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Teoremas de dualidad para C^*-álgebras, álgebra de multiplicadores en los contextos de dualidad, teorema de extensión de Tietze y resultados relacionados

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RESUMEN

En esta tesis estudiamos, entre otras cosas, dos teoremas de dualidad para C^*-álgebras, en espíritu versiones no conmutativas de la dualidad de Gelfand. El primero de ellos, el teorema de Takesaki-Bichteler ([34], [6]), afirma que una C^*-álgebra A se puede expresar como el álgebra de "campos continuos" sobre el espacio de representaciones de A.

La totalidad de los "campos" forma el álgebra de von Neumann universal de A. Desarrollamos este hecho con un enfoque categórico, que nos permite clarificar algunos aspectos del concepto de campo. En este sentido, aportamos los siguientes dos resultados: proposición 4.3 y proposición 4.8/corolario 4.9. El mismo concepto de campo permite construir el álgebra de von Neumann universal de un grupo topológico arbitrario (generalizando así la construcción de [13]). Analizamos esta construcción y probamos que se obtiene un funtor adjunto a izquierda del funtor "grupo de unitarios". En cuanto a la dualidad de Takesaki, probamos un teorema (5.9) que es más fuerte que el teorema de dualidad. Por otra parte, hallamos una descripción de los distintos tipos de multiplicadores de A en el contexto de esta dualidad.

En el segundo capítulo, desarrollamos de manera autocontenida la teoría necesaria para probar el siguiente enunciado original (teorema de extensión de Tietze para C^*-álgebras) "el espectro $\widehat{A}$ de una C^*-álgebra A es normal si y sólo si para todo cociente $A \xrightarrow{f} B$ el morfismo inducido entre los centros de las álgebras de multiplicadores $ZM(A) \xrightarrow{\tilde{f}} ZM(B)$ es suryectivo". Para esto requerimos la teoría básica sobre el espectro de las C^*-álgebras, el teorema de Dauns-Hofmann y otros resultados.

En el tercer capítulo nos ocupamos de la teoría de C^*-fibrados necesaria para demostrar un teorema de dualidad (teorema 13.8; [26] teorema 2, generaliza teoremas anteriores similares, por ejemplo en [14], [36], [8]) que permite expresar una C^*-álgebra como secciones continuas que se anulan en el infinito, $\Gamma_0(p)$, de un C^*-fibrado $E \xrightarrow{p} X$, cuyo espacio base es la hausdorffización del espectro y cuyas fibras son cocientes del álgebra original. En este contexto demostramos que el álgebra de multiplicadores de $\Gamma_0(p)$ es igual al álgebra de secciones continuas acotadas de un fibrado asociado, que se obtiene tomando el álgebra de multiplicadores en cada fibra.

Palabras claves: álgebras de operadores - dualidad de Takesaki-Bichteler - álgebra de multiplicadores - teorema de extensión de Tietze - C^*-fibrados.

Duality theorems for C^*-algebras, multiplier algebra in duality contexts, Tietze extension theorem and related results

ABSTRACT

In this thesis we study, among other things, two different known duality theorems for C^*-álgebras in the spirit of noncommutative Gelfand duality. The first of them, due to Takesaki and Bichteler ([34], [6]), asserts that a C^*-algebra A is equal to the algebra of "continuous fields" over the representation space of A.

The set of "fields" form the universal von Neumann algebra of A. We develop this fact from a categorical point of view that is useful to clarify some aspects on the concept of field. Namely, we provide the following two results: proposition 4.3 and proposition 4.8/corollary 4.9. The same concept of field allows the construction of the universal von Neumann algebra of an arbitrary topological group (thus generalizing the construction from [13]). We analyse this construction and prove that it gives a functor that is left adjoint to the "unitary group" functor. Regarding Takesaki duality, we prove an interesting theorem (5.9) that is stronger than the duality theorem. Moreover, we give a description of the different types of multipliers of A in the context of this duality.

In the second chapter, we develop in a self-contained manner the necessary theory to prove the following original statement: (Tietze extension theorem for C^*-algebras) "the spectrum $\widehat{A}$ of a C^*-algebra A is normal if and only if for every quotient $A \xrightarrow{f} B$ the induced morphism between the centers of the multiplier algebras $ZM(A) \xrightarrow{\tilde{f}} ZM(B)$ is surjective". We require the basic theory on spectra of C^*-álgebras, the Dauns-Hofmann theorem and other results.

In the third chapter, we deal with the theory of C^*-bundles required to prove a duality theorem (theorem 13.8; [26] theorem 2, generalizing previous similar theorems, for example in [14], [36], [8]) that allows to represent any C^*-algebra as the continuous sections vanishing at infinity, $\Gamma_0(p)$, of a C^*-bundle $E \xrightarrow{p} X$, whose base space is the hausdorffization of the spectrum and whose fibers are quotients of the original algebra. In this context, we show that the multiplier algebra of $\Gamma_0(p)$ is equal to the algebra of bounded sections of an associated bundle, obtained by taking the multiplier algebra on each fiber.

Keywords: operator algebras - Takesaki-Bichteler duality - multiplier algebra - Tietze extension theorem - C^*-bundles.

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INTRODUCCIÓN

Las C^*-álgebras son normalmente consideradas una versión no conmutativa de las álgebras $C_0(X)$, donde X es un espacio localmente compacto y Hausdorff. Es, por lo tanto, un tema muy importante dentro de la teoría de las C^*-algebras la búsqueda de una buena versión no conmutativa del teorema de dualidad de Gelfand, aquel que afirma que toda C^*-álgebra conmutativa es isomorfa a $C_0(X)$, siendo X el espacio de caracteres del álgebra. Takesaki [34] dio un paso en esta dirección al probar que toda C^*-álgebra separable A es igual al conjunto de campos continuous sobre el espacio de representaciones $rep(A : H)$ en un espacio de Hilbert H suficientemente grande.

Un "campo" significa una función acotada $rep(A : H) \xrightarrow{T} B(H)$ que satisface cierta condición de compatibilidad. La topología en $rep(A : H)$ está dada por la convergencia puntual respecto de la topología *wot* (o equivalentemente *sot*, σ-débil, etc.) mientras que la topología en $B(H)$ puede ser, indistintamente, la *wot*, *sot*, etc. Bichteler luego extendió el teorema para C^*-álgebras arbitrarias en [6]. En este contexto, hallamos el siguiente resultado: para una C^*-álgebra unital A y un vector de norma uno $\xi \in H$, la aplicación $rep(A : H) \to Q(A)$, $\pi \mapsto \langle \pi(-)\xi, \xi \rangle$ es un cociente topológico ($Q(A)$ es el espacio de funcionales positivas de norma menor o igual a 1). La dualidad de Takesaki-Bichteler es un corolario inmediato de este teorema junto con el análisis previo. Luego damos una descripción de los multiplicadores de A como campos que satisfacen cierta noción de continuidad (teorema 5.15).

La razón por la cual la dualidad de Takesaki-Bichteler no es exactamente una versión no conmutativa de la dualidad de Gelfand, es que utiliza todas las representaciones en lugar de sólo las irreducibles. Por lo tanto, queremos mencionar una secuela de este teorema que utiliza efectivamente sólo las representaciones irreducibles: Fujimoto [15] teorema 2.3, A es recuperada como el espacio de campos uniformemente continuos sobre $Irr(A : H) \cup \{0\}$.

Hacemos otras contribuciones a la base conceptual de la dualidad de Takesaki al estudiar la noción de campo en términos categóricos. Un campo, según este punto de vista, es una función que asigna a cada representación no degenerada π de A un operador $T \in B(H_\pi)$ de manera tal que $\{||T(\pi)||\}$ es acotado y T es compatible con morfismos de representaciones. Este enfoque nos permite explicar dos hechos: 1) la categoría de representaciones cíclicas es suficiente para definir a los campos; esto es la proposición 4.3. 2) compatibilidad con equivalencias unitarias y sumas directas implica compatibilidad con todo morfismo de representaciones (proposición 4.8 y corolario 4.9). Este último resultado explica por qué las distintas definiciones de campo (la de Takesaki, la de Bichteler y la categórica) son equivalentes, lo cual no es sorprendente, ya que en todos los casos se prueba que los campos forman la W^*-álgebra universal de A. Mostramos que la W^*-álgebra universal da un funtor de la categoría de C^*-álgebras a la categoría de W^*-álgebras, que es adjunto a izquierda del funtor olvido.

El tratamiento categórico de las representaciones para definir este tipo de campos es usual en el contexto de la dualidad de Tannaka para grupos compactos. Resulta natural extenderlo a grupos topológicos arbitrarios para estudiar la W^*-álgebra universal de grupos topológicos (el artículo [13] de J. Ernest es la referencia histórica para el caso de grupos separables localmente compactos). Con exactamente las mismas técnicas que para el caso de C^*-álgebras, llegamos al siguiente resultado, aparentemente ausente en la literatura: la W^*-álgebra universal define un funtor de la categoría de grupos topológicos a la categoría de W^*-álgebras con morfismos unitales, adjunto a izquierda del funtor "grupo de unitarios" (en particular G y $W^*(G)$ tienen las mismas representaciones). Cuando G es localmente compacto y Hausdorff, G es un subespacio topológico de $W^*(G)$ y se aplica la dualidad de Tatsuuma, que es una generalización de la dualidad de Tannaka para grupos no necesariamente compactos. Introducimos la dualidad de Tatsuuma junto con algunos comentarios.

En el capítulo 2, damos dos resultados sobre el teorema de extensión de Tietze para C^*-algebras (11.3 y 11.10) luego de introducir la teoría necesaria. Parte de esta teoría (espectro y la versión de Glimm del teorema de Stone-Weierstrass no conmutativo) es necesaria para el capítulo 3. Ahora resumiremos los enunciados principales del capítulo 2 y sus conexiones lógicas. El primero de ellos es la proposición 7.7, debida a Kadison, sobre representaciones de C^*-álgebras unitales como funciones escalares sobre un espacio compacto: "sea A una C^*-álgebra unital y X un espacio compacto. Si tenemos una función lineal $A \to C(X)$ que preserva el orden, la unidad, la norma para elmentos positivos y separa los puntos de X, entonces, salvo homeomorfismo, vale $\overline{P(A)} \subset X \subset S(A)$ y la función está definida por $a \mapsto (\varphi \mapsto \varphi(a))$". Este resultado fue tomado de Kadison [21] (página 328, ver página 311 para una definición relevante) y levemente modificado por nosotros. La demostración utiliza un resultado (7.5) sobre extensión de estados de espacios vectoriales parcialmente ordenados ([20] también por Kadison), de modo que incluimos esta interesante pieza teórica pero considerablemente simplificada. Uno de los objetivos de la proposición 7.7 es probar que un ideal esencial de una C^*-álgebra ($I \subset A$) induce un subespacio denso $P(I) \subset P(A)$ (proposición 8.3). El caso particular $A \subset M(A)$ es relevante al estudiar una posible generalización del teorema de extensión de Tietze no conmutativo (proposición 11.3), y también al probar que $ZM(A) \simeq C_b(\widehat{A})$ a partir de del teorema de Dauns-Hofmann, lo cual es crucial para nuestro "teorema de extensión de Tietze para C^*-álgebras" 11.10. Explicaremos estos resultados más adelante.

Luego de estudiar la relación entre los espacios de estados (puros) de una C^*-álgebra A y un ideal I (proposición 8.3) y mostrar que hay una correspondencia biyectiva entre los ideales de A y los subespacios cerrados saturados de $P(A)$ (saturados respecto de la relación de equivalencia dada por

$\varphi \sim \varphi' \iff \pi_\varphi$ es unitariamente equivalente a $\pi_{\varphi'}$) estamos en condiciones de definir $\widehat{A}$, el espectro de A, como el cociente de $P(A)$ por esa relación de equivalencia. Así, $\widehat{A}$ es el conjunto de clases de representaciones irreducibles con una topología que puede ser expresada en términos de la relación de orden (dada por la inclusión) entre los núcleos. Esta topología es conocida como "hull-kernel" o topología de Jacobson cuando se la considera en el espectro de ideales primitivos $prim(A)$ (un ideal es primitivo si es el núcleo de una representación irreducible). Las topologías en $\widehat{A}$ y $prim(A)$ son esencialmente el mismo objeto, ya que $prim(A) = T_0(\widehat{A})$ (el cociente de Kolmogorov de $\widehat{A}$). Luego estudiamos algunas propiedades topológicas del espectro.

Seguidamente incluimos una sección sobre el problema de Stone-Weierstrass no conmutativo, ya que utilizaremos un resultado en la proposición 11.3, y también en en capítulo 3, teorema 13.7. El problema permanece abierto en su forma más general: "una C^*-subálgebra A de una C^*-álgebra B que separa los puntos del conjunto $P(B) \cup \{0\}$ debe ser igual a B" (incluir el cero equivale a la hipótesis "la inclusión $A \hookrightarrow B$ es propia"). Sin embargo, se sabe que el enunciado es válido si reemplazamos $P(B)$ con $\overline{P(B)}$ (ver el artículo de Glimm [16], o el libro de Dixmier [10] teorema 11.3.1, corolario 11.5.2). También existe una versión con estados factoriales ("factor states") pero no será tratada aquí (ver [22] y [30]). Combinando ideas de orígenes diversos probamos un resultado básico importante sobre este tema: bajo las hipótesis del problema general de Stone-Weierstrass no conmutativo, tenemos un homeomorfismo $P(B) \leftrightarrow P(A)$ que preserva la relación de equivalencia en ambos sentidos. Este homeomorfismo figura como comentario en [33] ("final remark") pero no conocemos ninguna otra referencia al respecto.

Nuestra investigación sobre el teorema de extensión de Tietze no conmutativo surgió con la simple observación de que la hipótesis sobre A utilizada en la versión conocida no se corresponde con la hipótesis correcta en el caso conmutativo. El teorema de extensión de Tietze no conmutativo de Pedersen ([29], teorema 10) afirma: "Sea A una C^*-algebra σ-unital. Para todo cociente $A \xrightarrow{f} B$, el morfismo inducido entre las álgebras de multiplicadores $M(A) \xrightarrow{\widetilde{f}} M(B)$ es suryectivo". Para un álgebra conmutativa $A = C_0(X)$, tenemos $M(A) = C_b(X)$, y el teorema se reduce al teorema de extensión de Tietze clásico para espacios Hausdorff σ-compactos. Como la hipótesis en la versión general es "X normal", intentamos generalizar el teorema de Pedersen en el mismo sentido. Comenzamos probando el teorema de Tietze clásico (11.1) de un modo aparentemente posible de trasladar al caso no conmutativo utilizando los espectros de las álgebras correspondientes. Esto nos condujo a la proposición 11.3: se trata del teorema de Pedersen pero reemplazando la hipótesis "σ-unital" por la siguiente hipótesis de separación para $P(A)$: "para todo par de conjuntos cerrados disjuntos de $P(A)$ las clausuras en $\overline{P(M(A))}$ son disjuntas". Para A conmutativa, la hipótesis se reduce a la normalidad del espacio, pero lamentablemente no pudimos chequear su validez en el caso separable no conmutativo. Por otra parte, luego de aprender sobre el teorema de Dauns-Hofmann, hallamos otra versión sensata del teorema de extensión de Tietze para C^*-algebras: “el espectro de una C^*-algebra es normal si y sólo si para todo cociente, el morfismo inducido entre los centros de las álgebras de multiplicadores es suryectivo”. La razón por la cual preferimos llamarlo “teorema de extensión de Tietze para C^*-algebras” en lugar de “teorema de extensión de Tietze no conmutativo” es que en definitiva se trata del teorema clásico aplicado al espectro de A e interpretado en términos de las álgebras.

En el tercer capítulo estudiamos el otro enfoque existente para la dualidad de Gelfand no conmutativa. La idea es representar fielmente toda C^*-álgebra como secciones continuas, que se anulan en el infinito, de un C^*-fibrado. Nuestros dos objetivos son: 1) proveer una exposición autocontenida de un teorema de J. Migda ([26], teorema 2 “noncommutative Gelfand-Naimark theorem”, teorema 13.8 en esta tesis) que generaliza teoremas de Fell [14], Tomiyama [36] y Dauns-Hofmann [8]. 2) Dar una descripción del álgebra de multiplicadores en este contexto de C^*-fibrados (13.11, 12.5). El teorema de Migda se reduce al teorema clásico de Gelfand cuando el álgebra es conmutativa, pero tiene una desventaja importante: el espacio base puede ser poco refinado, ya que la mejor opción es la hausdorffización de espectro, mientras que las fibras son cocientes de la C^*-álgebra original, no necesariamente primitivos. Sin embargo, como muestra de fortaleza, el teorema de Dauns-Hofmann es un corolario inmediato. A decir verdad, éste era el espíritu de la demostración original por Dauns y Hofmann ([8] corolario 8.16) aunque aquí se obtiene más fácilmente. Para desarrollar la teoría necesaria de C^*-fibrados, partimos de la definición en [11] y proveemos nuestras propias demostraciones. El teorema 12.5 (descripción del álgebra de multiplicadores de la C^*-álgebra de secciones continuas que se anulan en el infinito de un C^*-fibrado) es una modificación del teorema 3.3 en [2]. La diferencia radica en el enfoque utilizado para las álgebras de secciones, en particular la definición de “continuidad estricta” para las secciones correspondientes.

En los preliminares introducimos el álgebra de multiplicadores, los morfismos propios, conceptos básicos de la teoría de categorías y una versión categórica de la dualidad de Gelfand: la categoría de C^*-álgebras conmutativas es dual a la categoría de espacios compactos y Hausdorff punteados.

Contents

1. Introduction

C^*-algebras are usually regarded as a noncommutative analogue of the algebras $C_0(X)$ where X is a locally compact Hausdorff space. Therefore, it is a very important topic in the theory of C^*-algebras the quest for a good noncommutative version of Gelfand duality theorem, which states that every commutative C^*-algebra is isomorphic to $C_0(X)$, being X the character space of the algebra. One step in that direction was given by Takesaki [34] who showed that a separable C^*-algebra A can be recovered as the set of continuous fields over the space of representations $rep(A:H)$ on a big enough Hilbert space H.

A field here means a bounded function $rep(A:H) \xrightarrow{T} B(H)$ satisfying certain compatibility condition. The topology in $rep(A:H)$ is given by pointwise *wot* convergence (or equivalently *sot*, σ-weak, etc.) while the topology in $B(H)$ can be taken to be the *wot*, *sot*, etc. Bichteler extended the theorem for arbitrary C^*-algebras in [6]. We provide a new proof for this theorem based on a result by us: for unital A and any unit vector $\xi \in H$, we have a topological quotient $rep(A:H) \to Q(A)$ defined by $\pi \mapsto \langle \pi(-)\xi, \xi \rangle$ ($Q(A)$ is the space of positive linear functionals of norm less or equal to 1). From this result and previous analysis, Takesaki-Bichteler duality follows as a corollary. In this duality context, we provide a description of the multipliers of A as fields satisfying certain notion of continuity (theorem 5.15).

The reason why Takesaki-Bichteler duality is not exactly a noncommutative version of Gelfand duality is that it makes use of all representations instead of just the irreducible ones. Thus, we want to mention a sequel of this theorem using only irreducible representations: Fujimoto's [15] theorem 2.3 (A is recovered as the space of uniformly continuous fields over $Irr(A:H) \cup \{0\}$).

We make other contributions to the conceptual basis of Takesaki duality when we study the notion of field starting from categorical terms. A field, in this setting, is defined as a function that assigns to each nondegenerate representation π of A an operator $T \in B(H_\pi)$ in such a way that $\{||T(\pi)||\}$ is bounded and T is compatible with intertwiners. This point of view allows us to explain two facts: 1) the category of cyclic representations is enough to define fields; this is proposition 4.3. 2) compatibility with unitary intertwiners and direct sums implies compatibility with every intertwiner (proposition 4.8 and corollary 4.9). This last result explains why the different definitions of field (Takesaki's, Bichteler's and categorical) give one and the same object. It is no surprise, since in all cases it is proven that the fields form the universal W^*-algebra of A. We show that the universal W^*-algebra gives a functor from the category of C^*-algebras to the category of W^*-algebras, that is left adjoint to the forgetful functor.

The categorical treatment of representations in order to define these kind of fields is actually mainstream in the context of Tannaka duality for compact groups. It is natural to extend it to arbitrary topological groups to study the universal W^*-algebra of topological groups (the article [13] by J. Ernest is the historical reference for locally compact separable groups). With the exact same techniques as for the C^*-algebra case, we reach the following nice result that is apparently absent in the literature: the universal W^*-algebra defines a functor from the category of topological groups to the category of W^*-algebras with unital morphisms, left adjoint to the "unitary group" functor (in particular G and $W^*(G)$ have the same representations). When G is locally compact and Hausdorff, G is topologically embedded in $W^*(G)$ and Tatsuuma duality theorem applies (it reduces to Tannaka's theorem for compact G). We introduce Tatsuuma's duality and make some comments.

In chapter 2, we give two results by us about Tietze extension theorem for C^*-algebras (11.3 and 11.10) after introducing all the necessary theory. Part of this theory (spectrum and Glimm's Stone-Weierstrass theorem) is also needed in chapter 3. We now summarize the main statements in chapter 2 and their logical dependence. The first of them is proposition 7.7, due to Kadison, about representations of unital C^*-algebras as functions on a compact space: "let A be a unital C^*-algebra and X a compact space. If we have a linear map $A \to C(X)$ that is unit and order preserving, separates the points of X and is norm preserving for positive elements, then, up to a homeomorphism, we have $\overline{P(A)} \subset X \subset S(A)$ and the morphism is defined by $a \mapsto (\varphi \mapsto \varphi(a))$". This result was taken from Kadison's [21] (page 328, and see page 311 for a relevant definition) and slightly changed by us. Its proof uses a result (7.5) on extension of states on partially ordered vector spaces ([20] also by Kadison), so we include this nice piece of theory about partially ordered vector spaces but considerably simplified. One of the aims of proposition 7.7 is to prove that an essential ideal of a C^*-algebra ($I \subset A$) induces a dense topological embedding $P(I) \subset P(A)$ (proposition 8.3). The particular case $A \subset M(A)$ plays a role when studying a possible generalization of the noncommutative Tietze extension theorem (proposition 11.3), and also when proving $ZM(A) \simeq C_b(\widehat{A})$ from Dauns-Hofmann theorem, which is an important piece for our "Tietze extension theorem for C^*-algebras" 11.10. We will explain these results later.

After studying the relation between the (pure) state spaces of a C^*-algebra A and an ideal I (proposition 8.3) and showing that there is a bijective correspondence between ideals of A and closed saturated subsets of $P(A)$ (saturated with respect to the equivalence relation given by $\varphi \sim \varphi' \iff \pi_\varphi$ is unitarily equivalent to $\pi_{\varphi'}$) we are in a good position to define $\widehat{A}$, the spectrum of A, as the quotient of $P(A)$ by that equivalence relation. Thus, $\widehat{A}$ is the set of classes of irreducible representations with a topology that can be expressed directly in terms of the inclusion order relation between its kernels. This is called hull-kernel or Jacobson topology when carried to the primitive ideal spectrum $prim(A)$ (a primitive ideal is the kernel of an irreducible representation). The topologies on $\widehat{A}$ and $prim(A)$ are essentially the same object, since $prim(A) = T_0(\widehat{A})$ (the Kolmogorov quotient of $\widehat{A}$). We then study further properties of these spectra.

Then we include a section on the very interesting noncommutative Stone-Weierstrass problem because we use it in proposition 11.3, and also in chapter 3, theorem 13.7. The most general form of the problem remains open: "a C^*-subalgebra A of a C^*-algebra B that separates the set $P(B) \cup \{0\}$ must be equal to B" (including the zero in the set is equivalent to the hypothesis "the inclusion $A \hookrightarrow B$ is proper"). However the statement is known to be true replacing $P(B)$ with $\overline{P(B)}$ (see Glimm's article [16], or Dixmier's book [10] theorem 11.3.1, corollary 11.5.2). There also exists a version with factor states but we won't treat it (see [22] and [30]). Combining ideas from different places we prove an important basic proposition on this subject: under the hypothesis of the general Stone-Weierstrass conjecture, we have a homeomorphism $P(B) \leftrightarrow P(A)$ preserving the equivalence relation. This homeomorphism is mentioned in [33] (final remark) but we don't know any other precise reference.

Our investigation on noncommutative Tietze extension theorem started from the simple observation that the hypothesis on A used in the known version doesn't correspond to the correct hypothesis in the commutative case. Pedersen's noncommutative Tietze extension theorem ([29], theorem 10) asserts: "Let A be a σ-unital C^*-algebra. For every quotient $A \xrightarrow{f} B$ the induced morphism between the multiplier algebras $M(A) \xrightarrow{\widetilde{f}} M(B)$ is surjective". For commutative $A = C_0(X)$, we have $M(A) = C_b(X)$, and the theorem reduces to Tietze extension theorem for σ-compact Hausdorff spaces. Since the hypothesis in the general version is "X normal" we tried to generalize Pedersen's theorem in the same direction. We started by proving classical Tietze extension theorem (11.1) in a way that seems possible to carry to the noncommutative case using the spectra of the corresponding algebras. This led to proposition 11.3: it is Pedersen's theorem but replacing σ-unitality by the following separation hypothesis for $P(A)$: "every two disjoint closed subsets of $P(A)$ have disjoint closures in $\overline{P(M(A))}$". It reduces to normality for commutative A but unfortunately we couldn't check this hypothesis in the separable noncommutative case. On the other hand, after learning about the Dauns-Hofmann theorem, we found out another sensible version of Tietze extension theorem for C^*-algebras: "the spectrum of a C^*-algebra is normal if and only if for every quotient, the induced morphism between the centers of the multiplier algebras is surjective". The reason why we prefer to call it "Tietze extension theorem for C^*-algebras" instead of "noncommutative Tietze extension theorem" is that it is nothing more than classical Tietze extension theorem applied to the spectrum but interpreted in terms of the algebras.

In the third chapter we study the other existing approach to noncommutative Gelfand duality. The idea is to represent faithfully any C^*-algebra by the continuous sections vanishing at infinity of a C^*-bundle. Our two objectives are: 1) to give a self-contained exposition of a theorem by Migda ([26], theorem 2 "noncommutative Gelfand-Naimark theorem"; theorem 13.8 for us) that generalizes theorems by Fell [14], Tomiyama [36] and Dauns-Hofmann [8]. 2) A description of the multiplier algebra in the context of C^*-bundles (13.11, 12.5). The theorem by Migda reduces to classical Gelfand duality when the algebra is commutative, but it has a disadvantage: the base space might be rather coarse (since the most refined choice is the hausdorffization of the spectrum), while the fibers are quotients of the original C^*-algebra, not necessarily primitive. Nevertheless, as an evidence of its strength, Dauns-Hofmann theorem follows immediately. Actually, this was the spirit of the original proof by Dauns and Hofmann ([8] corollary 8.16) though here it is simpler. For the development of the necessary theory on C^*-bundles, we started from the definition in [11] and made our own proofs. Theorem 12.5 (description of the multipliers of the algebra of continuous sections vanishing at infinity of a C^*-bundle) is a modification of theorem 3.3 in [2]. The difference consists in the theoretical setting for sectional algebras, particularly the definition of "strict continuity" for the relevant sections.

In the preliminaries we introduce the multiplier algebra, proper morphisms, basic concepts of category theory and a categorical version of Gelfand duality, stating that the category of commutative C^*-algebras is dual to the category of pointed compact Hausdorff topological spaces.

2. Preliminaries

2.1. **Notation and conventions.**

A, B will always denote C^*-algebras.

$S(A)$ is the state space of A with the w^*-topology. $P(A)$ is the subspace of pure states.

$\widehat{A}$ is the space of irreducible representations modulo unitary equivalence, $prim(A)$ the primitive ideal spectrum. Both with the usual topology (see section 9).

H will be used for a Hilbert space and $B(H)$ its linear bounded endomorphisms.

X^* means the dual of the Banach space X.

For $\varphi \in A^*_{\geq 0}$, $(\pi_\varphi, H_\varphi, \xi_\varphi)$ will be the GNS triple: $A \xrightarrow{\pi_\varphi} B(H_\varphi)$ with cyclic vector ξ_φ, $||\xi_\varphi||^2 = ||\varphi||$.

$\widetilde{A}$ is the minimal unitization of A and $M(A)$ the multiplier algebra (remark: if A is unital, $\widetilde{A} = A$).

$Q(A) := \{\varphi \in A^*_{\geq 0} / ||\varphi|| \leq 1\}$, with the w^*-topology. For nonunital A, $S(\widetilde{A}) \simeq Q(A)$ by restriction (see 8.4).

<u>Convention:</u> an ideal of a C^*-algebra will mean a norm-closed two-sided ideal. Recall that these ideals are also self-adjoint.

2.2. **Multiplier algebra and proper morphisms.**

2.1. **Definition.** *An ideal I of a C^*-algebra A is called essential if for an element $a \in A$, $aI = 0$ implies $a = 0$.*

2.2. **Definition.** *A unitization of a C^*-algebra A is a unital C^*-algebra M containing A as an essential ideal.*

If A is unital, its only possible unitization is A itself.

The minimal unitization of a C^*-algebra A is usually constructed considering the isometric embedding of A into its linear bounded endomorphisms: $A \hookrightarrow \mathcal{B}(A)$ (by left multiplication) and taking the subspace generated by A and the identity element. We denote this algebra with $\widetilde{A}$. In case A is nonunital, $\widetilde{A} \simeq A \oplus \mathbb{C}$ as vector spaces. $\widetilde{A}$ is a subalgebra of any other unitization.

Recall that a representation of a C^*-algebra A on a Hilbert space H is said to be nondegenerate if $\overline{\pi(A)H} = H$, where $\pi(A)H$ denotes the generated subspace. Every representation decomposes as a direct sum of a nondegenerate representation on a closed subspace of H and the zero representation on the orthogonal complement. Besides, a representation π on H is nondegenerate if and only if $\pi(u_\lambda) \xrightarrow[\lambda]{sot} Id_H$ for every (u_λ) approximate unit, or equivalently, just for one approximate unit.

The multiplier algebra is the biggest possible unitization: it contains any other unitization as a subalgebra. It can be constructed from the set of double centralizers $\mathcal{DC}(A)$. A double centralizer for A is a pair (L, R) of functions $L, R : A \to A$ satisfying $R(x)y = xL(y)$ for all $x, y \in A$. It is proved (see [40]) that this set coincides with the idealizer of A with respect to any faithful nondegenerate representation. Explicitly: let $A \overset{j}{\hookrightarrow} B(H)$ be such a representation. The idealizer $\mathcal{I}(A)$ is the set of those $x \in B(H)$ such that $xA \subset A$ and $Ax \subset A$. Consider $L_x, R_x : A \to A$, left and right multiplication by $x \in \mathcal{I}(A)$. (L_x, R_x) is a double centralizer. The application $x \mapsto (L_x, R_x)$ turns out to be a bijection. The multiplier algebra is then, by definition: $M(A) = \mathcal{I}(A) = \mathcal{DC}(A)$.

A is an essential ideal in $\mathcal{I}(A)$: if $x \in \mathcal{I}(A)$ makes $xa = 0$ $\forall a \in A$, we can take an approximate unit of A. Since $\mathcal{I}(A)$ is the idealizer with respect to a nondegenerate representation, the approximate unit converges strongly to the identity, so $x = 0 \in B(H)$.

Maximality of $M(A)$ is an immediate consequence of the following proposition:

2.3. **Proposition.** *If A is an ideal of a C^*-algebra B, then there exists a unique morphism $B \xrightarrow{\mu} M(A)$ such that the triangle commutes:*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, hook] & B \arrow[d, "\mu"] \\ & M(A) \end{tikzcd}$$

μ is injective if and only if A is essential in B.

Proof. Existence follows defining $\mu(b) = (L_b, R_b)$, left and right multiplication by $b \in B$. When $b \in A$, this is the regular inclusion $A \subset M(A)$. For uniqueness, assume $B \xrightarrow{\nu} M(A)$ also makes the triangle commute.

$$\nu(b)a = \nu(ba) = \mu(ba) = \mu(b)a$$

Then $\nu(b) = \mu(b)$ because A is essential in $M(A)$.

If A is essential in B: $\mu(b) = 0$ implies $L_b(a) = ba = 0 \; \forall a \in A$. Then $b = 0$, so μ is injective.

Conversely, if μ is a monomorphism and $bA = 0$, we can do $b^*bA = 0$. Taking adjoint, $Ab^*b = 0$. $L_{b^*b} = R_{b^*b} = 0$, $\mu(b^*b) = 0$, $b^*b = 0$, $b = 0$. $\square$

Basic examples of multiplier algebras are: $M(K(H)) = B(H)$ ($K(H)$ is the algebra of compact operators on H) and $M(C_0(X)) = C_b(X)$ for a locally compact Hausdorff space X. Of course, $A = M(A)$ if and only if A is unital.

2.4. **Definition.** *A proper morphism between C^*-algebras is a morphism $A \xrightarrow{f} B$ satisfying any of the following equivalent statements.*

2.5. **Proposition.** *Let $A \xrightarrow{f} B$ be a morphism of C^*-algebras. The following are equivalent:*

1) For every (u_λ) approximate unit of A, $(f(u_\lambda))$ is an approximate unit of B.

2) There exist an approximate unit of A (u_λ) such that $(f(u_\lambda))$ is an approximate unit of B.

3) $\overline{f(A)B} = B$, where $f(A)B$ is the subspace generated by the elements $f(a)b$, $a \in A$, $b \in B$ and the closure is the norm closure.

4) For every nondegenerate representation $B \xrightarrow{\pi} B(H)$, the induced representation πf is nondegenerate.

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) follows immediately from $b = \lim_\lambda f(u_\lambda)b$.

(3) $\Rightarrow$ (1). Since f is contractive, $f(u_\lambda)f(a)b \to f(a)b$. And convergence holds taking linear combinations and norm limits, so $f(u_\lambda)b \to b \; \forall b \in B$.

(1) $\Rightarrow$ (4). For every (u_λ) approximate unit of A, $(\pi f(u_\lambda))$ is an approximate unit. Thus πf is nondegenerate.

(4) $\Rightarrow$ (1). We want to thank Leonel Robert for providing this proof. Without loss of generality we assume f injective, because f can be replaced

by ${}^{A}/_{ker(f)} \xrightarrow{\tilde{f}} B$. So $A \subset B$. Let L be the closed left ideal of B generated by A. If $L \subsetneq B$, by theorem 3.10.7 of [28], there is a state $\varphi \in S(B)$ such that $\varphi(L) = 0$, but this φ produces a GNS representation that is degenerate when restricted to A, absurd. Hence $L = B$. Now take (u_λ) an approximate unit of A. The set $C = \{b \in B / bu_\lambda \to b\}$ is a closed left ideal containing A, thus $C = B$ and (u_λ) is an approximate unit of B.

$\square$

2.6. **Remarks.**

• The name "proper" comes from the commutative case, since $C_0(X) \to C_0(Y)$ is proper iff it is induced by a proper continuous map $Y \to X$ (i.e.: preimage of a compact subspace is compact).

• If A is unital, proper is equivalent to $f(1) = 1$.

• If a representation morphism $A \xrightarrow{\pi} B(H)$ is proper, then π is nondegenerate. If A is unital, proper and nondegenerate are equivalent to $\pi(1) = 1$, but for nonunital A, π might be nondegenerate but not proper. Consider $C_0(\mathbb{R}) \overset{\pi}{\hookrightarrow} B(L^2(\mathbb{R}))$ acting by multiplication. It is nondegenerate but the image of an approximate unit doesn't converge to $1 \in B(L^2(\mathbb{R}))$ in norm, so it is not an approximate unit.

• For an ideal $I \subset A$, if the inclusion is proper then, by assertion (3) $A = \overline{I} = I$. So the inclusion is not proper for a proper ideal.

2.7. **Proposition.** *Let $A \xrightarrow{f} B$ be a proper morphism of C^*-algebras. There exists a unique extension $M(A) \xrightarrow{\widetilde{f}} M(B)$. It satisfies $\widetilde{f}(1) = 1$.*

Proof. Take (u_λ) an approximate unit in A. f proper implies that $(f(u_\lambda))$ is an approximate unit. Consider π a faithful nondegenerate representation of B. It extends uniquely to a faithful nondegenerate representation $\widetilde{\pi}$ of $M(B)$. Besides, πf extends uniquely to a nondegenerate $M(A) \xrightarrow{\widetilde{\pi f}} B(H)$. Now take $m \in M(A)$.

$$\widetilde{\pi f}(m)\pi(b) = \widetilde{\pi f}(m)\pi(\lim_\lambda f(u_\lambda)b) = \widetilde{\pi f}(m) \lim_\lambda \pi f(u_\lambda)\pi(b) =$$

$$\lim_\lambda \widetilde{\pi f}(m u_\lambda)\pi(b) = \lim_\lambda \pi f(m u_\lambda)\pi(b) \in \pi(B)$$

Thus $\widetilde{\pi f}(m)$ belongs to the idealizer of B, so $\widetilde{\pi f}(m) \in \widetilde{\pi}(M(B)) = \mathcal{I}(\pi(B))$. This allows the definition of $M(A) \xrightarrow{\widetilde{f}} M(B)$, because $\widetilde{\pi}$ is injective. Uniqueness follows from uniqueness of $\widetilde{\pi f}$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & B \arrow[d, hook] \arrow[dr, hook, "\pi"] & \\
M(A) \arrow[r, dashed, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{\pi f}"'] & M(B) \arrow[r, hook, "\widetilde{\pi}"] & B(H)
\end{tikzcd}$$

□

We now give definitions for the spaces of left multipliers, right multipliers and quasi-multipliers because they will appear in 5.15. If $A \overset{\pi}{\hookrightarrow} B(H)$ is a faithful nondegenerate representation

$$LM(A) = \{x \in B(H)/xA \subset A\}$$
$$RM(A) = \{x \in B(H)/Ax \subset A\}$$
$$QM(A) = \{x \in B(H)/AxA \subset A\}$$

It is possible to give intrinsic definitions of these spaces. For example, $QM(A) = \{A \times A \xrightarrow{q} A$ bilinear $/q(ab, cd) = aq(b, c)d\}$. This allows to prove that previous definitions don't depend on the chosen faithful representation π. Observe that $LM(A)^* = RM(A)$, $LM(A) \cap RM(A) = M(A)$ and $LM(A) + RM(A) \subset QM(A)$. Left multipliers (resp. right multipliers) form a Banach algebra having A as a closed left (resp. right) ideal, while the set of quasi-multipliers isn't in general closed under multiplication but it is a *-closed subspace.

The set of quasi-multipliers lies inside the bicommutant of A with respect to the considered faithful nondegenerate representation π: if $q \in QM(A)$, $S \in \pi(A)'$, $a, b \in \pi(A)$ then $aqSb = aqbS = Saqb = aSqb$, so $qS = Sq$.

There are several articles dealing with quasi-multipliers. In [1] Akemann and Pedersen define $QM(A)$ and prove that it is the subspace of A^{**} formed by the elements $A^* \xrightarrow{f} \mathbb{C}$ continuous in $S(A)$. This is closely related to 5.2. Another interesting reference is [25], where the author studies the quasi-multipliers for Banach algebras. He finds a sufficient condition that allows to define a product in $QM(A)$. For example, he obtains, for a locally compact Hausdorff group G, $QM(L^1(G)) = \mathcal{M}(G)$, where $\mathcal{M}(G)$ is the measure algebra.

Other articles study whether $QM(A)$ is equal to $LM(A) + RM(A)$.

2.3. **Categories.** One feature of this thesis (specifically in the first chapter and the following version of Gelfand duality) is to provide categorical frameworks for some constructions and results in operator algebras which are usually presented without these terms. Therefore, we include a brief summary of the fundamental concepts of category theory.

We start with an undetailed set-theoretic definition of category.

2.8. **Definition.** *A category is a set of objects and a set of arrows (or morphisms). Each arrow has two associated objects called domain and codomain. There is an associative composition operation between arrows:* $f \circ g$ *is defined if and only if the codomain of* g *is equal to the domain of* f*. Each object has an identity arrow, i.e. a neutral element for the composition.*

Large categories, such as "the category of sets" or "the category of C^*-algebras" can be more technically defined as the category of sets that belong to certain universe $\mathcal{U}$ or, respectively, the category of C^*-algebras that belong to $\mathcal{U}$. A universe is a set closed by every set theoretic operation between its elements. Most traditional "concrete" mathematical objects belong to the smallest universe containing an infinite countable set as an element. So, at least for us, the expression "all sets" (as in "the category of (all) sets") conventionally refers to the elements of this universe. The collection of objects of a large category form a set that belongs to a bigger universe.[1]

2.9. **Definition.** *A functor is a morphism of categories. It consists of a function between the objects and a function between the arrows, preserving composition, identities, domains and codomains.*

2.10. **Definition.** *Consider two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *and two functors between them:* $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$. *A natural transformation* $F \overset{\eta}{\to} G$ *is a map that assigns to each object* $A \in \mathcal{C}$ *an arrow* $F(A) \xrightarrow{\eta(A)} G(A)$ *in such a way that for every arrow* $A \overset{f}{\to} B$ *it holds* $\eta(B)F(f) = G(f)\eta(A)$.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta(A)"'] & F(B) \arrow[d, "\eta(B)"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

If all $\eta(A)$ *are isomorphisms (i.e.: invertible),* η *is called a natural isomorphism.*

Natural transformations are the morphisms between functors.

2.11. **Notation.**

- *If* $\mathcal{C}$ *is a category,* $A \in \mathcal{C}$ *means that* A *is an object of* $\mathcal{C}$.
- $[A, B]_{\mathcal{C}}$ *denotes the set of arrows in the category* $\mathcal{C}$ *whose domain is* A *and codomain is* B.
- *For* F, G *functors,* $F \simeq G$ *means that there is a natural isomorphism between them.*

[1] Consider the axiom "for every set there exists a universe which contains it as an element" (see [5], the appendix to *exposé I*).

2.12. **Definition.** *Given two functors* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *, F is a left adjoint to G (or G is a right adjoint to F) if there exists a natural family of bijections* $[F(C), D]_{\mathcal{D}} \simeq [C, G(D)]_{\mathcal{C}}$*. It is denoted* $F \dashv G$*.*

The word "natural" in previous definition means that the given family of bijections induce natural isomorphisms between $[F(-), D]$ and $[(-), G(D)]$ $(\forall D)$, and between $[F(C), -]$ and $[C, G(-)]$ $(\forall C)$. In other words, we have a natural isomorphism between the following functors from the product category $\mathcal{C} \times \mathcal{D}$ to the category of sets.

$$\begin{tikzcd} \mathcal{C} \times \mathcal{D} \arrow[r, bend left, "{[F(-),-]}"] \arrow[r, bend right, "{[-,G(-)]}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{S}ets \end{tikzcd}$$

The fundamental property of adjoint functors is the following:

2.13. **Proposition.** *If $F \dashv G$, F preserves all colimits and G preserves all limits.*

The notion of limit generalizes constructions such as products, categorical monomorphisms, pullbacks and inverse limits, while colimits generalize coproducts, categorical epimorphisms, pushouts and direct limits. See [24].

2.14. **Definition.** *An equivalence of categories is a pair of functors* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *such that* $FG \simeq Id_{\mathcal{D}}$ *and* $GF \simeq Id_{\mathcal{C}}$*.*

With F and G as in the definition we have $F \dashv G$ and $G \dashv F$.

Examples. The following categories will appear:

• Top, the category of all topological spaces with continuous functions as arrows.

• $Top^*_{cT_2}$ the category of pointed compact Hausdorff spaces. Arrows are continuous functions preserving the distiguished point.

• $\mathcal{C}^*$, the category of C^*-algebras with linear $*$-multiplicative morphisms.

• $\mathcal{C}^*_p$, the category of C^*-algebras with proper morphisms.

• $\mathcal{C}^*_1$, the category of unital C^*-algebras with unital morphisms.

• $\mathcal{C}^*_{comm}$, the category of commutative C^*-algebras with C^*-morphisms.

• $\mathcal{W}^*$, the category of W^*-algebras with normal morphisms.

• $\mathcal{W}^*_1$, the category of W^*-algebras with unital normal morphisms.

• $\mathcal{G}r$, topological groups, with continuous homomorphisms.

• $rep(A)$. Objects: nondegenerate representations of the C^*-algebra A on Hilbert spaces. Morphisms: bounded intertwiners.

• $rep(M)$. Objects: nondegenerate normal representations of the W^*-algebra M on Hilbert spaces. Morphisms: bounded intertwiners.

• $rep(G)$. Objects: unitary weakly continuous representations of the group $G \in \mathcal{G}r$. Morphisms: bounded intertwiners.

• $rep_0(A)$, $rep_0(M)$. Possibly degenerate representations.

• $cyc(A)$, $cyc(M)$, $cyc(G)$. The full subcategories of $rep(A)$, $rep(M)$, $rep(G)$ formed by cyclic representations. A full subcategory is the category formed by certain objects and all the arrows between them from the original category.

2.15. **Remark.** The category $\mathcal{C}^*$ is complete and cocomplete (see [39], theorem 3.15), i.e. it has arbitrary limits and colimits, like other common categories: $\mathcal{S}ets$, Top, etc. The coproduct of a family of C^*-algebras (A_i) is the free product $\ast A_i$ and can be constructed as the completion of the free $*$-algebra generated by the A_i, with respect to the largest possible C^*-norm ([39] lemma 3.7). In other words, this is the enveloping C^*-algebra of the free $*$-algebra generated by the A_i. A general colimit is just a free product with amalgamation.

The coproduct in $\mathcal{C}_1^*$ is a different free product, because the units from the original algebras A_i must map to the unit in the coproduct algebra. It can be defined as the enveloping C^*-algebra of the free $*$-algebra with unit generated by the unital $*$-algebras A_i (see [37], definitions 1.2.1 and 1.4.1).

2.3.1. **Example of adjoint functors: Stone-Cech compactification.** For every topological space X, the Stone-Cech compactification is a compact Hausdorff space denoted by βX with a continuous map $X \to \beta X$ such that for every continuous map $X \to K$ with K compact Hausdorff, there is a unique $\beta X \to K$ continuous such that the triangle commutes. It will play a role in the section about noncommutative Tietze extension theorem, chapter 2. A continuous map $X \to Y$ between arbitrary topological spaces induce a map $\beta X \to \beta Y$. This follows from previous universal property applied to:

$$\begin{tikzcd} \beta X \arrow[r, dotted, "\exists!"] & \beta Y \\ X \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

Thus the Stone-Cech compactification defines a functor $Top \xrightarrow{\beta} Top_{cT_2}$. There is also a forgetful functor (or inclusion) $Top_{cT_2} \xrightarrow{i} Top$. Now the universal property of βX can expressed as $[\beta X, K]_{Top_{cT_2}} \simeq [X, K]_{Top}$, or simply $\beta \dashv i$.

Since by 2.7 every proper morphism between C^*-algebras can be extended to the corresponding multiplier algebras, we have a functor $\mathcal{C}_p^* \xrightarrow{M} \mathcal{C}_1^*$. In the other direction we have the forgetful functor $\mathcal{C}_1^* \xrightarrow{\mathcal{F}} \mathcal{C}_p^*$. The multiplier algebra of a C^*-algebra is usually regarded as a noncommutative version of the Stone-Cech compactification. This is because $M(C_0(X)) = C_b(X) = C(\beta X)$. In addition to the universal property for M, this suggests that M

might be left adjoint to $\mathcal{F}$. However, a straightforward attempt fails even at $[M(A), M(A)]_{\mathcal{C}_1^*} \overset{?}{\simeq} [A, M(A)]_{\mathcal{C}_p^*}$ because the restriction of the identity map $M(A) \to M(A)$ to $A \to M(A)$ is not proper. Actually, we can prove that M is not a left adjoint to any functor. In case it was, it should preserve epimorphisms (2.13). By 8.2 and 11.1 this would imply that every locally compact Hausdorff space is normal, which is false.

2.4. **Gelfand duality, categorical version.**

Gelfand duality theorem classifies commutative C^*-algebras as the family formed by $C_0(X)$, the continuous complex-valued functions which vanish at infinity on locally compact and Hausdorff spaces X. This correspondence can be extended to morphisms, i.e. it can be stated as an equivalence between the relevant categories. If we restrict to the category of unital commutative C^*-algebras with unital morphisms, it is equivalent to ${Top_{cT_2}}^{op}$ (^{op} means the "opposite" category: arrows and composition are reversed). If we want to consider nonunital algebras as well, with proper morphisms, the corresponding topological category has the locally compact Hausdorff spaces as objects and proper continuous maps as arrows. Here we will do with some detail the wider case of not necessarily unital commutative algebras with not necessarily proper morphisms. Call this category $\mathcal{C}^*_{comm}$. The corresponding topological category turns out to be that of compact Hausdorff spaces with a distinguished point, $Top^*_{cT_2}$. The arrows are those continuous maps which preserve the distiguished point. In this context it is reasonable to use ∞ for the distiguished point. Consider the following functors:

$$\mathcal{C}^*_{comm} \underset{C_0}{\overset{char_0}{\rightleftarrows}} Top^*_{cT_2}$$

$$char_0(A) = (\{\varphi \in A^* / \varphi(ab) = \varphi(a)\varphi(b)\}, 0)$$

$$C_0(K, \infty) = \{K \xrightarrow{h} \mathbb{C} \text{ continuous} / h(\infty) = 0\}$$

Elements in $char_0(A)$ have norm one or zero and this set is w^*-closed, so it is compact. It is clear that a morphism $A \xrightarrow{f} B$ induces a map $char_0(B) \xrightarrow{f^*} char_0(A)$ thus giving a functor. Actually, it is a contravariant functor, for it interchanges domains with codomains and reverses composition. $C_0(K, \infty)$ is the algebra of continuous functions $K \to \mathbb{C}$ mapping ∞ to 0. It is also clear that a continuous function $(K, \infty) \xrightarrow{g} (L, \infty)$ produces a morphism $C_0(L, \infty) \xrightarrow{g^*} C_0(K, \infty)$.

There is a natural transformation $Id_{\mathcal{C}^*_{comm}} \xrightarrow{\eta} C_0 \circ char_0$, the Gelfand transform:

$$A \xrightarrow{\eta(A)} C_0(char_0(A))$$

$$a \longmapsto ev_a$$

that is a natural isomorphism thanks to Gelfand duality theorem.

We also have a natural transformation $Id_{Top^*_{cT_2}} \xrightarrow{\gamma} char_0 \circ C_0$

$$(K,\infty) \xrightarrow{\gamma(K)} char_0(C_0(K,\infty))$$
$$x \longmapsto ev_x$$

Let $x, y \in K$ with $x \neq \infty$. Since a compact Hausdorff space is always normal, by Urysohn's lemma we can separate $\{x\}$ and $\{y,\infty\}$ by a continuous function $h \in C(K)$: $h(x) = 1$, $h(y) = h(\infty) = 0$, so $h \in C_0(K,\infty)$. This means $\gamma(K)$ is injective. It is also continuous and therefore closed. With surjectivity, provided by next proposition, we conclude that γ is a natural isomorphism

2.16. **Proposition.** *Let $(K,\infty) \in Top^*_{cT_2}$. Every character $C_0(K,\infty) \xrightarrow{\chi} \mathbb{C}$ is equal to ev_x for some $x \in K$.*

Proof. Extend χ to $C(K) \xrightarrow{\chi} \mathbb{C}$ by the formula $\chi(1) = 1$. Now χ is a character of the algebra $C(K)$. $H = ker\ \chi$ is a maximal ideal of $C(K)$. Take a finite subset of H, whose elements are $f_1, f_2, ..., f_n$. If these functions don't have any common zero, then we can consider the element $f = f_1^2 + f_2^2 + ... + f_n^2 \in K$, which have no zeroes at all, so it is invertible and $H = C(K)$. Absurd. So any nonempty finite subset of K have at least a common zero.

Now consider the family $\{Z(f)\}_{f\in H}$, where $Z(f) \subset K$ is the set of zeroes of f. Any finite subfamily has nonempty intersection. Since K is compact, $\bigcap_{f\in H} Z(f)$ is not empty. Take $x \in \bigcap_{f\in H} Z(f)$. We have $H \subset ker(ev_x)$, so $H = ker(ev_x)$ and $\chi = ev_x$ because they are both normalized and have the same kernel. □

Thus, we have:

2.17. **Theorem** (Gelfand duality, categorical version)**.**

The functors $\mathcal{C}^*_{comm} \underset{C_0}{\overset{char_0}{\rightleftarrows}} Top^*_{cT_2}$ *define a contravariant equivalence of categories.*

2.18. **Remarks.**

1) An interesting feature of this setting is that the equivalence functors are representable, i.e. they are given by certain object of each category. Explicitly:

$$char_0 = [-,\mathbb{C}]_{\mathcal{C}^*_{comm}}$$
$$C_0 = [-,(\mathbb{C},0)]_{Top^*_{cT_2}}$$

adding the corresponding structure to the *hom* sets $[A,\mathbb{C}]_{\mathcal{C}^*_{comm}}$ and $[(K,\infty),(\mathbb{C},0)]_{Top^*_{cT_2}}$.

2) $A \in \mathcal{C}^*_{comm}$ is unital if and only if $0 \in char_0(A)$ is isolated.

Chapter 1

The main subject of this chapter is Takesaki-Bichteler duality theorem, subject to which we contribute with theorems 5.9 and 5.15. Takesaki-Bichteler theorem describes any C^*-algebra A as certain continuous fields over the representations of A. It is an easy consequence of 5.9, while theorem 5.15 describes the multipliers (also left, right and quasi multipliers) in that context, i.e. in terms of fields over the representations of A. First we need to develop three different points of view for the universal (or enveloping) W^*-algebra of a C^*-algebra, showing their equivalences. These are: the bidual approach, the bicommutant of the universal representation and the approach by fields over the category of representations. We focus on the latter, since it is the one needed to formulate Takesaki-Bichteler duality (the others play a role in the proof). The enveloping W^*-algebra gives a functor $\mathcal{C}^* \xrightarrow{W^*} \mathcal{W}^*$ and its universal property is expressed as the adjunction $W^* \dashv \mathcal{F}$, where $\mathcal{W}^* \xrightarrow{\mathcal{F}} \mathcal{C}^*$ is the forgetful functor. These same techniques allow us to study the universal W^*-algebra of a general topological group. We define a functor $\mathcal{G}r \xrightarrow{W^*} \mathcal{W}_1^*$ and prove the adjunction $W^* \dashv U$ where $\mathcal{W}_1^* \xrightarrow{U} \mathcal{G}r$ is the unitary group functor.

3. Enveloping W^*-algebra of a C^*-algebra

Here we show the equivalence between the bidual and bicommutant approaches. First we see that they are isometrically isomorphic Banach spaces and then we describe the operations in A^{**} that make the isomorphism a $*$-algebra isomorphism.

Recall Sakai's characterization of W^*-algebras as those C^*-algebras with a predual (see for example [32]). For a concrete W^*-algebra $M \subset B(H)$, the predual is obtained as the set of ultraweakly continuous linear functionals (the ultraweak topology in $B(H)$ can be defined as the initial topology with respect to the functionals $\sum_{i\in\mathbb{N}}\langle(-)\alpha_i, \beta_i\rangle$ for collections of vectors (α_i), (β_i) such that $\sum_i ||\alpha_i||^2 < \infty$ and $\sum_i ||\beta_i||^2 < \infty$).

The universal representation of A is

$$A \xrightarrow{\pi} B(\bigoplus_{\varphi\in S(A)} H_\varphi)$$

where $\pi = \bigoplus_{\varphi\in S(A)} \pi_\varphi$ is the direct sum of all the GNS representations.

3.1. **Proposition.** *Let A be a C^*-algebra and π its universal representation. The application $A^{**} \xrightarrow{\widetilde{\pi}} \pi(A)''$ defined by $\langle\widetilde{\pi}(f)\alpha, \beta\rangle = f(\langle\pi(-)\alpha, \beta\rangle)$ is an isometric isomorphism of Banach spaces.*

The proof was essentially taken from [32], theorem 1.17.2.

Proof. The isometric embedding $A \hookrightarrow \pi(A)''$ allows to take a restriction map $(\pi(A)'')_* \xrightarrow{R} A^*$. Every state $\varphi \in S(A)$ gives a vector state of the von Neumann algebra $\pi(A)''$ so it belongs to $(\pi(A)'')_*$. Hence R is surjective, since every element of A^* is a linear combination of states. Let's see that it is an isometry in order to have a Banach space isomorphism. Let $\phi \in \pi(A)''_*$.

$$||\phi|| = \sup_{\substack{||x|| \leq 1 \\ x \in \pi(A)''}} \phi(x) = \sup_{\substack{||x|| \leq 1 \\ x \in \pi(A)}} \phi(x) = ||R(\phi)||$$

The equality in the middle holds because the unit ball of A is weakly dense in the unit ball of $\pi(A)''$, this is Kaplansky density theorem.

Taking duals, we have $\pi(A)'' \simeq A^{**}$ as Banach spaces. Now we check that the dual application R^* is given by $\widetilde{\pi}$ as in the statement: $\langle \widetilde{\pi}(f)\alpha, \beta\rangle = f(\langle \pi(-)\alpha, \beta\rangle)$. Let $f \in A^{**}$. $R^*(f) \in (\pi(A)''_*)^*$. Call $\widehat{R^*(f)}$ the corresponding element in $\pi(A)''$.

$$\langle \widehat{R^*(f)}\alpha, \beta\rangle = R^*(f)\Big(\langle (-)\alpha, \beta\rangle\Big) =$$
$$= f \circ R\Big(\langle (-)\alpha, \beta\rangle\Big) = f(\langle \pi(-)\alpha, \beta\rangle)$$

.

□

3.2. **Observation.** We can also describe the inverse of $\widetilde{\pi}$, $\pi(A)'' \xrightarrow{\Psi} A^{**}$. Let $T \in \pi(A)''$ and $\phi \in A^*$. Decomposing $\phi = r_1\varphi_1 - r_2\varphi_2 + ir_3\varphi_3 - ir_4\varphi_4$ with $r_i \in \mathbb{R}_{\geq 0}$ and $\varphi_i \in S(A)$ it holds:

$$\Psi(T)(\phi) = r_1\langle T\xi_{\varphi_1}, \xi_{\varphi_1}\rangle - r_2\langle T\xi_{\varphi_2}, \xi_{\varphi_2}\rangle + ir_3\langle T\xi_{\varphi_3}, \xi_{\varphi_3}\rangle - ir_4\langle T\xi_{\varphi_4}, \xi_{\varphi_4}\rangle$$

This is seen easily with the following calculation:

$$\widetilde{\pi}\Psi(T) = T$$
$$\langle \widetilde{\pi}\Psi(T)\alpha, \beta\rangle = \langle T\alpha, \beta\rangle$$
$$\Psi(T)\Big(\langle \pi(-)\alpha, \beta\rangle\Big) = \langle T\alpha, \beta\rangle$$

and taking $\alpha = \beta = \xi_\varphi$ for a state φ,

$$\Psi(T)(\varphi) = \langle T\xi_\varphi, \xi_\varphi\rangle$$

Arens multiplication. Given a normed algebra $\mathcal{A}$, there are two natural product operations induced in the bidual which extend the product of $\mathcal{A}$, called Arens multiplications. When A is a C^*-algebra these two operations coincide with the product in $\pi(A)''$ through the isomorphism previously described. To define Arens multiplications we follow the steps of [27] page 2.

For $a, b \in A$, $\phi \in A^*$,

$${}_a\phi(b) := \phi(ba) \quad \text{ and } \quad \phi_a(b) := \phi(ab)$$

${}_a\phi$ and ϕ_a are elements of A^* which depend linearly and continuously on ϕ and a. Continuity is due to: $||{}_a\phi|| \leq ||a||.||\phi||$, $||\phi_a|| \leq ||a||.||\phi||$. Besides, ${}_a({}_b\phi) = {}_{ab}\phi$, $(\phi_a)_b = \phi_{ab}$ and $({}_a\phi)_b = {}_a(\phi_b)$.

For $f \in A^{**}$ and $\phi \in A^*$ let:

$$_f\phi(a) = f(\phi_a) \quad \text{and} \quad \phi_f(a) = f(_a\phi)$$

$_f\phi$ and ϕ_f are elements of A^*. They depend linearly and continuously on ϕ and f, satisfying $||_f\phi|| \leq ||f||.||\phi||$, $||\phi_f|| \leq ||f||.||\phi||$. These definitions extend the previous ones because $_{\widehat{a}}\phi = {}_a\phi$ and $\phi_{\widehat{a}} = \phi_a$, for $a \in A$ and $\widehat{a}$ the corresponding element in A^{**}

Now we can define the two product operations on A^{**}. For $f, g \in A^{**}$,

$$fg(\phi) := f(_g\phi) \qquad f\cdot g(\phi) := g(\phi_f)$$

fg and $f\cdot g$ belong to A^{**}. Any of these multiplications extend the product of A and turn A^{**} into a Banach algebra. Linearity in each variable and submultiplicativity of the norm follow directly from the definition. The identities $_{fg}\phi = {}_f(_g\phi)$, $\phi_{f\cdot g} = (\phi_f)_g$ are useful to prove associativity.

3.3. **Proposition.** *For a C^*-algebra A, both product operations on A^{**} defined above coincide with the regular composition on $\pi(A)''$ through the isomorphism $\widetilde{\pi}$ from proposition 3.1. Besides, the involution on A^{**} induced by the involution of A coincides with the adjoint operation in $\pi(A)''$.*

Proof. We must check $\widetilde{\pi}(fg) = \widetilde{\pi}(f)\widetilde{\pi}(g)$. It holds if and only if

$$\langle\widetilde{\pi}(fg)\alpha, \beta\rangle \stackrel{?}{=} \langle\widetilde{\pi}(f)\widetilde{\pi}(g)\alpha, \beta\rangle$$
$$(fg)\Big(\langle\pi(-)\alpha, \beta\rangle\Big) \stackrel{?}{=} f\Big(\langle\pi(-)\widetilde{\pi}(g)\alpha, \beta\rangle\Big)$$
$$f\Big(_g\langle\pi(-)\alpha, \beta\rangle\Big) \stackrel{?}{=} f\Big(\langle\pi(-)\widetilde{\pi}(g)\alpha, \beta\rangle\Big)$$

But $_g\langle\pi(-)\alpha, \beta\rangle = \langle\pi(-)\widetilde{\pi}(g)\alpha, \beta\rangle$, because:

$$_g\langle\pi(-)\alpha, \beta\rangle(a) = g\Big(\langle\pi(-)\alpha, \beta\rangle_a\Big) = g\Big(\langle\pi(a(-))\alpha, \beta\rangle\Big) =$$
$$= g\Big(\langle\pi(-)\alpha, \pi(a^*)\beta\rangle\Big) = \langle\widetilde{\pi}(g)\alpha, \pi(a^*)\beta\rangle = \langle\pi(a)\widetilde{\pi}(g)\alpha, \beta\rangle$$

For the other product:

$$\widetilde{\pi}(f\cdot g) \stackrel{?}{=} \widetilde{\pi}(f)\widetilde{\pi}(g)$$
$$(f\cdot g)\Big(\langle\pi(-)\alpha, \beta\rangle\Big) \stackrel{?}{=} \langle\widetilde{\pi}(f)\widetilde{\pi}(g)\alpha, \beta\rangle = \langle\widetilde{\pi}(g)\alpha, \widetilde{\pi}(f)^*\beta\rangle$$
$$g(\langle\pi(-)\alpha, \beta\rangle_f) \stackrel{?}{=} g\Big(\langle\widetilde{\pi}(f)\pi(-)\alpha, \beta\rangle\Big)$$

and we see $\langle\pi(-)\alpha, \beta\rangle_f = \langle\widetilde{\pi}(f)\pi(-)\alpha, \beta\rangle$ without difficulty.

Regarding the involution, let us first recall the definitions of the induced involutions on A^* and A^{**}. For $\phi \in A^*$, $\phi^*(a) \stackrel{def}{=} \overline{\phi(a^*)}$. For $f \in A^{**}$, $f^*(\phi) \stackrel{def}{=} \overline{f(\phi^*)}$. Now:

$$\widetilde{\pi}(f^*) \stackrel{?}{=} \widetilde{\pi}(f)^*$$
$$\langle\widetilde{\pi}(f^*)\alpha, \beta\rangle \stackrel{?}{=} \langle\widetilde{\pi}(f)^*\alpha, \beta\rangle$$

$$f^*(\langle\pi(-)\alpha,\beta\rangle) \stackrel{?}{=} \langle\alpha,\widetilde{\pi}(f)\beta\rangle$$
$$\overline{f(\langle\pi(-)\alpha,\beta\rangle^*)} \stackrel{?}{=} \overline{\langle\widetilde{\pi}(f)\beta,\alpha\rangle}$$
$$f(\langle\pi(-)\alpha,\beta\rangle^*) \stackrel{?}{=} f(\langle\pi(-)\beta,\alpha\rangle)$$

but $\langle\pi(-)\alpha,\beta\rangle^* = \langle\pi(-)\beta,\alpha\rangle$, so the equality holds. □

3.4. **Notation.** *To unify we denote $W^*(A)$ the enveloping von Neumann algebra of a C^*-algebra A.*

4. Fields over the category of representations

In this section we give the third point of view for the enveloping von Neumann algebra: the elements are fields over the representations of the algebra. It was developed in [34] and [6] in order to obtain the duality theorem for C^*-algebras that we study in the following section. For this part we consider appropriate to deal with representations as forming a category, thus slightly modifying the original definition of field. Proposition 4.3 and proposition 4.8/corollary 4.9 fill small gaps in the literature about the nature of these fields. Then we show that the universal W^*-algebra functor $\mathcal{C}^* \xrightarrow{W^*} \mathcal{W}^*$ is left adjoint to the forgetful functor $\mathcal{W}^* \xrightarrow{\mathcal{F}} \mathcal{C}^*$.

4.1. **Definition.** *For a C^*-algebra A, we call "field" over $rep(A)$ a function T assigning to each $\pi \in rep(A)$, $A \xrightarrow{\pi} B(H_\pi)$, an element $T(\pi) \in B(H_\pi)$ in a bounded and coherent with morphisms way. Explicitly: $\sup_\pi ||T(\pi)|| < \infty$, and if $H_{\pi_1} \xrightarrow{S} H_{\pi_2}$ is an intertwiner ($S\pi_1(a) = \pi_2(a)S$) then $ST(\pi_1) = T(\pi_2)S$. In other words, fields are bounded endomorphisms of the forgetful functor $rep(A) \to \mathcal{H}$, where $\mathcal{H}$ is the category of Hilbert spaces.*

4.2. **Definition.** *We call A^F the set of fields over $rep(A)$.*

A^F is a C^*-algebra with the operations defined pointwise and the norm $||T|| = \sup_{\pi \in rep(A)} ||T(\pi)||$. For proving completeness a typical argument applies. We will show that A^F is the enveloping von Neumann algebra of A. Notice that A is a subalgebra of A^F: an element $a \in A$ induces the field $\widehat{a}$ defined by $\widehat{a}(\pi) = \pi(a)$.

The following proposition allows to replace $rep(A)$ by the small category $cyc(A)$ of cyclic representations.

4.3. **Proposition.** *The set of fields over $rep(A)$ is equal to the set of fields over $cyc(A)$.*

Of course, a field over $cyc(A)$ is a bounded endomorphism of the forgetful functor $cyc(A) \to \mathcal{H}$.

Proof. Clearly, a field over $rep(A)$ can be restricted to a field over $cyc(A)$. Now let T be a field over $cyc(A)$, and $(\pi, H) \in rep(A)$. π can be expressed as a direct sum of cyclic representations, so we define $T(\pi)$ as the direct sum of the operators associated to these subrepresentations. This definition is correct because of the following. Assume we have two decompositions into cyclic subrepresentations: $H = \bigoplus A_i = \bigoplus B_j$. Consider P_i and Q_j the orthogonal projections to the subspaces A_i and B_j. We have the following morphisms of cyclic representations, $B_j \xrightarrow{P_i|_{B_j}} A_i$. Compatibility of T says $T(A_i)P_i|_{B_j} = P_i|_{B_j}T(B_j)$ (we abuse harmlessly identifying the subspace with the subrepresentation).

$$\sum_i T(A_i)P_i = (\sum_i T(A_i)P_i)(\sum_j Q_j) = \sum_{i,j} T(A_i)P_iQ_j =$$

$$= \sum_{i,j} P_iT(B_j)Q_j = (\sum_i P_i)(\sum_j T(B_j)Q_j) = \sum_j T(B_j)Q_j$$

The sums converge strongly (*sot* in $B(H)$). It is valid to interchange the order of summation because composition of operators is jointly continuous for the strong operator topology when restricted to bounded sets. This proves that T is well defined.

The extended field is clearly bounded. To see compatibility, take a morphism between π_1 and π_2, $H_1 \xrightarrow{S} H_2$, and any vector $\alpha \in H_1$. Now take decompositions of these representations as sum of cyclic subrepresentations, containing the cyclic representations generated by α and $S(\alpha)$ respectively. S restricts to an intertwiner between these cyclic representations. Because of the original compatibility in $cyc(A)$, we have $ST(\pi_1)(\alpha) = T(\pi_2)S(\alpha)$. (In case $S(\alpha) = 0$ then $S(\overline{\pi_1(A)\alpha}) = 0$, and since $\overline{\pi_1(A)\alpha}$ is $T(\pi_1)$-invariant, we have $ST(\pi_1)(\alpha) = 0 = T(\pi_2)S(\alpha)$). □

4.4. **Proposition.** *A^F is a von Neumann algebra.*

Proof. We already know that it is a C^*-algebra. Consider the following representation:

$$H = \bigoplus_{\pi \in cyc(A)} H_\pi \quad , \quad A^F \xrightarrow{\Pi} B(H) \quad , \quad \Pi(T) = \bigoplus_{\pi \in cyc(A)} T(\pi)$$

where we can take just one π for each unitary equivalence class. Π is clearly a faithful representation. Let us see that the image is strongly closed. Assume $\Pi(T_\mu) \to S$ for the *sot*. If $\alpha \in H_\pi$, $\Pi(T_\mu)\alpha = T_\mu(\pi)\alpha \to S\alpha$, then $S\alpha \in H_\pi$, i.e. H_π is S-invariant. We write $S = \bigoplus S_\pi$. Putting $T(\pi) = S_\pi$, it follows easily that T is a field (using $T(\pi_\mu) \to T(\pi)$ strongly) and $\Pi(T) = S$. □

4.5. **Observation.** For $T \in A^F$ it holds $T(\Pi|_A) = \Pi(T)$. This can be easily checked for $\alpha \in H_\pi$ applying compatibility with the inclusion morphism $H_\pi \hookrightarrow H$.

4.6. **Proposition.** *If T is a field over $rep(A)$, $T(\pi)$ belongs to the von Neumann algebra generated by $\pi(A)$ for every $\pi \in rep(A)$.*

Proof. An operator $S \in \pi(A)'$ is an endomorphism of π, so it commutes with $T(\pi)$ because of compatibility with morphisms. Therefore, $T(\pi) \in \pi(A)''$. $\square$

4.7. **Proposition.** $A^F = W^*(A)$

Proof. Consider A^F acting on $\bigoplus_{\varphi \in \mathcal{S}(A)} H_\varphi$. We call $\widetilde{\Pi}$ this faithful representation and the restriction to A, $\widetilde{\Pi}|_A = \Pi_U$, is the universal representation of A. So we must prove $\widetilde{\Pi}(A^F) = \Pi_U(A)''$

For a field $T \in A^F$, compatibility with the inclusion intertwiner $H_{\varphi_0} \hookrightarrow \bigoplus H_\varphi$ implies $T(\Pi_U)\alpha = T(\pi_{\varphi_0})\alpha = \widetilde{\Pi}(T)\alpha$ for $\alpha \in H_{\varphi_0}$. Then $\widetilde{\Pi}(T) = T(\Pi_U)$.

If $S \in \Pi_U(A)'$, it is an endomorphism of Π_U. Compatibility says:

$$ST(\Pi_U) = T(\Pi_U)S$$

$$S\widetilde{\Pi}(T) = \widetilde{\Pi}(T)S$$

which means $\widetilde{\Pi}(T) \in \Pi_U(A)''$ and therefore $\widetilde{\Pi}(A^F) \subset \Pi_U(A)''$. The other inclusion holds because the bicommutant is the smallest von Neumann algebra containing $\Pi_U(A) = \widetilde{\Pi}(A)$ (we use 4.4). $\square$

The following proposition and its corollary explain why the different definitions for "field" from [34] and [6] coincide between them and with ours. Informally, let us say that Takesaki ask the fields to be compatible only with finite direct sums and unitary equivalences. Here we see that this implies compatibility with every intertwiner. Bichteler adds to the definition compatibility with intertwiners that are partial isometries (instead of just unitary equivalences) and he considers important this change in order to prove the duality theorem in the nonseparable case.

4.8. **Proposition.** *Let T be a map that assigns to each $\pi \in rep(A)$ a bounded operator on H_π in a compatible way with those intertwiners that are partial isometries. Then T is compatible with every intertwiner.*

Proof. Given an arbitrary intertwiner $\pi_1 \xrightarrow{S} \pi_2$, let $S = UP$ be its polar decomposition. $P = (S^*S)^{1/2}$ is an intertwiner $\pi_1 \xrightarrow{P} \pi_1$ and the partial isometry U is an intertwiner as well (to check this we used that U maps $(S^*S)^{1/2}y$ to Sy and the orthogonal complement to 0). Therefore, T is compatible with U by hypothesis and it only remains to prove that T is compatible with any positive intertwiner P of a representation with itself. Taking $r > 0$ small enough, rP has its spectrum inside $[0, 2\pi)$. e^{irP} is a unitary equivalence, so it is compatible (commutes) with T. But rP is the logarithm of e^{irP}, so rP also commutes with T. $\square$

4.9. **Corollary.** *For a map T that assigns to each $\pi \in rep(A)$ a bounded operator on H_π, it suffices that it preserves finite direct sums and unitary equivalences to be compatible with every intertwiner.*

Proof. Let $H_1 \xrightarrow{S} H_2$ be a partial isometry intertwiner. Taking the kernel and its orthogonal complement in the domain, and the image and its orthogonal complement in the codomain, S decomposes as $H_3 \oplus H_4 \to H_5 \oplus H_6$ where H_3 maps isometrically onto H_5 and H_4 goes to 0. By hypothesis T is compatible with this morphism and by previous proposition this finishes the proof. $\square$

4.10. **Observation.** A field over $rep(A)$ is equivalent to a field over $rep_0(A)$ such that $T(0) = 0$, where $0 \in rep_0(A)$ is the zero representation on a unidimensional Hilbert space.

The proof is straightforward and not very interesting. It allows to take $T(\pi)$ for a possibly degenerate representation π.

Universal property: $W^* \dashv \mathcal{F}$**.** A morphism of C^*-algebras $A \xrightarrow{f} B$ induces a functor $rep_0(B) \xrightarrow{f^*} rep_0(A)$ and this induces a $*$-algebra morphism between the fields $A^F \xrightarrow{f^F} B^F$. In terms of biduals, this morphism is $A^{**} \xrightarrow{f^{**}} B^{**}$, that is w^*-w^*-continuous, so it is normal. This proves that the enveloping W^*-algebra defines a functor $\mathcal{C}^* \xrightarrow{W^*} \mathcal{W}^*$. The W^*-algebra $W^*(A)$ has a universal property commonly expressed in terms of representations: normal representations of $W^*(A)$ are in bijective correspondence (via restriction) with representations of A. A more general statement is "$\mathcal{W}^*$ is left adjoint to the forgetful functor".

4.11. **Proposition.** *Let M be a von Neumann algebra and $A \xrightarrow{f} M$ a C^*-morphism. There exists a unique morphism of W^*-algebras $A^F \xrightarrow{\widetilde{f}} M$ such that the triangle commutes.*

$$\begin{tikzcd} A^F \arrow[r, dotted, "\exists! \widetilde{f}"] & M \\ A \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. Uniqueness is clear, since A generates A^F as a W^*-algebra. Let us prove existence. We first consider the case $M = B(H)$ (H is any Hilbert space). f is a representation, so we can define $\widetilde{f}(T) = T(f)$. $\widetilde{f}$ clearly preserves the operations of sum, product and involution. We must prove that it is continuous for the σ-weak topologies. In order to do so, take an element of the predual of $B(H)$. We write it as $tr(R(-))$, where R is trace class.

$$\begin{tikzcd} A^F \arrow[r, "\widetilde{f}"] \arrow[drr] & B(H) \arrow[dr, "tr(R(-))"] & \\ A \arrow[u, hook] \arrow[ur, "f"'] & & \mathbb{C} \end{tikzcd}$$

If we show that $tr(A\widetilde{f}(-))$ is a normal functional we are done. But this is clear if we think A^F through the faithful representation on the Hilbert space

$$\widetilde{H} = (\bigoplus_{\pi \in cyc(A)} H_\pi) \oplus H$$

where an element $T \in A^F$ acts on each H_π and H according to $T(\pi)$ and $T(f)$ respectively. Just like in 4.4, this representation is faithful and the image is strongly closed. So, through this faithful representation, $tr(A\widetilde{f}(-)) = tr(\widetilde{R}(-))$, where $\widetilde{R} \in B(\widetilde{H})$ is $0 \oplus R$.

If we now have a general W^*-algebra M, we can take a faithful representation $M \overset{j}{\hookrightarrow} B(H)$ and extend $j \circ f$ to $\widetilde{j \circ f}$. By 4.6 $\widetilde{j \circ f}(T)$ belongs to the von Neumann algebra generated by $f(A)$, so $\widetilde{j \circ f}(T) \in M$.

$$\begin{tikzcd}
A^F \arrow[r, dotted, "\widetilde{f}"] \arrow[rr, bend left, "\widetilde{j \circ f}"] & M \arrow[r, hook, "j"] & B(H) \\
A \arrow[u, hook] \arrow[ur, "f"] \arrow[urr, "j \circ f"'] & &
\end{tikzcd}$$

$\square$

4.12. **Corollary.** *$W^* \dashv \mathcal{F}$. I.e.: $\mathcal{C}^* \xrightarrow{W^*} \mathcal{W}^*$ is left adjoint to the forgetful functor $\mathcal{W}^* \xrightarrow{\mathcal{F}} \mathcal{C}^*$.*

Proof. A C^*-morphism $A \to M$ induces a W^*-morphism $W^*(A) \to M$ by previous proposition, and viceversa by restriction. These maps $[W^*(A), M] \rightleftarrows [A, \mathcal{F}(M)]$ are clearly mutually inverse and natural in both variables. $\square$

A direct consequence of the adjunction $W^* \dashv \mathcal{F}$ is that limits in $\mathcal{W}^*$ must be computed as the corresponding limits of the underlying C^*-algebras. See [18] for a complete description of limits and colimits of W^*-algebras ($\mathcal{W}^*$ is complete and cocomplete). We can also consider the category of unital C^*-algebras with unital morphisms $\mathcal{C}_1^*$ and the category of W^*-algebras with unital morphisms $\mathcal{W}_1^*$. We have the functors $\mathcal{C}_1^* \underset{\mathcal{F}_1}{\overset{W_1^*}{\rightleftarrows}} \mathcal{W}_1^*$, where W_1^* is the usual universal W^*-algebra, and $\mathcal{F}_1$ the corresponding forgetful functor. Using the universal property that we already know, we see that $W_1^* \dashv \mathcal{F}_1$.

5. Takesaki-Bichteler duality theorem for C^*-algebras

The theorem asserts that any C^*-algebra can be recovered as the continuous fields over its representations. Before going into the details of the meaning of "continuous field" we introduce the necessary lemmas.

The first of them, taken from Bichteler's article ([6], first lemma, parts (iii) and (iv)) consists of two isomorphisms: $A^{**} \simeq AN_0(Q(A))$ and $A \simeq AC_0(Q(A))$ (see next definition). We give a more detailed proof of the first isomorphism and a simpler proof for the second one. Recall that any Banach space V can be recovered from the bidual as those elements $V^* \to \mathbb{C}$ that are continuous for the w^*-topology. The lemma, in particular says that for a C^*-algebra A it suffices with continuity on $Q(A)$ instead of all A^*.

5.1. **Definition.** *Let $AN_0(Q(A))$ be the set of affine bounded $\mathbb{C}$-valued functions on $Q(A)$ taking the value 0 at 0. It is a normed space for the supremum norm. $AC_0(Q(A))$ will be the subspace of $AN_0(Q(A))$ of continuous functions.*

5.2. **Lemma.** *There is a Banach spaces isomorphism $A^{**} \to AN_0(Q(A))$ that restricts to a bijection $A \to AC_0(Q(A))$.*

Proof. The map is defined simply by restriction from A^* to $Q(A)$. It is linear, injective and contractive. To see that it is onto, we must extend an $f \in AN_0(Q(A))$ to A^*. We write a functional $\varphi \in A^*$ as

$$\varphi = a_1\varphi_1 - a_2\varphi_2 + ia_3\varphi_3 - ia_4\varphi_4$$

with $a_i \geq 0$, φ_i positive functionals of norm 1 and define

$$\tilde{f}(\varphi) = a_1 f(\varphi_1) - a_2 f(\varphi_2) + ia_3 f(\varphi_3) - ia_4 f(\varphi_4)$$

This is well defined, for if

$$\varphi = a_1\varphi_1 - a_2\varphi_2 + ia_3\varphi_3 - ia_4\varphi_4 = a_1'\varphi_1' - a_2'\varphi_2' + ia_3'\varphi_3' - ia_4'\varphi_4'$$

taking hermitian part we get:

$$a_1\varphi_1 - a_2\varphi_2 = a_1'\varphi_1' - a_2'\varphi_2'$$
$$a_1\varphi_1 + a_2'\varphi_2' = a_1'\varphi_1' + a_2\varphi_2$$

Evaluating at an approximate unit we have $a_1 + a_2' = a_1' + a_2$. If this number is not 0, we can use affinity of f to obtain:

$$a_1 f(\varphi_1) - a_2 f(\varphi_2) = a_1' f(\varphi_1') - a_2' f(\varphi_2')$$

if $a_1 + a_2' = a_1' + a_2 = 0$, then $a_1 = a_2 = a_1' = a_2' = 0$ and the equality also holds. We do the same for the antihermitian part. The restriction of this extension back to $Q(A)$ gives the original f thanks to $f(0) = 0$. $\tilde{f}$ is linear, and bounded: $|f(\varphi)| \leq ||f||_\infty$ for $\varphi \in S(A)$. Now take $\varphi \in A^*$. $\varphi = \varphi_h + i\varphi_{ah}$, the hermitian and antihermitian part. Both φ_h and φ_{ah} are hermitian and have norm less or equal than φ because $\varphi_h = \frac{1}{2}(\varphi + \varphi^*)$ and $||\varphi|| = ||\varphi^*||$. Then we need to decompose $\varphi_h = \varphi_1 - \varphi_2$ where φ_1, φ_2 are positive and $||\varphi_h|| = ||\varphi_1|| + ||\varphi_2||$ and similarly $\varphi_{ah} = \varphi_3 - \varphi_4$.

$$|\tilde{f}(\varphi)| \leq |\tilde{f}(\varphi_1)| + |\tilde{f}(\varphi_2)| + |\tilde{f}(\varphi_3)| + |\tilde{f}(\varphi_4)| \leq$$
$$\leq ||f||_\infty(||\varphi_1|| + ||\varphi_2|| + ||\varphi_3|| + ||\varphi_4||) \leq ||f||_\infty . 2||\varphi||$$

So $||\tilde{f}|| \leq 2||f||_\infty$, and clearly $||f||_\infty \leq ||\tilde{f}||$.

Under this bijection, $A \subset AC_0(Q(A))$. We now prove that equality holds. Take $f \in AC_0(Q(A))$. We have continuous maps $Q(A) \times Q(A) \xrightarrow{\bar{f}} \mathbb{C}$, $\bar{f}(\varphi, \psi) = f(\varphi) - f(\psi)$, and $Q(A) \times Q(A) \xrightarrow{m} A_h^*$, $m(\varphi, \psi) = \varphi - \psi$ (A_h^* is the hermitian part of A^* with the w^*-topology). Since $Q(A) \times Q(A)$ is compact and A_h^* Hausdorff, m is closed, and therefore a quotient if we restrict the codomain to the image.

$$\begin{tikzcd} Q(A) \times Q(A) \arrow[r, "m"] \arrow[dr, "\bar{f}"'] & A_h^* \arrow[d, "\tilde{f}"] \\ & \mathbb{C} \end{tikzcd}$$

The image of m contains the unit ball, because every $\varphi \in A_h^*$ can be written as $\varphi_1 - \varphi_2$ with $\varphi_1, \varphi_2 \geq 0$ and $||\varphi|| = ||\varphi_1|| + ||\varphi_2||$. Thus, $\tilde{f}$ is w^*-continuous on the unit ball, so it is w^*-continuous on A_h^*, analogously on A_{ah}^* and therefore on A^*. Hence we conclude that f comes from an element of A. □

5.3. **Remark.** Taking $S(A)$ instead of $Q(A)$ we have: $A^{**} \simeq AN(S(A))$ and, for unital A, $A \simeq AC(S(A))$ (where $AN(S(A))$ is the space of affine bounded $\mathbb{C}$-valued functions on $S(A)$ and $AC(S(A))$ the subspace of continuous functions).

Proof. It is straightforward to check that $AN_0(Q(A)) = AN(S(A))$. Alternatively, the previous proof applies to $A^{**} \simeq AN(S(A))$ with no trouble. To obtain $AC_0(Q(A)) = AC(S(A))$ for unital A, we must prove that continuity on $S(A)$ implies continuity on $Q(A)$. So take $f \in AN_0(Q(A))$ continuous on $S(A)$ and $\varphi_\mu \to \varphi$ in $Q(A)$. Evaluating at 1, we have $||\varphi_\mu|| \to ||\varphi||$. If $\varphi = 0$ we have $|f(\varphi_\mu)| = ||\varphi_\mu||.|f(\frac{\varphi_\mu}{||\varphi_\mu||})| \leq ||\varphi_\mu||.||f||_\infty \to 0$ for those μ such that $\varphi_\mu \neq 0$ and $f(\varphi_\mu) = 0$ if $\varphi_\mu = 0$; so $f(\varphi_\mu) \to 0$. If $\varphi \neq 0$, for large enough μ we have $\varphi_\mu \neq 0$ and

$$f(\varphi_\mu) = ||\varphi_\mu|| f(\frac{\varphi_\mu}{||\varphi_\mu||}) \to ||\varphi|| f(\frac{\varphi}{||\varphi||}) = f(\varphi)$$

□

Next lemma says that if two n-tuples of vectors in a Hilbert space have similar orthogonality relations (inner products between the vectors of each tuple) then, up to an isometry, the n-tuples are close in norm. For the proof see lemma 3.5.6 of Dixmier's book [10].

5.4. **Lemma.** *Let $v_1, ..., v_n \in H$, H a Hilbert space. For every $\epsilon > 0$ there is a $\delta > 0$ such that for every $w_1, ..., w_n \in H$ with $|\langle w_i, w_j\rangle - \langle v_i, v_j\rangle| < \delta$ $\forall i, j$ there is a unitary operator $H \xrightarrow{U} H$ such that $||U(w_i) - v_i|| < \epsilon$ $\forall i$.*

5.5. **Lemma.** *Let H be a Hilbert space and $\alpha, \beta \in H$ unit vectors. Then there is a unitary $U_{\alpha\to\beta}$ such that $||U_{\alpha\to\beta} - Id|| = ||\alpha - \beta||$.*

Proof. In case $\beta = k\alpha$ for $k \in \mathbb{C}$, then $|k| = 1$ and $U_{\alpha\to\beta} := k.Id$. Otherwise, we define $U_{\alpha\to\beta}$ as the identity on $[\alpha]^\perp \cap [\beta]^\perp = [\alpha,\beta]^\perp$. On the subspace $[\alpha,\beta]$ we take an orthonormal basis (α,α'). Write $\beta = r\alpha + s\alpha'$, and $\beta' := -\overline{s}\alpha + \overline{r}\alpha'$, obtaining an orthonormal basis (β,β'). Now define $U_{\alpha\to\beta}|_{[\alpha,\beta]}$ by $\alpha \mapsto \beta$, $\alpha' \mapsto \beta'$. For $x \in H$, let $\lambda\alpha + \mu\alpha'$ be the projection of x to $[\alpha,\beta]$. We have:

$$||x - U_{\alpha\to\beta}(x)||^2 = ||\lambda\alpha + \mu\alpha' - \lambda\beta - \mu\beta'||^2 =$$

$$= ||\lambda\alpha + \mu\alpha' - \lambda(r\alpha + s\alpha') - \mu(-\overline{s}\alpha + \overline{r}\alpha')||^2 = ... = (|\lambda|^2 + |\mu|^2)||\alpha - \beta||^2$$

So $||x - U_{\alpha\to\beta}(x)|| = ||\alpha - \beta||.||p_{[\alpha,\beta]}(x)|| \leq ||\alpha - \beta||.||x||$. □

To deal with continuous fields we follow the original framework by Takesaki, where representations are considered on a fixed big enough Hilbert space. So H will be an infinite dimensional Hilbert space such that any cyclic representation of A has a lower or equal dimension than H. Fields will be defined on $rep(A:H)$, the set of (possibly degenerate) representations of A on H.

5.6. **Definition.** *A field over $rep(A:H)$ is a bounded map $rep(A:H) \xrightarrow{T} B(H)$ such that:*

a) for every intertwiner $\pi_1 \xrightarrow{S} \pi_2$ ($\pi_1, \pi_2 \in rep(A:H)$, $S \in B(H)$) that is a partial isometry, it holds $ST(\pi_1) = T(\pi_2)S$.

b) $T(0) = 0$.

By proposition 4.8, it is not hard to check that fields over $rep(A:H)$ coincide with fields over the category $cyc(A)$.

The topology on $rep(A:H)$ will be the pointwise convergence topology with respect to the *wot*, *sot*, σ-weak or σ-strong topologies in $B(H)$. Next lemma shows that these topologies coincide.

5.7. **Lemma.** *Let π be a representation of A on a Hilbert space H and (π_j) a net of such representations. Convergence $\pi_j(a) \to \pi(a)$ for all $a \in A$ is equivalent for the σ-weak, σ-strong, wot and sot on $B(H)$.*

Recall the following relations between these topologies on $B(H)$:

$$\begin{array}{ccc} sot & \subset \sigma\text{-str} & \subset \tau_{||\ ||} \\ \cup & \cup & \\ wot & \subset \sigma\text{-w} & \end{array}$$

Proof. It is sufficient to show that pointwise *wot* convergence implies pointwise σ-strong convergence. Take a net (π_j) such that $\pi_j(a) \xrightarrow{wot} \pi(a)\ \forall a \in A$. Let $\xi \in H$.

$$||\pi_j(a)\xi - \pi(a)\xi||^2 =$$

$$= \langle \pi_j(a^*a)\xi, \xi\rangle + \langle \pi(a^*a)\xi, \xi\rangle - \langle \pi_j(a)\xi, \pi(a)\xi\rangle - \langle \pi(a)\xi, \pi_j(a)\xi\rangle \to 0$$

so $\pi_j(a) \xrightarrow{sot} \pi(a)$. But $(\pi_j(a))$ is a bounded net in $B(H)$, and the *sot* coincides with the σ-strong on bounded subsets, so $\pi_j(a) \xrightarrow{\sigma-\text{str}} \pi(a)$.

□

In other words, the topology we consider on $rep(A : H)$ is that inherited from the product topology on $B(H)^A$, where the topology on $B(H)$ can equivalently be the σ-weak, σ-strong, *wot* or *sot*. It is Hausdorff because it is a subspace of a product of Hausdorff spaces.

5.8. **Definition.** *A field T over $rep(A : H)$ is τ-continuous if the map $rep(A : H) \xrightarrow{T} B(H)$ is continuous with respect to the τ-topology in $B(H)$, where τ stands for σ-weak, σ-strong, wot or sot.*

However we will see that all these choices for τ are equivalent: continuous fields will be exactly the elements of A. Elements in A clearly induce continuous fields for every election of τ. Since *wot* is the weakest between these topologies, we have that *sot*-continuous, σ-weak-continuous and σ-strong-continuous fields are *wot*-continuous. Hence, it will suffice to prove that *wot*-continuous fields are elements of A. We denote the set of *wot*-continuous fields by $\mathcal{C}_0(rep(A : H))$. The subindex 0 emphasize that they annihilate on the zero representation.

The following result, due to us, easily implies Takesaki-Bichteler duality theorem. Its proof involves an argument similar to the one used by Bichteler for the proof of the duality, but it also requires more work.

5.9. **Theorem.** *Let A be a unital C^*-algebra and H a Hilbert space of infinite dimension d, greater or equal than the dimension of any cyclic representation of A. Let $\xi \in H$ be a unit vector. Then,*

a) the application

$$Rep(A : H) \xrightarrow{\theta_\xi} Q(A)$$
$$\pi \longmapsto \langle \pi(-)\xi, \xi \rangle$$

is a quotient map.

b) The restriction $Rep_\xi(A : H) \xrightarrow{\theta_\xi} S(A)$ is a quotient, where $Rep_\xi(A : H) = \{\pi \in Rep(A : H) / \xi \in \pi(1)H\}$. (Recall $\pi(1)H = \overline{\pi(A)H}$).

Proof.

a) Continuity is trivial. Each $\varphi \in Q(A)$ has many preimages. To produce a preimage of φ_0 we must embed a GNS representation of φ_0 in H in such a way that the orthogonal projection of ξ to the essential space is the cyclic vector of the GNS. To achieve this, take a unit vector η orthogonal to ξ, define $\xi_0 = ||\varphi_0||\xi + (||\varphi_0|| - ||\varphi_0||^2)^{\frac{1}{2}}\eta$. This ξ_0 satisfies $||\xi_0||^2 = ||\varphi_0||$ and $\xi - \xi_0 \perp \xi_0$. Now take a closed subspace S of $[\xi - \xi_0]^\perp$ with dimension d containing ξ_0 (notice that it is possible to choose S in a way such that its codimension is also d). Now embed the GNS space H_{φ_0} into S taking ξ_{φ_0}

to ξ_0, and define $\pi_0 \in rep(A:H)$ as π_{φ_0} through the isometry $H_{\varphi_0} \hookrightarrow H$, being 0 on the orthogonal to the image of H_{φ_0}. We have $\theta_\xi(\pi_0) = \varphi_0$.

We divide in three parts the proof that $Q(A)$ has the quotient topology.

Part 1:

Take $D \subset Q(A)$ such that $\theta_\xi^{-1}(D)$ is open. We must see that D is open to conlude that θ_ξ is a quotient. Let $\varphi_0 \in D$. Take a preimage π_0 of φ_0 as before, such that $\pi_0(1)H = \overline{\pi_0(A)\xi_0}$ has codimension d. Now take a basic open neighborhood V of π_0 contained in $\theta_\xi^{-1}(D)$:

$$V = \{\pi \in rep(A:H) / ||\pi(c_j)\alpha_i - \pi_0(c_j)\alpha_i|| < \epsilon \ \forall i = 1, ..., m;\ j = 1, ..., n\}$$

We shall find an open neighborhood of φ_0 such that every element φ in that neighborhood is the image of an element $\pi \in V$. This would finish the proof.

Part 2: We decompose $\alpha_i = \beta_i + \gamma_i$ where $\beta_i \in \pi_0(1)H$, $\gamma_i \in (\pi_0(1)H)^\perp$. To obtain $\pi \in V$ we will satisfy $||\pi(c_j)\beta_i - \pi_0(c_j)\beta_i||$ arbitrarily small, and the same for γ_i.

To obtain $||\pi(c_j)\beta_i - \pi_0(c_j)\beta_i||$ arbitrarily small, we can approximate β_i with $\pi_0(b_i)\xi_0$, so it will suffice with $||\pi(c_j)\pi_0(b_i)\xi_0 - \pi_0(c_j)\pi_0(b_i)\xi_0||$ arbitrarily small.

Now take the following open neighborhood of φ_0

$$W = \{\varphi \in Q(A) / |(\varphi - \varphi_0)(b_k^* c_l^* c_j b_i)| < \delta \ \forall i, k = 0, ..., m; j, l = 0, ..., n\}$$

where $b_0 = c_0 = 1$. Thus, for $\varphi \in W$, the orthogonality relations of the set $\{\pi_\varphi(c_j)\pi_\varphi(b_i)\xi_\varphi\}_{\substack{i=0,...,m \\ j=0,...,n}}$ are similar to those of $\{\pi_0(c_j)\pi_0(b_i)\xi_0\}_{\substack{i=0,...,m \\ j=0,...,n}}$. Hence, there is an isometric embedding $H_\varphi \hookrightarrow H$ such that the images of $\pi_\varphi(c_j b_i)\xi_\varphi$ are close in norm to $\pi_0(c_j b_i)\xi_0$ (first choose any isometric embedding and then use lemma 5.4). Let us call $\pi' \in rep(A:H)$ the representation π_φ through the embedding, being 0 on the orthogonal complement of the image of H_φ; let ξ' be the image of ξ_φ. We have:

$$\begin{aligned} &||\pi'(c_j)\pi_0(b_i)\xi_0 - \pi_0(c_j)\pi_0(b_i)\xi_0|| \leq \\ &\leq ||\pi'(c_j)\pi_0(b_i)\xi_0 - \pi'(c_j)\pi'(b_i)\xi'|| + ||\pi'(c_j b_i)\xi' - \pi_0(c_j b_i)\xi_0|| \leq \\ &\leq \max_j ||c_j||.||\pi_0(b_i)\xi_0 - \pi'(b_i)\xi'|| + ||\pi'(c_j b_i)\xi' - \pi_0(c_j b_i)\xi_0|| \leq \epsilon \end{aligned}$$

So far we achieved "$||\pi(c_j)\beta_i - \pi_0(c_j)\beta_i||$ arbitrarily small". Notice that π' (on its essential space) is a GNS for φ but it is not necessarily true that $\theta_\xi(\pi') = \varphi$. We will correct this in part 3.

To obtain "$||\pi(c_j)\gamma_i - \pi_0(c_j)\gamma_i||$ arbitrarily small", we can simply embed $H_\varphi \hookrightarrow H' \subset H$ in the previous procedure, where $H' = H \cap [\gamma_i]^\perp$, so the resulting π' has essential space orthogonal to every γ_i, and $\pi'(c_j)\gamma_i - \pi_0(c_j)\gamma_i = 0$ $(j = 1, ..., n)$. Actually, for the next part, we will also need $\xi - \xi_0 \perp \pi'(1)H$, so instead of H' we embed H_φ inside $H'' = H' \cap [\xi - \xi_0]^\perp$ for previous procedure (this works because $\pi_0(1)H \subset H''$).

Part 3:

We will rotate π' slightly to a representation π so that $\theta_\xi(\pi) = \varphi$. First we need a unit vector η close to ξ such that $\eta - \xi' \perp \pi'(1)H$.

In case $\xi = \xi_0$, since ξ' is close to ξ_0 and $||\xi'|| \leq ||\xi|| = 1$, we can take a small $v \in H$ orthogonal to $\pi'(A)H$ such that $||\xi' + v|| = 1$, and define $\eta := \xi' + v$.

In case $\xi \neq \xi_0$, we take $\eta = \lambda(\xi - \xi_0) + \xi'$. To determine λ:

$$||\eta||^2 = |\lambda|^2.||\xi - \xi_0||^2 + ||\xi'||^2 = |\lambda|^2.(1 - ||\xi_0||^2) + ||\varphi|| = |\lambda|^2.(1 - ||\varphi_0||) + ||\varphi||$$

so we choose $\lambda = \frac{1-||\varphi||}{1-||\varphi_0||}$ to obtain $||\eta|| = 1$. Since $\varphi(1)$ is close to $\varphi_0(1)$, λ is close to 1 and therefore η is close to ξ (i.e.: η is arbitrarily close to ξ as long as δ is sufficiently small).

Now, having η we just apply $U := U_{\eta \to \xi} \in U(H)$ (lemma 5.5), and take $\pi(-) := U^{-1}\pi'(-)U$. Since $||U - Id|| = ||\eta - \xi||$, we still have "$||\pi(c_j)\alpha_i - \pi_0(c_j)\alpha_i||$ arbitrarily small", so $\pi \in V$ and $\theta_\xi(\pi) = \varphi$.

b) Clearly we have the restriction $rep_\xi(A : H) \xrightarrow{\theta_\xi} S(A)$. Furthermore $\theta_\xi^{-1}(S(A)) = rep_\xi(A : H)$. Let $D \subset S(A)$ be a set such that $\theta_\xi^{-1}(D)$ is open in $rep_\xi(A : H)$, so $\theta_\xi^{-1}(D) = U \cap rep_\xi(A : H)$ with U open in $rep(A : H)$. Let $\varphi_0 \in D$. We take a preimage π_0 as before. We already know that $\theta_\xi(U)$ contains an open neighborhood $W \ni \varphi_0$, W open in $Q(A)$. Now it is easy to check:

$$W \cap S(A) \subset \theta_\xi(U \cap rep_\xi(A : H)) \subset D$$

Thus, D is open in $S(A)$. $\square$

5.10. **Theorem** (Takesaki-Bichteler duality). *Every C^*-algebra A is isomorphic to the set of continuous fields $\mathcal{C}_0(rep(A : H))$, where H is a Hilbert space whose dimension is greater or equal to the dimension of any cyclic representation of A.*

Proof. We already know that an element of A defines a continuous field. Now take a *wot*-continuous field T, and assume $1 \in A$. Since fields form the universal W^*-algebra of A (4.7), by lemma 5.2 we have a $f_T \in AN_0(Q(A))$. Using the description from 3.2 we can see that $\langle T(\pi)\xi, \xi\rangle = f_T(\langle \pi(-)\xi, \xi\rangle)$, $\forall \pi \in rep(A : H)$. In other words, the following diagram commutes:

$$\begin{tikzcd} rep(A : H) \arrow[r, "\theta_\xi"] \arrow[d, "T"'] & Q(A) \arrow[d, dotted, "f_T"] \\ B(H) \arrow[r, "\langle (-)\xi, \xi \rangle"] & \mathbb{C} \end{tikzcd}$$

Since θ_ξ is a quotient, f_T is continuous, so, by lemma 5.2 it is an element of A.

For the case $1 \notin A$ we consider the minimal unitization $A \overset{i}{\hookrightarrow} \widetilde{A}$. Let T be a *wot*-continuous field on $rep(A : H)$. T defines a *wot*-continuous field $\widetilde{T}$ over $rep(\widetilde{A} : H)$, simply restricting representations of $\widetilde{A}$ to A

(it is not hard to check that $\widetilde{T}$ is compatible with intertwiners). So $\widetilde{T} = (a, \lambda) \in \widetilde{A}$. But $\widetilde{T}(\rho) = 0$, where ρ is the trivial representation obtained by $\widetilde{A} \to \widetilde{A}/A \simeq \mathbb{C} \xrightarrow{(-)Id} B(H)$.

$$0 = T(\rho) = \rho\big((a, \lambda)\big) = \lambda.Id$$

Thus $\lambda = 0$ and $T \in A$.

□

5.11. **Remark.** For unital A, a field over $Rep(A : H)$ only needs to be continuous on $Rep_{\xi}(A : H)$ to be an element of A. This is because of part (b) of theorem 5.9 and remark 5.3.

5.12. **Remark.** The reason why this duality theorem is not exactly a noncommutative version of Gelfand duality, is that it uses all of the representations of A instead of just the irreducible ones. This motivated a lot of subsequent work, like [4], [3], [15]. In the latter, Fujimoto obtained (theorem 2.3) that A can be recovered as the set of uniformly continuous fields on $Irr(A : H) \cup \{0\}$ vanishing at 0, where $Irr(A : H) \cup \{0\} \subset Rep(A : H)$. $Irr(A : H)$ is the space of irreducible representations on a sufficiently large H.

5.1. Multipliers in terms of fields.

We want to give a description of $M(A)$ using fields. The multiplier algebra of a commutative C^*-algebra is given by $M(C_0(X)) = C_b(X)$. The point at infinity corresponds to the zero representation. So the multipliers should be fields that are continuous except maybe at the 0 representation. We would like to use a definition of continuity for fields over $rep(A)$ (directly avoiding nondegenerate representations) to obtain $M(A) =$ "$\mathcal{C}_b(rep(A))$", but it is not clear to us whether this is possible. We will use instead a slightly more intrincate notion: a field T will be nondegenerately continuous if for every convergent net of representations $\pi_j \to \pi$, $T(\pi_j)$ converges to $T(\pi)$ on the nondegenerate part of π, $\overline{\pi(A)H}$. More precisely:

5.13. **Definition.** *A field T over $rep(A : H)$ is called:*

- *s-nondegenerately continuous if for every convergent net $\pi_j \to \pi$ in $rep(A : H)$ and $\alpha \in \overline{\pi(A)H}$ we have $T(\pi_j)\alpha \to T(\pi)\alpha$.*
- *as-nondegenerately continuous if for every convergent net $\pi_j \to \pi$ in $rep(A : H)$ and $\alpha \in \overline{\pi(A)H}$ we have $T(\pi_j)^*\alpha \to T(\pi)^*\alpha$ ("as" stands for "antistrong").*
- *s^*-nondegenerately continuous if it is both s and as-nondegenerately continuous.*
- *w-nondegenerately continuous if for every convergent net $\pi_j \to \pi$ in $rep(A : H)$ and $\alpha, \beta \in \overline{\pi(A)H}$ we have $\langle T(\pi_j)\alpha, \beta\rangle \to \langle T(\pi)\alpha, \beta\rangle$.*

We denote with $\mathcal{C}^s(rep(A : H))$, $\mathcal{C}^{as}(rep(A : H))$, $\mathcal{C}^{s^*}(rep(A : H))$ and $\mathcal{C}^w(rep(A : H))$ the respective sets of nondegenerately continuous fields.

5.14. **Remark.** These notions of continuity cannot apparently be expressed as continuity of certain maps between topological spaces. However, we can describe them as continuity of certain maps at certain points. For example, a field T over $rep(A:H)$ is s^*-nondegenerately continuous if and only if for every $\pi \in rep(A:H)$ and $\alpha \in \overline{\pi(A)H}$, the maps

$$rep(A:H) \longrightarrow H$$
$$\pi' \longmapsto T(\pi')\alpha$$
$$\pi' \longmapsto T(\pi')^*\alpha$$

are continuous at π. Thus, they are genuine continuity notions.

5.15. **Theorem.**
1) $\mathcal{C}^s(rep(A:H)) = LM(A)$
2) $\mathcal{C}^{as}(rep(A:H)) = RM(A)$
3) $\mathcal{C}^{s*}(rep(A:H)) = M(A)$
4) $\mathcal{C}^w(rep(A:H)) = QM(A)$

Proof. As we remarked after its definition in the preliminaries, $QM(A)$ lies inside the bicommutant of A with respect to any faithful nondegenerate representation. Therefore $QM(A) \subset W^*(A)$ and we have the following characterizations for the multiplier spaces:
$LM(A) = \{T \in W^*(A)/\ Ta \in A\ \forall a \in A\}$
$RM(A) = \{T \in W^*(A)/\ aT \in A\ \forall a \in A\}$
$M(A) = \{T \in W^*(A)/\ Ta \in A \wedge aT \in A\ \forall a \in A\}$
$QM(A) = \{T \in W^*(A)/\ aTb \in A\ \forall a, b \in A\}$

1) Take $T \in LM(A)$. We must see that T is s-nondegenerately continuous. If $\pi_j \to \pi$ is a convergent net in $rep(A:H)$, and $w = \pi(a)v \in \pi(A)H$,

$$||(T(\pi_j) - T(\pi))w|| \leq$$
$$\leq ||(T(\pi_j)\pi(a)v - T(\pi_j)\pi_j(a)v|| + ||T(\pi_j)\pi_j(a)v - T(\pi)\pi(a)v|| \leq$$
$$||T||\epsilon' + ||(T\widehat{a}(\pi_j) - T\widehat{a}(\pi))v|| \leq \epsilon$$

In the last line we used $Ta \in A$. Elements $\pi(a)v$ generate $\overline{\pi(A)H}$, therefore we can reach $||(T(\pi_j) - T(\pi))w|| \leq \epsilon$ for any $w \in \overline{\pi(A)H}$, i.e. $T \in \mathcal{C}^s(rep(A:H))$.

For the reverse inclusion, the proof is very similar: take $T \in \mathcal{C}^s(rep(A:H))$, and $a \in A$. Let us see that $Ta \in \mathcal{C}_0(rep(A:H))$, which is equal to A by 5.10. Let $\pi_j \to \pi$ be a convergent net of representations.

$$||(T\widehat{a}(\pi) - T\widehat{a}(\pi_j))v|| = ||(T(\pi)\pi(a) - T(\pi_j)\pi_j(a))v|| \leq$$
$$\leq ||(T(\pi)\pi(a) - T(\pi_j)\pi(a))v|| + ||T(\pi_j)(\pi(a) - \pi_j(a))v|| \leq$$
$$\epsilon_1 + ||T||\epsilon_2 \leq \epsilon$$

where the first term is small thanks to the continuity hypothesis for T and the second because $\pi_j \to \pi$. This shows $Ta \in A$, or $T \in LM(A)$.

2) follows from $LM(A)^* = RM(A)$, $\mathcal{C}^s(rep(A:H))^* = \mathcal{C}^{as}(rep(A:H))$ and 1).

3) follows from 1), 2) and

$$M(A) = LM(A) \cap RM(A)$$
$$\mathcal{C}^{s^*}(rep(A:H)) = \mathcal{C}^s(rep(A:H)) \cap \mathcal{C}^{as}(rep(A:H))$$

4) The idea from 1) applies. Start with a $T \in QM(A)$ and consider $\pi_j \to \pi$, $\pi(a)v$, $\pi(b)w$. w-nondegenerately continuity of T follows from next computation.

$$\begin{gathered}|\langle T(\pi_j)\pi(a)v, \pi(b)w\rangle - \langle T(\pi)\pi(a)v, \pi(b)w\rangle| \leq \\ |\langle T(\pi_j)\pi(a)v, \pi(b)w\rangle - \langle T(\pi_j)\pi_j(a)v, \pi(b)w\rangle| + \\ +|\langle T(\pi_j)\pi_j(a)v, \pi(b)w\rangle - \langle T(\pi_j)\pi_j(a)v, \pi_j(b)w\rangle| + \\ +|\langle \widehat{b^*}T\widehat{a}(\pi_j)v, w\rangle - \langle \widehat{b^*}T\widehat{a}(\pi)v, w\rangle| \leq \\ ||T||.\epsilon_1.||b||.||w|| + ||T||.||a||.||v||.\epsilon_2 + \epsilon_3\end{gathered}$$

And similarly, if $T \in \mathcal{C}^w(rep(A:H))$, we get $aTb \in A$, for every $a, b \in A$. □

6. Universal W^*-algebra for topological groups

For a locally compact Hausdorff topological group G, there are many interesting associated algebras, such as $L^1(G)$, $\mathcal{M}(G)$[2] and $C^*(G)$ the universal group C^*-algebra (or full C^*-algebra). $L^1(G)$ is a Banach algebra having the same representations as G. $C^*(G)$ is usually constructed starting from $L^1(G)$, changing the norm to:

$$||f||_{C^*(G)} := \sup_{\pi \in rep(G)} ||\pi(f)||$$

and taking the completion with respect to this norm. The result is a C^*-algebra whose nondegenerate representations coincide with the unitary representations of G (see [9], chapter VII). A morphism $G \xrightarrow{f} G'$ doesn't induce $C^*(G) \to C^*(G')$ in general, but it does if f is surjective, an open inclusion, or a finite composition of such morphisms.

The universal von Neumann algebra $W^*(G)$ $(= W^*(C^*(G)))$ contains $L^1(G)$, $C^*(G)$, $\mathcal{M}(G)$ as subalgebras and G as a subgroup of its unitary group. $W^*(G)$ also has the same representations as G.

$$\begin{array}{ccccc} G & \subset & M(G) & \subset & W^*(G) \\ & & \cup & & \cup \\ & & L^1(G) & \subset & C^*(G) \end{array}$$

This algebra along with these properties has been studied by John Ernest in [13] for second countable locally compact Hausdorff groups. He defined

[2] $\mathcal{M}(G)$ is the algebra of all complex valued finite regular measures on G.

$W^*(G)$ as the set of fields over the representations of G. Ernest's procedure can actually be applied to every topological group to obtain $W^*(G)$, as we shall do now. However, for non locally compact groups some properties might be lost. For example, the canonical application $G \to W^*(G)$ won't be injective. The importance of removing locally compactness from the hypothesis, for us, is that it allows to take the W^*-algebra of $U(M)$, the unitary group of a von Neumann algebra M, so we can prove that W^* is left adjoint to U.

The construction of $W^*(G)$ is very similar to that of A^F. We consider fields over the category of representations of G.

6.1. **Definition.** *We call "field" over $rep(G)$ a function T assigning to each $\pi \in rep(G)$, $G \xrightarrow{\pi} U(H_\pi)$, an element $T(\pi) \in B(H_\pi)$ in a bounded and coherent way with morphisms. Explicitly: $\sup_\pi ||T(\pi)|| < \infty$, and if $H_{\pi_1} \xrightarrow{S} H_{\pi_2}$ is an intertwiner ($S\pi_1(g) = \pi_2(g)S$) then $ST(\pi_1) = T(\pi_2)S$. In other words, fields are bounded endomorphisms of the forgetful functor $rep(G) \to \mathcal{H}$, where $\mathcal{H}$ is the category of Hilbert spaces.*

6.2. **Definition.** *$W^*(G)$ is the set of fields over $rep(G)$ with $*$-algebra operations defined pointwise and the norm given by $||T|| = \sup_{\pi \in rep(G)} ||T(\pi)||$.*

6.3. **Proposition.** *For a locally compact Hausdorff group G, we have $W^*(G) = W^*(C^*(G))$ as W^*-algebras.*

Proof. Since $rep(G) = rep(C^*(G))$, the set of fields over these categories coincide. □

However, for general topological groups we don't have $C^*(G)$, so it is technically necessary to redo some work in order to obtain the basic properties for $W^*(G)$. We write down all the statements but omit some of the proofs. First of all, it is clear that $W^*(G)$ is a C^*-algebra.

6.4. **Proposition.** *If T is a field over $rep(G)$, $T(\pi)$ belongs to the von Neumann algebra generated by $\pi(G)$ for every $\pi \in rep(G)$.*

Proof. An operator $S \in \pi(G)'$ is an endomorphism of π, so it commutes with $T(\pi)$ because of compatibility with morphisms. Therefore, $T(\pi) \in \pi(G)''$. □

The following proposition allows to replace $rep(G)$ by the essentially small category $cyc(G)$ of cyclic representations (if G is compact, we might replace $cyc(G)$ by the finite dimensional representations). The statement and also the proof are the same as 4.3.

6.5. **Proposition.** *The set of fields over $rep(G)$ is equal to the set of fields over $cyc(G)$.*

6.6. **Proposition.** *$W^*(G)$ is a von Neumann algebra.*

Proof. Take the Hilbert

$$H = \bigoplus_{\pi \in cyc(G)} H_\pi \quad , \quad W^*(G) \xrightarrow{\Pi} B(H) \quad , \quad \Pi(T) = \bigoplus_{\pi \in cyc(G)} T(\pi)$$

Π is clearly a faithful representation. Let us see that the image is strongly closed. Assume $\Pi(T_\mu) \to S$ for the *sot*. If $\alpha \in H_\pi$, $\Pi(T_\mu)\alpha = T_\mu(\pi)\alpha \to S\alpha$, then $S\alpha \in H_\pi$. This means $S = \bigoplus S_\pi$. Putting $T(\pi) = S_\pi$, it follows easily that T is a field and $\Pi(T) = S$. □

6.7. **Observation.** Consider Π, the faithful representation from proposition 6.6, and $\Pi_G = \bigoplus_{\pi \in cyc(G)} \pi$ the representation of G acting on the same Hilbert as Π, $H = \bigoplus_{\pi \in cyc(G)} H_\pi$. For $T \in W^*(G)$ we have $T(\Pi_G) = \Pi(T)$. This can easily be checked on a vector $\alpha \in H_\pi$ by applying compatibility of T with the inclusion morphism $H_\pi \hookrightarrow H$, and this is sufficient.

6.8. **Proposition.** *There exists a canonical continuous function $G \xrightarrow{\wedge} W^*(G)$. The elements $\hat{g}$ are unitaries and generate $W^*(G)$ as a von Neumann algebra.*

Proof. $\hat{g}(\pi) := \pi(g)$ defines a unitary field. Next we show continuity. σ-weak topology, *wot* and *sot* coincide on the unitary group for every von Neumann algebra. $g_\mu \to g$ implies $\Pi(\hat{g_\mu})\alpha = \hat{g_\mu}(\pi)\alpha = \pi(g_\mu)\alpha \to \pi(g)\alpha$ for $\alpha \in H_\pi$ (because π is continuous), and this is easily generalized for all $\alpha \in H$, so $\Pi(\hat{g_\mu}) \xrightarrow{sot} \Pi(\hat{g})$. Now we'll see $\Pi(W^*(G)) = \Pi(\hat{G})''$. Let $T \in W^*(G)$. If $S \in \Pi(\hat{G})'$, it is an endomorphism of Π_G. Compatibility says:

$$ST(\Pi_G) = T(\Pi_G)S$$

$$S\Pi(T) = \Pi(T)S$$

proving $\Pi(T) \in \Pi(\hat{G})''$ and therefore $\Pi(W^*(G)) \subset \Pi(\hat{G})''$. The other inclusion holds because the bicommutant is the smallest von Neumann algebra containing $\Pi(\hat{G})$. □

The definition of field we use presents the same differences with the historic definition in Ernest's [13] as in the case for C^*-algebras with its corresponding references. Here we rewrite the facts that explain why the definitions are equivalent (4.8, 4.9). The proofs are the same.

6.9. **Proposition.** *Let T be a function that assigns to each $\pi \in rep(G)$ a bounded operator on H_π in a compatible way with those intertwiners that are partial isometries. Then T is compatible with every intertwiner.*

6.10. **Corollary.** *For a map T that assigns to each $\pi \in rep(G)$ a bounded operator on H_π, it suffices that it preserves finite direct sums and unitary equivalences to be compatible with every intertwiner.*

6.1. $\mathcal{G}r \xrightarrow{W^*} \mathcal{W}_1^*$ **is left adjoint to** $\mathcal{W}_1^* \xrightarrow{U} \mathcal{G}r$**.**

If M is a von Neumann algebra, the set of unitaries $U(M)$ with the σ-weak topology is a Hausdorff topological group. The product operation $U(M) \times U(M) \to U(M)$ is continuous. Proof: consider a faithful representation of M. The *sot* coincides with the *wot* and σ-weak in $U(M)$; composition is jointly continuous for the *sot* over bounded sets. The "inverse" application $U(M) \xrightarrow{(-)^{-1}} U(M)$ is equal to the involution $*$, continuous for the weak topology.

Functoriality of U is clear, so we have a functor $\mathcal{W}_1^* \xrightarrow{U} \mathcal{G}r$.

6.11. **Proposition.** *Let M be a von Neumann algebra and $G \xrightarrow{f} U(M)$ a continuous morphism of groups. There exists a unique morphism of W^*-algebras $W^*(G) \xrightarrow{\widetilde{f}} M$ such that the triangle commutes.*

$$\begin{tikzcd}
W^*(G) \arrow[r, dotted, "\exists ! \widetilde{f}"] & M \\
G \arrow[u, "\wedge"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof. Uniqueness is clear, since G generates $W^*(G)$ as a W^*-algebra. The rest of the proof is also equal to that of 4.11

$\square$

6.12. **Observation.** From previous proposition it follows
$rep(G) = rep(W^*(G))$.

6.13. **Corollary.** *W^* is a functor $\mathcal{G}r \xrightarrow{W^*} \mathcal{W}_1^*$ left adjoint to $\mathcal{W}_1^* \xrightarrow{U} \mathcal{G}r$.*[3]

Proof. Functoriality is a direct consequence of the previous proposition applied to:

$$\begin{tikzcd}
W^*(G) \arrow[r, dotted, "\widetilde{f}"] & W^*(K) \\
G \arrow[u] \arrow[r, "f"] & K \arrow[u]
\end{tikzcd}$$

A composition $G \to K \to L$ is preserved thanks to uniqueness.

The adjunction $W^* \dashv U$ also follows immediately: a morphism $G \to U(M)$ induces $W^*(G) \to M$, and a morphism $W^*(G) \to M$ can be restricted to $G \to U(M)$ composing with the canonical map $G \to W^*(G)$. Again uniqueness from last proposition allows to prove that these correspondences $[W^*(G), M] \rightleftarrows [G, U(M)]$ are mutually inverse, and natural in both variables. $\square$

[3]We want to thank Martin Wanvik for his previous announcement of a result related to this one.

6.14. **Example.** $W^*(\mathbb{F}_n) = \mathop{\ast}\limits_{i=1}^{n} W^*(\mathbb{Z}) = \mathop{\ast}\limits_{i=1}^{n} \mathcal{M}(S^1)^*$, where $\mathbb{F}_n$ is the free group of n generators. The free product of W^*-algebras is the colimit in the category $\mathcal{W}_1^*$.

Proof. The first equality follows from the fact that the free product is a coproduct in both categories: $\mathcal{G}r$ and $\mathcal{W}_1^*$, and W^* preserves colimits for being a left adjoint functor. For the second equality we use the fact $C^*(G) = C_0(G^*)$ for abelian G ([9], proposition VII.1.1; G^* is the dual group). Then $W^*(\mathbb{Z}) = C^*(\mathbb{Z})^{**} = C(S^1)^{**} = \mathcal{M}(S^1)^*$. □

6.2. **Locally compact Hausdorff groups.**

In this case we will see that G is a topological subspace of $W^*(G)$. Since cyclic representations of G separate points, the canonical map $G \to W^*(G)$ is injective.

6.15. **Lemma.** *For a locally compact Hausdorff group G, the topology of G is the initial topology with respect to the family of positive type functions.*

Proof. Let τ_p be the topology generated by the positive type functions. Every element in τ_p is an open set of G. So it is enough to prove that for every $x \in G$, U open set of G containing x, there exists an open set $W \in \tau_p$ such that $x \in W \subset U$. First we assume $x = 1$. Let V be an open set of G with compact closure such that $V^2 \subset U$ y $V^{-1} = V$. The function $\chi_V * \chi_V$ is continuous, positive type, it annihilates outside U and takes the value $|V| > 0$ on 1. With this function it is easy to find a W as required. If we now take any $x \in G$, we can translate it to 1. A translation of a function of positive type is a linear combination of positive type functions, as the following calculation shows:

$$\begin{aligned}
&\langle \pi(g^{-1}x)\xi, \xi\rangle = \langle \pi(x)\xi, \pi(g)\xi\rangle = \langle \pi(x)\alpha, \beta\rangle \\
&= {}^1\!/_4\Big(\langle \pi(x)(\alpha+\beta), \alpha+\beta\rangle - \langle \pi(x)(\alpha-\beta), \alpha-\beta\rangle + \\
&+ i\langle \pi(x)(\alpha+i\beta), \alpha+i\beta\rangle - i\langle \pi(x)(\alpha-i\beta), \alpha-i\beta\rangle\Big)
\end{aligned}$$

where $\alpha = \xi$ y $\beta = \pi(g)\xi$. □

6.16. **Proposition.** *Let G be a locally compact Hausdorff group. G is a topological subspace of $W^*(G)$ through the canonical inclusion $G \overset{\wedge}{\hookrightarrow} W^*(G)$.*

Proof. The σ-weak topology of $W^*(G)$ is, by definition, initial with respect to $C^*(G)^*$. Since $C^*(G)^*$ is linearly generated by the positive functionals, these suffice to generate the topology. The topological inclusion $\widehat{G} \hookrightarrow W^*(G)$ is of course initial, so if we compose it with the positive functionals we have an initial family that is, as we will check now, equal to the class of all positive type functions. For a $\varphi \in C^*(G)^*$, $0 \neq \varphi \geq 0$, we have the representation $\widetilde{\pi_\varphi}$ of $W^*(G)$ such that $\varphi = \langle \widetilde{\pi_\varphi}(-)\xi, \xi\rangle$ on $W^*(G)$. The restriction to G is the positive type function f associated to the representation $\pi = \widetilde{\pi_\varphi} \circ \wedge$.

Conversely, for a positive type function $f \neq 0$ over G, there is an associated representation whose extention to $W^*(G)$ gives a positive $\varphi \in C^*(G)^*$ extending f.

Now the result follows from previous lemma. □

6.2.1. *Duality for locally compact Hausdorff groups.*

Consider the following set:

6.17. **Definition.**

$$G_{\otimes} := \{T \in W^*(G) \setminus \{0\} / T(\pi_1 \otimes \pi_2) = T(\pi_1) \otimes T(\pi_2)\, \forall \pi_1, \pi_2 \in rep(G)\}$$

Elements in $G_{\otimes}$ have norm less or equal than 1, because if $T \in G_{\otimes}$, $||T|| > 1$, then $||T(\pi) > 1||$ for some $\pi \in rep(G)$, and we obtain

$$||T(\pi \otimes ... \otimes \pi)|| = ||T(\pi) \otimes ... \otimes T(\pi)|| = ||T(\pi)||^n$$

that is unbounded. The unit ball in $W^*(G)$ is compact by Banach-Alaoglu's theorem, therefore closed because the σ-weak topology is Hausdorff. $G_{\otimes}$ is closed for that topology, and for the product and involution operations. Thus $G_{\otimes}$ is a compact semitopological semigroup. "Semitopological" means that the product is separately continuous in each variable but not necessarily jointly continuous. $G_{\otimes}$ contains G.

Take an invertible element $T \in G_{\otimes}$. The inverse T^{-1} also belongs to $G_{\otimes}$. $1 = ||TT^{-1}|| \leq ||T||.||T^{-1}|| \leq 1$. Then equality holds and $||T|| = ||T^{-1}|| = 1$. Thinking T and T^{-1} as operators on a Hilbert space, they must be isometries, i.e. unitaries. Thus, invertible elements of $G_{\otimes}$ are unitaries. Tatsuuma duality theorem [35] asserts that the set of invertible elements of $G_{\otimes}$ is exactly G (it seems interesting the question of whether $\overline{G} = G_{\otimes}$). If G is compact, we can reduce $rep(G)$ to the category of finite dimensional representations of G (6.5 and previous remark) so that Tatsuuma's theorem becomes Tannaka duality theorem.

An interesting point of view for this, due to M. E. Walter [38], is the following. An element $f \in C^*(G)^*$ is a linear combination of states of $C^*(G)$. States of $C^*(G)$ are equivalent to positive definite functions on G. A pointwise multiplication of two positive definite functions on G is positive definite (actually it is associated to the tensor product of the corresponding GNS representations). Thus this multiplication defines a product on $C^*(G)^*$ that makes it a commutative semisimple Banach algebra with unit and involution. This algebra is called "Fourier-Stieljes algebra of G". It is not hard to see that the characters of this algebra are exactly the elements of $G_{\otimes}$.

Chapter 2

We begin this chapter by studying several aspects of the state space, pure state space and spectrum of C^*-algebras that are necessary for our results on Tietze extension theorem (theorem 11.10, proposition 11.3) and for chapter 3.

Proposition 7.7 (from [21], page 328) deals with representations of unital C^*-algebras as functions on a compact space X: $A \to C(X)$. These representations preserve the linear structure, the involution, the unit and the order structure, but forget the product. That is why we need a result on partially ordered vector spaces. More precisely, we need to extend a state on a subspace of a partially ordered vector space to the whole space (proposition 7.5; [20] corollary 2.1). Thus, we summarize those needed results from [20]. Having this, we can prove proposition 7.7: if such a representation $A \to C(X)$ is an order isomorphism between A and its image, then, essentially, X is a subspace of $S(A)$, it contains $P(A)$ and the morphism $A \hookrightarrow C(X)$ is the natural one.

We then study the maps between state spaces induced by C^*-morphisms in these two different cases: a proper morphism $A \to B$ and an ideal $I \subset A$. Proposition 7.7 allows to prove that an essential ideal $I \subset A$ induces a dense subspace $P(I) \subset P(A)$. After showing the bijection between ideals of A and saturated closed subsets of $P(A)$, we define the spectrum $\widehat{A}$, the primitive spectrum $prim(A)$ (whose topologies result isomorphic to the lattice of ideals of A) and show some other important properties of their topologies.

In the next section we state without proof a noncommutative version of Stone-Weierstrass theorem due to Glimm (theorem 10.2) for its reference in proposition 11.3 and theorem 13.7. We include an interesting proposition related to the general noncommutative Stone-Weierstrass problem, proposition 10.3: if a C^*-subalgebra $A \subset B$ separates $P(B) \cup \{0\}$ then $P(A) \simeq P(B)$ through the restriction map.

The last section of this chapter is about Tietze extension theorem for C^*-algebras. Proposition 11.3 aims at generalizing Pedersen's noncommutative Tietze extension theorem (11.2) while theorem 11.10 presents a different statement that we call "Tietze extension theorem for C^*-algebras". We include simple proofs for the Dauns-Hofmann theorem (theorem 11.4) and its corollary 11.6, which are needed in 11.10.

7. C^*-ALGEBRAS REPRESENTED AS SETS OF FUNCTIONS ON A COMPACT SPACE

7.1. **States on partially ordered vector spaces.** The content of this subsection essentially comes from [20].

7.1. **Definition.** *A "partially ordered vector space" will be a vector space over the reals, V, with a partial ordering, satisfying:*

(1) $0 \le \lambda \in \mathbb{R}$, $0 \le a, b \in V \Rightarrow a + b \ge 0$ *and* $\lambda a \ge 0$.

(2) There is an element $e \in V$, the "order unit", such that for each $a \in V$ there exists a real $\lambda \geq 0$ with $-\lambda e \leq a \leq \lambda e$.

The only examples we are interested in are:

- The set of self adjoint elements of a unital C^*-algebra.
- A subspace (containing 1) of $C(X, \mathbb{R})$, the continuous real valued functions on a compact topological space X.

7.2. **Definition.** *A state on a partially ordered vector space is a normalized positive linear functional $V \xrightarrow{\varphi} \mathbb{R}$ (i.e: $\varphi(a) \geq 0$ for $a \geq 0$ and $\varphi(e) = 1$). We call $S(V)$ the state space of V with the w^*-topology.*

Note: if $V \xrightarrow{\varphi} \mathbb{R}$ is a positive linear functional with $\varphi(e) = 0$, then $\varphi = 0$. It follows from condition (2).

Having a state over the self adjoint part of a C^*-algebra A in this sense is equivalent to having a state of A. In particular, a state on $C(X, \mathbb{R})$ in this sense is equivalent to a state of $C(X)$ as a complex C^*-algebra. Moreover, the corresponding convex structures also coincide, along with the derived notion of "pure state".

7.3. **Proposition.** *$S(V)$ is compact for any partially ordered vector space V.*

Proof. The usual proof for Banach-Alaoglu theorem works:

For every $a \in V$, taking $\lambda \in \mathbb{R}_{\geq 0}$ as in condition (2) of the definition, we get an interval $I_a = [-\lambda, \lambda]$ such that $\varphi(a) \in I_a$ for every $\varphi \in S(V)$. Thus, we have a topological embedding $S(V) \hookrightarrow \prod_{a \in V} I_a$. Since $\prod_{a \in V} I_a$ is compact by Tychonoff theorem and the image is closed, $S(V)$ is compact. □

Krein-Milman theorem applies: $S(V) = \overline{co(P(V))}$, i.e, $S(V)$ is equal to the closure of the convex set generated by the pure states $P(V)$.

7.4. **Definition.** *An ideal of a partially ordered vector space V is a linear subspace I with the property that $-a \leq b \leq a$ for $a \in I$ implies $b \in I$.*

Ideals are the kernels of morphisms between ordered vector spaces: if $I \subset V$ is a proper ideal, ${}^{V}\!/_{I}$ has naturally a structure of ordered vector space. The kernel of a morphism $V \to \mathbb{R}$ (a state) is a maximal ideal. Next we will show that if I is a maximal ideal, then ${}^{V}\!/_{I} \simeq \mathbb{R}$.

$U = {}^{V}\!/_{I}$ is simple, i.e. it has no proper ideals. Let us see that U is totally ordered. Take $0 \neq a \in U$. $J = \{x \in U / \exists \alpha, \beta \in \mathbb{R} \text{ such that } \alpha a \leq x \leq \beta a\}$ is an ideal of U containing a, so $J = U$. From $\alpha a \leq e \leq \beta a$, it follows $0 \leq (\beta - \alpha)a$. If $\beta = \alpha$ ($\neq 0$ otherwise $e = 0$) $e = \alpha a$, and a is positive or negative. If $\beta \neq \alpha$ we also have a positive or negative. Thus, U is totally ordered. Now, for each $x \in U$, the line λe is divided in two half-lines: the ray of elements lower than x, and the ray of elements greater than x. So we can assign to each $x \in U$ the only number $f(x)$ (the division point) such that $x \leq \alpha e$ for every $\alpha \geq f(x)$ and $x \geq \beta e$ for every $\beta \leq f(x)$. f is linear, and it is a state. The kernel is trivial since U is simple, so it is an isomorphism of partially ordered vector spaces.

Besides, the isomorphism $U \simeq \mathbb{R}$ is unique, since there is only one isomorphism $\mathbb{R} \simeq \mathbb{R}$.

7.5. **Proposition.** *Let V be a partially ordered vector space and W a subspace containing the order unit e. Any $W \xrightarrow{\varphi} \mathbb{R}$ state of W can be extended to a state of V. If φ is pure as a state of W, there is a pure extension.*

Proof. Let $I = \{x \in V / \exists u \in ker(\varphi) / -u \leq x \leq u\}$. I is an ideal of V. Actually it is the ideal generated by $ker(\varphi)$. $e \notin I$ because $e \notin ker(\varphi)$. By Zorn's lemma we can take a maximal ideal containing I. By the argument before the proposition, the quotient gives a state $V \xrightarrow{\tilde{\varphi}} \mathbb{R}$ that restricted to W vanishes at $ker(\varphi)$. Therefore, by uniqueness, $\tilde{\varphi}|_W = \varphi$. Assume φ is pure. The set of possible extensions $C = \{\phi \in S(V)/\phi|_W = \varphi\}$ is convex and w^*-closed, therefore compact. Let ϕ be an extremal point of C. We will show that ϕ is a pure state of V. Assume

$$\phi = \lambda\phi_1 + (1-\lambda)\phi_2$$

with $\lambda \neq 0, 1$. Restricting to W, we obtain $\phi_1|_W = \phi_2|_W = \varphi$ because φ is pure. Now, by extremality of ϕ in C we get $\phi_1 = \phi_2 = \phi$. □

7.2. **Application to unital C^*-algebras.** Let A be a C^*-algebra with unit. If we take any closed set of states X such that $P(A) \subset X \subset S(A)$, we have a canonical application

$$A \xrightarrow{\Phi} C(X)$$
$$a \mapsto (\varphi \mapsto \varphi(a))$$

It has the following properties: it is linear; unit, order[4] and star preserving. Given $\varphi_1 \neq \varphi_2 \in X$, there exists $a \in A$ such that $\Phi(a)(\varphi_1) \neq \Phi(a)(\varphi_2)$, so Φ separates the points of X. It preserves the norm of every $a \geq 0$ (actually it is isometric on the selfadjoint part because for every selfadjoint $a \in A$ there is a pure state φ such that $\varphi(a) = ||a||$, as a consequence Φ is injective).

In other words, if we think the elements of A as continuous functions on the state space (or a subspace containing the pure states), we keep the linear structure, the unit, the star, and the partial order structure of A (see next lemma). We will show that any compact space X with $A \to C(X)$ having those properties must be of the form $\overline{P(A)} \subset X \subset S(A)$.

7.6. **Lemma.** *Let $X \subset S(A)$ be a compact set of states which realizes the norm of every $a \geq 0$, i.e. $A \xrightarrow{\Phi} C(X)$ preserves the norm of positive elements. Then $A \xrightarrow{\Phi} \Phi(A)$ is an order isomorphism.*

[4] In $C(X)$, $f \leq g$ means $f(x) \leq g(x)$ for every $x \in X$.

Proof. We already know that Φ is order and star preserving. We must only prove that $\Phi(a) \geq 0$ implies $a \geq 0$. Assume first a selfadjoint. Write $a = a_+ - a_-$ with $a_+, a_- \geq 0$ and $a_+ a_- = 0$. Assume $a_- \neq 0$. Take $\varphi \in X$ such that $\varphi(a_-) = ||a_-||$.

$$||a_-|| = \varphi(a_-) = \langle \pi_\varphi(a_-)\xi_\varphi, \xi_\varphi \rangle \leq ||\pi_\varphi(a_-)|| \leq ||a_-||$$

Then the inequalities are equalities. Since we have an equality in Cauchy-Schwartz inequality, we deduce $\pi_\varphi(a_-)\xi_\varphi = \lambda\xi_\varphi$, and λ must be nonzero. Now:

$$\langle \pi_\varphi(a_+)\xi, \pi_\varphi(a_-)\xi \rangle = 0 = \overline{\lambda}\varphi(a_+)$$

Therefore $\Phi(a)(\varphi) = \varphi(a) < 0$, absurd. So $a_- = 0$, $a \geq 0$.

For a not necessarily selfadjoint, write $a = b + ic$, b and c selfadjoint. $\Phi(b + ic) = \Phi(b) + i\Phi(c) \geq 0$. Since $\Phi(b)$ and $\Phi(c)$ are real valued, we get $\Phi(c) = 0$. By the previous case, $c \geq 0$ and $-c \geq 0$, so $c = 0$. Now a is selfadjoint. □

7.7. **Proposition.** *Let A be a unital C^*-algebra. If for a compact space X we have a linear map $A \xrightarrow{T} C(X)$ that is unit and order preserving, separates the points of X and is norm preserving for positive elements (or alternatively $A \xrightarrow{T} T(A)$ is an order isomorphism) then, up to a homeomorphism, we have the topological inclusions $\overline{P(A)} \subset X \subset S(A)$ and through the homeomorphism, T is the canonical application Φ previously treated.*

Proof. Define $X \xrightarrow{j} S(A)$ sending an element x to the linear functional $ev_x \circ T = (a \mapsto T(a)(x))$. $j(x)$ is a state since $ev_x \circ T(1) = 1$ and T is order preserving. j is injective since $T(A)$ separates points and it is continuous because for a convergent net $x_\mu \to x$ we have $ev_{x_\mu} \circ T(a) = T(a)(x_\mu) \to T(a)(x) = ev_x \circ T(a)$. Besides j is closed since X is compact and $S(A)$ is Hausdorff, so $X \subset S(A)$.

Now take $\varphi \in P(A)$. Compose it with $T(A) \xrightarrow{T^{-1}} A$ (T^{-1} is order preserving thanks to previous lemma) and extend $\varphi \circ T^{-1}$ by 7.5 to a pure state $\phi \in P(C(X))$. A pure state of a commutative C^*-algebra ($C(X)$ in this case) is a character; this is because the GNS representation must be irreducible and therefore unidimensional. $\phi = ev_x$ for some $x \in X$. This shows that any pure state of A is equal to evaluation at some $x \in X$, i.e. $P(A) \subset X$ and also $\overline{P(A)} \subset X$ because X is closed. □

In his article on noncommutative Stone-Weierstrass theorem [16], J. Glimm states the following lemma:

"If X is a set of states of a C^*-algebra A such that for each non-zero $a \in A$ there is a $\varphi \in X$ with $\varphi(a) \neq 0$, then the w^*-closure of X contains the pure state space of A."

For the proof he invokes Kadison's [21] page 328. This is where we found essentially our proposition 7.7. It is not clear how this weak condition by

Glimm suffices to conclude $P(A) \subset X$.[5] Even more, for finite dimensional A, say $A = M_2(\mathbb{C})$, Glimm's lemma is false:

Take $A = M_2(\mathbb{C})$. Consider the four vector states corresponding to $(1,0), (0,1), \frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,i)$, in the usual representation on $\mathbb{C}^2$. Applying these states to a general element $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we get:

$$a_{11} \ , \ a_{22} \ , \ \frac{1}{2}(a_{11} + a_{12} + a_{21} + a_{22}) \ , \ \frac{1}{2}(a_{11} - ia_{12} + ia_{21} + a_{22})$$

So the four states form a basis for $M_2(\mathbb{C})^*$ and therefore taking X equal to this four elements set we satisfy this lemma's hypothesis. However, the w^*-closure still has four elements and doesn't contain the (infinite) pure state space of $M_2(\mathbb{C})$.

8. Lemmas on maps between state spaces induced by C^*-morphisms

8.1. **Proposition.** *Let $A \xrightarrow{f} B$ be a proper morphism of C^*-algebras.*

1) The transpose $B^* \xrightarrow{f^*} A^*$ is w^*-w^*-continuous and restricts to an affine continuous map $S(B) \xrightarrow{f^*} S(A)$.

2) f is surjective if and only if $S(B) \xrightarrow{f^*} S(A)$ is injective. In this case φ is pure iff $f^*(\varphi)$ is pure. Besides $S(B) \xhookrightarrow{f^*} S(A)$ and $P(B) \xhookrightarrow{f^*} P(A)$ are closed.

3) f is injective if and only if f^* surjective.

Proof.

1) w^*-w^*-continuity of f^* is a basic fact of Banach spaces. If $\varphi \in S(B)$, $\varphi \circ f$ is positive. It is normalized because f is proper. $S(B) \xrightarrow{f^*} S(A)$ is affine trivially.

2) If f is surjective, injectivity of f^* is clear. If $S(B) \xrightarrow{f^*} S(A)$ is injective, consider the closed subalgebra $A' := f(A) \subset B$. If $A' \neq B$, take $b \in B \setminus A'$ selfadjoint and a functional $\phi \in B^*$ such that $\phi(b) = 1$, $\phi(A') = 0$, whose existence is guaranteed by Hahn-Banach theorem on ${}^{B}\!/\!{}_{A'}$. We keep with the hermitian part of ϕ (that is $\frac{1}{2}(\phi + \phi^*)$) and call it ϕ. We write ϕ as a difference of positive functionals $\phi = \varphi_1 - \varphi_2$. Restricting to A', $\varphi_1|_{A'} = \varphi_2|_{A'}$. Since we have a restriction map $S(B) \to S(A')$, the restriction of any positive functional doesn't shrink the norm.. This implies $||\varphi_1|| = ||\varphi_1|_{A'}|| = ||\varphi_2|_{A'}|| = ||\varphi_2||$. These norms cannot be 0, for this would contradict $\phi(b) = 1$. So we can normalize φ_1 and φ_2 obtaining states whose restrictions to A' are equal, thus $\varphi_1 = \varphi_2$ by hypothesis, reaching again the contradiction $0 = \phi(b) \neq 0$.

[5]According to our analysis we have two possible alternative conditions: that $\varphi(a) \geq 0$ $\forall \varphi \in X$ implies $a \geq 0$, or for every $a \geq 0$ there is a $\varphi \in X$ such that $\varphi(a) = ||a||$.

When f is surjective, if $B \xrightarrow{\pi} B(H)$ is a representation of B, a closed subspace of H is invariant if and only if it is invariant for $\pi \circ f$ (in other words, it is B-invariant iff it is A-invariant). Looking at the GNS representations $\varphi \in S(B)$ is pure iff $f^*(\varphi)$ is pure.

Take $C \subset S(B)$ closed. If $f^*(\varphi_\mu) = \varphi_\mu f \to \phi$ with $\varphi_\mu \in C$ and $\phi \in S(A)$, we have $\phi|_{ker(f)} = 0$, so $\phi = \varphi f$ for some linear functional φ. Clearly $\varphi_\mu \to \varphi$, implying $||\varphi|| \leq 1$. Since f is proper, φ attains the value 1 at an approximate unit, so $\varphi \in S(B)$, and also $\varphi \in C$ because C is closed. Thus $f^*(C)$ is closed, proving $S(B) \overset{f^*}{\hookrightarrow} S(A)$ is closed. For $P(B) \overset{f^*}{\hookrightarrow} P(A)$ closed we apply the same argument but with $C \subset P(B)$ relatively closed and $\phi \in P(A)$. From $\varphi f = \phi$ it results φ pure so we can conclude $\varphi \in C$.

3) If f is injective, A is a subalgebra of B. Any $\varphi \in S(A)$ extends by Hahn-Banach to a linear functional on B of norm 1 attaining its norm at an approximate unit, so it is a state of B. Conversely, if f^* is surjective, take $0 \neq a \in A$, and $\varphi \in S(A)$ such that $\varphi(a) \neq 0$. Now take $\phi \in S(B)$ such that $f^*(\phi) = \varphi$. We have $\phi(f(a)) = \varphi(a) \neq 0$, so $f(a) \neq 0$. □

8.2. **Remark.** Part 2) of previous proposition allows to prove that categorical epimorphisms[6] in $\mathcal{C}^*$ are exactly surjective morphisms (as a consequence, the same holds for $\mathcal{C}^*_p$ and $\mathcal{C}^*_1$). This fact is the content of [19] by K.H. Hofmann and Karl-H. Neeb. Interestingly, this problem had been solved much earlier: see [31] and [23]. Let us summarize the proof from [19] and simplify it a little. Surjectivity clearly implies categorical epimorphism. For the converse, take $A \xrightarrow{f} B$ a categorical epimorphism. Nice lemma 2 from [19] shows that a cyclic representation of B induces, by composition with f, a cyclic representation of A with the same cyclic vector. As a consequence, if (π, H, ξ) is the GNS representation associated to a state $\varphi \in S(B)$, then $(\pi f, H, \xi)$ is the GNS for $\varphi f \in S(A)$. By (the proof of[7]) previous proposition (item 2) it suffices to show that $\varphi_1 f = \varphi_2 f \Rightarrow \varphi_1 = \varphi_2$ for $\varphi_1, \varphi_2 \in S(B)$. Let (π_1, H_1, ξ_1) and (π_2, H_2, ξ_2) be the GNS representations of φ_1 and φ_2. $(\pi_1 f, H_1, \xi_1)$ and $(\pi_2 f, H_2, \xi_2)$ are then the GNS representations of $\varphi_1 f$ and $\varphi_2 f$. Since $\varphi_1 f = \varphi_2 f$, there is a unitary $H_1 \xrightarrow{u} H_2$ such that $u(\xi_1) = \xi_2$ and $\pi_2(f(a)) = u\pi_1(f(a))u^*$. Calling $ad\, u = u(-)u^*$ the C^*-morphism $B(H_1) \xrightarrow{ad\, u} B(H_2)$, we have $(ad\, u)\pi_1 f = \pi_2 f$.

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[6] An arrow $A \xrightarrow{f} B$ is a categorical epimorphism iff having $B \overset{a}{\underset{b}{\rightrightarrows}} Z$ in the category such that $af = bf$ implies $a = b$.

[7] this is because at this point we don't know that f is proper, but we don't care because we already have $S(B) \xrightarrow{f^*} S(A)$, i.e. the restriction of a positive functional doesn't shrink the norm.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & B(H_1) \arrow[d, "ad\,u"] \\ & & B(H_2) \end{tikzcd}$$

f epi implies $(ad\,u)\pi_1 = \pi_2$. Since $(ad\,u)\pi_1$ is a GNS representation for φ_1 with cyclic vector ξ_2, we conclude $\varphi_1 = \varphi_2$. Or by direct computation:

$$\varphi_2 = \langle \pi_2(-)\xi_2, \xi_2\rangle = \langle (ad\,u)\pi_1(-)\xi_2, \xi_2\rangle =$$
$$= \langle u\pi_1(-)u^*\xi_2, \xi_2\rangle = \langle \pi_1(-)\xi_1, \xi_1\rangle = \varphi_1$$

8.3. **Proposition.** *Let I be an ideal of a C^*-algebra A.*
1) $S(I)$ is a topological subspace of $S(A)$.
2) $S(A) = S(A/I) \oplus S(I)$ as convex sets.
3) $P(A) = P(A/I) \overset{d}{\cup} P(I)$ with $P(I)$ open.
4) $P(I)$ is dense in $P(A)$ if and only if I is essential.

Proof. 1) Let $\varphi \in S(I)$. The GNS π_φ extends uniquely to A and allows to extend φ by the formula $\widetilde{\varphi}(a) = \langle \widetilde{\pi_\varphi}(a)\xi_\varphi, \xi_\varphi\rangle$.

$$S(I) \overset{j}{\hookrightarrow} S(A)$$
$$\varphi \mapsto \widetilde{\varphi}$$

j is clearly injective. Claim: $\widetilde{\varphi}$ is the unique element in $Q(A)$ such that its restriction to I is φ. Assume $\phi \in Q(A)$ with $\phi|_I = \varphi$. By restricting π_ϕ to I we will obtain the GNS representation of φ: take the Hilbert space $H' = \overline{\pi_\phi(I)\xi_\phi}$ and $p_{H'}(\xi_\phi)$ the orthogonal projection of ξ_ϕ to H'.

$$\varphi(i) = \langle \pi_\phi(i)\xi_\phi, \xi_\phi\rangle = \langle \pi_\phi(i)p_{H'}(\xi_\phi), p_{H'}(\xi_\phi)\rangle$$

Thus $(\pi_\phi|_I, H', p_{H'}\xi_\phi)$ is the GNS of φ, and this implies $||p_{H'}\xi_\phi||^2 = ||\varphi|| = 1$, $p_{H'}\xi_\phi = \xi_\phi$. Since H' is A-invariant, $H = \overline{\pi_\phi(A)\xi_\phi} \subset H'$, so $H = H'$. Now we know $\phi = \widetilde{\varphi}$.

To see that j is a subspace map we will show $\varphi_\mu \to \varphi$ iff $\tilde{\varphi}_\mu \to \tilde{\varphi}$. If $\varphi_\mu \to \varphi$, take an accumulation point ϕ of $(\widetilde{\varphi}_\mu)$ in the compact space $Q(A)$ (there is always at least one accumulation point), so there is a subnet $\widetilde{\varphi}_{\mu_l} \to \phi$. The restriction $\phi|_I$ is φ. Thus $\phi = \tilde{\varphi}$ and this is the only accumulation point of $\tilde{\varphi}_\mu$, implying $\tilde{\varphi}_\mu \to \tilde{\varphi}$. The other implication is trivial.

2) $S(A/I)$ is a subspace of $S(A)$ (8.1, item 2), it is the set of states annihilating on I. $S(A) = S(A/I) \oplus S(I)$ means that every $\varphi \in S(A)$ decomposes uniquely as a convex combination $r\varphi_I + s\varphi_{A/I}$ with $\varphi_I \in S(I)$ and $\varphi_{A/I} \in S(A/I)$. This is achieved as follows. Consider the subspace $H' = \overline{\pi_\varphi(I)H_\varphi}$ of H_φ. It is a subrepresentation of π_φ. We have a decomposition $H_\varphi = H' \oplus H'^\perp$, $\xi_\varphi = \xi' + \xi^\perp$. The vector states associated with ξ' and $\xi^\perp$ give the desired convex combination ($r = ||\xi'||^2$, $s = ||\xi^\perp||^2$).

3) $P(A) = P({}^{A}\!/\!{}_{I}) \overset{d}{\cup} P(I)$ follows immediately from 2) taking the extremal points of the convex sets. $P({}^{A}\!/\!{}_{I})$ is closed because if $\varphi_\mu \to \varphi$ with $\varphi_\mu \in P({}^{A}\!/\!{}_{I})$, then $\varphi|_I = 0$, $\varphi \in P({}^{A}\!/\!{}_{I})$.

4) Assume $P(I)$ dense in $P(A)$. If $a \in A$ is such that $aI = 0$ and $a \neq 0$, take a pure state $\varphi \in P(A)$ such that $\varphi(a) \neq 0$. Now take a neighborhood of φ such that $\phi(a) \neq 0$ for every ϕ in that neighborhood. For $\varphi' \in P(I)$ we have:

$$\widetilde{\varphi'}(a) = \langle \widetilde{\pi_{\varphi'}}(a)\xi_{\varphi'}, \xi_{\varphi'}\rangle = \lim_{\lambda} \langle \widetilde{\pi_{\varphi'}}(a)\pi_{\varphi'}(\lambda)\xi_{\varphi'}, \xi_{\varphi'}\rangle = \lim_{\lambda} \varphi'(a\lambda) = 0$$

where λ is an approximate unit in I. This proves that our neighborhood of φ and $P(I)$ are disjoint, contradicting the hypothesis.

For the converse, first notice that if J is an essential ideal of a C^*-algebra B, then the extension of any faithful representation $J \xrightarrow{\pi} B(H)$ to $B \xrightarrow{\widetilde{\pi}} B(H)$ is also faithful. For if $\widetilde{\pi}(b) = 0$, then $\pi(bj) = 0\ \forall j \in J$. Then $bJ = 0$, $b = 0$.

Consider the faithful representation of I,

$$I \xrightarrow{\pi} \bigoplus_{\varphi \in P(I)} B(H_\varphi) \qquad \pi = \bigoplus_{\varphi \in P(I)} \pi_\varphi \qquad H = \bigoplus_{\varphi \in P(I)} H_\varphi$$

We can extend π faithfully to A and then extend it faithfully to $\widetilde{A}$. We call $\widetilde{\pi}$ this extension. Now consider the application

$$\widetilde{A} \xrightarrow{\Phi} C(\overline{P(I)})$$
$$a \mapsto (\varphi \mapsto \varphi(a))$$

where the closure $\overline{P(I)}$ is taken inside the compact space $S(\widetilde{A})$. If $\Phi(a) \geq 0$, for every $v \in H$ we have:

$$\langle \widetilde{\pi}(a)v, v\rangle = \sum_{\varphi \in P(I)} \langle \widetilde{\pi}(a)v_\varphi, v_\varphi\rangle \geq 0$$

where v_φ is v projected to H_φ. Each $\langle \widetilde{\pi}(a)v_\varphi, v_\varphi\rangle$ is positive because if $v_\varphi \neq 0$, we can normalize it and define a vector state of I: $\rho = \langle \pi(-)v_\varphi, v_\varphi\rangle$, that is pure since its GNS is $(\pi_\varphi, H_\varphi, v_\varphi)$, so $\langle \widetilde{\pi}(a)v_\varphi, v_\varphi\rangle = \Phi(a)(\widetilde{\rho}) \geq 0$. Thus $\widetilde{\pi}(a) \geq 0$, $a \geq 0$. By 7.7 we conclude $\overline{P(I)} \supset P(\widetilde{A}) \supset P(A)$, so $P(I)$ is dense in $P(A)$. □

8.4. **Observation.** If A is nonunital, it is an essential ideal of its minimal unitization $\widetilde{A}$. By item (3) we know $P(\widetilde{A}) = P(A) \overset{d}{\cup} \{*\}$, with $P(A)$ open and dense ($*$ here is the unique state of ${}^{\widetilde{A}}\!/\!{}_{A} \simeq \mathbb{C}$). Since $*$ is the 0 functional when restricted to A, we obtain $0 \in \overline{P(A)}$, the closure taken in A^*.

By item (2), states of $\widetilde{A}$ have the form $\phi = \lambda\varphi + (1-\lambda)*$, where $0 \leq \lambda \leq 1$, $\varphi \in S(A)$. We have a continuous map $S(\widetilde{A}) \to Q(A)$, assigning to each state its restriction to A. This map is bijective and it is closed because the spaces are compact and Hausdorff. So it is a homeomorphism preserving the convex structures.

9. Spectrum of a C^*-algebra

We first show the bijection between the ideals of a C^*-algebra A and the saturated closed subspaces of $P(A)$. This gives an order isomorphism between the lattice of ideals and the topology of the spectrum of A. In the commutative case, this reduces to the characterization of ideals of $C_0(X)$ as those sets of functions annihilating on a closed subset of X.

9.1. **Saturated subsets of $P(A)$.** Consider the equivalence relation in $P(A)$ given by

$$\varphi_1 \sim \varphi_2 \Longleftrightarrow \pi_{\varphi_1} \text{ is unitarily equivalent to } \pi_{\varphi_2}$$

A saturated subset of $P(A)$ will be just a saturated set with respect to this equivalence relation. Just in case:

9.1. **Definition.** *We call a subset E of $P(A)$ saturated if $\forall \varphi_1 \in E$, $\varphi_2 \sim \varphi_1$ implies $\varphi_2 \in E$.*

Next proposition shows the more or less clear fact that the class of a $\varphi \in P(A)$ can be described as the set of unit vectors of the corresponding irreducible representation, modulo unit scalars. For a Hilbert space H, we denote with $U(H)$ the set of unit vectors and $U(H)/\sim$ is the quotient by the equivalence relation given by the action of unit scalars.

9.2. **Proposition.** *Let $\varphi \in P(A)$. We have a bijection*

$$U(H_\varphi)/\sim \, \to class(\varphi)$$

$$\overline{v} \mapsto \langle \pi_\varphi(-)v, v\rangle$$

Proof. The application is clearly well defined. If $\varphi' \in class(\varphi)$, then, after applying the unitary equivalence, the GNS of φ' is $(\pi_\varphi, H_\varphi, v)$ for some unit vector $v \in H_\varphi$. I.e., our application is surjective. If $v, w \in U(H_\varphi)$ give the same state: $\langle \pi_\varphi(-)v, v\rangle = \langle \pi_\varphi(-)w, w\rangle$, by the uniqueness of GNS representation, there is a unitary equivalence $H_\varphi \to H_\varphi$ between π_φ and π_φ sending v to w. But, since π_φ is irreducible, $\pi_\varphi(A)' = \mathbb{C}$, the unitary equivalence is just a unit scalar. □

For $\varphi \in P(A)$, $a \in A$ such that $||\pi_\varphi(a)\xi_\varphi|| = 1$, we obtain $\varphi(a^*(-)a) \sim \varphi$. Since $\{\pi_\varphi(a)\xi_\varphi\}_{a\in A}$ is a dense subspace of H_φ, every $\varphi' \in class(\varphi)$ can be approximated (in the norm topology) by states of this form. Actually, since topologically irreducible representations are also algebraically irreducible (see 2.8.5, [10]) $\{\pi_\varphi(a)\xi_\varphi\} = H_\varphi$. So every state equivalent to φ is of that form (compare with 2.8.6 of [10]). For our purposes (next proposition) it suffices with our previous weaker statement (density).

9.3. **Proposition.** *Let $E \subset P(A)$ be saturated. Its closure in $P(A)$ is saturated.*

Proof. Take $\varphi \in \overline{E} \cap P(A)$, $\varphi_\mu \to \varphi$ with $\varphi_\mu \in E$. If $\psi \sim \varphi$, by our previous observation we can approximate ψ in the norm topology by elements of the form $\varphi(a^*(-)a) \in P(A)$ (or simply $\psi = \varphi(a^*(-)a)$). We must show that $\varphi(a^*(-)a) \in \overline{E}$ to conclude $\psi \in \overline{E}$. We have $\varphi_\mu(a^*(-)a) \to \varphi(a^*(-)a)$. We must normalize $\varphi_\mu(a^*(-)a)$ to obtain elements in E. This is possible because $\varphi_\mu(a^*a) \to \varphi(a^*a) = 1 \neq 0$, so we can write $\frac{\varphi_\mu(a^*(-)a)}{\varphi_\mu(a^*a)} \to \varphi(a^*(-)a)$. □

9.4. **Definition.** *For $E \subset P(A)$, we denote:*

$sat(E)$ the set of those $\varphi \in P(A)$ such that φ is equivalent to some element of E. This is just the smallest saturated set containing E.

$\langle E \rangle := \overline{sat(E)} \cap P(A)$, being this the smallest closed and saturated subset of $P(A)$ containing E.

$\pi_E := \bigoplus_{\varphi \in E} \pi_\varphi$ is the sum of the GNS representations associated to each $\varphi \in E$.

9.5. **Proposition.** *If $E \subset P(A)$, π_E is faithful if and only if $P(A) = \langle E \rangle$.*

Proof. If $P(A) = \langle E \rangle$, take $a \in A$, $a \neq 0$ and $\varphi_0 \in P(A)$ such that $\varphi_0(a) = ||a||$. We can approximate this value at a by a $\varphi_1 \in sat(E)$. Now, $\varphi_1 = \langle \pi_E(-)v, v\rangle$ for some $v \in H_\varphi$, $\varphi \in E$, $\varphi \sim \varphi_1$. Thus $\pi_E(a) \neq 0$, π_E is faithful.

Now assume π_E faithful. Case $1 \in A$: the closure $\overline{sat(E)}$ in $S(A)$ is compact because it is bounded and w^*-closed. Consider the canonical application $A \xrightarrow{j} C(\overline{sat(E)})$. Let's see that j is norm preserving for positive elements. Take $a \in A$ positive. $||a|| = ||\pi_E(a)|| = \sup_{\varphi \in E} ||\pi_\varphi(a)||$ and $||\pi_\varphi(a)|| = \sup_{v \in U(H_\varphi)} \langle \pi_\varphi(a)v, v\rangle$. Thus, we can approximate $||a||$ by states in $sat(E)$, so $||j(a)|| = ||a||$. By 7.7 $P(A) \subset \overline{sat(E)}$, or $P(A) = \langle E \rangle$.

If $1 \notin A$ recall that $P(\widetilde{A}) = P(A) \cup \{*\}$, so $E \subset P(\widetilde{A})$. We can extend π_E to $\widetilde{A}$ and denote it $\widetilde{\pi_E}$. This extension is faithful because A is essential in $\widetilde{A}$ (this was pointed out in the proof of 8.3, item 4). Now the unital case tells us that $P(\widetilde{A}) = \langle E \rangle_{\widetilde{A}}$. Having in mind that $sat_A(E) = sat_{\widetilde{A}}(E)$, we see that every $\varphi \in P(A)$ is a w^*-limit of elements in $sat(E)$. □

9.2. Ideals $\longleftrightarrow$ Saturated closed subsets.

9.6. **Definition.** *Let A be a C^*-algebra. For any subsets $J \subset A$ and $E \subset P(A)$ define*

$$V(J) = \{\varphi \in P(A) / \varphi(bac) = 0 \;\; \forall a \in J, \;\forall b, c \in A\}$$
$$I(E) = \{a \in A / \varphi(bac) = 0 \;\; \forall \varphi \in E, \;\forall b, c \in A\}$$

9.7. **Observation.** $V(J)$ is a saturated closed subset of $P(A)$, $I(E)$ is an ideal of A. If J is an ideal, we can replace $\varphi(bac) = 0$ by just $\varphi(a) = 0$ in the definition of $V(J)$. Besides $J \subset IV(J)$ and $E \subset VI(E)$.

9.8. **Proposition.** $V(J) = P({}^{A}/_{\langle J \rangle})$ *and* $I(E) = ker\,\pi_E$*, where* $\langle J \rangle$ *is the ideal generated by* J*.*

Proof. By 8.1 or 8.3 we have an inclusion $P(A/\langle J\rangle) \subset P(A)$. The elements of $P(A/\langle J\rangle)$ seen as elements of $P(A)$ clearly annihilate on every bac with $a \in J$, $b, c \in A$, so $P(A/\langle J\rangle) \subset V(J)$. Conversely, if $\varphi \in P(A)$ annihilate on such elements, it defines a state of $A/\langle J\rangle$ that is pure (8.1). Thus, $V(J) = P(A/\langle J\rangle)$.

For the other equality, take $a \in I(E)$. To conclude $\pi_E(a) = 0$ it suffices with $\langle \pi_E(a)\alpha, \beta\rangle = 0$ for α, β belonging to the same H_φ, $\varphi \in E$. But H_φ is generated by the cyclic vector ξ_φ and

$$\langle \pi_E(a)\pi_E(c)\xi_\varphi, \pi_E(b^*)\xi_\varphi\rangle = \varphi(bac) = 0$$

Conversely, by the same equality, if $a \in ker\, \pi_E$, then $\varphi(bac) = 0$ for every b, $c \in A$, $\varphi \in E$. ☐

9.9. **Proposition.** *For a C^*-algebra A, $J \subset A$ and $E \subset P(A)$:*
$IV(J) = \langle J\rangle$. In particular $IV(J) = J$ if J is an ideal
$VI(E) = \langle E\rangle$, the saturated closure in $P(A)$.

Proof. As we already said, $\langle J\rangle \subset IV(J)$ and $\langle E\rangle \subset VI(E)$ trivially. Let $a \in IV(J)$. If $a \notin \langle J\rangle$, then $0 \neq \overline{a} \in A/\langle J\rangle$. There is a $\overline{\varphi} \in P(A/\langle J\rangle)$ such that $\overline{\varphi}(\overline{a}) \neq 0$. From previous proposition, $\overline{\varphi}$ can be seen as $\varphi \in V(J)$ so $\overline{\varphi}(\overline{a}) = \varphi(a) = 0$, absurd.

For the second part, consider the faithful representation

$$A/I(E) \overset{\overline{\pi_E}}{\hookrightarrow} B(\bigoplus_{\varphi\in E} H_\varphi)$$

The pure states $\varphi \in E$ are also pure states of $A/I(E)$, and since $\overline{\pi_E}$ is faithful, by 9.5 we have $P(A/I(E)) \subset \langle E\rangle_{A/I(E)}$. This saturated closure with respect to $A/I(E)$ coincides with the saturated closure with respect to A. This is because $P(A/I(E))$ is closed in $P(A)$ and the equivalence relation $\sim$ inside $P(A/I(E))$ is the same with respect to both algebras. Thus $VI(E) = P(A/I(E)) \subset \langle E\rangle$. ☐

So we have proved that there is an order reversing bijection between ideals and saturated closed subspaces of $P(A)$. Equivalently, we can express the bijection using saturated open subspaces:

$$\begin{aligned}\{\text{Ideals of } A\} &\longleftrightarrow \{\text{Saturated open subspaces of } P(A)\}\\ I &\longmapsto P(I)\end{aligned}$$

and we have $P(I \cap J) = P(I) \cap P(J)$, $P(\overline{I + J}) = P(I) \cup P(J)$ for being an order isomorphism. More generally: $P(\overline{\sum I_k}) = \bigcup P(I_k)$ and $P(\bigcap I_k) = \big(\bigcap P(I_k)\big)^\circ$. As we showed in 8.3, I essential corresponds to $P(I)$ dense.

9.3. **Spectrum.**

Saturated open and closed sets in $P(A)$ are nothing else than the open and closed sets of the quotient $\widehat{A} = P(A)/\sim$. We call $\widehat{A}$ the spectrum of A. It is the set of irreducible representations (modulo unitary equivalence) equipped with the quotient topology. Call $P(A) \xrightarrow{q} \widehat{A}$ the quotient map.

It is easy to read the applications I and V from previous section using subsets of $\widehat{A}$ instead of $P(A)$. For any subset $J \subset A$ we have the closed set $V'(J) = qV(J) = \{\pi \in \widehat{A}/J \subset ker\,\pi\} = \{\pi \in \widehat{A}/\pi(J) = 0\}$, and for any subset $S \subset \widehat{A}$ we have the ideal $I'(S) = Iq^{-1}(S) = \bigcap_{\pi \in S} ker\,\pi$.

It follows from 9.9 that the closure in $\widehat{A}$ can be described by

$$\overline{S} = V'I'(S) = \{\pi' \in \widehat{A}/ \bigcap_{\pi \in S} ker\,\pi \subset ker\,\pi'\}$$

We also have:

9.10. **Proposition.** *I' and V' define an order reversing bijection between the set of ideals of A and that of closed sets of $\widehat{A}$. An ideal J is essential iff $V'(J)^c$ is dense.*

Notice that the only relevant information from the representations $\pi \in \widehat{A}$ to determine the topology is their kernels. If two points $\pi_1, \pi_2 \in \widehat{A}$ have the same kernel, then they belong to the same open/closed sets. Conversely, if π_1 and π_2 belong to the same closed sets, since closed sets are $C_I = \{\pi \in \widehat{A}/ker(\pi) \supset I\}$, the kernels $ker(\pi_1)$, $ker(\pi_2)$ contain the same ideals. In particular $ker(\pi_1) \supset ker(\pi_2)$ and $ker(\pi_2) \supset ker(\pi_1)$, so $ker(\pi_1) = \ker(\pi_2)$. This proves that identifying representations with the same kernel amounts to take the Kolmogorov quotient of $\widehat{A}$, denoted by $T_0(\widehat{A})$. We write this quotient more explicitly as follows. Let $prim(A)$ be the set of kernels of irreducible representations. These kernels are also called "primitive ideals", and $prim(A)$ the primitive spectrum.

$$\widehat{A} \xrightarrow{p} prim(A)$$

$$\pi \longmapsto ker\,\pi$$

$prim(A) = T_0(\widehat{A})$. Since every closed subset of $\widehat{A}$ is saturated with respect to this quotient, and therefore every open set too, p is open and closed. In other words, the topologies are the same but there might be multiple points in $\widehat{A}$ for each primitive ideal (this comment is valid for a general Kolmogorov quotient). The closure of $S \subset prim(A)$ can be described as:

$$\overline{S} = \{I \in prim(A)/I \supset \bigcap_{J \in S} J\}$$

and this is the Jacobson topology in $prim(A)$. An easy way to think of this topology is the following: closed sets are indexed by the ideals $I \subset A$, and the points of each closed set are those primitive ideals J such that $J \supset I$.

Each of the following five results about the topology of the spectrum are clearly valid for both $prim(A)$ and $\widehat{A}$.

9.11. **Proposition.** *For unital A, $prim(A)$ is compact.*

Proof. We show that if (F_k) is a family of closed subsets such that any finite intersection is not empty, then $\bigcap F_k \neq \emptyset$. Considering the order isomorphism between closed subsets of $prim(A)$ and ideals of A, the intersection of a family of closed sets is not empty if and only if the closure of the sum of the corresponding ideals is not A. Let (I_k) be a family of ideals such that the sum of any finite subfamily is not A. Without loss of generality we assume (I_k) filtering (just add all finite sums). Now $\overline{\sum I_k} = \overline{\bigcup I_k}$. If $\overline{\bigcup I_k} = A$, then 1 can be approximated by elements of I_k, but $\overline{1} \in {}^{A}/_{I_k}$ is a unit and therefore has norm one, so $dist(1, I_k) = 1$, absurd. $\square$

For nonunital A, we already said that $P(\widetilde{A}) = P(A) \cup \{*\}$ with $P(A)$ open and dense. For the spectrum we therefore have $\widehat{\widetilde{A}} = \widehat{A} \cup \{*\}$ with $\widehat{A}$ open and dense.

9.12. **Lemma.** *Let $x \in A$, $\alpha \in \mathbb{R}_{\geq 0}$. $\{\pi \in \widehat{A} / ||\pi(x)|| \leq \alpha\}$ is a closed subset of $\widehat{A}$.*

Proof. The equality $||\pi(x)||^2 = ||\pi(x^*x)||$ justifies the reduction to the case $x \geq 0$.

Since $||\pi(x)|| = \sup_{||\xi||=1} \langle \pi(x)\xi, \xi \rangle$, $\varphi \in P(A)$ is in the preimage of our set by $P(A) \xrightarrow{q} \widehat{A}$ if and only if $\frac{\varphi(a^*xa)}{\varphi(a^*a)} \leq \alpha$ $\forall a \in A$ such that $\varphi(a^*a) \neq 0$. In case $\varphi(a^*a) = 0$, $\pi_\varphi(a)\xi_\varphi = 0$, so $\varphi(a^*xa) = 0$ and $\varphi(a^*xa) \leq \alpha\varphi(a^*a)$ holds.

Thus the preimage of our set is:

$$\bigcap_{a \in A} \{\varphi \in P(A) / \varphi(a^*xa) - \alpha\varphi(a^*a) \leq 0\}$$

which is closed for being an intersection of closed sets. $\square$

Notation: if $I \subset A$ is an ideal and $x \in A$, x_I will denote the image of x in the quotient ${}^{A}/_{I}$.

9.13. **Lemma.** *Let $x \in A$, $\alpha \in \mathbb{R}_{>0}$. $Z = \{J \in prim(A) / ||x_J|| \geq \alpha\}$ is compact.*

Proof. For a closed set $S \subset prim(A)$, the corresponding ideal is $I_S = \bigcap_{I \in S} I$. Notice that if S has a point $J \in Z$, we have $A \to {}^{A}/_{I_S} \to {}^{A}/_{J}$ and $||x_{I_S}|| \geq ||x_J|| \geq \alpha$. Conversely, if $||x_{I_S}|| \geq \alpha$, we can take an irreducible representation of ${}^{A}/_{I_S}$ preserving the norm of x_{I_S}, and this (actually the kernel) will be a point $J \in Z \cap S$. Conclusion: a closed set $S \subset prim(A)$ intersects Z if and only if $||x_{I_S}|| \geq \alpha$.

Consider a decreasing filtering family (Z_i) of relatively closed nonempty subsets of Z. Take $F_i \subset prim(A)$ closed such that $F_i \cap Z = Z_i$. The ideal which corresponds to the closed set $\bigcap_i F_i$ is the least upper bound of the I_{F_i}. This is $I = \overline{\bigcup_i I_{F_i}}$.

$$||x_I|| = dist(x, \overline{\bigcup_i I_{F_i}}) = \inf_i dist(x, I_{F_i}) = \inf_i ||x_{I_{F_i}}|| \geq \alpha$$

Thus $\bigcap_i F_i$ intersects Z and $\bigcap Z_i$ is nonempty. $\square$

9.14. **Proposition.** *$\widehat{A}$ is locally compact.*

Proof. Let $\pi \in \widehat{A}$ and $U \subset \widehat{A}$ an open neighborhood of π.
Let $I = \bigcap_{\rho \in U^c} ker\, \rho$ the ideal associated to U^c. Since $\pi \notin U^c$, we have $I \nsubseteq ker\, \pi$ so there is an $x \in I$ (i.e. $\rho(x) = 0$ for $\rho \in U^c$) such that $\pi(x) \neq 0$.
Now let

$$V = \{\pi' \in \widehat{A} / ||\pi'(x)|| > \frac{1}{2}||\pi(x)||\}$$

$$W = \{\pi' \in \widehat{A} / ||\pi'(x)|| \geq \frac{1}{2}||\pi(x)||\}$$

We have $\pi \in V \subset W \subset U$ and by previous lemmas V is open and W compact, the compact neighborhood we need. □

A C^*-algebra is said to be σ-unital if it has a countable approximate unit. In the commutative case this is equivalent to have a σ-compact spectrum. A C^*-algebra is σ-unital if and only if it has a strictly positive element (Blackadar II.4.2.4) but we won't need this characterization. Note: a separable C^*-algebra is σ-unital.

9.15. **Proposition.** *If A is σ-unital, $\widehat{A}$ is σ-compact.*

Proof. Let (u_n) be a countable approximate unit. Consider

$$K_n = \{\pi' \in \widehat{A} / ||\pi'(u_n)|| \geq \frac{1}{2}\}$$

These are compact by 9.13. If there is a $\pi \in \widehat{A}$ that doesn't belong to any of these sets, take $\pi(a) \neq 0$,

$$\frac{1}{2}||\pi(a)|| > ||\pi(u_n)||.||\pi(a)|| \geq ||\pi(u_n a)|| \to ||\pi(a)||$$

absurd. □

10. Stone-Weierstrass problem

Classical Stone-Weierstrass theorem states that for a locally compact Hausdorff X, a sub-C^*-algebra $B \subset C_0(X)$ that separates the points of $X \cup \{\infty\}$ (i.e.: for $x \neq y \in X \cup \{\infty\}$, there is an $f \in B$ such that $f(x) \neq f(y)$) must be equal to the whole algebra $C_0(X)$. Since X is the set of pure states it is reasonable to conjecture:

10.1. **Conjecture.** *Let B be a C^*-algebra and A a C^*-subalgebra. If A separates $P(B) \cup \{0\}$ then $A = B$.*

A weaker but known version is:

10.2. **Theorem** (Glimm's noncommutative Stone-Weierstrass theorem)**.** *Let B be a C^*-algebra and A a C^*-subalgebra. If A separates $\overline{P(B)} \cup \{0\}$ then $A = B$.*

See [10] (theorem 11.3.1, corollary 11.5.2) or [16] for the proof.

If B is nonunital, $0 \in \overline{P(B)}$ (8.4). The condition of separating the zero from $P(B)$ implies (see the proof of next proposition) that A is a proper subalgebra. If B is unital, this means $1 \in A$.

As a good starting point for thinking conjecture 10.1, the following proposition shows that under those hypothesis A is a proper subalgebra and the spaces $P(B)$ and $P(A)$ are equal.

10.3. **Proposition.** *Let B be a C^*-algebra and A a C^*-subalgebra that separates $P(B) \cup \{0\}$. Then the inclusion $A \hookrightarrow B$ is proper and induces a homeomorphism $P(B) \to P(A)$. Besides, two elements in $P(B)$ are equivalent if and only if they are equivalent in $P(A)$.*

Proof. We start showing that the condition of separating 0 from the pure states of B implies that the inclusion is proper. We will do this through condition 4 of proposition 2.5. Consider $F := \{\varphi \in S(\widetilde{B}) / \varphi|_A = 0\} \subset S(\widetilde{B})$. This set is convex and closed in $S(\widetilde{B})$. Besides, it is a face of $S(\widetilde{B})$: if we have a nontrivial convex combination $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2 \in F$, then φ_1 and φ_2 vanish at positive elements of A, so they vanish on A and $\varphi_1, \varphi_2 \in F$. This implies that an extremal point of F is an extremal point of $S(\widetilde{B})$, which restricted to B must be a pure state or 0 (recall $P(\widetilde{B}) = P(B) \cup \{*\}$ in case $1 \notin B$). If there is $\varphi \in F$ such that $\varphi|_B \neq 0$, by Krein-Milman theorem there will be an element of $P(B)$ in F, i.e. a pure state of B that is 0 on A. But there is no such thing, by hypothesis; so there is no state of B that vanishes on A. Now it follows that if π is a nondegenerate representation of B, then $\pi|_A$ is nondegenerate: if there is a unit vector $v \in H$ such that $\pi(A)v = 0$, then $\langle \pi(a)v, v\rangle = 0\ \forall a \in A$ but $0 \neq \langle \pi(-)v, v\rangle \in S(B)$. By 2.5, the inclusion $A \subset B$ is proper.

From 8.1 we know that restriction to A induces a continuous map $S(B) \to S(A)$. We next see that it restricts to $P(B) \to P(A)$. Take $\varphi \in P(B)$ and its GNS representation π_φ. Suppose that $\pi_\varphi|_A$ is reducible: $H_\varphi = H_1 \oplus H_2$ with H_i A-invariant. Take non-zero $\xi_1 \in H_1$, $\xi_2 \in H_2$ such that $||\xi_1 + \xi_2|| = 1$. Consider the vector states of B induced by $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$. They are different pure states of B (see 9.2) coinciding on A, but this is absurd since A separates $P(B)$. Thus, $(\pi_\varphi|_A, H_\varphi)$ is irreducible, and it is the GNS associated to $\varphi|_A$.

$P(B) \to P(A)$ is injective by hypothesis. It is also surjective: a state $\varphi \in P(A)$ can be extended by Hahn-Banach to a state of B. The set of states of B that extend φ is convex, closed and not empty (inside the compact space $S(\widetilde{B})$). An extremal point of this set belongs to $P(\widetilde{B})$ (just like in 7.5), so there is an element of $P(B)$ that gives φ when restricted to A. Observe that since there is only one element of $P(B)$ extending $\varphi \in P(A)$, the set of states of B extending φ has only one extremal point. By Krein-Milman theorem, this set must have only this point.

So we have a continuous bijection. To conclude that it is a homeomorphism, we take a convergent net in the image: $\varphi_\mu|_A \to \varphi_0|_A$ $(\varphi_\mu, \varphi_0 \in P(B))$

and deduce $\varphi_\mu \to \varphi_0$. The net (φ_μ) lives in the compact space $S(\widetilde{B})$, so it will suffice to show that the only accumulation point in $S(\widetilde{B})$ is φ_0. Assume $\varphi_\mu \to \varphi$. We have $\varphi|_A = \varphi_0|_A \in P(A)$. This implies $\varphi \in S(B)$. By the observation at the end of previous paragraph, $\varphi = \varphi_0$.

If $\varphi_1 \sim \varphi_2 \in P(B)$, i.e. the GNS representations are unitarily equivalent, then their restrictions to A are clearly equivalent (remember that we proved in a previous paragraph that for $\varphi \in P(B)$, we have that $(\pi_\varphi|_A, H_\varphi, \xi_\varphi)$ is the GNS for $\varphi|_A$, where the nontrivial point is that the Hilbert space is not smaller than H_φ). Conversely, if $\varphi_1|_A \sim \varphi_2|_A$, after considering the unitary equivalence, we can assume that π_{φ_1} and π_{φ_2} act on the same Hilbert space H and coincide on A. Hence

$$\langle \pi_{\varphi_1}(a)\xi_{\varphi_2}, \xi_{\varphi_2}\rangle = \langle \pi_{\varphi_2}(a)\xi_{\varphi_2}, \xi_{\varphi_2}\rangle = \varphi_2(a)$$

for $a \in A$. We conclude $\langle \pi_{\varphi_1}(b)\xi_{\varphi_2}, \xi_{\varphi_2}\rangle = \varphi_2(b)$ for $b \in B$. Thus, $(\pi_{\varphi_1}, H, \xi_{\varphi_2})$ is a GNS representation for φ_2, so $\varphi_1 \sim \varphi_2$.

$\square$

10.4. **Remark.** With little more effort we see that the homeomorphism $P(B) \to P(A)$ extends to a homeomorphism $P(B) \cup \{0\} \to P(A) \cup \{0\}$. The application is well defined, it is injective, surjective, continuous, and if $\varphi_\mu|_A \to 0$ ($\varphi_\mu \in P(B) \cup \{0\}$), then the only accumulation point of (φ_μ) in $S(\widetilde{B})$ can be 0, so $\varphi_\mu \to 0$.

A direct consequence from the fact that $A \overset{i}{\hookrightarrow} B$ induce a homeomorphism between $P(B)$ and $P(A)$ which preserve the equivalence of states is that $\widehat{B}$ and $\widehat{A}$ are homeomorphic via $\pi \mapsto \pi i$. The primitive spectra are homeomorphic through $J \mapsto J \cap A$. Also the lattices of ideals are isomorphic because of 9.10. An ideal $I \subset B$ is equal to $I = \bigcap_{J \in S} J$ for some closed set $S \subset prim(B)$, hence $I \cap A = \bigcap_{J\in S} J \cap A$ is the ideal of A which corresponds to I through the bijection.

The only fact that $P(A)$ is homeomorphic to $P(B)$ preserving the equivalence relation (or, even more, preserving "transition probabilities" and "orientation" [33]) doesn't guarantee that A and B are isomorphic (without assuming $A \subset B$). See [33] for a counterexample, and for the proof that it is sufficient for that purpose to have a homeomorphism uniformly continuous in both directions preserving transition probabilities and orientation.

As we mentioned in remark 5.12, I. Fujimoto proved that any C^*-algebra A is isomorphic to the uniformly continuous fields on $Irr(A : H) \cup \{0\}$. Under the hypothesis of the general Stone-Weierstrass problem, we have a homeomorphism $Irr(B : H) \cup \{0\} \to Irr(A : H) \cup \{0\}$ that is uniformly continuous. If we had that it is biuniformly continuous, then the sets of uniformly continuous fields would be the same, so $A = B$. This simple train of thought may adapt well for Glimm's version 10.2, considering that the resulting homeomorphism $\overline{P(B)} \leftrightarrow \overline{P(A)}$ is biuniformly continuous because they are compact Hausdorff spaces.

Actually, Fujimoto addresses the Stone-Weierstrass problem and proves the following theorem: ([15], theorem 3.5) "Let A be a unital C^*-algebra, B be a quasi-perfect C^*-subalgebra of A which contains the identity of A, and suppose that B separates $P(A)$. Then, $A = B$."

11. Noncommutative Tietze extension theorem

We begin by stating the classical Tietze extension theorem and providing a proof that was meant to approach the noncommutative case.

Recall that a topological space is said to be normal iff every pair of disjoint closed subspaces can be separated by two disjoint open sets. Remark: a normal space is not necessarily Hausdorff or even T_0.

11.1. **Theorem** (Tietze extension theorem)**.** *Let X be a topological space. The following conditions are equivalent:*

1) X is normal.

2) For every closed subspace $Y \subset X$, the canonical map between the Stone-Cech compactifications $\beta Y \to \beta X$ is injective.

3) For every closed subspace $Y \subset X$, the restriction map $C_b(X) \to C_b(Y)$ is surjective.

Proof. 1) $\Rightarrow$ 2)

$$\begin{tikzcd} \beta Y \arrow[r, "\beta i"] & \beta X \\ Y \arrow[u, "j_Y"] \arrow[r, hook, "i"] & X \arrow[u, "j_X"'] \end{tikzcd}$$

Take different points $y_1, y_2 \in \beta Y$. We can separate them by two disjoint compact neighbourhoods $K_1 \ni y_1$ and $K_2 \ni y_2$. The preimages $j_Y^{-1}(K_1)$ and $j_Y^{-1}(K_2)$ are disjoint closed sets, and they satisfy $y_i \in \overline{j_Y(j_Y^{-1}(K_i))}$. To see this, take $j_Y(y_\mu) \to y_i$. For big enough μ, $j_Y(y_\mu) \in K_i$, so $j_Y(y_\mu) \in j_Y(j_Y^{-1}(K_i))$, and $y_i \in \overline{j_Y(j_Y^{-1}(K_i))}$. Thus, it suffices to show that $\beta i(\overline{j_Y(j_Y^{-1}(K_1))})$ and $\beta i(\overline{j_Y(j_Y^{-1}(K_2))})$ are disjoint. Since i is closed and injective, $i(j_Y^{-1}(K_1))$ and $i(j_Y^{-1}(K_2))$ are closed and disjoint. By normality of X, there is a continuous bounded function with scalar values over X separating these two sets, taking the constant values 0 and 1 on each set. But these function extends to the Stone-Cech compactification βX, separating $\overline{j_X i(j_Y^{-1}(K_1))} = \overline{\beta i(j_Y(j_Y^{-1}(K_1)))}$ from $\overline{j_X i(j_Y^{-1}(K_1))} = \overline{\beta i(j_Y(j_Y^{-1}(K_2)))}$. By continuity of βi, the sets we've just separated are actually bigger than those we had to separate.

2) $\Rightarrow$ 3)

$C_b(X) = C(\beta X)$, $C_b(Y) = C(\beta Y)$. We must show that the restriction map $C(\beta X) \to C(\beta Y)$ is surjective. We apply Stone-Weierstrass theorem. The image is a closed subalgebra containing the constants and it separates points: given $y_1 \neq y_2 \in \beta Y \subset \beta X$, they are different characters of $C(\beta X)$, so they differ at some $g \in C(\beta X)$.

3) $\Rightarrow$ 1)

Let C and D be two disjoint closed subspaces of X. $C \cup D$ is closed. The function $C \cup D \to \mathbb{C}$ taking the values 0 in C and 1 in D is continuous because every preimage is closed. By hypothesis we can extend it to X and then take preimage of $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$ to separate C and D with disjoint open sets. $\square$

Noncommutative formulation

Traditionally, Tietze extension theorem is (1) $\Leftrightarrow$ (3) in previous theorem. A good noncommutative analog of "a closed subspace of X" is "a closed subset of the spectrum $\widehat{A}$", or equivalently, a quotient $A \xrightarrow{f} B$. Since $C_b(X) = M(C_0(X))$, we can write the restriction map $C_b(X) \to C_b(Y)$ as $M(C_0(X)) \to M(C_0(Y))$. This suggests the following noncommutative version of condition (3): "for every quotient $A \xrightarrow{f} B$, the induced map between its multiplier algebras $M(A) \xrightarrow{\tilde{f}} M(B)$ is surjective", which led to the following theorem by Pedersen:

11.2. **Theorem** (Noncommutative Tietze extension theorem)**.** *If A is a σ-unital C^*-algebra, for every quotient $A \to B$ the induced morphism $M(A) \to M(B)$ is surjective.*

See [29] theorem 10 or [40] 2.3.9 for the proof. If A is σ-unital, $\widehat{A}$ is σ-compact (9.15) and in the commutative case the converse holds.

We considered the reasonable hypothesis "$\widehat{A}$ normal" (or equivalently $prim(A)$ normal). Apparently, it is not clear whether the theorem is true under this condition. However, it does hold (with converse) if in addition we replace $M(A) \to M(B)$ with $ZM(A) \to ZM(B)$, as we show in 11.10. This "Tietze extension theorem for C^*-algebras" doesn't generalize Pedersen's theorem. An attempt in that direction is the next proposition. It provides a sufficient separation condition on $P(A)$ that guarantees $M(A) \to M(B)$ surjective for every quotient $A \to B$. Unfortunately we couldn't check this condition for any nontrivial case.

11.3. **Proposition.** *Let A be a C^*-algebra such that every two disjoint closed subsets of $P(A)$ have disjoint closures on $\overline{P(M(A))}$. Then, for every epimorphism $A \xrightarrow{f} B$ the induced function $M(A) \xrightarrow{\tilde{f}} M(B)$ is an epimorphism.*

Proof. According to 8.1, we have a closed inclusion $P(B) \overset{f^*}{\hookrightarrow} P(A)$. By 8.3, $P(A)$ is a dense subspace of the compact and Hausdorff space $\overline{P(M(A))} \subset S(M(A))$ and the same for $P(B)$.

$$\begin{tikzcd}
\overline{P(M(B))} \arrow[r, dotted, "\tilde{f}^*"] & \overline{P(M(A))} \\
P(B) \arrow[u, hook, "j_B"] \arrow[r, hook, "f^*"] & P(A) \arrow[u, hook, "j_A"']
\end{tikzcd}$$

If we compose a $\varphi \in \overline{P(M(B))} \subset S(M(B))$ with $\tilde{f}$, we obtain a state of $M(A)$ that belongs to $\overline{P(M(A))}$ because

$$\tilde{f}^*\left(\overline{P(M(B))}\right) = \tilde{f}^*\left(\overline{P(B)}\right) \subset \overline{\tilde{f}^*(P(B))} \subset \overline{P(A)} = \overline{P(M(A))}$$

Thanks to the noncommutative version of the Stone-Weierstrass theorem 10.2 applied to $\tilde{f}^*(M(A)) \subset M(B)$, it is sufficient to prove that the induced $\overline{P(M(B))} \xrightarrow{\tilde{f}^*} \overline{P(M(A))}$ is injective. Now the proof follows like 1) $\Rightarrow$ 2) in 11.1.

$\square$

For the theorem 11.10, it is important to know the relation between $\widehat{A}$ and $\widehat{M(A)}$. While in the commutative case $\widehat{M(A)}$ is the Stone-Cech compactification of $\widehat{A}$, for general A, $\widehat{M(A)}$ won't be Hausdorff but it is still true that the inclusion $\widehat{A} \hookrightarrow \widehat{M(A)}$ extends every continuous function with compact image. This very nice property follows from the Dauns-Hofmann theorem.

11.1. **Dauns-Hofmann theorem.** An element of a C^*-algebra $a \in A$ can be thought as a field over $\widehat{A}$: $\widehat{a}(\pi) = \pi(a)$. If we multiply the field by a continuous bounded function $\widehat{A} \xrightarrow{f} \mathbb{C}$, we obtain, according to the Dauns-Hofmann theorem a field that also comes from an element of A. More precisely: there is a bijection between bounded continuous scalar-valued functions on the spectrum of a C^*-algebra and central multipliers, as we will now show in detail.

11.4. **Theorem** (Dauns-Hofmann)**.** *Let $\widehat{A} \xrightarrow{f} \mathbb{C}$ be a bounded continuous function and $a \in A$, where A is a C^*-algebra. There exists a unique $f \cdot a \in A$ such that $f(\pi)\pi(a) = \pi(f \cdot a)$ for every $\pi \in \widehat{A}$.*

Notice that since $prim(A) = T_0(\widehat{A})$, we have $C_b(prim(A)) = C_b(\widehat{A})$. Besides, the equality $f(\pi)\pi(a) = \pi(f \cdot a)$ is equivalent to $f(\pi)a \overset{mod\ P}{=} f \cdot a$ where $P = ker(\pi) \in prim(A)$. Therefore, it is clear how to formulate the theorem for $prim(A)$ instead of $\widehat{A}$, and it is immediately equivalent. We imitate the proof from [12] simplifying where possible.

11.5. **Lemma.** *Let J_0,...,J_n be ideals of A and $c \in \mathbb{R}$, $c > 2$. Any element $a \in J_0 + ... + J_n$ can be decomposed as $a = a_0 + ... + a_n$, $a_i \in J_i$, in a way such that $||a_i|| \leq c||a||$.*

Proof. Consider

$$a \in J_0 + ... + J_n \to {}^{J_0+...+J_n}/_{J_n} \to {}^{J_0+...+J_{n-1}}/_{(J_0+...+J_{n-1})\cap J_n}$$

The last arrow is an isometry because it is a bijective morphism of C^*-algebras. Hence, the image of a in ${}^{J_0+...+J_{n-1}}/_{(J_0+...+J_{n-1})\cap J_n}$ has a norm not bigger than $||a||$, and we can take a representative $b \in J_0 + ... + J_{n-1}$ such that $||b|| \leq (1+\epsilon)||a||$ and $a - b \in J_n$. $||a-b|| \leq ||a|| + ||b|| \leq 2||a|| + \epsilon$. By the inductive hypothesis we can decompose $b = a_1 + ... + a_{n-1}$ with $||a_i|| \leq (2+\epsilon')||b|| \leq (2+\epsilon'')||a||$. For $n = 0$ the statement is trivial. □

Proof of the theorem. Uniqueness follows from the faithfulness of

$$A \xrightarrow{\bigoplus_{\pi\in\widehat{A}} \pi} B(\bigoplus_{\pi\in\widehat{A}} H_\pi)$$

Existence: taking real and imaginary parts, multiplying by a constant and translating if necessary, we may assume $\widehat{A} \xrightarrow{f} [0,1]$. We will produce an approximation to the desired element $f \cdot a \in A$. Take the following open covering of $\widehat{A}$:

$$U_i := f^{-1}(\Big(\frac{i-1}{n}, \frac{i+1}{n}\Big)) \qquad i = 0, 1, ..., n$$

Consider J_i the ideals associated to U_i (or U_i^c, according to 9.10). Since $\bigcup_{i=0}^n U_i = \widehat{A}$, it holds $A = \sum_{i=0}^n J_i$ (9.10, the bijection is an order isomorphism). Decompose $a = a_0 + ... + a_n$ as in the lemma ($a_i \in J_i$ and $||a_i|| \leq 3||a||$). Consider

$$d_n = \sum_{i=0}^n \frac{i}{n} a_i \in A$$

The value of $\pi(d_n)$ approximates $f(\pi)\pi(a)$: for i such that $\pi \in U_i^c$ we have $\pi(a_i) = 0$ and if $\pi \in U_i$, $f(\pi) \simeq \frac{i}{n}$ (π belongs to at most two open sets U_i, say U_{i_0} and eventually U_{i_0+1}) so

$$||f(\pi)\pi(a) - \pi(d_n)|| = ||\sum_{i=0}^n \Big(f(\pi) - \frac{i}{n}\Big)\pi(a_i)|| \leq \sum_{i=0}^n |f(\pi) - \frac{i}{n}|.||\pi(a_i)|| \leq$$

$$\leq \frac{1}{n}(||\pi(a_{i_0})|| + ||\pi(a_{i_0+1})||) \leq \frac{6}{n}||a||$$

(d_n) is Cauchy: $||d_n - d_m|| = \max_{\pi\in\widehat{A}} ||\pi(d_n - d_m)|| \leq \frac{6}{n}||a|| + \frac{6}{m}||a||$ and the limit $d \in A$ satisfies $f(\pi)\pi(a) = \pi(d)$, so $d = f \cdot a$. □

Consider an irreducible representation $\pi \in \widehat{M(A)}$. If we restrict π to the center $ZM(A)$, we obtain $\pi(ZM(A)) \subset \pi(M(A)) \cap \pi(M(A))' = \mathbb{C}.1$, so $\pi|_{ZM(A)}$ acts by scalars and can be identified with an irreducible representation of the commutative algebra $ZM(A)$. This gives a map $\widehat{M(A)} \to \widehat{ZM(A)}$. To show continuity of this map, let us consider it at the level of pure states $P(M(A)) \to P(ZM(A))$, where it is clearly continuous, and then pass to the quotient. Moreover, these maps are surjective, since we can extend by Hahn-Banach any character of $ZM(A)$ to a pure state of $M(A)$, choosing an extremal extension.

11.6. **Corollary.** *The maps $\widehat{A} \overset{j}{\hookrightarrow} \widehat{M(A)} \overset{r}{\to} \widehat{ZM(A)}$ induce isomorphisms $C_b(\widehat{A}) \simeq C(\widehat{M(A)}) \simeq C(\widehat{ZM(A)})$. As a direct consequence, $\beta\widehat{A} \simeq T_2(\widehat{M(A)}) \simeq \widehat{ZM(A)}$.*

Proof. We have morphisms

$$C(\widehat{ZM(A)}) \xrightarrow{r^*} C(\widehat{M(A)}) \xrightarrow{j^*} C_b(\widehat{A})$$

that are injective because j and r are dense and surjective respectively. Now take $f \in C_b(\widehat{A})$. Define $A \overset{L}{\to} A$ and $A \overset{R}{\to} A$ as $L(a) = R(a) = f \cdot a$. (L, R) is a double centralizer, since $aL(b) = R(a)b = f \cdot (ab)$. The formula $L(a) = R(a)$ says that $z_f := (L, R) \in M(A)$ commutes with every element in A. If $m \in M$, $a \in A$:

$$z_f ma = z_f(ma) = (ma)z_f = mz_f a$$

Then $z_f m = mz_f$, i.e. $z_f \in ZM(A)$.

$$\begin{tikzcd}
\widehat{A} \arrow[r, hook, "j"] \arrow[dr, "f"'] & \widehat{M}(A) \arrow[r, "r"] & \widehat{ZM(A)} = char(ZM(A)) \arrow[dl, dotted, "ev_{z_f}"] \\
& \mathbb{C} &
\end{tikzcd}$$

Notice that $f \cdot a = z_f a$. The equation $\pi(f \cdot a) = f(\pi)\pi(a)$ can be written as $\widetilde{\pi}(z_f)\pi(a) = f(\pi)\pi(a)$ or simply $\widetilde{\pi}(z_f) = f(\pi)$. This means that the diagram commutes. Thus j^*r^* is surjective, implying j^* surjective. But j^*r^* and j^* were also injective, so they are bijections. Hence r^* is a bijection too.

It is a topological fact that if a continuous map $X \to Y$ with Y compact and T_2 extends uniquely every scalar continuous and bounded function $X \to \mathbb{R}$ then it also extends uniquely every continuous map with compact and T_2 image. Since this is the universal property of the Stone-Cech compactification, $Y = \beta X$. This is the case for $\widehat{A} \longrightarrow \widehat{ZM(A)}$ and $\widehat{A} \longrightarrow T_2(\widehat{M(A)})$: the isomorphism $C_b(\widehat{A}) \simeq C(\widehat{ZM(A)})$ implies $\widehat{ZM(A)} \simeq \beta\widehat{A}$, while the isomorphisms $C_b(\widehat{A}) \simeq C(\widehat{M(A)}) \simeq C(T_2(\widehat{M(A)}))$ imply $T_2(\widehat{M(A)}) \simeq \beta\widehat{A}$. □

11.7. **Corollary.** *For unital A, $T_2(\widehat{A}) = \widehat{ZA}$.*

For nonunital A, the dense inclusion $\widehat{A} \hookrightarrow \widehat{\widetilde{A}}$ induces a dense continuous map

$$T_2(\widehat{A}) \to T_2(\widehat{\widetilde{A}}) \simeq \widehat{Z\widetilde{A}} \simeq \widehat{\widetilde{ZA}} \simeq \widehat{ZA} \cup \{\infty\}$$

11.8. **Corollary.** *We have an isomorphism*

$$ZM(A) \xrightarrow{\phi_A} C_b(\widehat{A}) \ , \ \phi_A(z)(\pi) = \widetilde{\pi}(z)$$

with a slight abuse of notation. $\widetilde{\pi}(z) \in \mathbb{C}$ because $ZM(A)$ acts by scalars for every $\pi \in \widehat{A}$.

11.9. **Proposition.** *Let A be a C^*-algebra. For every quotient $A \xrightarrow{f} B$ the induced morphism $M(A) \xrightarrow{\widetilde{f}} M(B)$ restricts to $ZM(A) \xrightarrow{\widetilde{f}} ZM(B)$.*

Proof. Take $z \in ZM(A)$. $\widetilde{f}(z)$ commutes with every $b \in B$ for being in the image of $\widetilde{f}$. For every $m \in M(B)$ we have:

$$f(z)mb = (mb)f(z) = mf(z)b \quad \forall b \in B$$

hence $f(z)m = mf(z)$, $f(z) \in ZM(B)$. □

11.10. **Theorem** (Tietze extension theorem for C^*-algebras)**.** *Let A be a C^*-algebra. $\widehat{A}$ is normal if and only if for every quotient $A \xrightarrow{f} B$ the induced morphism $ZM(A) \xrightarrow{f} ZM(B)$ is surjective.*

Proof. Having a quotient $A \xrightarrow{f} B$ of A is the same as having a closed subspace $\widehat{B} \overset{f^*}{\hookrightarrow} \widehat{A}$. This f^* induces $C_b(\widehat{A}) \xrightarrow{f^{**}} C_b(\widehat{B})$. Let $ZM(A) \xrightarrow{\phi_A} C_b(\widehat{A})$ and $ZM(B) \xrightarrow{\phi_B} C_b(\widehat{B})$ be the isomorphisms from corollary 11.8. Consider the square:

$$\begin{tikzcd}
C_b(\widehat{A}) \arrow[r, "f^{**}"] & C_b(\widehat{B}) \\
ZM(A) \arrow[u, "\phi_A"] \arrow[r, "\widetilde{f}"] & ZM(B) \arrow[u, "\phi_B"]
\end{tikzcd}$$

We need to check that it is commutative. Let $z \in ZM(A)$, $\pi \in \widehat{B}$.

$$f^{**}\big(\phi_A(z)\big)(\pi) = \phi_A(z)\big(f^*(\pi)\big) = \phi_A(z)\big(\pi f\big) = \widetilde{\pi f}(z)$$

On the other side:

$$\phi_B\big(\widetilde{f}(z)\big)(\pi) = \widetilde{\pi}(\widetilde{f}(z))$$

But $\widetilde{\pi}\widetilde{f} = \widetilde{\pi f}$, because it is the unique extension of πf to $M(A)$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & B \arrow[d, hook] \arrow[dr, "\pi"] & \\
M(A) \arrow[r, "\widetilde{f}"] & M(B) \arrow[r, "\widetilde{\pi}"'] & B(H)
\end{tikzcd}$$

Thus $\phi_B \widetilde{f} = f^{**}\phi_A$.

Now, by classical Tietze extension theorem 11.1, $\widehat{A}$ is normal if and only if for every closed subspace $\widehat{B} \overset{f^*}{\hookrightarrow} \widehat{A}$ the map

$$ZM(A) \simeq C_b(\widehat{A}) \xrightarrow{f^{**}} C_b(\widehat{B}) \simeq ZM(B)$$

is surjective. $\square$

Trivial examples of C^*-algebras with normal spectrum are $C_0(X)$ with X normal and finite dimensional C^*-algebras, which have finite discrete spectrum. We can give a more interesting family of C^*-algebras with $\widehat{A}$ normal: σ-unital C^*-algebras A with $prim(A)$ Hausdorff. By 9.14 $prim(A)$ is locally compact. Locally compact plus Hausdorff implies completely regular, which added to σ-compactness (9.15) implies $prim(A)$ normal. For concrete examples of C^*-algebras with $prim(A)$ Hausdorff see [7], [41], [17]. However, there might be C^*-algebras with normal spectrum with $prim(A)$ not necessarily Hausdorff.

Notice that "$prim(A)$ Hausdorff" is less restrictive than "$\widehat{A}$ Hausdorff". $\widehat{A}$ Hausdorff is clearly equivalent to $\widehat{A}$ T_0 and $prim(A)$ Hausdorff. $\widehat{A}$ T_0, in the separable case, is equivalent to A postliminal (mentioned in [7]). A theorem from [7] shows that if A is a unital separable C^*-algebra with $\widehat{A}$ Hausdorff, then every irreducible representation is finite dimensional, so it is a very restrictive condition.

For unital A, $prim(A)$ Hausdorff is equivalent to $prim(A) \to T_2(prim(A))$ injective. $T_2(prim(A)) = T_2(\widehat{A}) = \widehat{ZA}$ (11.7), so $prim(A)$ Hausdorff means that for $I,J \in prim(A)$, $I \cap ZA = J \cap ZA$ implies $I = J$, i.e.: "A is central" (according to [7]).

Chapter 3

12. C^*-BUNDLES

A usual approach to noncommutative Gelfand duality consists in expressing C^*-algebras as the algebras of continuous sections vanishing at infinity of a C^*-bundle ([14],[8],[11]). The base space is a topological space related to the spectrum of the algebra and the fibers are C^*-algebras that are quotients of the original algebra. We focus on the most general version of this theorem to our knowledge, due to J. Migda ([26], theorem 2; theorem 13.8 here) and we give a description (theorem 12.5) of the multiplier algebra of $\Gamma_0(p)$ ($\Gamma_0(p)$ is the C^*-algebra of continuous sections vanishing at infinity of a C^*-bundle $E \xrightarrow{p} X$) similar to the one in [2] (theorem 3.3). The difference is that we define the notion of "strictly continuous section" as mere continuity with respect to certain topology on the corresponding bundle instead of the *ad hoc* definition in [2].

We develop a self-contained exposition of the necessary theory on C^*-bundles, starting from the definition in [11]. A salient point of this definition of C^*-bundle is the requirement of upper semicontinuity of the norm, instead of continuity. This more general condition, actually first introduced by K.H. Hofmann[8], is important in order to reach theorem 13.8.

12.1. **Notation.** *For a continuous map $E \xrightarrow{p} X$ we will denote with $E \vee E$ the fibered product:*

$$\begin{tikzcd} E \vee E \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & E \arrow[d, "p"] \\ E \arrow[r, "p"] & X \end{tikzcd}$$

Recall that $E \vee E = \{(\alpha, \beta) \in E \times E / p(\alpha) = p(\beta)\}$ and it has the following universal property: for every pair of continuous maps $Z \xrightarrow{s} E$, $Z \xrightarrow{t} E$ such that $ps = pt$ there is a unique $Z \xrightarrow{u} E \vee E$ such that $\pi_1 u = s$ and $\pi_2 u = t$.

12.2. **Definition.** *A C^*-bundle is a continuous open and surjective map $E \xrightarrow{p} X$ such that the fibers $p^{-1}(x)$ have a structure of C^*-algebra, the operations $E \vee E \xrightarrow{+,\times} E$, $E \xrightarrow{*} E$ are continuous and the norm $E \xrightarrow{||\ ||} \mathbb{R}$ upper semicontinuous. It also satisfies the following condition:*

Let $0(x)$ be the zero of $p^{-1}(x)$ and $U \ni 0(x)$, an open subset of E. Then there is an $\epsilon > 0$ and an open set V, $x \in V \subset X$ such that $\{\alpha \in p^{-1}(V) / ||\alpha|| < \epsilon\} \subset U$.

Upper semicontinuity is equivalent to continuity with respect to the topology on $\mathbb{R}$ generated by the subsets $(-\infty, t)$, $t \in \mathbb{R}$. The open sets of this topology are simply $\emptyset$, $\mathbb{R}$, and $(-\infty, t)$, $t \in \mathbb{R}$. Let us call this topology τ^+.

[8]Gillette-Dupré [11] denote "(H) C^*-bundle" what we call, following [26], just "C^*-bundle". The "H" stands for "Hofmann".

The last condition in the definition is crucial for the next fact:

12.3. **Proposition.** *For a C^*-bundle $E \xrightarrow{p} X$, the topology induced on the fibers coincides with the norm topology.*

Proof. Let $E_x = p^{-1}(x)$. With the sum, it is an abelian topological group for both topologies. Thus, it will suffice to show that every neighborhood of $0 \in E_x$ in one topology contains a neighborhood of the other. Let $B_\epsilon := \{\alpha \in E_x / ||\alpha|| < \epsilon\}$. These form a base of neighborhoods at 0 for the norm topology. By semicontinuity of the norm for C^*-bundles, B_ϵ is an open set for the topology induced by the bundle. Conversely, take $0 \in W \subset E_x$, W open for the topology induced by the bundle. $W = U \cap E_x$ for $U \subset E$ open. Now the last condition in the definition applies to find a ball inside W. $\square$

Next we prove that for a C^*-bundle $E \xrightarrow{p} X$, the set of bounded continuous sections $\Gamma_b(p) := \{X \xrightarrow{s} E / ps = Id_X,\ s \text{ continuous}, \sup\limits_{x \in X} ||s(x)|| < \infty\}$ is a C^*-algebra.

$\Gamma_0(p)$ is the set of continuous sections vanishing at infinity, i.e. those continuous sections s such that $\{x \in X / ||s(x)|| \geq \epsilon\}$ is compact $\forall \epsilon \in \mathbb{R}$. Notice that $\Gamma_0(p) \subset \Gamma_b(p)$: if $s \in \Gamma_0(p)$ then $K = \{x \in X / ||s(x)|| \geq 1\}$ is compact and $||s(K)||$ is compact for the τ^+ topology. It is easy to check that compact sets in this topology are bounded from above (actually, the compact sets are exactly those sets with a maximum) so s is bounded.

12.4. **Proposition.** *If $E \xrightarrow{p} X$ is a C^*-bundle, the set of continuous bounded sections $\Gamma_b(p)$ is a C^*-algebra. The set of continuous sections vanishing at infinity $\Gamma_0(p)$ is an ideal of $\Gamma_b(p)$.*

Proof. The 0 section is continuous thanks to the last condition in the definition of C^*-bundle. The operations are defined fiberwise: if $s, t \in \Gamma_b(p)$, $(s+t)(x) := s(x) + t(x)$, etc. We get $s+t$ trivially bounded, and continuous because by the universal property of the pullback we have $X \xrightarrow{(s,t)} E \vee E$ continuous and composing with $E \vee E \xrightarrow{+} E$ we obtain $s+t$. It is similar for the other operations. The norm in $\Gamma_b(p)$ is defined by $||s|| = \sup_{x \in X} ||s(x)||$. It is clearly a norm. Besides:

$$||st|| = \sup_x ||st(x)|| \leq \sup_x (||s(x)||.||t(x)||) \leq \sup_x ||s(x)||. \sup_x ||t(x)|| = ||s||.||t||$$

$$||ss^*|| = \sup_x ||ss^*(x)|| = \sup_x ||s(x)||^2 = ||s||^2$$

We must prove completeness. Let (s_n) be a Cauchy sequence. For each $x \in X$, $(s_n(x))$ is Cauchy, so $s_n(x) \to s(x)$. Let $\epsilon > 0$. For $m, n \in \mathbb{N}$, $m, n \geq n_\epsilon$, we have $||s_m(x) - s_n(x)|| < \epsilon$ for every $x \in X$. Since the norm is continuous on each fiber, we can take limit and obtain $||s(x) - s_n(x)|| \leq \epsilon$ for every $x \in X$. This says $s_n \to s$ but still we need to see that s is continuous. Let $U \subset E$ be a neighborhood of $s(x_0)$. Take $n \in \mathbb{N}$ such that $||s(x) - s_n(x)|| < \epsilon \ \forall x \in X$ and $s_n(x_0) \in U$ (this is possible since by previous proposition the topology induced on the fiber is equal to the norm topology). Consider the set $U - s_n = \{\alpha \in E/\alpha = \beta - s_n(p(\beta)), \beta \in U\}$. It is open because it is the image of U by the homeomorphism $E \xrightarrow{Id - s_n p} E$ (whose inverse is $Id + s_n p$). For $\beta = s_n(x_0)$, we obtain $0(x_0) \in U - s_n$. Let $x_0 \in V \subset X$ an open set such that $\{\beta \in p^{-1}(V)/||\beta|| < \epsilon\} \subset U - s_n$. For $y \in V$ we have $s(y) - s_n(y) \in U - s_n$. Hence, $s(y) - s_n(y) = \beta - s_n(p(\beta))$ for some $\beta \in U$. We have $p(\beta) = y$, so $s(y) = \beta \in U$. Thus, s is continuous (and clearly bounded).

We now check that $\Gamma_0(p)$ is a closed ideal of $\Gamma_b(p)$. If $s_n \to s$ with $s_n \in \Gamma_0(p)$, $C_\epsilon := \{x \in X/||s(x)|| \geq \epsilon\}$ is a closed set and $C_\epsilon \subset \{x \in X/||s_n(x)|| \geq \frac{\epsilon}{2}\}$ for n such that $||s - s_n|| < \frac{\epsilon}{2}$. Therefore C_ϵ is compact. If $s \in \Gamma_0(p)$ and $t \in \Gamma_b(p)$,

$$C_\epsilon := \{x \in X/||ts(x)|| \geq \epsilon\} \subset \{x \in X/||s(x)|| \geq \frac{\epsilon}{||t||}\}$$

because $||t||.||s|| \geq ||ts||$. Again C_ϵ is a closed set inside a compact. $\square$

In the following theorem we express the multiplier algebra of $\Gamma_0(p)$ for a C^*-bundle $E \xrightarrow{p} X$ as the continuous bounded sections of an associated bundle whose fibers are $M(A_x)$, $x \in X$. This won't be a C^*-bundle because the topology induces the strict topology on each $M(A_x)$ instead of the norm topology. Recall that the strict topology in the multiplier algebra $M(A)$ of a C^*-algebra A is the initial topology with respect to the maps $M(A) \xrightarrow{l_a, r_a} A$, $l_a(m) = am$, $r_a(m) = ma$.

We need to assume that all the evaluations $\Gamma_0(p) \xrightarrow{ev_x} A_x$ are surjective, i.e. for every $\alpha \in E$ there is a global section $s \in \Gamma_0(p)$ such that $s(p(\alpha)) = \alpha$. However, after the theorem we show that this condition can always be fulfilled by restricting the C^*-bundle, keeping the same Γ_0. When representing a C^*-algebra as in theorem 13.8, this condition is automatically true.

12.5. **Theorem.** *Let $E = \coprod_{x \in X} A_x \xrightarrow{p} X$ be a C^*-bundle with surjective evaluations $\Gamma_0(p) \xrightarrow{ev_x} A_x$. Let $ME := \coprod_{x \in X} M(A_x)$ with the initial topology with respect to the functions $ME \xrightarrow{L_s} E$ and $ME \xrightarrow{R_s} E$, defined by $L_s(m) = s(q(m))m \in A_{q(m)}$ where $s \in \Gamma_0(p)$ and $\coprod_{x \in X} M(A_x) \xrightarrow{q} X$ is the canonical projection. R_s is defined analogously by right multiplication. Then:*

1) $M(\Gamma_0(p)) = \Gamma_b(q)$, the bounded continuous sections of q

2) (a) q is continuous, (b) the inclusion $E \hookrightarrow ME$ is continuous-, (c) the topology induced on $M(A_x)$ is the strict topology.

$$\begin{tikzcd}
\coprod_{x \in X} A_x \arrow[r, hook] \arrow[d, "p"] & \coprod_{x \in X} M(A_x) \arrow[dl, "q"] \\
X &
\end{tikzcd}$$

Proof.

1) Let $\mu \in \Gamma_b(q)$. Define $\Gamma_0(p) \xrightarrow{\lambda_\mu} \Gamma_0(p)$, $\lambda_\mu(s)(x) := \mu(x)s(x) \in A_x$. We must check that $\lambda_\mu(s) \in \Gamma_0(p)$. But $\lambda_\mu(s) = R_s \circ \mu$, so it is continuous. It vanishes at infinity because μ is bounded and s vanishes at infinity. Analogously we define ρ_μ by right multiplication, and obtain a double centralizer $(\lambda_\mu, \rho_\mu) \in M(\Gamma_0(p))$. Conversely, take $m \in M(\Gamma_0(p))$. The morphism $\Gamma_0(p) \xrightarrow{ev_x} A_x$ is surjective by hypothesis, thus it induces a morphism $M(\Gamma_0(p)) \xrightarrow{M(ev_x)} M(A_x)$. We can define pointwisely $\mu_m(x) := M(ev_x)(m) \in M(A_x)$. The map $X \xrightarrow{\mu_m} \coprod_{x\in X} M(A_x)$ is continuous as long as its composition with L_s and R_s is continuous:

$$L_s \circ \mu_m(x) = L_s(M(ev_x)(m)) = s(x).M(ev_x)(m) =$$

$$= M(ev_x)(sm) = ev_x(sm) = (sm)(x)$$

where $sm \in \Gamma_0(p)$ (remark: $s(x)\mu_m(x) = (sm)(x)$). Besides μ_m is bounded, $||\mu_m(x)|| = ||M(ev_x)(m)|| \le ||m||$, so $\mu_m \in \Gamma_b(q)$. Thus we have defined

$$\Gamma_b(q) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} M(\Gamma_0(p)) \quad \Psi(\mu) = (\lambda_\mu, \rho_\mu) \quad \Phi(m) = \mu_m$$

With the equalities $s(x)\Phi(m)(x) = (sm)(x)$ or $\Phi(m)(x)s(x) = (ms)(x)$ we see that $\Phi\Psi$ and $\Psi\Phi$ are the identities:

$$\Phi(\Psi(\mu))(x).s(x) = (\Psi(\mu)s)(x) = \mu(x).s(x) \qquad \Rightarrow \Phi(\Psi(\mu)) = \mu$$

$$(\Psi(\Phi(m)).s)(x) = \Phi(m)(x).s(x) = (ms)(x) \qquad \Rightarrow \Psi(\Phi(m)) = m$$

2)

(a) $q = p \circ L_s$ for any $s \in \Gamma_0(p)$, for example $s = 0$, so q is continuous.

(b) We must show that the composition $E \hookrightarrow ME \xrightarrow{L_s \text{ or } R_s} E$ is continuous. Choosing L_s, that composition maps $\alpha \mapsto s(p(\alpha)).\alpha$ and we can write it as a composition of continuous maps: $E \xrightarrow{(s \circ p)\vee Id} E \vee E \xrightarrow{\cdot} E$.

(c) We denote $M(A_x)_ß$ the multiplier algebra with the strict topology and simply $M(A_x)$ the same algebra with the topology inherited from ME. Consider the composition $M(A_x) \xrightarrow{id} M(A_x)_ß \xrightarrow{a.(-)} A_x$ for $a \in A_x$. It is equal to the restriction of $ME \xrightarrow{L_s} E$ for $s \in \Gamma_0(p)$ such that $s(x) = a$. Thus it is continuous, and this (along with the corresponding right multiplication) implies $M(A_x) \xrightarrow{id} M(A_x)_ß$ continuous. Now consider:

$$M(A_x)_ß \xrightarrow{id} M(A_x) \overset{i_x}{\hookrightarrow} \coprod_{x\in X} M(A_x) \xrightarrow{L_s \text{ or } R_s} \coprod_{x\in X} A_x$$

Since the long composition is continuous, $i_x \circ id$ is continuous, and therefore $M(A_x)_ß \xrightarrow{id} M(A_x)$ is continuous, and this finishes the proof. $\square$

When $A_x = \mathbb{C}\ \forall x \in X$, X locally compact and Hausdorff guarantees that the evaluations $\Gamma_0(p) \xrightarrow{ev_x} A_x$ are surjective and the theorem reduces to $M(C_0(X)) = C_b(X)$.

12.6. **Remark.** In case a C^*-bundle $E \xrightarrow{p} X$ doesn't satisfy that all the evaluations $\Gamma_0(p) \xrightarrow{ev_x} A_x$ are surjective, we can take the images of these morphisms, $A'_x := ev_x(\Gamma_0(p))$, and show that they form a C^*-bundle $E' \xrightarrow{p'} X$. This bundle satisfies $\Gamma_0(p') = \Gamma_0(p)$ and all the evaluations $\Gamma_0(p') \xrightarrow{ev_x} A'_x$ are surjective, so we can apply last theorem to this C^*-bundle to obtain $M(\Gamma_0(p))$.

Proof. A'_x are C^*-subalgebras of A_x. We give $E' = \coprod_{x\in X} A'_x$ the subspace topology from E. The topology on $E' \vee E'$ is the subspace topology from $E' \times E'$ which is the subspace topology from $E \times E$, thus $E' \vee E'$ is a subspace of $E \vee E$. This is important to conclude that the operations are continuous on E'. The projection $E' \xrightarrow{p'} X$ is clearly continuous and surjective (openness: coming soon).

Let $U' \subset E'$ be an open neighborhood of $0(x) \in A'_x$. We have $U' = U \cap E'$ where $U \subset E$ is open. Since $E \xrightarrow{p} X$ is a C^*-bundle, there is an open neighborhood V of x and $\epsilon > 0$ such that $\{\alpha \in p^{-1}(V)/||\alpha|| < \epsilon\} \subset U$. Hence, $\{\alpha \in p'^{-1}(V)/||\alpha|| < \epsilon\} \subset U'$.

Openness of p'. Let $U' \subset E'$ be an open set. Let $x \in p'(U')$, $x = p'(\alpha)$. Since $\alpha \in E'$, there is an $s \in \Gamma_0(p)$ such that $s(x) = \alpha$. The set $U' - s_n$ (defined as in the proof of 12.4) is an open subset of E' containing $0(x)$. By previous property we have an open $V \subset X$ such that $x \in V \subset p'(U' - s_n) = p'(U')$. $\square$

13. A as the continuous sections vanishing at ∞ of a C^*-bundle

Following Migda [26] we introduce the notion of "H-family of a C^*-algebra" as a tool for the main theorem of this section, theorem 13.8. We give a direct proof of the fact that an H-family induces a C^*-bundle.

13.1. **Definition.** *An H-family of a C^*-algebra A is a family of C^*-epimorphisms $A \xrightarrow{f_x} A_x$ with $x \in X$ a topological space, such that for each $a \in A$ the map $x \mapsto ||f_x(a)||$ is upper semicontinuous and vanishing at infinity, i.e. $\{x \in X/||f_x(a)|| \geq \epsilon\}$ is compact and closed in X for every $\epsilon > 0$.*

Given an H-family as in definition 13.1, define:

$$E = \coprod_{x\in X} A_x \xrightarrow{p} X$$

The topology on E is that generated by the tubes:
$T(V, a, \epsilon) = \coprod_{x\in V} B(f_x(a), \epsilon)$ (disjoint union of open balls) where $V \subset X$ is an open set, $a \in A$, $\epsilon > 0$. p is continuous because the preimage of an open set $V \subset X$ is a union of tubes centered at $0 \in A$.

13.2. **Remark.**
• The tubes form a basis for the topology.
• For every $\alpha \in E$, $\alpha = f_x(a)$ the tubes $T(V, a, \epsilon)$ $(x \in V)$ form a basis of neighborhoods at α

For the proof, the key is the following fact: if $\alpha \in T(V', a', \epsilon')$ and $\alpha = f_x(a)$, then there is a tube centered at a, $T(V, a, \epsilon)$ contained in $T(V', a', \epsilon')$. To see this, take ϵ_0 such that $||\alpha - f_x(a')|| < \epsilon_0 < \epsilon'$. Consider the open set $W = \{y \in X / ||f_y(a - a')|| < \epsilon_0\}$. Now define $V = V' \cap W$ and $\epsilon > 0$ satisfying $\epsilon < \epsilon' - \epsilon_0$. We check that $T(V, a, \epsilon) \subset T(V', a', \epsilon')$. Let $\beta \in T(V, a, \epsilon)$. $p(\beta) \in V \subset V'$.

$$||\beta - f_{p(\beta)}(a')|| \leq ||\beta - f_{p(\beta)}(a)|| + ||f_{p(\beta)}(a) - f_{p(\beta)}(a')|| < \epsilon + \epsilon_0 < \epsilon'$$

Having this, now it is easy to check that for a point $\beta \in E$ in the intersection of two tubes, there is a tube contained in that intersection and centered at some $b \in f^{-1}_{p(\beta)}(\beta)$. This proves the first assertion. Now the second assertion also follows easily. □

We will need the following simple topological lemma:

13.3. **Lemma.** *Let $X \xrightarrow{f} Y$ be a continuous and open map between topological spaces and $S \subset X$, $T \subset Y$ such that $f^{-1}(T) = S$. Then the restriction $S \xrightarrow{f} T$ is open.*

Proof. Let $U \subset X$ be an open set. We show that $f(U \cap S) = f(U) \cap T$. The inclusion $f(U \cap S) \subset f(U) \cap T$ is clear. If $t \in f(U) \cap T$, then $t = f(u)$ with $u \in U$. But $u \in f^{-1}(T) = S$, so $u \in U \cap S$. □

13.4. **Proposition.** *Given an H-family $\{A \xrightarrow{f_x} A_x / x \in X\}$, the bundle $E = \coprod_{x \in X} A_x \xrightarrow{p} X$ with the topology given by the tubes, is a C^*-bundle.*

Proof. Continuity of the involution is clear from the equality $*^{-1}(T(V, a, \epsilon)) = T(V, a^*, \epsilon)$. For upper semicontinuity of the norm, take any point of E, say $\alpha \in A_{p(\alpha)}$, $\alpha = f_{p(\alpha)}(a)$. We shall find a neighbourhood of α such that the norm in that neighbourhood is less than $||\alpha|| + \epsilon$. Consider the open set $V = \{x \in X / ||f_x(a)|| < ||\alpha|| + \frac{\epsilon}{2}\} \ni p(\alpha)$. If $\beta \in T(V, a, \frac{\epsilon}{2})$,

$$||\beta|| \leq ||\beta - f_{p(\beta)}(a)|| + ||f_{p(\beta)}(a)|| < \frac{\epsilon}{2} + ||\alpha|| + \frac{\epsilon}{2} = ||\alpha|| + \epsilon$$

Regarding continuity of the other operations, we will do it first for the trivial case $A_x = A$, $A \xrightarrow{f_x = Id} A$. In this case $E = A \times X$ with the product topology. Considering the natural homeomorphism $(A \times X) \vee (A \times X) \simeq A \times A \times X$ we conclude that addition and multiplication are continuous, because they are on A. Clearly, multiplication by scalars $\mathbb{C} \times A \times X \to A \times X$ is continuous too.

For the general case, the key is that the natural application $A \times X \to E$ is surjective, continuous and open.

$$A \times X \xrightarrow{\tilde{f}} \coprod_{x \in X} A_x$$

$$(a, x) \longmapsto f_x(a)$$

Surjectivity is clear. Continuity: for a point $(a, x) \in A \times X$, take a neighborhood of $f_x(a)$. By the remark before the lemma, it contains a neighborhood of the form $T(V, a, \epsilon)$. The neighborhood of (a, x), $B(a, \epsilon) \times V$ satisfies $\tilde{f}(B(a, \epsilon) \times V) \subset T(V, a, \epsilon)$ because the quotients f_x are contractive. Thus, $\tilde{f}$ is continuous. Actually, it holds the equality $\tilde{f}\big(B(a, \epsilon) \times V\big) = T(V, a, \epsilon)$ because for a quotient of C^*-algebras the image of the open unit ball is the open unit ball, and this proves $\tilde{f}$ open. As a consequence $(A \times X) \times (A \times X) \xrightarrow{\tilde{f} \times \tilde{f}} E \times E$ is open, and since $(\tilde{f} \times \tilde{f})^{-1}(E \vee E) = (A \times X) \vee (A \times X)$, we have $(A \times X) \vee (A \times X) \xrightarrow{\tilde{f} \times \tilde{f}} E \vee E$ open by previous lemma, and therefore it is a topological quotient. Now the diagram

$$\begin{tikzcd}
(A \times X) \vee (A \times X) \arrow[r, "+"] \arrow[d, "\tilde{f} \times \tilde{f}"'] & A \times X \arrow[d, "\tilde{f}"] \\
E \vee E \arrow[r, "+"] & E
\end{tikzcd}$$

shows that the sum is continuous in the general case. And similarly for the product and multiplication by scalars. $\square$

We will need the following two cases of H-families.

13.5. **Proposition.** *Let $prim(A) \xrightarrow{c} X$ be a surjective continuous map with X Hausdorff. For $x \in X$ define $A_x = {}^{A}\!/\!_{\bigcap_{I \in c^{-1}(x)} I}$, and $A \xrightarrow{f_x} A_x$ simply the quotient map. This is an H-family.*

Proof. Let $\alpha \in A$ and $\epsilon > 0$. By lemma 9.13, the set
$Z_\epsilon = \{I \in prim(A) / ||a_I|| \geq \epsilon\}$ is compact. If we show that
$c(Z_\epsilon) = \{x \in X / ||f_x(a)|| \geq \epsilon\}$ we are done. Let $I \in Z_\epsilon$. Since $I \in c^{-1}(c(I))$, we have $A \to A_{c(I)} \to {}^{A}\!/\!_{I}$ and $||f_{c(I)}(a)|| \geq ||a_I|| \geq \epsilon$. Conversely, take $x \in X$ such that $||f_x(a)|| \geq \epsilon$. We can take an irreducible representation of A_x, $A_x \xrightarrow{\pi} B(H)$, that preserves the norm of $f_x(a)$. This gives by composition an irreducible representation of A whose kernel $I \in Z_\epsilon$ and satisfies $\bigcap_{J \in c^{-1}(x)} J \subset I$ (because $ker(f_x) \subset ker(\pi f_x)$). Since $c^{-1}(x)$ is closed, $I \in c^{-1}(x)$, so $x \in c(Z)$. $\square$

13.6. **Proposition.** *If $E \xrightarrow{p} X$ is a C^*-bundle with surjective evaluations $\Gamma_0(p) \xrightarrow{ev_x} A_x$, then $\Gamma_0(p) \xrightarrow{ev_x} A_x$ is an H-family.*

Proof. For $s \in \Gamma_0(p)$, $\epsilon > 0$, the set $\{x \in X/||ev_x(s)|| \geq \epsilon\}$ is closed by the upper semicontinuity of the norm and compact because s vanishes at infinity. $\square$

The following theorem is "theorem 1" in [26], where it is called "Stone-Weierstrass theorem for H-families". It makes use of the known noncommutative version of Stone-Weierstrass theorem, 10.2.

13.7. **Theorem.** *Let $A \xrightarrow{f_x} A_x$ $(x \in X)$ be an H-family of C^*-algebras and $B \subset A$ a subalgebra. If $B + ker(f_x) \cap ker(f_y) = A$ for every $x, y \in X$, then $B + \bigcap_{x\in X} ker(f_x) = A$.*

Proof. We divide the proof in four parts.

Part 1: reduction to the case $\bigcap_{x\in X} ker(f_x) = 0$.

Consider $A \xrightarrow{\pi} \bigcap_{x\in X} ker(f_x)$. We have ${}^{A}/_{\bigcap_{x\in X} ker(f_x)} \xrightarrow{\overline{f_x}} A_x$ an H-family. Since $B + ker(f_x) \cap ker(f_y) = A$, it holds $\pi(B) + ker(\overline{f_x}) \cap ker(\overline{f_y}) = {}^{A}/_{\bigcap_{x\in X} ker(f_x)}$. If we had the theorem for the case $\bigcap_{x\in X} ker(f_x) = 0$, we would conclude $\pi(B) = {}^{A}/_{\bigcap_{x\in X} ker(f_x)}$, and therefore $A = B + \bigcap_{x\in X} ker(f_x)$.

Part 2. Here we show that it suffices with proving that every $\varphi \in \overline{P(A)} \cup \{0\}$ can be written as $A \xrightarrow{f_x} A_x \xrightarrow{\tilde{\varphi}} \mathbb{C}$ ($\tilde{\varphi}$ a function, but it results necessarily a positive functional). If we have this, the theorem follows by the known noncommutative version of Stone-Weierstrass theorem (10.2): take $\varphi_1, \varphi_2 \in \overline{P(A)} \cup \{0\}$, $\varphi_1 \neq \varphi_2$. For some $a \in A$, $\varphi_1(a) \neq \varphi_2(a)$. Now we write $\varphi_1 = \tilde{\varphi}_1 \circ f_{x_1}$ and $\varphi_2 = \tilde{\varphi}_2 \circ f_{x_2}$ as before, and $a = b + k$ with $b \in B$ and $k \in ker(f_{x_1}) \cap ker(f_{x_2})$. We have

$$\varphi_1(b) = \varphi_1(a) \neq \varphi_2(a) = \varphi_2(b)$$

so B separates $\overline{P(A)} \cup \{0\}$. Hence $B = A$.

Part 3. Applying the order preserving bijection between ideals of A and closed saturated sets of $P(A)$ (9.9) we get:

$$V(\bigcap_{x\in X} ker(f_x)) = V(0)$$

$$\overline{\bigcup_{x\in X} V(ker(f_x))} = P(A)$$

By 9.8 ("$V(J) = P({}^{A}/_{\langle J\rangle})$") we have $V(ker(f_x)) = f_x^*(P(A_x))$. So the conclusion of this part is that the set of those pure states of A induced by pure states of A_x is dense in $\overline{P(A)}$.

Part 4. Take $\varphi \in \overline{P(A)} \cup \{0\}$. If $\varphi = 0$, we have $\varphi = \tilde{\varphi} \circ f_x$ for $\tilde{\varphi} = 0$. If $\varphi \in \overline{P(A)}$, by previous part, we have a net $\varphi_i \to \varphi$, with $\varphi_i \in f_{x_i}^*(P(A_{x_i}))$, i.e. $\varphi_i = g_i \circ f_{x_i}$ where $g_i \in P(A_{x_i})$. If $\{x_i\}$ has no accumulation points, we can show that $\varphi = 0$: take $a \in A$, $\epsilon > 0$ and a compact $K \subset X$ such that $||f_x(a)|| < \epsilon$ $\forall x \in K^c$. Then, choosing a sufficiently large i such that $x_i \notin K$,

$$|\varphi(a)| \leq |\varphi(a) - \varphi_i(a)| + |\varphi_i(a)| < \epsilon + |g_i(f_{x_i})| < 2\epsilon$$

Now assume there is an accumulation point $x \in X$ of the net $\{x_i\}$. Relabeling to avoid an extra subindex, we are left with $x_i \to x$. The existence of $A_x \xrightarrow{g} \mathbb{C}$ with $\varphi = g \circ f_x$ is guaranteed if we prove $ker(f_x) \subset ker(\varphi)$. Take $a \in ker(f_x)$. If $|\varphi(a)| > 0$, say $\varphi(a) = 2\delta$, since $\varphi_i(a) \to \varphi(a)$, there is an i_1 such that $|\varphi_i(a)| > \delta$ for $i \geq i_1$. Hence $||f_{x_i}(a)|| \geq ||g_i f_{x_i}(a)|| = \varphi_i(a) > \delta$ for $i \geq i_1$. On the other hand, the set $\{y \in X / ||f_y(a)|| < \delta\}$ is an open neighborhood of x, so there is i_2 such that $f_{x_i}(a) < \delta$ for $i \geq i_2$. This is a contradiction. ☐

13.8. **Theorem.** *Let $prim(A) \xrightarrow{c} X$ be a continuous map onto a Hausdorff space X. Let $A \xrightarrow{f_x} A_x$ be the H-family induced by this map as in 13.5. Let $E \xrightarrow{p} X$ be the C^*-bundle induced by this H-family (13.4). Then $A = \Gamma_0(p)$.*

Proof. First, we have the map:

$$A \longrightarrow \Gamma_0(p)$$

$$a \longmapsto \hat{a} = f_{(-)}(a)$$

$\hat{a}$ is continuous in $x \in X$ because, for a basic neighborhood $T(V, a, \epsilon)$ of $f_x(a)$, we have $\hat{a}(V) \subset T(V, a, \epsilon)$. Also $\hat{a}$ vanishes at infinity because $\{x \in X / ||f_x(a)|| \geq \epsilon\}$ is compact. Besides $A \to \Gamma_0(p)$ is injective: for $a \neq 0$, take $I \in prim(A)$ such that $a \notin I$ (or even $||a_I|| = ||a||$). We have $f_{c(I)}(a) \neq 0$.

This inclusion implies that the evaluations $\Gamma_0(p) \xrightarrow{ev_x} A_x$ are surjective. By proposition 13.6, $\{\Gamma_0(p) \xrightarrow{ev_x} A_x\}_{x \in X}$ is an H-family. Since $\bigcap_{x \in X} ker(ev_x) = 0$, by theorem 13.7 it suffices with $A + ker(ev_x) \cap ker(ev_y) = \Gamma_0(p)$ $\forall x \neq y \in X$ to conclude $A = \Gamma_0(p)$.

If $ker(f_x) + ker(f_y) \subsetneq A$, we would have a primitive ideal $I \supset ker(f_x) + ker(f_y)$. From $I \supset ker(f_x) = \bigcap\limits_{J \in c^{-1}(x)} J$ it follows $I \in c^{-1}(x)$, for $c^{-1}(x)$ is closed. Analogously we have $I \in c^{-1}(y)$. This is absurd, since $c^{-1}(x) \cap c^{-1}(y) = \emptyset$. Thus, $ker(f_x) + ker(f_y) = A$.

Now consider $\alpha \in A_x$, $\beta \in A_y$. Take a and b such that $f_x(a) = \alpha$ and $f_y(b) = \beta$. Now write $a = a_x + a_y$, $b = b_x + b_y$ with $a_x, b_x \in ker(f_x)$ and $a_y, b_y \in ker(f_y)$. The element $d = a_y + b_x$ satisfies:

$$f_x(d) = f_x(a_y + b_x) = f_x(a_y) = f_x(a) = \alpha$$

$$f_y(d) = f_y(a_y + b_x) = f_y(b_x) = f_y(b) = \beta$$

Applying this for $\alpha = s(x)$, $\beta = s(y)$, where $s \in \Gamma_0(p)$, we obtain an element $d \in A$ such that $s - \hat{d} \in ker(ev_x) \cap ker(ev_y)$. ☐

13.9. **Remarks.** If $X = \{*\}$ in the theorem, we have a trivial bundle and the trivial identity $A = A$. The other extreme, the most refined choice, is $X = T_2(prim(A))$. If A is commutative, $A = C_0(X)$ for a locally compact

Hausdorff X, $T_2(prim(A)) = prim(A) = X$ and we get Gelfand duality. If A has a unit, $T_2(prim(A)) = \widehat{Z(A)}$ (11.7) and this is a central decomposition. In case A doesn't have a unit we still have a map $prim(A) \to \widehat{ZA} \cup \{\infty\}$, where the codomain is the one-point compactification of the spectrum of the center of A (see remark after 11.7). This map can be made surjective simply by restricting the codomain to the image.

13.10. **Corollary** (Dauns-Hofmann theorem)**.** *See 11.4 and subsequent commentary.*

Proof. We start from a continuous and bounded function $prim(A) \xrightarrow{f} \mathbb{C}$. This is the same as a continuous bounded $T_2(prim(A)) \xrightarrow{f} \mathbb{C}$. We apply last theorem with $X = T_2(prim(A))$. Since multiplication by scalars $\mathbb{C} \times E \to E$ is continuous for a C^*-bundle, we have $f \cdot a \in \Gamma_0(p)$ for $a \in A = \Gamma_0(p)$. $\square$

This proof of the theorem is close in spirit to the original proof by Dauns and Hofmann. As a matter of fact, the theorem came out from the study of representation of more general rings by sections ([8], corollary 8.16).

A direct application of 13.8 and 12.5 gives:

13.11. **Corollary.** *For a C^*-algebra A and a continuous map $prim(A) \xrightarrow{c} X$ onto a Hausdorff space X, let $\coprod_{x\in X} A_x \xrightarrow{p} X$ be the associated C^*-bundle ($A = \Gamma_0(p)$ by 13.8). The multiplier algebra $M(A)$ is equal to the continuous sections of $ME = \coprod_{x\in X} M(A_x) \xrightarrow{q} X$ where the topology on ME is the initial topology with respect to the family of left and right multiplication by elements of $\Gamma_0(p) = A$ as in 12.5.*

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