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Juegos de tipo Tug-of-War y soluciones viscosas

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Juegos de tipo Tug-of-War y soluciones viscosas

La motivación de esta tesis es el estudio de los juegos llamados Tug-of-War en la literatura, y su conexión con ecuaciones en derivadas parciales (EDP). Consideramos diferentes variantes de juegos de dos jugadores, con suma cero, que dependen de un parámetro que controla el tamaño del paso que se da cuando se actualiza la posición del juego. Se demuestra que el valor de estos juegos converge (cuando el parámetro tiende a cero) a una solución de una EDP (que debe ser interpretada en sentido viscoso). De esta forma nos encontramos con una nueva herramienta, basada en teoría de probabilidad, para obtener soluciones de problemas no-variacionales como por ejemplo:

(i) $\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = 0$,

(ii) $\min\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = 0$,

(iii) $\lambda_j(D^2 u) = 0$.

Aquí $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ es el operador conocido como $p-$laplaciano y $\lambda_j(D^2u)$ es ej $j-$ésimo autovalor de D^2u.

También presentamos resultados relacionados con estos operadores que no están directamente conectados con los juegos que motivaron su estudio. Obtuvimos una interpretación geométrica de las soluciones viscosas de la ecuación $\lambda_j(D^2u) = 0$ en términos de envolventes cóncavo/convexas sobre espacios afines de dimensión j. Esta caracterización geométrica nos permitió dar condiciones necesarias y suficientes sobre el dominio para asegurar el buen planteo del problema de Dirichlet asociado a la ecuación.

Motivados por las ecuaciones (i) y (ii) consideramos ecuaciones de la forma

$$\max\{L_1 u, L_2 u\} = 0.$$

Presentamos un nuevo esquema iterativo usando el problema del obstáculo, que converge a una solución de esta ecuación.

Finalmente, encontramos nuevas cotas para el primer autovalor de un operador elíptico totalmente no-lineal. Esta nueva cota inferior nos permite probar que

$$\lim_{p\to\infty} \lambda_{1,p} = \lambda_{1,\infty} = \left(\frac{\pi}{2R}\right)^2,$$

donde $\lambda_{1,p}$ y $\lambda_{1,\infty}$ son el autovalor principal del p-laplaciano homogénero y del infinito laplaciano homogéneo respectivamente.

Palabras clave: Juegos de tipo Tug-of-War, Soluciones Viscosas, Condición de frontera de Dirichlet.

Tug-of-War games and viscosity solutions

This thesis is motivated by the study of Tug-of-War games in connection with partial differential equations (PDE). We consider different variants of two-player zero-sum games that depend on a parameter that control the size of the step that actualizes the position of the game. We show that the value functions of these games converge (as the parameter goes to zero) to a solution of a PDE (that has to be interpreted in the viscosity sense). In this way we found a new tool, based in probability theory, to obtain solutions to non-variational problems like

(i) $\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = 0,$

(ii) $\min\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = 0,$

(iii) $\lambda_j(D^2 u) = 0.$

Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$laplacian and $\lambda_j(D^2u)$ stands for the $j-$th eigenvalue of D^2u.

We also present results related to these operators that are not directly connected to the games that motivated their study. We obtained a geometric interpretation of the viscosity solutions to the equation $\lambda_j(D^2u) = 0$ in terms of convex/concave envelopes over affine spaces of dimension j. This geometric interpretation of the solutions allows us to give necessary and sufficient conditions on the domain in order to guarantee the well posedness of the Dirichlet problem associated to this equation.

Motivated by equations (i) and (ii) we were lead to consider equations of the form

$$\max\{L_1 u, L_2 u\} = 0.$$

We present a new iterative scheme using the obstacle problem that converges to a solution of this equation.

Finally, we also discuss new bounds for the first eigenvalue of fully nonlinear elliptic operators. These new bounds allow us to prove that

$$\lim_{p\to\infty} \lambda_{1,p} = \lambda_{1,\infty} = \left(\frac{\pi}{2R}\right)^2,$$

where $\lambda_{1,p}$ and $\lambda_{1,\infty}$ are the principal eigenvalue for the homogeneous p-laplacian and the homogeneous infinity laplacian respectively.

Keywords: Tug-of-War games, Viscosity solutions, Dirichlet boundary conditions.

Contents

Chapter 1

Introducción

La motivación de esta tesis es el estudio de diferentes versiones de los juegos conocidos como de tipo Tug-of-War. Estos juegos constituyen un nuevo capítulo en la rica historia de resultados que conectan la teoría de ecuaciones diferenciales con la teoría de probabilidad. Los trabajos de Doob, Feller, Hunt, Kakutani, Kolmogorov y muchos otros muestran la profunda raíz común entre la teoría clásica del potencial y la teoría de probabilidad. La idea que subyace en esta relación es que las funciones armónicas y las martingalas tiene un punto en común: las fórmulas de valor medio. Esta relación también es fructífera en el caso no lineal y los juegos del tipo Tug-of-War dan muestra de ello.

A fines de la década del '80 el matemático David Ross Richman propuso un nuevo tipo de juego a mitad de camino entre los clásicos de la teoría de juegos de Von Neumann y Morgenstern, y los juegos combinatorios estudiados por Zermelo, Lasker y Conway, entre otros. Aquí dos jugadores se enfrentan en un juego combinatorio arbitrario (tatetí, ajedrez, damas, etc.), pero cada uno posee cierta suma de dinero, que modifica las reglas del juego: cada movida se licita y el que oferta más gana el derecho a mover. En caso de empate en las ofertas, se puede decidir aleatoriamente tirando una moneda.

A fines de los '90 en [50] y [51] se estudió esta clase de juegos, conocidos como Richman games. Se los tradujo a un problema de difusión sobre un grafo: los nodos son las posiciones del juego, los links equivalen a las movidas permitidas, y una ficha en cierto nodo es desplazada por uno u otro jugador según quien gane la licitación. El juego termina cuando la ficha llega a los nodos del grafo marcados como terminales y un dato de borde indica cuánto gana el primer jugador, monto pagado por el otro jugador.

Entre las distintas modificaciones del juego, un caso interesante es cuando el turno se decide aleatoriamente tirando una moneda justa (con la misma probabilidad de cara o cruz), eliminando así el proceso de licitación en cada turno. Esta idea dio lugar al juego Tug-of-War cuyo origen está en el trabajo [70] de Peres, Schramm, Sheffield y

Wilson. Allí se estudia ese juego y se muestra su conexión con ecuaciones diferenciales. Más concretamente con el $\infty-$laplaciano, un operador que surge naturalmente asociado al problema de extensión Lipschitz absolutamente minimal, ver [6].

Al trabajar con ecuaciones diferenciales cabe hacer mención al tipo de soluciones a considerar. La teoría de operadores de segundo orden en forma de divergencia se asocia usualmente al concepto de soluciones débiles; sin embargo, cuando se trata de ecuaciones fuertemente no lineales que no están en forma de divergencia, el uso de soluciones en sentido viscoso es más apropiado.

Este tipo de soluciones fue introducida por Crandall y Lions en la década del '80. Al hablar de soluciones viscosas es inevitable mencionar la clásica referencia [33]. Incluimos en esta tesis un breve repaso sobre la teoría de soluciones viscosas en el Apéndice A. Allí se pueden encontrar algunos comentarios generales sobre la teoría y detalles de algunos resultados utilizados a lo largo de este manuscrito. Más allá de que en la tesis utilicemos de manera exclusiva este tipo de soluciones, esto no es un limitante en cuanto a los resultados. Por ejemplo, notemos que para el $p-$Laplaciano, $\mathrm{div}(|Du|^{p-2}Du) = 0$, fue probado en [41] y [44] la equivalencia entre soluciones en sentido viscoso y soluciones débiles.

Por ser uno de los disparadores de esta tesis, describamos aquí el juego introducido en [70]. El juego conocido como Tug-of-War es un juego de dos jugadores de suma cero, es decir, dos jugadores compiten uno contra el otro y las ganancias de uno de ellos son las perdidas del otro. Entonces, uno de ellos, digamos el Jugador I, juega tratando de maximizar su ganancia esperada, mientras el otro, digamos el Jugador II, trata de minimizar la ganancia del Jugador I (o, como el juego es de suma cero, trata de maximizar su propia ganancia).

Consideremos un dominio acotado $\Omega \subset \mathbb{R}^N$ y un $\varepsilon > 0$ fijo. Inicialmente, una ficha se encuentra en un punto $x_0 \in \Omega$. Los dos jugadores, Jugador I y Jugador II, juegan de acuerdo a las siguientes reglas: se tira una moneda equilibrada (con la misma probabilidad de cara o cruz) y el jugador que gana la tirada de la moneda mueve la ficha a cualquier punto x_1 de su elección a distancia menor a ε del anterior, $x_1 \in B_\varepsilon(x_0)$. A partir de este punto x_1 continúan jugando con las mismas reglas. En cada turno, se tira nuevamente la moneda y el ganador elige la nueva posición del juego $x_k \in B_\varepsilon(x_{k-1})$.

Este procedimiento nos da una sucesión de posiciones del juego $x_0, x_1, \ldots$. Cuando la posición de la ficha sale del dominio Ω, digamos en el paso τ, el juego termina. En esta última posición del juego la ficha se encuentra en $\mathbb{R}^N \setminus \Omega$. Dada una *función de pago final*, $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$, al final del juego el Jugador II le paga al Jugador I la cantidad dada por $g(x_\tau)$, es decir, el Jugador I obtuvo $g(x_\tau)$ mientras que el Jugador II obtuvo $-g(x_\tau)$.

Una estrategia para el Jugador I, S_{I}, es la colección de funciones medibles $S_{\mathrm{I}} = \{S_{\mathrm{I}}^k\}_{k=0}^\infty$ tales que la siguiente posición del juego es

$$S_{\mathrm{I}}^k(x_0, x_1, \ldots, x_k) = x_{k+1} \in B_\varepsilon(x_k)$$

dado que el Jugador I ganó la tirada de la moneda dada la historia del juego hasta esa movida $(x_0, x_1, \ldots, x_k)$. Análogamente, el Jugador II juega de acuerdo a una estrategia $S_{\rm II}$.

Para cada $x_0 \in \Omega$ podemos considerar el valor esperado del juego empezando en ese punto x_0 asumiendo que ambos jugadores juegan de forma óptima, denotamos este valor por $u^\varepsilon(x_0)$. Esta función es la que llamaremos el *valor del juego*. Para cada ε, tenemos entonces una función $u^\varepsilon : \overline{\Omega} \to \mathbb{R}$. En [70] se demuestra que existe una función continua $u : \overline{\Omega} \to \mathbb{R}$ tal que $u^\varepsilon \to u$ cuando $\varepsilon \to 0$, y que u satisface

$$-\Delta_\infty u = -(\nabla u)^t D^2 u \nabla u = 0 \qquad \text{en } \Omega,$$

junto con la condión de borde $u = g$ en $\partial\Omega$.

Luego de este innovador trabajo, se consideraron diversas versiones de este juego y se obtuvieron diferentes resultados. En [71] se estudió una versión del juego relacionada con el $p-$laplaciano. Una versión no local del juego fue propuesta en [20] y en [21]. Diferentes condiciones de contorno también pueden ser consideradas: condiciones de tipo Neumann, [3], y condiciones de tipo mixto, [32]. Un juego a tiempo continuo fue introducido en [9].

En este punto debemos mencionar [61], [62] y [63] donde se estudia una versión del juego relacionada con el p-laplaciano. Estos trabajos introdujeron un marco teórico que fue aprovechado en trabajos posteriores. Un juego relacionado con el problema del obstáculo fue estudiado en [64], uno relacionado con un operador que involucra una restricción de gradiente fue considerado en [45], un juego para el $p(x)$-laplaciano en [7], juegos para problemas parabólicos en [58], [37] y [10], y otras variantes en [38] y [66]. Esta forma de abordar problemas de EDP también se ha usado para probar resultados de regularidad de formas diferentes a las usuales (por ejemplo, la desigualdad de Harnack's y regularidad Hölder), citamos [56], [57], [73], [8] y [69].

Motivados por estos resultados previos, en esta tesis consideramos una variante distinta del juego. En el Capítulo 3 introducimos el juego que llamamos *Tug-of-War desbalanceado con ruido*. El formato general es el mismo que el del juego original, pero las reglas cambian ligeramente. En cada turno el Jugador I elige una moneda entre dos posibles. Se tira esta moneda, que está desbalanceada, con probabilidades α_i y β_i, con $\alpha_i + \beta_i = 1$ y $1 \geq \alpha_i, \beta_i \geq 0$, $i = 1, 2$. Ahora se juega el juego Tug-of-War con ruido descripto en [63] con probabilidades dadas por α_i, β_i. Si sale cara (probabilidad α_i), entonces se tira una moneda balanceada (con probabilidades iguales de cara y ceca) y el jugador ganador de la tirada de esta última moneda mueve la posición de la ficha a cualquier $x_1 \in B_\varepsilon(x_0)$ de su elección. Por otra parte, si en la tirada de la moneda original desbalanceada sale ceca (probabilidad β_i) la posición del juego se mueve al azar con probabilidad uniforme a un punto $x_1 \in B_\varepsilon(x_0)$.

Cuando la posición de la ficha cae fuera del dominio Ω, digamos en la jugada τ, el juego termina. El pago total está dado por una *función de pago parcial* $f : \Omega \to \mathbb{R}$ y una *función de pago final* $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$. Con estas dos funciones de pago, el pago final que el Jugador II le paga al Jugador I viene dado por la fórmula $g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$.

Primero demostramos que este juego tiene un valor y que este valor satisface el Principio de Programación Dinámico dado por:

$$u^\varepsilon(x) = \varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{y\in B_\varepsilon(x)} u^\varepsilon(y) + \inf_{y\in B_\varepsilon(x)} u^\varepsilon(y) \right\} + \beta_i \fint_{B_\varepsilon(x)} u^\varepsilon(y)dy \right)$$

para todo $x \in \Omega$, con $u^\varepsilon(x) = g(x)$ para $x \notin \Omega$. A continuación, probamos que existe una función continua u tal que

$$u^\varepsilon \to u \qquad \text{uniformemente en} \quad \overline{\Omega}.$$

Este límite u resulta ser una solución viscosa de

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = \bar{f} & \text{en } \Omega, \\ u = g & \text{en } \partial\Omega, \end{cases} \tag{$*$}$$

donde $\bar{f} = 2f$, $-\Delta_p u = |\nabla u|^{2-p} \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ es el $p-$Laplaciano $1-$homogéneo y finalmente p_1, p_2 están dados por

$$\alpha_i = \frac{p_i - 2}{p_i + N} \quad , \quad \beta_i = \frac{2+N}{p_i+N}, \qquad i = 1,2.$$

Para este problema límite probamos existencia y unicidad de solución viscosa. Un resultado similar se puede obtener para el operador $\min\{-\Delta_{p_1} u, -\Delta_{p_2} u\}$. Notemos que una solución u de $(*)$ con $\bar{f} = 0$ nos da una cota uniforme para todas las funciones $p-$armónicas con $p_1 \leq p \leq p_2$, es decir, si v es solución de

$$\begin{cases} -\Delta_p v = 0 & \text{en } \Omega, \\ v = g & \text{en } \partial\Omega, \end{cases}$$

se tiene $v \geq u$.

Cuando este juego se juega con ruido en todos los turnos, es decir, cuando los dos β_i son estrictamente positivos, el juego termina casi seguramente independientemente de las estrategias elegidas por los dos jugadores. Cuando f es estrictamente positiva o estrictamente negativa, uno de los dos jugadores tiene una fuerte motivación para terminar el juego rápidamente. En ambos casos, este hecho simplifica notablemente los argumentos usados en las demostraciones. Cuando f es cero y uno de los α_i es igual a uno (y en consecuencia el correspondiente β_i es igual a cero) los argumentos se vuelven más delicados. Para probar que el juego tiene un valor en este caso nos hace falta desarrollar un nuevo argumento que es diferente a los usados en los trabajos previos.

Motivados por el estudio de la ecuación $\max\{-\Delta_{p_1}u, -\Delta_{p_2}u\} = 0$ fuimos llevados a considerar ecuaciones de la forma

$$\max\{L_1 u, L_2 u\} = 0.$$

Aquí L_1 y L_2 son dos operadores para los que vale el principio del máximo. En el Capítulo 4 estudiamos este tipo de problemas y los relacionamos con soluciones del problema del obstáculo, es decir, soluciones de

$$\begin{cases} u \geq \Phi & \text{en } \Omega, \\ Lu \geq 0 & \text{en } \Omega, \\ Lu = 0 & \text{en } \{u > \phi\}, \\ u = g & \text{en } \partial\Omega. \end{cases} \tag{$**$}$$

Aquí las soluciones están sobre el obstáculo Φ en Ω. Una forma de entender este problema es interpretar su solución como la menor supersolución de $Lu = 0$ que se encuentra por encima del obstáculo Φ. Nos referiremos al problema del obstáculo como $P_L(\Phi, g)$.

Sean L_1 y L_2 dos operadores diferenciales y g definida en $\partial\Omega$ un dato de borde fijo. Definimos una sucesión de funciones continuas inductivamente. Tomamos u_1 como la solución del problema de Dirichlet para L_1. Luego, u_n está dada por la solución al problema del obstáculo para L_i ($i = 1, 2$ alternadamente) con obstáculo dado por el termino previo u_{n-1} en el dominio Ω. Es decir, definimos

$$u_n \text{ como la solución de } \begin{cases} P_{L_2}(u_{n-1}, g) & \text{para } n \text{ par}, \\ P_{L_1}(u_{n-1}, g) & \text{para } n \text{ impar}. \end{cases}$$

Demostramos que de esta manera obtenemos un sucesión creciente que converge uniformemente a una solución viscosa del operador minimal asociado a L_1 y L_2, esto es, el límite u verifica $\min\{L_1 u, L_2 u\} = 0$ en Ω con $u = g$ en $\partial\Omega$.

Al considerar el problema del obstáculo por arriba (esto es, tomamos $u \leq \Phi$ y $Lu \leq 0$ en Ω en $(**)$) obtenemos, con las mismas ideas, una solución de $\max\{L_1 u, L_2 u\} = 0$. También incluimos algunas extensiones de este resultado. De manera similar podemos obtener una construción para una familia finita o numerable de operadores. También proponemos una construcción diferente que nos permite obtener un resultado similar para una familia arbitraria de operadores.

En el Capítulo 5 presentamos resultados que fueron inspirados por el estudio del $\infty-$laplaciano en el contexto de los juegos tipo Tug-of-War. Obtenemos una cota inferior para el autovalor principal de Dirichlet de un operador elíptico fuertemente no lineal. La cota obtenida depende del radio de la bola más grande incluida en Ω, esto es

$$R = \max_{x \in \bar{\Omega}} \operatorname{dist}(x, \Omega^c).$$

Dado un operador L, para obtener la cota debemos construir un función radial creciente $\phi(r)$ definida en B_R con $\phi(0) = 0$ tal que

$$L\phi + \lambda\phi \leq 0$$

en $B_R \setminus \{0\}$ para algún $\lambda \in \mathbb{R}$. Luego obtenemos

$$\lambda_1(\Omega) \geq \lambda.$$

Ilustramos la construción requerida para obtener la cota en varios ejemplos. En particular usamos el resultado para probar que

$$\lim_{p\to\infty} \lambda_{1,p} = \lambda_{1,\infty} = \left(\frac{\pi}{2R}\right)^2$$

donde $\lambda_{1,p}$ y $\lambda_{1,\infty}$ son los autovalores principales para el p-laplaciano homogéneo y para el ∞-laplaciano respectivamente.

En el Capítulo 6 presentamos un juego que llamamos *caminata aleatoria para* λ_j. Aquí $\lambda_1 \leq ... \leq \lambda_N$ son los autovalores (ordenados de menor a mayor) de la matriz Hesiana D^2u. Como antes el juego se desarrolla en un dominio abierto acotado $\Omega \subset \mathbb{R}^N$. Se fija un número real $\varepsilon > 0$. Una ficha se coloca en $x_0 \in \Omega$. El Jugador I, que busca minimizar el pago final, elige un subespacio S de dimensión j y luego en Jugador II (que intenta maximizar el pago final) elige un vector unitario, v, en el subespacio previamente elegido S. Luego la ficha es movida a $x \pm \varepsilon v$ con igual probabilidad. Después de la primera ronda, el juego continua desde x_1 con las mismas reglas.

Denotamos $x_\tau \in \mathbb{R}^N \setminus \Omega$ la primer posición fuera de Ω. En este momento el juego termina con pago final dado por $g(x_\tau)$, donde $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ es una función continua. El Jugador I gana $-g(x_\tau)$ y el Jugador II obtiene $g(x_\tau)$.

Los valores del juego satisfacen

$$u^\varepsilon(x) = \inf_{dim(S)=j} \sup_{v\in S, |v|=1} \left\{ \frac{1}{2}u^\varepsilon(x + \varepsilon v) + \frac{1}{2}u^\varepsilon(x - \varepsilon v) \right\}$$

y convergen uniformemente cuando $\varepsilon \to 0$ a una solución de

$$\lambda_j(D^2u) = 0,$$

en Ω, con $u = g$, en $\partial\Omega$.

El juego nos motivó a estudiar la ecuación $\lambda_j(D^2u) = 0$. Como referencias sobre este problema mencionamos [19, 18, 27, 39, 40, 74, 76]. En la tesis, damos una interpretación geométrica de las soluciones viscosas del problema en términos de envolvente cóncavas/convexas sobre espacios afines de dimensión j. Consideramos H_j, el conjunto de las funciones v tales que

$$v \leq g \qquad \text{en } \partial\Omega,$$

y tienen la siguiente propiedad: para todo espacio afín S de dimension j y todo dominio j-dimensional $D \subset S \cap \Omega$ vale que

$$v \leq z \qquad \text{en } D$$

donde z es la envolvente cóncava de $v|_{\partial D}$ en D.

Obtuvimos el siguiente resultado: una función semi-continua superior v pertenece a H_j si y solo si es una subsolución de $\lambda_j(D^2u) = 0$. Más aún, probamos que la función

$$u(x) = \sup_{v \in H_j} v(x)$$

es la solución viscosa más grande de $\lambda_j(D^2u) = 0$, en Ω, con $u \leq g$ en $\partial\Omega$.

Con esta caracterización para las soluciones damos condiciones necesarias y suficientes en el dominio tal que el problema tiene una solución continua para todo dato g. Dado $y \in \partial\Omega$ asumimos que para todo $r > 0$ existe $\delta > 0$ tal que para todo $x \in B_\delta(y) \cap \Omega$ y $S \subset \mathbb{R}^N$ subespacio de dimensión j, existe $v \in S$ de norma 1 tal que

$$\{x + tv\}_{t \in \mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset. \qquad (G_j)$$

Con esta definición, probamos que la ecuación $\lambda_j(D^2u) = 0$ tiene una solución continua para todo dato continuo g si y solo si Ω satisface (G_j) y (G_{N-j+1}). Notar que este es un resultado del tipo "si y solo si", algo que no es usual al dar condiciones de resolubilidad sobre los dominios.

En el Apéndice A incluimos algunos resultados de la teoría de soluciones viscosas; en el Apéndice B algunos resultados de la teoría de probabilidades que usamos a lo largo de los capítulos.

Los resultados de esta tesis están contenidos en los siguientes artículos:

1. P. Blanc – J. P. Pinasco – J. D. Rossi. *Obstacle problems and maximal operators.* Advanced Nonlinear Studies. Vol. 16(2), 355–362, (2016).

2. P. Blanc – J. P. Pinasco – J. D. Rossi. *Maximal operators for the p-Laplacian family.* Pacific Journal of Mathematics. Vol. 287(2), 257–295, (2017).

3. P. Blanc – J. D. Rossi. *Games for eigenvalues of the Hessian and concave/convex envelopes.* To appear in Journal de Mathematiques Pures et Appliquees.

4. P. Blanc. *A lower bound for the principal eigenvalue of fully nonlinear elliptic operators.* Submitted.

Chapter 2

Introduction

This thesis was motivated by the study of different variants of Tug-of-War games. These games have lead to a new chapter in the rich history of results connecting differential equations and probability theory. The fundamental works by Doob, Feller, Hunt, Kakutani, Kolmogorov and many others show the deep connection between classical potential theory and probability theory. The main idea that is behind this relation is that harmonic functions and martingales have something in common: the mean value formulas. This relation is also quite fruitful in the non-linear case and the Tug-of-War games are a clear evidence of this fact.

At the end of the decade of the 80s the mathematician David Ross Richman proposed a new kind of game that lies in between the classical games introduced by Von Neumann and Morgenstern, and the combinatoric games studied by Zermelo, Lasker and Conway among others. Here two players are in contest in an arbitrary combinatoric game (Tic-tac-toe, chess, checkers, etc.), but each one of them has a certain amount of money that modifies the rules of the game: at each turn the players bid and the one who offered more win the right to make the next move. In case that both players bid the same amount the turn can be decided by a coin toss.

At the end of the 90s in [50] and [51] this kind of games, known as Richman's games were studied. They have been translated into a diffusion problem on a graph: the nodes stand for the positions of the game and the links between the nodes are the allowed moves, a token is moved by one of the players according to who wins the bidding. The game ends when the token arrives to the nodes of the graph labelled as terminal ones and there a certain boundary datum says how much the first player gets (that is the amount of money that the other player pays).

Among the several variants of these games, an interesting case is when the turn is decided at random tossing a fair coin at each turn, getting rid in this way of the bidding mechanism. This idea gives rise to the game called Tug-of-War introduced in [70] by Peres, Schramm, Sheffield and Wilson. In that reference this game was studied and a connection with PDEs was found. More concretely, with the $\infty-$laplacian, an operator that appears naturally in a completely different context, it is associated to the minimal Lipschitz extension problem, see [6].

When we deal with differential equations we have to mention the concept of solution that we are considering. The theory for second order operators in divergence form is associated to the concept of weak solutions; however, when one deals with fully nonlinear equations that are not in divergence form, the use of viscosity solution seems more appropriate.

This notion of solution was introduced by Crandall and Lions in the 80s. Here we have to mention the classical reference [33]. We included a brief summary of the viscosity solutions theory in Appendix A. There you will find some general comments concerning the theory and some results that will be used in this thesis. Here, we restrict ourselves to this notion of solution. However, this is not a limitation. For example, for the $p-$Laplacian, $\mathrm{div}(|Du|^{p-2}Du) = 0$, we remark that it was proved in [41] and [44] the equivalence between solutions in the viscosity sense and in the weak sense.

Being one of the triggers of this thesis, we describe here the game introduced in [70]. Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one of them are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome).

Consider a bounded domain $\Omega \subset \mathbb{R}^N$ and a fixed $\varepsilon > 0$. At an initial time, a token is placed at a point $x_0 \in \Omega$. Players I and II play as follows. They toss a fair coin (with the same probability for heads and tails) and the winner of the toss moves the game token to any point x_1 of his choice at distance less than ε of the previous position, $x_1 \in B_\varepsilon(x_0)$. Then, they continue playing from x_1. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in B_\varepsilon(x_{k-1})$.

This procedure yields a sequence of game states $x_0, x_1, \ldots$. Once the game position leaves Ω, let say at the τ-th step, the game ends. At that time the token will be on $\mathbb{R}^N \setminus \Omega$. A *final payoff function* is given in $\mathbb{R}^N \setminus \Omega$, $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$. At the end of the game Player II pays Player I the amount given by $g(x_\tau)$, that is, Player I have earned $g(x_\tau)$ while Player II have earned $-g(x_\tau)$.

A strategy S_{I} for Player I is a collection of measurable mappings $S_{\mathrm{I}} = \{S_{\mathrm{I}}^k\}_{k=0}^\infty$ such that the next game position is

$$S_{\mathrm{I}}^k(x_0, x_1, \ldots, x_k) = x_{k+1} \in B_\varepsilon(x_k)$$

if Player I wins the toss given a partial history $(x_0, x_1, \ldots, x_k)$. Similarly Player II plays according to a strategy S_{II}.

For each $x_0 \in \Omega$ we can consider the expected payoff $u^\varepsilon(x_0)$ for the game starting at x_0 assuming that both players play optimally. This is what we call the *game value.*

For each ε, we have a function $u^\varepsilon : \bar{\Omega} \to \mathbb{R}$. In [70] it is proved that there exists a continuous function $u : \bar{\Omega} \to \mathbb{R}$ such that $u^\varepsilon \to u$ as $\varepsilon \to 0$, and that u satisfies

$$-\Delta_\infty u = -(\nabla u)^t D^2 u \nabla u = 0 \qquad \text{in } \Omega,$$

with the boundary condition $u = g$ on $\partial\Omega$.

After this seminal work many versions of the game were considered and many results obtained. In [71] a version of the game related to the $p-$laplacian is studied. Non-local version of the game where proposed in [20] and [21]. Different boundary conditions where considered: Neumann boundary conditions in [3] and mixed boundary conditions in [32]. A continuous time game was presented in [9].

Let us mention [61], [62] and [63] where a version of the game related to the p-laplacian is studied. These works provided a framework that was exploited in later works. A game related to the obstacle problem was studied in [64], one related to an operator with a gradient constrain was considered in [45], a game related to the $p(x)$-laplacian in [7], games related to parabolic problems in [58], [37] and [10], and we can find other variants in [38] and [66]. This approach was also useful to found different proofs of regularity results (such us Harnack's inequality and Hölder regularity), we refer to [56], [57], [73], [8] and [69].

Motivated by these results, here we consider a different variant of the game. In Chapter 3 we introduce the game that we call *unbalanced Tug-of-War game with noise*. The set-up is the same as in the original game. At every round Player I chooses a coin between two possible ones. They toss the chosen coin which is biased with probabilities α_i and β_i, $\alpha_i + \beta_i = 1$ and $1 \geq \alpha_i, \beta_i \geq 0$, $i = 1, 2$. Now, they play the Tug-of-War with noise game described in [63] with probabilities α_i, β_i. If they get heads (probability α_i), they toss a fair coin (with equal probability of heads and tails) and the winner of the toss moves the game position to any $x_1 \in B_\varepsilon(x_0)$ of his choice. On the other hand, if they get tails (probability β_i) the game state moves according to the uniform probability density to a random point $x_1 \in B_\varepsilon(x_0)$.

Once the game position leaves Ω, say at the τ-th step, the game ends. The total payoff is given by a *running payoff function* $f : \Omega \to \mathbb{R}$ and a *final payoff function* $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$. At the end Player II pays to Player I the amount given by the formula $g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$.

We prove that the game has a value and that the value satisfies the Dynamic Programming Principle, given by:

$$u^\varepsilon(x) = \varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{y\in B_\varepsilon(x)} u^\varepsilon(y) + \inf_{y\in B_\varepsilon(x)} u^\varepsilon(y) \right\} + \beta_i \fint_{B_\varepsilon(x)} u^\varepsilon(y)dy \right)$$

for $x \in \Omega$, with $u^\varepsilon(x) = g(x)$ for $x \notin \Omega$. Then we prove that there exists a continuous function u such that

$$u^\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega}.$$

This limit u turns out to be a viscosity solution to

$$\begin{cases} \max\{-\Delta_{p_1}u, -\Delta_{p_2}u\} = \bar{f} & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where $\bar{f} = 2f$, $-\Delta_p u = |\nabla u|^{2-p}\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $1-$homogeneous $p-$Laplacian and p_1, p_2 are given by

$$\alpha_i = \frac{p_i - 2}{p_i + N} \quad , \quad \beta_i = \frac{2+N}{p_i+N}, \qquad i = 1, 2.$$

For this limit problem, we prove existence and uniqueness of viscosity solutions. A similar result can be obtained for $\min\{-\Delta_{p_1}u, -\Delta_{p_2}u\}$. Note that a solution u to $(*)$ when $\bar{f} = 0$ gives a uniform bound for every $p-$harmonic function with $p_1 \leq p \leq p_2$, that is, if

$$\begin{cases} -\Delta_p v = 0 & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

we have $v \geq u$.

When the game is played with some noise at every turn, that is, when the two β_i are strictly positive, the game ends almost surely independently of the strategies adopted by the players. When f is strictly positive or negative, one of the players is motivated to finish the match quickly. In both cases, this fact simplifies the arguments used in the proofs. When f is zero and one of the α_i is one (and therefore the corresponding β_i is zero) the argument is more delicate. To prove that the game has a value we need to develop a new argument different from the ones used in the previous works.

Motivated by the study of the equation $\max\{-\Delta_{p_1}u, -\Delta_{p_2}u\} = 0$ we were lead to consider equations of the form

$$\max\{L_1u, L_2u\} = 0.$$

Here L_1 and L_2 are two operators that have a maximum principle. In Chapter 4 we study that kind of problem. We relate them to solutions of the obstacle problem, that is

$$\begin{cases} u \geq \Phi & \text{in } \Omega, \\ Lu \geq 0 & \text{in } \Omega, \\ Lu = 0 & \text{in } \{u > \phi\}, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{$**$}$$

Here solutions are above the obstacle Φ inside Ω. A possible way to interpret the solution to this problem is to look for the smallest supersolution to $Lu = 0$ that is above the obstacle Φ. We will refer to the obstacle problem as $P_L(\Phi, g)$.

Let L_1 and L_2 be two differential operators and g defined on $\partial\Omega$ a fixed boundary datum. We define a sequence of continuous functions inductively. We take u_1 as the solution to the Dirichlet problem for L_1. Then, u_n is given by the solution to the obstacle problem for an operator L_i ($i = 1, 2$ alternating them) with obstacle given by the previous term u_{n-1} in the domain Ω. That is, we define

$$u_n \text{ as the solution to } \begin{cases} P_{L_2}(u_{n-1}, g) & \text{for } n \text{ even,} \\ P_{L_1}(u_{n-1}, g) & \text{for } n \text{ odd.} \end{cases}$$

We show that in this way we obtain an increasing sequence that converge uniformly to a viscosity solution to the minimal operator associated with L_1 and L_2, that is, the limit u verifies $\min\{L_1 u, L_2 u\} = 0$ in Ω with $u = g$ on $\partial\Omega$.

When we consider the obstacle problem from above (that is we take $u \leq \Phi$ and $Lu \leq 0$ in Ω in $(**)$) we get, using the same ideas, a solution to $\max\{L_1 u, L_2 u\} = 0$. We also propose some extensions of this result. In a similar way we can obtain a construction for a finite number of operators or even for countably many operators. We also propose a different construction that allow us to obtain a similar result for an arbitrary family of operators.

In Chapter 5 we present a result inspired in the study of the $\infty-$laplacian in the context of Tug-of-War games. We obtain a lower bound for the principal Dirichlet eigenvalue of a fully nonlinear elliptic operator. The bound obtained depends on the largest radius of a ball included in Ω, that is

$$R = \max_{x \in \bar{\Omega}} \operatorname{dist}(x, \Omega^c).$$

Given an operator L, to obtain the lower bound we need to construct a radial increasing function $\phi(r)$ defined in B_R with $\phi(0) = 0$ such that

$$L\phi + \lambda\phi \leq 0$$

in $B_R \setminus \{0\}$ for certain $\lambda \in \mathbb{R}$. Then we obtain

$$\lambda_1(\Omega) \geq \lambda.$$

We illustrate the construction required to obtain the bound in several examples. In particular we use our results to prove that

$$\lim_{p \to \infty} \lambda_{1,p} = \lambda_{1,\infty} = \left(\frac{\pi}{2R}\right)^2$$

where $\lambda_{1,p}$ and $\lambda_{1,\infty}$ are the principal eigenvalue for the homogeneous p-laplacian and the homogeneous infinity laplacian respectively.

In Chapter 6 we present a game that we call *a random walk for* λ_j. Here $\lambda_1 \leq ... \leq \lambda_N$ are the eigenvalues (ordered form the smaller to the largest) of the Hessian matrix

D^2u. As before the game is played in a bounded open set $\Omega \subset \mathbb{R}^N$. A real number $\varepsilon > 0$ is given. A token is placed at $x_0 \in \Omega$. Player I, the player seeking to minimize the final payoff, chooses a subspace S of dimension j and then Player II (who wants to maximize the expected payoff) chooses a unitary vector, v, in the subspace S. Then the position of the token is moved to $x \pm \varepsilon v$ with equal probabilities. After the first round, the game continues from x_1 according to the same rules.

We denote by $x_\tau \in \mathbb{R}^N \setminus \Omega$ the first point outside Ω. At this time the game ends with the final payoff given by $g(x_\tau)$, where $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ is a continuous function. Player I earns $-g(x_\tau)$ while Player II earns $g(x_\tau)$.

The game value satisfies

$$u^\varepsilon(x) = \inf_{dim(S)=j} \sup_{v \in S, |v|=1} \left\{ \frac{1}{2} u^\varepsilon(x + \varepsilon v) + \frac{1}{2} u^\varepsilon(x - \varepsilon v) \right\}$$

and converges uniformly as $\varepsilon \to 0$ to a solution of

$$\lambda_j(D^2u) = 0,$$

in Ω, with $u = g$, on $\partial\Omega$.

This game motivated us to study the equation $\lambda_j(D^2u) = 0$. As references for this problem and related ones we mention [19, 18, 27, 39, 40, 74, 76]. In the thesis, we gave a geometric interpretation of the viscosity solutions to the problem in terms of convex/concave envelopes over affine spaces of dimension j. We consider H_j, the set of functions v such that

$$v \leq g \qquad \text{on } \partial\Omega,$$

and have the following property: for every S affine of dimension j and every $j-$ dimensional domain $D \subset S \cap \Omega$ it holds that

$$v \leq z \qquad \text{in } D,$$

where z is the concave envelope of $v|_{\partial D}$ in D.

We obtained the following result: an upper semi-continuous function v belongs to H_j if and only if it is a viscosity subsolution to $\lambda_j(D^2u) = 0$. Even more, we prove that the function

$$u(x) = \sup_{v \in H_j} v(x)$$

is the largest viscosity solution to $\lambda_j(D^2u) = 0$, in Ω, with $u \leq g$ on $\partial\Omega$.

With this characterization of the solutions we give necessary and sufficient conditions on the domain so that the problem has a continuous solution for every continuous datum g. Given $y \in \partial\Omega$ we assume that for every $r > 0$ there exists $\delta > 0$ such that for every

$x \in B_\delta(y) \cap \Omega$ and $S \subset \mathbb{R}^N$ a subspace of dimension j, there exists $v \in S$ of norm 1 such that

$$\{x + tv\}_{t\in\mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset. \qquad (G_j)$$

With this definition, we prove that the equation $\lambda_j(D^2u) = 0$ has a continuous solution for every continuous data g if and only if Ω satisfies both (G_j) and (G_{N-j+1}). Note that this is an "if and only if" result, something that is not usual when dealing with solvability conditions on the domain.

In Appendix A we include results from general viscosity theory; while in Appendix B we collect some probability results that we use along the chapters.

The results of this thesis are contained in the following articles:

1. P. Blanc – J. P. Pinasco – J. D. Rossi. *Obstacle problems and maximal operators.* Advanced Nonlinear Studies. Vol. 16(2), 355–362, (2016).

2. P. Blanc – J. P. Pinasco – J. D. Rossi. *Maximal operators for the p-Laplacian family.* Pacific Journal of Mathematics. Vol. 287(2), 257–295, (2017).

3. P. Blanc – J. D. Rossi. *Games for eigenvalues of the Hessian and concave/convex envelopes.* To appear in Journal de Mathematiques Pures et Appliquees.

4. P. Blanc. *A lower bound for the principal eigenvalue of fully nonlinear elliptic operators.* Submitted.

Chapter 3

Maximal operators for the p-laplacian family

3.1 Introduction

In this chapter we study a variant of the tug-of-war game introduced in [62]. The version of the game that we consider here is related to the PDE

$$\max\{-\Delta_{p_1}u(x),\, -\Delta_{p_2}u(x)\} = f(x).$$

We will explore the interplay between probability theory and partial differential equations that arises. Here we include the results obtained in [24], a joint work with Juan Pablo Pinasco and Julio Daniel Rossi.

Our first goal is to show existence and uniqueness of viscosity solutions to the Dirichlet problem for the maximal operator associated with the family of $p-$Laplacian operators, $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $2 \leq p \leq \infty$. We refer to Appendix A for details on viscosity solutions.

When one considers the family of uniformly elliptic second order operators of the form $-tr(AD^2u)$ and look for maximal operators one finds the so-called Pucci maximal operator, $P^+_{\lambda,\Lambda}(D^2u) = \max_{A\in\mathcal{A}} -tr(AD^2u)$, where $\mathcal{A}$ is the set of uniformly elliptic matrices with ellipticity constant between λ and Λ. This maximal operator plays a crucial role in the regularity theory for uniformly elliptic second order operators and has the following properties, see [30]:

1. (Monotonicity) If $\lambda_1 \leq \lambda_2 \leq \Lambda_2 \leq \Lambda_1$ then $P^+_{\lambda_2,\Lambda_2}(D^2u) \leq P^+_{\lambda_1,\Lambda_1}(D^2u)$.

2. (Positively homogeneous) If $\alpha \geq 0$, then $P^+_{\lambda,\Lambda}(\alpha D^2u) = \alpha P^+_{\lambda,\Lambda}(D^2u)$.

3. (Subsolutions) If u verifies $P^+_{\lambda,\Lambda}(D^2u) \leq 0$ in the viscosity sense, then $-tr(AD^2u) \leq 0$ for every matrix A with ellipticity constants λ and Λ (that is, a subsolution to

the maximal operator is a subsolution for every elliptic operator in the class). Therefore, from the comparison principle we get that a solution to $P^+_{\lambda,\Lambda}(D^2u) \leq 0$ provides a lower bound for every solution of any elliptic operator in the class with the same boundary values.

If we try to reproduce these properties for the family of $p-$Laplacians we are lead to consider the operator $\max_{p_1 \leq p \leq p_2} -\Delta_p u(x)$. This operator has similar properties to the ones that hold for the Pucci maximal operator, but with respect to the $p-$Laplacian family.

Hence, it is natural to consider the Dirichlet problem for the partial differential equation

$$\max_{p_1 \leq p \leq p_2} -\Delta_p u(x) = f(x) \tag{3.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ for $2 \leq p_1, p_2 \leq \infty$. Here we have normalized the $p-$Laplacian and considered the operator

$$\Delta_p u = \frac{\operatorname{div}(|\nabla u|^{p-2}\nabla u)}{(N+p)|\nabla u|^{p-2}},$$

that is called the 1-homogeneus p-Laplacian. We will assume that $f \equiv 0$ or that f is strictly positive or negative in Ω. We will call solutions to this problem with $f \equiv 0$, u, as p_1-p_2-harmonic functions.

Note that, formally, the 1-homogeneus p-laplacian can be written as

$$\Delta_p u = \frac{p-2}{N+p}\Delta_\infty u + \frac{1}{N+p}\Delta u$$

where Δu is the usual Laplacian and $\Delta_\infty u$ is the normalized $\infty-$Laplacian, that is,

$$\Delta u = \sum_{i=1}^N u_{x_i x_i} \qquad \text{and} \qquad \Delta_\infty u = \frac{1}{|\nabla u|^2}\sum_{i,j=1}^N u_{x_i} u_{x_i x_i} u_{x_j}.$$

Therefore, we can think about the 1-homogeneus p-laplacian as a convex combination of the laplacian divided by $N+2$ and the ∞-laplacian, in fact,

$$\Delta_p u = \frac{p-2}{N+p}\Delta_\infty u + \frac{N+2}{N+p}\frac{\Delta u}{N+2} = \alpha\Delta_\infty u + \theta\Delta u$$

with $\alpha = \frac{p-2}{N+p}$ and $\theta = \frac{1}{N+p}$ (we reserve β for a different constant) for $2 \leq p < \infty$, and $\alpha = 1$ and $\theta = 0$ for $p = \infty$.

Since we are dealing with convex combinations, equation (3.1) becomes

$$\max_{p_1 \leq p \leq p_2} -\Delta_p u(x) = \max\left\{-\Delta_{p_1} u(x),\, -\Delta_{p_2} u(x)\right\} = f(x) \tag{3.2}$$

with $2 \leq p_1, p_2 \leq \infty$.

Our main result concerning viscosity solutions to (3.2) reads as follows:

Theorem 3.1.1. *Let Ω be a bounded domain such that the exterior ball condition holds when $p_1 \le N$ or $p_2 \le N$. Assume that $\inf_\Omega f > 0$, $\sup_\Omega f < 0$ or $f \equiv 0$. Then, given g a continuous function defined on $\partial\Omega$, there exists a unique viscosity solution $u \in C(\bar{\Omega})$ of (3.2) with $u = g$ in $\partial\Omega$.*

Moreover, a comparison principle holds, if $u, v \in C(\bar{\Omega})$ are such that

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f \qquad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge f$$

in Ω and $v \ge u$ on $\partial\Omega$, then $v \ge u$ in Ω.

In addition, we have a Hopf's lemma: let u be a supersolution to (3.2) and $x_0 \in \partial\Omega$ be such that $u(x_0) > u(x)$ for all $x \in \Omega$, then we have

$$\limsup_{t\to 0^+} \frac{u(x_0 - t\nu) - u(x_0)}{t} < 0.$$

where ν is exterior normal to $\partial\Omega$.

Remark 3.1.2. An analogous result holds for the equation $\min_{p_1 \le p \le p_2} -\Delta_p u(x) = f$.

Remark 3.1.3. For the homogeneous case, $f \equiv 0$, we have that viscosity sub and supersolutions to the 1-homogeneus p-Laplacian, $-\frac{p-2}{N+p}\Delta_\infty u - \frac{1}{N+p}\Delta u = 0$, coincide with viscosity sub and supersolutions to the usual ($p-1$ homogeneous) $p-$Laplacian $-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = 0$, see [63].

Therefore, for $f \equiv 0$ we are providing existence and uniqueness of viscosity solutions to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) = 0$, being $\Delta_p u$ the usual $p-$Laplacian that comes from calculus of variations.

Remark 3.1.4. This maximal operator for the $p-$Laplacian family has the following properties that are analogous to the ones described above for Pucci's operator:

1. (Monotonicity) If $p_{1,1} \le p_{2,1} \le p_{2,2} \le p_{1,2}$ then

$$\max_{p_{2,1} \le p \le p_{2,2}} -\Delta_p u \le \max_{p_{1,1} \le p \le p_{1,2}} -\Delta_p u.$$

2. (Positively homogeneous) If $\alpha \ge 0$, then

$$\max_{p_1 \le p \le p_2} -\Delta_p(\alpha u) = \alpha \max_{p_1 \le p \le p_2} -\Delta_p u.$$

3. (Subsolutions) A viscosity solution u to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) \le 0$, is a viscosity solution to $-\Delta_p u(x) \le 0$ for every $p_1 \le p \le p_2$. Hence, from the comparison principle we get that a solution to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) \le 0$, provides a lower bound for every solution of any elliptic operator in the class with the same boundary values.

We have two different approaches for this problem. The first one is based in PDE tools in the framework of viscosity solutions. The second one is related to probability theory (game theory) using the game that we describe below.

Let us introduce thea game that we call *unbalanced Tug-of-War game with noise*. It is a two-player (Players I and II) zero-sum stochastic game. The game is played in a bounded open set $\Omega \subset \mathbb{R}^N$. Fix an $\varepsilon > 0$. At the initial time, the players place a token at a point $x_0 \in \Omega$ and Player I chooses a coin between two possible ones. They toss the chosen coin which is biased with probabilities α_i and β_i, $\alpha_i + \beta_i = 1$ and $1 \geq \alpha_i, \beta_i \geq 0$, $i = 1, 2$. Now, they play the Tug-of-War with noise game described in [63] with probabilities α_i, β_i. If they get heads (probability α_i), they toss a fair coin (with equal probability of heads and tails) and the winner of the toss moves the game position to any $x_1 \in B_\varepsilon(x_0)$ of his choice. On the other hand, if they get tails (probability β_i) the game state moves according to the uniform probability density to a random point $x_1 \in B_\varepsilon(x_0)$. Once the game position leaves Ω, let say at the τ-th step, the game ends. The payoff is given by a *running payoff function* $f : \Omega \to \mathbb{R}$ and a *final payoff function* $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ (note that we only use the values of g in a strip of width ε around $\partial\Omega$). At the end Player II pays to Player I the amount given by the formula $g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$. Note that the positions of the game depend on the strategies adopted by Players I and II. From this procedure we get two extreme functions, $u_{\mathrm{I}}(x_0)$ (the value of the game for Player I) and $u_{\mathrm{II}}(x_0)$ (the value of the game for Player II), that are in a sense the best expected outcomes that each player can expect choosing a strategy when the game starts at x_0. When $u_{\mathrm{I}}(x_0)$ and $u_{\mathrm{II}}(x_0)$ coincide at every $x_0 \in \Omega$ this function $u_\varepsilon := u_{\mathrm{I}} = u_{\mathrm{II}}$ is called the *value of the game*.

Theorem 3.1.5. *Assume that f is a Lipschitz function with $\sup_\Omega f < 0$ or $\inf_\Omega f > 0$ or $f \equiv 0$. The unbalanced Tug-of-War game with noise with $\{\alpha_1, \alpha_2\} \neq \{0, 1\}$ when $f \equiv 0$ has a value and that value satisfies the Dynamic Programming Principle, given by:*

$$u_\varepsilon(x) = \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{y \in B_\varepsilon(x)} u_\varepsilon(y) + \inf_{y \in B_\varepsilon(x)} u_\varepsilon(y) \right\} + \beta_i \fint_{B_\varepsilon(x)} u_\varepsilon(y)dy \right)$$

for $x \in \Omega$, with $u_\varepsilon(x) = g(x)$ for $x \notin \Omega$.

Moreover, if g is Lipschitz and Ω satisfies the exterior ball condition, then there exists a uniformly continuous function u such that

$$u_\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega}.$$

This limit u is a viscosity solution to

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = \bar{f} & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\bar{f} = 2f$ and p_1, p_2 are given by

$$\alpha_i = \frac{p_i - 2}{p_i + N} \quad , \quad \beta_i = \frac{2 + N}{p_i + N}, \qquad i = 1, 2.$$

Remark 3.1.6. When f is strictly positive or negative we have that the game ends almost surely. The same is true (regardless the strategies adopted by the players) when they play with some noise at every turn, that is, when the two β_i are positive. This fact simplifies the arguments used in the proofs.

When one of the α_i is one (and therefore the corresponding β_i is zero) the argument is more delicate, see Section 3.4.

Remark 3.1.7. The proof of Theorem 3.1.5 follows from the results in sections 4 and 5. In section 4 we establish that the game has a value and that the value is the unique function that satisfies the Dynamic Programming Principle (DPP). In section 5 we prove the convergence part of the theorem. In Proposition 3.4.4 we establish the existence of a function satisfying the DPP. In Theorem 3.4.6 we prove that the function satisfying the DPP is unique and coincide with the game value, in the case $\beta_1, \beta_2 > 0$, $\sup f < 0$ or $\inf f > 0$. The same result is obtained in the remaining cases in Theorems 3.4.8 and 3.4.9. Here is where we had to assume that $\{\alpha_1, \alpha_2\} \neq \{0, 1\}$. Finally, the convergence is established in Corollaries 3.5.8 and 3.5.9.

Remark 3.1.8. Note that in the limit problem one only considers the values of g on $\partial\Omega$ while in the game one needs g to be defined in a bigger set. Given a Lipschitz function defined on $\partial\Omega$ we can just extend it to this larger set without affecting the Lipschitz constant. For simplicity but making an abuse of notation we also call such extension as g.

Remark 3.1.9. We also prove uniqueness of solutions to the DPP, see Section 3.4. That is, there exists a unique function verifying

$$v(x) = \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{y \in B_\varepsilon(x)} v(y) + \inf_{y \in B_\varepsilon(x)} v(y) \right\} + \beta_i \fint_{B_\varepsilon(x)} v(y) dy \right)$$

for $x \in \Omega$, with $v(x) = g(x)$ for $x \notin \Omega$.

Remark 3.1.10. When Player II (recall that this player wants to minimize the expected outcome) has the choice of the probabilities α and β we end up with a solution to

$$\begin{cases} \min\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

Finally, we finish the introduction with a comment on the main technical novelties contained in this chapter. To obtain existence and uniqueness for our maximal PDE we first use ideas and techniques from viscosity solutions theory. This part follows the usual steps (first one shows a comparison principle and then applies Perron's method, including the construction of barriers near the boundary), but here some extra care is needed to deal with points at which the gradient of a test function vanishes. Concerning the game theoretical approach we want to emphasize that when $p_2 = \infty$ we don't know a priori that the game terminates almost surely and this fact introduces some extra difficulties. The argument that shows that there is a unique solution to the dynamic programming principle in this case is delicate, see Theorem 3.4.8. The proof of convergence of the values of the game as the size of the steps goes to zero is also different from previous results in the literature since here one has to take care of the strategy of the player who chooses the parameters of the game. In particular, the proof of the fact that when any of the two players pull in a fix direction the expectation of the exit time is bounded above $C\epsilon^2$ is more involved, see Lemma 3.5.2.

The rest of the chapter is organized as follows: In Section 3.2 we prove the comparison principle and then existence and uniqueness for our problem using Perron's method; in Section 3.3 we introduce a precise description of the game; in Section 3.4 we show that the game has a value and that this value is the solution to the Dynamic Programming Principle; finally, in Section 3.5 we collect some properties of the value function of the game and show that these values converge to the unique viscosity solution of our problem.

3.2 Existence and uniqueness

First, let us state the definition of a viscosity solution. We refer to Appendix A for general comments on viscosity solutions. We have to handle some technical difficulties as the 1−homogeneous ∞-laplacian is not well defined when the gradient vanish. Observing that

$$\Delta u = tr(D^2u) \qquad \text{and} \qquad \Delta_\infty u = \frac{\nabla u}{|\nabla u|} D^2u \frac{\nabla u}{|\nabla u|},$$

we can write (3.2) as $F(\nabla u, D^2u) = f$ where

$$F(v, X) = \max_{i\in\{1,2\}} \left\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} - \theta_i tr(X) \right\}$$

Note that F is degenerate elliptic, that is,

$$F(v, X) \leq F(v, Y) \text{ for } v \in \mathbb{R}^N \setminus \{0\} \text{ and } X, Y \in S^N \text{ provided } X \geq Y,$$

as it is generally requested to work in the context of viscosity solutions.

This function $F : \mathbb{R}^N \times \mathbb{S}_N \mapsto \mathbb{R}$ is not well defined at $v = 0$. Therefore, we need to consider the lower semicontinous F_* and upper semicontinous F^* envelopes of F. This functions coincide with F for $v \neq 0$ and for $v = 0$ are given by

$$F^*(0, X) = \max_{i \in \{1,2\}} \{-\alpha_i \lambda_{\min}(X) - \theta_i tr(X)\}$$

and

$$F_*(0, X) = \max_{i \in \{1,2\}} \{-\alpha_i \lambda_{\max}(X) - \theta_i tr(X)\}$$

where

$$\lambda_{\min}(X) = \min\{\lambda : \lambda \text{ is an eigenvalue of } X\}$$

and

$$\lambda_{\max}(X) = \max\{\lambda : \lambda \text{ is an eigenvalue of } X\}.$$

Now we are ready to give the definition for a viscosity solution to our equation.

Definition 3.2.1. For $2 \leq p_1, p_2 \leq \infty$ consider the equation

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f$$

in Ω.

1. A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below, we have

$$F^*(\nabla\phi(x), D^2\phi(x)) \geq f(x).$$

2. An upper semi-continuous function u is a subsolution if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above, we have

$$F_*(\nabla\psi(x), D^2\psi(x)) \leq f(x).$$

3. Finally, u is a viscosity solution if it is both a sub- and supersolution.

In the case $f \equiv 0$ comparison holds for our equation as a consequence of the main result of [49]. See also [11]. Note that the comparison principle obtained in [49] is slightly more general than the one obtained in [11]. We need this more general result here as our F is not necessarily continuous when the gradient vanishes. In [49] a different notion of viscosity solution is considered. We remark that when a function is a viscosity sub or super-solution in the sense of Definition 5.2.1 it is also that in the sense considered in [49]. Therefore we can use the comparison result established there once we check their hypotheses.

Proposition 3.2.2. *Let $u \in USC(\Omega)$ and $v \in LSC(\Omega)$ be, respectively, a viscosity subsolution and a viscosity supersolution of* (3.2) *with $f \equiv 0$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Proof. We just apply the main result in [49]. To this end we need to check some conditions (we refer to [49] for notations and details). First, let us show that F is elliptic, in fact we have

$$\begin{aligned} F(v, X - \mu v \otimes v) &= \max_{i \in \{1,2\}} \left\{ -\alpha_i \frac{v}{|v|}(X - \mu v \otimes v)\frac{v}{|v|} - \theta_i tr(X - \mu v \otimes v) \right\} \\ &= \max_{i \in \{1,2\}} \left\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} + \alpha_i \mu |v|^2 - \theta_i tr(X) + \theta_i \mu |v|^2 \right\} \\ &= \max_{i \in \{1,2\}} \left\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} - \theta_i tr(X) + \theta_i \right\} + \mu |v|^2 \\ &= F(v, X) + \mu |v|^2. \end{aligned}$$

Moreover, F is invariant by rescaling in v and 1-homogeneous in X.

So, we can take $\sigma_0(v) = |v|^2$, $\sigma_1(t) = t$ and $\rho \equiv 0$ (using the notation from [49]) that satisfy the conditions imposed in [49] to obtain the comparison result. $\square$

Now we deal with the case where f is assumed to be nontrivial and does not change sign. In fact, we assume that $\inf f > 0$ or $\sup f < 0$. We follow similar ideas to the ones in [55].

Lemma 3.2.3. *If we have $u, v \in C(\bar{\Omega})$ such that*

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \leq f \qquad \text{and} \qquad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \geq g$$

where $g > f$ and $v \geq u$ in $\partial\Omega$, then we have $v \geq u$ in Ω.

Proof. By adding a constant if necessary we can assume that $u, v > 0$. Arguing by contradiction we assume that

$$\max_{\overline{\Omega}}(u - v) > 0 \geq \max_{\partial\Omega}(u - v).$$

Now we double the variables and consider

$$\sup_{x,y \in \Omega} \left\{ u(x) - v(y) - \frac{j}{2}|x - y|^2 \right\}.$$

For large j the supremum is attained at interior points x_j, y_j such that $x_j \to \hat{x}$, $y_j \to \hat{x}$, where $\hat{x}$ is an interior point (that $\hat{x}$ cannot be on the boundary can be obtained as in [52]).

Now, we observe that there exists a constant C such that $j|x_j - y_j| \leq C$. The theorem of sums (see Theorem 3.2 from [33]) implies that there are symmetric matrices $\mathbb{X}_j$, $\mathbb{Y}_j$, with $\mathbb{X}_j \leq \mathbb{Y}_j$ such that $(j|x_j - y_j|, \mathbb{X}_j) \in \overline{J^{2,+}}(u)(x_j)$ and $(j|x_j - y_j|, \mathbb{Y}_j) \in \overline{J^{2,-}}(v)(y_j)$, where $\overline{J^{2,+}}(u)(x_j)$ and $\overline{J^{2,-}}(v)(y_j)$ are the closures of the super and subjets of u and v respectively. Using the equations, assuming that $x_j \neq y_j$, we have

$$\max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i tr(\mathbb{X}_j) \right\} \leq f(y_j)$$

and

$$\max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i tr(\mathbb{Y}_j) \right\} \geq g(y_j)$$

Now we observe that, since $\mathbb{X}_j \leq \mathbb{Y}_j$ we get

$$-tr(\mathbb{X}_j) \geq -tr(\mathbb{Y}_j)$$

and

$$-\left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle \geq -\left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle.$$

Hence

$$\begin{aligned} f(y_j) &\geq \max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i tr(\mathbb{X}_j) \right\} \\ &\geq \max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i tr(\mathbb{Y}_j) \right\} \geq g(x_j). \end{aligned}$$

This gives a contradiction passing to the limit as $j \to \infty$.

When $x_j = y_j$ we obtain

$$\max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\max}(\mathbb{Y}_j) - \theta_i tr(\mathbb{Y}_j)\} \leq f(y_j)$$

and

$$\max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\min}(\mathbb{X}_j) - \theta_i tr(\mathbb{X}_j)\} \geq g(x_j)$$

that also leads to a contradiction since $\lambda_{\max}(\mathbb{Y}_j) \geq \lambda_{\max}(\mathbb{X}_j) \geq \lambda_{\min}(\mathbb{X}_j)$.

Hence we have obtained that $u \leq v$, as we wanted to prove. □

Lemma 3.2.4. *If $u, v \in C(\bar{\Omega})$ are such that*

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \leq f, \qquad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \geq f$$

in Ω with $\inf_\Omega f > 0$ and $v \geq u$ on $\partial\Omega$, then we have $v \geq u$ in Ω .

Proof. By adding a constant if necessary we can assume that $u, v > 0$. Lets consider $v_\delta = (1+\delta)v$

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \leq f < (1+\delta)f \leq \max\{-\Delta_{p_1} v_\delta, -\Delta_{p_2} v_\delta\}$$

and $v_\delta \geq v \geq u$ in $\partial\Omega$. Then by the preceding lemma we conclude that and $v_\delta \geq u$ in Ω for all $\delta > 0$. Making $\delta \to 0$, we get $v \geq u$ in Ω as we wanted to show. $\square$

Remark 3.2.5. The above lemma is also true when $\sup_\Omega f < 0$. So, we have comparison for the cases $\inf_\Omega f > 0$, $\sup_\Omega f < 0$ and $f \equiv 0$. From this comparison result we get uniqueness of solutions.

Now we deal with the existence of solutions. In the proof of this result we are only using that the exterior ball condition holds for Ω when $p_1 \leq N$ or $p_2 \leq N$.

Theorem 3.2.6. *Assume that* $\inf f > 0$*,* $\sup f < 0$ *or* $f \equiv 0$*. Then, given* g *a continuous function defined on* $\partial\Omega$*, there exists* $u \in C(\bar{\Omega})$ *a viscosity solution of* (3.2) *such that* $u = g$ *in* $\partial\Omega$*.*

Proof. We consider the set

$$\mathcal{A} = \Big\{ v \in C(\bar{\Omega}) : \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \geq f \ \textit{in } \Omega \textit{ and } v \geq g \textit{ on } \partial\Omega \Big\},$$

where the inequality for the equation inside Ω is verified in the viscosity sense and the inequality on $\partial\Omega$ in the pointwise sense. Since $\Delta|x|^2 = 2n$ and $\Delta_\infty |x|^2 = 2$ we have that $\max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} > 0$ for $v(x) = -|x|^2$. Hence we can choose K_1 such that the operator applied to $-K_1|x|^2$ is grater than $\sup f$ and then we can choose K_2 such that $K_2 - K_1|x|^2 \geq g(x)$ in $\partial\Omega$. We conclude that the function $K_2 - K_1|x|^2 \in \mathcal{A}$ for suitable K_1, K_2. Therefore the set $\mathcal{A}$ is not empty.

We define

$$u(x) = \inf_{v \in \mathcal{A}} v(x), \quad x \in \bar{\Omega}.$$

This infimum is finite since, as comparison holds, we have $u(x) \geq -L_2 + L_1|x|^2$ for all $u \in \mathcal{A}$ for large L_1, L_2. The function u, being the infimum of supersolutions, is a supersolution. We already know that u is upper semi-continuous, as it is the infimum of continuous functions. Let us see it is indeed a solution. Suppose not, then there exist $\phi \in C^2$ such that ϕ touches u at $x_0 \in \Omega$ strictly from above but

$$\max\{-\Delta_{p_1}\phi(x_0), -\Delta_{p_2}\phi(x_0)\} > f(x_0).$$

Lets write

$$\phi(x) = \phi(x_0) + \nabla\phi(x_0)\cdot(x - x_0) + \frac{1}{2}\langle D^2\phi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2)$$

We define $\hat{\phi}(x) = \phi(x) - \delta$ for a small positive number δ. Then $\hat{\phi} < u$ in a small neighborhood of x_0, contained in the set $\{x : \max\{-\Delta_{p_1}\phi(x), -\Delta_{p_2}\phi(x)\} > f(x)\}$, but $\hat{\phi} \geq u$ outside this neighborhood, if we take δ small enough.

Now we can consider $v = \min\{\hat{\phi}, u\}$. Since u is a viscosity supersolution in Ω and $\hat{\phi}$ also is a viscosity supersolution in the small neighborhood of x_0, it follows that v is a viscosity supersolution. Moreover, on $\partial\Omega$, $v = u \geq g$. This implies $v \in \mathcal{A}$, but $v = \hat{\phi} < u$ near x_0 , which is a contradiction with the definition of u as the infimum in $\mathcal{A}$.

Finally, we want to prove that $u = g$ on $\partial\Omega$ and that boundary values are attained with continuity. To this end, we have to construct barriers for our operator. It is enough to prove that for every $x_0 \in \partial\Omega$ and $\varepsilon > 0$ there exists a supersolution such that $v \geq g$ on $\partial\Omega$ and $v(x_0) \leq g(x_0) + \varepsilon$, and that there exists a subsolution such that $v \leq g$ on $\partial\Omega$ and $v(x_0) \geq g(x_0) - \varepsilon$. We prove now the existence of the supersolution, the subsolution can be obtained in a similar way.

Let us consider ϕ a radial function, $\phi(x) = \psi(r)$ with $\psi'(r) > 0$. Then

$$\Delta_\infty \phi = \psi'' \qquad \text{and} \qquad \Delta\phi = \psi'' + \frac{N-1}{r}\psi'$$

and we get

$$\begin{aligned}
\max_{i\in\{1,2\}} \{-\Delta_{p_i}\phi\} &= \max_{i\in\{1,2\}} \{-\alpha_i \Delta_\infty \phi - \theta_i \Delta\phi\} \\
&= \max_{i\in\{1,2\}} \left\{-\alpha_i \psi'' - \theta_i \left(\psi'' + \frac{N-1}{r}\psi'\right)\right\} \\
&= \max_{i\in\{1,2\}} \left\{-\frac{p_i-2}{N+p_i}\psi'' - \frac{1}{N+p_i}\left(\psi'' + \frac{N-1}{r}\psi'\right)\right\} \\
&= \max_{i\in\{1,2\}} \left\{-\frac{p_i-1}{N+p_i}\psi'' - \frac{1}{N+p_i}\frac{N-1}{r}\psi'\right\}.
\end{aligned}$$

We want this last expression to be greater than a positive constant.

To have a function of the form $\psi(r) = r^\gamma$ with $\gamma > 0$ that fulfills this, we need

$$\max_{i\in\{1,2\}} \left\{-\frac{p_i-1}{N+p_i}\gamma(\gamma-1) - \frac{N-1}{N+p_i}\gamma\right\} r^{\gamma-2} \geq c > 0.$$

Hence we have to choose γ according to

$$0 < \gamma < 1 - \frac{N-1}{p_i-1}.$$

We have that such γ exists if $N < p_1$ or $N < p_2$. We will require that $\min\{p_1, p_2\} > N$, that is, $N < p_1, p_2$.

In this case we can consider $v(x) = K\phi(x - x_0) + g(x_0) + \varepsilon$ with K big enough. If $Kc > \sup f$, then v is a supersolution. We have that $v(x_0) = g(x_0) + \varepsilon$, it remains to prove that $v \geq g$ on $\partial\Omega$. Since g is continuous at x_0, there exists $\delta > 0$ such that $|g(x) - g(x_0)| < \varepsilon$ for every $x \in B_\delta(x_0)$. Then we have that $v \geq g$ on $\partial\Omega \cap B_\delta(x_0)$. Finally we can pick K such that $K\delta^\gamma + g(x_0) + \varepsilon > \sup g$, and we obtain $v \geq g$ on $\partial\Omega \cap B_\delta(x_0)^c$.

When $N \geq p_1$ or $N \geq p_2$, we can find (with similar computations) a barrier of the form $\psi(r) = -r^\gamma$ with $\gamma < 0$. Note that this function is not well defined at 0. In this case, we have a barrier if the exterior ball condition holds. Given $x_0 \in \partial\Omega$ there exist $\lambda > 0$ and $y_0 \in \Omega^c$ such that $|x_0 - y_0| = \lambda$ and $B_\lambda(y_0) \subset \Omega^c$. We can consider $v(x) = K(\phi(x - y_0) - \phi(x_0 - y_0)) + g(x_0) + \varepsilon$ and pick K in a similar way as above. □

Now, we prove a version of the Hopf lemma for our equation. Note that since we deal with viscosity solutions the normal derivative may not exists in a classical sense.

Lemma 3.2.7. *Let $\Omega \subset \mathbb{R}^N$ be a domain with the interior ball condition and u subsolution to* (3.2) *whith $f \equiv 0$. Given $x_0 \in \partial\Omega$ such that $u(x_0) > u(x)$ for all $x \in \Omega$, we have*

$$\limsup_{t \to 0^+} \frac{u(x_0 - t\nu) - u(x_0)}{t} < 0.$$

where ν is exterior normal to $\partial\Omega$.

Proof. As the interior ball condition holds, we can assume there exist a ball centered at 0, contained in Ω that has x_0 in its boundary, that is, we have $B_r(0) \subset \Omega$ and $x_0 \in \partial B_r(0)$. Let us consider $\phi(x) = \frac{1}{|x|^{N-2}} - \frac{1}{r^{N-2}}$ if $N > 2$ and $\phi(x) = -ln|x| + ln(r)$ for $N = 2$. It easy to check that

$$\Delta\phi = 0, \qquad \Delta_\infty\phi \geq 0, \qquad \text{in } B_r(0) \setminus \{0\}.$$

So we have

$$\begin{aligned} \max\{-\Delta_{p_1}\phi, -\Delta_{p_2}\phi\} &\leq 0 && \text{in } B_r(0) \setminus \{0\}, \\ \phi &\equiv 0 && \text{on } \partial B_r(0). \end{aligned}$$

As $u(x_0) > u(x)$ for all $x \in \Omega$, in particular on $\partial B_{\frac{r}{2}}(0)$, then there exists $\varepsilon > 0$ such that $u(x_0) - \varepsilon\phi \geq u$ on $\partial B_{\frac{r}{2}}(0)$. Therefore, by the comparison principle, we get $u(x_0) - \varepsilon\phi \geq u$ in $B_r(0) \setminus B_{\frac{r}{2}}(0)$ and the result follows. □

3.3 Unbalanced Tug-of-War games with noise

In this section we introduce the game that we call *Unbalanced Tug-of-War game with noise*. First, let us describe the game without entering in mathematical details. It is a two-player zero-sum stochastic game. The game is played over a bounded open set $\Omega \subset \mathbb{R}^N$. An $\varepsilon > 0$ is given. Players I and II play as follows. At an initial time, they place a token at a point $x_0 \in \Omega$ and Player I choose a coin between two possible ones (each of the two coins have different probabilities of getting a head). We think she chooses $i \in \{1, 2\}$. Now they play the *Tug-of-War with noise* introduced in [63] starting with the chosen coin. They toss the chosen coin which is biased with probabilities α_i and β_i, $\alpha_i + \beta_i = 1$ and $1 \geq \alpha_i, \beta_i \geq 0$. If they get heads (probability α_i), they toss a fair coin (with the same probability for heads and tails) and the winner of the toss moves the game position to any $x_1 \in B_\varepsilon(x_0)$ of his choice. On the other hand, if they get tails (probability β_i) the game state moves according to the uniform probability density to a random point $x_1 \in B_\varepsilon(x_0)$. Note that Player I chooses the probability of playing the usual Tug-of-War game or moving at random with the choice of the first coin between two possibilities. Then they continue playing from x_1. At each turn Player I may change the choice of the coin.

This procedure yields a sequence of game states $x_0, x_1, \ldots$. Once the game position leaves Ω, let say at the τ-th step, the game ends. At that time the token will be on the compact boundary strip around Ω of width ε that we denote

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \leq \varepsilon\}.$$

The payoff is given by a *running payoff function* $f : \Omega \to \mathbb{R}$ and a *final payoff function* $g : \Gamma_\varepsilon \to \mathbb{R}$. At the end Player II pays Player I the amount given by a $g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$, that is, Player I have earned $g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$ while Player II have earned $-g(x_\tau) - \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$. We can think that when the token leaves x_i Player II pays Player I $\varepsilon^2 f(x_i)$, and $g(x_\tau)$ when the game ends.

A strategy S_{I} for Player I is a pair of collections of measurable mappings $S_{\mathrm{I}} = \big(\{\gamma^k\}_{k=0}^\infty, \{S_{\mathrm{I}}^k\}_{k=0}^\infty\big)$, such that, given a partial history $(x_0, x_1, \ldots, x_k)$, Player I choose coin 1 with probability

$$\gamma^k(x_0, x_1, \ldots, x_k) = \gamma \in [0, 1]$$

and the next game position is

$$S_{\mathrm{I}}^k(x_0, x_1, \ldots, x_k) = x_{k+1} \in B_\varepsilon(x_k)$$

if Player I wins the toss. Similarly Player II plays according to a strategy $S_{\mathrm{II}} = \{S_{\mathrm{II}}^k\}_{k=0}^\infty$. Then, the next game position $x_{k+1} \in B_\varepsilon(x_k)$, given a partial history $(x_0, x_1, \ldots, x_k)$, is distributed according to the probability

$$\begin{aligned}
&\pi_{S_{\mathrm{I}}, S_{\mathrm{II}}}(x_0, x_1, \ldots, x_k, A) \\
&\qquad = \frac{\beta\, |A \cap B_\varepsilon(x_k)|}{|B_\varepsilon(x_k)|} + \frac{\alpha}{2} \delta_{S_{\mathrm{I}}^k(x_0, x_1 \ldots, x_k)}(A) + \frac{\alpha}{2} \delta_{S_{\mathrm{II}}^k(x_0, x_1, \ldots, x_k)}(A),
\end{aligned}$$

where $\gamma = \gamma^k(x_0, x_1 \ldots, x_k)$, $\alpha = \alpha_1\gamma + \alpha_2(1-\gamma)$, $\beta = \beta_1\gamma + \beta_2(1-\gamma)$ and A is any measurable set (note that α and β depend on S_{I} and $(x_0, x_1, \ldots, x_k)$, we do not make this explicit to avoid overloading the notation). From now on, we shall omit k and simply denote the strategies by γ, S_{I} and S_{II}.

Let $\Omega_\varepsilon = \Omega \cup \Gamma_\varepsilon \subset \mathbb{R}^n$ be equipped with the natural topology, and the σ-algebra $\mathcal{B}$ of the Lebesgue measurable sets. The space of all game sequences

$$H^\infty = \{x_0\} \times \Omega_\varepsilon \times \Omega_\varepsilon \times \ldots,$$

is a product space endowed with the product topology.

Let $\{\mathcal{F}_k\}_{k=0}^\infty$ denote the filtration of σ-algebras, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ defined as follows: $\mathcal{F}_k$ is the product σ-algebra generated by cylinder sets of the form $\{x_0\} \times A_1 \times \ldots \times A_k \times \Omega_\varepsilon \times \Omega_\varepsilon \ldots$ with $A_i \in \mathcal{B}$. For

$$\omega = (x_0, \omega_1, \ldots) \in H^\infty,$$

we define the coordinate processes

$$X_k(\omega) = \omega_k, \quad X_k : H^\infty \to \mathbb{R}^n,\ k = 0, 1, \ldots$$

so that X_k is an $\mathcal{F}_k$-measurable random variable. Moreover, $\mathcal{F}_\infty = \sigma(\bigcup \mathcal{F}_k)$ is the smallest σ-algebra so that all X_k are $\mathcal{F}_\infty$-measurable. To denote the time when the game state reaches Γ_ε, we define a random variable

$$\tau(\omega) = \inf\{k \, : \, X_k(\omega) \in \Gamma_\varepsilon, k = 0, 1, \ldots\},$$

which is a stopping time relative to the filtration $\{\mathcal{F}_k\}_{k=0}^\infty$.

A starting point x_0 and the strategies S_{I} and S_{II} define (by Kolmogorov's extension theorem) a unique probability measure $\mathbb{P}^{x_0}_{S_I, S_{\mathrm{II}}}$ in H^∞ relative to the σ-algebra $\mathcal{F}^\infty$. We denote by $\mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}$ the corresponding expectation.

Then, if S_{I} and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for Player I as

$$V_{x_0,\mathrm{I}}(S_{\mathrm{I}}, S_{\mathrm{II}}) = \begin{cases} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[g(X_\tau) + \varepsilon^2 \sum_{n=1}^{\tau-1} f(x_n)] & \text{if the game ends a.s.} \\ -\infty & \text{otherwise,} \end{cases}$$

and then the expected payoff for Player II as

$$V_{x_0,\mathrm{II}}(S_{\mathrm{I}}, S_{\mathrm{II}}) = \begin{cases} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[g(X_\tau) + \varepsilon^2 \sum_{n=1}^{\tau-1} f(x_n)] & \text{if the game ends a.s.} \\ +\infty & \text{otherwise.} \end{cases}$$

Note that we penalize both players when the games doesn't end a.s.

The *value of the game for Player I* is given by

$$u_{\mathrm{I}}(x_0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} V_{x_0,\mathrm{I}}(S_{\mathrm{I}}, S_{\mathrm{II}})$$

while the *value of the game for Player II* is given by

$$u_{\mathrm{II}}(x_0) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} V_{x_0,\mathrm{II}}(S_{\mathrm{I}}, S_{\mathrm{II}}).$$

When $u_{\mathrm{I}} = u_{\mathrm{II}}$ we say the game has a value $u := u_{\mathrm{I}} = u_{\mathrm{II}}$. The values $u_{\mathrm{I}}(x_0)$ and $u_{\mathrm{II}}(x_0)$ are in a sense the best outcomes each player can expect when the game starts at x_0. For the measurability of the value functions we refer to [59] and [60].

Remark 3.3.1. It seems natural to consider a more general protocol to determine α in a prescribed closed set. It is clear that there are only two possible scenarios: At each turn Player I wants to maximize the value of α and Player II wants to minimize it, or the converse. An expected value for α is obtained in each case assuming each player plays optimal. Depending on the value of α in each case, we are considering a game equivalent to the one that we described previously or another one where Player II gets the choice of the first coin, for certain values of α_i.

3.4 The game value function and the Dynamic Programming Principle

In this section, we prove that the game has a value, that is, $u_{\mathrm{I}} = u_{\mathrm{II}}$ and that this value function satisfies the Dynamic Programming Principle (DPP) given by:

$$\begin{aligned} u(x) =& \varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \right\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\, dy \right), \quad x \in \Omega, \\ u(x) =& g(x), \qquad x \in \Gamma_\varepsilon. \end{aligned}$$

Let see intuitively why this holds. At each step we have that Player I chooses $i \in \{1,2\}$ and then we have three possibilities:

- With probability $\frac{\alpha_i}{2}$, Player I moves the token, she will try to maximize the expected outcome.
- With probability $\frac{\alpha_i}{2}$, Player II moves the token, he will try to minimize the expected outcome.
- With probability β_i, the token moves at random.

Since Player I chooses i trying to maximize the expected outcome we obtain a $\max_{i\in\{1,2\}}$ in the DPP. Finally, the expected payoff at x is given by $\varepsilon^2 f(x)$ plus the expected payoff for the rest of the game.

Similar results are proved in [4], [53], [57], [62], [70] and [73]. Note that when $\alpha_1 = \alpha_2$ (and hence $\beta_1 = \beta_2$) player I has no choice to make and we recover known results for Tug-of-War games (with or without noise), see [70] and [63]. We follow [73] where the idea is to prove the existence of a function satisfying the DPP and then that this function gives the game value. For the existence of a solution to the DPP we borrow some ideas from [4], and for the uniqueness of such a solution and the existence of the value of the game we use martingales as in [62]. However we will have two different cases. One where the noise assures us that the game ends almost surely independently of the strategies adopted by the players or where the strictly positivity (or negativity) of f helps us in this sense. And another one where we have to handle the problem of getting strategies for the players to play almost optimal and to make sure that the game ends almost surely.

In what follows $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\varepsilon > 0$, $g : \Gamma_\varepsilon \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ bounded Borel functions such that $f \equiv 0$, $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$.

Definition 3.4.1. A function u is sub-p_1-p_2-harmonious if

$$
\begin{aligned}
&u(x) \leq \varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \right\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\, dy \right), \quad x \in \Omega,\\
&u(x) \leq g(x), \qquad x \in \Gamma_\varepsilon
\end{aligned}
$$

Analogously, a function u is super-p_1-p_2-harmonious if

$$
\begin{aligned}
&u(x) \geq \varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \right\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\, dy \right), \quad x \in \Omega,\\
&u(x) \leq g(x), \qquad x \in \Gamma_\varepsilon
\end{aligned}
$$

Finally, u is p_1-p_2-harmonious if it is both sub- and super-p_1-p_2-harmonious (i.e. the equality holds).

Here α_i and β_i are given by

$$
\alpha_i = \frac{p_i - 2}{p_i + N} \quad \text{and} \quad \beta_i = \frac{N+2}{p_i + N} \qquad i = 1, 2.
$$

Our next task is to prove uniform bounds for these functions.

Lemma 3.4.2. *Sub-p_1-p_2-harmonious functions are uniformly bounded from above.*

Proof. We will consider the space partitioned along the x_N axis in strips of width $\frac{\varepsilon}{2}$. To this end we define

$$
D = \frac{|\{y \in B_\varepsilon : y_N < -\frac{\varepsilon}{2}\}|}{|B_\varepsilon|} = \frac{|\{y \in B_1 : y_N < -\frac{1}{2}\}|}{|B_1|} \quad \text{and} \quad C = 1 - D.
$$

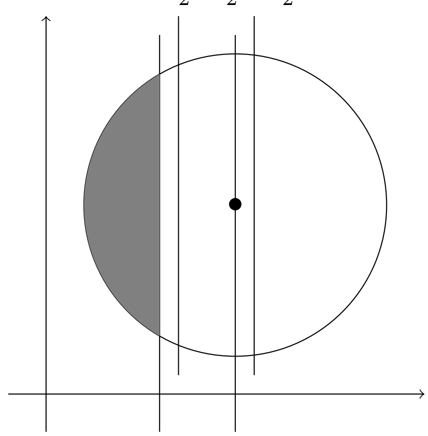


Figure 3.1: The partition considered in the proof of Lemma 3.4.2.

The constant D gives the fraction of the ball $B_\varepsilon(x)$ covered by the shadowed section in Figure 3.1, $\{y \in B_\varepsilon : y_N < x_N - \frac{\varepsilon}{2}\}$, and C the fraction occupy by its complement.

Given $x \in \Omega$, let us consider $t \in \mathbb{R}$ such that $x_N < t\frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. We get

$$\left\{y \in B_\varepsilon(x) : y_N < x_N - \frac{\varepsilon}{2}\right\} \subset \left\{z \in \mathbb{R}^N : z_N < t\frac{\varepsilon}{2}\right\}.$$

Now, given u a sub-p_1-p_2-subharmonious function, we have that

$$u(x) \leq \varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \right\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\, dy \right).$$

Now we can bound the terms in the RHS considering the partition given above, see Figure 3.1. We have

$$\sup_{B_\varepsilon(x)} u \leq \sup_{\Omega_\varepsilon} u,$$

$$\inf_{B_\varepsilon(x)} u \leq \sup_{\{y\in B_\varepsilon(x): y_N < x_N - \frac{\varepsilon}{2}\}} u \leq \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u,$$

and

$$\begin{aligned}
\fint_{B_\varepsilon(x)} u(y)\, dy \leq & \left| \left\{ y \in B_\varepsilon(x) : y_N \geq x_N - \frac{\varepsilon}{2} \right\} \right| \sup_{\{y\in B_\varepsilon(x): y_N \geq x_N - \frac{\varepsilon}{2}\}} u \\
& + \left| \left\{ y \in B_\varepsilon(x) : y_N < x_N - \frac{\varepsilon}{2} \right\} \right| \sup_{\{y\in B_\varepsilon(x): y_N < x_N - \frac{\varepsilon}{2}\}} u \\
\leq & \ C \sup_{\Omega_\varepsilon} u + D \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
u(x) &\le \varepsilon^2 \sup_{\Omega} f + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{\Omega_\varepsilon} u + \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u \right\} \right. \\
&\qquad \left. + \beta_i \left\{ C \sup_{\Omega_\varepsilon} u + D \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u \right\} \right) \\
&= \varepsilon^2 \sup_{\Omega} f + \max_{i\in\{1,2\}} \left(\left\{ \frac{\alpha_i}{2} + \beta_i C \right\} \sup_{\Omega_\varepsilon} u + \left\{ \frac{\alpha_i}{2} + \beta_i D \right\} \sup_{\Omega_\varepsilon \cap \{z_n < t\frac{\varepsilon}{2}\}} u \right) \\
&= \varepsilon^2 \sup_{\Omega} f + \max_{i\in\{1,2\}} \left\{ \frac{\alpha_i}{2} + \beta_i C \right\} \sup_{\Omega_\varepsilon} u + \min_{i\in\{1,2\}} \left\{ \frac{\alpha_i}{2} + \beta_i D \right\} \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u \\
&= \varepsilon^2 \sup_{\Omega} f + K \sup_{\Omega_\varepsilon} u + (1-K) \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u,
\end{aligned}$$

where $K = \max_{i\in\{1,2\}} \left\{ \frac{\alpha_i}{2} + \beta_i C \right\}$. We conclude that

$$\sup_{\Omega_\varepsilon \cap \{z_N < (t+1)\frac{\varepsilon}{2}\}} u_k \le \varepsilon^2 \sup_{\Omega} f + K \sup_{\Omega_\varepsilon} u_k + (1-K) \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u_k.$$

Then, inductively, we get

$$\sup_{\Omega_\varepsilon \cap \{z_N < (t+n)\frac{\varepsilon}{2}\}} u \le \left(\varepsilon^2 \sup_{\Omega} f + K \sup_{\Omega_\varepsilon} u \right) \sum_{i=0}^{n-1} (1-K)^i + (1-K)^n \sup_{\Omega_\varepsilon \cap \{z_N < t\frac{\varepsilon}{2}\}} u.$$

We assume without lost of generality that $\Omega \subset \{x \in \mathbb{R}^N : 0 < x_N < R\}$ for some $R > 0$. Now, we apply the formula for $t = 0$ and n such that $\frac{n\varepsilon}{2} > R$, we get

$$\begin{aligned}
\sup_{\Omega_\varepsilon} u &\le \left(\varepsilon^2 \sup_{\Omega} f + K \sup_{\Omega_\varepsilon} u \right) \sum_{i=0}^{n-1} (1-K)^i + (1-K)^n \sup_{\Gamma_\varepsilon} g \\
&= \left(\varepsilon^2 \sup_{\Omega} f + K \sup_{\Omega_\varepsilon} u \right) \frac{1-(1-K)^n}{1-(1-K)} + (1-K)^n \sup_{\Gamma_\varepsilon} g \\
&= \frac{1-(1-K)^n}{K} \varepsilon^2 \sup_{\Omega} f + (1-(1-K)^n) \sup_{\Omega_\varepsilon} u + (1-K)^n \sup_{\Gamma_\varepsilon} g.
\end{aligned}$$

Hence, we obtain

$$(1-K)^n \sup_{\Omega_\varepsilon} u \le \frac{1-(1-K)^n}{K} \varepsilon^2 \sup_{\Omega} f + (1-K)^n \sup_{\Gamma_\varepsilon} g,$$

that gives the desired upper bound,

$$\sup_{\Omega_\varepsilon} u \le \frac{1-(1-K)^n}{K(1-K)^n} \varepsilon^2 \sup_{\Omega} f + \sup_{\Gamma_\varepsilon} g.$$

□

Analogously, there holds that super-p_1-p_2-harmonious functions are uniformly bounded from below.

Now with this results we can show that there exists a p_1-p_2-harmonious function as in [54] applying Perron's Method. Remark that when f and g are bounded we can easily obtain the existence of sub-p_1-p_2-harmonious and super-p_1-p_2-harmonious functions.

We prefer a constructive argument (since we will use again this construction in what follows). Let $u_k : \Omega_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $u_k = g$ on Γ_ε for all $k \in \mathbb{N}$, u_0 is sub-p_1-p_2-harmonious and

$$u_{k+1}(x) = \varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u_k + \inf_{B_\varepsilon(x)} u_k \right\} + \beta_i \fint_{B_\varepsilon(x)} u_k(y)\, dy \right)$$

for $x \in \Omega$.

Now, our main task is to show that this sequence converges uniformly. To this end, let us prove an auxiliary lemma where we borrow some ideas from [4].

Lemma 3.4.3. *Let $x \in \Omega$, $n \in \mathbb{N}$ and fix λ_i for $i = 1, \dots, 4$ such that*

$$\begin{aligned} u_{n+1}(x) - u_n(x) &\geq \lambda_1, \\ \|u_n - u_{n-1}\|_\infty &\leq \lambda_2, \\ \fint_{B_\varepsilon(x)} u_n - u_{n-1} &\leq \lambda_3, \end{aligned}$$

$\lambda_3 < \lambda_1$ and $\lambda_4 > 0$. Then, for $\alpha := \max\{\alpha_1, \alpha_2\} > 0$, there exists $y \in B_\varepsilon(x)$ such that

$$\inf_{B_\varepsilon(x)} u_n \geq u_{n-1}(y) + \frac{2\lambda_1}{\alpha} - \lambda_2 - \frac{2(1-\alpha)\lambda_3}{\alpha} - \lambda_4.$$

Proof. Given $u_{n+1}(x) - u_n(x) \geq \lambda_1$, by the recursive definition, we have

$$\begin{aligned} &\varepsilon^2 f(x) + \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n \right\} + \beta_i \fint_{B_\varepsilon(x)} u_n(y)\, dy \right) \\ &\quad -\varepsilon^2 f(x) - \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u_{n-1} + \inf_{B_\varepsilon(x)} u_{n-1} \right\} + \beta_i \fint_{B_\varepsilon(x)} u_{n-1}(y)\, dy \right) \\ &\qquad \geq \lambda_1. \end{aligned}$$

Since $\max\{a,b\} - \max\{c,d\} \leq \max\{a-c, b-d\}$, we get

$$\begin{aligned} \max_{i\in\{1,2\}} \Bigg(\frac{\alpha_i}{2} &\left\{ \sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n - \sup_{B_\varepsilon(x)} u_{n-1} - \inf_{B_\varepsilon(x)} u_{n-1} \right\} + \\ &\qquad\qquad + \beta_i \fint_{B_\varepsilon(x)} u_n(y) - u_{n-1}(y)\, dy \Bigg) \geq \lambda_1. \end{aligned}$$

Using that $\fint_{B_\varepsilon(x)} u_n - u_{n-1} \leq \lambda_3$ we get

$$\max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \left\{ \sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n - \sup_{B_\varepsilon(x)} u_{n-1} - \inf_{B_\varepsilon(x)} u_{n-1} \right\} + \beta_i \lambda_3 \right) \geq \lambda_1.$$

Now $\lambda_3 < \lambda_1$ implies

$$\frac{\alpha}{2} \left\{ \sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n - \sup_{B_\varepsilon(x)} u_{n-1} - \inf_{B_\varepsilon(x)} u_{n-1} \right\} + (1-\alpha)\lambda_3 \geq \lambda_1.$$

We bound the difference between the suprema using $\|u_n - u_{n-1}\|_\infty \leq \lambda_2$ and we obtain

$$\frac{\alpha}{2} \left\{ \inf_{B_\varepsilon(x)} u_n - \inf_{B_\varepsilon(x)} u_{n-1} \right\} + \frac{\alpha\lambda_2}{2} + (1-\alpha)\lambda_3 \geq \lambda_1,$$

that is,

$$\inf_{B_\varepsilon(x)} u_n \geq \inf_{B_\varepsilon(x)} u_{n-1} + \frac{2\lambda_1}{\alpha} - \lambda_2 - \frac{2(1-\alpha)\lambda_3}{\alpha}.$$

Finally we can choose $y \in B_\varepsilon(x)$ such that

$$u_{n-1}(y) \leq \inf_{B_\varepsilon(x)} u_{n-1} + \lambda_4$$

which gives the desired inequality. □

Now we are ready to prove the uniform convergence and, therefore, the existence of a p_1-p_2-harmonious function.

Proposition 3.4.4. *The sequence u_k converges uniformly and the limit is a solution to the DPP.*

Proof. Since u_0 is sub-p_1-p_2-harmonious we have $u_1 \geq u_0$. In addition, if $u_k \geq u_{k-1}$, by the recursive definition, we have $u_{k+1} \geq u_k$. Then, by induction, we obtain that the sequence of functions is an increasing sequence. Replacing $u_k \leq u_{k+1}$ in the recursive definition we can see that u_k is a sub-p_1-p_2-harmonious function for all k. This gives us a uniform bound for u_k (independent of k). Hence, u_k converge pointwise to a bounded Borel function u.

In the case $\alpha_1 = \alpha_2 = 0$ we can pass to the limit on the recursion because of Fatou's Lemma. Hence we assume $\alpha := \max\{\alpha_1, \alpha_2\} > 0$.

Now we show that the convergence is uniform. Suppose not. Observe that if $\|u_{n+1} - u_n\|_\infty \to 0$ we can extract a uniformly Cauchy subsequence, thus this subsequence converges uniformly to a limit u. This implies that u_k converge uniformly to u, because of the monotonicity. By the recursive definition we have $\|u_{n+1}-u_n\|_\infty \geq \|u_n-u_{n-1}\|_\infty \geq 0$. Then, as we are assuming the convergence is not uniform, we have

$$\|u_{n+1} - u_n\|_\infty \to M \qquad \text{and} \qquad \|u_{n+1} - u_n\|_\infty \geq M$$

for some $M > 0$.

Let us observe that by Fatou's Lemma it follows that

$$\lim_{n\to\infty} \int_\Omega u(y) - u_n(y)\, dy = 0$$

so we can bound $⨍_{B_\varepsilon(x)} u_{n+1} - u_n$ uniformly on x.

Given $\delta > 0$, let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\|u_{n+1} - u_n\|_\infty \leq M + \delta \quad \text{and} \quad ⨍_{B_\varepsilon(x)} u_{n+1} - u_n < \delta$$

for all $x \in \Omega$. We fix $k \geq 0$. Let $x_0 \in \Omega$ such that $u_{n_0+k}(x_0) - u_{n_0+k-1}(x_0) \geq M - \delta$. Now we apply Lemma 3.4.3 for $\lambda_1 = M - \delta$, $\lambda_2 = M + \delta$, $\lambda_3 = \delta$ and $\lambda_4 = \delta$ and we get

$$\begin{aligned}
u_{n_0+k-1}(x_0), u_{n_0+k-1}(x_1) &\geq \inf_{B_\varepsilon(x_0)} u_{n_0+k-1} \\
&\geq u_{n_0+k-2}(x_1) + \frac{2(M-\delta)}{\alpha} - (M+\delta) - \frac{2(1-\alpha)}{\alpha} - \delta \\
&= u_{n_0+k-2}(x_1) + M(\frac{2}{\alpha} - 1) - \delta\frac{4}{\alpha} \\
&\geq u_{n_0+k-2}(x_1) + M - \delta\frac{4}{\alpha}.
\end{aligned}$$

for some $x_1 \in B_\varepsilon(x_0)$. Let us define $\xi = \frac{4}{\alpha}$. If we repeat the argument for x_1, but now with $\lambda_1 = M - \delta\xi$, we obtain

$$u_{n_0+k-2}(x_1), u_{n_0+k-2}(x_2) \geq u_{n_0+k-3}(x_2) + M - \delta\left(\xi^2 + \xi\right).$$

Inductively, we obtain a sequence x_l, $1 \leq l \leq k-1$ such that

$$u_{n_0+k-l}(x_{l-1}), u_{n_0+k-l}(x_l) \geq u_{n_0+k-l-1}(x_l) + M - \delta\sum_{t=1}^{l} \xi^t.$$

In Lemma 3.4.3 we require $\lambda_3 < \lambda_1$, so we need $k(\delta)$ to satisfy

$$M - \delta\sum_{t=1}^{l} \xi^t > \delta,$$

that is,

$$M > \delta \sum_{t=0}^{l} \xi^t$$

for $1 \le l \le k-1$, as the right hand side term grows with l, it is enough to check it for $l = k-1$. Since

$$\sum_{t=1}^{l} \xi^t = \xi \frac{\xi^l - 1}{\xi - 1} \le \xi^{l+1} - 1 \le \xi^{l+1},$$

we obtain

$$u_{n_0+k-l}(x_{l-1}) \ge u_{n_0+k-l-1}(x_l) + M - \delta \xi^{l+1}.$$

Adding this inequalities for $1 \le l \le k-1$, and $u_{n_0+k}(x_0) - u_{n_0+k-1}(x_0) \ge M - \delta$ we get

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + kM - \delta \sum_{l=0}^{k-1} \xi^{l+1}.$$

From the last inequality and the condition for $k(\delta)$, since

$$\sum_{l=0}^{k-1} \xi^{l+1} = \sum_{l=1}^{k} \xi^l \le \xi^{k+1},$$

we have

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + kM - \delta \xi^{k+1}$$

for all k such that $M > \delta \xi^{k+1}$. For $k+1 = \left\lfloor \frac{\log \frac{M}{\delta}}{\log \xi} \right\rfloor$ this gives

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + \left(\frac{\log \frac{M}{\delta}}{\log \xi} - 3 \right) M$$

which is a contradiction since

$$\lim_{\delta \to 0^+} \frac{\log \frac{M}{\delta}}{\log \xi} = \infty$$

and the sequence u_n is bounded. We have that $u_n \to u$ uniformly, therefore the result follows by passing to the limit in the recursive definition of u_n. In fact, that the uniform limit of the sequence u_n is a solution to the DPP is immediate since from the uniform convergence we can pass to the limit as $n \to \infty$ in all the terms of the DPP formula. $\square$

Now we want to prove that this solution to the DPP, u, is unique and that it gives the value of the game. To this end we have to take special care of the fact that the game ends (or not) almost surely. First, we deal with the case $\beta_1, \beta_2 > 0$, $\sup_\Omega f < 0$ or $\inf_\Omega f > 0$. We apply a martingales argument to handle these cases. In other cases we also use the construction of the sequence u_k.

Lemma 3.4.5. *Assume that $\beta_1, \beta_2 > 0$, $\sup f < 0$ or $\inf f > 0$. Then, if v is a p_1-p_2-harmonious function for g_v and f_v such that $g_v \le g_{u_I}$ and $f_v \le f_{u_I}$, then $v \le u_I$.*

Proof. By choosing a strategy according to the points where the maximal values of v are attained, we show that Player I can obtain that a certain process is a submartingale. The optional stopping theorem then implies that the expectation of the process under this strategy is bounded by v. Moreover, this process provides a lower bound for u_{I}.

Player II follows any strategy and Player I follows a strategy S_{I}^0 such that at $x_{k-1} \in \Omega$ he chooses Γ as follows:

$$\begin{aligned}\gamma = 1 \text{ if } \quad & \frac{\alpha_1}{2}\left\{\sup_{y\in B_\varepsilon(x)} u(y) + \inf_{y\in B_\varepsilon(x)} u(y)\right\} + \beta_1 \fint_{B_\varepsilon(x)} u(y)\,dy \\ & > \frac{\alpha_2}{2}\left\{\sup_{y\in B_\varepsilon(x)} u(y) + \inf_{y\in B_\varepsilon(x)} u(y)\right\} + \beta_2 \fint_{B_\varepsilon(x)} u(y)\,dy \\ \text{and } \gamma = 0 \text{ othewise,}\end{aligned}$$

and to step to a point that almost maximize v, that is, to a point $x_k \in B_\varepsilon(x_{k-1})$ such that

$$v(x_k) \ge \sup_{B_\varepsilon(x_{k-1})} v - \eta 2^{-k}$$

for some fixed $\eta > 0$. We start from the point x_0. It follows that

$$\begin{aligned}&\mathbb{E}^{x_0}_{S_{\mathrm{I}}, S^0_{\mathrm{II}}}[v(x_k) + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \,|\, x_0, \dots, x_{k-1}] \\ &\ge \max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2}\left\{\inf_{B_\varepsilon(x_{k-1})} v - \eta 2^{-k} + \sup_{B_\varepsilon(x_{k-1})} v\right\} + \beta_i \fint_{B_\varepsilon(x_{k-1})} v\,dy\right) \\ &\qquad\qquad\qquad\qquad + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \\ &\ge v(x_{k-1}) - \varepsilon^2 f(x_{k-1}) - \eta 2^{-k} + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \\ &= v(x_{k-1}) + \varepsilon^2 \sum_{n=0}^{k-2} f(x_n) - \eta 2^{-k+1}\end{aligned}$$

where we have estimated the strategy of Player II inf and used the fact that v is p_1-p_2-harmonious. Thus

$$M_k = v(x_k) + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k}$$

is a submartingale.

Now we observe the following: if $\beta_1, \beta_2 > 0$ then the game ends almost surely and we can continue (see below). If $\sup f < 0$ we have that the fact that M_k is a submartingale implies that the game ends in a finite number of moves (that can be estimated). In the case $\inf f > 0$ if the game does not end in a finite number of moves then we have to play until the accumulated payoff (recall that f gives the running payoff) is greater than v and then choose a strategy that ends the game almost surely (for example pointing to some prescribed point x_0 outside Ω).

Since $g_v \le g_{u_{\mathrm{I}}}$ and $f_v \le f_{u_{\mathrm{I}}}$, we deduce

$$\begin{aligned} u_{\mathrm{I}}(x_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[g_{u^\varepsilon_{\mathrm{I}}}(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[g_v(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n) - \eta 2^{-\tau}] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[M_0] = v(x_0) - \eta, \end{aligned}$$

where we used the optional stopping theorem for M_k. Since η is arbitrary this proves the claim. $\square$

A symmetric result can be proved for u_{II}, hence we obtain the following result:

Theorem 3.4.6. *Assume that $\beta_1, \beta_2 > 0$, $\sup f < 0$ or $\inf f > 0$. Then there exists a unique p_1-p_2-harmonious function. Even more the game has a value, that is $u_I = u_{II}$, which coincides with the unique p_1-p_2-harmonious function.*

Proof. Let u be a p_1-p_2-harmonious function, that we know that exits by Proposition 3.4.4. From the definition of the game values we know that $u_I \le u_{II}$. Then by Lemma 3.4.5 we have that

$$u_I \le u_{II} \le u \le u_I.$$

This is $u_I = u_{II} = u$. Since we can repeat the argument for any p_1-p_2-harmonious function, uniqueness follows. $\square$

Remark 3.4.7. Note that if we have a sub-p_1-p_2-harmonious function u, then v given by $v = u - C$ in Ω and $v = u$ in Γ_ε is sub-p_1-p_2-harmonious for every constant $C > 0$. In this way we can obtain a sub-p_1-p_2-harmonious function smaller that any super-p_1-p_2-harmonious function, and then if we start the above construction with this function we get the smallest p_1-p_2-harmonious function. That is, there exists a minimal p_1-p_2-harmonious function. We can do the analogous construction to get the larger p_1-p_2-harmonious function (the maximal p_1-p_2-harmonious function).

We now tackle the remaining case in which $f \equiv 0$ and one of the β_i is zero (that is the same as saying that one of the α_i is equal to one).

Theorem 3.4.8. *There exists a unique p_1-p_2-harmonious function when $\alpha_1 = 1$, $\alpha_2 > 0$ and $f \equiv 0$.*

Proof. Supposed not, this is, we have u,v, such that

$$v(x) = \max\left\{\frac{1}{2}\left(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\right), \frac{\alpha}{2}\left(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\right) + \beta \fint_{B_\varepsilon(x)} v\right\}$$

$$u(x) = \max\left\{\frac{1}{2}\left(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\right), \frac{\alpha}{2}\left(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\right) + \beta \fint_{B_\varepsilon(x)} u\right\}$$

in Ω and

$$u = v = g$$

on Γ_ε with

$$\|u - v\|_\infty = M > 0.$$

As we observed in Remark 3.4.7 we can assume $u \geq v$ (just take v the minimal solution to the DPP). Now we want to build a point where the difference between u and v is almost attained and v has a large variation in the ball of radius ε around this point (all this has to be carefully quantified). First, we apply a compactness argument. We know that

$$\bar{\Omega}_{\frac{\varepsilon}{4}} \subset \bigcup_{x\in\Omega} B_{\frac{\varepsilon}{2}}(x).$$

As $\bar{\Omega}_{\frac{\varepsilon}{4}}$ is compact there exists y_i such that

$$\bar{\Omega}_{\frac{\varepsilon}{4}} \subset \bigcup_{i=1}^{k} B_{\frac{\varepsilon}{2}}(y_i).$$

Let $A = \{i \in \{1, \ldots, k\} : u$ or v are not constant on $B_{\frac{\varepsilon}{2}}(y_i)\}$ and let $\lambda > 0$ such that for every $i \in A$

$$\sup_{B_\varepsilon(y_i)} u - \inf_{B_\varepsilon(y_i)} u > \left(4 + \frac{4\beta}{\alpha}\right)\lambda \quad \text{or} \quad \sup_{B_\varepsilon(y_i)} v - \inf_{B_\varepsilon(y_i)} v > 2\lambda.$$

We fix this λ. Now, for every $\delta > 0$ such that $\lambda > \delta$ and $M > \delta$, let $z \in \Omega$ such that $M - \delta < u(z) - v(z)$. Let

$$O = \{x \in \Omega : u(x) = u(z) \text{ and } v(x) = v(z)\} \subset \Omega.$$

Take $\bar{z} \in \partial O \subset \bar{\Omega}$. Let i_0 such that $\bar{z} \in B_{\frac{\varepsilon}{2}}(y_{i_0})$, we have

$$B_{\frac{\varepsilon}{2}}(y_{i_0}) \cap O \neq \emptyset \quad \text{and} \quad B_{\frac{\varepsilon}{2}}(y_{i_0}) \cap O^c \neq \emptyset$$

hence $i_0 \in A$. Let $x_0 \in B_{\frac{\varepsilon}{2}}(y_{i_0}) \cap O$. In this way we have obtained x_0 such that $u(x_0) - v(x_0) > M - \delta$ and one of the following holds:

1. $$\sup_{B_\varepsilon(x_0)} u - \inf_{B_\varepsilon(x_0)} u > \left(4 + \frac{4\beta}{\alpha}\right)\lambda$$

2. $$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda.$$

Let us show that in fact the second statement must hold. Supposed not, then the first holds and we have

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v \leq 2\lambda.$$

Given that

$$v(x_0) \geq \frac{1}{2}\left(\sup_{B_\varepsilon(x_0)} v + \inf_{B_\varepsilon(x_0)} v\right)$$

we get

$$v(x_0) + \lambda \geq \sup_{B_\varepsilon(x_0)} v.$$

Hence

$$v(x_0) + \lambda + M \geq \sup_{B_\varepsilon(x_0)} v + M \geq \sup_{B_\varepsilon(x_0)} u.$$

But we have more, since

$$u(x_0) - v(x_0) > M - \delta > M - \lambda,$$

we get

$$u(x_0) + 2\lambda > \sup_{B_\varepsilon(x_0)} u,$$

and

$$\sup_{B_\varepsilon(x_0)} u > \inf_{B_\varepsilon(x_0)} u + \left(4 + \frac{4\beta}{\alpha}\right)\lambda.$$

Hence

$$u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda > \inf_{B_\varepsilon(x_0)} u.$$

If we bound the integral by the value of the supremum we can control all the terms in the DPP in terms of $u(x_0)$. We have

$$\begin{aligned}
u(x_0) &= \max\left\{\frac{1}{2}\left(\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u\right), \frac{\alpha}{2}\left(\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u\right) + \beta \fint_{B_\varepsilon(x_0)} u\right\} \\
&< \max\left\{\frac{1}{2}\left(u(x_0) + 2\lambda + u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda\right),\right. \\
&\qquad \left.\frac{\alpha}{2}\left(u(x_0) + 2\lambda + u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda\right) + \beta(u(x_0) + 2\lambda)\right\} \\
&< \max\left\{u(x_0) - \frac{4\beta}{\alpha}\lambda, u(x_0)\right\} = u(x_0),
\end{aligned}$$

which is a contradiction. Hence we obtain that the second condition must hold, that is, we have

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda.$$

Applying the DPP we get

$$v(x_0) \geq \frac{1}{2}\left(\sup_{B_\varepsilon(x_0)} v + \inf_{B_\varepsilon(x_0)} v\right)$$

together with the fact that

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda,$$

then we conclude that

$$v(x_0) > \inf_{B_\varepsilon(x_0)} v + \lambda.$$

We have proved that there exists x_0 such that

$$v(x_0) > \inf_{B_\varepsilon(x_0)} v + \lambda \quad \text{and} \quad u(x_0) - v(x_0) > M - \delta.$$

Now we are going to build a sequence of points where the difference between u and v is almost maximal and where the value of v decrease at least λ in every step. Applying the DPP to $M - \delta < u(x_0) - v(x_0)$ and bounding the difference of the suprema by M we get:

$$M - \frac{2}{\alpha}\delta + \inf_{B_\varepsilon(x_0)} v < \inf_{B_\varepsilon(x_0)} u.$$

Let x_1 be such that $v(x_0) > v(x_1) + \lambda$ and $\inf_{B_\varepsilon(x_0)} v + \delta > v(x_1)$. We get

$$M - \left(1 + \frac{2}{\alpha}\right)\delta + v(x_1) < u(x_1).$$

To repeat this construction we need two things:

- In the last inequality if δ is small enough we have $u(x_1) \neq v(x_1)$, hence $x_1 \in \Omega$.
- We know that $2v(x_1) \geq \inf_{B_\varepsilon(x_1)} v + \sup_{B_\varepsilon(x_1)} v > v(x_0) + \inf_{B_\varepsilon(x_1)} v$. Hence, since $v(x_0) > v(x_1) + \lambda$, we get $v(x_1) > \inf_{B_\varepsilon(x_1)} v + \lambda$.

Then we get

$$v(x_{n-1}) > v(x_n) + \lambda$$

and

$$M - \left(\sum_{k=0}^{n} \left(\frac{2}{\alpha} \right)^k \right) \delta + v(x_n) < u(x_n).$$

We can repeat this argument as long as

$$M - \left(\sum_{k=0}^{n} \left(\frac{2}{\alpha} \right)^k \right) \delta > 0,$$

which is a contradiction with the fact that we know that v is bounded. $\square$

Now we want to show that this unique function that satisfies the DPP is the game value. The key point of the proof is to construct an strategy based on the approximating sequence that we used to construct the solution.

Theorem 3.4.9. *Given $f \equiv 0$, the game has a value, that is $u_I = u_{II}$, which coincides with the unique p_1-p_2-harmonious function.*

Proof. Let u be the unique p_1-p_2-harmonious function (the uniqueness is given by Theorem 3.4.6 and Theorem 3.4.8). We will show that $u \leq u_I$. The analogous result can be proved for u_{II} completing the proof.

Let us consider a function u_0, sub-p_1-p_2-harmonious smaller that $\inf_\Omega g$ at every point in Ω. Starting with this u_0 we build the corresponding u_k as in Proposition 3.4.4. We have that $u_k \to u$ as $k \to \infty$.

Now, given $\delta > 0$ let $n > 0$ be such that $u_n(x_0) > u(x_0) - \frac{\delta}{2}$. We build an strategy S_I^0 for Player I, in the firsts n moves, given x_{k-1} he will choose to move to a point that almost maximize u_{n-k}, that is, he chooses $x_k \in B_\varepsilon(x_{k-1})$ such that

$$u_{n-k}(x_k) > \sup_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n}.$$

and choose γ in order to maximize

$$\frac{\alpha_i}{2} \left\{ \inf_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n} + \sup_{B_\varepsilon(x_{k-1})} u_{n-k} \right\} + \beta_i \fint_{B_\varepsilon(x_{k-1})} u_{n-k} \, dy.$$

After the first n moves he will follow a strategy that ends the game almost surely (for example pointing in a fix direction).

We have

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[u_{n-k}(x_k)+\frac{k\delta}{2n}|\, x_0,\ldots,x_{k-1}]\\
&\geq \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\left\{\inf_{B_\varepsilon(x_{k-1})}u_{n-k}-\frac{\delta}{2n}+\sup_{B_\varepsilon(x_{k-1})}u_{n-k}\right\}\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.+\beta_i\fint_{B_\varepsilon(x_{k-1})}u_{n-k}\,dy\right)+\frac{k\delta}{2n}\\
&\geq u_{n-k+1}(x_{k-1})+\frac{(k-1)\delta}{2n},
\end{aligned}$$

where we have estimated the strategy of Player II by inf and used the construction for the u_k's. Thus

$$M_k=\begin{cases} u_{n-k}(x_k)+\dfrac{k\delta}{2n}-\dfrac{\delta}{2} & \text{for } 0\leq k\leq n,\\ M_k=\inf_\Omega g & \text{for } k>n,\end{cases}$$

is a submartingale.

Now we have

$$\begin{aligned}
u_{\mathrm{I}}(x_0)&=\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[g(x_\tau)]\\
&\geq \inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[g(x_\tau)]\\
&\geq \inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[M_\tau]\\
&\geq \inf_{S_{\mathrm{II}}}\mathbb{E}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[M_0]=u_n(x_0)-\frac{\delta}{2}>u(x_0)-\delta,
\end{aligned}$$

where we used the optional stopping theorem for M_k. Since δ is arbitrary this proves the claim. □

As an immediate corollary of our results in this section we obtain a comparison result for solutions to the DPP.

Corollary 3.4.10. *If v and u are p_1-p_2-harmonious functions for g_v, f_v and g_u, f_u, respectively such that $g_v\geq g_u$ and $f_v\geq f_u$, then $v\geq u$.*

3.5 Properties of harmonious functions and convergence

First, we show some properties of p_1-p_2-harmonious functions that we need to prove convergence as $\varepsilon \to 0$. We want to apply the following Arzela-Ascoli type lemma. For its proof see Lemma 4.2 from [63].

Lemma 3.5.1. *Let $\{u_\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ be a set of functions such that*

1. *there exists $C > 0$ such that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*
2. *given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x - y| < r_0$ it holds*

$$|u_\varepsilon(x) - u_\varepsilon(y)| < \eta.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by $\{u_\varepsilon\}$ such that

$$u_\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega},$$

as $\varepsilon \to 0$.

So our task now is to show that the family u_ε satisfies the hypotheses of the previous lemma. To this end we need some bounds on the expected exit time in the case a player choose a certain strategy.

Let us start showing that u_ε are uniformly bounded. In Lemma 3.4.2 we obtained a bound for the value of the game for a fixed ε, here we need a bound independent of ε. To this end, let us define what we understand by pulling in one direction: We fix a direction, that is, a unitary vector v and at each turn of the game the Player strategy is given $S(x_{k-1}) = x_{k-1} + (\varepsilon - \varepsilon^3/2^k)v$.

Lemma 3.5.2. *In a game where a player pulls in a fix direction the expectation of the exit time is bounded above by*

$$\mathbb{E}[\tau] \leq C\varepsilon^{-2}$$

for some $C > 0$ independent of ε.

Proof. First, let us assume without lost of generality that

$$\Omega \subset \{x \in \mathbb{R}^n : 0 < x_n < R\}$$

and that the direction that the player is pulling to is $-e_n$. Then

$$M_k = (x_k)_n + \frac{\varepsilon^3}{2^k}$$

is a supermartingale. Indeed, if the random move occurs, then we know that the expectation of $(x_{k+1})_n$ is equal to $(x_k)_n$. If the tug-of-war game is played we know that with probability one half $(x_{k+1})_n = (x_k)_n - \varepsilon + \varepsilon^3/2^k$ and if the other player moves $(x_{k+1})_n \leq (x_k)_n + \varepsilon$, so the expectation is less or equal to $(x_k)_n + \frac{\varepsilon^3}{2^{k+1}}$.

Let us consider the expectation for $(M_{k+1} - M_k)^2$. If the random walk occurs, then the expectation is $\frac{\varepsilon^2}{n+2} + o(\varepsilon^2)$. Indeed,

$$\fint_{B_\varepsilon} x_n^2 = \frac{1}{n} \fint_{B_\varepsilon} |x|^2 = \frac{1}{\varepsilon^n n|B_1|} \int_0^\varepsilon r^2 |\partial B_r| \, dr = \frac{|\partial B_1|}{\varepsilon^n n |B_1|} \int_0^\varepsilon r^{n+1} \, dr = \frac{\varepsilon^2}{n+2}.$$

If the tug-of-war occurs we know that with probability one half $(x_{k+1})_n = (x_k)_n - \varepsilon + \varepsilon^3/2^k$, so the expectation is greater than or equal to $\frac{\varepsilon^2}{3}$.

Let us consider the expectation for $M_k^2 - M_{k+1}^2$. We have,

$$\mathbb{E}[M_k^2 - M_{k+1}^2] = \mathbb{E}[(M_{k+1} - M_k)^2] + 2\mathbb{E}[(M_k - M_{k+1})M_{k+1}].$$

As $(x_k)_n$ is positive, we have $2\mathbb{E}[(M_k - M_{k+1})M_{k+1}] \geq 0$. Then $\mathbb{E}[M_k^2 - M_{k+1}^2] \geq \frac{\varepsilon^2}{n+2}$, so $M_k^2 + \frac{k\varepsilon^2}{n+2}$ is a supermartingale. According to the optional stopping theorem for supermartingales

$$\mathbb{E}\left[M_{\tau\wedge k}^2 + \frac{(\tau \wedge k)\varepsilon^2}{n+2}\right] \leq M_0^2.$$

We have

$$\mathbb{E}[(\tau \wedge k)] \frac{\varepsilon^2}{n+2} \leq M_0^2 - E[M_{\tau\wedge k}^2] \leq M_0^2.$$

Taking limit in k, we get a bound for the expected exit time,

$$\mathbb{E}[\tau] \leq (n+2) M_0^2 \varepsilon^{-2},$$

so, the statement holds for $C = (n+2)R^2$. □

Lemma 3.5.3. *A f-p_1-p_2-harmonious function u_ε with boundary values g satisfies*

$$\inf_{y \in \Gamma_\varepsilon} g(y) + C \inf_{y \in \Omega} f(y) \leq u_\varepsilon(x) \leq \sup_{y \in \Gamma_\varepsilon} g(y) + C \sup_{y \in \Omega} f(y). \tag{3.1}$$

Proof. We use the connection to games. Let one of the players choose a strategy of pulling in a fix direction. Then

$$\mathbb{E}[\tau] \leq C\varepsilon^{-2}$$

and this gives the upper bound

$$\mathbb{E}[g(X_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(X_n)] \leq \sup_{y \in \Gamma_\varepsilon} g(y) + E[\tau]\varepsilon^2 \sup_{y \in \Omega} f(y) \leq \sup_{y \in \Gamma_\varepsilon} g(y) + C \sup_{y \in \Omega} f(y).$$

The lower bound follows analogously. □

Let us show now that the u_ε are asymptotically uniformly continuous. First we need a lemma that bound the expectation for the exit time when one player is pulling towards a fixed point.

Let us consider an annular domain $B_R(y)\setminus\overline{B}_\delta(y)$ and a game played inside. In each round the token starts at a certain point x, an ε step *tug-of-war* is played inside $B_R(y)$ or the token moves at random with uniform probability in $B_R(y)\cap B_\varepsilon(x)$. If an ε-step *tug-of-war* is played, with probability $1/2$ each player moves the token to a point of his choice in $B_R(y)\cap B_\varepsilon(x)$. We can think there is a third player choosing whether the ε-step *tug-of-war* or the random move occurs. The game ends when the position reaches $\overline{B}_\delta(y)$, that is, when $x_{\tau^*}\in\overline{B}_\delta(y)$.

Lemma 3.5.4. *Assume that one of the players pulls towards y in the game described above. Then, no mater how many times the tug-of-war is played or the random move is done the exit time verifies*

$$\mathbb{E}^{x_0}(\tau^*)\le\frac{C(R/\delta)\operatorname{dist}(\partial B_\delta(y),x_0)+o(1)}{\varepsilon^2},\tag{3.2}$$

for $x_0\in B_R(y)\setminus\overline{B}_\delta(y)$. Here τ^* is the exit time in the previously described game and $o(1)\to 0$ as $\varepsilon\to 0$ can be taken depending only on δ and R.

Proof. Let us denote

$$h_\varepsilon(x)=\mathbb{E}^x(\tau).$$

By symmetry we know that h_ε is radial and it is easy to see that it is increasing in $r=|x-y|$. If we assume that the other player wants to maximize the expectation for the exit time and that the random move or *tug-of-war* is chosen in the same way, we have that the function h_ε satisfies a dynamic programming principle

$$h_\varepsilon(x)=\max\left\{\frac{1}{2}\left(\max_{B_\varepsilon(x)\cap B_R(y)}h_\varepsilon+\min_{B_\varepsilon(x)\cap B_R(y)}h_\varepsilon\right),\fint_{B_\varepsilon(x)\cap B_R(y)}h_\varepsilon\,dz\right\}+1$$

by the above assumptions and that the number of steps always increases by one when making a step. Further, we denote $v_\varepsilon(x)=\varepsilon^2h_\varepsilon(x)$ and obtain

$$v_\varepsilon(x)=\max\left\{\frac{1}{2}\left(\sup_{B_\varepsilon(x)\cap B_R(y)}v_\varepsilon+\inf_{B_\varepsilon(x)\cap B_R(y)}v_\varepsilon\right),\fint_{B_\varepsilon(x)\cap B_R(y)}v_\varepsilon\,dz\right\}+\varepsilon^2$$

This induces us to look for a function v such that

$$\begin{gathered}v(x)\ge\fint_{B_\varepsilon(x)}v\,dz+\varepsilon^2\\ \text{and}\\ v(x)\ge\frac{1}{2}\left(\sup_{B_\varepsilon(x)}v+\inf_{B_\varepsilon(x)}v\right)+\varepsilon^2.\end{gathered}\tag{3.3}$$

Note that for small ε this is a sort of discrete version to the following inequalities

$$\begin{cases} \Delta v(x) \leq -2(n+2), & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y), \\ \Delta_\infty v(x) \leq -2, & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y). \end{cases} \tag{3.4}$$

This leads us to consider the problem

$$\begin{cases} \Delta v(x) = -2(n+2), & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta}(y), \\ v(x) = 0, & x \in \partial B_{\delta}(y), \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial B_{R+\varepsilon}(y), \end{cases} \tag{3.5}$$

where $\frac{\partial u}{\partial \nu}$ refers to the normal derivative. The solution to this problem is radially symmetric and strictly increasing in $r = |x-y|$. It takes the form

$$v(r) = -ar^2 - br^{2-N} + c,$$

if $N > 2$ and

$$v(r) = -ar^2 - b\log(r) + c,$$

if $N = 2$. If we extend this v to $B_\delta(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$, it satisfies $\Delta v(x) = -2(N+2)$ in $B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$. We know that

$$\Delta_\infty v = v_{rr} \leq v_{rr} + \frac{N-1}{r} v_r = \Delta v.$$

Thus, v satisfy the inequalities (3.4). Then, the classical calculation shows that v satisfies (3.3) for each $B_\varepsilon(x) \subset B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$.

In addition, as v is increasing in r, it holds for each $x \in B_R(y) \setminus \overline{B}_\delta(y)$ that

$$\fint_{B_\varepsilon(x) \cap B_R(y)} v \, dz \leq \fint_{B_\varepsilon(x)} v \, dz \leq v(x) - \varepsilon^2$$

and

$$\frac{1}{2}\left(\sup_{B_\varepsilon(x) \cap B_R(y)} v + \inf_{B_\varepsilon(x) \cap B_R(y)} v \right) \leq \frac{1}{2}\left(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v \right) \leq v(x) - \varepsilon^2.$$

It follows that

$$\begin{aligned} &\mathbb{E}[v(x_k) + k\varepsilon^2 | \, x_0, \ldots, x_{k-1}] \\ &\quad \leq \max\left\{ \frac{1}{2}\left(\sup_{B_\varepsilon(x_{k-1}) \cap B_R(y)} v + \inf_{B_\varepsilon(x_{k-1}) \cap B_R(y)} v \right), \fint_{B_\varepsilon(x_{k-1}) \cap B_R(y)} v \, dz \right\} \\ &\quad \leq v(x_{k-1}) + (k-1)\varepsilon^2, \end{aligned}$$

if $x_{k-1} \in B_R(y) \setminus \overline{B}_\delta(y)$. Thus $v(x_k) + k\varepsilon^2$ is a supermartingale, and the optional stopping theorem yields

$$\mathbb{E}^{x_0}[v(x_{\tau^* \wedge k}) + (\tau^* \wedge k)\varepsilon^2] \le v(x_0). \tag{3.6}$$

Because $x_{\tau^*} \in \overline{B}_\delta(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$, we have

$$0 \le -\mathbb{E}^{x_0}[v(x_{\tau^*})] \le o(1).$$

Furthermore, the estimate

$$0 \le v(x_0) \le C(R/\delta)\,\mathrm{dist}(\partial B_\delta(y), x_0)$$

holds for the solutions of (3.5). Thus, by passing to the limit as $k \to \infty$, we obtain

$$\varepsilon^2 \mathbb{E}^{x_0}[\tau^*] \le v(x_0) - \mathbb{E}[u(x_{\tau^*})] \le C(R/\delta)(\mathrm{dist}(\partial B_\delta(y), x_0) + o(1)).$$

This completes the proof. □

Next we derive a uniform bound and estimate for the asymptotic continuity of the family of p_1-p_2-harmonious functions.

We assume here that Ω satisfies an exterior sphere condition: For each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$.

Lemma 3.5.5. *Let g be Lipschitz continuous in Γ_ε and f Lipschitz continuous in Ω such that $f \equiv 0$, $\inf f > 0$ or $\sup f < 0$. The p_1-p_2-harmonious function u_ε with data g and f satisfies*

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(y)| \le{}& \mathrm{Lip}(g)(|x-y| + \delta) \\ &+ C(R/\delta)(|x-y| + o(1))(1 + \|f\|_\infty) + \widetilde{C} Lip(f)|x-y|, \end{aligned} \tag{3.7}$$

for every small enough $\delta > 0$ and for every two points $x, y \in \Omega \cup \Gamma_\varepsilon$. Here $o(1)$ can be taken depending only on δ and R.

Proof. The case $x, y \in \Gamma_\varepsilon$ is clear. Thus, we can concentrate on the cases $x \in \Omega$ and $y \in \Gamma_\varepsilon$ as well as $x, y \in \Omega$.

We use the connection to games. Suppose first that $x \in \Omega$ and $y \in \Gamma_\varepsilon$. By the exterior sphere condition, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. Now Player I chooses a strategy of pulling towards z, denoted by S_I^z. Then

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for a constant C large enough independent of ε. Indeed,

$$
\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[|x_k-z| \mid x_0,\dots,x_{k-1}] \\
&\le \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\left\{|x_{k-1}-z|+\varepsilon-\varepsilon^3+|x_{k-1}-z|-\varepsilon\right\}+\beta_i \fint_{B_\varepsilon(x_{k-1})}|x-z|\,dx\right) \\
&\quad \le |x_{k-1}-z|+C\varepsilon^2.
\end{aligned}
$$

The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x_{k-1})}|x-z|\,dx \le |x_{k-1}-z|+C\varepsilon^2.$$

By the optional stopping theorem, this implies that

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[|x_\tau-z|] \le |x_0-z|+C\varepsilon^2\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau]. \tag{3.8}$$

Next we can estimate $\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau]$ by the stopping time of Lemma 3.5.4. Let $R>0$ be such that $\Omega\subset B_R(z)$. Thus, by (3.2),

$$\varepsilon^2\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau] \le \varepsilon^2\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau^*] \le C(R/\delta)(\operatorname{dist}(\partial B_\delta(z),x_0)+o(1)).$$

Since $y\in\partial B_\delta(z)$,

$$\operatorname{dist}(\partial B_\delta(z),x_0)\le |y-x_0|\,,$$

and thus, (3.8) implies

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[|x_\tau-z|] \le C(R/\delta)(|x_0-y|+o(1)).$$

We get

$$g(z)-C(R/\delta)(|x-y|+o(1)) \le \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[g(x_\tau)].$$

Thus, we obtain

$$
\begin{aligned}
&\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[g(x_\tau)+\varepsilon^2\sum_{n=0}^{\tau-1}f(x_n)] \\
&\qquad \ge \inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[g(x_\tau)+\varepsilon^2\sum_{n=0}^{\tau-1}f(x_n)] \\
&\qquad \ge g(z)-C(R/\delta)(|x_0-y|+o(1))-\varepsilon^2\inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau]\|f\|_\infty \\
&\qquad \ge g(y)-\operatorname{Lip}(g)\delta-C(R/\delta)(|x_0-y|+o(1))(1+\|f\|_\infty).
\end{aligned}
$$

The upper bound can be obtained by choosing for Player II a strategy where he points to z, and thus, (3.7) follows.

Finally, let $x, y \in \Omega$ and fix the strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$ for the game starting at x. We define a virtual game starting at y: we use the same coin tosses and random steps as the usual game starting at x. Furthermore, the players adopt their strategies $S_{\mathrm{I}}^v, S_{\mathrm{II}}^v$ from the game starting at x, that is, when the game position is y_{k-1} a player chooses the step that would be taken at x_{k-1} in the game starting at x. We proceed in this way until for the first time $x_k \in \Gamma_\varepsilon$ or $y_k \in \Gamma_\varepsilon$. At that point we have $|x_k - y_k| = |x - y|$, and we may apply the previous steps that work for $x_k \in \Omega$, $y_k \in \Gamma_\varepsilon$ or for $x_k, y_k \in \Gamma_\varepsilon$.

If we are in the case $f \equiv 0$ we are done. In the case $\inf_{y\in\Omega} |f(y)| > 0$, as we know that the u_ε are uniformly bounded according to Lemma 3.5.3, we have that the expected exit time is bounded by

$$\widetilde{C} = \frac{\max_{y\in\Gamma_\varepsilon} |g(y)| + C \max_{y\in\Omega} |f(y)|}{\inf_{y\in\Omega} |f(y)|}.$$

So the expected difference in the running payoff in the game starting at x and the virtual one is bounded by $\widetilde{C} Lip(f)|x - y|$, because $|x_i - y_i| = |x - y|$ for all $0 \leq i \leq k$. $\square$

Corollary 3.5.6. *Let $\{u_\varepsilon\}$ be a family of p_1-p_2-harmonious. Then there exists a uniformly continuous u and a subsequence still denoted by $\{u_\varepsilon\}$ such that*

$$u_\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega}.$$

Proof. Using Lemmas 3.5.3 and 3.5.5 we get that the family u_ε satisfies the hypothesis of the compactness Lemma 6.5.3. $\square$

Theorem 3.5.7. *The function u obtained as a limit in Corollary 3.5.6 is a viscosity solution to* (3.2) *when we consider the game with $f/2$ as the running pay-off function.*

Proof. First, we observe that $u = g$ on $\partial\Omega$ due to $u_\varepsilon = g$ on $\partial\Omega$ for all $\varepsilon > 0$. Hence, we can focus our attention on showing that u is p_1-p_2-harmonic inside Ω in the viscosity sense. To this end, we recall from [61] an estimate that involves the regular Laplacian ($p = 2$) and an approximation for the infinity Laplacian ($p = \infty$). Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Note that since ϕ is continuous then we have

$$\min_{y\in\overline{B}_\varepsilon(x)} \phi(y) = \inf_{y\in B_\varepsilon(x)} \phi(y)$$

for all $x \in \Omega$. Let x_1^ε be the point at which ϕ attains its minimum in $\overline{B}_\varepsilon(x)$

$$\phi(x_1^\varepsilon) = \min_{y\in\overline{B}_\varepsilon(x)} \phi(y).$$

It follows from the Taylor expansions in [61] that

$$
\begin{aligned}
&\frac{\alpha}{2}\left(\max_{y\in\overline{B}_\varepsilon(x)}\phi(y)+\min_{y\in\overline{B}_\varepsilon(x)}\phi(y)\right)+\beta\fint_{B_\varepsilon(x)}\phi(y)\,dy-\phi(x)\\
&\geq\frac{\varepsilon^2}{2(n+p)}\left\{(p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon-x}{\varepsilon}\right),\left(\frac{x_1^\varepsilon-x}{\varepsilon}\right)\right\rangle+\Delta\phi(x)\right\}\\
&\quad+o(\varepsilon^2).
\end{aligned}
\tag{3.9}
$$

Suppose that ϕ touches u at x strictly from below. We want to prove that

$$F^*(\nabla\phi(x),D^2\phi(x))\geq f(x).$$

By the uniform convergence, there exists sequence $\{x_\varepsilon\}$ converging to x such that $u_\varepsilon-\phi$ has an approximate minimum at x_ε, that is, for $\eta_\varepsilon>0$, there exists x_ε such that

$$u_\varepsilon(x)-\phi(x)\geq u_\varepsilon(x_\varepsilon)-\phi(x_\varepsilon)-\eta_\varepsilon.$$

Moreover, considering $\tilde{\phi}=\phi-u_\varepsilon(x_\varepsilon)-\phi(x_\varepsilon)$, we can assume that $\phi(x_\varepsilon)=u_\varepsilon(x_\varepsilon)$. Thus, by recalling the fact that u_ε is p_1-p_2-harmonious, we obtain

$$\eta_\varepsilon\geq\varepsilon^2\frac{f(x_\varepsilon)}{2}-\phi(x_\varepsilon)+\max_{i\in\{1,2\}}\left\{\frac{\alpha_i}{2}\left(\max_{\overline{B}_\varepsilon(x_\varepsilon)}\phi+\min_{\overline{B}_\varepsilon(x_\varepsilon)}\phi\right)+\beta_i\fint_{B_\varepsilon(x_\varepsilon)}\phi(y)\,dy\right\},$$

and thus, by (3.9), and choosing $\eta_\varepsilon=o(\varepsilon^2)$, we have

$$
\begin{aligned}
0\geq\frac{\varepsilon^2}{2}\max_{i\in\{1,2\}}&\left\{\alpha_i\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon-x_\varepsilon}{\varepsilon}\right),\left(\frac{x_1^\varepsilon-x_\varepsilon}{\varepsilon}\right)\right\rangle+\theta_i\Delta\phi(x_\varepsilon)\right\}\\
&+\varepsilon^2\frac{f(x_\varepsilon)}{2}+o(\varepsilon^2).
\end{aligned}
$$

Next we need to observe that

$$\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon-x_\varepsilon}{\varepsilon}\right),\left(\frac{x_1^\varepsilon-x_\varepsilon}{\varepsilon}\right)\right\rangle$$

converge to $\Delta_\infty\phi(x)$ when $\nabla\phi(x)\neq 0$ and always is bounded in the limit by $\lambda_{\min}(D^2\phi(x))$ and $\lambda_{\max}(D^2\phi(x))$. Dividing by ε^2 and letting $\varepsilon\to 0$, we get

$$F^*(\nabla\phi(x),D^2\phi(x))\geq f(x).$$

Therefore u is a viscosity supersolution.

To prove that u is a viscosity subsolution, we use a reverse inequality to (3.9) by considering the maximum point of the test function and choose a smooth test function that touches u from above. $\square$

Now, we just observe that this probabilistic approach provides an alternative existence proof of viscosity solutions to our PDE problem.

Corollary 3.5.8. *Any limit function obtained as in Corollary 3.5.6 is a viscosity solution to the problem*

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f & \textit{on } \Omega, \\ u = g & \textit{on } \partial\Omega. \end{cases}$$

In particular, the problem has a solution.

We proved that the problem has an unique solution using PDE methods, therefore we conclude that we have convergence as $\varepsilon \to 0$ of u_ε (not only along subsequences).

Corollary 3.5.9. *It holds that*

$$u_\varepsilon \to u \qquad \textit{uniformly in} \quad \overline{\Omega},$$

being u the unique solution to the problem

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f & \textit{on } \Omega, \\ u = g & \textit{on } \partial\Omega. \end{cases}$$

Chapter 4

Obstacle problems and maximal operators

4.1 Introduction

Motivated by the work in the previous chapter we were lead to consider PDEs of the form

$$\max\{L_1 v,\, L_2 v\} = 0.$$

That gave origin to the work that we present in this chapter. Here we include the results obtained in [23], a joint work with Juan Pablo Pinasco and Julio Daniel Rossi.

Both the obstacle problem and maximal operators are classical subjects in the theory of PDEs and have brought the attention of many researchers for many years. For example, if one considers the family of uniformly elliptic second order operators of the form $-tr(AD^2u)$ and look for maximal operators one finds the so-called Pucci maximal operator, $P^+_{\lambda,\Lambda}(D^2u) = \max_{A\in\mathcal{A}} -tr(AD^2u)$, where $\mathcal{A}$ is the set of uniformly elliptic matrices with ellipticity constant between λ and Λ, we refer to [30] for properties of these operators and details of its crucial role in regularity theory. On the other hand, the obstacle problem is a well known and widely studied free boundary problem, [72].

In this chapter we show that one can obtain solutions to maximal or minimal operators by taking the limit of a sequence constructed iterating the obstacle problem alternating the involved operators with the previous term in the sequence as obstacle.

We will look for solutions to the Dirichlet problem for the maximum or the minimum of two operators. To this end, let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth, domain and $g : \partial\Omega \to \mathbb{R}$ a smooth boundary condition. We want to point out that here we are not dealing with regularity issues of the solutions, therefore to simplify the presentation we set the domain and the boundary datum to be smooth.

Given an operator L (notice that here we can consider fully nonlinear problems of the form $Lu = F(x, u, Du, D^2u)$) we consider the obstacle problem (here solutions are assumed to be above the obstacle)

$$\begin{cases} u \geq \Phi & \text{in } \Omega, \\ Lu \geq 0 & \text{in } \Omega, \\ Lu = 0 & \text{in } \{u > \phi\}, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

or equivalently

$$\begin{cases} \min\{Lu, u - \Phi\} = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The obstacle problem can be also stated as follows: we look for the smallest super solution of L (with boundary datum g) that is above the obstacle. This formulation is quite convenient when dealing with fully nonlinear problems. We will refer to the obstacle problem as $P_L(\Phi, g)$.

We will assume here that the problem (4.1) has a unique viscosity solution. This is guaranteed if L has a comparison principle and one can construct barriers close to the boundary so that the boundary datum g is taken continuously. For general references on the obstacle problem (including regularity of solutions that are proved to be $C^{1,1}$) we just mention [2], [28], [29], [36], [72] and references therein.

Now, we define a sequence of continuous functions as follows: given two continuous operators L_1 and L_2 we start with u_1 the viscosity solution to

$$\begin{cases} L_1 u_1 = 0 & \text{in } \Omega, \\ u_1 = g & \text{on } \partial\Omega, \end{cases}$$

and then inductively we set

$$u_n \text{ as the solution to } \begin{cases} P_{L_2}(u_{n-1}, g) & \text{for } n \text{ even}, \\ P_{L_1}(u_{n-1}, g) & \text{for } n \text{ odd}. \end{cases}$$

That is, we define u_n as the solution to the obstacle problem alternating the involved operator L_i and using u_{n-1} as obstacle. Note that since u_{n-1} stands as the obstacle for u_n then we trivially have $u_{n-1} \leq u_n$. Hence this sequence is increasing with n.

We will also require that there exists a function U that is a viscosity super solution for both operators L_1 and L_2 with boundary datum g simultaneously, that is, we require that

$$L_1 U \geq 0, \qquad L_2 U \geq 0 \qquad \text{and} \qquad U|_{\partial\Omega} \geq g. \tag{4.2}$$

This function U will be used to obtain a uniform upper bound for the increasing sequence u_n. Hypothesis (4.2) holds, for example, if the maximum principle holds for the operators or when L_1 and L_2 are proper (uniformly degenerated elliptic and non decreasing in u). In the first case we can consider $U = \max g$. While in the second one we can consider $U = c - k|x|^2$ where k is the maximum of the ellipticity constants for L_1 and L_2 and c is large enough so that $U = c - k|x|^2 \geq g$ on $\partial\Omega$.

Note that when we consider the obstacle problem looking for sub solutions that are below the obstacle (that is, we reverse the inequalities in (4.1)) we can produce, iterating this obstacle problem starting with $v_1 = u_1$ as above, a sequence that we call v_n. With this procedure the obtained sequence is decreasing with n. When one considers this decreasing sequence v_n we need the existence of a function V such that

$$L_1 V \leq 0, \qquad L_2 V \leq 0 \qquad \text{and} \qquad V|_{\partial\Omega} \leq g.$$

As before one can show that this holds if the minimum principle holds for the operators or when L_1 and L_2 are proper.

As u_n is monotone and bounded we have that there exists the limit,

$$u(x) := \lim_n u_n(x).$$

We will assume that the limit u is continuous, or equivalently (by Dini's theorem) that the limit is uniform. This assumption can be checked by tracking the constants that appear in the regularity results for the obstacle problem in such a way that there is a uniform modulus of continuity for the sequence u_n (this holds, for example, when the Lipschitz constant remains uniformly bounded), see Remark 4.3.4. Now we are ready to state our main result that reads as follows:

Theorem 4.1.1.

1. *The increasing sequence u_n converges uniformly in $\overline{\Omega}$ to a viscosity solution of*

$$\begin{cases} \min\{L_1 u,\ L_2 u\} = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

2. *The decreasing sequence v_n converges uniformly in $\overline{\Omega}$ to a viscosity solution of*

$$\begin{cases} \max\{L_1 v,\ L_2 v\} = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The rest of the chapter is organized as follows: In Section 4.2 we prove Theorem 6.1.1 and in Section 4.3 we gather some remarks concerning extensions of our results.

4.2 Proof of the main result

We will prove part *(1)* of Theorem 6.1.1. The proof of part *(2)* is entirely analogous.

Recall that we deal with viscosity solutions to the obstacle problem (4.1) and that we assume that there is a comparison principle for the involved operators, L_1 and L_2. Note that we can consider only test functions that touch u *strictly*, see Definition A.1.1.

We assumed here that the operators L_1 and L_2 are continuous in the usual sense. We will comment on how to relax this hypothesis in Remark 4.3.5.

Also recall that the sequence u_n is constructed as follows: We take u_1 the solution to

$$\begin{cases} L_1 u_1 = 0 & \text{in } \Omega, \\ u_1 = g & \text{on } \partial\Omega, \end{cases}$$

and u_n is given inductively by

$$u_n \text{ is the solution to } \begin{cases} P_{L_2}(u_{n-1}, g) & \text{for } n \text{ even}, \\ P_{L_1}(u_{n-1}, g) & \text{for } n \text{ odd}. \end{cases}$$

This sequence u_n is increasing, bounded (since, using the comparison principle one can show by induction that $u_n \leq U$, where U is such that (4.2) holds) and we are assuming that there exists a continuous function u such that

$$u_n \to u$$

uniformly in $\overline{\Omega}$. We will comment on this assumption in Remark 4.3.4.

Now we observe that $L_1 u_n \geq 0$ if n is even and $L_2 u_n \geq 0$ when n is odd and since u_{2n} and u_{2n+1} converge uniformly to the same limit u we conclude that

$$L_1 u \geq 0 \qquad \text{and} \qquad L_2 u \geq 0,$$

in the viscosity sense in Ω.

On the other hand, we claim that

$$\min\{L_1 u_n, L_2 u_n\} \leq 0$$

for every n in the viscosity sense in Ω. Let us show this claim by induction. First, let us point out that it is clear that $\min\{L_1 u_1, L_2 u_1\} \leq 0$ since $L_1 u_1 \leq 0$.

Now assume that the claim holds for u_n and let us prove it for u_{n+1}. In the set $\{u_{n+1} > u_n\}$ it holds because $L_1 u_{n+1}$ or $L_2 u_{n+1}$ is zero. It remains to look in the coincidence set $\{u_{n+1} = u_n\}$. Let $x \in \{u_{n+1} = u_n\} \cap \Omega$, then we have $u_{n+1}(x) = u_n(x)$ and $u_{n+1} \geq u_n$ in the whole Ω. To conclude the argument we want to show that $\min\{L_1\psi(x), L_2\psi(x)\} \leq 0$ for every $\psi \in C^2$ that touches u_{n+1} strictly from above at x, but this follows since ψ also touches u_n from above at x.

As we have that $\min\{L_1 u_n, L_2 u_n\} \leq 0$ and u_n converges uniformly to u we conclude that

$$\min\{L_1 u, L_2 u\} \leq 0, \qquad \text{in } \Omega.$$

As we also have $L_1 u \geq 0$ and $L_2 u \geq 0$ in Ω we get that

$$\min\{L_1 u, L_2 u\} = 0 \qquad \text{in } \Omega.$$

The boundary datum $u = g$ is taken in a pointwise sense since $u_n = g$ on $\partial\Omega$ and we have uniform convergence.

4.3 Remarks and extensions

4.3.1 The maximum/minimum of two $p-$Laplacians.

In the previous chapter, the Dirichlet problem for the maximal operator associated with the $p-$Laplacian family was studied.

Let $\Delta_p u = |Du|^{2-p}\text{div}(|Du|^{p-2}Du)$ and consider

$$\max\{-\Delta_{p_1}u(x),\, -\Delta_{p_2}u(x)\} = f(x)$$

for $2 \leq p_1 < p_2 \leq \infty$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with $u = g$ on $\partial\Omega$. We proved existence and uniqueness of a viscosity solution using PDE techniques combined with game theoretical arguments. Now we remark that we can use the previously described iterations with the obstacle problems for $-\Delta_{p_1}u$ and $-\Delta_{p_2}u$ to obtain a decreasing (or increasing) sequence u_n (or v_n) that converges uniformly (see Remark 4.3.4) to the unique viscosity solution to the Dirichlet problem for $\max\{-\Delta_{p_1}u(x),\, -\Delta_{p_2}u(x)\} = f(x)$ (or for $\min\{-\Delta_{p_1}u(x),\, -\Delta_{p_2}u(x)\} = f(x)$).

4.3.2 Parabolic Problems.

Our results can be also extended to parabolic problems. In fact we can consider the parabolic obstacle problem for a parabolic operator of the form

$$L(u) = F(u_t, t, x, u, Du, D^2u),$$

that is,

$$\begin{cases} u \geq \Phi & \text{in } \Omega \times (0,T), \\ Lu \geq 0 & \text{in } \Omega \times (0,T), \\ Lu = 0 & \text{in } \{u > \Phi\}, \\ u = g & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega. \end{cases}$$

Note that now the obstacle Φ is a function of x and t. As before, we assume here that the problem (4.1) has a unique viscosity solution, that the involved operators L have a comparison principle and that there exists a simultaneous supersolution, U, valid for every L. In this way we obtain an increasing and bounded sequence that (assuming continuity of the limit) converge to a viscosity solution to

$$\begin{cases} \min\{L_1u(x,t),\ L_2u(x,t)\} = 0 & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega. \end{cases}$$

Combining previous remarks we obtain existence of a viscosity solution to

$$\begin{cases} \min\left\{u_t - \Delta_{p_1} u,\ u_t - \Delta_{p_1} u\right\}(x,t) = 0 & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega. \end{cases}$$

4.3.3 Maximum/minimum of more than two operators.

Let us mention that this idea works for a finite number of operators, $L_1, ..., L_K$. We just have to consider the obstacle problem for $L_1, ..., L_K$ and iterate to produce an increasing (or decreasing) sequence that converges uniformly to a viscosity solution to $\min_j \{L_j u(x)\} = 0$ (or to $\max_j \{L_j u(x)\} = 0$).

This procedure can be also extended to a sequence of operators $\{L_j\}_{j\in\mathbb{N}}$ the only point is that the obstacle problem for every operator in the sequence has to appear infinitely many times. This can be done just by considering the sequence

$$P_{L_1}, P_{L_2}, P_{L_1}, P_{L_2}, P_{L_3}, P_{L_1}, \ldots.$$

We can also consider an arbitrary family of operators $\{L_i\}_{i\in I}$ (here the set of indexes I is not assumed to be numerable). To this end we have to modify slightly our previous procedure. Before proceeding with this extension we have to recall some definitions for two reasons: first, to complete the technical details omitted in the proof of Theorem 6.1.1 related to the fact that we can consider non continuous operators (as we do in Remark 4.3.5) and second because, even when all the operators $\{L_i\}_{i\in I}$ are continuous, the $\inf_{i\in I}\{L_i\}$ it is not necessarily so.

We refer to Appendix A for the definition of the lower semicontinous and upper semicontinous envelopes of F. And in particular to Definition A.1.2, for the definition of viscosity solutions for a discontinuous F.

Now, let us construct our sequence. We start by solving the Dirichlet problem for all the operators L_i, that is, we let u_1^i be such that

$$\begin{cases} L_i u_1^i = 0 & \text{in } \Omega, \\ u_1^i = g & \text{on } \partial\Omega, \end{cases}$$

and then take

$$u_1 = \sup_{i\in I} u_1^i.$$

Now we define inductively u_n by taking the supremum of the solutions to the obstacle problem for the operators L_i with obstacle u_{n-1}, that is, we take u_n^i the solution to

$$\begin{cases} u_n^i \geq u_{n-1} & \text{in } \Omega, \\ L_i u_n^i \geq 0 & \text{in } \Omega, \\ L_i u_n^i = 0 & \text{in } \{u_n^i > u_{n-1}\}, \\ u_n^i = g & \text{on } \partial\Omega, \end{cases}$$

and then we let

$$u_n = \sup_{i \in I} u_n^i.$$

As was argued previously, we assume that there exists a continuous function u such that

$$u_n \to u$$

uniformly in $\overline{\Omega}$.

Now our aim is to show that u is a viscosity solution to

$$Lu = \inf_{i \in I} L_i u = 0. \tag{4.3}$$

First, we observe that, given $i \in I$, we have $u_n^i \to u$ because $u_n \leq u_n^i \leq u_{n+1}$. As for each $i \in I$ we know that $L_i u_n^i \geq 0$ in Ω, we get that u is a supersolution of $L_i u = 0$, this is $L_i u \geq 0$. Hence it is a supersolution of (4.3), in the sense that $Lu \geq 0$ in the viscosity sense.

Let us now show that u_n is a subsolution of (4.3) for all $n \in \mathbb{N}$. We proceed by induction. For $n = 1$, we consider an arbitrary $x_0 \in \Omega$. Let ψ be an arbitrary smooth function that touches from above u_1 at x_0, then there exist $\{i_k\}_{k \in \mathbb{N}} \subset I$ such that $u_1(x_0) = \lim_k u_1^{i_k}(x_0)$. Then let x_k be a point where $\psi - u_1^{i_k}$ attains its minimum, we know that $x_k \to x_0$ (note that we ask that ψ touches u_1 strictly from above). We have that $\psi - \psi(x_k) + u_1^{i_k}(x_k)$ touches from above $u_1^{i_k}$ at x_k. Then, since $L_{i_k} u_1^{i_k} = 0$, we get

$$L_{i_k}(x_k, u_1^{i_k}(x_k), \nabla\psi(x_k), D^2\psi(x_k)) \leq 0.$$

Hence

$$L(x_k, u_1^{i_k}(x_k), \nabla\psi(x_k), D^2\psi(x_k)) \leq 0$$

and we can conclude that

$$L_* \psi(x_0) \leq \lim_k L(D^2\psi(x_k), \nabla\psi(x_k), \psi - \psi(x_k) + u_1^{i_k}(x_k), x_k) \leq 0$$

We have proved that u_1 is a viscosity subsolution of (4.3).

Analogously, we can show that the claim holds for u_{n+1} assuming that it holds for u_n. We consider an arbitrary $x_0 \in \Omega$. Let ψ be an arbitrary smooth function that strictly touches from above u_{n+1} at x_0, then, as before, there exist $\{i_k\}_{k\in\mathbb{N}} \subset I$ such that $u_{n+1}(x_0) = \lim_k u_{n+1}^{i_k}(x_0)$. Then let x_k be a point where $\phi - u_{n+1}^{i_k}$ attains its minimum, we know that $x_k \to x_0$. We have that $\psi(z) - \psi(x_k) + u_{n+1}^{i_k}(x_k) + |z - x_k|^4$ strictly touches from above $u_{n+1}^{i_k}$ at x_k. Then, since $L_{i_k} u_{n+1}^{i_k} = 0$ in $\{u_{n+1}^{i_k} > u_n\}$ and $L_* u_{n+1}^{i_k} \leq 0$ in $\{u_{n+1}^{i_k} = u_n\}$ by the inductive hypothesis, we get

$$L_* u_{n+1}^{i_k} \leq 0.$$

Hence

$$L_*(x_k, u_1^{i_k}(x_k), \nabla\psi(x_k), D^2\psi(x_k)) \leq 0$$

and we can conclude that

$$L_*\psi(x_0) \leq \lim_k L_*(x_k, u_1^{i_k}(x_k), \nabla\psi(x_k), D^2\psi(x_k)) \leq 0$$

We have proved that u_{n+1} is a subsolution of (4.3), that is, $L_* u_{n+1} \leq 0$.

Finally, being the limit of subsolutions, we conclude that u (the limit of the sequence u_n) is a subsolution and therefore a solution of (4.3).

4.3.4 Continuity of the limit and uniform convergence hypothesis.

Let us recall a result from Section 6 of [33] that we used in the proof of the main result. Given u_n subsolutions of an equation, we have that

$$\bar{U} := \limsup_n {}^* u_n = \limsup_j \left\{ u_n(z) : n \geq j, z \in \Omega, |z - x| \leq \frac{1}{j} \right\}$$

is a subsolution of the same equation. An analogous result holds for super solutions.

Hence if the limit of the sequence is continuous we can conclude it is a subsolution. Even more, if we have that the limit is continuous, we can conclude that the convergence is uniform by Dini's theorem.

On the other hand, we want to be able to obtain the continuity of the limit (that we assumed) imposing conditions on the involved operators. In this direction, we can require that there is a uniform modulus of continuity for the obstacle problems involved, that is, if the obstacle and the boundary datum have a modulus of continuity, then the solution to the obstacle problem also has the same modulus of continuity. This holds, for example, for the obstacle problem for the fractional Laplacian (a quite popular operator nowadays), see Theorem 3.2.3 in [75]. This also holds for the $p-$laplacian, the solution is holder-continuous (for a specific exponent) with the same constant, see Theorem 6 in [31]. Then, inductively, we conclude that all the u_n have the same modulus of continuity, and hence they are equicontinous. By Arzelà-Ascoli theorem we get that the sequence converges uniformly. Then, of course, the limit u is continuous (and even more, we get a modulus of continuity for the limit).

4.3.5 On the hypothesis of continuity of the operators.

In the proof of the main theorem and Remark 4.3.3 we assumed that the involved operators are continuous. We used this fact in two steps.

We conclude that u was a supersolution of $L_i u = 0$ (as being the limit of supersolutions) and then, because of this, that it was a super solution of $\inf_{i\in I} L_i u = 0$. But this fact is not necessarily true when the operators are not continuous. We have that $L_i^* u \geq 0$ and hence we can conclude that $\inf_{i\in I} L_i^* u \geq 0$ but we want to conclude that $(\inf_{i\in I} L_i)^* u \geq 0$. So we need to require that $(\inf_{i\in I} L_i)^* \geq \inf_{i\in I} L_i^*$, that holds when the operator are continuous.

On the other hand we need that $(\inf_{i\in I} L_i)_* \leq \inf_{i\in I} L_{i*}$. In the proof of the main theorem we need this fact to conclude that u_{n+1} is a subsolution of $\min\{L_1, L_2\}u = 0$ on the set $\{u_{n+1} > u_n\}$ where we know $L_1 u_{n+1}$ or $L_2 u_{n+1}$ is zero. In Remark 4.3.3 we need it in a similar way when we conclude that $L_* u_{n+1}^{i_k} \leq 0$. In this case we have that this inequality always holds.

In conclusion, when we have that the involved operators are not continuous we need to require that

$$\left(\inf_{i\in I} L_i\right)^* \geq \inf_{i\in I} L_i^*.$$

Let us present two simple examples where this assumption does not hold. We consider $\Omega = (0,1)$ and the boundary datum $g(0) = g(1) = 0$ in both examples.

Example 1. We consider $L_1 = -u'' - \chi_{[0,1/2)}$ and $L_2 = -u'' - \chi_{[1/2,1]}$, then

$$(\inf\{L_1, L_2\})^* = \left(-u'' - \chi_{[0,1]}\right)^* = -u'' - \chi_{[0,1]}$$

while

$$\inf\{L_1^*, L_2^*\} = \inf\{-u'' - \chi_{[0,1/2)}, -u'' - \chi_{(1/2,1]}\} = -u'' - \chi_{[0,1/2)\cup(1/2,1]}.$$

Hence, $(\inf_{i\in I} L_i)^* \geq \inf_{i\in I} L_i^*$ does not hold pointwise. However, remark that in this example we have the same solutions for $\inf_{i\in I} L_i^* u = 0$ and for $\inf_{i\in I} L_i u = 0$.

Example 2. Now we consider $L_i = -u'' - \delta_i$ for $i \in [0,1]$ (remark that in this example we have an uncountable family of operators), then

$$\left(\inf_{i\in I} L_i\right)^* = \left(-u'' - \chi_{[0,1]}\right)^* = -u'' - \chi_{[0,1]}$$

while

$$\inf_{i\in I} L_i^* = \inf_{i\in I} -u'' = -u''.$$

Again in this example the hypothesis $(\inf_{i\in I} L_i)^* \geq \inf_{i\in I} L_i^*$ does not hold pointwise. Note that now the equations $\inf_{i\in I} L_i^* u = 0$ and $\inf_{i\in I} L_i u = 0$ are really different.

Chapter 5

A lower bound for the principal eigenvalue

5.1 Introduction

In this chapter we present the results obtained in [22]. These results where obtained while considering different variants of Tug-of-War games. However, the results are not directly related to games and the presentation does not refer to them.

Let $\Omega \subset \mathbb{R}^N$ be a domain and $Lu = F(u, \nabla u, D^2u)$ a differential operator. We consider the Dirichlet eigenvalue problem

$$\begin{cases} Lu + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

We are interested in the principal eigenvalue of $-L$, that is the smallest number $\lambda \in \mathbb{R}$ for which the Dirichlet eigenvalue problem (5.1) has a non-trivial solution. Our goal here is to introduce a novel technique to obtain a lower bound for this value.

We will consider solutions in the viscosity sense, see Appendix A. This will allow us to consider fully nonlinear operators like $Lu = F(u, \nabla u, D^2u)$. In this general framework we define the principal eigenvalue through the maximum principle as in [12]. That is, we let

$$\lambda_1(\Omega) = \sup\{\lambda \in \mathbb{R} : \exists v \in C(\Omega) \text{ satisfying } v(x) > 0\ \forall x \in \Omega \text{ and } Lv + \lambda v \leq 0\}.$$

This definition allows us to consider operators in non-divergence form. In [12] the authors proved that for uniformly elliptic linear operators the value $\lambda_1(\Omega)$ defined above is indeed the principal eigenvalue of $-L$. This work opened the path to develop an eigenvalue theory for nonlinear operators.

Let us mention some previous work that deal with the operators that we will consider as examples to illustrate our general result. The Pucci extremal operators were studied in [26]. In [16, 17] it is proved that the number defined above is the principal eigenvalue for a class of homogeneous fully nonlinear operators which includes the homogeneous p-laplacian (see also [47] and [65]). In [42] this was done for the homogeneous infinity laplacian. The eigenvalue problem that arises as limit of the problem for the p-laplacian was considered in [43]. Other questions were addressed in more recent work as problems in non-smooth domains [14], unbounded domains [13] and simplicity of the first eigenvalue [15].

The lower bound that we obtain here depends on the largest radius of a ball included in Ω. We define

$$R = \max_{x \in \bar{\Omega}} \operatorname{dist}(x, \Omega^c).$$

From the definition of λ_1 it is clear that the first eigenvalue is monotone with respect to the domain, that is

$$\Omega_1 \subset \Omega_2 \Rightarrow \lambda_1(\Omega_2) \leq \lambda_1(\Omega_1).$$

Then

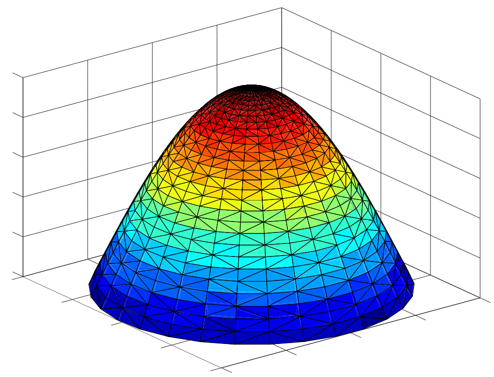
$$\lambda_1(\Omega) \leq \lambda_1(B_R)$$

and hence we can obtain an upper bound for the principal eigenvalue by computing this value for a ball. We can do this by constructing a radial positive eigenfunction. Therefore, we have to provide a radial solution $\phi(r)$ to the equation (5.1) such that $\phi(R) = 0$ and $\phi'(0) = 0$. The eigenfunction will look like the one in Figure 5.1a. In this way we can obtain an upper bound for the principal eigenvalue by solving certain ODE.

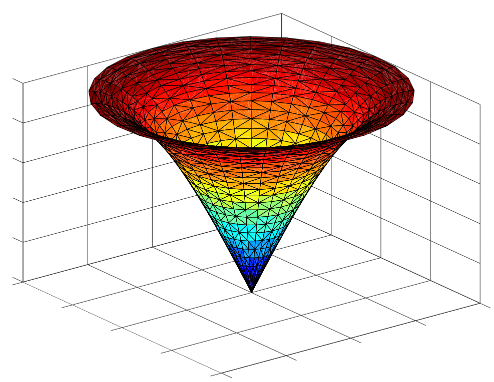
Our main result provides an analogous construction to obtain a lower bound for the principal eigenvalue. This time we require a radial solution $\phi(r)$ to the equation $Lu + \lambda u = 0$ defined in the punctured ball $B_R \setminus \{0\}$ such that $\phi'(R) = 0$ and $\phi(0) = 0$. The function will look like the one shown in Figure 5.1b. In this way we can obtain a lower bound for the principal eigenvalue by solving an ODE. The lower bound will be the value of λ for which we can solve the ODE.

Since our bound only depends on R, our technique is well suited for example for L shape domains where considering a ball or and strip that contains Ω gives poorer results or can't be done for example if the L shape domain is unbounded. We also compare our result with the classical Rayleigh-Faber-Krahn inequality in Example 5.3.3. Even more, our technique is well suited to obtain sharp bounds for certain operators as will be shown in the example section.

In the next section we state and prove our main result of this chapter and then we outline some extensions. Later, in Section 5.3, we compute the bound explicitly for the homogeneous infinity laplacian, for the homogeneuos p-laplacian and for other operators. We prove that for the homogeneous infinity laplacian the principal eigenvalue



(a) A eigenfunction in a ball.



(b) The radial function required in the main theorem.

Figure 5.1: Radial functions that allow us to obtain bounds for the principal eigenvalue

is $\lambda_{1,\infty} = \left(\frac{\pi}{2R}\right)^2$. In addition, our bound for the homogeneous p-laplacian proves that $\lim_{p\to\infty} \lambda_{1,p} = \lambda_{1,\infty}$.

5.2 Main theorem

Let $\Omega \subset \mathbb{R}^N$ be a domain (not necessarily bounded) and $Lu := F(u, \nabla u, D^2u)$ a fully nonlinear operator. As we are interested in operators like the homogeneous infinity laplacian and p-laplacian which are not well defined where the gradient vanishes we will give a suitable definition of solution that includes these operators as in Definition A.1.2 but for the eigenvalue problem.

Definition 5.2.1. We consider the equation

$$F(u, \nabla u, D^2u) + \lambda u = 0.$$

1. A lower semi-continuous function u is a viscosity supersolution if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from below, we have

$$F_*(\psi(x), \nabla\psi(x), D^2\psi(x)) + \lambda\psi(x) \leq 0.$$

2. An upper semi-continuous function u is a subsolution if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above, we have

$$F^*(\psi(x), \nabla\psi(x), D^2\psi(x)) + \lambda\psi(x) \geq 0.$$

3. Finally, u is a viscosity solution if it is both a sub- and supersolution.

As we have mentioned in the introduction, we want to obtain a lower bound for the principal eigenvalue of $-L$ given by

$$\lambda_1(\Omega) = \sup\{\lambda \in \mathbb{R} : \exists v \in C(\Omega) \text{ satisfying } v(x) > 0 \ \forall x \in \Omega \text{ and } Lv + \lambda v \leq 0\},$$

where the last inequality holds in the viscosity sense. Let us recall that here

$$R = \max_{x\in\bar{\Omega}} \operatorname{dist}(x, \Omega^c).$$

We are ready to state and prove the main theorem of this Chapter.

Theorem 5.2.2. *Suppose $\phi(r)$ is an increasing radial function defined in B_r for some $r > R$ with $\phi(0) = 0$ and $\lambda \in \mathbb{R}$ is such that*

$$L\phi + \lambda\phi \leq 0$$

in $B_r \setminus \{0\}$. Then $\lambda_1(\Omega) \geq \lambda$.

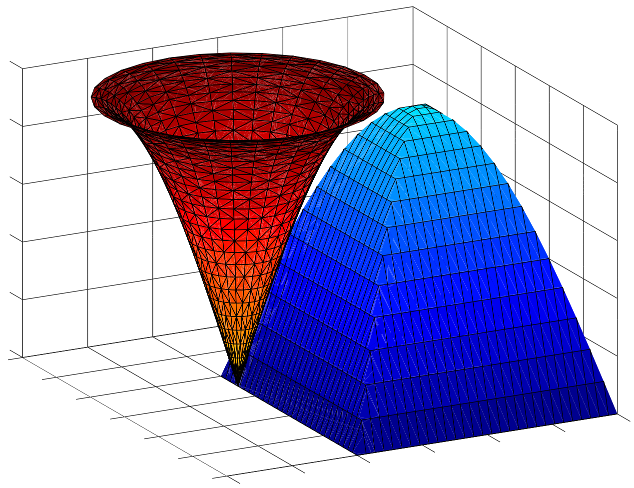


Figure 5.2: Functions v (blue) and ϕ_{y_0} (red) defined in the proof of Theorem 5.2.2 for a square.

Proof. Let us consider $v : \Omega \to \mathbb{R}$ given by

$$v(x) = \phi(\operatorname{dist}(x, \Omega^c)).$$

Since ϕ is positive outside the origin so is v inside Ω. If we prove that $Lv + \lambda v \leq 0$ for the given value of λ, we obtain the desired inequality.

Let us consider $x_0 \in \Omega$ and $\psi \in C^2$ such that it touches v at x_0 strictly from below. Since Ω is an open set, there exists $y_0 \in \partial\Omega$ such that $\operatorname{dist}(x_0, \Omega^c) = \operatorname{dist}(x_0, y_0)$ and we can consider $\phi_{y_0}(x) = \phi(|x - y_0|)$. Then, since ϕ is radial increasing, one get $v \leq \phi_{y_0}$ and coincides with it at x_0. So ψ touches ϕ_{y_0} at x_0 strictly from below and hence ψ satisfies the inequality. This shows that $Lv + \lambda v \leq 0$ in the viscosity sense as desired. $\square$

Remark 5.2.3. Given r if we are able to construct ϕ for certain $\lambda(r)$ that depends continuously on r since $\lambda_1(\Omega) \geq \lambda(r)$ for all $r > R$, we obtain $\lambda_1(\Omega) \geq \lambda(R)$.

It may be the case that we could not construct ϕ as required above (see Example 5.3.3). In that case we can modify our construction in order to obtain the lower bound as follows. Given $\delta > 0$, we consider

$$\Omega_\delta = \{x : \operatorname{dist}(x, \Omega) < \delta\}$$

and

$$R_\delta = \max_{x \in \Omega_\delta} \operatorname{dist}(x, \Omega_\delta^c).$$

Theorem 5.2.4. *Suppose $\phi(r)$ is an increasing radial function defined in $B_r \setminus B_\delta$ for some $r > R_\delta$ with $\phi = 0$ on ∂B_δ and λ is such that*

$$L\phi + \lambda\phi \leq 0$$

in $B_r \setminus B_\delta$. Then $\lambda_1(\Omega) \geq \lambda$.

Proof. The proof is completely analogous to that of Theorem 5.2.2. We have to consider $v(x) = \phi(\mathrm{dist}(x, \Omega_\delta^c))$ which is positive in Ω since $\mathrm{dist}(x, \Omega_\delta^c) \geq \delta$ for all $x \in \Omega$. And we prove that v is a supersolution at x_0 by considering $\phi_{y_0}(x) = \phi(|x - y_0|)$ for $y_0 \in \partial\Omega_\delta$ such that $\mathrm{dist}(x_0, \Omega_\delta^c) = \mathrm{dist}(x_0, y_0)$. □

Let us make some comments regarding R_δ which will be useful when applying Theorem 5.2.4, see Example 5.3.3. We observe that $R_\delta \geq R + \delta$ but equality is not true in general. This can be seen by considering an U shaped domain, if δ is big enough the 'hole' inside the domain is covered and then R_δ is strictly bigger than $R + \delta$. Let us prove that the equality holds for convex domains.

Lemma 5.2.5. *When Ω is convex, $R_\delta = R + \delta$.*

Proof. Let $y \in \Omega_\delta$ such that $B_{\tilde{R}}(y) \subset \Omega_\delta$. Let us show that $B_{\tilde{R}-\delta}(y) \subset \Omega$, and hence $R_\delta - \delta \leq R$ as desired.

Suppose not, let $x \in B_{\tilde{R}-\delta}(y) \setminus \Omega$. As $x \notin \Omega$ and Ω is convex there exists a plane though x such that one of the half-spaces defined by this plane is disjoint with Ω. Now, points in that half-space at distance greater that δ from the plane are not in Ω_δ but this is a contradiction since $B_{\tilde{R}-|x-y|} \subset \Omega_\delta$ and $\tilde{R} - |x - y| > \delta$. □

Remark 5.2.6. We have considered the Dirichlet eigenvalue problem given by

$$Lu + \lambda u = 0$$

but we can consider a more general version of the problem $Lu + \lambda Mu = 0$, where M is a given differential operator, or even more generally

$$G(D^2u, \nabla u, u, \lambda) = 0.$$

As examples of this general situation we can consider $M = |u|^\alpha u$ as in [17] and $G(D^2u, \nabla u, u, \lambda) = \min\{-\Delta_\infty u, |\nabla u| - \lambda u\}$ as in [43]. Theorems 5.2.2 and 5.2.4 also hold in this more general case.

5.3 Examples

In this section we compute explicitly the bound for the principal eigenvalue of the homogeneous infinity laplacian, the homogeneuos p-laplacian, the eigenvalue problem that rises when considering the limit as $p \to \infty$ of the problem for the p-laplacian and Pucci extremal operator. We denote $\lambda_{1,\infty}$ and $\lambda_{1,p}$ the principal eigenvalue of the homogeneous infinity laplacian and the homogeneuos p-laplacian, respectively. For the homogeneous infinity laplacian we prove that the principal eigenvalue is given by $\lambda_{1,\infty} = \left(\frac{\pi}{2R}\right)^2$. For the homogeneous p-laplacian our bound allows us to prove that $\lim_{p\to\infty} \lambda_{1,p} = \lambda_{1,\infty}$, see [65] for a different proof of this result.

Example 5.3.1. Here we consider the homogeneous infinity laplacian, which is given by

$$\Delta_\infty^H u = \left(\frac{\nabla u}{|\nabla u|}\right)^t D^2 u \frac{\nabla u}{|\nabla u|}.$$

The eigenvalue problem for this operator was studied in [42]. We want to prove that

$$\lambda_{1,\infty}(\Omega) = \left(\frac{\pi}{2R}\right)^2,$$

which gives us an explicit new characterization of the eigenvalue.

On the one hand we have that $\lambda_{1,\infty}(B_R) = \left(\frac{\pi}{2R}\right)^2$. It is easy to check that

$$u(x) = \sin\left(\frac{(R - ||x||)\pi}{2R}\right)$$

is the corresponding eigenfunction. On the other hand it is easy to verify that

$$\phi(x) = \sin\left(\frac{||x||\pi}{2R}\right)$$

satisfies $L\phi + \lambda_{1,\infty}\phi \leq 0$ in $B_R \setminus \{0\}$, it is radially increasing in B_R and $\phi(0) = 0$. Hence Theorem 5.2.2 allows us to conclude the desired result.

Moreover $v(x) = \phi(\text{dist}(x, \Omega^c))$ is an eigenfunction for stadium like domains. As can be seen in the proof of Theorem 5.2.2, it is a supersolution to the equation. In the same way it can be shown that it is a subsolution by considering the eigenfunction in balls of radius R contained in Ω. Let us mention that in [35] stadium like domains are characterized by considering a Serrin-type problem for the homogeneous infinity laplacian.

Example 5.3.2. We consider the homogeneous p-laplacian, that is

$$\Delta_p^H u = \frac{1}{p}|\nabla u|^{2-p} div(|\nabla u|^{p-2}\nabla u) = \frac{p-2}{p}\Delta_\infty^H u + \frac{1}{p}\Delta u.$$

When we look for radial solutions to the equation $\Delta_p^H v + \lambda v = 0$ in B_R, we obtain the equation

$$v_{rr} + \frac{N-1}{p-1}\frac{v_r}{r} + \frac{p\lambda}{p-1}v = 0. \tag{5.1}$$

The general solution is given by

$$v(r) = c_1 r^\alpha J_\alpha(\eta r) + c_2 r^\alpha Y_\alpha(\eta r),$$

where

$$\alpha = \frac{1 - \frac{N-1}{p-1}}{2} = \frac{p-N}{2(p-1)} \quad , \quad \eta = \sqrt{\lambda \frac{p}{p-1}}$$

and J_α and Y_α are Bessel functions.

In [48] the eigenvalue for a ball B_R is computed,

$$\lambda_p(B_R) = \frac{p-1}{p} \left(\frac{\mu_1^{(-\alpha)}}{R} \right)^2,$$

where $\mu_1^{(-\alpha)}$ is the first zero of the Bessel function $J_{-\alpha}$. This implies that

$$\lambda_{1,p}(\Omega) \leq \frac{p-1}{p} \left(\frac{\mu_1^{(-\alpha)}}{R} \right)^2.$$

We want to construct an appropriate function to apply Theorem 5.2.2. We consider the case $p > N$ (we analyse the case $p \leq N$ in the following example). We observe that

$$0 < \alpha = \frac{p-N}{2(p-1)}.$$

As we require $v(0) = 0$, we have to take

$$v(r) = c r^\alpha J_\alpha(\eta r).$$

Then

$$v'(r) = c \eta r^\alpha J_{\alpha-1}(\eta r)$$

and we can take v increasing up to the first zero of the derivative. We impose $v'(R) = 0$, that is

$$\frac{\mu_1^{(\alpha-1)}}{R} = \eta = \sqrt{\lambda \frac{p}{p-1}}.$$

Then,

$$\frac{p-1}{p} \left(\frac{\mu_1^{(\alpha-1)}}{R} \right)^2 \leq \lambda_{1,p}(\Omega),$$

and we have that

$$\frac{p-1}{p} \left(\frac{\mu_1^{(\alpha-1)}}{R} \right)^2 \leq \lambda_{1,p}(\Omega) \leq \frac{p-1}{p} \left(\frac{\mu_1^{(-\alpha)}}{R} \right)^2.$$

Now let us consider the limit as $p \to \infty$. Since $\alpha \to \frac{1}{2}^-$, we have that

$$\alpha - 1 \to -\frac{1}{2}^- \text{ and } -\alpha \to -\frac{1}{2}^+$$

and hence

$$\frac{p-1}{p}\left(\frac{\mu_1^{(\alpha-1)}}{R}\right)^2 \to \left(\frac{\pi}{2R}\right)^2 \quad \text{and} \quad \frac{p-1}{p}\left(\frac{\mu_1^{(-\alpha)}}{R}\right)^2 \to \left(\frac{\pi}{2R}\right)^2.$$

We have proved that

$$\lim_{p\to\infty} \lambda_{1,p}(\Omega) = \lambda_{1,\infty}(\Omega).$$

Example 5.3.3. If we consider the case $p \le N$ in the previous example, the ordinary differential equation (5.1) has no non-trivial solution with $v(0) = 0$. Hence we apply Theorem 5.2.4.

We can take

$$v(r) = cr^{\alpha} J_{\alpha}(\eta r).$$

Then

$$v'(r) = c\eta r^{\alpha} J_{\alpha-1}(\eta r).$$

If $x < y$ are zeros of J_α and $J_{\alpha-1}$ respectively, we can choose the sign of c such that v is an increasing positive function in the interval $(x/\eta, y/\eta)$. Let us assume that we can choose δ such that $\delta/R_\delta = x/y$ (we can do this when Ω is convex and hence $R_\delta = R+\delta$ as stated in Lemma 5.2.5). If we take $\eta = x/\delta = y/R_\delta$ we obtain that v is an increasing positive function in the interval (δ, R_δ) and we can apply Theorem 5.2.4. In the case that $R_\delta = R + \delta$, $\delta/R_\delta = x/y$ implies that $\delta = \frac{Rx}{y-x}$. Then $\eta = \frac{y-x}{R}$, and we obtain

$$\frac{p-1}{p}\left(\frac{y-x}{R}\right)^2 \le \lambda_{1,p}(\Omega).$$

Let us observe that the same can be done with Y_α instead of J_α.

Let us make some explicit computation in a particular case, for the Laplacian in dimension 3. We avoid the term $1/p$ in the operator and consider the equation $\Delta u + \lambda u = 0$ in $\Omega \subset \mathbb{R}^3$. We have $\alpha = -1/2$,

$$x^{-1/2} J_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\frac{cos(x)}{x}.$$

The distance between the zeros of the function and the subsequent zero of its derivative increases and approaches $\pi/2$. Hence, we obtain

$$\left(\frac{\pi}{2R}\right)^2 \le \lambda_{1,2}(\Omega).$$

Let us compare our result to the classical Rayleigh-Faber-Krahn inequality which states

$$\lambda_1(\Omega) \ge |\Omega|^{-\frac{2}{N}} C_N^{\frac{2}{N}} (\mu_1^{\frac{N}{2}-1})^2$$

where C_N is the volume of the N-dimensional unit ball. This inequality is sharp for the unit ball, in $\mathbb{R}^3$ we have

$$\lambda_{1,2}(B_1) = \left(\frac{\pi}{R}\right)^2.$$

If $|\Omega| \geq 8|B_R|$, we have

$$|\Omega|^{-\frac{2}{3}} C_3^{\frac{2}{3}} (\mu_1^{\frac{n}{2}-1})^2 \leq |8B_R|^{-\frac{2}{3}} |B_1|^{\frac{2}{3}} (\mu_1^{\frac{1}{2}})^2 = \left(\frac{\pi}{2R}\right)^2,$$

hence our inequality is sharper in this case. For example for a cylinder tall enough.

Example 5.3.4. We consider the equation

$$\min\{-\Delta_\infty u, |\nabla u| - \lambda u\} = 0,$$

where

$$\Delta_\infty u = (\nabla u)^t D^2 u \nabla u$$

is the infinity laplacian. This equation arises when considering the limit as $p \to \infty$ in the eigenvalue problem for the p-laplacian, see [43].

In this case the principal eigenvalue is $\frac{1}{R}$, we can prove this fact in the same way as in Example 5.3.1 by considering $u(x) = R - ||x||$ and $\phi(x) = ||x||$.

Example 5.3.5. We consider Pucci's extremal operator, that is

$$M^+_{\gamma,\Gamma}(D^2 u) = \Gamma \sum_{e_i > 0} e_i + \gamma \sum_{e_i < 0} e_i,$$

where e_i are the eigenvalues of $D^2 u$.

When $u(x) = \phi(r)$ is radial, the eigenvalues are $\phi''(r)$ with multiplicity one and $\phi'(r)/r$ with multiplicity $n-1$. Since we require the function to be increasing, we have $\phi'(r)/r > 0$, let us consider the case $\phi''(r) < 0$. We obtain the equation

$$\phi'' + \Gamma(N-1)\gamma \frac{\phi'}{r} + \frac{\lambda}{\gamma}\phi = 0.$$

Again, the general solution is given by

$$v(r) = c_1 r^\alpha J_\alpha(\eta r) + c_2 r^\alpha Y_\alpha(\eta r),$$

where

$$\alpha = \frac{1 - \frac{\Gamma(N-1)}{\gamma}}{2} = \frac{\gamma - \Gamma(N-1)}{2\gamma} \quad \text{and} \quad \eta = \sqrt{\frac{\lambda}{\gamma}}.$$

We can obtain the bound as in the previous examples. Let us illustrate this with a particular case. With $\gamma = 1$, $\Gamma = 2$ in dimension 2 we have $\alpha = -1/2$ as in the end of Example 5.3.3, we obtain $\left(\frac{\pi}{2R}\right)^2 \leq \lambda_1(\Omega)$.

Chapter 6

Games for eigenvalues of the Hessian and concave/convex envelopes

6.1 Introduction

In this chapter, we study the boundary value problem

$$\begin{cases} \lambda_j(D^2u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{λ_j, g}$$

Here Ω is a domain in $\mathbb{R}^N$ and for the Hessian matrix of a function $u : \Omega \mapsto \mathbb{R}$, D^2u, we denote by

$$\lambda_1(D^2u) \leq ... \leq \lambda_N(D^2u)$$

the ordered eigenvalues. Thus our equation says that the $j-$st smaller eigenvalue of the Hessian is equal to zero inside Ω. We include here the results obtained in [25], a joint work with Julio Daniel Rossi.

The uniqueness and a comparison principle for the equation were proved in [39]. For the existence, in [39] it is assumed that the domain is smooth (at least C^2) and such that $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_{N-1}$, the main curvatures of $\partial\Omega$, verify

$$\kappa_j(x) > 0 \text{ and } \kappa_{N-j+1}(x) > 0, \qquad \forall x \in \partial\Omega. \tag{H}$$

Our main goal here is to improve the previous result and give sufficient and necessary conditions on the domain (without assuming smoothness of the boundary) so that the problem has a continuous solution for every continuous data g. Our geometric condition on the domain reads as follows: Given $y \in \partial\Omega$ we assume that for every $r > 0$ there exists $\delta > 0$ such that for every $x \in B_\delta(y) \cap \Omega$ and $S \subset \mathbb{R}^N$ a subspace of dimension j, there exists $v \in S$ of norm 1 such that

$$\{x + tv\}_{t\in\mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset. \qquad (G_j)$$

We say that Ω satisfies condition (G) if it satisfies both (G_j) and (G_{N-j+1}).

Theorem 6.1.1. *The equation (λ_j, g) has a continuous solution for every continuous data g if and only if Ω satisfies condition* (G)*.*

As part of the proof of this theorem we use the following geometric interpretation of solutions to (λ_j, g). Let H_j be the set of functions v such that

$$v \leq g \qquad \text{on } \partial\Omega,$$

and have the following property: for every S affine of dimension j and every $j-$dimensional domain $D \subset S \cap \Omega$ it holds that

$$v \leq z \qquad \text{in } D$$

where z is the concave envelope of $v|_{\partial D}$ in D. Then we have the following results:

Theorem 6.1.2. *An upper semi-continuous function v belongs to H_j if and only if it is a viscosity subsolution to (λ_j, g).*

Theorem 6.1.3. *The function*

$$u(x) = \sup_{v\in H_j} v(x).$$

is the largest viscosity solution to $\lambda_j(D^2u) = 0$, in Ω, with $u \leq g$ on $\partial\Omega$.

Notice that, for $j = N$, we have that the equation for the concave envelope of $u|_{\partial\Omega}$ in Ω is just $\lambda_N = 0$; while the equation for the convex envelope is $\lambda_1 = 0$. See [68] for the convex envelope of a boundary datum and [67] for the convex envelope of a function $f : \Omega \mapsto \mathbb{R}$. Notice that our condition (G) in these two extreme cases is just saying that the domain is strictly convex. Hence, Theorem 6.1.1 implies that for a strictly convex domain the concave or the convex envelope of a continuous datum g on its boundary is attached to g continuously. Note that the concave/convex envelope of g inside Ω is well defined for every domain (just take the infimum/supremum of concave/convex functions that are above/below g on $\partial\Omega$). The main point of Theorem 6.1.1 is the continuity up to the boundary of the concave/convex envelope of g if and only if (G) holds. Remark that Theorem 6.1.2 says that the equation $\lambda_j(D^2u) = 0$ for $1 < j < N$ is also related to concave/convex envelopes of g, but in this case we consider concave/convex functions restricted to affine subspaces. Also in this case Theorem 6.1.1

gives a necessary and sufficient condition on the domain in order to have existence of a solution that is continuous up to the boundary.

Remark that we have that u is a continuous solution to (λ_j, g) if and only if $-u$ is a solution to $(\lambda_{N-j+1}, -g)$. This fact explains why we have to include both (G_j) and (G_{N-j+1}) in condition (G).

Our original motivation to study the problem (λ_j, g) comes from game theory. Let us describe the game that we propose to approximate solutions to the equation. It is a two-player zero-sum game. Fix a domain $\Omega \subset \mathbb{R}^N$, $\varepsilon > 0$ and a final payoff function $g : \mathbb{R}^N \setminus \Omega \mapsto \mathbb{R}$. The rules of the game are the following: the game starts with a token at an initial position $x_0 \in \Omega$, one player (the one who wants to minimize the expected payoff) chooses a subspace S of dimension j and then the second player (who wants to maximize the expected payoff) chooses one unitary vector, v, in the subspace S. Then the position of the token is moved to $x \pm \epsilon v$ with equal probabilities. The game continues until the position of the token leaves the domain and at this point x_τ the first player gets $-g(x_\tau)$ and the second player $g(x_\tau)$. When the two players fix their strategies S_I (the first player chooses a $j-$dimensional subspace S at every step of the game) and S_{II} (the second player chooses a unitary vector $v \in S$ at every step of the game) we can compute the expected outcome as

$$\mathbb{E}^{x_0}_{S_I,S_{II}}[g(x_\tau)].$$

Then the values of the game for any $x_0 \in \Omega$ for the two players are defined as

$$u^\varepsilon_{\mathrm{I}}(x_0) = \inf_{S_{\mathrm{I}}} \sup_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}} \left[g(x_\tau)\right], \qquad u^\varepsilon_{\mathrm{II}}(x_0) = \sup_{S_{\mathrm{II}}} \inf_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}} \left[g(x_\tau)\right].$$

When the two values coincide we say that the game has a value.

Next, we state that this game has a value and the value verifies an equation (called the Dynamic Programming Principle (DPP) in the literature).

Theorem 6.1.4. *The game has value*

$$u^\epsilon = u^\epsilon_I = u^\epsilon_{II}$$

that verifies

$$\begin{cases} u^\epsilon(x) = \displaystyle\inf_{dim(S)=j} \sup_{v \in S, |v|=1} \left\{ \frac{1}{2} u^\epsilon(x + \epsilon v) + \frac{1}{2} u^\epsilon(x - \epsilon v) \right\} & x \in \Omega, \\ u^\epsilon(x) = g(x) & x \notin \Omega. \end{cases} \tag{DPP}$$

Our next goal is to look for the limit as $\varepsilon \to 0$. To this end we need another geometric assumption on $\partial\Omega$. Given $y \in \partial\Omega$ we assume that there exists $r > 0$ such

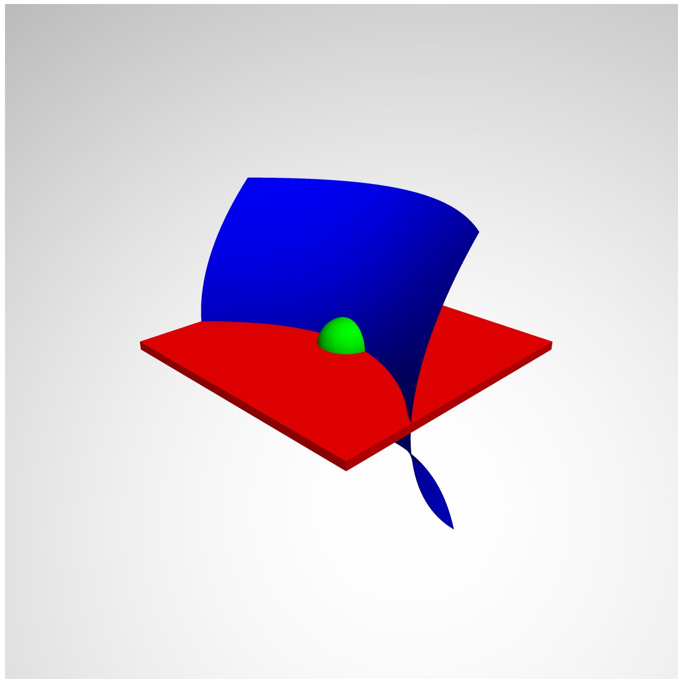


Figure 6.1: Condition (F_2) in $\mathbb{R}^3$. We have $\partial\Omega$ in blue, $B_\delta(y)$ in green and T_λ in red.

that for every $\delta > 0$ there exists $T \subset \mathbb{R}^N$ a subspace of dimension j, $v \in \mathbb{R}^N$ of norm 1, $\lambda > 0$ and $\theta > 0$ such that

$$\{x \in \Omega \cap B_r(y) \cap T_\lambda : \langle v, x - y\rangle < \theta\} \subset B_\delta(y) \tag{F_j}$$

where

$$T_\lambda = \{x \in \mathbb{R}^N : d(x, T) < \lambda\}.$$

For our game with a given j we will assume that Ω satisfies both (F_j) and (F_{N-j+1}), in this case we will say that Ω satisfy condition (F).

For example, if we consider the equation $\lambda_2 = 0$ in $\mathbb{R}^3$, we will require that the domain satisfy (F_2) as illustrated in Figure 6.1.

Theorem 6.1.5. *Assume that Ω satisfies (F) and let u^ε be the values of the game. Then,*

$$u^\varepsilon \to u, \qquad \text{as } \epsilon \to 0,$$

uniformly in $\overline{\Omega}$. Moreover, the limit u is characterized as the unique viscosity solution to

$$\begin{cases} \lambda_j(D^2u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

We regard condition (F) as a geometric way to state (H) without assuming that the boundary is smooth. In section 6.6, we discuss the relation within the different conditions on the boundary in detail, we have that

$$(\mathrm{H}) \Rightarrow (\mathrm{F}) \Rightarrow (\mathrm{G}).$$

Our results can be easily extended to cover equations of the form

$$\sum_{i=1}^{k} \alpha_i \lambda_{j_i} = 0 \tag{6.1}$$

with $\alpha_1 + ... + \alpha_k = 1$, $\alpha_i > 0$ and $\lambda_{j_1} \leq ... \leq \lambda_{j_k}$ any choice of k eigenvalues of D^2u (not necessarily consecutive ones). In fact, once we fixed indexes $j_1, ..., j_k$, we can just choose at random (with probabilities $\alpha_1, ..., \alpha_k$) which game we play at each step (between the previously described games that give λ_{j_i} in the limit). In this case the DPP reads as

$$u^\epsilon(x) = \sum_{i=1}^{k} \alpha_i \left(\inf_{dim(S)=j_i} \sup_{v\in S, |v|=1} \left\{ \frac{1}{2} u^\epsilon(x+\epsilon v) + \frac{1}{2} u^\epsilon(x-\epsilon v) \right\} \right).$$

Passing to the limit as $\epsilon \to 0$ we obtain a solution to (6.1).

In particular, we can handle equations of the form

$$P_k^+(D^2u) := \sum_{i=N-k+1}^{N} \lambda_i(D^2u) = 0, \quad \text{and} \quad P_k^-(D^2u) := \sum_{i=1}^{k} \lambda_i(D^2u) = 0,$$

or a convex combination of the previous two

$$P_{k,l,\alpha}^{\pm}(D^2u) := \alpha \sum_{i=N-k+1}^{N} \lambda_i(D^2u) + (1-\alpha) \sum_{i=1}^{l} \lambda_i(D^2u) = 0.$$

These operators appear in [19, 18, 39, 40] and in [74, 76] with connections with geometry. See also [27] for uniformly elliptic equations that involve eigenvalues of the Hessian.

Remark 6.1.6. We can interchange the roles of Player I and Player II. In fact, consider a version of the game where the player who chooses the subspace S of dimension j is the one seeking to maximize the expected payoff while the one who chooses the unitary vector wants to minimize the expected payoff. In this case the game values will converge to a solution of the equation

$$\lambda_{N-j+1}(D^2u) = 0.$$

Notice that the geometric condition on Ω, (F_j) and (F_{N-j+1}), is also well suited to deal with this case.

The chapter is organized as follows: in Section 6.2 we collect some preliminary results; in Section 6.3 we obtain the geometric interpretation of solutions to (λ_j, g) stated in Theorem 6.1.3 and Theorem 6.1.2; in Section 6.4 we prove Theorem 6.1.1; in Section 6.5 we prove our main results concerning the game, Theorem 6.1.4 and Theorem 6.1.5; and, finally, in Section 6.6 we discuss the relation between the different geometric conditions on Ω.

6.2 Preliminaries

We begin by stating the usual definition of a viscosity solution to (λ_j, g). We refer to Appendix A for the definitions of the lower semicontinuous envelope, u_*, and the upper semicontinuous envelope, u^*, of u.

Definition 6.2.1. A function $u : \Omega \mapsto \mathbb{R}$ verifies

$$\lambda_j(D^2u) = 0$$

in the viscosity sense if

1. for every $\phi \in C^2$ such that $u_* - \phi$ has a strict minimum at the point $x \in \Omega$ with $u_*(x) = \phi(x)$, we have
$$\lambda_j(D^2\phi(x)) \leq 0.$$
2. for every $\psi \in C^2$ such that $u^* - \psi$ has a strict maximum at the point $x \in \Omega$ with $u^*(x) = \psi(x)$, we have
$$\lambda_j(D^2\psi(x)) \geq 0.$$

We refer to [39] for the following existence and uniqueness result for viscosity solutions to (λ_j, g).

Theorem 6.2.2 ([39])**.** *Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Assume that condition* (H) *holds at every point on $\partial\Omega$. Then, for every $g \in C(\partial\Omega)$, the problem*

$$\begin{cases} \lambda_j(D^2u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

has a unique viscosity solution $u \in C(\overline{\Omega})$.

We remark that for the equation $\lambda_j(D^2u) = 0$ there is a comparison principle. A viscosity supersolution $\overline{u}$ (a lower semicontinuous function that verifies *(1)* in Definition 6.2.1) and viscosity subsolution $\underline{u}$ (an upper semicontinuous function that verifies *(2)* in Definition 6.2.1) that are ordered as $\underline{u} \leq \overline{u}$ on $\partial\Omega$ are also ordered as $\underline{u} \leq \overline{u}$ inside Ω. This comparison principle holds without assuming condition (H).

Condition (H) allows us to construct a barrier at every point of the boundary. This implies the continuity up to the boundary as stated above. For the reader's convenience, let us include some details on the constructions of such barriers. This calculations may help the reader to understand the interplay between the different conditions on the boundary of Ω that will be discussed in Section 6.6.

For a given point on the boundary (that we assume to be $x = 0$) we take coordinates according to x_N in the direction of the normal vector and $(x_1, ..., x_{N-1})$ in the tangent plane in such a way that the main curvatures of the boundary $\kappa_1 \leq ... \leq \kappa_{N-1}$ corresponds to the directions $(x_1, ..., x_{N-1})$. That is, locally the boundary of Ω can be described as

$$x_N = f(x_1, ..., x_{N-1})$$

with

$$f(0, ..., 0) = 0, \qquad \nabla f(0, ..., 0) = 0.$$

That is, locally we have that the boundary of Ω is given by

$$x_N - \frac{1}{2}\sum_{i=1}^{N-1} \kappa_i x_i^2 = o\left(\sum_{i=1}^{N-1} x_i^2\right),$$

and

$$\begin{aligned} \Omega \cap B_r(0) &= \Big\{(x_1, ..., x_N) \in B_r(0) \, : \, x_N - f(x_1, ..., x_{N-1}) > 0\Big\} \\ &= \left\{(x_1, ..., x_N) \in B_r(0) \, : \, x_N - \frac{1}{2}\sum_{i=1}^{N-1} \kappa_i x_i^2 > o\left(\sum_{i=1}^{N-1} x_i^2\right)\right\}. \end{aligned}$$

for some $r > 0$.

Now we take as candidate for a barrier a function of the form

$$\overline{u}(x_1, ..., x_N) = x_N - \frac{1}{2}\sum_{i=1}^{N-1} a_i x_i^2 - \frac{1}{2} b x_N^2,$$

with

$$a_i = \kappa_i - \eta \qquad \text{and} \qquad b = \kappa_{N-j+1} - \eta.$$

We have that

$$D^2(\overline{u}) = \begin{pmatrix} -a_1 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & -a_{N-1} & 0 \\ 0 & \dots & 0 & -b \end{pmatrix},$$

and then the eigenvalues of $D^2(\overline{u})$ are given by

$$\begin{aligned} \lambda_1 = -\kappa_{N-1} + \eta \leq \dots \leq \lambda_{j-1} = -\kappa_{N-j+1} + \eta = \\ \lambda_j = -\kappa_{N-j+1} + \eta \leq \dots \leq \lambda_N = -\kappa_1 + \eta. \end{aligned}$$

We asked that condition (H) holds, that implies, in particular, that

$$\kappa_{N-j+1} > 0,$$

and therefore,

$$\lambda_j(D^2\overline{u}) = -\kappa_{N-j+1} + \eta < 0$$

for $\eta > 0$ small enough.

We also have

$$\overline{u}(x_1, ..., x_N) > 0 \qquad \text{for } (x_1, ..., x_N) \in \Omega \cap B(0, r)$$

for r small enough. To see this fact we argue as follows:

$$\begin{aligned}
\overline{u}(x_1, ..., x_N) &= x_N - \frac{1}{2}\sum_{i=1}^{N-1} a_i x_i^2 - \frac{1}{2} b x_N^2 \\
&= x_N - f(x_1, ..., x_{N-1}) + f(x_1, ..., x_{N-1}) - \frac{1}{2}\sum_{i=2}^{n} a_i x_i^2 - \frac{1}{2} b x_1^2 \\
&\geq f(x_1, ..., x_{N-1}) - \frac{1}{2}\sum_{i=1}^{N-1} \kappa_i x_i^2 + \frac{\eta}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}\kappa_{N-j+1} x_N^2 \\
&\geq \frac{\eta}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}\kappa_{N-j+1} x_N^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right).
\end{aligned}$$

Since we assumed that $\kappa_{N-j+1} > 0$ we have

$$\begin{aligned}
\overline{u}(x_1, ..., x_N) &\geq \frac{\eta}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}\kappa_{N-j+1} x_N^2 + o(\sum_{i=1}^{N-1} x_i^2) \\
&\geq \frac{\eta}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}\kappa_{N-j+1}\left(\frac{1}{2}\sum_{i=1}^{N-1}\kappa_i x_i^2\right)^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right) \\
&\geq \frac{\eta}{2}\sum_{i=1}^{N} x_i^2 - C\left(\sum_{i=1}^{N-1} x_i^2\right)^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right) > 0
\end{aligned}$$

for $(x_1, ..., x_N) \in \Omega \cap B(0, r)$ with r small enough. We also have that $\overline{u}(0) = 0$ and at a point on $\partial\Omega \setminus \{0\}$

$$\begin{aligned}
\overline{u}(x_1, ..., x_N) &= x_N - \frac{1}{2}\sum_{i=1}^{N-1} a_i x_i^2 - \frac{1}{2} b x_N^2 \\
&= \frac{\eta}{2}\sum_{i=1}^{N} x_i^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right) > 0.
\end{aligned}$$

When looking for a subsolution we can do an analogous construction. In this case we will use the condition $\kappa_j > 0$.

6.3 The geometry of convex/concave envelopes and the equation $\lambda_j = 0$

Let us describe a geometric interpretation of being a solution (the largest) to the equation

$$\lambda_j(D^2u) = 0, \qquad \text{in } \Omega$$

with $u \le g$ on $\partial\Omega$.

We begin with two special cases of Theorem 6.1.3.

6.3.1 $j = 1$ and the convex envelope.

Let us start with the case $j = 1$. We let H_1 be the set of functions v such that

$$v \le g \qquad \text{on } \partial\Omega,$$

and have the following property: for every segment $D = (x_1, x_2) \subset \Omega$ it holds that

$$v \le z \qquad \text{in } D$$

where z is the linear function in D with boundary values $v|_{\partial D}$. In this case, the graph of z is just the segment that joins $(x_1, v(x_1))$ with $(x_2, v(x_2))$ and then we get

$$v(tx_1 + (1-t)x_2) \le tv(x_1) + (1-t)v(x_2) \qquad t \in (0,1).$$

That is, H_1 is the set of convex functions in Ω that are less or equal that g on $\partial\Omega$.

Now we have

Theorem 6.3.1. *Let*

$$u(x) = \sup_{v \in H_1} v(x).$$

It turns out that u is the largest viscosity solution to

$$\lambda_1(D^2u) = 0 \qquad \text{in } \Omega,$$

with $u \le g$ on $\partial\Omega$.

Notice that u is just the convex envelope of g in Ω and that this function is known to be twice differentiable almost everywhere inside Ω, [1].

6.3.2 $j = N$ and the concave envelope.

Similarly, when one deals with $j = N$, we consider

$$\lambda_N(D^2u) = 0 \qquad \text{in } \Omega,$$

with $u = g$ on $\partial\Omega$. We get that $v = -u$ is a solution to

$$\lambda_1(D^2v) = 0 \qquad \text{in } \Omega,$$

with $v = -g$ on $\partial\Omega$. Hence $v = -u$ is the convex envelope of $-g$, that is, u is the concave envelope of g.

6.3.3 $1 < j < N$ and the convcave/convex envelope in affine spaces.

Let us consider H_j the set of functions v such that

$$v \leq g \qquad \text{on } \partial\Omega,$$

and have the following property: for every S affine of dimension j and every $j-$dimensional domain $D \subset S \cap \Omega$ it holds that

$$v \leq z \qquad \text{in } D$$

where z is the concave envelope of $v|_{\partial D}$ in D. Notice that, from our previous case, $j = N$, we have that the equation for the convex envelope of g in a $j-$dimensional domain D is just $\lambda_j = 0$.

Now we proceed with the proof of Theorem 6.1.2.

Proof of Theorem 6.1.2. First, let us show that every upper semi-continuous $v \in H_j$ is a viscosity subsolution to our problem. In fact, we start mentioning that $v \leq g$ on $\partial\Omega$. Concerning the equation, let $\phi \in C^2$ such that $\phi - v$ has a strict minimum at $x_0 \in \Omega$ with $v(x_0) = \phi(x_0)$ (ϕ touches v from above at x_0) and assume, arguing by contradiction, that

$$\lambda_j(D^2\phi(x_0)) < 0.$$

Therefore, there are j orthogonal directions $v_1, ..., v_j$ such that

$$\langle D^2\phi(x)v_i, v_i\rangle < 0.$$

Notice that $\lambda_1(D^2\phi(x_0)) \leq ... \leq \lambda_j(D^2\phi(x_0)) < 0$, therefore the matrix $D^2\phi(x_0))$ has at least j negative eigenvalues. Let us call S the affine variety generated by $v_1, ..., v_j$ that passes trough x_0.

Then we have, for any vector $w \in B_\delta(x_0) \cap S$ not null (δ small)

$$v(x_0 + w) \leq \phi(x_0 + w) < \phi(x_0) + \langle \nabla\phi(x_0), w - x_0 \rangle.$$

Therefore, we obtain that

$$w \mapsto \phi(x_0) + \langle \nabla\phi(x_0), w - x_0 \rangle - \varepsilon$$

describes a function z over the ball $B_\delta(x_0) \cap S$ with $v|_{\partial B_\delta(x_0)\cap S} \leq z|_{\partial B_\delta(x_0)\cap S}$ (for ε small), such that

$$z(x_0) = \phi(x_0) - \varepsilon < \phi(x_0) = v(x_0).$$

A contradiction since $v \in H_j$ and z is linear and hence concave. This shows that every $v \in H_j$ upper semi-continuous is a subsolution.

Now we show that every upper semi-continuous subsolution v belongs to H_j. Suppose, arguing again by contradiction, that there exist S an affine space of dimension j, a $j-$dimensional domain $D \subset S \cap \Omega$ and $x_0 \in D$ such that

$$v(x_0) > z(x_0)$$

where z is the concave envelope of $v|_{\partial D}$ in D. Since z is a concave envelope, there exists a linear function L defined in D such that $L \geq v$ on ∂D and $v(x_0) > L(x_0)$. By considering $L' = L + \frac{v(x_0)-L(x_0)}{2}$ we obtain a linear function L' such that $L' > v$ on ∂D and $v(x_0) > L'(x_0)$.

Let us consider

$$D_\varepsilon = \{x \in \mathbb{R}^n : p(x) \in D \text{ and } \operatorname{dist}(x, D) < \varepsilon\}$$

where p is the orthogonal projection over S. We split the boundary of this set, ∂D_ε, into two regions, the bases and the sides. Let

$$A = \{x \in \mathbb{R}^n : p(x) \in \partial D \text{ and } \operatorname{dist}(x, D) \leq \varepsilon\}$$

and

$$B = \{x \in \mathbb{R}^n : p(x) \in D \text{ and } \operatorname{dist}(x, D) = \varepsilon\}.$$

We have $\partial D_\varepsilon = A \cup B$.

We extend L' to $\mathbb{R}^n$ by considering $L'' = L' \circ p$. Since L'' is continuous, v is upper semi-continuous and $L'' > v$ on ∂D we have that $L'' > v$ on A for ε small enough. We can consider $\phi(x) = L''(x) + K[\operatorname{dist}(x, S)]^2$. Since v is upper semi-continuous it is bounded on B, hence $\phi > v$ on B for K large enough. We have $\phi > v$ on ∂D_ε and $v(x_0) > \phi(x_0)$. We consider $\phi'(x) = \phi(x) - \delta \operatorname{dist}(x, x_0)^2$ for δ small enough such that $\phi' > v$ on ∂D_ε and $v(x_0) > \phi'(x_0)$. The function $\phi' - v$ must attain a minimum inside D_ε, which is a contradiction since v is a subsolution and the second derivatives of ϕ' are strictly negative in the j directions spanned by S. $\square$

Before we proceed with the proof of Theorem 6.1.3 we need to show the next lemma. Notice that for a function $v \in H_j$ it could happen that v^* does not satisfy $v^* \leq g$ on $\partial\Omega$, nevertheless the main condition in the definition of the set H_j still holds for v^*.

Lemma 6.3.2. *If $v \in H_j$ then for every S affine of dimension j and every $j-$dimensional domain $D \subset S \cap \Omega$ it holds that*

$$v^* \leq z \qquad \text{in } D$$

where z is the concave envelope of $v^*|_{\partial D}$ in D.

Proof. Suppose not. Then, there exist $x \in \Omega$, an affine space S of dimension j and a $j-$dimensional domain $D \subset S \cap \Omega$ such that $x \in D$ and $v^*(x) > z(x)$, where $z : \overline{D} \to \mathbb{R}$ is the concave envelope of $v^*|_{\partial D}$ in $\overline{D}$. We consider $w = z + \varepsilon$ for $\varepsilon > 0$ such that $v^*(x) > w(x)$. We have that $w(y) > v^*(y)$ for every $y \in \partial D$. We assume, without lost of generality, that $x = 0$.

We know that there exists $x_k \in \Omega$ such that $x_k \to 0$ and $v(x_k) \to v^*(0)$. We let $S_k = x_k + S$ and $D_k = (D + x_k) \cap \Omega$. Now, we consider $r > 0$ such that $B_r(0) \cap S \subset D$ and $B_{2r}(0) \subset \Omega$, if $|x_k| < r$ then $B_r(x_k) \subset D_k$. Hence, D_k is not empty for k large enough, since we have that $x_k \in D_k$.

We consider $w_k : D_k \to \mathbb{R}$ given by $w_k(x) = w(x - x_k)$. Since $v^*(0) > w(0) = w_k(x_k)$ and $v(x_k) \to v^*(0)$ we know that $v(x_k) > w_k(x_k)$ for k large enough. Since w_k is concave, $v \in H_j$ and $v(x_k) > w_k(x_k)$ there exists $y_k \in \partial D_k$ such that $v(y_k) > w_k(y_k)$. As $\partial D_k \subset \partial(D + x_k) \cup \partial\Omega$, by considering a subsequence we can assume that there exists y such that $y_k \to y$, and $y_k \in \partial(D + x_k)$ for every k or $y_k \in \partial\Omega$ for every k.

When $y_k \in \partial(D + x_k)$, we have that $y_k - x_k \in \partial D$ and hence $y \in \partial D$. Since $v(y_k) > w_k(y_k) = w(y_k - x_k)$ and w is continuous we obtain that

$$v^*(y) \geq \limsup_k v(y_k) \geq \limsup_k w(y_k - x_k) \geq w(y),$$

which is a contradiction.

Now we consider the case when $y_k \in \partial\Omega$. Since $y_k \in \overline{D}_k$, we have that $y \in \overline{D}$. If $y \in \partial D$ we can arrive to a contradiction as before. If $y \in D$ then $y \in \Omega$ which is a contradiction since $y_k \in \partial\Omega$ and $y_k \to y$. □

Now, we are ready to prove the main theorem of this section.

Proof of Theorem 6.1.3. First, let us observe that every $v \in H_j$ is a viscosity subsolution to our problem. This holds as a direct consequence of Lemma 6.3.2 and Theorem 6.1.2. Hence

$$u(x) = \sup_{v \in H_j} v(x)$$

is also a subsolution.

Now, to show that u is a supersolution we let $\phi \in C^2$ such that $\phi - u_*$ has a strict maximum at $x_0 \in \Omega$ with $u_*(x_0) = \phi(x_0)$ (ϕ touches u_* from below at x_0) and assume, arguing by contradiction, that

$$\lambda_j(D^2\phi(x_0)) > 0.$$

Therefore, all the eigenvalues $\lambda_j(D^2\phi(x_0)) \leq ... \leq \lambda_N(D^2\phi(x_0))$ of $D^2\phi(x_0)$ are strictly positive. Hence $\phi \in H_j$ in a small neighborhood of x_0 (every affine S of dimension j contains a direction v such that $\langle D^2\phi(x_0)v, v\rangle > 0$).

Now, we take (for ε small)

$$\hat{u}(x) = \max\{u(x), \phi(x) + \varepsilon\}$$

and we obtain a function $\hat{u} \in H_j$ that verifies

$$\hat{u}(z) = \max\{u(z), \phi(z) + \varepsilon\} > u(z) = \sup_{v \in H_j} v(z)$$

for some z close to x_0, a contradiction. □

Hence, for a general j we can say that the largest solution to our problem

$$\lambda_j(D^2u) = 0, \qquad \text{in } \Omega$$

with $u \leq g$ on $\partial\Omega$, is the $j-$**dimensional affine convex envelope** of g inside Ω.

Remark 6.3.3. Notice that we can look at the equation

$$\lambda_j = 0$$

from a dual perspective.

Now, we consider V_{N-j+1} the set of functions w that are greater or equal than g on $\partial\Omega$ and verify the following property, for every T affine of dimension $N-j+1$ and any domain $D \subset T$, w to be bigger or equal than z for every z a convex function in D that is less or equal than w on ∂D.

Let

$$u(x) = \inf_{w \in V_{N-j+1}} w(x).$$

Arguing as before, it turns out that u is the smallest viscosity solution to

$$\lambda_j(D^2u) = 0, \qquad \text{in } \Omega$$

with $u \geq g$ on $\partial\Omega$.

6.4 Existence of continuous solutions

In the previous section we showed existence and uniqueness of the largest/smallest viscosity solution to the PDE problem

$$\lambda_j(D^2u) = 0, \qquad \text{in } \Omega$$

with

$$u \leq g \ / \ u \geq g, \qquad \text{on } \partial\Omega.$$

Our main goal in this section is to show that under condition (G) on $\partial\Omega$ these functions coincide and then we have a solution u that is continuous up to the boundary. Uniqueness and continuity inside Ω follow from the comparison principle for the equation $\lambda_j(D^2u) = 0$ proved in [39]. In fact, for a solution that is continuous on $\partial\Omega$, we have that u^* is a subsolution and u_* is a supersolution that verify $u^* = u_* = g$ on $\partial\Omega$ and then the comparison principle gives $u^* \leq u_*$ in Ω. This fact proves that $u = u^* = u_*$ is continuous.

Let us start by pointing out that when Ω does not satisfy condition (G) then we have that (G_j) or (G_{N-j+1}) does not hold.

If Ω does not satisfy (G_j) then there exist $y \in \partial\Omega$, $r > 0$, a sequences of points $x_n \in \Omega$ such that $x_n \to y$ and S_n a sequence of affine subspaces of dimension j such that $x_n \in S_n$ and

$$S_n \cap \partial\Omega \cap B_r(y) = \emptyset.$$

Example 6.4.1. The half-ball, that is, the domain

$$\Omega = B_1(0) \cap \{x_2 > 0\}$$

in $\mathbb{R}^3$ does not satisfy (G). In fact, if we take $y = 0 \in \partial\Omega$, $r = \frac{1}{2}$, $x_n = (0, \frac{1}{n}, 0)$ and $S_n = x_n + \langle (1,0,0), (0,0,1) \rangle$ we have

$$S_n \cap \Omega \cap B_r(y) = \emptyset$$

for every n.

Now, let us show that (λ_j, g) with $j = 2$ does not have a continuous solution for a certain continuous boundary datum g. We consider g such that $g(x) \equiv 0$ for $x \in \partial B_1(0) \cap \{x_2 > 0\}$ and $g(0) = 1$. Then, from our geometric characterization of solutions to the equation $\lambda_2 = 0$ we obtain that there is no continuous solution to the Dirichlet problem in Ω with datum $u = g$ on $\partial\Omega$. In fact, if such solution exists, then it must hold that

$$u(0, a, 0) \leq 0$$

for every $a > 0$. To see this, just observe that u has to be less or equal than $z \equiv 0$ that is the concave envelope of g on the boundary of $\Omega \cap \{x_2 = a\}$. Now, as u is continuous we must have

$$0 \geq \lim_{a \searrow 0} u(0, a, 0) = u(0, 0, 0) = g(0) = 1$$

a contradiction.

With this example in mind we are ready to prove our main theorem.

Proof of Theorem 6.1.1. Our goal is to show that (λ_j, g) has a continuous solution for every boundary data g if and only if Ω satisfy (G).

Let us start by proving that the condition is necessary. We assume that Ω does not satisfies condition (G), hence (G_j) or (G_{N-j+1}) does not hold.

If Ω does not satisfy (G_j) then there exist $y \in \partial\Omega$, $r > 0$, a sequences of points $x_n \in \Omega$ such that $x_n \to y$ and S_n a sequence of affine subspaces of dimension j such that $x_n \in S_n$ and

$$S_n \cap \partial\Omega \cap B_r(y) = \emptyset.$$

We consider a continuous g such that $g(y) = 1$ and $g \equiv 0$ in $\partial\Omega \setminus B_r(y)$. We assume there exists a solution u. We have that $g \equiv 0$ in $S_n \cap \partial\Omega$ and hence $z \equiv 0$ is concave in $S_n \cap \overline{\Omega}$, we conclude that $u(x_n) \leq 0$ for every $n \in \mathbb{N}$. Since $u(y) = g(y) = 1$ we obtain that u is not continuous.

If Ω does not satisfy (G_{N-j+1}) then we consider a continuous g such that $g(y) = -1$ and $g \equiv 0$ in $\partial\Omega \setminus B_r(y)$. As before we arrive to a contradiction by considering the characterization given in Remark 6.3.3.

We have proved that condition (G) is necessary. Now, let us show that if condition (G) holds we have a continuous solution for every continuous boundary datum g. To this end, we consider the largest viscosity solution to the our PDE, $\lambda_j(D^2u) = 0$ in Ω with $u \leq g$ on $\partial\Omega$ that was constructed in the previous section.

We fix $y \in \partial\Omega$. Given $\varepsilon > 0$, we want to prove that there exists $\delta > 0$ such that $u(x) > g(y) - \varepsilon$ for every $x \in \Omega \cap B_\delta(y)$. To prove this, we will show there exists $\delta > 0$ such that for every $x \in \Omega \cap B_\delta(y)$ and for every affine space S of dimension j through x, if we consider $D = \Omega \cap S$ and the concave envelope z of $g|_{\partial D}$ in D, it holds that

$$z(x) > g(y) - \varepsilon.$$

Since g is continuous, there exists $\overline{\delta} > 0$ such that $|g(x) - g(y)| < \frac{\varepsilon}{2}$ for every $x \in \partial\Omega \cap B_{\overline{\delta}}(y)$. We consider $r \leq \overline{\delta}$ and $\delta > 0$ such that condition (G_j) is verified. Given $x \in \Omega \cap B_\delta(y)$, for every affine space S of dimension j through x there exists v of norm one, a direction in S such that

$$\{x + tv\}_{t \in \mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset. \tag{6.2}$$

We can consider the line segment $\overline{AB}$ contained in $\{x+tv\}_{t\in\mathbb{R}}$ such that $x \in \overline{AB}$, the interior of the segment is contained in Ω and $A, B \in \partial\Omega$. Due to (6.2) we can assume that $A \in B_r(y) \cap \partial\Omega$.

If $B \in B_{\overline{\delta}}(y)$, then, recalling that $A \in B_r(y) \subset B_{\overline{\delta}}(y)$, we have

$$z(x) \geq \min\{g(A), g(B)\} > g(y) - \frac{\varepsilon}{2} > g(y) - \varepsilon.$$

If $B \notin B_{\overline{\delta}}(y)$, then $\operatorname{dist}(x, B) \geq \overline{\delta} - \delta$. We have

$$\begin{aligned} z(x) &\geq \frac{g(A)\operatorname{dist}(x,B) + g(B)\operatorname{dist}(x,A)}{\operatorname{dist}(x,A)+\operatorname{dist}(x,B)} \\ &\geq g(y) + \frac{(g(A)-g(y))\operatorname{dist}(x,B) + (g(B)-g(y))\operatorname{dist}(x,A)}{\operatorname{dist}(x,A)+\operatorname{dist}(x,B)} \\ &\geq g(y) - \frac{|g(A)-g(y)|\operatorname{dist}(x,B)}{\operatorname{dist}(x,A)+\operatorname{dist}(x,B)} - \frac{|g(B)-g(y)|\operatorname{dist}(x,A)}{\operatorname{dist}(x,A)+\operatorname{dist}(x,B)} \\ &\geq g(y) - \frac{\varepsilon}{2} - 2\max|g|\frac{\operatorname{dist}(x,A)}{\operatorname{dist}(x,B)}. \end{aligned}$$

We know that $\operatorname{dist}(x, A) \leq r + \delta$. If we take $\delta \leq r$, then, for r small enough

$$z(x) \geq g(y) - \frac{\varepsilon}{2} - 2\max|g|\frac{2r}{\overline{\delta} - r} > g(y) - \varepsilon$$

as we wanted.

Analogously, taking into account that Ω verifies (G_{N-j+1}) and employing the characterization given in Remark 6.3.3, we can show that there exists $\delta > 0$ such that $u(x) < g(y) + \varepsilon$ for every $x \in \Omega \cap B_\delta(y)$. In this way we obtain that u is continuous on $\partial\Omega$ and hence in the whole $\overline{\Omega}$. $\square$

Example 6.4.2. The domain $\Omega = B_{1.4}(0,0,1) \cup B_{1.4}(0,0,-1)$ in $\mathbb{R}^3$ that can be seen in Figure 6.2 satisfy (G_2). Hence, we have that the equation $\lambda_2 = 0$ has a solution in such domain. Observe that the boundary is not smooth.

6.5 Games

In this section, we describe in detail the two-player zero-sum game that we call *a random walk for λ_j*.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and fix $\varepsilon > 0$. A token is placed at $x_0 \in \Omega$. Player I, the player seeking to minimize the final payoff, chooses a subspace S of dimension j and then Player II (who wants to maximize the expected payoff) chooses

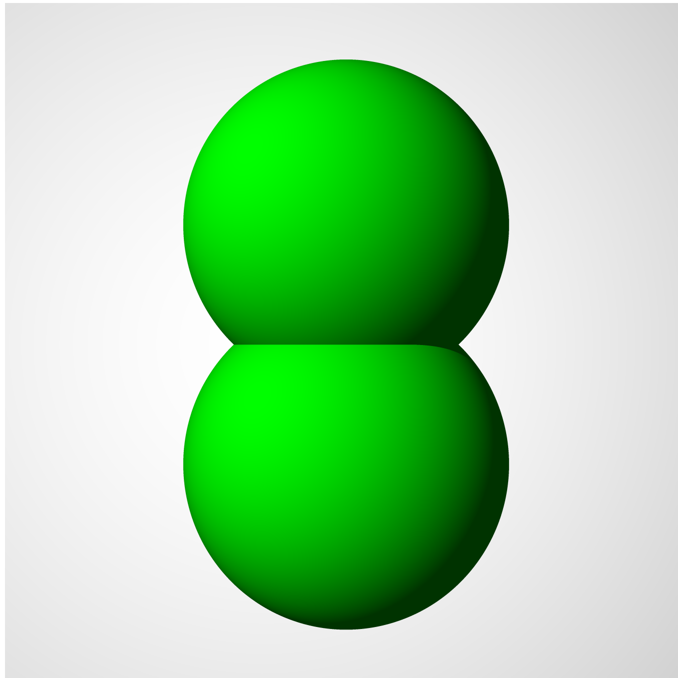


Figure 6.2: The domain $\Omega = B_{1.4}(0,0,1) \cup B_{1.4}(0,0,-1)$.

one unitary vector, v, in the subspace S. Then the position of the token is moved to $x \pm \varepsilon v$ with equal probabilities. After the first round, the game continues from x_1 according to the same rules.

This procedure yields a possibly infinite sequence of game states $x_0, x_1, \dots$ where every x_k is a random variable. The game ends when the token leaves Ω, at this point the token will be in the boundary strip of width ϵ given by

$$\Gamma_\epsilon = \{x \in \mathbb{R}^N \setminus \Omega \,:\, \mathrm{dist}(x, \partial\Omega) < \epsilon\}.$$

We denote by $x_\tau \in \Gamma_\varepsilon$ the first point in the sequence of game states that lies in Γ_ε, so that τ refers to the first time we hit Γ_ε. At this time the game ends with the final payoff given by $g(x_\tau)$, where $g : \Gamma_\varepsilon \to \mathbb{R}$ is a given continuous function that we call *payoff function*. Player I earns $-g(x_\tau)$ while Player II earns $g(x_\tau)$.

A strategy S_{I} for Player I is a function defined on the partial histories that gives a $j-$dimensional subspace S at every step of the game

$$S_{\mathrm{I}}(x_0, x_1, \dots, x_k) = S \in Gr(j, \mathbb{R}^N).$$

A strategy S_{II} for Player II is a function defined on the partial histories that gives a unitary vector in a prescribed $j-$dimensional subspace S at every step of the game

$$S_{\mathrm{II}}(x_0, x_1, \dots, x_k, S) = v \in S.$$

When the two players fix their strategies S_I (the first player chooses a subspace S at every step of the game) and S_{II} (the second player chooses a unitary vector $v \in S$ at every step of the game) we can compute the expected outcome as follows: Given the sequence $x_0, \dots, x_k$ with $x_k \in \Omega$ the next game position is distributed according to the probability

$$\begin{aligned}
&\pi_{S_{\rm I},S_{\rm II}}(x_0,\dots,x_k,A) \\
&\qquad = \frac{1}{2}\delta_{x_k+\varepsilon S_{\rm II}(x_0,\dots,x_k,S_{\rm I}(x_0,\dots,x_k))}(A) + \frac{1}{2}\delta_{x_k-\varepsilon S_{\rm II}(x_0,\dots,x_k,S_{\rm I}(x_0,\dots,x_k))}(A).
\end{aligned}$$

By using the Kolmogorov's extension theorem and the one step transition probabilities, we can build a probability measure $\mathbb{P}^{x_0}_{S_{\rm I},S_{\rm II}}$ on the game sequences. The expected payoff, when starting from x_0 and using the strategies $S_{\rm I}, S_{\rm II}$, is

$$\mathbb{E}^{x_0}_{S_{\rm I},S_{\rm II}}\left[g(x_\tau)\right] = \int_{H^\infty} g(x_\tau)\, d\mathbb{P}^{x_0}_{S_{\rm I},S_{\rm II}}. \tag{6.3}$$

The *value of the game for Player I* is given by

$$u^\varepsilon_{\rm I}(x_0) = \inf_{S_{\rm I}} \sup_{S_{\rm II}} \mathbb{E}^{x_0}_{S_{\rm I},S_{\rm II}}\left[g(x_\tau)\right]$$

while the *value of the game for Player II* is given by

$$u^\varepsilon_{\rm II}(x_0) = \sup_{S_{\rm II}} \inf_{S_{\rm I}} \mathbb{E}^{x_0}_{S_{\rm I},S_{\rm II}}\left[g(x_\tau)\right].$$

Intuitively, the values $u_{\rm I}(x_0)$ and $u_{\rm II}(x_0)$ are the best expected outcomes each player can guarantee when the game starts at x_0. If $u^\varepsilon_{\rm I} = u^\varepsilon_{\rm II}$, we say that the game has a value.

Let us observe that the game ends almost surely, then the expectation (6.3) is well defined. If we consider the square of the distance to a fix point in Γ_ε, at every step, this values increases by at least ε^2 with probability $\frac{1}{2}$. As the distance to that point is bounded with a positive probability the game ends after a finite number of steps. This implies that the game ends almost surely.

To see that the game has a value, we can consider u^ε, a function that satisfies the DPP

$$\begin{cases} u^\varepsilon(x) = \displaystyle\inf_{dim(S)=j} \sup_{v\in S, |v|=1} \left\{ \frac{1}{2}u^\varepsilon(x+\varepsilon v) + \frac{1}{2}u^\varepsilon(x-\varepsilon v) \right\} & x \in \Omega, \\ u^\varepsilon(x) = g(x) & x \not\in \Omega. \end{cases}$$

The existence of such a function can be seen by Perron's method. The operator given by the RHS of the DPP is in the hipoteses of the main result of [54].

Now, we want to prove that $u^\varepsilon = u^\varepsilon_{\rm I} = u^\varepsilon_{\rm II}$. We know that $u^\varepsilon_{\rm I} \geq u^\varepsilon_{\rm II}$, to obtain the desired result, we will show that $u^\varepsilon \geq u^\varepsilon_{\rm I}$ and $u^\varepsilon_{\rm II} \geq u^\varepsilon$.

Given $\eta > 0$ we can consider the strategy S^0_{II} for Player II that at every step almost maximize $u^\varepsilon(x_k + \varepsilon v) + u^\varepsilon(x_k - \varepsilon v)$, that is

$$S^0_{\mathrm{II}}(x_0, x_1, \dots, x_k, S) = w \in S$$

such that

$$\begin{aligned}\left\{\frac{1}{2}u^\varepsilon(x_k+\varepsilon w) + \frac{1}{2}u^\varepsilon(x_k-\varepsilon w)\right\} \geq \\ \sup_{v\in S, |v|=1}\left\{\frac{1}{2}u^\varepsilon(x_k+\varepsilon v) + \frac{1}{2}u^\varepsilon(x_k-\varepsilon v)\right\} - \eta 2^{-(k+1)}\end{aligned}$$

We have

$$\begin{aligned}&\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[u^\varepsilon(x_{k+1}) - \eta 2^{-(k+1)} |\, x_0, \dots, x_k] \\ &\quad \geq \inf_{S, dim(S)=j}\sup_{v\in S, |v|=1}\left\{\frac{1}{2}u^\varepsilon(x_k+\varepsilon v) + \frac{1}{2}u^\varepsilon(x_k-\varepsilon v)\right\} \\ &\qquad - \eta 2^{-(k+1)} - \eta 2^{-(k+1)} \\ &\quad \geq u^\varepsilon(x_k) - \eta 2^{-k},\end{aligned}$$

where we have estimated the strategy of Player I by inf and used the DPP. Thus

$$M_k = u^\varepsilon(x_k) - \eta 2^{-k}$$

is a submartingale. Now, we have

$$\begin{aligned}u^\varepsilon_{\mathrm{II}}(x_0) &= \sup_{S_{\mathrm{II}}}\inf_{S_{\mathrm{I}}}\, \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[g(x_\tau)] \\ &\geq \inf_{S_{\mathrm{I}}}\, \mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[g(x_\tau)] \\ &\geq \inf_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[M_0] = u^\varepsilon(x_0) - \eta,\end{aligned}$$

where we used the optional stopping theorem for M_k. Since η is arbitrary this proves that $u^\varepsilon_{\mathrm{II}} \geq u^\varepsilon$. An analogous strategy can be consider for Player I to prove that $u^\varepsilon \geq u^\varepsilon_{\mathrm{I}}$.

We can obtain the following result similar to Lemma 4 in [5]. Given a function u, defined in the set

$$\Omega^\varepsilon = \{x \in \Omega : \overline{B}_\varepsilon(x) \subset \Omega\},$$

we define

$$T_\varepsilon u(x) = \inf_{dim(S)=j}\sup_{w\in S, |w|=1}\left\{\frac{1}{2}u(x+\epsilon w) + \frac{1}{2}u(x-\epsilon w)\right\},$$

for $x \in \Omega^\varepsilon$.

Lemma 6.5.1. *Let u a lower semi-continuous function and v an upper semi-continuous function such that*

$$u(x) \leq T_\varepsilon u(x) \quad \textit{and} \quad v(x) \geq T_\varepsilon v(x) \tag{6.4}$$

for every $x \in \Omega^\epsilon$. Then

$$\sup_{\Omega}(u - v) = \sup_{\Omega \setminus \Omega^\varepsilon}(u - v).$$

Proof. Let us suppose, arguing by contradiction, that

$$\sup_{\Omega}(u - v) > \sup_{\Omega \setminus \Omega_\varepsilon}(u - v).$$

We define

$$E = \Big\{ x \in \Omega : (u - v)(x) = \sup_{\Omega}(u - v) \Big\}.$$

This subset of Ω^ε, E, is not empty and closed (hence compact). We consider $x_0 \in E$ the first point in lexicographic order. That is,

$$x_0 = ((x_0)_1, ..., (x_0)_N) \in E$$

minimizes the first coordinate, $(x_0)_1$, in E then the second coordinate among minimizers of the first one and so on.

Since $v(x) \geq T_\varepsilon v(x)$ and v is upper semi-continuous, there exists S of dimension j such that

$$v(x_0) \geq \sup_{w \in S, |w| = 1} \left\{ \frac{1}{2} v(x + \epsilon w) + \frac{1}{2} v(x - \epsilon w) \right\}.$$

For that S,

$$u(x_0) \leq \sup_{w \in S, |w| = 1} \left\{ \frac{1}{2} u(x + \epsilon w) + \frac{1}{2} u(x - \epsilon w) \right\}.$$

Then, since u is lower semi-continuous, there exists $w \in S, |w| = 1$ such that

$$u(x_0) \leq \frac{1}{2} u(x + \epsilon w) + \frac{1}{2} u(x - \epsilon w) \tag{6.5}$$

and for that w, we have

$$v(x_0) \geq \frac{1}{2} v(x + \epsilon w) + \frac{1}{2} v(x - \epsilon w). \tag{6.6}$$

By subtracting equation (6.5) from equation (6.6), we obtain

$$2(u(x_0) - v(x_0)) \leq u(x + \epsilon w) - v(x + \epsilon w) + u(x - \epsilon w) - v(x - \epsilon w).$$

Since $u(x+\epsilon w) - v(x+\epsilon w)$ and $u(x-\epsilon w) - v(x-\epsilon w)$ are less or equal than $u(x_0) - v(x_0)$, equality must hold. Hence, $x + \epsilon w$ and $x - \epsilon w$ belong to E. This contradicts the choice of x_0 as the first point in E in lexicographic order. $\square$

Remark 6.5.2. The same result can be obtained for

$$\widetilde{T}_\varepsilon u(x) = \max_{dim(S)=N-j+1} \min_{w\in S,|w|=1} \left\{ \frac{1}{2}u(x+\epsilon w) + \frac{1}{2}u(x-\epsilon w) \right\}.$$

Now, if u is a subsolution to $\lambda_j = 0$ then we have that for every S affine of dimension j and every $j-$dimensional domain $D \subset S \cap \Omega$ it holds that

$$u \leq z \qquad \text{in } D$$

where z is the concave envelope of $u|_{\partial D}$ in D. Hence

$$u(x) \leq T_\varepsilon u(x).$$

In the same way we can prove that for a supersolution to $\lambda_j = 0$, v, we have

$$v(x) \geq \widetilde{T}_\varepsilon v(x).$$

But, we can not obtain $v(x) \geq T_\varepsilon v(x)$ or $u(x) \leq \widetilde{T}_\varepsilon u(x)$. Any of these two inequalities would allow us to obtain a comparison result for viscosity solutions to the equation $\lambda_j = 0$ with a proof similar to that in [5].

Now our aim is to pass to the limit in the values of the game

$$u^\varepsilon \to u, \qquad \text{as } \varepsilon \to 0$$

and obtain in this limit process a viscosity solution to (λ_j, g).

To obtain a convergent subsequence $u^\varepsilon \to u$ we will use the following Arzela-Ascoli type lemma. For its proof see Lemma 4.2 from [63].

Lemma 6.5.3. *Let $\{u^\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ be a set of functions such that*

1. *there exists $C > 0$ such that $|u^\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*
2. *given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x - y| < r_0$ it holds*

$$|u^\varepsilon(x) - u^\varepsilon(y)| < \eta.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by $\{u^\varepsilon\}$ such that

$$u^\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega},$$

as $\varepsilon \to 0$.

So our task now is to show that the family u^ε satisfies the hypotheses of the previous lemma.

Lemma 6.5.4. *There exists $C>0$ independent of ε such that*

$$|u^\varepsilon(x)| < C$$

for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$.

Proof. We just observe that

$$\min g \leq u^\varepsilon(x) \leq \max g$$

for every $x \in \overline{\Omega}$. □

To prove that u^ε satisfies second hypothesis we will have to make some geometric assumptions on the domain. For our game with a given j we will assume that Ω satisfies both (F_j) and (F_{N-j+1}).

Let us observe that for $j=1$ we assume (F_N), this condition can be read as follows. Given $y \in \partial\Omega$ we assume that there exists $r>0$ such that for every $\delta > 0$ there exists $v \in \mathbb{R}^N$ of norm 1 and $\theta > 0$ such that

$$\{x \in \Omega \cap B_r(y) : \langle v, x-y \rangle < \theta\} \subset B_\delta(y). \tag{6.7}$$

Lemma 6.5.5. *Given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x-y| < r_0$ it holds*

$$|u^\varepsilon(x) - u^\varepsilon(y)| < \eta.$$

Proof. The case $x, y \in \Gamma_\varepsilon$ follows from the uniformity continuity of g in Γ_ε. For the case $x, y \in \Omega$ we argue as follows. We fix the strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$ for the game starting at x. We define a virtual game starting at y. We use the same random steps as the game starting at x. Furthermore, the players adopt their strategies $S_{\mathrm{I}}^v, S_{\mathrm{II}}^v$ from the game starting at x, that is, when the game position is y_k a player make the choices that would have taken at x_k in the game starting at x. We proceed in this way until for the first time $x_k \in \Gamma_\varepsilon$ or $y_k \in \Gamma_\varepsilon$. At that point we have $|x_k - y_k| = |x-y|$, and the desired estimate follow from the one for $x_k \in \Omega$, $y_k \in \Gamma_\varepsilon$ or for $x_k, y_k \in \Gamma_\varepsilon$.

Thus, we can concentrate on the case $x \in \Omega$ and $y \in \Gamma_\varepsilon$. Even more, we can assume that $y \in \partial\Omega$. If we have the bound for those points we can obtain a bound for a point $y \in \Gamma_\varepsilon$ just by considering $z \in \partial\Omega$ in the line segment between x and y.

In this case we have

$$u_\varepsilon(y) = g(y),$$

and we need to obtain a bound for $u_\varepsilon(x)$.

First, we deal with $j = 1$. To this end we just observe that, for any possible strategy of the players (that is, for any possible choice of the direction v at every point) we have that the projection of x_n in the direction of the a fixed vector w of norm 1,

$$\langle x_n - y, w \rangle$$

is a martingale. We fix $r > 0$ and consider x_τ, the first time x leaves Ω or $B_r(y)$. Hence

$$\mathbb{E}\,\langle x_\tau - y, w \rangle \leq \langle x - y, w \rangle \leq d(x, y) < r_0.$$

From the geometric assumption on Ω, we have that $\langle x_n - y, w \rangle \geq -\varepsilon$. Therefore

$$\mathbb{P}\left(\langle x_\tau - y, w \rangle > r_0^{1/2}\right) r_0^{1/2} - \left(1 - \mathbb{P}\left(\langle x_\tau - y, w \rangle > r_0^{1/2}\right)\right) \varepsilon < r_0.$$

Then, we have (for every ε small enough)

$$\mathbb{P}\left(\langle x_\tau - y, w \rangle > r_0^{1/2}\right) < 2r_0^{1/2}.$$

Then, (6.7) implies that given $\delta > 0$ we can conclude that

$$\mathbb{P}(d(x_\tau, y) > \delta) < 2r_0^{1/2}.$$

by taking r_0 small enough and a appropriate w.

When $d(x_\tau, y) \leq \delta$, the point x_τ is actually the point where the process have leaved Ω. Hence,

$$\begin{aligned}
&|u_\varepsilon(x) - g(y)| \\
&\leq \mathbb{P}(d(x_\tau, y) \leq \delta)|g(x_\tau) - g(y)| + \mathbb{P}(d(x_\tau, y) > \delta) 2 \max g \\
&\leq \sup_{x_\tau \in B_\delta(y)} |g(x_\tau) - g(y)| + 4r_0^{1/2} \max g < \eta
\end{aligned}$$

if r_0 and δ are small enough.

For a general j we can proceed in the same way. We have to make some extra work to argue that the points x_n that appear along the argument belong to T_λ. If $r_0 < \lambda$ we have that $x \in T_\lambda$, so if we make sure that at every move $v \in T$ we will have that the game sequence will be contained in $x + T \subset T_\lambda$.

Recall that here we are assuming both (F_j) and (F_{N-j+1}) are satisfied. We can separate the argument into two parts. We will prove on the one hand that $u_\varepsilon(x) - g(y) < \eta$ and on the other that $g(y) - u_\varepsilon(x) < \eta$. For the first inequality we can make extra assumptions on the strategy for Player I, and for the second one we can do the same with Player II.

Since Ω satisfies (F_j), Player I can make sure that at every move v belongs to T by selecting $S = T$. This proves the upper bound $u_\varepsilon(x) - g(y) < \eta$. On the other hand, since Ω satisfy (F_{N-j+1}), Player II will be able to select v in a space S of dimension j and hence he can always choose $v \in S \cap T$ since

$$\dim(T) + \dim(S) = N - j + 1 + j = N + 1 > N.$$

This shows the lower bound $g(y) - u_\varepsilon(x) < \eta$. $\square$

From Lemma 6.5.4 and Lemma 6.5.5 we have that the hypotheses of the Arzela-Ascoli type lemma, Lemma 6.5.3, are satisfied. Hence we have obtained uniform convergence of u^ε along a subsequence.

Corollary 6.5.6. *Let u^ε be the values of the game. Then, along a subsequence,*

$$u^\varepsilon \to u, \qquad \text{as } \varepsilon \to 0, \tag{6.8}$$

uniformly in $\overline{\Omega}$.

Now, let us prove that any possible limit of u^ε is a viscosity solution to the limit PDE problem.

Theorem 6.5.7. *Any uniform limit of the values of the game u^ε, u, is a viscosity solution to*

$$\begin{cases} \lambda_j(D^2u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{6.9}$$

Proof. First, we observe that since $u^\varepsilon = g$ on $\partial\Omega$ we obtain, form the uniform convergence, that $u = g$ on $\partial\Omega$. Also, notice that Lemma 6.5.3 gives that a uniform limit of u^ε is a continuous function. Hence, we avoid the use of u^* and u_* in what follows.

To check that u is a viscosity solution to $\lambda_j(D^2u) = 0$ in Ω, in the sense of Definition 6.2.1, let $\phi \in C^2$ be such that $u - \phi$ has a strict minimum at the point $x \in \Omega$ with $u(x) = \phi(x)$. We need to check that

$$\lambda_j(D^2\phi(x)) \leq 0.$$

As $u^\varepsilon \to u$ uniformly in $\overline{\Omega}$ we have the existence of a sequence x_ε such that $x_\varepsilon \to x$ as $\varepsilon \to 0$ and

$$u^\varepsilon(z) - \phi(z) \geq u^\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \varepsilon^3$$

(remark that u^ϵ is not continuous in general). As u^ε is a solution to

$$u^\epsilon(x) = \inf_{dim(S)=j} \sup_{v\in S, |v|=1} \left\{ \frac{1}{2} u^\varepsilon(x + \varepsilon v) + \frac{1}{2} u^\varepsilon(x - \varepsilon v) \right\}$$

we obtain that ϕ verifies the inequality

$$0 \geq \inf_{dim(S)=j} \sup_{v \in S, |v|=1} \left\{ \frac{1}{2}\phi(x_\varepsilon + \varepsilon v) + \frac{1}{2}\phi(x_\varepsilon - \varepsilon v) - \phi(x_\varepsilon) \right\} - \varepsilon^3.$$

Now, consider the Taylor expansion of the second order of ϕ

$$\phi(y) = \phi(x) + \nabla\phi(x) \cdot (y-x) + \frac{1}{2}\langle D^2\phi(x)(y-x), (y-x)\rangle + o(|y-x|^2)$$

as $|y-x| \to 0$. Hence, we have

$$\phi(x + \varepsilon v) = \phi(x) + \varepsilon \nabla\phi(x) \cdot v + \varepsilon^2 \frac{1}{2}\langle D^2\phi(x)v, v\rangle + o(\varepsilon^2) \tag{6.10}$$

and

$$\phi(x - \varepsilon v) = \phi(x) - \varepsilon \nabla\phi(x) \cdot v + \varepsilon^2 \frac{1}{2}\langle D^2\phi(x)v, v\rangle + o(\epsilon^2). \tag{6.11}$$

Hence, using these expansions we get

$$\frac{1}{2}\phi(x_\varepsilon + \varepsilon v) + \frac{1}{2}\phi(x_\varepsilon - \varepsilon v) - \phi(x_\varepsilon) = \frac{\varepsilon^2}{2}\langle D^2\phi(x_\varepsilon)v, v\rangle + o(\varepsilon^2),$$

and then we conclude that

$$0 \geq \varepsilon^2 \inf_{dim(S)=j} \sup_{v \in S, |v|=1} \left\{ \frac{1}{2}\langle D^2\phi(x_\varepsilon)v, v\rangle \right\} + o(\varepsilon^2).$$

Dividing by ε^2 and passing to the limit as $\varepsilon \to 0$ we get

$$0 \geq \inf_{dim(S)=j} \sup_{v \in S, |v|=1} \left\{ \langle D^2\phi(x)v, v\rangle \right\},$$

that is equivalent to

$$0 \geq \lambda_j(D^2\phi(x))$$

as we wanted to show.

The reverse inequality when a smooth function ψ touches u from below can be obtained in a similar way. □

Remark 6.5.8. Since there is uniqueness of viscosity solutions to the limit problem (6.9) (uniqueness holds for every domain without any geometric restriction once we have existence of a continuous solution) we obtain that the uniform limit

$$\lim_{\varepsilon \to 0} u^\varepsilon = u$$

exists (not only along a subsequence).

6.6 Geometric conditions on $\partial\Omega$

Now, our goal is to analyze the relation between the different conditions on $\partial\Omega$. We have introduced in this Chapter three different conditions:

(H) that involve the curvatures of $\partial\Omega$ and hence requires smoothness, this condition was used in [39] to obtain existence of a continuous viscosity solution to (λ_j, g).

(F) that is given by (F_j) and (F_{N-j+1}). This condition was used to obtain convergence of the values of the game.

(G) that was proved to be equivalent to the solvability of (λ_j, g) for every continuous datum g.

We will show that

$$(\text{H}) \Rightarrow (\text{F}) \Rightarrow (\text{G}).$$

6.6.1 (H) implies (F_j)

Let us show that the condition $\kappa_{N-j+1} > 0$ in (H) implies (F_j). We consider $T = \langle x_{N-j+1}, \dots, x_N \rangle$ (note that this is a subspace of dimension j), $v = x_N$ and r as above. We want to show that for every $\delta > 0$ there exists $\lambda > 0$ and $\theta > 0$ such that

$$\{x \in \Omega \cap B_r(y) \cap T_\lambda : \langle v, x - y \rangle < \theta\} \subset B_\delta(y). \tag{6.12}$$

We have to choose λ and θ such that for x with $\|x\| > \delta$,

$$\|(x_1, \dots, x_{N-j})\| < \lambda$$

and

$$x_N - \frac{1}{2}\sum_{i=1}^{N-1} \kappa_i x_i^2 > o\left(\sum_{i=1}^{N-1} x_i^2\right),$$

it holds that

$$x_N > \theta.$$

Let us prove this fact. We have

$$\begin{aligned}
x_N &> \frac{1}{2}\sum_{i=1}^{N-1} \kappa_i x_i^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right) \\
&\geq \frac{1}{2}\sum_{i=1}^{N-j} \kappa_i x_i^2 + \frac{1}{2}\sum_{i=N-j+1}^{N-1} \kappa_i x_i^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right) \\
&\geq -C_1 \sum_{i=1}^{N-j} x_i^2 + C_2 \sum_{i=1}^{N-1} x_i^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right) \\
&\geq -C_1 \lambda^2 + C_2 \delta^2 + o\left(\sum_{i=1}^{N-1} x_i^2\right) > \theta
\end{aligned}$$

for r, λ and θ small enough (for a given δ).

6.6.2 (F) implies (G)

We proved that (F) implies existence of a continuous viscosity solution to (λ_j, g) (that was obtained as the limit of the values of the game described in Section 6.5). Notice that we have proved that (G) is equivalent to the existence of a continuous solution to (λ_j, g) for every continuous datum g. Then, we deduce that (F) implies (G).

The same argument can be used to show that (H) implies (G) directly.

6.6.3 (H) implies (G)

We use again that (G) is equivalent to the existence of a continuous solution to (λ_j, g) for every continuous datum g and that in [39] it is proved that (H) implies existence of a continuous viscosity solution to (λ_j, g) thanks to the construction of the barriers described in Section 6.2. Hence we can deduce that (H) implies (G).

Appendix A

Viscosity Solutions

A.1 Definition

In this Appendix we give a brief introduction to the theory of viscosity solutions. We base the presentation in the introductory text [46] and the classical reference [33].

Viscosity solutions were first introduced in the 1980s by Crandall and Lions [34]. The term "viscosity solutions" originate from the "vanishing viscosity method", but it is not necessarily related to this method. Viscosity solutions constitute a general theory of "weak" (i.e. non-differentiable) solutions which applies to certain fully nonlinear Partial Differential Equations (PDE) of 1st and 2nd order.

Consider the PDE

$$F(\cdot, u, Du, D^2u) = 0$$

where

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}_N \to \mathbb{R}$$

and $\mathbb{S}_N$ denotes the set of symmetric $N \times N$ matrices.

The idea behind Viscosity Solutions is to use the maximum principle in order to "pass derivatives to smooth test functions". This idea allows us to consider operators in non divergence form. We will assume that F is degenerate elliptic, that is, F satisfy

$$X \leq Y \text{ in } \mathbb{S}_N \implies F(x, r, p, X) \geq F(x, r, p, Y)$$

for all $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

Now, let us motivate the definition of viscosity solution. Suppose that $u \in C^2(\Omega)$ is a classical solution of the PDE

$$F(x, u(x), Du(x), D^2u(x)) = 0, \quad x \in \Omega.$$

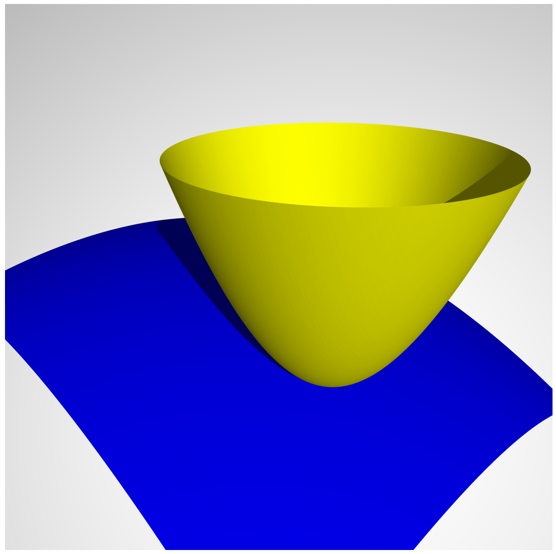


Figure A.1: ψ (in yellow) touches u (in blue) from above

Assume further that at some $x_0 \in \Omega$, u can be "touched from above" by some smooth function $\psi \in C^2(\mathbb{R}^N)$ at x_0. That is

$$\psi - u \geq 0 = (\psi - u)(x_0)$$

on a ball $B_r(x_0)$. Since $\psi - u$ attains a minimum at x_0 we have

$$D(\psi - u)(x_0) = 0 \quad \text{ and } \quad D^2(\psi - u)(x_0) \leq 0.$$

By using that u is a solution and the ellipticity of F, we obtain

$$0 = F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \geq F(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)).$$

We have proved that if u is a solution to the equation and ψ "touches from above" u then

$$0 \geq F(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)).$$

Analogously, it can be seen that if ϕ "touches from below" u then

$$0 \leq F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)).$$

Now, with this result in mind, we are ready to give the definition of viscosity solution to the equation

$$F(\cdot, u, \nabla u, D^2u) = 0. \tag{A.1}$$

Definition A.1.1. A lower semi-continuous function u is a viscosity supersolution of (A.1) if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below (that is, $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$), we have

$$F(x, \phi(x), \nabla\phi(x), D^2\phi(x)) \geq 0.$$

An upper semi-continuous function u is a subsolution of (A.1) if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above (that is, $u - \psi$ has a strict maximum at x with $u(x) = \psi(x)$), we have

$$F(x, \phi(x), \nabla\phi(x), D^2\phi(x)) \leq 0.$$

Finally, u is a viscosity solution of (A.1) if it is both a sub- and a supersolution.

Observe that we have required $u - \phi$ to have a strict minimum. We have done this since in general this is the definition that we use along the thesis. If we only require the difference to have a minimum we obtain an equivalent definition.

In general we assume that F is continuous, that is, for sequences $x_k \to x$ in Ω, $u_k \to u$ in $\mathbb{R}$, $\xi_k \to \xi$ in $\mathbb{R}^N$ and $M_k \to M$ in $\mathbb{S}_N$, we have

$$F(x_k, r_k, p_k, X_k) \to F(x, r, p, X) \qquad \text{as } k \to \infty.$$

Although, discontinuous operators arise along the thesis and we are interested in operators as the homogeneous p-laplacian and the ∞-laplacian that are not defined when the gradient vanishes. In order to be able to handle these cases, we need to consider the lower semicontinous, F_*, and upper semicontinous, F^*, envelopes of F. These functions are given by

$$\begin{aligned} F^*(x, r, p, X) &= \limsup_{(y,s,w,Y)\to(x,r,p,X)} F(y, s, w, Y), \\ F_*(x, r, p, X) &= \liminf_{(y,s,w,Y)\to(x,r,p,X)} F(y, s, w, Y). \end{aligned}$$

These functions coincide with F at every point of continuity of F and are lower and upper semicontinous respectively.

Definition A.1.2. A lower semi-continuous function u is a viscosity supersolution of (A.1) if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below (that is, $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$), we have

$$F^*(x, \phi(x), \nabla\phi(x), D^2\phi(x)) \geq 0.$$

An upper semi-continuous function u is a subsolution of (A.1) if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above (that is, $u - \psi$ has a strict maximum at x with $u(x) = \psi(x)$), we have

$$F_*(x, \phi(x), \nabla\phi(x), D^2\phi(x)) \leq 0.$$

Finally, u is a viscosity solution of (A.1) if it is both a sub- and supersolution.

Here we have required supersolutions to be lower semi-continuous and subsolutions to be upper semi-continuous. To extend this concept we consider the lower semicontinuous envelope, u_*, and the upper semicontinuous envelope, u^*, of u, that is,

$$u_*(x) = \sup_{r>0} \inf_{y \in B_r(x)} u(y) \quad \text{and} \quad u^*(x) = \inf_{r>0} \sup_{y \in B_r(x)} u(y).$$

As stated before for F, these functions coincide with u at every point of continuity of u and are lower and upper semicontinous respectively. Now we give the more general definition of viscosity solution involving these functions.

Definition A.1.3. A function u is a viscosity supersolution of (A.1) if for every $\phi \in C^2$ such that ϕ touches u_* at $x \in \Omega$ strictly from below (that is, $u_* - \phi$ has a strict minimum at x with $u_*(x) = \phi(x)$), we have

$$F^*(x, \phi(x), \nabla\phi(x), D^2\phi(x)) \geq 0.$$

A function u is a subsolution of (A.1) if for every $\psi \in C^2$ such that ψ touches u^* at $x \in \Omega$ strictly from above (that is, $u^* - \psi$ has a strict maximum at x with $u^*(x) = \psi(x)$), we have

$$F_*(x, \phi(x), \nabla\phi(x), D^2\phi(x)) \leq 0.$$

Finally, u is a viscosity solution of (A.1) if it is both a sub- and supersolution.

The definitions given above are going to be consider depending on the context (whether we are considering a continuous F or not, if u is continuous or not know a *priori*, etc). Another possible way to state the definition of viscosity solution, that we do not include here, is to define the Super-Jets and Sub-Jets, that play the role of the derivatives of u, and give later the definition of viscosity solution referring to them.

For a bounded domain $\Omega \subset \mathbb{R}^N$, we consider the Dirichlet problem

$$\begin{cases} F(\cdot, u, Du, D^2u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

where g is a continuous boundary condition. In what remains of this Appendix we will comment on the question of existence and uniqueness of solutions for the Dirichlet problem.

A.2 Uniqueness

In this section we address the question of uniqueness of solutions of the Dirichlet problem. Uniqueness can be obtained as an immediate consequence of the comparison principle for solutions to the equation. Let us start by giving a proof of comparison for smooth viscosity solutions. We will assume that F is degenerate elliptic and satisfies

$$r < s \text{ in } \mathbb{R} \implies F(x, r, p, X) < F(x, s, p, X)$$

for all $(x, p, X) \in \Omega \times \mathbb{R}^N \times \mathbb{S}_N$.

Our goal is to show that if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a subsolution and $v \in C^2(\Omega) \cap C(\bar{\Omega}))$ is a supersolution such that $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω. Observe that if v and u are smooth, then we can use them in the definition of super and subsolution as tests functions. We obtain

$$F(x, u, \nabla u, D^2 u) \leq 0 \leq F(x, v, \nabla v, D^2 v).$$

Suppose, arguing by contradiction, that $u > v$ somewhere in Ω. Then, since $u \leq v$ on $\partial\Omega$, there exists $x_0 \in \Omega$ such that

$$(u - v)(x_0) \geq u - v,$$

on Ω.

Hence $\nabla(u - v)(x_0) = 0$ and $D^2(u - v)(x_0) \leq 0$. We have that $u(x_0) > v(x_0)$, $\nabla u(x_0) = \nabla u(x_0)$ and $D^2 u(x_0) \leq D^2 v(x_0)$. By our assumptions on F, we have

$$\begin{aligned} F(x_0, u(x_0), \nabla u(x_0), D^2 u(x_0)) &\geq F(x_0, u(x_0), \nabla u(x_0), D^2 v(x_0)) \\ &\geq F(x_0, u(x_0), \nabla v(x_0), D^2 v(x_0)) \\ &> F(x_0, v(x_0), \nabla v(x_0), D^2 v(x_0)). \end{aligned}$$

Which is a contradiction since u is a subsolution and v is a supersolution.

We can not apply this idea to only continuous solutions since we may not be able to touch the functions at the points of maxima of $u - v$. The idea to overcome this difficulty is to double the number of variables and in the place of $u - v$, to consider instead the maximization of the function of two variables

$$(x, y) \to u(x) - v(y).$$

Then we penalize the doubling of variables, in order to push the maxima to the diagonal $\{x = y\}$. The idea is to maximize the function

$$W_\alpha(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2$$

and let $\alpha \to +\infty$. We used this idea in the proof of Lemma 3.2.3.

In [46] a comparison principle for the equation $F(u, \nabla u, D^2 u) = f$ is proved under the assumptions of F and f being continuous, F degenerate elliptic and

$$F(r, p, X) \geq F(s, p, X) + \gamma(s - r)$$

for some $\gamma > 0$. Of course the result holds in grater generality. For example, let us mention the classical reference [11].

A.3 Existence

In this section we include a proof of existence via Perron's Method. We assume that F is continuous and *proper*, that is F is degenerate elliptic and satisfies

$$r \geq s \text{ in } \mathbb{R} \implies F(x, r, p, X) \geq F(x, s, p, X)$$

for all $(x, p, X) \in \Omega \times \mathbb{R}^N \times \mathbb{S}_N$. We also assume that the equation satisfies the comparison principle.

Theorem A.3.1. *If there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of the Dirichlet problem such that $\underline{u} = \overline{u} = g$ on $\partial\Omega$, then*

$$u(x) = \inf \{v(x) : v \textit{ is a supersolution and } \underline{u} \leq v \leq \overline{u}\}$$

is a solution of the Dirichlet problem.

Proof. Being the infimum of supersolutions, the function u is a supersolution. We already know that u is upper semi-continuous, as it is the infimum of upper semi-continuous functions. Let us see it is indeed a solution. Suppose not, then there exists $\phi \in C^2$ such that ϕ touches u at $x_0 \in \Omega$ strictly from above but

$$F(x_0, u(x_0), \nabla u(x_0)u, D^2u(x_0)) > 0.$$

Let us write

$$\phi(x) = \phi(x_0) + \nabla\phi(x_0) \cdot (x - x_0) + \frac{1}{2}\langle D^2\phi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2).$$

We define $\hat{\phi}(x) = \phi(x) - \delta$ for a small positive number δ. Then $\hat{\phi} < u$ in a small neighborhood of x_0, contained in the set $\{x : F(x, u, \nabla u, D^2u) > 0\}$, but $\hat{\phi} \geq u$ outside this neighborhood, if we take δ small enough.

Now we can consider $v = \min\{\hat{\phi}, u\}$. Since u is a viscosity supersolution in Ω and $\hat{\phi}$ also is a viscosity supersolution in the small neighborhood of x_0, it follows that v is a viscosity supersolution. Moreover, on $\partial\Omega$, $v = u \geq g$. This implies $v \in \{v(x) : v \text{ is a supersolution and } \underline{u} \leq v \leq \overline{u}\}$, but $v = \hat{\phi} < u$ near x_0 , which is a contradiction with the definition of u as the infimum of that set. □

Let us remark that in the same way we can prove that

$$u(x) = \max \{v(x) : v \text{ is a subsolution and } \underline{u} \leq v \leq \overline{u}\}$$

is a solution to the Dirichlet problem.

Appendix B

Probability Theory

B.1 Stochastic processes

In this appendix we include some definitions and the proof of some results that are used along the thesis. As we will not refer to the games explicitly we use a notation slightly different to the one that we used in the game context. Although the general setting will be very similar.

Let $\Omega \subset \mathbb{R}^N$ be equipped with the natural topology, and the σ-algebra $\mathcal{B}$ of the Lebesgue measurable sets. Let $x_0 \in \Omega$ be a given point. We want to define a stochastic process in Ω starting in x_0. To that end, we consider the space of all sequences

$$H^\infty = \{x_0\} \times \Omega \times \Omega \times \dots,$$

which is a product space endowed with the product topology.

Let $\{\mathcal{F}_k\}_{k=0}^\infty$ denote the filtration of σ-algebras, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ defined as follows: $\mathcal{F}_k$ is the product σ-algebra generated by cylinder sets of the form

$$\{x_0\} \times A_1 \times \dots \times A_k \times \Omega \times \Omega \dots$$

with $A_i \in \mathcal{B}(\Omega)$. For

$$\omega = (x_0, \omega_1, \dots) \in H^\infty,$$

we define the coordinate processes

$$X_k(\omega) = \omega_k, \quad X_k : H^\infty \to \mathbb{R}^n, \ k = 0, 1, \dots$$

so that X_k is an $\mathcal{F}_k$-measurable random variable. Moreover, $\mathcal{F}_\infty = \sigma(\bigcup \mathcal{F}_k)$ is the smallest σ-algebra so that all X_k are $\mathcal{F}_\infty$-measurable.

Given the sequence $x_0, \dots, x_k$ with $x_k \in \Omega$ the next position is distributed according to the probability $\pi(x_0, \dots, x_k, A)$ for $A \in \mathcal{B}(\Omega)$. By using the Kolmogorov's extension theorem and the one step transition probabilities, we can build a probability measure $\mathbb{P}$ in H^∞ relative to the σ-algebra $\mathcal{F}^\infty$. We denote by $\mathbb{E}$ the corresponding expectation with respect to $\mathbb{P}$.

Definition B.1.1. We say that $M = (M_k)_{k \geq 0}$ is a stochastic process if it is a collection of random variables such that M_k is $\mathcal{F}_k$-measurable for every $k \geq 0$.

The coordinate process defined above is a stochastic process. To define a process we have to specify the probability $\pi(x_0, \dots, x_k, A)$. In other words, given the history $(x_0, \dots, x_k)$, we have to specify how is x_{k+1} chosen. Let us give two examples to which we are going to refer to illustrate the definitions and results that we are going to introduce in the next section.

Example B.1.2. Let us consider $\Omega = \mathbb{R}^N$. Suppose that at every time a random unitary vector v is selected and then $x_{k+1} = x_k + v$ with probability $\frac{1}{2}$ or $x_{k+1} = x_k - v$ with probability $\frac{1}{2}$. Then, given a fixed $y \in \mathbb{R}^N$, we can consider $M_k = ||x_k - y||^2$. Observe that M_k depends only on $(x_0, x_1, \dots, x_k)$ and hence it is $\mathcal{F}_k$-measurable, it is a stochastic process.

Example B.1.3. Suppose that you are playing at a roulette (without the zero) in a casino, starting with $x_0 = 0$ pesos (you are allow to get credit to play). At every round you bet certain amount of money (that may depend on the result of the previous rounds). If you start the round with X_k pesos and you bet v_k, then $X_{k+1} = X_k + v_k$ with probability $\frac{1}{2}$ and $X_{k+1} = X_k - v_k$ with probability $\frac{1}{2}$. In our setting, we can consider $\Omega = \mathbb{Z}$ to model this situation.

B.2 Optional stopping theorem

Definition B.2.1. A stopping time with respect to the filtration $\{\mathcal{F}_k\}_{k=0}^\infty$ is a random variable $\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$ such that $\{\tau \leq k\} \in \mathcal{F}_k$ for all $k \in \mathbb{N}$.

In particular we will be interested in the hitting times. Suppose $\Gamma \subset \Omega$ is a given set. To denote the time when the process state reaches Γ, we define a random variable

$$\tau(\omega) = \inf\{k \geq 0 \,:\, X_k(\omega) \in \Gamma\}.$$

This random variable is a stopping time relative to the filtration $\{\mathcal{F}_k\}_{k=0}^\infty$.

In Example B.1.2, for a given $R > 0$, we can consider $\Gamma = B_R(0)^c$. Then τ refers to the first time the process leaves $B_R(0)$. In Example B.1.3, we can consider that the player leaves the casino the first time he finds himself with a profit. If $\Gamma = \mathbb{N}$, the hitting time τ is by definition that moment.

Definition B.2.2. Let $M = (M_k)_{k\geq 0}$ be a stochastic process such that $\mathbb{E}[M_k] < \infty$.

- We say that M is a submartingale if $\mathbb{E}[M_k|\mathcal{F}_{k-1}] \geq M_{k-1}$ for every $k \in \mathbb{N}$.
- We say that M is a supermartingale if $\mathbb{E}[M_k|\mathcal{F}_{k-1}] \leq M_{k-1}$ for every $k \in \mathbb{N}$.
- We say that M is a martingale if $\mathbb{E}[M_k|\mathcal{F}_{k-1}] = M_{k-1}$ for every $k \in \mathbb{N}$.

In Example B.1.2, we have $M_k \leq k^2$ and hence $\mathbb{E}[M_k] < \infty$. Since

$$\begin{aligned}\mathbb{E}[M_{k+1}|\mathcal{F}_k] &= \frac{||x_k + v - y||^2 + ||x_k - v - y||^2}{2} \\ &= ||x_k - x_0||^2 + ||v||^2 \\ &= ||x_k - x_0||^2 + 1 \\ &\geq ||x_k - x_0||^2 = M_k,\end{aligned}$$

M_k is a supermartingale.

In Example B.1.3,

$$\mathbb{E}[X_{k+1}|\mathcal{F}_k] = \frac{X_k + v_k}{2} + \frac{X_k - v_k}{2} = X_k.$$

If the strategy used by the player guaranties $\mathbb{E}[X_k] < \infty$, X_k is a martingale.

Theorem B.2.3. *(Optional stopping theorem) Let $M = (M_k)_{k\geq 0}$ be a supermartingale and τ a stopping time. Suppose there exists a constant c such that $|M_{\tau\wedge k}| \leq c$ almost surely for every $k \geq 0$ where $\wedge$ denotes the minimum operator. Then,*

$$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_0].$$

Analogously, if M is a submartingale it holds that $\mathbb{E}[M_\tau] \geq \mathbb{E}[M_0]$. And hence the equality holds for M a martingale.

There are different versions of the theorem. The hypothesis of the uniform bound for the variables $|X_{\tau\wedge k}| \leq c$ can be substituted by: $\tau \leq c$ almost surely, or by $\mathbb{E}[\tau] < \infty$ and $\mathbb{E}[M_{k+1} - M_k|\mathcal{F}_k] \leq c$.

In Example B.1.3, suppose that the player use the *martingale betting system*. That is, he bets 1 peso the first round, 2 the second round, 4 the third round, etc. In the n-th round he bets 2^{n-1} pesos until he wins for the first time. Observe that at that moment he will have

$$-1 - 2 - \cdots - 2^{n-2} + 2^{n-1} = 1$$

pesos. We have, $X_0 = 0$ and $X_\tau = 1$, hence the *optional stopping theorem* does not hold in this case. Observe that x_k is not bounded (from below). At every round with probability one half the player will won and hence stop. Then, $\mathbb{P}(\tau = k) = 2^{-k}$, and so τ is not bounded almost surely. On the other hand, we have

$$\mathbb{E}[\tau] = \sum_{k=1}^{\infty} k2^{-k} = 2.$$

We have $\mathbb{E}[\tau] < \infty$ but $\mathbb{E}[X_{k+1} - X_k|\mathcal{F}_k]$ is not bounded. As we can see none of the possible sets of hypothesis for the validity of the optional stopping theorem is fulfilled.

In Example B.1.2, suppose that $x_0 = 0$. We consider $N_k = ||x_k||^2 - k$. With a similar computation as the one done before we can show that N_k is a martingale. We consider $\Gamma = B_R(0)^c$ and the corresponding hitting time τ. If we apply the optional stopping theorem we obtain that

$$\mathbb{E}[N_\tau] = N_0 = ||x_0||^2 - 0 = 0$$

Since at every step the process makes a jump of distance 1 and before living $x_{\tau-1} \in B_R(0)$, we have $||x_\tau|| \leq R + 1$. Hence $\mathbb{E}[||x_\tau||^2] \leq (R+1)^2$. Since

$$\mathbb{E}[N_\tau] = \mathbb{E}[||x_k||^2 - k] = 0,$$

we obtain

$$\mathbb{E}[\tau] = \mathbb{E}[||x_\tau||^2] \leq (R+1)^2.$$

That is, we have proved that the expected time for the process to exit the ball of radius R is bounded by $(R+1)^2$.

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