$$d^0(m) = Dm\hat{D} \oplus Em\hat{E};$$
$$d^1(n\hat{D} + m\hat{E}) = (Dm - En + rn)\hat{D} \wedge \hat{E}.$$

The following observations describe the cohomology of this complex.

- If $n \in M$ is homogeneous then $d^1(n\hat{D}) = (r - |n|)n\hat{D} \wedge \hat{E}$. This means that $\operatorname{Im} d^1$ contains all homogeneous components $M_i\hat{D} \wedge \hat{E}$ with $i \neq r$. On the other hand, $d^1(m\hat{E}) = Dm\hat{D} \wedge \hat{E}$. As D is homogeneous of degree r, we see that

  $$H^2(L|S, M) = \operatorname{coker} d^1 = \operatorname{coker}(D : M_0 \to M_r).$$

- If $m \in M$ is homogeneous then the component of $d^0(m)$ in $\hat{E}$ is $|m|m$, and therefore $\ker d^1 \subset M_0$. In fact,

  $$H^0(L|S, M) = \ker d^0 = \ker(D : M_0 \to M_r).$$

- A 1-cocycle is, up to adding coboundaries of elements of nonzero degree, cohomologous to one of the form $\omega = n\hat{D} + m\hat{E}$ with $n \in M$ and $m \in M_0$. What is more, using now coboundaries of degree zero we can assume that n is not in the image of $D : M_0 \to M_r$. As $D(m) \in M_r$ and

  $$d^1(\omega) = Dm + (-En + rn) \in M_r \oplus \bigoplus_{i \neq r} M_i,$$

  we must have $Dm = 0$ and also $n \in M_r$. We conclude in this way that

  $$H^1(L|S, M) = \operatorname{coker}(D : M_0 \to M_r)\,\hat{D} \oplus \ker(D : M_0 \to M_r)\,\hat{E}.$$

We notice that the cohomology $H^\bullet(L|S, M)$ depends only on the map $M_0 \to M_r$ given by multiplication by D.

6.34. Let W be the $\Bbbk$-vector space with basis $\{x, y\}$. It is well-known that the complex $P_\bullet = S^e \otimes \Lambda^\bullet W$ —which we sometimes identify with $S \otimes W^\bullet \otimes S$— with Koszul differentials $k_\bullet : P_\bullet \to P_{\bullet-1}$ such that for $s, t \in S$ and $w \in W$

$$k_1(s|w|t) = sw|t - s|wt,$$
$$k_2(s|x \wedge y|t) = sx|y|t - s|y|xt - sy|x|t + s|x|yt$$

is a resolution of S by free S^e-modules. Applying $\mathsf{hom}_{S^e}(-, U)$ and using standard identifications we obtain the complex

$$U \xrightarrow{\delta^0} U \otimes \mathsf{hom}(W, \Bbbk) \cong U\hat{x} \oplus U\hat{y} \xrightarrow{\delta^1} U \otimes \mathsf{hom}_{\Bbbk}(\Bbbk x \wedge y, \Bbbk) \tag{6.10}$$

with differentials

$$\delta^0(u) = [x, u]\hat{x} + [y, u]\hat{y}$$
$$\delta^1(a\hat{x} + b\hat{y}) = ([x, b] - [y, a])\,\hat{x} \wedge \hat{y},$$

where $\{\hat{x}, \hat{y}\}$ is the dual basis of $\{x, y\}$ and $\hat{x} \wedge \hat{y}$ is the linear morphism $\Bbbk x \wedge y \to \Bbbk$ that sends $x \wedge y$ to one. The cohomology of the complex (6.10) is $H^\bullet(S, U)$.

6.35. We now describe the U-module structure on $H^\bullet(S, U)$ following Subsection 6.3. In order to do that we fix the Koszul resolution we described in **6.34** and recall from Example 6.19 that if $\alpha \in L$ and we regard $H^0(S, U)$ as a submodule of U then $\nabla^0_\alpha(u) = \alpha(u)$.

We first deal with the action of E; let E^e be the induced derivation on S^e and let, for $p \geq 0$, E_p be linear endomorphism of $S^e \otimes \Lambda^p W$ such that

$$E_p(s|z|t) = (|s| + |z| + |t|)s|z|t$$

for homogeneous s, $t \in S$ and $z \in \Lambda^p W$. It is immediate to see that the sequence (E_p) is an E^e-lifting of $E : S \to S$; with this at hand we obtain that the endomorphism $E_1^\sharp$, defined by equation (6.2) in Section 6.3, is given by

$$E_1^\sharp(a\hat{x} + b\hat{y}) = (|a| - 1)a\hat{x} + (|b| - 1)b\hat{y}$$

whenever $a, b \in U$ are homogeneous, and that $E_2^\sharp$ is given by

$$E_2^\sharp(u\hat{x} \wedge \hat{y}) = (|u| - 2)u\hat{x} \wedge \hat{y}.$$

for homogeneous $u \in U$.

We now study the action of D: it is enough to give a D^e-lifting (D_p) of $D : S \to S$. Recall, again from Example 6.19, that we may take D_0 equal to D^e, the derivation of S^e induced by D. The unique D^e-operator D_1 of $S^e \otimes W$ such that $D_1(1|x|1) = 0$ and

$$D_1(1|y|1) = \nabla(F)$$

satisfies $D_0 \circ k_1 = k_1 \circ D_1$, for their evaluation in $1|x|1$ is zero, and $\nabla(F)$ was defined precisely so that $k_1(\nabla(F)) = y|1 - 1|y$.

We define the remaining D^e-operator, that is, the endomorphism D_2 of $S^e \otimes \Lambda W$, by $D_2(1|x \wedge y|1) = x \wedge \nabla(F)$. It is not difficult to see that $k_2 \circ D_2 = D_1 \circ k_2$ by computing directly on both sides.

6.36. The action of E induces a $\mathbb{Z}$-grading on the complex (6.10) such that $|\hat{x}| = |\hat{y}| = -1$ and $|\hat{x} \wedge \hat{y}| = -2$, and, as the differentials preserve this grading, $H^\bullet(S, U)$ inherits a $\mathbb{Z}$-graded structure. In view of the description of the action of E that we gave in **6.35**, the U-modules $H^p(S, U)$ satisfy the hypothesis in **6.33**. As a consequence of this, to get $H^\bullet(L|S, H^\bullet(S, U))$ we need only to compute the homogeneous components of degree 0 and r of $H^\bullet(S, U)$ and then to describe the map given by the action of D.

6.37. Our plan is not difficult to execute for $H^0(S, U)$ and $H^2(S, U)$, but for of $H^1(S, U)$ the calculations are more involved. In particular, the cases in which $r \leq 2$ and $r \geq 3$ are different: we reserve a section for each of those situations. We take on the easy part here.

- It is proven in Lemma 4.4 that $H^0(S, U)$, the kernel of δ^0, is precisely S. The homogeneous component of S of degree zero is $\Bbbk$ and the action of D is zero.
- Let us denote by $S_{\geq 1}$ the space of polynomials with no constant term. We claim that $S_{\geq 1} D^k T$ is contained in the image of δ^1 for every $k \geq 0$. Indeed, if $f, g \in S$ and $\psi \in T$ then

  $$\delta^1(g\varphi\hat{x} + f\psi\hat{y}) = (xf\psi' - yg\varphi')\hat{x} \wedge \hat{y},$$

  so that our claim is true if $k = 0$. Assume now that $k > 0$ and that for every $j < k$ the inclusion $S_{\geq 1} D^j T \subset \operatorname{Im} \delta^1$ holds. Given $f \in S$ and $\psi \in T$, we have that

  $$\delta^1(fD^k\psi\hat{y}) = xfD^k\psi'\hat{x} \wedge \hat{y}$$

  and

  $$\begin{aligned}\delta^1(fD^k\psi\hat{x}) &= (-f[y, D^k]\psi - fD^k y\psi')\hat{x} \wedge \hat{y} = (-f[y, D^k](\psi - \psi') - fyD^k\psi')\hat{x} \wedge \hat{y} \\ &\equiv -fyD^k\psi'\hat{x} \wedge \hat{y} \mod \operatorname{Im} \delta^1,\end{aligned}$$

  which proves the claim. We easily see, on the other hand, that the intersection of $\Bbbk[D]T$ with $\operatorname{Im} \delta^1$ is trivial, so that

  $$H^2(S, U) \cong \Bbbk[D]T\hat{x} \wedge \hat{y}. \tag{6.11}$$

6.5.2 THE CASE $r \geq 3$

We assume that $r \geq 3$, so that we are in the situation of Section 3.2. Following **6.36**, for each $i \in \{0, 1, 2\}$ we compute the homogeneous components of degree 0 and r of $H^i(S, U)$ and then the action of D.

6.38. According to our calculation of $H^2(S,U)$ in (6.11), the only non-zero homogeneous components of $H^2(S,U)$ have degrees $-2+tr$, for $t \in \mathbb{N}_0$. For the component of degree zero to be non-trivial we need that $-2+tr=0$, which never happens if $r \geq 3$. On the other hand, for the component of degree r to be zero we need that $-2+tr=r$, which, again, cannot happen if $r \geq 3$. We conclude in this way that the components of $H^2(S,U)$ in degree zero and in degree r are both trivial and therefore that $H^\bullet(L|S, H^2(S,U)) = 0$.

6.39. Let us now compute the homogeneous component of $H^1(S,U)$ of degree 0. Let ω be a 1-cocycle of degree zero in the complex (6.10) of **6.34** and write it in the form $\omega = a\hat{x} + b\hat{y}$ for a and b in U of degree one, so that they belong to $xT \oplus yT$. Up to adding coboundaries we can assume that the component of a in xT is zero, so that there exist α, β and γ in T such that $a = y\alpha$ and $b = x\beta + y\gamma$.

The condition $\delta^1(\omega) = 0$, which amounts to $[x,b] = [y,a]$, implies that α, β and γ are scalars. Moreover, if ω were a coboundary, there should be a $\psi \in T$ such that $x\psi' = \alpha y$ and $y\psi' = \beta x + \gamma y$, leaving only the possibility that $\alpha = \beta = \gamma = 0$. We conclude from these calculations that the component of degree 0 of $H^1(S,U)$ is isomorphic to $\Bbbk y\hat{x} \oplus \Bbbk x\hat{y} \oplus \Bbbk y\hat{y}$.

6.40. Suppose now that ω is a 1-cocycle of degree r in the complex (6.10) of **6.34** and write $\omega = a\hat{x} + b\hat{y}$, with both a and b homogeneous of degree $r+1$. Up to coboundaries, we can assume that there is no monomial in a divisible by x. We write

$$\omega = \left(y^{r+1}\psi_1 + yD\psi_2\right)\hat{x} + \left(\sum_{i+j=r+1} x^i y^j \phi_{ij} + xD\phi_1 + yD\phi_2\right)\hat{y},$$

with the ψ's and ϕ's in T. The coboundary $\delta^1(\omega)$ belongs to $U_{r+2}\hat{x} \wedge \hat{y} = (S_{r+2}T \oplus S_2DT)\hat{x} \wedge \hat{y}$ and its component in S_2DT is $x^2D\phi_1' + xyD\phi_2' - y^2D\psi_2' = 0$, so that ψ_2, ϕ_1 and ϕ_2 are in $\Bbbk$. As we now have

$$\delta^1(\omega) = \left(\sum_{i+j=r+1} x^{i+1}y^j\phi_{ij}' - y^{r+2}\psi_1' + \psi_2 yF\right)\hat{x} \wedge \hat{y}$$

and $F = y^{r+1} + x\bar{F}$, since $\delta^1(\omega) = 0$ we must have $\psi_1' = \psi_2$ and

$$\sum_{i+j=r+1} x^{i+1}y^j\phi_{ij}' = -\psi_2 yx\bar{F}.$$

These equalities imply that $\psi_1 = -E\psi_2 + \mu$ for some $\mu \in \Bbbk$ and that $\sum_{i+j=r+1} x^i y^j \phi_{ij} = \psi_2 y\bar{F}E + f$, with $f \in S_{r+1}$. This means that

$$\omega = \psi_2\eta + \mu\hat{x} + f\hat{y} + hD\hat{y}, \tag{6.12}$$

where $\eta = (yD - y^{r+1}E)\hat{x} + y\bar{F}E\hat{y}$ and $h \in S_1$.

Let us now determine when it is possible that ω be a coboundary. Suppose now that there exists $u \in U_r$ such that $\delta^0(u) = \omega$; write $u = \sum x^i y^j \rho_{ij} + D\rho$ with ρ's in T and the sum taken

$$\begin{array}{c|ccc}
q & & & \\
2 & 0 & 0 & 0 \\
1 & 0 & r+2 & r+2 \\
0 & 1 & r+2 & r+1 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

Figure 6.1. Dimensions of the second page of the spectral sequence for $r \geq 3$.

over all i, j such that $i + j = r$. We equal

$$\delta^0(u) = \left(\sum x^{i+1}y^j\rho'_{ij} + xD\rho'\right)\hat{x} + \left(\sum x^i y^{j+1}\rho'_{ij} + yD\rho' + F(\rho' - \rho)\right)\hat{y}$$

to ω: looking at the component $\hat{x}$ we deduce that all ρ's must be zero — that leaves us only with $\delta^0(u) = -F\rho\hat{y}$.

We thus see that the only cocycles ω of the form (6.12) that are coboundaries are the scalar multiples of $F\hat{y}$. We therefore have that

$$H^1(S, U)_r \cong \Bbbk\eta \oplus \Bbbk\hat{x} \oplus \frac{S_{r+1}}{F}\hat{y} \oplus S_2 D\hat{y}.$$

6.41. We now describe the map $\nabla^1_D : H^1(S,U)_0 \to H^1(S,U)_r$. Let α, β and γ be scalars and consider the cocycle of degree zero $\xi = \alpha y\hat{x} + (\beta x + \gamma y)\hat{y}$. Using the formula for the D^e-lifting of $D : S \to S$ that we found in **6.35**, we see that $\nabla^1_D(\xi)(x) = \alpha F$ and that

$$\nabla^1_D(\xi)(y) = \gamma F - (\alpha y F_x + \beta x F_y + \gamma y F_y) = \gamma x F_x - \alpha y F_x - \beta x F_y,$$

thanks to Euler's identity. It follows from this that the map $\nabla^1_D : H^1(S,U)_0 \to H^1(S,U)_r$ is injective and has cokernel

$$\mathsf{coker}\left(\nabla^1_D : (H^1(S,U))_0 \to (H^1(S,U))_r\right) \cong \Bbbk\eta \oplus \frac{S_{r+1}}{\langle xF_x, yF_x, xF_y\rangle}\hat{y} \oplus S_1 D\hat{y}.$$

6.42. Collecting the information we have obtained so far about the dimensions of each vector space appearing in the second page of our spectral sequence we see that it must degenerate, for there is no possible non-zero arrow —see Figure 6.1. We conclude in this way that there is an isomorphism of vector spaces

$$HH^i(\mathsf{Diff}\,\mathcal{A}) \cong \begin{cases} \Bbbk, & \text{if } i = 0; \\ S_r\hat{D} \oplus \Bbbk\hat{E}, & \text{if } i = 1; \\ \left(\Bbbk\eta \oplus \dfrac{S_{r+1}}{\langle xF_x, yF_x, xF_y\rangle}\hat{y} \oplus S_1 D\hat{y}\right)\hat{D} \oplus S_r\hat{D} \wedge \hat{E}, & \text{if } i = 2; \\ \left(\Bbbk\eta \oplus \dfrac{S_{r+1}}{\langle xF_x, yF_x, xF_y\rangle}\hat{y} \oplus S_1 D\hat{y}\right)\hat{D} \wedge \hat{E}, & \text{if } i = 3; \\ 0, & \text{otherwise,} \end{cases} \tag{6.13}$$

where, we recall, $\eta = (yD - y^{r+1}E)\hat{x} + y\bar{F}E\hat{y}$. The dimensions in each cohomological degree agree with those found in Section 3.2, where the calculation was performed using the resolution **P** of U constructed in **3.5**. Moreover, there seems to be a correspondence between each cohomology class in (6.13) and one in Proposition 3.15: without this identification, we would not know how to relate this description of $HH^\bullet(U)$ with the cohomology of the Hochschild complex, so it could be difficult to describe the Gerstenhaber algebra structure on $HH^\bullet(U)$, and neither would it be clear how to relate $HH^2(U)$ with the deformations of U: both of these issues are well addressed when the cohomology is computed as in Section 3.2.

6.5.3 THE CASE $r = 1$

We may assume, without losing any generality, that the defining polynomial of our arrangement is $Q = xF$ with $F = y(tx + y)$, for some $t \in \Bbbk$. We adopt the strategy of **6.36** to compute $H^\bullet(L|S, H^\bullet(S, U))$, which is the second page of our spectral sequence of Corollary 6.8. This case was excluded in our computations of Chapter 3.

THE SECOND PAGE

6.43. We see from equation (6.11) that the homogeneous components of degree 0 and 1 of $H^2(S, U)$ are $D^2T\hat{x} \wedge \hat{y}$ and $D^3T\hat{x} \wedge \hat{y}$, respectively. Let us compute the kernel and the cokernel of $\nabla^2_D : H^2(S, U)_0 \to H^2(S, U)_1$ using the description we obtain in **6.35**. We have

$$D_2^\sharp(D^2\varphi\hat{x} \wedge \hat{y}) = \Big([D, D^2\varphi] - D^2\varphi\hat{x} \wedge \hat{y}\,(D_2(1|x \wedge y|1))\Big)$$

and, as in the second term there never appears a higher power of D than D^2,

$$D_2^\sharp(D^2\varphi\hat{x} \wedge \hat{y}) \equiv D^3\dot{\varphi}\hat{x} \wedge \hat{y} \mod \operatorname{Im}\delta_1^1.$$

We thus see that the kernel of $\nabla^2_D : H^2(S, U)_0 \to H^2(S, U)_1$ is $\Bbbk D\hat{x} \wedge \hat{y}$ and its cokernel is 0.

6.44. We now compute the component of degree zero of $H^1(S, U)$. The homogeneous component of degree zero of the complex (6.11) in **6.34** is

$$U_0 \xrightarrow{\delta_0^0} U_1\hat{x} \oplus U_1\hat{y} \xrightarrow{\delta_0^1} U_2\hat{x} \wedge \hat{y}$$

with $U_0 = T$, $U_1 = S_1T \oplus DT$,

$$U_2 = S_2T \oplus S_1DT \oplus D^2T \tag{6.14}$$

and differentials

$$\begin{aligned}
&\delta_0^0 : \phi \mapsto x\phi'\hat{x} + y\phi'\hat{y} \\
&\delta_0^1 : (x\varphi_1 + y\varphi_2 + D\varphi_3)\hat{x} \mapsto \Big(-xy\varphi_1' - y^2\varphi_2' - yD\varphi_3' - Q(\varphi_3' - \varphi_3)\Big)\,\hat{x} \wedge \hat{y}, \\
&\qquad (x\psi_1 + y\psi_2 + D\psi_3)\hat{y} \mapsto (x^2\psi_1' + xy\psi_2' + xD\psi_3')\hat{x} \wedge \hat{y},
\end{aligned}$$

where all Greek letters denote elements of T.

Let $a, b \in U_1$ and let $\omega = a\hat{x} + b\hat{y}$ be a 1-cocycle. Up to adding a coboundary we may suppose that the component of a in xT is zero: we may therefore write

$$a = y\varphi_2 + D\varphi_3, \qquad b = x\psi_1 + y\psi_2 + D\psi_3,$$

with Greek letters in T. The coboundary $\delta_0^1(\omega)$ belongs to $U_2\hat{x} \wedge \hat{y}$, which decomposes as in (6.14). The vanishing of the component in D^2T does not give any information, that of the one in S_1DT tells us that $\varphi_3' = \psi_3' = 0$ and, finally, that of S_2T that

$$x^2\psi_1' + xy\psi_2' = y^2\varphi_2' - F\varphi_3'. \tag{6.15}$$

Let us put $\lambda := \varphi_3$. Looking at the component on y^2T of Equation (6.15) and keeping in mind that $F = y^2 + txy$ we see that $\varphi_2' = \lambda$ and, using this, that $x\psi_1' + y\psi_2' = -\lambda ty$. In this way we obtain that

$$\varphi_2 = -\lambda E + \mu, \qquad x\psi_1 + y\psi_2 = \lambda tyE + f,$$

for certain $\mu \in \Bbbk$ and $f_1 \in S_1$. We conclude that

$$H^1(S, U)_0 \cong \Bbbk\eta_0 \oplus \Bbbk y\hat{x} \oplus (S_1 \oplus \Bbbk D)\hat{y} \tag{6.16}$$

with $\eta_0 = (-yE + D)\hat{x} + tyE\hat{y}$.

6.45. The homogeneous component of degree 1 of the complex (6.11) in **6.34** is

$$U_1 \xrightarrow{\delta_0^1} U_2\hat{x} \oplus U_2\hat{y} \xrightarrow{\delta_1^1} U_3\hat{x} \wedge \hat{y}$$

where $U_3 = S_3T \oplus S_2DT \oplus S_1D^2T \oplus D^3T$ and the differentials are such that

$$\begin{aligned}
&\delta_1^0(x\phi_1 + y\phi_2 + D\rho) \\
&\qquad = (x^2\phi_1' + xy\phi_2' + xD\rho')\hat{x} + (xy\phi_1' + y^2\phi_2' + yD\rho' + F(\rho' - \rho)\hat{y}, \\
&\delta_1^1\left(\left(\sum x^iy^j\varphi_{ij} + xD\varphi_1 + yD\varphi_2 + D^2\varphi\right)\hat{x}\right) \\
&\qquad = -\sum x^iy^{j+1}\varphi_{ij}' - xyD\varphi_1' - xF(\varphi_1' - \varphi_1) - y^2D\varphi_2' - yF(\varphi_2' - \varphi_2) \\
&\qquad\quad - yD^2\varphi' - 2FD(\varphi_2' - \varphi_2) - FF_y(\varphi' - \varphi), \\
&\delta_1^1\left(\left(\sum x^iy^j\psi_{ij} + xD\psi_1 + yD\psi_2 + D^2\psi\right)\hat{y}\right) \\
&\qquad = \sum x^{i+1}y^j\psi_{ij}' + x^2D\psi_1' + xyD\psi_2' + xD^2\psi',
\end{aligned}$$

In all the sums that appear here the indices i and j are such that $i + j = 2$ and we have omitted the factor $\hat{x} \wedge \hat{y}$. Again, all Greek letters lie in T.

Let us write, once again, $\omega = a\hat{x} + b\hat{y}$, this time with a and b in U_2. Up to coboundaries, we write, with the same conventions as before,

$$a = y^2\varphi_{02} + yD\varphi_2 + D^2\varphi, \qquad b = \sum x^iy^j\psi_{ij} + xD\psi_1 + yD\psi_2 + D^2\psi.$$

Let us examine the condition $\delta_1^1(\omega) = 0$ component by component according to our description of U_2 above.

- In D^3T there is no condition at all.
- In S_1D^2T we have $xD^2\psi' - yD^2\varphi' = 0$, so that ψ and φ are scalars.
- In S_2DT the condition reads

$$x^2D\psi_1' + xyD\psi_2' = y^2D\varphi_2' + 2FD(\varphi' - \varphi). \tag{6.17}$$

  Writing $F = y^2+txy$ and looking at the terms that are in y^2T we find $0 = \varphi_2'-2\varphi$, and then $\varphi_2 = -2\varphi E + \lambda$ for some $\lambda \in \Bbbk$. What remains of (6.17) implies that $x\psi_1' + y\psi_2' = -2ty\varphi$ and therefore there exists $h \in S_1$ such that

$$xD\psi_1 + yD\psi_2 = 2\varphi tyDE + hD.$$

- Finally, we look at S_3T: we have

$$\sum x^{i+1}y^j\psi_{ij}' = y^3\varphi_{02}' + yF(\varphi_2' - \varphi_2) + FF_y(\varphi' - \varphi).$$

  In particular, using that $F_y = 2y + tx$ and looking at the terms in y^3T, we find that $0 = \varphi_{02}' + (\varphi_2' - \varphi_2) + 2(\varphi' - \varphi)$, or, rearranging, $\varphi_{02}' = 2\varphi E + \lambda$. "Integrating", we see there exists a $\mu \in \Bbbk$ such that

$$\varphi_{02} = \varphi(E - E^2) - \lambda E + \mu.$$

  Now, as $FF_y = 2y^3 + 3txy^2 + t^2x^2y$, we must have

$$\sum x^iy^j\psi_{ij}' = txy(\varphi_2' - \varphi_2) - (3ty^2 + t^2xy)\varphi,$$

  and, integrating yet another time, we get $\sum x^iy^j\psi_{ij} = f_2E^2+f_1E+f_0$, for some polynomials f_1 and f_2 in S_2 that depend only and linearly on φ and λ.

We conclude in this way that there exist a cocycle ζ such that every 1-cocyle of degree 1 is cohomologous to one of the form

$$\omega = \varphi\zeta + \lambda\eta + f\hat{y} + hD\hat{y} + \psi D^2\hat{y} + \mu y^2\hat{x} \tag{6.18}$$

with $\eta = (-y^2E + yD)\hat{x} + ty^2E\hat{y}$, φ, λ, ψ and μ in $\Bbbk$ and h and f in S_1.

It is easy to see from the expression we have for δ_1^0 that such a cocycle is a coboundary if and only if it is a scalar multiple of $F\hat{y}$. The upshot of all this is that

$$H^1(S,U)_1 \cong \langle\zeta,\eta\rangle \oplus \Bbbk y^2\hat{x} \oplus \left(S_2/(F) \oplus S_1D \oplus \Bbbk D^2\right)\hat{y}.$$

6.46. We now study the action of D on the first cohomology group. We will give explicit formulas for the evaluation of $\nabla_D^1 : H^1(S,U)_0 \to H^1(S,U)_1$ and, at the same time, compute its cokernel. Suppose that ω is a representative of a class in $H^1(S,U)$ chosen as in (6.18).

- As $D_1^\sharp(D\hat{y}) = -D\hat{y}(\nabla(F))\hat{y} = -F_y D + \boxed{S_2}$, we see that up to adding to ω an element in the image of ∇_D^1 we may suppose that $h = h_0 x$, for some $h_0 \in \Bbbk$.
- Let α, β and γ in $\Bbbk$ and define $\phi = \alpha y\hat{x} + (\beta x + \gamma y)\hat{y}$. Since $\phi(\nabla(F))$ is equal to $\gamma x F_x - \alpha y F_x - \beta x F_y$, we have

$$\begin{aligned} D_1^\sharp(\phi) &= ([D, \alpha y] - \phi(D_1(1|x|1)))\,\hat{x} + ([D, \beta x + \gamma y] - \phi(D_1(1|y|1)))\,\hat{y} \\ &= \alpha F\hat{x} + (\gamma y F_y + \alpha y F_x + \beta x F_y)\hat{y}. \end{aligned}$$

  In view of this, it is easy to see that we may choose α, β and γ in such a way that $\omega + D_1^\sharp(\phi)$, which is a cocycle of the form (6.18), has $\mu = 0$ and $f = 0$, since $\{yF_x, xF_y, F\}$ spans S_2.
- Let us see that the 1-cocycle η belongs to the image of ∇_D^1. Consider the 1-cocycle $\eta_0 = (-yE + D)\hat{x} + tyE\hat{y}$. Using that $D_1(1|y|1) = \nabla(F) = t|x|y + y|y|1 + 1|y|y + tx|y|1$, we find

$$D_1^\sharp(\eta_0)(1|x|1) = [D, -yE + D] = -FE + yD,$$

  and

$$\begin{aligned} D_1^\sharp(\eta_0)(1|y|1|) &= [D, tyE] - \eta_0(\nabla(F)) \\ &= tFE - tyD - t(-yE + D)y - (tx + y)tyE - tyEy \\ &= ty^2E + t^2xyE - tyD + ty^2E + ty^2 - tyD - t(y^2 + txy) \\ &\quad - t^2xyE - ty^2E - ty^2E - ty^2 \\ &= -2tyD + ty^2 + t(y^2 + txy), \end{aligned}$$

  which belongs to $S_2 + \Bbbk yD$. We already know that the elements of $(S_2 + \Bbbk yD)\,\hat{y}$ are coboundaries: it follows that $D_1^\sharp(\eta_0) \equiv (-FE + yD)\hat{x}$ modulo coboundaries. Now, the difference between $D_1^\sharp(\eta_0)$ and η is cohomologous to $txyE\hat{x} + ty^2E\hat{y}$, which is in turn equal to $\delta_1^0(-tyE)$. As a consequence of this, we have that $\nabla_D^1(\eta_0)$ is equal to η in cohomology.

We conclude from the preceding calculation that $\operatorname{coker}\left(\nabla_D^1 : H^1(S,U)_0 \to H^1(S,U)_1\right)$ is generated by the classes of ζ, $xD\hat{y}$, and $D^2\hat{y}$. As these classes are linearly independent, the dimension of this cokernel is 3. Finally, we can use the dimension theorem to see that $\nabla_D^1 : H^1(S,U)_0 \to H^1(S,U)_1$ is a monomorphism.

6.47. We have already made all the computations required in **6.36**; the results are displayed in Figure 6.2 on the next page. As opposed to what happens when $r \geq 3$, the differential in the second page could be non-zero, since neither the domain nor the codomain of the map $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ are. As $\dim E_2^{0,2} = 1$, the differential $d_2^{0,2}$ is or zero or a monomorphism. If it is zero, the sequence degenerates and using Corollary 6.8 we obtain —among other things— that $\dim HH^3(U) = 4$; if not, we have $\dim HH^3(U) = 3$. It follows from this observation that to

$$
\begin{array}{c|ccc}
q & & & \\
2 & \Bbbk D^2\hat{x}\wedge\hat{y} & \Bbbk(D^2\hat{x}\wedge\hat{y})\hat{E} & 0 \\
1 & 0 & \langle\eta, xD\hat{y}, D^2\hat{y}\rangle\hat{D} & \langle\eta, xD\hat{y}, D^2\hat{y}\rangle\hat{D}\wedge\hat{E} \\
0 & \Bbbk & \Bbbk\hat{E}\oplus S_1\hat{D} & S_1\hat{D}\wedge\hat{E} \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

Figure 6.2. The second page for $r = 1$.

see whether the sequence degenerates or not it is enough to compute $HH^3(U)$. We will now do this using our complex of Chapter 3: we will find that $\dim HH^3(U) = 4$, so that Figure 6.2 actually describes the Hochschild cohomology of U.

The third Hochschild cohomology group for $r = 1$

As we saw in Section 3.2, the cohomology of the complex of **3.9** is $HH^\bullet(U)$. We will use this complex again to compute $HH^3(U)$: let us take a generic 3-cocycle

$$\omega = a\hat{x}\hat{y}\hat{D} + b\hat{x}\hat{y}\hat{E} + c\hat{x}\hat{D}\hat{E} + d\hat{y}\hat{D}\hat{E}$$

with $a \in A_3$ and b, c and d in A_2, where, we recall, we have $A_k = \sum_{i+j=k} S_i D^j T$ for $k \geq 0$.

6.48. We use the image of the second differential to simplify ω.

- We may suppose that $a = 0$: indeed, we have

$$d^2(A_2\hat{x}\hat{D}) = [y, A_2]\hat{x}\hat{y}\hat{D} = yA_2\hat{x}\hat{y}\hat{D},$$
$$d^2(A_2\hat{y}\hat{D}) = [x, A_2]\hat{x}\hat{y}\hat{D} = xA_2\hat{x}\hat{y}\hat{D}$$

  and the only coefficient of $d^2(D^2\psi\hat{x}\hat{y})$ is in $\hat{x}\hat{y}\hat{D}$ and it is congruent to $D^3\dot{\psi}$ modulo $SD^{\leq 2}T$.
- We may suppose that $b \in D^2T$. This follows from the facts that the components in $\hat{x}\hat{y}\hat{E}$ of $d^2(A_1\hat{x}\hat{E})$ and of $d^2(A_1\hat{y}\hat{E})$ are $[y, A_1]$ and $[x, A_1]$ and that their components in $\hat{x}\hat{y}\hat{D}$ are zero.
- We may suppose that the scalar components of c in y^2 and in yD are zero. The first assumption follows from the equality $d^2(-y\hat{x}\hat{E}) = F\hat{x}\hat{D}\hat{E} + FF_x\hat{y}\hat{D}\hat{E}$ and the second one from

$$d^2(\eta_0 \wedge E) = (FE - yD)\hat{x}\hat{D}\hat{E} + \boxed{A_2}\,\hat{y}\hat{D}\hat{E},$$

  where η_0 is the cocycle found in (6.16) and $\eta_0 \wedge E$ is formally obtained from it. Moreover, as $d^2(a\hat{D}\hat{E}) = [x, a]\hat{x}\hat{D}\hat{E} + [y, a]\hat{y}\hat{D}\hat{E}$, we may as well assume that c has no monomials from xA.

- We may assume that d has zero scalar component in yD, for

  $$d^2(D\hat{y}\hat{E}) = (F_yD + \boxed{S_2})\hat{y}\hat{D}\hat{E},$$

  and, finally, we may suppose that d has no monomials that involve only x and y, since

  $$d^2(x\hat{y}\hat{E}) = xF_y\hat{y}\hat{D}\hat{E}, \qquad d^2(y\hat{y}\hat{E}) = (yF_y - F)\hat{y}\hat{D}\hat{E},$$
  $$d^2(D\hat{D}\hat{E}) = -F\hat{y}\hat{D}\hat{E}.$$

6.49. Taking all these assumptions into account, we can write

$$\begin{aligned} b &= D^2\rho, \\ c &= y^2\varphi_2 + yD\varphi_1 + D^2\varphi, \\ d &= \sum_{i+j=2} x^iy^j\psi_{ij} + xD\psi_1 + yD\psi_2 + D^2\psi, \end{aligned}$$

with all Greek letters in T. We examine the equation $d^3(\omega) = 0$ looking at each of its components in $S_{3-i}D^iT$, for $0 \le i \le 3$. The equation we have to solve is

$$[x, d] = [y, c] + \nabla_y^{D^2\rho}(F). \tag{6.19}$$

- Looking at the components in D^3T, we immediately obtain that $D^3\dot{\rho} = 0$ and hence that $\rho \in \Bbbk$.
- The component in S_1D^2T of equation (6.19) is $xD^2\psi' = yD^2\varphi' + \rho(2y + tx)D^2$, from which we deduce that $\psi' = \rho tx$ and that $0 = \varphi' + 2\rho$. We may thus write

  $$\varphi = 2\rho E + \varphi_0, \qquad \psi = -\rho txE + \psi_0,$$

  with φ_0 and ψ_0 in $\Bbbk$.
- Using the information we have obtained thus far, we see that the component of the equation in S_2DT is

  $$x^2D\psi_1' + xyD\psi_2' = y^2D\varphi_1' + 2FD(\varphi' - \varphi) + 2\rho FD. \tag{6.20}$$

  Let us write $S_2DT = x^2DT \oplus xyDT \oplus y^2DT$. We look at equation (6.20) in y^2DT: it reads $0 = \varphi_1' + 2(\varphi' - \varphi) + 2\rho$ and from this we may write φ' in terms of ρ and φ_0. Integrating and recalling that φ has its scalar component in yD equal to zero, we see that

  $$\varphi_1 = 2\rho(E - E^2) - 2(\rho - \varphi_0)E.$$

  Next, we quickly look at x^2DT to get $\psi_1' = 0$ and therefore that $\psi_1 \in \Bbbk$. Finally, we look at xyD^2. The equation there reads $\psi_2' = 2t(\varphi' - \varphi) + 2t\rho$ and, again, as d has zero scalar component in yD, this determines uniquely that $\psi_2 = -t\psi_1$.

- The only remaining component of our equation is the one in S_3T,

$$\sum_{i+j=2} x^{i+1} j^j \psi'_{ij} = y^3\varphi'_2 + yF(\varphi'_1 - \varphi_1) + FF_y(\varphi' - \varphi) + \rho FF_y. \tag{6.21}$$

  Let us take $\{x^k y^l : k + l = 3\}$ as a basis of S_3T as a T-right module: the component in y^3 of our equation is

$$\varphi'_2 = (\varphi'_1 - \varphi_1) + 2(\varphi - \varphi') - 2\rho.$$

  As the scalar component of φ_2 is zero, this equation determines φ_2. Using this, the equation (6.21) gives us an expression for $\sum x^i y^j \psi'_{ij}$ in terms of the already known parameters and, integrating, we obtain the same for the ψ_{ij}'s.

6.50. We have already seen at this point that $\dim HH^3(A) \leq 4$. A computation very similar to the one in **3.13** and which we omit shows that actually the equality holds.

Proposition. *The spectral sequence for $r = 1$ degenerates at the second page and therefore*

$$HH^i(\mathsf{Diff}\,\mathcal{A}) \cong \begin{cases} \Bbbk, & \text{if } i = 0; \\ \Bbbk\hat{E} \oplus S_1\hat{D} & \text{if } i = 1; \\ S_1\hat{D} \wedge \hat{E} \oplus \langle \eta, xD\hat{y}, D^2\hat{y}\rangle \hat{D} \oplus \Bbbk D^2\hat{x} \wedge \hat{y}, & \text{if } i = 2; \\ \langle \eta, xD\hat{y}, D^2\hat{y}\rangle \hat{D} \wedge \hat{E} \oplus \Bbbk(D^2\hat{x} \wedge \hat{y})\hat{E} & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As $\dim HH^3(U) = 4$, our argument from **6.47** implies that the spectral sequence degenerates at E_2. The isomorphisms in the statement are a consequence of the convergence and the information in Figure 6.2. □

6.5.4 Resemblance and dissemblance

We end this chapter with a comparison between the cases in which $r \geq 3$ and that in which r is 1 or 2. In both situations, to compute the second page of the spectral sequence $E_\bullet$ of Corollary 6.8 we used the Koszul resolution $P_\bullet$ of S, which is an S^e-projective resolution of length 2, and computed the cohomology of $\hom_{S^e}(P_\bullet, S)$ to obtain $H^\bullet(S, U)$. We then used the complex of Proposition 6.1, which also has lenght 2, to obtain, for each $0 \leq q \leq 2$, the Lie–Rinehart cohomology of the pair (S, L) with values on $H^q(S, U)$. Since each of the complexes we used has lenght 2, the second page has $E_2^{p,q} = 0$ for every $p, q \geq 3$.

It is at this point that the case $r \geq 3$ is different to the case $r = 1, 2$. Let us consider the first case, depicted in Figure 6.3, when $r \geq 3$. We have

$$E_2^{p,q} = 0 \qquad \text{if } p \geq 3 \text{ and } q \geq 2,$$

and, moreover, $E_2^{0,1} = 0$. As the differential on the second page has bidegree $(2, -1)$, the spectral sequence degenerates at E_2, thus immediately giving us a description of $HH^\bullet(U)$. A problem

q			
2	0	0	0
1	0	$r+2$	$r+2$
0	1	$r+2$	$r+1$
	0	1	2 p

Figure 6.3. Dimensions of E_2 for $r \geq 3$

q			
2	1	1	0
1	0	$r+2$	$r+2$
0	1	$r+2$	$r+1$
	0	1	2 p

Figure 6.4. Dimensions of E_2 for $r = 1, 2$

with this is that it is not obvious how to compute the Gerstenhaber algebra structure on $HH^\bullet(U)$: in Chapter 3 we obtained explicit cocycles and this allowed us to compute cup products and Gerstenhaber brackets. Here, we still do not know the relation between our spectral sequence and the multiplicative structure of $HH^\bullet(U)$. Another consequence of the lack of explicitness of this procedure is that it is difficult to describe the formal deformations of U as in Chapter 5 even though we do know $HH^2(U)$.

Let us now consider the case in which r is equal to 1 or 2. The dimensions of the components of the second page of the spectral sequence are tabulated in Figure 6.4. As opposed to the first case, it is not evident that the spectral sequence degenerates at its second page: the differential $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ could be non-zero. Computing $HH^3(U)$ from the U^e-projective resolution of U described in **3.5** we were able to check that, in fact, $d_2^{0,2}$ is zero, thus allowing us to obtain the dimensions of $HH^\bullet(U)$ as a graded vector space. The end result is that the Hilbert series of $HH^\bullet(U)$ is

$$h_{HH^\bullet(U)}(t) = \begin{cases} 1 + (r+2)t + (2r+4)t^2 + (r+3)t^3, & \text{if } r = 1, 2; \\ 1 + (r+2)t + (2r+3)t^2 + (r+2)t^3, & \text{if } r \geq 3. \end{cases}$$

This shows that the case in which r is 1 or 2 is *genuinely* different to that in which $r \geq 3$.

6.6 Resumen

En el Capítulo 2 vimos que si $\mathcal{A}$ es un arreglo de hiperplanos libre, el par $(S, \operatorname{Der} \mathcal{A})$ determinado por el álgebra de polinomios S y el álgebra de Lie $\operatorname{Der} \mathcal{A}$ es un par de Lie–Rinehart y su álgebra envolvente es isomorfa a $\operatorname{Diff}(\mathcal{A})$. En este capítulo desarrollamos una herramienta que permite abordar el problema, más general, de determinar la cohomología de Hochschild del álgebra envolvente $U = U(S, L)$ de un par de Lie–Rinehart (S, L).

Precisamente, siguiendo las ideas de Th. Lambre y P. Le Meur en [LLM18], construimos una sucesión espectral que reduce el problema del cálculo de la cohomología de Hochschild del álgebra conmutativa S a valores en U y de la cohomología de Lie–Rinehart del par (S, L). Explícitamente, obtenemos el siguiente resultado en el Corolario 6.8.

Teorema. *Para cada U-bimódulo M hay una sucesión espectral $E_\bullet$ en el primer cuadante que converge a $HH^\bullet(U,M)$ tal que*

$$E_2^{p,q} \cong H^p(L|S, H^q(S,M)).$$

Para poder utilizar esta sucesión espectral en el cálculo de la cohomología de Hochschild del álgebra $\mathsf{Diff}(\mathcal{A})$ asociada a un arreglo $\mathcal{A}$ es necesario contar con una descripción práctica de la estructura de U-módulo en la cohomología de Hochschild de S a valores en U: nos ocupamos exitosamente de este problema en la Sección 6.3, siguiendo [SÁ17].

A continuación, dedicamos la Sección 6.4 a dar una descripción de los diferenciales de la página E_2. Para hacer esto, basándonos en [SÁ07], estudiamos primero los llamados operadores cohomológicos estables y vemos que nuestra diferencial se corresponde con uno de ellos. El resultado de esta sección es el Teorema 6.30.

Teorema. *Para cada $q \geq 0$ existe $\zeta_q(M) \in \mathsf{Ext}^2_U(H^q(S,M), H^{q-1}(S,M))$ tal que la diferencial de la segunda página de la sucesión espectral del Corolario 6.8*

$$d_2^{p,q} : H^p(L|S, H^q(S,M)) \to H^{p+2}(L|S, H^{q-1}(S,M))$$

está dada por $d_2^{p,q}(\xi) = (-1)^p \zeta_q(M) \circ \xi$.

Para terminar la tesis, nos ocupamos en la Sección 6.5 de mostrar que nuestra sucesión espectral hace posible determinar completamente $HH^\bullet(U)$ y mostramos cómo este método se aplica al caso especial del cálculo de la cohomología del álgebra $\mathsf{Diff}(\mathcal{A})$ asociada a un arreglo de rectas. Primero recuperamos nuestros resultados de la cohomología de Hochschild en tanto espacio vectorial graduado para arreglos con al menos cinco rectas y, a continuación, extendemos estos resultados a arreglos con 3 o 4 rectas, que habían sido excluidos anteriormente. Este resultado aparece en el texto como la Proposición 6.50 para el caso de 3 rectas. Observamos los casos en que tenemos más o menos de cinco rectas son genuinamente diferentes: si l denota la cantidad de rectas del arreglo, la serie de Hilbert de $HH^\bullet(\mathsf{Diff}(\mathcal{A}))$ es

$$h_{HH^\bullet(U)}(t) = \begin{cases} 1 + lt + 2lt^2 + (l+1)t^3, & \text{si } l = 3, 4; \\ 1 + lt + (2l-1)t^2 + lt^3, & \text{si } l \geq 5. \end{cases}$$

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