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Topología y geometría aplicada al estudio de algunas ecuaciones diferenciales de segundo orden

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Resumen: En esta tesis estudiamos la existencia y multiplicidad de soluciones a algunas ecuaciones diferenciales de segundo orden con condiciones de Dirichlet o periódicas. Los resultados de existencia se deducen principalmente de la teoría de grado topológico de Leray-Schauder. Usando métodos de geometría diferencial en espacios de funciones se consigue complementar estos resultados con dependencia continua y genericidad. Las herramientas utilizadas involucran tanto el análisis como la topología algebraica y diferencial, y también hay resultados que usan teoría de nudos.

Mostramos que hay profundas conexiones entre la existencia de soluciones y la topología de algunos espacios relacionados con la ecuación.

Palabras claves: ecuaciones diferenciales, teoría de grado, teoría de Morse, teoría de nudos, teorema de Sard-Smale

Topology and geometry applied to the study of some second order differential equations

Abstract: In this thesis we study existence and multiplicity of solutions to some differential equations of second order, with dirichlet or periodic boundary conditions. The existence results are inferred mainly from the topological degree theory of Leray and Schauder. Using methods from differential geometry in function spaces we may complement these results with continuous dependence and genericity. The tools we use involve analysis as much as differential and algebraic topology, and also there are results using knot theory.

We show that there are deep connections between the existence of solutions and the topology of some spaces related to the equation.

Key Words: differential equations, degree theory, Morse theory, knot theory, Sard-Smale theorem

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Chapter 1

Introduction

Topology provides effective methods to describe qualitative aspects of dynamical systems, where elementary analysis fails. This is the case, for example, of weakly nonlinear equations with boundary conditions where the nonlinearity has no useful algebraic properties. Using abstract and powerful tools like topological degree theory, variational methods and Conley theory (among others) one may easily obtain some amount of information about very general type of equations.

A prototypical example which we shall treat in detail in the next chapter is the now classical theorem of Landesman and Lazer which states that a scalar equation of the form $u''(t) = g(u(t)) + p(t)$ with p a periodic function, admits at least one periodic solution when the average of p lies between the limits of g at $\pm\infty$. The importance of this theorem relies in the fact that the condition depends only on the average, so p is allowed to have a very nasty behaviour like rapid and ample oscilations, or fail to be differentiable.

An other nice example is a theorem of Mawhin and Bereanu [11] which states the existence of solutions for the homogeneous relativistic acceleration equation

$$\begin{cases} \phi(u')' = f(t, u, u') \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

for $\phi(x) = \frac{x}{\sqrt{1-|x|^2}}$ and f continuous, under no other condition!

An advantage of these tools is that they give information which is robust and global so they help to describe the solution set more than a single solution.

On the other hand, geometry provides methods to describe things locally. For example, using the inverse function theorem on Banach spaces we can deduce continuous (or differentiable) dependence on the parameters and local uniqueness of solutions. Combining degree theory with the differentiability properties of functionals a lot of information can be obtained.

This thesis focuses mainly on the problem of existence and multiplicity of periodic motions of weakly nonlinear second order systems and is organized as follows:

In the next chapter we introduce some of the basic notions and techniques which are involved in the results presented in this work.

Chapter 3 treats the general equation $u''(t) = f(t, u(t), u'(t))$ under some geometric assumptions on f, which are generalizations of the Hartman-Nagumo conditions for second order systems. This generalization puts in evidence the geometry behind these conditions which are often presented as a complicated set of inequalities.

Chapters 4, 5 and 6 deal with singular problems of Kepler type $u''(t) \pm \frac{u(t)}{|u(t)|^{q+1}} = \lambda h(t)$. We study the existence and multiplicity of periodic solutions in relation to some topological data which depends on the parameters of the equation. For example in dimension 2 we deduce existence of solutions by counting the connected components of a space and the winding numbers of a curve. In dimension 3 the existence of solutions is related to the first homology group of a space and, in some cases, to the knot type of a curve. In the same lines of these results we also study the restricted N-body problem and obtain similar results considering links in $\mathbb{R}^3$.

Finally in Chapter 7 we study an elliptic partial differential ecuation with Neumann boundary conditions.

Some of the results in this thesis have been published in research articles in [4], [5] and [6]

Contents

Chapter 2

Preliminaries

2.1 Degree Theory

Since one of the common lines along the chapters of this work is the use of the Leray-Schauder degree, we start with a short overview on the subject.

A precise and elementary definition of the Brouwer degree would take too long so for that purpose we refer the reader to the book of N.G. Lloyd [32]. Here we will give just the idea of the definition and list its most important properties. Curiously enough, these properties are undoubtedly true knowing just the idea of what the degree should be, and the consequences of its existence are numerous and non trivial.

2.1.1 The Brouwer degree

Let us give an informal definition of the Brouwer degree in a very specific context:

Definition 1. *Let $B \subseteq \mathbb{R}^n$ be an open ball and $f : \overline{B} \to \mathbb{R}^n$ a continuous function. Take $p \in \mathbb{R}^n$ and assume that $f(\partial B) \not\ni p$. Then the Brouwer degree $deg(f, B, p) \in \mathbb{Z}$ is the signed number of laps that $f(\partial B)$ turns around the point p.*

Without worrying about the precise meaning of this, the following properties are obviously satisfied:

1. The degree depends only on the values of the function f in ∂B.
2. $deg(f, B, p) = deg(f - p, B, 0)$.
3. If $x_0 \neq p$ and $f : \overline{B} \to \mathbb{R}^n$ is the constant function $f(x) = x_0$ then $deg(f, B, p) = 0$.
4. Let id be the identity function, then $deg(id, B, p) = \begin{cases} 1 \text{ if } p \in B \\ 0 \text{ if } p \notin \bar{B} \end{cases}$

A continuous function of two variables $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ can be considered as a deformation of the function $f_0(x) = f(0, x)$. Using the notation $f_t(x) = f(t, x)$ we also have:

5. If $f_t : \overline{B} \to \mathbb{R}^n$ is a deformation of f such that $f_t(\partial B) \not\ni p$ for all t then $deg(f_t, B, p)$ is independent of t.

It is clear that if $p \notin f(\overline{B})$ then we can take a deformation f_t such that $f_1 = f$ and f_0 is constantly a point x_0: just take $f_t(x) = f(tx)$. Then using the properties 5 and 3 it follows that $deg(f, B, 0) = 0$. We obtain the following important property:

6. If $deg(f, B, p) \neq 0$ then there exists an $x \in B$ such that $f(x) = p$.

Also by using property 5 and by taking the linear homotopy between two continuous functions f, g, we obtain:

7. If $p \notin f(\partial B)$ and $|f(x) - g(x)| < dist(f(\partial B), p)$ for all $x \in \partial B$, then $p \notin g(\partial B)$ and $deg(g, B, p) = deg(f, B, p)$.

For the special case $n = 2$ the degree is just the winding number of a curve, a basic notion from complex analysis:

Assume $f : \overline{B} \to \mathbb{R}^2$ is C^1, B is a ball of radius r centered at s_0 and let us identify $\mathbb{R}^2$ with the complex plane. Parametrize ∂B with the curve $\gamma(t) = x_0 + re^{2\pi it}$ and then define

$$deg(f, B, p) = \frac{1}{2\pi i} \int_{f\gamma} \frac{1}{z-p} dz = \frac{1}{2\pi i} \int_0^1 \frac{\frac{\partial}{\partial t} f(\gamma(t))}{f(\gamma(t)) - p} dt. \tag{2.1}$$

While the second expression provides us an explicit formula for the calculation of the degree, it depends on the first derivatives of f. On the other side, the first expression doesn't need the differentiability of f and is well defined for continuous functions.

Let us discuss some toy examples showing the usefulness of the Brouwer degree.

Example 2. *Fundamental Theorem of Algebra*

Let p be a polynomial of the form $p(z) = z^d + q(z)$ with q of degree less than d and $d \geq 1$. Clearly there exists a constant $R > 0$ such that

$$R^d > |q(z)| \text{ for } |z| = R. \tag{2.2}$$

We take the deformation $p_t(z) = z^d + tq(z)$ for $t \in [0, 1]$ and set $B = B(R, 0)$. The inequality (2.2) implies that $p_t(\partial B) \not\ni 0$ for all $t \in [0, 1]$. Then using property 5 we obtain $deg(p, B, 0) = deg(z^d, B, 0) = d \neq 0$ and conclude using 6 that p must have at least one zero.

Example 3. *Non-retraction of the n-sphere*

Let D^n, S^{n-1} denote the closed unit ball of $\mathbb{R}^n$ and its boundary, respectively. The non-retraction theorem says that there is no continuous function $D^n \to S^{n-1}$ leaving the boundary fixed.

This classic theorem from topology is highly non-trivial and has many consequences like the Brouwer fixed point theorem and the hairy ball theorem. For a beautiful proof by Milnor and Rogers using only analysis we refer to [41] and [57]. The proof using topological degree is as follows:

Suppose by contradiction that there exists a retraction $f : D^n \to S^{n-1}$ (i.e. f is continuous and $f|_{S^{n-1}}$ is the identity). By properties 1 and 4, $deg(f, D^n, 0) = deg(id, D^n, 0) = 1$. Using 6 we obtain a zero of f which is a contradiction.

Example 4. *Global implicit function theorem*

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a function satisfying the hypotheses of the implicit function theorem, namely that f is C^1, $f(0,0) = 0$ and $\partial_2 f$ is invertible at $(0,0)$. It is easy to see that we can find a small ball B centered at 0 for which $f_0(\partial B) \not\ni 0$ and in that case $deg(f_0, B, 0) = \pm 1$ (the sign depends on the orientation properties of $\partial_2 f(0,0)$) roughly speaking, because f_0 is close to its derivative. Since the deformation f_t is continuous we have that $f_t(\partial B) \not\ni 0$ for small t. Using properties 5 and 6 we deduce that the equation $f(t,x) = 0$ admits at least one solution $x(t) \in D$.

Although the conclusion is weaker than the one in the classic implicit function theorem (neither local uniqueness nor continuous dependence are deduced), we don't use deeply the differentiable structure of f; we only need $deg(f_0, B, 0) \neq 0$ for some open set B, which is a condition a lot easier to verify than the invertibility of $\partial_2 f$. This principle will be widely used in chapters 4 and 6.

We also obtain a solution $x(t)$ for t far from 0, as long as $f_t(\partial B) \not\ni 0$. So this technique provides global information, in contrast to the local information given by the implicit function theorem.

The notion of degree can be extended to continuous functions defined in the closure of any open bounded set. In this exposition we have chosen B to be a ball because otherwise property 3 would be far from obvious.

2.1.2 Definition using algebraic topology

For the special case we are dealing with (namely that B is a ball) we can easily give a definition of deg using the sophisticated machinery of homology theory.

As before, let $B \subseteq \mathbb{R}^n$ be the unit ball and let $f : \overline{B} \to \mathbb{R}^n$ be a continuous function such that $f(\partial B) \not\ni 0$. Restrict f to ∂B and normalize it as $\overline{f}(x) = \frac{f(x)}{|f(x)|}$. We obtain a (well defined) continuous map $S^{n-1} \to S^{n-1}$. Now applying the singular homology functor $H^{n-1}(-, \mathbb{Z})$ we obtain a group homomorphism $\overline{f}_* : \mathbb{Z} \to \mathbb{Z}$. This morphism is obviously a multiplication by an integer d which depends only on f. We define $deg(f, B, 0) = d = \bar{f}_*(1)$.

All properties from 1 to 6 are immediate from the functoriality of H^* and its homotopy invariance.

Kronecker gave a definition of degree for C^1 mappings based on the integration of differential forms in [29], see also [38]. This is an elegant generalization of formula (2.1) for dimension n but ultimately it is just the image of a generator through the morphism $\bar{f}_*$ when one applies the de Rham co-homology functor instead of singular homology.

When $f : \bar{D} \to \mathbb{R}^n$ is C^1, D is an open bounded set and $p \in \mathbb{R}^n \setminus \partial D$ is a regular value of f it's easy to see that the equation $f(x) = p$ can only have finitely many solutions $x_1, \ldots, x_k$. In this case there is an explicit formula

$$deg(f, D, p) = \sum_{i=1}^{k} sg(\det D_{x_i} f) \tag{2.3}$$

which is well defined presicely because of the regularity of p.

2.1.3 The Leray-Schauder degree

A degree theory for a class of mappings between Banach spaces was defined for the first time by Leray and Schauder in 1933 in a foundational paper [59]. Several applications of this tool were found in the field of differential equations and non-linear analysis in general.

The key notion to develop this theory is the following:

Definition 5. *Let X be a Banach space.*

A map $T : X \to X$ is said to be compact if it is continuous and $\overline{T(B)}$ is compact for every bounded set B.

A map $F : X \to X$ of the form $F(x) = x - T(x)$ where T is a compact map, is said to be a compact perturbation of the identity.

The Leray-Schauder degree $deg_{LS}(F, D, 0) \in \mathbb{Z}$ is defined for D an open bounded set in X, and for $F : \overline{D} \to X$ a compact perturbation of the identity. As always we require that $F(\partial D) \not\ni 0$. It fulfills the following set of properties:

I The degree depends only on the values of the function f in ∂D.

II $deg(f, D, p) = deg(f - p, D, 0)$.

III If $x_0 \neq p$ and $f : \overline{D} \to \mathbb{R}^n$ is the constant function $f(x) = x_0$ then $deg(f, D, p) = 0$.

IV Let id be the identity function, then $deg(id, D, p) = \begin{cases} 1 \text{ if } p \in D \\ 0 \text{ if } p \notin \bar{D} \end{cases}$

V If $f_t : \overline{D} \to \mathbb{R}^n$ is a deformation of f such that for all t, f_t is a compact perturbation of the identity and $f_t(\partial D) \not\ni p$ then $deg(f_t, D, p)$ is independent of t.

VI If $deg(f, D, p) \neq 0$ then there exists an $x \in D$ such that $f(x) = p$.

VII If D_1, D_2 are disjoint open bounded sets and $f(\partial D_i) \not\ni p$ for $i = 1, 2$ then $deg(f, D_1 \cup D_2, p) = deg(f, D_1, p) + deg(f, D_2, p)$.

VIII If $E \subset D$ are open bounded sets and $f(\overline{E}) \not\ni p$ then $deg(f, D, p) = deg(f, D \setminus \overline{E}, p)$.

IX If $T : X \to V \subset X$ is compact and its range lies in a closed linear subspace V containing p, and if $f = id_X - T$ then $f|_V : V \to V$ and $deg(f, D, p) = deg(f|_V, D \cap V, p)$.

X If f is a diffeomorphism from D onto a neighbourhood of p then $deg(f, D, p) = \pm 1$.

The feature that makes compact mappings relevant to this theory is that when restricted to bounded sets they can be uniformly approximated by functions of finite rank.

The construction of the Leray-Schauder degree is as follows: Take a compact map $T : D \subset X \to X$ defined on a bounded domain. For each $\varepsilon > 0$ there exists a function T_ε whose image lies in a finite dimensional subspace V_ε and such that $\|T - T_\varepsilon\|_\infty \leq \varepsilon$. Then using property IX for finite dimensional mappings one can prove that the degrees of the restricted functions $id_{V_\varepsilon} - T_\varepsilon : D \cap V_\varepsilon \to V_\varepsilon$ are independent of the approximation provided that ε is sufficiently small. In this way a degree for compact perturbations of the identity may be defined. Afterwards the properties I to X are proved in the finite dimensional case, and extended to the general case by taking uniform limits.

2.2 Setting differential equations as point equations in function spaces

2.2.1 Non-resonant problems

Consider the scalar two point boundary value problem

$$\begin{cases} u''(t) = f(t, u(t)) \\ u(0) = u(1) = 0 \end{cases} \tag{2.4}$$

with f continuous.

The general form of this equation is

$$L(u) = N(u) \tag{2.5}$$

where L is the linear operator $L(u) = u''$ and N is the non-linear operator $N(u)(t) = f(t, u(t))$. The boundary conditions are encoded in the space where we define these functions. Take X to be the Banach space of C^2 functions $u : [0,1] \to \mathbb{R}$ such that $u(0) = u(1) = 0$ and Y the space of C^0 functions on $[0,1]$. The expression "non-resonant" refers to the fact that the operator $L : X \to Y$ is invertible. We call $\mathcal{K} : Y \to Y$ the composition of L^{-1} with the inclusion $X \hookrightarrow Y$. Using the Arzelá-Ascoli theorem one can see that $\mathcal{K}$ is a compact mapping (indeed, L^{-1} is continuous and the inclusion is compact). Equation (2.5) transforms to $\mathcal{F}(u) = u - \mathcal{K}(N(u)) = 0$ and now $\mathcal{F} : Y \to Y$ is a compact perturbation of the identity.

The same approach can be used to state very different problems like vector equations and elliptic partial differential equations, as point equations in function spaces.

2.2.2 Resonant problems

Consider the same equation as (2.4) but with periodic conditions

$$\begin{cases} u''(t) = f(t, u(t)) \\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases} \tag{2.6}$$

The general form of the equation is still the one in (2.5) but this time L is not invertible and the problem is called resonant. Take X to be the space of C^2 functions $u : [0,1] \to \mathbb{R}$ satisfying the periodic conditions in (2.6) and Y the space of C^0 functions on $[0,1]$. Now L is a Fredholm operator of index 0 and $dim(Ker(L)) = codim(R(L)) = 1$, namely $Ker(L)$ is the space of constant functions and $R(L)$ is the space of C^0 functions of zero average. We decompose as a direct sum $X = \overline{X} \oplus \widetilde{X}$ and $Y = \overline{Y} \oplus \widetilde{Y}$, where $\overline{Y} = \overline{X} = Ker(L), \widetilde{Y} = R(L)$ and $\widetilde{X}$ is the space of periodic C^2 functions of zero average. Write $z = (\overline{z}, \widetilde{z})$ for z in X or Y. Under this decomposition equation (2.5) looks like

$$(0, L(\widetilde{u})) = (\overline{N(u)}, \widetilde{N(u)}) \tag{2.7}$$

Now we may take $\mathcal{K}$ to be the partial inverse $L^{-1} : \widetilde{Y} \to \widetilde{X}$ composed with the inclusion $\widetilde{X} \hookrightarrow \widetilde{Y}$, then the equation takes the form

$$(0, \widetilde{u}) = (\overline{N(u)}, \mathcal{K}\widetilde{N(u)}) \tag{2.8}$$

Finally this is equivalent to $\mathcal{F}(u) := (\overline{u}, \widetilde{u}) - (\overline{u} - \overline{N(u)}, \mathcal{K}\widetilde{N(u)}) = 0$ which is clearly a compact perturbation of the identity.

A similar strategy will be employed in section 3 in a different context.

An other kind of resonant equation is $u'' + u = f(t, u(t))$ with periodic conditions. The linear operator $L(u) = u'' + u$ is still Fredholm of index 0 but this time its kernel is 2 dimensional, namely it is the linear space generated by the functions $\sin(2\pi t), \cos(2\pi t)$.

2.2.3 An application: the Landesman-Lazer theorem

Theorem 2.2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, bounded, with finite limits at $\pm\infty$ and $p : [0,1] \to \mathbb{R}$ a continuous periodic function. Consider the equation*

$$\begin{cases} u''(t) = g(u(t)) + p(t) \\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases} \tag{2.9}$$

and denote $\overline{p} = \int_0^1 p(t)dt$ the average of p. Then the condition

$$g(-\infty) < \overline{p} < g(+\infty) \tag{2.10}$$

is sufficient for the existence of a solution of (2.9).

Proof. Clearly we are in the conditions of the previous discussion on resonant problems. We take the deformation $\mathcal{F}_\lambda(u) = (\overline{u}, \widetilde{u}) - (\overline{u} - \overline{N(u)}, \lambda\mathcal{K}\widetilde{N(u)})$. For every $\lambda \in [0,1]$, a solution of $\mathcal{F}_\lambda(u) = 0$ is a solution of the differential equation $u'' = \lambda(g(u) + p)$. Notice that from the first coordinate of the equality (2.8), such a function must satisfy $\overline{g(u)} = \overline{p}$.

The boundedness of g ensures that the diameter of the image of any periodic solution is bounded by a constant $C > 0$ depending only on g and p. Condition (2.10) implies that for $|x|$ larger than a constant, say M, we must have $g(x) > \overline{p}$ or $g(x) < \overline{p}$ depending on the sign of x. This fact together with the previous observation implies that the solutions of the equation for $\lambda \in [0,1]$ are uniformly bounded by a constant $R > 0$ depending only on g and p. Now if $D = B(0,R) \subseteq Y$ we conclude that $\mathcal{F}_\lambda(\partial D) \not\ni 0$ for all $\lambda \in [0,1]$ so we may apply property V, and $deg(F, B, 0) = deg(F_0, D, 0)$.

Since $\mathcal{F}_0(u) = (\overline{N(u)}, \tilde{u})$ we may apply property IX where V is the space of constant functions which we now identify with $\mathbb{R}$. We see that $deg(F, B, 0) = deg(g - \overline{p}, [-R, R], 0) = deg(g, [-R, R], \overline{p})$ which is obviously 1, again because of condition (2.10).

Lastly by property VI we obtain a $u_0 \in D$ a zero of $\mathcal{F}$ which is a solution of the problem. $\square$

This theorem may be proved using only elementary analysis by considering the Poincaré map in the phace space. But this technique has an advantage. It uses almost no information about the equation itself or about the solutions (we don't speak here about regularity, stability or continuity respect of initial values). It only involves the properties of the functionals associated to the equation. Thus the proof has more range of applicability. In fact the original theorem of Landesman and Lazer is stated with much more generality, see [31].

2.3 Morse theory

We give here a brief exposition on Morse theory without any proof. For a complete and detailed introduction on the subject we refer to the books of Milnor [43] and Banyaga [10].

Morse theory studies the homotopy type of the level sets of a real smooth function f defined on a manifold M. Roughly speaking, the central result of the theory is that the topology of M is closely related to the structure of the critical points of f.

The typical example is the two-dimensional tours embedded vertically in $\mathbb{R}^3$ like in figure 2.1 and $f : M \to \mathbb{R}$ the height function $f(x_1, x_2, x_3) = x_3$.

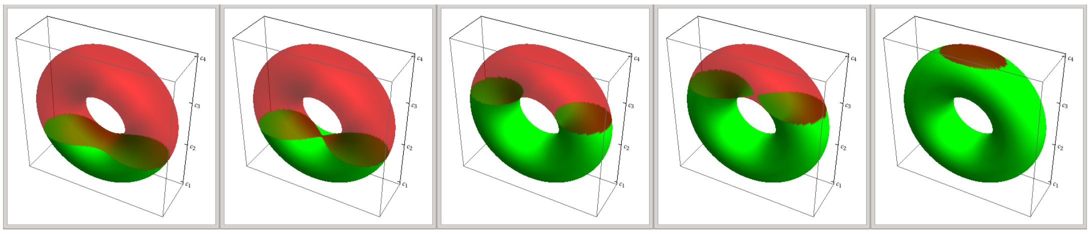


Figure 2.1: Level sets of the height function in the torus

Let us denote the level sets of f as $\{f \le \alpha\} := \{x \in M / f(x) \le \alpha\}$.

The critical points of f are the points where the tangent space of M is horizontal, thus orthogonal to the gradient of f in $\mathbb{R}^3$ which is allways $(0, 0, 1)$. We see that there are 4 critical values $c_1, \cdots, c_4$ corresponding to 4 critical points. A global maximum c_4, a global minimum c_1 and two saddle points corresponding to the values c_2, c_3.

Let us consider the level sets $\{f \le \alpha\}$ as α varies along $\mathbb{R}$. For $\alpha < c_1$ the level set is empty while for $\alpha = c_1$ we have only one point. As α varies from c_1 to c_2 the level sets are all homeomorphic to a 2-disk which is in turn homotopy equivalent to a point. When α crosses c_2, we see that two points of the boundary of our 2-disk touch each other, now changing the homotopy type of $\{f \le \alpha\}$ (figure 2.1, second diagram from the left). This change in the homotopy type is equivalent to attaching a 1-handle i.e., for small ε there is an embeded curve $e^1 \subset M$ such that $\{f \le c_2 + \varepsilon\}$ is homotopy equivalent to $\{f \le c_2 - \varepsilon\} \cup e^1$. The same happens when α crosses c_3 (figure 2.1, fourth diagram from the left). When α approaches c_4 from below the level set is like M with a hole at the top (diagram of the right of figure 2.1). Finally when α reaches c_4 this hole is filled. This filling operation is equivalent to attaching a 2-cell.

Unfortunately this nice behaviour of passing from one level set to an other by attaching cells is valid for almost, but not all critical values. We can see in figure 2.2 several examples of critical values for which the level sets have a very complicated relation.

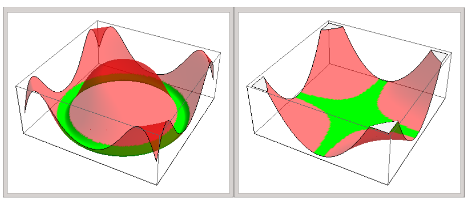


Figure 2.2: functions with degenerate critical points

We need then to restrict our attention to a special class of functions called the Morse functions. Observe for example in the function in the left of figure 2.2 that the level set passes from the empty set to a one dimensional circle directly. However if we tilt the graphic of the function a little to the right (see figure 2.3) then the circle now appears in two steps. Firstly a local minimum forces a disk-like shape to appear, then a saddle point makes a 1 dimensional connection between points on the boundary of the disk. Again we construct the homotopy type of the level set by attaching handles.

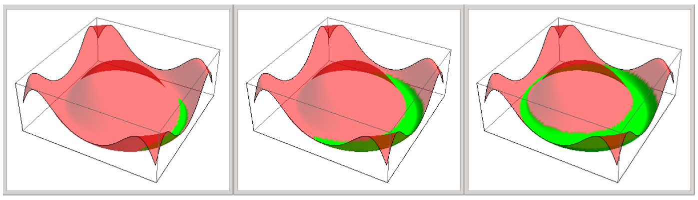


Figure 2.3: A suitable perturbation of a function with degenerate critical points

Let us formalize the above discusion. Assume from now on that M is a differentiable manifold of dimension m without boundary and that $f : M \to \mathbb{R}$ is a C^2 function.

Definition 1. 1. *A point $x \in M$ is a critical point of f if the differential vanishes at x, i.e $d_x f = 0$. In that case the second differential $d_x^2 f : T_xM \times T_xM \to \mathbb{R}$ is a well defined symmetric bilinear form.*

2. *For a critical point $x \in M$ we define its index λ_x as the negative signature of $d_x^2 f$ i.e. the largest dimension of a subspace of T_xM where $d_x^2 f$ is definite negative.*

3. *A critical point $x \in M$ is said to be non degenerate if $d_x^2 f$ is non degenerate.*

4. *A function is called a "Morse function" if all critical points of f are non degenerate.*

Morse theory relies in the following 4 basic lemmas:

Lemma 2 (Morse functions are generic)**.** *The set of Morse functions on M is a countable intersection of open and dense subspaces with the C^1 topology.*

In particular Morse functions are dense in the space $C^1(M)$.

Lemma 3 (Morse lemma)**.** *Let $f : M \to \mathbb{R}$ be C^2 and $p \in M$ a non-degenerate critical point of index $\lambda_p = k$. Then there exists a coordinate chart $(x_1, \dots, x_m) : U \subseteq M \to \mathbb{R}^m$ sending p to 0 and such that*

$$f = f(p) - (x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_m^2)$$

in U.

Lemma 4 (Deformation lemma)**.** *Assume that $f : M \to \mathbb{R}$ is a Morse function, all values in $[a, b]$ are regular and $f^{-1}([a, b])$ is compact.*

Then $\{f \leq a\}$ is a strong deformation retract of $\{f \leq b\}$. Furthermore, the deformation is an isotopy and the sets are thus diffeomorphic.

Lemma 5 (Handle attachment)**.** *Let $f : M \to \mathbb{R}$ be a C^2 smooth function. Suppose that $f^{-1}([a, b])$ is compact and inside $f^{-1}((a, b))$ there is exactly one critical point. Assume that this critical point is non-degenerate and of index k. Then $\{f \leq b\}$ has the homotopy type of $\{f \leq a\}$ with one k-cell attached. Actually, there exists a set $e^k \subset \{f \leq b\}$ diffeomorphic to the closed k-disk D^k such that $\{f \leq a\} \cup e^k$ is a deformation retract of $\{f \leq b\}$.*

Let us look for a moment at these lemmas. There is a list of m types of singularities that characterize the non-degenerate critical points, up to change of coordinates. Once we know this, lemma 5 has to be proved only in this finite number of cases. For example a critical point of index 1 of a function in $\mathbb{R}^2$ looks like $x^2 - y^2$. Its level sets are diagramed in figure 2.4.

Reasoning by induction we deduce from lemmas 4 and 5 that the whole manifold M is homotopy equivalent to a CW-complex with a cell of dimension $\lambda = 0, \cdots, m$ for each critical point of index λ.

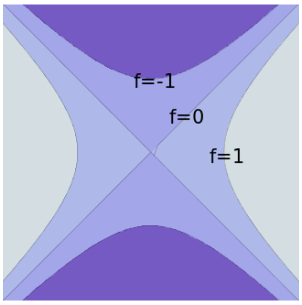


Figure 2.4: Level sets of the function $x^2 - y^2$ in the plane

Morse theory is usually employed in the following fashion:

A Morse function with few critical points gives a description of the manifold as a CW-complex with few cells which, in turn, provides a bound for the ranks of the cellular homology groups. On the other hand, a Morse function defined in a manifold with a very complicated homology must have lots of critical points. Both points of view will be used in this work, specifically in theorems 3.5.1, 3.5.2 and 6.6.3

Consider the cellular complex of the CW-structure induced by a Morse function $f : M \to \mathbb{R}$

$$H_m(X_m, X_{m-1}) \to \cdots \to H_2(X_2, X_1) \to H_1(X_1, X_0) \to H_0(X_0) \to 0$$

where X_k is the k-skeleton. We know from cellular homology theory that the k-th term of this complex is isomorphic to $\mathbb{Z}^{n_k}$ where n_k is the number of cells of the k-skeleton.

Since the alternated sum of the ranks of a complex equals the alternated sum of the ranks of the homology groups, we deduce the following beautiful fact about Morse functions:

Theorem 2.3.1. *Let M be a compact manifold and $f : M \to \mathbb{R}$ a Morse function. Let n_i be the number of critical points of index i, for $i = 0, \ldots, m$. Then $\sum_{i=0}^{m} (-1)^i n_i = \chi(M)$ is the Euler characteristic of M.*

Going back to deree theory, take an open bounded set D in $\mathbb{R}^n$ with smooth boundary. Consider the normal map $n : \partial D \to \mathbb{R}^n$. The degree $deg(n, D, 0)$ is defined as the degree of any continuous extension to $\bar{D}$. Since this degree is the same as for any vector field homotopic to n, it is easy to construct a non-positive Morse function $f : \bar{D} \to \mathbb{R}$ for which $deg(n, D, 0) = deg(\nabla f, D, 0)$ and $f \equiv 0$ at the boundary. Consider formula 2.3 for the degree. The sign of the determinant of a hessian at a critical point is exactly $(-1)^i$ where i is the Morse index. Then formula 2.3 applied to the gradient vector field ∇f equals the sum on the left hand side in the previous theorem and thus, the Euler characteristic. We deduce that

$$deg(n, D, 0) = \chi(D).$$

This is a formula we will be using repeatedly in the present work.

2.4 Some classical problems from differential equations

A number of differential equations motivated the development of global analysis techniques during the 20th century. These equations can be approached from several distinct angles and using very different theories covering analysis, geometry, chaotic dynamics, differential and algebraic topology, etc. There are many interesting open problems around them and they are a fruitful source of challenging problems.

2.4.1 The pendulum equation

The equation

$$x''(t) + cx'(t) + a\sin(x(t)) = h(t) \tag{2.11}$$

describes the evolution of a pendulum of length $a > 0$ with friction determined by the parameter c, under a time-dependent force h.

This apparently simple equation led to one century of research in analysis on Banach spaces, and constitutes a paradigmatic example of a system with chaotic motions. The survey paper [34] has a good recount of results and open problems around the pendulum equation.

Some of the first results on existence of periodic solutions of period 1 of equation (2.11) for the frictionless case ($c = 0$) where achieved using the direct method of calculus of variations. One can prove that the "action functional"

$$A(x) := \int_0^1 \frac{x'(t)^2}{2} + a\cos(x(t)) + x(t)h(t)dt$$

has a global minimum in an appropriate function space and that this minimum is a periodic solution (see [16]). The existence of a second solution was proved by Mawhin and Willem using a variant of the Ambrosetti-Rabinowitz mountain pass lemma in [48].

The problem of describing the set of functions h for which (2.11) has a periodic solution can be stated as describing the range $\mathcal{R}$ of the operator $x \mapsto x'' + cx' + a\sin(x)$. Over the space of periodic functions of class C^2 of a period 1.

Integrating the equation, we see that a necessary condition for the existence of a periodic solution is

$$|\bar{h}| = \left|\int_0^1 h(t)dt\right| \le a.$$

Using a Lyapunov-Schmidt argument, topological degree and the method of upper and lower solutions, it is shown in the papers [19, 48, 33] that for $\tilde{h}$ a function of zero average, the set

$$\{\bar{h} \in \mathbb{R}/\bar{h} + \tilde{h} \in \mathcal{R}\} = I_{\tilde{h}}$$

is a compact interval whose endpoints depend continuously on $\tilde{h} \in \widetilde{Y}$ (the Banach space defined in 2.2.2). In fact there exist at least two different periodic solutions when $\bar{h} \in I^{\circ}_{\tilde{h}}$.

An interesting open question around the pendulum equation is whether the interval $I_{\tilde{h}}$ may or may not be a single point. If this happens, the equation is called degenerate and a very special behaviour of the solution set is to be expected (see [55]).

2.4.2 The forced Kepler problem

The equation

$$x''(t) + \frac{x(t)}{|x(t)|^3} = h(t) \tag{2.12}$$

describes the motion of a particle $x : \mathbb{R} \to \mathbb{R}^3$ under a singular central force, and a time-dependent forcing h. This equation and some variations of it, is studied in chapters 4, 5 and 6 when the forcing term h is significant.

The two-body problem, where two masses orbit around each other under gravitational force, is solved by reducing it to equation (2.12) with $h = 0$. Here the center of mass is translated to the origin.

The term $\frac{x(t)}{|x(t)|^3}$ may be replaced by a more general singular gradient vector field $\nabla V(x,t)$ to describe other kinds of forces.

The most commonly used tools for studying this problem are variational techniques. A natural difficulty arises in the fact that a critical point of the functional may in fact be a collision orbit, meaning an orbit which crosses the singularity of the potential V at some time. This leads to the "Gordon strong force" condition which requires the potential V to grow at a certain rate at the singularity to guarantee that the "action functional" blows up at collision orbits (see [22]). Unfortunately this condition excludes the case $V(x) = \frac{1}{|x|}$, which is the potential of the Kepler problem Even in the scalar case, a weak singularity may lead to collisions.

2.4.3 The N-body problem

The N-body problem is the system of equations

$$x_k''(t) = -\sum_{i \neq k} \frac{x_k(t) - x_i(t)}{|x_k(t) - x_i(t)|^3} \text{ for } k = 1, 2, \ldots N \tag{2.13}$$

that describes the motion of N particles at positions $x_1(t), \ldots, x_N(t)$ moving under the forces of mutual gravitational attraction.

The existence, stability and bifurcation of periodic solutions has been the center of attention of many mathematicians since the works of Poincaré, who studied the N body problem, specially the case $N = 3$ in his work "Les méthodes nouvelles de la mécanique céleste" [56] where he develops a number of techniques still used today.

A particular simplification of this equation is called the "restricted N-body problem" in which the trajectories of $N-1$ bodies $x_1(t), \cdots, x_{N-1}(t)$ are known and the mass of the

n-th body is negligible with respect to the other masses. The equations take the simpler form

$$x_n''(t) = -\sum_{i=1}^{N-1} \frac{x_n(t) - x_i(t)}{|x_n(t) - x_i(t)|^3}. \tag{2.14}$$

Here $x_1(t), \cdots, x_{N-1}(t)$ are given functions and $x_n(t)$ is the only unknown.

Many different methods have been used to establish the existence of periodic solutions for the N-body problem, specially for the restricted 3-body problem, for example: the averaging method of Moser [45], special fixed-point theorems (see [49]), and recurrence theorems.

Also the use of critical point theory may provide multiple solutions to the N-body problem but sometimes the "Gordon strong force" condition must be present. The work of Faddel and Husseini [18] is a nice example of abstract theory applied to this concrete example.

Some results of section 6 will be applied to prove existence of periodic solutions to the restricted N-body problem.

Now we turn to the actual results of this thesis.

Chapter 3

A Hartman-Nagumo type condition for a class of contractible domains

3.1 Introduction

In 1960, Hartman [24] showed that the second order system in $\mathbb{R}^n$ for a vector function $x : I = [0, 1] \to \mathbb{R}^n$ satisfying

$$\begin{cases} x'' = f(t, x, x') \\ \quad x(0) = x_0 \\ \quad x(1) = x_1 \end{cases} \tag{3.1}$$

with $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ continuous, has at least one solution when f satisfies the following conditions:

$$\begin{gathered} \langle f(t, x, y), x\rangle + |y|^2 > 0 \text{ for all } (t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n, \\ \text{with } |x| = R,\ \langle x, y\rangle = 0 \end{gathered} \tag{3.2}$$

for some $R \geq |x_0|, |x_1|$.

$$|f(t, x, y)| \leq \phi(|y|) \text{ where } \phi : [0, \infty) \to \mathbb{R}^+ \text{ and } \int_0^\infty \frac{x}{\phi(x)} dx = \infty, \tag{3.3}$$

$$|f(t, x, y)| \leq \alpha(\langle f(t, x, y), x\rangle + |y|^2) + C, \text{ where } \alpha, C > 0. \tag{3.4}$$

A stronger version of (3.2) is easier to understand:

$$\begin{gathered} \langle f(t, x, y), x\rangle > 0 \text{ for all } (t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n, \\ \text{with } |x| = R,\ \langle x, y\rangle = 0. \end{gathered} \tag{3.5}$$

Indeed, this condition means that whenever $x \in \partial B(0, R)$, the vector field f points outwards the ball $B(0, R)$. Condition (3.2) allows f to point inwards, but not too much if the velocity is small.

The proof basically uses the Schauder fixed point theorem. It can also be proved using Leray-Schauder continuation theorem [35] in the open set of curves lying inside $B(0, R)$. The key argument is that a solution u cannot be tangent to the ball of radius R from inside because (from (3.2)) the second derivative of $|u|^2$ is positive when $|u|$ is close to R. Conditions (3.3) and (3.4) guarantee that the C^1 norm of the solutions remains bounded during the continuation.

Hartman's result has been extended in several ways, for different boundary conditions (see [28] for a first result of this type under periodic conditions) and for more general second order operators (see e.g. [39], [46] and the references therein). However, less generalizations are known if one replaces the ball $B(0, R)$ by an arbitrary domain D.

In view of the geometrical interpretation of (3.5), it is not difficult to prove existence of solutions using (3.3), (3.4) and (3.5) when D is convex. Condition (3.5) takes, in consequence, the following form:

$$\langle f(t,x,y), n_x\rangle > 0 \text{ for all } (t,x,y) \in I \times \mathbb{R}^n \times \mathbb{R}^n,$$
$$\text{with } x \in \partial D,\ \langle n_x, y\rangle = 0$$

where n_x is an outer normal of ∂D at the point x. For periodic conditions, this result has been obtained by Bebernes and Schmitt in [12] assuming, instead of (3.4) and (3.5), that f has some specific subquadratic growth on y. In this work, we extend the result for a more general (not necessarily convex) domain $D \subset \mathbb{R}^n$.

Some results in this direction have been given in [37] and [20] (see also [21] and the survey [40]), where the concept of *curvature bound set* is introduced in order to ensure that solutions starting inside an appropriate domain remain there all the time, thus allowing the use of the continuation method. Roughly speaking, at any point of the boundary of such a set D there exists a smooth surface that is tangent from outside and measures the curvature of the solutions touching ∂D from inside.

In this work we shall show that, in some sense, if D has C^2 boundary and the role of the surfaces in the previous definition is assumed by ∂D itself, then a precise geometric condition involving its second fundamental appears. In this context, our version of (3.2) reads as follows:

$$\langle f(t,x,y), n_x\rangle > \mathrm{II}_x(y) \text{ for all } (t,x,y) \in I \times T\partial D \tag{3.6}$$

where $T\partial D$ is the tangent vector bundle identified, as usual, with a subset of $\mathbb{R}^n \times \mathbb{R}^n$, $\mathrm{II}_x(y)$ is the second fundamental form of the hypersurface and n_x is the outer-pointing normal unit vector field. This condition requires f to point outside D as much as ∂D is "bended outside" in the direction of the velocity. In particular, when $D = B(0, R)$ its curvature is constantly $\frac{1}{R^{n-1}}$; moreover, $\mathrm{II}_x(y) = -\frac{1}{R}|y|^2$ and $n_x = \frac{x}{R}$, so our new Hartman condition takes the form of the original one.

The chapter is organized as follows. In the next section, we recall the basic facts about Gaussian curvature and state some preliminary results concerning the generalized Hartman condition (3.6). In section 3.3, we introduce some growth conditions that extend (3.3) and (3.4) on the one hand, and the growth condition in [12] (used also in [21]), on the other hand. In section 3.4 we establish and prove our main results on existence of solutions under Dirichlet and periodic conditions using the Leray-Schauder continuation method. Finally, in section 3.5 we prove that the growth conditions force the domain D to be contractible, thus restricting the class of examples to which the main theorems are applicable.

3.2 Curvature

Let D be an open subset of $\mathbb{R}^n$ such that $M := \partial D$ is a C^2 oriented manifold and let n_x be the outer unit normal vector at $x \in M$. The application $x \mapsto n_x$ defines a smooth function $n : M \to S^{n-1}$ and its differential defines a linear map $T_xM \to T_{n_x}S^{n-1}$. Since both linear spaces are orthogonal to n_x, they may be identified and we obtain a linear endomorphism known as the Gauss map $g_x : T_xM \to T_xM$. This map is easily seen to be self-adjoint with respect to the inner product inherited from $\mathbb{R}^n$. The associated quadratic form $\mathrm{II}_x(v) = -\langle g_x(v), v\rangle$ is called the second fundamental form of the hypersurface. It is important to remark that II_x is independent -up to a sign- of the orientation given by n.

The next lemma is essentially proved in do Carmo's book [13]:

Lemma 1. *Let $\alpha : \mathbb{R} \to \overline{D}$ be a C^2 curve such that $\alpha(0) = p \in M$. Let n_p be the outer unit normal vector at p. Then $\langle \alpha''(0), n_p\rangle \leq \mathrm{II}_p(\alpha'(0))$*

Proof. As a direct application of the inverse function theorem, we obtain near p a coordinate system of the form $(m, \lambda) \in M \times \mathbb{R}$ given by $x = m(x) + \lambda(x)n_{m(x)}$. The curve α may be written as $\alpha(t) = \gamma(t) + \lambda(t)n_{\gamma(t)}$ for some C^2 functions γ, λ. Let $n(t) = n_{\gamma(t)}$ so $n'(t) = g_{\gamma(t)}(\gamma'(t))$ and compute

$$\langle n(t), \alpha'(t)\rangle = \langle n(t), \gamma'(t)\rangle + \lambda'(t) + \lambda(t)\langle n(t), n'(t)\rangle = \lambda'(t),$$

since $\gamma'(t)$ and $n'(t)$ are orthogonal to $n(t)$.

Observe that $\lambda'(0) = \langle n_p, \alpha'(0)\rangle = 0$. Moreover, as the image of α is contained in D, its λ-coordinate is always non-positive. But $\lambda(0) = 0$, and hence $\lambda''(0) \leq 0$. We deduce that $\frac{d}{dt}\langle n(t), \alpha'(t)\rangle |_{t=0} \leq 0$.

Now $\langle n(t), \alpha''(t)\rangle = \langle n(t), \alpha'(t)\rangle' - \langle n'(t), \alpha'(t)\rangle$; thus,

$$\langle n(0), \alpha''(0)\rangle \leq -\langle g_{\alpha(0)}(\alpha'(0)), \alpha'(0)\rangle = \mathrm{II}_p(\alpha'(0)).$$

□

Corollary 3.2.1. *Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be such that (3.6) holds.*

Then there are no solutions of the differential equation $x'' = f(t, x, x')$ inside $\overline{D}$ touching the boundary.

The following definition will be useful.

Definition 2. *We define $curv(D, x) = \lambda_1$, where $\lambda_1 \leq \ldots \leq \lambda_{n-1}$ are the eigenvalues of the self-adjoint operator g_x.*

Remark 3.

1. *It is clear from the definition that $-|y|^2 curv(D, x) \geq \mathrm{II}_x(y)$, and equality holds when y is an eigenvector of g_x associated with the eigenvalue λ_1.*

2. *If x is a point of convexity of the surface, then $curv(D, x) \geq 0$.*

3. *It may be deduced, as in the proof of Lemma 1, that:*

   - *If $curv(D, x) < 0$ then $-curv(D, x)^{-1}$ is the radius of the largest ball which is tangent from outside to $\overline{D}$ at the point x.*
   - *If $curv(D, x) > 0$ then $curv(D, x)^{-1}$ is the radius of the smallest ball B such that D is tangent from inside to ∂B at x.*

3.3 Growth conditions

In order to apply the Leray-Schauder method [35], it is necessary to find *a priori* bounds of the solutions during the continuation. As the nonlinear term depends on u and u', we will need estimates for the C^1 norm.

To this end, following [24], we shall impose some growth conditions on f. These conditions must be compatible with the deformations used in the main theorems.

We shall make use of the next two lemmas, proved in [24] (Lemmas 2 and 3 respectively), conveniently adapted to our situation. Without loss of generality, we may assume that $D = \eta^{-1}(-\infty, 0)$ where $\eta : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function and 0 is a regular value of η, then $M = \partial D = \eta^{-1}(0)$. We can use the function η itself to replace r in Lemma 3 of [24].

Lemma 1. *Let R, C be non-negative constants and $\phi : [0, \infty) \to (0, \infty)$ a continuous function such that*

$$\int_0^\infty \frac{s}{\phi(s)} ds = \infty. \tag{3.7}$$

Then there exists a constant $N = N(R, C, \|\eta\|_\infty, \phi)$ such that if $x \in C^2(I, \mathbb{R}^n)$ satisfies

$$|x| \leq R,\ |x''| \leq \eta(x(t))'' + C \quad \text{and} \quad |x''| \leq \phi(|x'|)$$

then

$$|x'| \leq N.$$

Now $\eta(x(t))''$ may be easily calculated as

$$d^2_{x(t)}\eta(x'(t), x'(t)) + d_{x(t)}\eta(x''(t)),$$

where $d^2_x\eta$ stands for the quadratic form induced by the Hessian. The condition thus obtained is

$$\begin{gathered} |f(t, x, y)| \leq d^2_x\eta(y, y) + d_x\eta(f(t, x, y)) + C \\ |f(t, x, y)| \leq \phi(|y|). \end{gathered} \tag{3.8}$$

This condition obviously generalizes the original assumptions (3.3) and (3.4) given in [24], setting $\eta(x) = |x|^2 - R^2$ and $D = B(0, R)$.

Remark 2. 1. *The function η might also depend on the time t, although the expression for $\eta(x(t))''$ in this case would become more complicated.*

2. *As $d^2_x\eta(y, y) + d_x\eta(f(t, x, y)) = (-\mathrm{II}_x(y) + \langle f(t, x, y), n_x\rangle).|\nabla_x\eta|$ for $x \in \partial D$, the fact that the expression $\langle f(t, x, y), x\rangle + |y|^2$ appears both in (3.2) and (3.4) is not a coincidence.*

From the discussion in [24, Corollary 1], we get a simpler (but somewhat more restrictive) growth condition on f. Let us firstly recall from [24] the following

Lemma 3. *Let R, γ, C be non-negative constants where $R\gamma < 1$. Then there exist $N = N(R, \gamma, C)$ such that if $x \in C^2(I, \mathbb{R}^n)$ satisfies*

$$|x| \leq R \text{ and } |x''| \leq \gamma|x'|^2 + C$$

then

$$|x'| \leq N.$$

With this last result in mind, we may impose, instead of (3.8), the condition:

$$|f(t, x, y)| \leq \gamma |y|^2 + C \tag{3.9}$$

for every $x \in \overline{D}$, where C and γ are constants with $\gamma R_D < 1$. Here, R_D is the radius of D, defined as the radius of the smallest ball containing D (notice that $diam(D) \leq 2R_D$). In particular, (3.9) generalizes the growth condition imposed in [12].

Unfortunately our version of the Hartman condition combined with (3.9) requires that $R_D^{-1}|y|^2 + C > \mathrm{II}_x(y)$ for all y, so the theorem is not applicable for arbitrary domains. For instance, taking y as an eigenvector of g_x such that $|y| \gg 0$, it is easy to see that (3.9) together with (3.6) implies that $R_D curv(D, x) > -1$. Thus, our results cannot be applied if for example $D = B(0, R) \backslash B(0, r)$ for some $r < R$.

In section 3.5 we shall prove that, furthermore, the domain D must be contractible. The same happens with condition (3.8), independently of (3.6).

Remark 4. *It is worth observing that as far as an a priori bound N is obtained for the derivative of the solutions, condition (3.6) can be relaxed to consider only those points $(x, y) \in T\partial D$ such that $|y| \leq N$. This shows that, in fact, condition (3.9) is not necessarily incompatible with (3.6) when the domain is non-contractible. We shall not pursue this direction here;*

The following result is a refinement of Lemma 3 that shall be needed for the proof of Theorem 3.4.1.

Lemma 5. *Let R, γ, C be non-negative constants where $R\gamma < 1$, and $0 < T \leq 1$. Then there exist $N = N(R, \gamma, C)$ (independent of T) such that if $x \in C^2([0, T], \mathbb{R}^n)$ satisfies*

$$|x| \leq R, \qquad |x''| \leq \gamma |x'|^2 + C$$

and

$$x(0) = x(T) = x_0$$

then

$$|x'| \leq N.$$

Proof. Following the remarks of [24, Lemma 3] and the proof of [24, Lemma 2], let $\rho : [0, T] \to \mathbb{R}$ be defined by

$$\rho(t) = \alpha |x|^2 + \frac{K}{2} t^2$$

where

$$\alpha = \frac{\gamma}{2(1 - \gamma R)}, \quad K = \frac{C}{1 - \gamma R}.$$

Then $\|\rho\|_\infty \leq M_1(C, \gamma, R)$, and $|x''| \leq \rho$, since

$$\rho''(t) = 2\alpha(\langle x'', x \rangle + |x'|^2) + K \geq 2\alpha(|x'|^2 - R(\gamma |x'|^2 + C)) + K$$

$$= |x'|^2 2\alpha(1 - \gamma R) - 2\alpha RC + K \geq \frac{|x''| - C}{\gamma} 2\alpha(1 - \gamma R) - 2\alpha RC + K$$

$$= |x''| - C - 2\alpha RC + K = |x''|.$$

From the discussion in [24] sections 3 to 5, we obtain the formula

$$|\Phi(|x'(t)|) - \Phi(|x'(T/2)|)| \leq \int_t^{T/2} M_2(C,\gamma,R)/T \pm \rho'(s)ds$$

for all $t \in [0,T]$ (the $\pm$ sign depends on whether $t < T/2$ or $t > T/2$). Hence,

$$|\Phi(|x'(t)|) - \Phi(|x'(T/2)|)| \leq M_2(C,\gamma,R) + 2\|\rho\|_\infty. \tag{3.10}$$

Unless x is constant, as $x(0) = x(T)$ there must exist a tangent ball $B \supseteq Im(x)$ of radius R to x, at a point $x(t_0) \neq x_0$. Now, let n be the outer unit normal vector of B at $x(t_0)$. Then

$$\gamma|x'(t_0)|^2 + C \geq |x''(t_0)| \geq -\langle x''(t_0), n\rangle \geq \frac{1}{R}|x'(t_0)|^2$$

$$|x'(t_0)| \leq \sqrt{\frac{C}{1/R - \gamma}}.$$

Using inequality (3.10) twice, for $t = t_0$ and for arbitrary t, we get

$$|\Phi(|x'(t_0)|) - \Phi(|x'(t)|)| \leq 2M_2(C,\gamma,R) + 4\|\rho\|_\infty$$

and hence

$$|\Phi(|x'(t)|)| \leq |\Phi(|x'(t_0)|)| + 2M_2(C,\gamma,R) + 4\|\rho\|_\infty \leq M_3(C,\gamma,R).$$

□

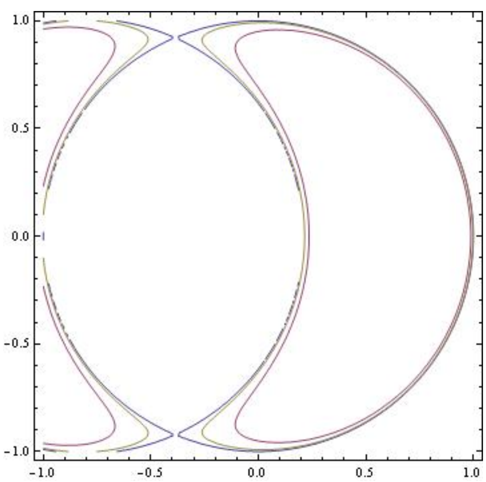
Next we show an example where the main theorems stated in section 3.4 are applicable for a function on a non-convex set in the plane $D \subset \mathbb{R}^2$. In this situation the boundary may be described locally by C^2 curves $b : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$. The second fundamental form is simply $\mathrm{II}_x(y) = -k(x)|y|^2$ where k is the curvature of b. When b is parametrized by arc-length, $|k|$ is the norm of the vector b''.

Example 6.

Let

$$\eta(x) = (|x|^2 - 1)(|x + (\delta, 0)|^2 - \frac{1}{\gamma^2}),$$

where $\gamma \in (0,1), \delta \in (1/\gamma - 1, 1/\gamma + 1)$ are fixed constants. Take D_ϵ to be the connected component of $\eta^{-1}(-\infty, -\epsilon)$ on the right side, where $\epsilon > 0$ is small enough.



Observe that $D_0 = B(0,1) \setminus \overline{B((-\delta,0),\frac{1}{\gamma}))}$. The singular points of η are the intersections of $\partial B(0,1)$ with $\partial B((-\delta,0),\frac{1}{\gamma}))$ and the centers of these circles.

Define

$$f(t,x,y) = |y|^2 \frac{\nabla\eta}{|\nabla\eta|} + p(t,x,y)$$

where p is bounded.

We have always $\overline{D_\epsilon} \subseteq D_0 \subseteq B(0,1)$, so $R_{D_\epsilon} < 1$ and (3.9) is satisfied since $\gamma < 1$. Next, we need to check condition (3.6).

Let $k(x)$ be the curvature of the curve $\partial D_{\eta(x)}$ in the point x. It is obvious that k is a continuous function of x for the regular points of η. From the choice of f, we only need to show that $k(x) > -1$. This is true for all regular points of ∂D_0 because of Remark 3.3. The differential $d^2\eta$ can be explicitly calculated to show that $k(x) > 0$ for points of ∂D_ϵ near the singular points of D_0.

Condition (3.6) takes the form

$$\langle n_x, p(t,x,y)\rangle > -|y|^2(k(x)+1)$$

Moreover, $k+1$ is strictly positive, so if for example $\langle n_x, p(t,x,y)\rangle \geq 0$, then the periodic and Dirichlet problems admit at least one solution.

Now we give an example of non-existence of solutions, showing that the growth conditions cannot be easily dropped.

Let $x_0, x_1 \in \mathbb{R}^n$ and $\epsilon, r > 0$ such that $x_1 \notin B(x_0, r+2\epsilon)$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a C^∞, bounded function such that

$$\eta(x) = \begin{cases} 1 & \text{if } ||x|-r| \leq \epsilon \\ 0 & \text{if } ||x|-r| \geq 2\epsilon. \end{cases} \tag{3.11}$$

Let $f_0(t,x,y) = -Ky(|y|^2+1)\eta(x)$ where $K > 2\frac{\pi}{\epsilon}$. Finally, let $g(t,x,y)$ be any function equal to 0 for $x \in B(x_0, r+2\epsilon)$.

Claim 7. *There is no solution of the problem*

$$\begin{cases} x'' = (f_0+g)(t,x,x') \\ x(0) = x_0 \\ x(1) = x_1 \end{cases} \tag{3.12}$$

Proof. Let x be a solution and $t_0 \in I$ such that $|x(t_0) - x_0| = r$. For all t such that $|x(t) - x(t_0)| \leq \epsilon$ we have

$$x'' = -Kx'(|x'|^2+1).$$

Let $w(t) = |x'|$. We compute

$$2ww' = (w^2)' = 2\langle x', x''\rangle = -2K|x'|^2(|x'|^2+1) = -2Kw^2(w^2+1)$$

$$w' = -w(w^2+1).$$

The unique solution of this differential equation is

$$w(t)^2 = ((1+w^{-2}(t_0))e^{2Kt} - 1)^{-1}$$

which satisfies

$$\int_{t_0}^{t} w(s)ds = \frac{1}{K}\ ArcTan\left(\sqrt{(1+w^{-2}(t_0))e^{2Kt}-1}\right)\Big|_{t_0}^{t} \leq \frac{\pi}{K}.$$

Since $K > 2\frac{\pi}{\epsilon}$ we have

$$|x(t)-x(t_0)| = \left|\int_{t_0}^{t} w(s)ds\right| \leq \frac{\epsilon}{2}.$$

Then we know that x lies in $B(x(t_0), \frac{\epsilon}{2})$ whenever x lies in $B(x(t_0), \epsilon)$. This means that x lies in $B(x(t_0), \frac{\epsilon}{2})$ for all $t \geq t_0$, a contradiction. □

For any domain D and points $x_0, x_1 \in D$ we may choose ϵ, r, g for which the counter-example applies and $f_0 + g$ satisfies condition (3.6). For example, for r, ϵ small and $g = 0$ we have a counter-example to the original Hartman condition when $D = B(0, R)$.

Other counter-examples for scalar equations (also with cubic growth in $|x'|$) can be found in [27].

A more delicate question is whether if condition $\gamma R_D < 1$ in (3.9) may be relaxed or dropped. The negative answer is illustrated in the following counterexample:

Let us use complex notation for $\mathbb{R}^2$. Define

$$f(t, x, y) = -\frac{x}{|x|^2}|y|^2 + 2ix$$

Claim 8. *The system $x'' = f(t, x, x')$ has no classical periodic solutions.*

Proof. Let x be a periodic solution. Notice that $\frac{d^2}{dt^2}|x|^2 = 2(\langle f(t,x,x'), x\rangle + |x'|^2) = 0$ so by periodicity, $|x| = r$ is constant. Writing $x(t) = re^{i\theta(t)}$ it follows that $\theta'' \equiv 2$, and hence $x'(t)$ cannot be equal to $x'(0)$ for any $t > 0$, a contradiction.

□

Also it is seen that $\langle f(t,x,y), \frac{x}{|x|}\rangle = -\frac{|y|^2}{|x|}$, so if

$$D = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\} \quad \text{for} \quad r_2 > r_1 > 0,$$

then (3.6) is satisfied for suitable perturbations of f. Condition (3.9) is not satisfied because of the requirement $\gamma R_D < 1$.

Now, the general theory of compact perturbations of the identity in Banach spaces implies that there are no periodic solutions in $\overline{D}$ of the equations $x'' = g(t, x, x')$ for g in a neighborhood of f. Indeed one may take $g : \mathbb{R}^n \to \mathbb{R}^n$ continuous such that

$$\langle g(u), u\rangle > 0 \qquad \text{if } |u| = r_2,$$

$$\langle g(u), u\rangle < 0 \qquad \text{if } |u| = r_1,$$

then (3.6) holds for $f_n := f + \frac{1}{n}g$. If x_n is a periodic solution of the problem $x'' = f_n(t, x, x')$ such that $x_n(t) \in D$ for all t, then $x_n = \overline{x}_n + K(f_n(x_n))$ where $\overline{x}$ denotes the average of the function x and K is the right inverse of the operator $Lx := x''$ satisfying $\overline{K\varphi} = 0$ for $\varphi \in C([0,1])$ with $\overline{\varphi} = 0$. As $\{x_n\}$ is bounded, passing to a subsequence we may assume that $\overline{x}_n + K(f_n(x_n))$ converges to some function x, so $x_n \to x$, and x is a solution of the problem, a contradiction.

We conclude that condition (3.9) is sharp.

3.4 Main theorems

3.4.1 Dirichlet conditions

Throughout this section, we shall use the following notations:

$T = C(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, equipped with the usual compact open topology.

$C^i = C^i(I, \mathbb{R}^n)$ as Banach spaces with the standard norms.

$X_0 = \{x \in X / x(0) = x(1) = 0\}$ for $X = C, C^1, \ldots$

It is well-known that the map $L : C_0^2 \to C, Lu = u''$ is a Banach space isomorphism; let $K : C \to C_0^2$ be its inverse and $\iota : C^2 \to C^1$ the compact inclusion.

Moreover, let $\mathcal{N} : T \times C^1 \to C$ be the nonlinear operator

$$\mathcal{N}(f, x)(t) = f(t, x(t), x'(t)),$$

which is clearly continuous. Finally, let $B : \mathbb{R}^n \times \mathbb{R}^n \to C^1$ be the segment $B(x, y)(t) = ty + (1 - t)x$ and $F : T \times C^1 \times \mathbb{R}^n \times \mathbb{R}^n \to C^1$ the operator defined by $F(f, u, x, y) = u - \iota K \mathcal{N}(f, u) - B(x, y)$.

Lemma 1. *$F(f, u, x, y) = 0$ if and only if u is a solution of the nonlinear problem*

$$\begin{cases} u'' = f(t, u, u') \\ u(0) = x \\ u(1) = y \end{cases} \tag{3.13}$$

Our main theorem for Dirichlet conditions reads as follows:

Theorem 3.4.1. *Let D be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $x_0, x_1 \in D$ and $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that (3.6) and (3.9) hold.*

Then there exists a solution of $x'' = f(t, x, x')$ satisfying the Dirichlet conditions $x(i) = x_i$.

Proof. Let N be the bound provided by (3.9) (see Lemma 5 in previous section) and define

$$\mathcal{D} = \{x \in C^1 / Im(x) \subset D \text{ and } \|x'\|_\infty < N + 1\}.$$

We shall construct a homotopy starting from the functional $u \mapsto F(f, u, x_0, x_1)$, so we may calculate its degree in the open set $\mathcal{D}$, provided that it does not vanish on $\partial\mathcal{D}$ along the homotopy.

The homotopy shall be constructed in three steps:

Step 1

Let $F^1_\lambda(u) := F(f, u, x_0, \gamma(\lambda))$ where $\gamma : [0, 1] \to D$ is a path joining x_0, x_1. It is obvious from Corollary 3.2.1 and from the choice of N that F^1_λ has no zeros on the boundary of $\mathcal{D}$. The problem is now homotopic to the same problem with boundary conditions $x(0) = x(1) = x_0$.

Step 2

Let $\lambda_0 > 0$ be such that $\overline{B(x_0, \lambda_0 N)} \subset D$ and set

$$f^2_\lambda(t, x, y) := \lambda^2 f(t, x, \lambda^{-1} y).$$

The function

$$\begin{array}{rcl} \mathbb{R} & \to & T \\ \lambda & \mapsto & f_\lambda^2 \end{array} \tag{3.14}$$

is continuous for $\lambda \in [\lambda_0, 1]$.

Next, define

$$F_\lambda^2(u) := F(f_\lambda^2, u, x_0, x_0)$$

so $F_1^2 = F_0^1$.

Remark 2. *The function f_λ^2 satisfies (3.6), because $\mathbb{I}_x(y)$ is quadratic in y. This means (from Corollary 3.2.1) that there are no solutions tangent to ∂D from inside. Also, as $x_0 \in D$ we know that solutions do not touch the boundary at $t = 0, 1$.*

Now we have to estimate the derivative of the solutions x_λ of the equation $F_\lambda^2(x) = 0$ satisfying $x_\lambda(t) \in \overline{D}$.

Let $y(t) = x_\lambda(\lambda^{-1}t)$ for $t \in [0, \lambda]$, then

$$\begin{aligned} y''(t) = \lambda^{-2}x_\lambda''(\lambda^{-1}t) &= \lambda^{-2}f_\lambda^2(\lambda^{-1}t, x_\lambda(\lambda^{-1}t), x_\lambda'(\lambda^{-1}t)) \\ &= f(\lambda^{-1}t, x_\lambda(\lambda^{-1}t), \lambda^{-1}x_\lambda'(\lambda^{-1}t)) \\ &= f(\lambda^{-1}t, y(t), y'(t)). \end{aligned}$$

So y is a solution of $y''(t) = f(\lambda^{-1}t, y(t), y'(t))$ for $t \in [0, \lambda]$ and (3.9) applies (here we use the fact that N in Lemma 5 does not depend on the interval of definition of y). Then we get

$$N > |y'(t)| = |\lambda^{-1}x(\lambda^{-1}t)| \text{ for } t \in [0, \lambda],$$

which implies

$$\|x_\lambda'\|_\infty < \lambda N. \tag{3.15}$$

Step 3

For $\lambda \in [0, \lambda_0]$, let us define

$$f_\lambda^3(t, x, y) := \lambda^2 f(t, x, y\lambda_0^{-1})$$

$$F_\lambda^3(u) := F(f_\lambda^3, u, x_0, x_0).$$

Let $x_\lambda \in \overline{\mathcal{D}}$ be a solution of $F_\lambda^3(x) = 0$. If $\lambda > 0$ it is clear that $|f_\lambda^3| \leq |f_{\lambda_0}^2|$ so $|x_\lambda'| \leq \lambda_0 N$ still holds. Hence $x_\lambda \in B(x_0, \lambda_0 N) \subset\subset D$ so $x_\lambda \notin \partial\mathcal{D}$.

Finally, $F_0^3(u) = F(0, u, x_0, x_0) = u - x_0$ so $deg(F_0^3, \mathcal{D}, 0) = 1$ and the proof is complete.

$\square$

3.4.2 Periodic conditions

The periodic case of a second order ecuation is in general much harder than the two point boundary value problem. The difference was explained in 2.2.2 in the introduction, roughly speaking is that now the operator $Lx = x''$ has non-trivial kernel. In order to state our existence result for periodic conditions, now we shall consider:

$T = \{f \in C(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)/f(0, x, y) = f(1, x, y)\}$.

$C^i_{per} = \{x \in C^i/x^{(j)}(0) = x^{(j)}(1), j < i\}$.

$\tilde{C^i} = \{x \in C^i/\overline{x} = 0\}$.

The map $L : C^2_{per} \cap \tilde{C} \to \tilde{C}$ given by $Lx = x''$ is an isomorphism, denote by $K : \tilde{C} \to C^2_{per} \cap \tilde{C}$ its inverse and $\iota : C^2 \to C^1$ the compact inclusion.

Let $P : C^1 \to \tilde{C}^1$ be the projection associated with the decomposition $C^1 = \mathbb{R}^n \oplus \tilde{C}^1$ and let $\mathcal{N} : T \times C^1 \to C$ as before. Following [35], define $F : T \times C^1 \to C^1$ as the operator given by

$$F(f, u) = u - \overline{u} - \overline{\mathcal{N}(f, u)} + \iota KP\mathcal{N}u$$

and $G : T \times C^1 \times \mathbb{R} \to C^1$ by

$$G(f, u, \mu) = u - \overline{u} - \overline{\mathcal{N}(f, u)} + \mu\iota KP\mathcal{N}u.$$

The following result is easily verified as in [35]:

Lemma 3. $F(f, u) = 0$ *if and only if* u *is a solution of the nonlinear problem*

$$\begin{cases} u'' = f(t, u, u') \\ u(0) = u(1) \\ u'(0) = u'(1). \end{cases} \tag{3.16}$$

Hence we may establish our main result for periodic conditions:

Theorem 3.4.2. *Let* $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be such that (3.6) holds, and either (3.8) or (3.9) is satisfied.*

Then there exists a periodic solution of $x'' = f(t, x, x')$.

Proof. The proof follows the same outline of Theorem 3.4.1, conveniently modified for this situation. In first place, it is obvious that we do not need to move the boundary conditions.

Let us set again $\mathcal{D} = \{x \in C^1/Im(x) \subset D$ and $\|x'\|_\infty < N+1\}$ with $N = N(C+1, 2\phi)$ where C and ϕ are as in (3.8) or (3.9), and let $F(u) = F(f, u)$ as before.

The problem of adapting the previous proof to this new context relies in the fact that, due to the resonance of the operator L, the bound for x'_λ does not force solutions to be far from the boundary. To overcome this difficulty we need f to point outwards the open set D, for every t and y.

Step 1

As M is a C^1 manifold, we may suppose n_x is a continuous and bounded function defined in $\mathbb{R}^n$. Furthermore, we may suppose that $|n_x| \leq 1$ (for example using Dugundji's extension theorem). In fact, as we are assuming that $D = \eta^{-1}(-\infty, 0)$ for some smooth η, we may choose the vector field $\nabla\eta(x)$, properly normalized in a neighborhood of M.

For $\lambda \in [0,2]$, set

$$f^1_\lambda(t,x,y) = f(t,x,y) + n_x\lambda(\max\{0, -\langle f(t,x,y), n_x\rangle\} + \frac{1}{2}\min\{1,\phi(y)\})$$

where, when (3.9) is assumed, $\phi(y) := \gamma|y|^2 + C$, and

$$F^1(\lambda,u) = F(f^1_\lambda, u)$$

so that $F^1(0,u) = F(f,u)$.

By Lemma 4 below, we know that $|f^1_\lambda| \leq |f| + \min\{1,\phi\}$. Thus, in both cases (3.8) and (3.9) it is easy to see that f_λ also satisfies it as well, with $\hat{C} = C + 1$ and $\hat{\phi} = 2\phi$. Indeed, for condition (3.8)

$$\begin{aligned} |f^1_\lambda| \leq |f| + \min\{1,\phi\} &\leq d^2_x\eta(y,y) + d_x\eta(f(t,x,y)) + C + 1 \\ &\leq d^2_x\eta(y,y) + d_x\eta(f^1_\lambda(t,x,y)) + \hat{C} \\ |f^1_\lambda| \leq |f| + \phi &\leq 2\phi = \hat{\phi} \end{aligned} \tag{3.17}$$

and for condition (3.9)

$$|f^1_\lambda| \leq |f| + 1 \leq \gamma|y|^2 + C + 1.$$

Moreover, condition (3.6) is trivially satisfied and it is easy to prove that there are no solutions of $x'' = f^1_\lambda(t,x,x')$ on the boundary of $\mathcal{D}$, for any $\lambda \in [0,2]$. Furthermore, $\langle f^1_\lambda(t,x,y), n_x\rangle =$

$$\begin{cases} \langle f(t,x,y), n_x\rangle(1-\lambda) + \frac{\lambda}{2}\min\{1,\phi(y)\} & \text{if } \langle f(t,x,y), n_x\rangle < 0 \\ \langle f(t,x,y), n_x\rangle + \frac{\lambda}{2}\min\{1,\phi(y)\} & \text{otherwise} \end{cases} \tag{3.18}$$

and hence

$$\langle f^1_2(t,x,y), n_x\rangle > 0 \tag{3.19}$$

for all t, y and $x \in M$. Thus,

$$deg(\overline{f^1_2(t,x,0)}, D, 0) = deg(n_x, D, 0). \tag{3.20}$$

Now the problem is homotopic to $x''(t) = f^1_2(t,x,x')$, where f^1_2 points outwards over the boundary (namely, it satisfies (3.19)).

Step 2

Following the idea from Step 2 in Theorem 3.4.1, let

$$f^2_\lambda(t,x,y) = \lambda^2 f^1_2(t,x,\lambda^{-1}y)$$

$$F^2(\lambda,u) = F(f^2_\lambda, u).$$

Both Remark 2 and the bound obtained in (3.15) apply here (in contrast with the Dirichlet case, Lemma 5 is not needed here since now we may extend solutions periodically) so we deduce that solutions of $x''_\lambda = f^2_\lambda(t, x_\lambda, x'_\lambda)$ are not on $\partial\mathcal{D}$.

Now we claim there exists $\lambda_0 > 0$ such that there are no solutions of $x''(t) = \mu f^2_{\lambda_0}(t,x,y)$ in $\partial\mathcal{D}$ for $\mu \in (0,1]$. Suppose again, by contradiction, that there exists a sequence $x_\lambda \in \partial\mathcal{D}$ of solutions of $x''_\lambda(t) = \mu_\lambda f^2_\lambda(t,x,y)$ with $\mu_\lambda \in (0,1]$ and $\lambda \to 0$. By (3.15), from Theorem

3.4.1 we know that $|x'_\lambda| \leq \lambda N$ and by compactness we can suppose that $x_\lambda \to p$ uniformly for some $p \in \partial D$. By periodicity, $\int_0^T x''_\lambda = 0$, so

$$0 = \int_0^T \langle \mu_\lambda^{-1} \lambda^{-2} x''_\lambda, n_p \rangle = \int_0^T \langle f_2^1(t, x_\lambda, \lambda^{-1} x'_\lambda), n_p) \rangle dt.$$

Passing to a subsequence, we may assume that

$$f_2^1(t, x_\lambda, \lambda^{-1} x'_\lambda) - f_2^1(t, p, \lambda^{-1} x'_\lambda) \to 0$$

uniformly, and we deduce:

$$\int_0^T \langle f_2^1(t, p, \lambda^{-1} x'_\lambda), n_p) \rangle dt \to 0.$$

This is a contradiction, because (3.19) implies that $\langle f_2^1(t, p, y), n_p \rangle$ has a positive minimum over the compact set $I \times \{p\} \times \overline{B(0, N)}$, which contains $(t, p, \lambda^{-1} x'_\lambda)$ for all t.

Step 3

Now we set

$$F_\mu^3(u) = G(f_{\lambda_0}^2, u, \mu)$$

where G is defined as before, and observe that a zero of F_μ^3 with $\mu \in (0, 1]$ is a solution of the equation $x''(t) = \mu \lambda_0^2 f_2^1(t, x, \lambda_0^{-1} y)$, so it does not belong to $\partial \mathcal{D}$.

Finally $F_0^3(u) = u - \overline{u} - \overline{\mathcal{N}(f_{\lambda_0}^2, u)}$ and its Leray-Schauder degree is equal to the Brouwer degree on D of the function $-\psi$, where $\psi : \overline{D} \longrightarrow \mathbb{R}^n$ is given by

$$\psi(p) := \int_0^T f_2^1(t, p, 0) dt.$$

Using (3.19), we deduce that $\int_0^T \langle f_2^1(t, x, 0), n_x \rangle dt > 0$ when $x \in \partial D$, so the function ψ is linearly homotopic to the normal unit vector field n. Due to a theorem by Hopf [26], the degree of n is equal to $\chi(\overline{D})$, where χ denotes the Euler characteristic. In section (3.5) we shall prove that $\overline{D}$ is contractible, so $\chi(\overline{D}) = 1$ and the proof is complete.

□

Lemma 4. *Let $v, n \in \mathbb{R}^d$ with $|n| \leq 1$. Then for $\lambda \in [0, 2]$ we have:*

$$|v - \lambda n \langle v, n \rangle| \leq |v|.$$

Proof. We compute

$$|v - \lambda n \langle v, n \rangle|^2 = |v|^2 - 2\lambda \langle v, n \rangle^2 + \lambda^2 \langle v, n \rangle^2 |n|^2$$

$$\leq |v|^2 - 2\lambda \langle v, n \rangle^2 + \lambda^2 \langle v, n \rangle^2$$

$$= |v|^2 + (\lambda^2 - 2\lambda) \langle v, n \rangle^2 \leq |v|^2$$

for $\lambda \in [0, 2]$.

□

3.5 Topology of the domain

In this section we will show that the conditions in the preceding results imply that D must be contractible. This is proved in Theorems 3.5.1 and 3.5.2 using two preliminary lemmas. Our main tool shall be Morse theory for manifolds with boundary.

Definition 1. *Let $M, N \subseteq \mathbb{R}^n$ be oriented C^2 manifolds with normal unit vector fields n_N and n_M. We shall say that M and N are outside tangent at $p \in N \cap M$ if $T_pM = T_pN$ and $n_N(p) = -n_M(p)$.*

Lemma 2. *Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be C^2 and suppose $S = \{\eta = \eta_0\}$ is a C^2 manifold oriented in the direction of $\nabla\eta$. Then $d^2_x\eta(y, y) = -|\nabla\eta|.\mathrm{II}^S_x(y)$.*

Proof. Let x be a curve such that $x(0) = x$ and $x'(0) = y$. As

$$\langle \frac{d}{dt}(\nabla\eta(x(t))), x'(t)\rangle = d^2_x\eta(x', x'),$$

and writing $\nabla_{x(t)}\eta = \delta(t).n_S(x(t))$ where $\delta = |\nabla\eta|$, we obtain:

$$\frac{d}{dt}(\nabla\eta(x(t))) = \delta'(t)n_S + \delta.g(x')$$

and hence

$$d^2_x\eta(x', x') = \delta\langle g(x'), x'\rangle = -|\nabla\eta|\mathrm{II}_x(x').$$

□

Lemma 3. *Let x be a curve in M, S as before and suppose M and N are outside tangent at $x(0)$. If $\mathrm{II}^M < -\mathrm{II}^S$ then $(\eta \circ x)''(0) > 0$.*

Proof. Using the previous lemma, we deduce

$$(\eta \circ x)'' = d^2_x\eta(x', x') + d_x\eta(x'') = -|\nabla\eta|\mathrm{II}^S_x(x') + |\nabla\eta|\langle n_S, x''\rangle$$

$$(\eta \circ x)''(0) = -(\mathrm{II}^S + \mathrm{II}^M)|\nabla\eta| > 0$$

□

Theorem 3.5.1. *Let D satisfy $R_D curv(D, p) > -1$ for all $p \in M = \partial D$. Then D is contractible.*

Proof. Without loss of generality, we may assume that D is centered at 0, that is: $\overline{D} \subseteq \overline{B}(R_D, 0)$. Let $R > R_D$ be such that $curv(D, p) > -\frac{1}{R}$ for every $p \in M$. Let $\eta : \overline{D} \to \mathbb{R}$

$$\eta(x, y) = x - \sqrt{R^2 - |y|^2}$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Clearly η is a C^∞ function defined also in a neighborhood of $\overline{D}$ which is an n-dimensional manifold with boundary ∂D. Consider $S(t) = \{\eta = t\} = \partial B((t, 0), R) \cap \{x \geq t\}$ with the orientation given by $\nabla\eta$.

Let us prove that η is a Morse function in $\overline{D}$. First of all, there are no critical points inside D. A critical point of $\eta|_M$ is a point p such that $S(t)$ is tangent to M in p for some t. A critical point of η as a Morse function in a manifold with boundary is a critical point of $\eta|_M$ such that $\nabla\eta$ points inwards D. In such a point p, $S(t)$ and M are outside tangent. Also

$$\mathrm{II}_p^{S(t)}(v) = -|v|^2\frac{1}{R} < |v|^2 curv(D,p) \leq -\mathrm{II}_p^M(v)$$

for every $t \in \mathbb{R}, v \in \mathbb{R}^n$.

The previous lemma applies and we get that p is a nondegenerate local minimum.

Morse theory implies now that $\overline{D}$ has the homotopy type of a disjoint union of points, one per each critical point. As D is connected, we deduce that $\overline{D}$ is contractible. □

Remark 4. *As we saw in Remark 3.3, $curv(D,p)$ can be calculated using tangent balls. This might suggest a generalization of the notion of curvature for arbitrary open sets if one defines the concept of exterior tangent ball in the following way: B is an exterior tangent ball at $p \in \partial D$ if $p \in \overline{B}$ and there exists a neighborhood V of p such that $D \cap V \subset \mathbb{R}^n \backslash B$.*

Then it is natural to ask if Theorem 3.5.1 is still valid in this context. The answer is negative:

Consider for example $n = 3$,

$$D = B(0,1) \setminus \big(\overline{B}((0,1,0), 1+\epsilon) \cup \overline{B}((0,-1,0), 1+\epsilon)\big)$$

for small $\epsilon > 0$. This set obviously satisfies $R_D curv(D,p) > -1$ because for every point in the boundary there is an external tangent ball of radius $1+\epsilon$, but it has the homotopy type of S^1.

However, if we approximate D by smooth domains, it is clear that the condition fails. This shows that the previous definition of curvature for arbitrary domains is not accurate.

Theorem 3.5.2. *Let D satisfy (3.8) for some η and some f. Then D is contractible.*

Proof. Using (3.8), let us firstly notice that if $|\nabla\eta| < 1$, then $0 \leq d_x^2\eta(y,y)+C$ for arbitrary y, so $d_x^2\eta$ must be positive semidefinite. Let $K_a = \{x \in D : d_x\eta = 0, \eta(x) = a\}$ be the critical set of level a and $\eta^a = \{x \in D : \eta(x) < a\}$ the level set.

As $\nabla\eta$ is continuous in $\overline{D}$ there is an $\epsilon > 0$ such that for all a, $d^2\eta$ is positive semidefinite in $O_a = B(K_a, \epsilon)$. Since $\eta \geq a$ in O_a, we deduce that if $b > a$ then $O_b \cap K_a = \emptyset$. This implies that there are only finite critical values. Also it is clear that $O_a \cap \eta^a = \emptyset$, so the Morse deformation Lemma shows that D has the homotopy type of the finite disjoint union of the level sets K_a. Again, since D is connected there is only one critical set K_a which is also the minimum set of η. Now let $\delta > 0$ be small enough such that $\eta^{a+\delta} \subseteq O_a$ (where a is the minimum). The set $\eta^{a+\delta}$ is a level set of a (non strict) convex function so it is a convex set. Then the critical set K_a is the intersection of all such convex sets so it is again convex. □

Chapter 4

Periodic motions in forced problems of Kepler type

4.1 Introduction and main results

Consider the second order equation in the plane

$$\ddot{z} \pm \frac{z}{|z|^{q+1}} = \lambda h(t), \quad z \in \mathbb{C} \setminus \{0\} \tag{4.1}$$

where $q \geq 2$, $\lambda \geq 0$ is a parameter and $h : \mathbb{R} \to \mathbb{C}$ is a continuous and 2π-periodic function satisfying

$$\int_0^{2\pi} h(t)dt = 0.$$

This equation models the motion of a particle under the action of a central force $F(z) = \mp \frac{z}{|z|^{q+1}}$ and an external force $\lambda h(t)$. The force F can be attractive or repulsive depending on the sign $+$ or $-$ in the equation (4.1). For $q = 2$ the vector field F becomes the classical gravitational or Coulomb force. For general information on this type of problems we refer to [2].

For the repulsive case it is known that (4.1) has no 2π-periodic solutions when λ is small enough (see [61] and [7]). In this chapter we will discuss the existence of 2π-periodic solutions when λ is large. Before stating the main result we recall the notion of index as it is usually employed in Complex Analysis (see [17]). Given a continuous and 2π-periodic function $\gamma : \mathbb{R} \to \mathbb{C}$ and a point z lying in $\mathbb{C} \setminus \gamma(\mathbb{R})$, the index of z with respect to the circuit γ is an integer denoted by $j(z, \gamma)$. When γ is smooth, say C^1, this index can be expressed as an integral,

$$j(z, \gamma) = \frac{1}{2\pi\imath} \int_\gamma \frac{d\xi}{z - \xi} = \frac{1}{2\pi\imath} \int_0^{2\pi} \frac{\dot{\gamma}(t)}{z - \gamma(t)} dt.$$

It is well known that $z \mapsto j(z, \gamma)$ is constant on each connected component Ω of $\mathbb{C} \setminus \gamma(\mathbb{R})$. From now on we write $j(\Omega, \gamma)$ for this index. Let $\phi(t)$ be a 2π-periodic solution of (4.1), the index $j(0, \phi)$ is well defined and can be interpreted as the winding number of the solution ϕ around the singularity $z = 0$.

Theorem 4.1.1. *Let $H(t)$ be a 2π-periodic solution of*

$$\ddot{H}(t) = -h(t)$$

and let $\Omega_1, \ldots, \Omega_r$ be bounded components of $\mathbb{C} \setminus H(\mathbb{R})$. Then there exists $\lambda_* > 0$ such that the equation (4.1) has at least r different solutions $\phi_1(t), \ldots, \phi_r(t)$ of period 2π if $\lambda \geq \lambda_*$. Moreover,

$$j(0, \phi_k) = j(\Omega_k, H), \quad k = 1, \ldots, r.$$

Next we discuss the applicability of the theorem in three simple cases.

Example 1. $h(t) \equiv 0$.

We also have $H(t) \equiv 0$ and so $\mathbb{C} \setminus H(\mathbb{R}) = \mathbb{C} \setminus \{0\}$. This set has no bounded components and so the theorem is not applicable. This is reasonable since the equation $\ddot{z} - \frac{z}{|z|^{q+1}} = 0$ has no periodic or even bounded solutions. This is easily checked since all solutions satisfy

$$\frac{1}{2} \frac{d^2}{dt^2}(|z|^2) = |\dot{z}|^2 + \frac{1}{|z|^{q-1}} > 0 \ .$$

On the contrary, in the attractive case the equation (4.1) has many periodic solutions for $h \equiv 0$. Notice that $\phi(t) = e^{\imath(t+c)}$ is a 2π periodic solution for any $c \in \mathbb{R}$.

Example 2. $h(t) = e^{\imath t}$

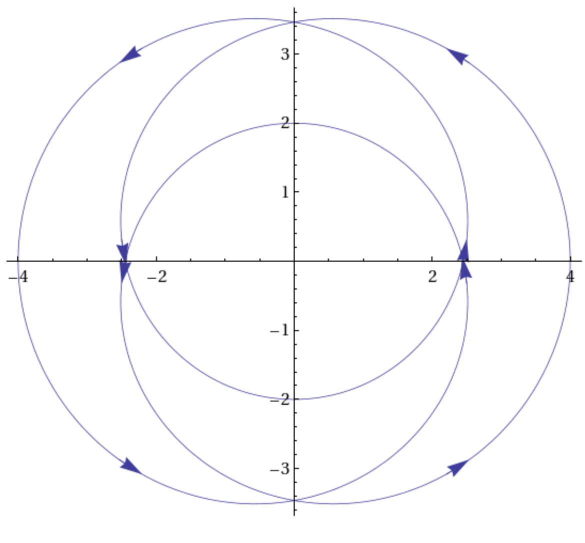
The second primitive of $-h$ is $H(t) = e^{\imath t}$ and $\mathbb{C} \setminus H(\mathbb{R})$ has one bounded component, the open disk $\{|z| < 1\}$. The theorem asserts the existence of a 2π-periodic solution $\phi_1(t)$ with $j(0, \phi_1) = 1$ for λ large enough. Indeed this result can be obtained using very elementary techniques. The change of variables $z = e^{\imath t} w$ transforms (4.1) into

$$\ddot{w} + 2\imath \dot{w} - w \pm \frac{w}{|w|^{q+1}} = \lambda \ .$$

This equation has, for large λ, two equilibria. These equilibria become 2π-periodic solutions with index one in the z-plane. After lengthy computations it is possible to find the spectrum of the linearization of the w equation around the equilibria. This allows to apply Lyapunov center theorem in some cases to deduce the existence of sub-harmonic and quasi-periodic solutions in the z-plane (see [49] for more details on this technique).

Example 3. $h(t) = e^{\imath t} + 27e^{3\imath t}$.

The function $H(t) = e^{\imath t} + 3e^{3\imath t}$ is a parametrization of an epicycloid.



We observe that $\mathbb{C} \setminus H(\mathbb{R})$ *has five bounded connected components with corresponding indices* $3, 2, 2, 1, 1$. *Hence we obtain five* 2π*-periodic solutions.*

For some forcings $h(t)$ the set $\mathbb{C} \setminus H(\mathbb{R})$ has infinitely many bounded components. In such a case the previous result implies that the number of 2π-periodic solutions grows arbitrarily as $\lambda \to \infty$.

4.2 Brouwer degree and weakly nonlinear systems

This section is devoted to describe a well known result on the existence of periodic solutions of the system

$$\dot{x} = \varepsilon g(t, x; \varepsilon), \quad x \in U \subseteq \mathbb{R}^d \tag{4.2}$$

where U is an open and connected subset of $\mathbb{R}^d$, $\varepsilon \in [0, \varepsilon_*]$ is a small parameter and $g : \mathbb{R} \times U \times [0, \varepsilon_*] \to \mathbb{R}^d$ is continuous and 2π-periodic with respect to t. Later it will be shown that our original system (4.1) can be transformed into a system of the type (4.2). Following the ideas of the averaging method (see [45]), we define the function

$$G(c) = \frac{1}{2\pi} \int_0^{2\pi} g(t, c; 0) dt, \quad c \in U \ .$$

Next we assume that G does not vanish on the boundary of a certain open set W, whose closure $\overline{W}$ is compact and contained in U. In such a case the degree of G on W is well defined.

Proposition 1. *In the above conditions assume that*

$$deg(G, W, 0) \neq 0.$$

Then the system (4.2) has at least one 2π*-periodic solution* $x_\varepsilon(t)$ *lying in* W *for* $\varepsilon > 0$ *sufficiently small.*

This result is essentially contained in Cronin's book [15]. We also refer to the more recent paper by Mawhin [36] containing more general results and some history.

Before applying this Proposition to (4.1) it will be convenient to have some information on the behaviour of $x_\varepsilon(t)$ as $\varepsilon \searrow 0$. The function g is bounded on the compact set $[0, 2\pi] \times \overline{W} \times [0, \varepsilon_*]$ and so

$$\|\dot{x_\varepsilon}\|_\infty = O(\varepsilon) \text{ as } \varepsilon \searrow 0.$$

Let $\varepsilon_n \searrow 0$ be a sequence such that $x_{\varepsilon_n}(0)$ converges to some point c in $\overline{W}$. Then $x_{\varepsilon_n}(t)$ converges uniformly to the constant c in $[0, 2\pi]$. Integrating the equation (4.2) over a period we obtain

$$\int_0^{2\pi} g(t, x_{\varepsilon_n}(t); \varepsilon_n) dt = 0$$

and letting $n \to \infty$ we deduce that $G(c) = 0$. In other words, as $\varepsilon \searrow 0$ the solutions $x_\varepsilon(t)$ given by the previous Proposition must accumulate on $G^{-1}(0)$, the set of zeros of G.

4.3 Reduction to a problem with small parameters

Let us start with the original equation (4.1) and consider the change of variables

$$z = \lambda(w - H(t))$$

where $w = w(t)$ is the new unknown. Then (4.1) is transformed into

$$\ddot{w} = \mp\varepsilon^2 \frac{w - H(t)}{|w - H(t)|^{q+1}} \tag{4.3}$$

with $\varepsilon^2 = \frac{1}{\lambda^{q+1}}$.

In principle, this equation can have solutions passing through $H(\mathbb{R})$ but we will look for solutions lying in one of the components Ω_k of $\mathbb{C} \setminus H(\mathbb{R})$. On this domain the equation (4.3) is equivalent to a first order system of the type (4.2) with $x = (w, \xi) \in \mathbb{C}^2$, $U = \Omega_k \times \mathbb{C}$ and

$$\dot{w} = \varepsilon\xi, \;\; \dot{\xi} = \mp\varepsilon \frac{w - H(t)}{|w - H(t)|^{q+1}}.$$

The averaging function is

$$G(c_1, c_2) = (c_2, \Phi(c_1)), \;\; c_1 \in \Omega_k, \; c_2 \in \mathbb{C}$$

and

$$\Phi(c_1) = \mp\frac{1}{2\pi}\int_0^{2\pi} \frac{c_1 - H(t)}{|c_1 - H(t)|^{q+1}} dt \; .$$

In the next section we will prove the following

Claim 1. *For each $k = 1, \ldots, r$ there exists an open and bounded set Ω_k^*, whose closure is contained in Ω_k, and such that*

$$\Phi(c_1) \neq 0 \;\, if \; c_1 \in \partial\Omega_k^*, \quad deg(\Phi, \Omega_k^*, 0) = 1.$$

Assuming for the moment that this claim holds, we notice that G does not vanish on the boundary of $W = \Omega_k^* \times B$ where B is the unit disk $|c_2| < 1$. Moreover G can be expressed as

$$G = L \circ (\Phi \times id)$$

where $L : \mathbb{C}^2 \to \mathbb{C}^2$ is the linear map $(c_1, c_2) \mapsto (c_2, c_1)$ and id is the identity in $\mathbb{C}$. The general properties of degree imply that

$$deg(G, W, (0,0)) = sign(detL) \cdot deg(\Phi \times id, \Omega_k^* \times B, (0,0))$$

$$= deg(\Phi, \Omega_k^*, 0) = 1.$$

In consequence Proposition 1 is applicable and we have proved the first part of Theorem 4.1.1. Namely, the existence of 2π-periodic solutions $\phi_1(t), \ldots, \phi_r(t)$ for large λ (or small ε).

Notice that $\phi_k(t) = \lambda(\psi_k(t) - H(t))$, where ψ_k is a 2π-periodic solution of (4.3) lying in Ω_k^*. For convenience we make explicit the dependence of ϕ_k with respect to ε and write $\phi_k(t) = \phi_k(t, \varepsilon)$.

To prove the identity

$$j(0, \phi_k(., \varepsilon)) = j(\Omega_k, H)$$

when ε is small enough, we proceed by contradiction. Let us assume that for some sequence $\varepsilon_n \searrow 0$, $j(0, \phi_k(., \varepsilon_n)) \neq j(\Omega_k, H)$. After extracting a subsequence of ε_n we can assume that $\psi_k(t, \varepsilon_n) \to z$, $\dot{\psi}_k(t, \varepsilon_n) \to 0$, uniformly in t, where z is some point in $\Omega_k^* \subset \Omega_k$ with $\Phi(z) = 0$. This is a consequence of the discussion after Proposition 1. Computing indexes via integrals and passing to the limit

$$j(0, \phi_k(\cdot, \varepsilon_n)) = -\frac{1}{2\pi\imath} \int_0^{2\pi} \frac{\dot{\psi}(t, \varepsilon_n) - \dot{H}(t)}{\psi(t, \varepsilon_n) - H(t)} dt \to$$

$$\to \frac{1}{2\pi\imath} \int_0^{2\pi} \frac{\dot{H}(t)}{z - H(t)} dt = j(z, H) = j(\Omega_k, H).$$

Since we are dealing with integer numbers, $j(0, \phi_k(., \varepsilon_n))$ and $j(\Omega_k, H)$ must coincide for large n. This is a contradiction with the definition of ε_n. By now the proof of the main theorem is complete excepting for the above claim.

4.4 Degree of gradient vector fields

The purpose of this section is to prove the claim concerning the function Φ. To do this we first prove a result valid for general gradient maps in the plane.

Proposition 1. *Let Ω be a bounded, open and simply connected subset of $\mathbb{C}$ and let $V : \Omega \to \mathbb{R}$ be a C^1 function (in the real sense). In addition assume that*

$$V(z) \to +\infty \text{ as } z \to \partial\Omega \ . \tag{4.4}$$

Then there exists an open set Ω^*, whose closure is contained in Ω, such that

1. *$\nabla V(z) \neq 0$ for each $z \in \partial\Omega^*$*
2. *$deg(\nabla V, \Omega^*, 0) = 1$.*

Remark. The condition (4.4) says that V blows up in the boundary of Ω. More precisely, given $r > 0$ there exist $\delta > 0$ such that if $z \in \Omega$ with $dist(x, \partial\Omega) < \delta$ then $V(z) > r$.

Notice also that, by the properties of degree in two dimensions,

$$deg(\nabla V, \Omega^*, 0) = deg(-\nabla V, \Omega^*, 0) \ .$$

Proof. By Sard's lemma we know that V has many regular values in the interval $(\min_\Omega V, +\infty)$. Let us pick one of these values, say α. Then the set $M = V^{-1}(\alpha)$ is a one-dimensional manifold of class C^1. Since V blows up at the boundary, M is compact and so it has to be composed by a finite number of disjoint Jordan curves. Let γ be one of these Jordan curves and let us define Ω^* as the bounded component of $\mathbb{C} \setminus \gamma$. Notice that the closure of Ω^* is contained in Ω because Ω is simply connected.

We know that

$$V(z) = \alpha \text{ and } \nabla V(z) \neq 0 \text{ if } z \in \gamma$$

and so $\nabla V(z)$ must be colinear to $n(z)$, the outward unitary normal vector to the curve γ. This implies that $\langle \nabla V(z), n(z) \rangle$ does not vanish on the curve γ. Assume for instance that

$$\langle \nabla V(z), n(z) \rangle > 0 \text{ if } z \in \gamma,$$

the other case being similar. Then it is easy to prove that $\nabla V(z)$ is linearly homotopic to any continuous vector field which is tangent to γ on every point of this curve. The proof is complete because it is well known that these tangent vector fields have degree one. See for instance Th. 4.3 (Ch. 15) of [14]. $\square$

We are ready to prove the claim concerning the function

$$\Phi : \Omega_k \to \mathbb{C},\ \Phi(z) = \pm\frac{1}{2\pi}\int_0^{2\pi} \frac{z - H(t)}{|z - H(t)|^{q+1}}dt$$

where Ω_k is a bounded component of $\mathbb{C} \setminus H(\mathbb{R})$.

To do this we will apply Proposition 1 and the crucial observation is that Φ is a gradient vector field. Namely

$$\Phi = \mp\nabla V \text{ on } \Omega_k$$

where V is the real analytic function on Ω_k,

$$V(z) = \frac{1}{2\pi(q-1)}\int_0^{2\pi} \frac{dt}{|z - H(t)|^{q-1}}.$$

Using very standard arguments of planar topology one can prove that Ω_k is simply connected and so we only have to check that (4.4) holds. We finish this chapter with a proof of this fact.

Lemma 2. *In the above setting,*

$$V(z) \to +\infty \text{ as } z \to \partial\Omega_k.$$

Proof. By a contradiction argument assume the existence of a sequence $\{z_n\}$ in Ω_k with $dist(z_n, \partial\Omega_k) \to 0$ and such that $V(z_n)$ remains bounded. Since Ω_k is bounded it is possible to extract a subsequence (again z_n) converging to some point $p \in \partial\Omega_k$. Let us define the set $A = \{t \in [0, 2\pi] : H(t) = p\}$ and the function

$$\mu(t) = \begin{cases} \frac{1}{|H(t)-p|^{q-1}}, & t \in [0, 2\pi] \setminus A \\ +\infty, & t \in A. \end{cases} \tag{4.5}$$

Then the sequence of functions $\frac{1}{|H(t)-z_n|^{q-1}}$ converges to μ pointwise. By Fatou's Lemma

$$\int_0^{2\pi} \mu(t)dt \le \liminf_{n\to\infty} \int_0^{2\pi} \frac{dt}{|H(t) - z_n|^{q-1}} = 2\pi(q-1)\liminf_{n\to\infty} V(z_n) < \infty.$$

Hence $\mu(t)$ is integrable in the sense of Lebesgue . In particular the set A has measure zero. Since the boundary of Ω_k is contained in $H(\mathbb{R})$, the set A is non-empty and we can fix $\tau \in [0, 2\pi]$ with $H(\tau) = p$. The previous discussion shows that

$$\mu(t) = \frac{1}{|H(t) - p|^{q-1}}, \text{ a.e. } t \in [0, 2\pi].$$

Let $L > 0$ be a Lipschitz constant for H, then

$$\mu(t) \ge \frac{1}{L^{q-1}|t - \tau|^{q-1}} \text{ a.e. } t \in [0, 2\pi].$$

At this point the condition $q \geq 2$ plays a role,

$$\int_0^{2\pi} \mu(t)dt \geq \frac{1}{L^{q-1}} \int_0^{2\pi} \frac{dt}{|t-\tau|^{q-1}} = +\infty$$

and this is a contradiction with the integrability of μ. $\square$

Chapter 5

Differentiable functions in Banach spaces

Now we turn to the subject of diferential calculus in Banach spaces.

The well known results stated in sections 5.1 and 5.3 are the cornerstone theorems for the study of local properties of differentiable functions in Banach spaces and Banach manifolds. For details on Banach manifolds and smooth functions we refer the reader to [30]. An excellent overview on the subject is [47] Chapter 3, and the references therein.

5.1 Local singularities

We noticed in example 2 of section 4 that in the repulsive case there exists a solution which does not lie in the corresponding bounded connected component. Its existence is not explained by theorem 4.1.1. We intend to show that this extra solution has its origins in the fact that the operator associated to the equation has a “fold” singularity. Later in section 6.2 we will give the global version of this fact.

Definition 1. *A bounded linear map between Banach spaces $T : X \to Y$ is called a Fredholm operator if $\mathrm{Im}(T) \subseteq Y$ is closed and $\mathrm{Ker}(T), coker(T)$ are finite dimensional. The Fredholm index of T is defined as $ind(T) = \dim(\mathrm{Ker}(T)) - \mathrm{codim}(\mathrm{Im}(T))$. The set of Fredholm operators is denoted $\mathcal{K} \subset L(X, Y)$.*

Let $f : \mathcal{U} \subseteq X \to Y$ be a C^1 function between Banach spaces, defined in an open set $\mathcal{U}$. We say that f is a (nonlinear) Fredholm map if for every $x \in X$ the differential $d_x f$ is a Fredholm operator.

Notice that since the index is continuous in the space of Fredholm operators $\mathcal{K} \subset L(X, Y)$ and f is C^1 then the index is constant in connected components of $\mathcal{U}$.

The goal of this section is to prove theorem 5.1.2 for which we will make use of the powerfull theorem 5.1.1.

Since all theorems are of local nature, we shall use terms like ‘local function at the point x’ or ‘local diffeomorphism at the point x’ refering to functions defined in a small open neighbourhood of x. For convenience, we will avoid mentioning this open set and denote simply $f : X \to Y$.

Lemma 2. *Let $f : X \to Y$ be a C^1 map and $x_0 \in X$ such that $d_{x_0} f$ is suryective and $K = \mathrm{Ker}(d_{x_0} f)$ is complemented in X.*

Then there exists a local diffeomorphism $\phi : K \oplus Y \to X$ sending 0 to x_0 such that $f\phi(k \oplus y) = y$.

Proof. Let $V \subset X$ be a (closed) direct sumand of K in X. Notice that the restriction $d_{x_0} f : V \to Y$ is a Banach space isomorphism (use the open mapping theorem).

Consider

$$\begin{array}{cccc} F : K \oplus V & \to & K \oplus Y \\ k \oplus v & \mapsto & k \oplus f(k+v) \end{array} \tag{5.1}$$

and compute the differential

$$d_{x_0} F = \begin{bmatrix} id_K & 0 \\ d^K f & d^V f \end{bmatrix}$$

which is also a Banach space isomorphism. By the inverse function theorem there exists a local diffeomorphism $\phi : K \oplus Y \to X = K \oplus V$ such that $F\phi = id_{K\oplus Y}$. By composing ϕ with a translation, we may assume that ϕ is defined in a neighbourhood of 0 and that $\phi(0) = x_0$. The conclusion follows. □

Some observations:

Remark 3. *1. Since $d_0\phi$ is invertible,* $\mathrm{Ker}(d_0(f\phi)) = K$.

2. From the definition of F we know that for x in the domain of ϕ, $d_x\phi|_K = id_K$

Lemma 4. *Let $f : X \to Y$ be a C^1 map such that $K = \mathrm{Ker}(d_{x_0} f)$ is complemented in X and $R = \mathrm{Im}(d_{x_0} f)$ is complemented in Y. Then there exists a Banach space C and local diffeomorphisms*
$\phi : K \oplus R \to X$
$\psi : Y \to C \oplus R$
such that $\psi f \phi(k, y) = (\tilde{f}(k, y) \oplus y)$ for some C^1 function $\tilde{f}$.

Proof. Let $R = \mathrm{Im}(d_{x_0} f)$, $K = \mathrm{Ker}(d_{x_0} f)$. Let V, C be the direct sumands of K and R respectively, let $P_R, P_C : Y \to R, C$ be the associated projections.

Obviously $d_{x_0}(P_R f) = P_R d_{x_0} f$ is surjective and $\mathrm{Ker}(d_{x_0}(P_R f)) = \mathrm{Ker}(d_{x_0} f)$ which is complemented so we may apply the previous theorem to $P_R f$ instead of f and obtain $\phi : K \oplus R \to X$ such that $P_R f \phi(k, r) = r$. Then

$$f\phi(k, r) = P_C f \phi(k, r) \oplus r = \tilde{f}(k, r) \oplus r$$

and the function ψ is just the canonical isomorphism. □

Remark 5. *Clearly* $\mathrm{Ker}(d_0(\psi f \phi)) = K$

Definition 6. *We say that two local functions $f : X \to Y$, $g : Z \to W$ at points $x \in X$, $z \in Z$ respectively are locally conjugated if there exist local diffeomorphisms ϕ, ψ at $x, f(x)$ such that the following diagram is commutative.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, "\psi"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \tag{5.2}$$

Local conjugacy is an equivalence relation. For a function $f : X \to Y$ we are interested in determining its local conjugacy class because this equivalence relation captures the structure of point preimages. That is, if f and g are locally conjugated at points x, z then for each y near $f(x)$, ϕ is a diffeomorphism between $f^{-1}(y)$ and $g^{-1}(\psi(y))$. The simplest example is the local conjugacy class of a diffeomorphism where the preimage of a point is always one point.

The next theorem characterizes the local conjugacy class of Fredholm mappings.

Theorem 5.1.1 (Local representative theorem for Fredholm maps)**.** *Let $f : X \to Y$ be a Fredholm map, let $n = \dim(\mathrm{Ker}(d_{x_0}f)), m = codim(\mathrm{Im}(d_{x_0}f))$.*

Then there exists a (closed) subspace $E \subset X$, and a C^1 function $\tilde{f} : \mathbb{R}^n \oplus E \to \mathbb{R}^m$ such that f is locally conjugated to the function

$$\begin{array}{ccc} \mathbb{R}^n \oplus E & \to & \mathbb{R}^m \oplus E \\ x \oplus e & \mapsto & \tilde{f}(x,e) \oplus e \end{array} \tag{5.3}$$

Proof. Since $\mathrm{Ker}(d_{x_0}f)$ and $\mathrm{Im}(d_{x_0}f)$ are finite dimensional and codimensional respectively they are complemented. The theorem therefore follows from the previous lemma. $\square$

Remark 7. *1.* $\mathrm{Ker}(d_0(\psi f \phi)) = \mathbb{R}^n$ *and* $\mathrm{Im}(d_0(\psi f \phi)) = E$

2. The preimage of a point (x, e) is the set of points $\{(p, e)/\tilde{f}(p, e) = x\}$.

Theorem 5.1.2 (Characterization of fold singularities)**.** *Let $f : X \to Y$ be a C^2 Fredholm map and $x_0 \in X$ such that*

1. *$ind(f) = 0$*

2. *$\mathrm{Ker}(d_{x_0}f)$ is one-dimensional*

3. *if ρ is a generator of the kernel then $d^2_{x_0}f(\rho,\rho) \notin \mathrm{Im}(d_{x_0}f)$*

Then there exists a (closed) subspace $E \subset X$ such that f is locally conjugated to the 'fold' function

$$\begin{array}{ccc} \mathbb{R} \oplus E & \to & \mathbb{R} \oplus E \\ t \oplus e & \mapsto & t^2 \oplus e \end{array} . \tag{5.4}$$

In this case the map f is said to have a local 'fold' singularity at x_0.

Remark 8.

The 'fold' function presents a particular structure with respect to preimages. The set of singular values (i.e. the images of points where the differential is not surjective) is the surface $M_1 = \{0\} \times E$ which divides the space $\mathbb{R} \times E$ into 2 connected components $M_0 = \{(t, e)/t < 0\}$ and $M_2 = \{(t, e)/t > 0\}$. Clearly a point $(t, e) \in \mathbb{R} \times E$ has exactly i preimages when $(t, e) \in M_i$ for $i = 0, 1, 2$.

The local conjugacy relation preserves this structure. Namely, if ϕ, ψ are the local conjugacy functions such that $\psi.f.\phi^{-1}$ is the fold map then the set $S_1 = \psi^{-1}(M_1)$ is a smooth manifold defined near $f(x_0)$ (and passing through $f(x_0)$) which divides a small neighbourhood of $f(x_0)$ in the two regions $S_0 = \psi^{-1}(M_0), S_2 = \psi^{-1}(M_2)$. The behaviour of the preimages of points near $f(x_0)$ is (locally) the same as for the 'fold' function. That is, $y \in S_i$ has exactly i preimages near x_0.

Proof of Theorem 5.1.2: In the proof we will show that the conjugacy functions ϕ, ψ are homeomorphisms but it is not hard to see that they are in fact diffeos.

By the previous theorem, f is conjugated to the function $t \oplus e \mapsto \tilde{f}(t,e) \oplus e$ and Remark 7 implies, $\frac{\partial \tilde{f}}{\partial t}(0,0) = 0$.

We make now an observation on the previous lemma: The function $\tilde{f} : K \oplus R \to C$ is $\tilde{f} = P_C f \phi$. So if we take $\rho, k \in K$ with k small,

$$d_{(k,0)}\tilde{f}(\rho) = P_C \circ d_{\phi(k,0)} f \circ d_{(k,0)}\phi(\rho) = P_C d_{\phi(k,0)} f(\rho).$$

(the last equality is consequence of the last item in Remark 3)

Using this expresion for the first derivative of $\tilde{f}$ we may calculate the second derivative

$$\frac{\partial^2 \tilde{f}}{\partial t^2}(0,0) = d^2_{(0,0)}\tilde{f}(\rho,\rho) = P_C d^2_{x_0} f(\rho, d_{(0,0)}\phi(\rho)) = P_C d^2_{x_0} f(\rho,\rho)$$

We deduce that $\frac{\partial^2 \tilde{f}}{\partial t^2}(0,0) \neq 0$ because of the third hipothesis, and we may assume that it is positive by composing if necessary with $(t \oplus e) \mapsto (-t \oplus e)$.

Using the implicit function theorem we obtain a function $t(e)$ such that $\frac{\partial \tilde{f}}{\partial t}(t(e), e) = 0$. Since the local map $t \oplus e \mapsto t + t(e) \oplus e$ is again a local diffeomorphism, f is conjugated to $t \oplus e \mapsto \tilde{g}(t,e) \oplus e$ where

$$\frac{\partial \tilde{g}}{\partial t}(0,e) = 0.$$

Now $t \oplus e \mapsto t - \tilde{g}(0,e) \oplus e$ is again a local diffeo so f is conjugated to $t \oplus e \mapsto \tilde{h}(t,e) \oplus e$ where

$$\tilde{h}(0,e) = 0, \quad \frac{\partial \tilde{h}}{\partial t}(0,e) = 0, \quad \frac{\partial^2 \tilde{h}}{\partial t^2}(0,0) > 0.$$

From the last inequality we obtain

$$\frac{\partial^2 \tilde{h}}{\partial t^2}(t,e) > 0$$

for all t, e in a small ball at 0. Since in this ball, the function $\frac{\partial \tilde{h}}{\partial t}(.,e)$ is strictly increasing and is 0 for $t = 0$, the function

$$t \oplus e \mapsto \sqrt{\tilde{h}(t,e)}.sg(t) \oplus e$$

is a local homeomorphism. We conclude that f is conjugated to the 'fold map' as desired. □

Corollary 5.1.1. *Let $f : X \to Y$ be a local fold map in $x_0 \in X$.*

Then through $f(x_0)$ passes a differentiable manifold M_1 which divides a small ball centered at $f(x_0)$ in two regions M_0, M_2 such that the equation $f(x) = y$ has exactly i solutions for $y \in M_i$, $i = 0, 1, 2$

5.2 An application to the repulsive singular problem

Let $X = C^2_{per}([0,2\pi], \mathbb{C})$, $Y = C^0([0,2\pi], \mathbb{C})$ be the Banach spaces with boundary conditions as defined in section 2.2.2. Namely $X = \{u \in C^2([0,2\pi], \mathbb{C}) / u(0) = u(2\pi), u'(0) = u'(2\pi)\}$.

Define $g(x) = \frac{x}{|x|^3}$ and consider the map $f : X \to Y,\ f(x) = x'' - g(x)$ (here $g(x)$ is an abuse of notation to denote the composition $g \circ x$). This functional is associated to the repulsive problem of Chapter 4. We will see below that f is a C^2 Fredholm map.

For $s \in \mathbb{R}$ define $z_s \in X$ as $z_s(t) = se^{it}$ so

$$f(z_s)(t) = (-s - s^{-2})e^{it}.$$

The equation of example 2, section 4 is

$$f(x) = \lambda e^{it}. \tag{5.5}$$

From the expression for f at the points z_s we only need to solve the equation $\lambda = -s - s^{-2}$ for s, which has $0, 1$ or 2 solutions depending on the value of λ. This suggests the presence of a fold singularity for f at $x_0 = z_\mu$ where $1 - 2\mu^{-3} = 0$. Let's prove this assertion.

Compute the differential

$$df(x_0)(v)(t) = v''(t) - dg(x(t))(v(t)).$$

This is the sum of a Fredholm operator of index 0 and a compact operator, so we deduce that f is a Fredholm map. We must see if the kernel is one-dimensional. Since g is rotation invariant we can compute for $r, v \in \mathbb{R}$

$$dg(re^{it})(ve^{it}) = e^{it} dg(r)(v)$$

$$dg(r)(1) = -2r^{-3}, dg(r)(i) = ir^{-3}$$

or in matrix notation

$$dg(r) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} r^{-3}.$$

To calculate the kernel of $df(x_0)$ we need to study the system $v''(t) - dg(x(t))(v(t)) = 0$ and check that the solution space is only one dimensional. Let $\tilde{v}(t)e^{it} = v(t)$. We compute the derivatives

$$v'(t) = (i\tilde{v}(t) + \tilde{v}'(t))e^{it}$$

$$v''(t) = (2i\tilde{v}'(t) + \tilde{v}''(t) - \tilde{v}(t))e^{it}.$$

The system transforms as

$$(\tilde{v}''(t) + 2i\tilde{v}'(t) - \tilde{v}(t)) - dg(\mu)(\tilde{v}(t)) = 0.$$

Let $\tilde{v}(t) = u_1(t) + iu_2(t)$. Taking real and imaginary parts we have

$$\begin{cases} u_1''(t) - 2u_2'(t) - (1 - 2\mu^{-3})u_1(t) = 0 \\ u_2''(t) + 2u_1'(t) - (1 + \mu^{-3})u_2(t) = 0 \end{cases}$$

and the system can be stated in the form of a first order equation

$$\begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ h & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & l & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix}$$

where

$$h = (1 - 2\mu^{-3}) = 0$$

$$l = (1 + \mu^{-3}) > 0$$

which is linear and autonomous (we write h instead of 0 to highlight the specific value of μ). The dimension of the space of periodic solutions is the number of purely imaginary eigenvalues of the 4×4 matrix.

The characteristic polynomial is

$$(X^2 - hX)(X^2 - l) + 4X = X^2(X^2 - l) + 4X$$

that has no nonzero, purely imaginary roots, and 0 has multiplicity one.

Now we have to look at the second derivative of f to check the third condition of Theorem 5.1.2.

$$d^2f(x)(v,w)(t) = d^2g(x(t))(v(t), w(t))$$

and after some computations we obtain

$$d^2g(r) = \begin{bmatrix} \begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix} \\ \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \end{bmatrix} r^{-4}$$

A generator of the kernel is the function x_0 itself.

$$d^2f(x_0)(x_0, x_0) = e^{it}d^2g(\mu)(\mu,\mu) = e^{it}\mu^{-2}6$$

Now notice that $df(x_0)$ is a self-adjoint operator with respect to the L^2 inner product. Since $\langle d^2f(x_0)(x_0,x_0), x_0\rangle_{L^2} \neq 0$ this implies that

$$d^2f(x_0)(x_0,x_0) \notin \langle x_0\rangle^\perp = Ker(df(x_0))^\perp \supseteq Im(df(x_0))$$

so the last condition is satisfied and now we can assert that f has a local fold in x_0.

The final conclusion is the following quite interesting result:

Corollary 5.2.1. *Let $\mu = 2^{-\frac{1}{3}}, \nu = -\mu - \mu^{-2}$ and $z_s(t) = se^{it}$ for $s \in \mathbb{R}$. As we mentioned at the beginning of the section, the functions z_s belong to the Banach space $X = C^2_{per}([0, 2\pi], \mathbb{C})$ and z_μ is a periodic solution of the equation*

$$z''(t) - g(z(t)) = z_\nu(t).$$

Then through z_ν passes a codimension 1 submanifold $M_1 \subseteq X$ which divides a small ball centered at z_ν in two regions M_0, M_2 in such a way that the equation

$$z''(t) - g(z(t)) = y(t)$$

has exactly i periodic solutions near z_μ for $y \in M_i$, $i = 0, 1, 2$. Of course we could have more solutions which are not close to z_μ.

5.3 The Sard-Smale theorem

Here we shall prove (Theorem 5.3.1) a generalization of the Sard theorem for infinite dimensions due to Smale in 1965 (see [60]). This theorem will be used repeatedly in sections 6.3 and 6.6. The present section has no results but is included in this chapter to make this thesis more legible and self-contained.

Definition 1. *Let $f : X \to Y$ be a differentiable function between Banach manifolds.*

1. *$x \in X$ is a regular point if $d_x f : T_x X \to T_{f(x)} Y$ is surjective. We say it is a singular point if it is not regular.*
2. *$y \in Y$ is a regular value if every preimage $x \in f^{-1}(y)$ is a regular point. We say it is a singular value if it is not regular. We denote the set of regular values of f by $\mathcal{R}(f)$*
3. *A subset of a topological space is said to be "meager" or "of first category" if it is a countable union of closed sets of X with empty interior. Conversely, a subset of a topological space is said to be "residual" or "of second category" if its complement is of first category, equivalently if it is a countable intersection of open and dense subsets.*

Recall the Baire category theorem which states that in a complete metric space, a second category subset is dense.

The conclusion of the Baire category theorem may usually be underestimated: If two sets A, B are of the second category then the intersection is (obviously from the definition) again of the second category, thus $A \cap B$ is dense. So the property of being a second category set is much stronger than being just dense, and in a sense behaves very much like the property of being a "full Lebesgue measure" set, only that the first notion makes sense in an abstract metric space.

Theorem 5.3.1 (Sard-Smale)**.** *Let $f : X \to Y$ be a C^∞ (nonlinear) Fredholm map and suppose the topology of X has a countable basis.*

Then the set of singular values of f is of the first category.

For the proof we shall need the following lemmas.

Lemma 2. *Fredholm maps are locally closed. Morover, the image of a closed and bounded subset is closed.*

Proof. Take charts such that f looks like the theorem of local representative of Fredholm maps.

Let (x_n, e_n) be a sequence such that $f(x_n, e_n) = (\tilde{f}(x_n, e_n), e_n)$ converges to some point (x, e). Obviously $e_n \to e$ and x_n is bounded. Then x_n has a convergent subsequence and we deduce that (x, e) is in the image.

□

Lemma 3. *The set of regular points of a Fredholm map is open.*

The proof of this lemma is quite simple, but it can be deduced from the continuity of $df : X \to L(X, Y)$ and a general and deep result known as the Graves theorem (see [30]):

Let X, Y be Banach spaces. The set of surjective linear continuous maps $Epi \subset L(X,Y)$ is open.

Now we are in conditions to prove the main theorem:

Proof of Theorem 5.3.1: Let $f : X \to Y$ be our nonlinear Fredholm map. Since X is second countable it is enough to show that there is a covering of X by open sets U so that the regular values of $f|_U$ are residual. In fact, we will show that we can find U so that the regular values of $f|_U$ are open and dense. Take some point x_0 in X. Since f is locally closed and the critical point set of f is closed, for any bounded U the regular values of $f|_U$ form an open set. Now choose charts about the point in question so that the local representative of f has the form guaranteed by Lemma 5.1.1. $f : E \oplus \mathbb{R}^n \to E \oplus \mathbb{R}^m$. The differential of the local representative of f has the form

$$\begin{bmatrix} I & 0 \\ * & d_{(e,x)}\tilde{f}|_{\mathbb{R}^n} \end{bmatrix}$$

so that $d_{(e,x)}f$ is surjective if and only if $d_{(e,x)}\tilde{f}|_{\mathbb{R}^n}$ is surjective. In other words (e,x) is a regular value for $f|_U$ if and only if x is a regular value of $k \mapsto \tilde{f}(e,k)$ (defined in $\{k \in \mathbb{R}^n/(e,k) \in U\}$). Now that the map in question is finite dimensional, we may apply the Sard theorem. Thus the intersection of $\mathcal{R}(f|_U)$ with each slice $\{e\} \times \mathbb{R}^n$ is dense and hence $\mathcal{R}(f|_U)$ is dense. This completes the proof. $\square$

Chapter 6

On existence of periodic solutions for Kepler type problems

6.1 Introduction

In this chapter we return to the periodic singular problem treated in Chapter 4,

$$\begin{cases} u''(t) \pm \frac{u(t)}{|u(t)|^{q+1}} = \lambda h(t) \\ u(0) = u(1) \\ u'(0) = u'(1) \end{cases} \tag{6.1}$$

for a vector function $u : I = [0,1] \to \mathbb{R}^n$, where $q \geq 2$ and $h \in C(I, \mathbb{R}^n)$ with $\overline{h} := \int_0^1 h(t)\, dt = 0$.

Here u describes the motion of a particle under a singular central force that can be attractive or repulsive depending on the sign $\pm$, and an arbitrary perturbation h.

In Chapter 4 we studied the case $n = 2$, for which we proved in Theorem 4.1.1 the existence of periodic solutions under a non-degeneracy condition. In more precise terms, we obtained a lower bound of the number of solutions that depends purely on a topological property of the second primitive of h. Here we extend Theorem 4.1.1 in several directions.

In section 6.2 we consider the repulsive case. We obtain at least one extra solution from the direct computation of the Leray-Schauder degree over the set of curves that are bounded away from the origin and from infinity. More precisely, we obtain a lower bound of the number of solutions that depends not only on the number of connected components of $\mathbb{R}^2 \setminus Im(H)$ but also on the winding number of H with respect to these components.

The main idea is the following: first, we prove that solutions of (6.1) are uniformly bounded for $\lambda \leq \lambda_*$ and that there are no solutions for $\lambda = 0$. Thus, if we picture the solution set $S = \{(u,\lambda) \in C^2([0,1],\mathbb{R}^2) \times \mathbb{R} / u \text{ is a solution of (6.1)}\}$ then S contains a continuum which starts at a solution (u_*, λ_*) given by Theorem 4.1.1 and is bounded both in the λ and the u directions. Then S must 'turn around' in the λ direction and intersect again the subspace $\lambda = \lambda_*$.

Furthermore, in section 6.3 we prove that for a 'generic' forcing term h the repulsive problem has in fact at least $2r$ periodic solutions. More specifically, take $\tilde{C}^0_{per}$ the Banach space of continuous periodic functions of zero average. Then there exists a residual set $\Sigma \in \tilde{C}^0_{per}$ such that if $\lambda h \in \Sigma$ then all solutions of (6.1) are non-degenerate. As a consequence, all of them have multiplicity one and depend differentiably on h.

In section 6.4 we give some examples illustrating existence and non existence of solutions in some particular situations.

In section 6.5 we extend Theorem 4.1.1 to higher dimensions. Our proofs make use of some classical results of algebraic topology. The case $n = 3$ is treated separately because the homology of open sets with smooth boundary is simple and easy to understand, while the case $n > 3$ needs more restrictive hypotheses.

In section 6.6 we obtain further results for the case $n = 3$, assuming that H is an embedded knot. The lower bounds for the number of solutions will depend on the knot type of H, specifically on a knot invariant called the tunnel number $t(H)$. For example, we prove existence of at least 3 solutions if H is a nontrivial knot and at least 5 solutions when it is a composite knot.

Finally, in section 6.7 we apply the methods of section 6.6 to the restricted N-body problem.

6.1.1 Preliminaries

Theorem 4.1.1 is proved in Chapter 4 using a result contained in Cronin's book [15] about the averaging method. However, for our purposes it shall be convenient to describe the procedure in a precise way. As before, let us make the change of variables

$$u(t) = \lambda(z(t) - H(t))$$

so equation (6.1) becomes

$$\begin{cases} z''(t) = \mp\epsilon\frac{z(t)-H(t)}{|z(t)-H(t)|^{q+1}} \\ z(0) = z(1) \\ z'(0) = z'(1) \end{cases} \tag{6.2}$$

where

$$\epsilon = \lambda^{-(q+1)}. \tag{6.3}$$

We shall associate a functional between Banach spaces

$$\mathcal{G}_\epsilon : C^2_{per}(I, \mathbb{R}^n) \to C^2_{per}(I, \mathbb{R}^n)$$

to this system, which is continuous (in fact, analytic) and such that solutions of (6.2) are characterized as points $z \in C^2_{per}(I, \mathbb{R}^n)$ which are zeros of $\mathcal{G}_\epsilon$. We shall use the Leray-Schauder degree in order to prove that the zeros of $\mathcal{G}_0$ can be continued to zeros of $\mathcal{G}_\epsilon$ for small values of ϵ. Then, it is enough to study the function $\mathcal{G}_0$, which can be identified with its restriction to the finite dimensional space of constant functions, namely the function $F : \mathbb{R}^n \setminus Im(H) \to \mathbb{R}^n$ given by

$$F(x) = \int_0^1 \frac{x - H(t)}{|x - H(t)|^{q+1}} dt. \tag{6.4}$$

Specifically, for each open set $D \subseteq \mathbb{R}^n \setminus Im(H)$ we may associate the open set of curves $\mathcal{D} = \{z \in C^2_{per}(I, \mathbb{R}^n)/Im(z) \subseteq D\}$ and hence $deg(\mathcal{G}_0, \mathcal{D}, 0) = deg(F, D, 0)$; thus, it suffices to look for open sets D such that the latter degree is different from zero. For each of these sets D there exists a solution of (6.2) for $\epsilon > 0$ small. In the situation of Theorem 4.1.1, if $\Omega_1, \ldots, \Omega_r$ are the bounded connected components of $\mathbb{R}^2 \setminus Im(H)$, we may construct as in section 4.4, open sets $\Omega_i^* \subset \overline{\Omega_i^*} \subset \Omega_i$ where all the respective degrees are equal to 1.

Moreover, there is also a functional $\mathcal{F}$ associated to equation (6.1) such that

$$\mathcal{G}_\epsilon(x) = \frac{1}{\lambda}\mathcal{F}(\lambda(x-H)) + H \tag{6.5}$$

so it is conjugated to $\mathcal{G}_\epsilon$ by affine homeomorphisms. The functional $\mathcal{F}$ is independent of λ. It shall be constructed explicitly in section 6.2 and then we shall define $\mathcal{G}_\epsilon$ by formula (6.5).

The relation between these two functionals is the following: the function u is a solution of (6.1) if and only if $\mathcal{F}(u) = -\lambda H$, if and only if $\mathcal{G}_\epsilon(\frac{u}{\lambda} + H) = \mathcal{G}_\epsilon(z) = 0$, if and only if $z = \frac{u}{\lambda} + H$ is a solution of (6.2). Also, the degrees are related by

$$deg(\mathcal{F}, \mathcal{E}, -\lambda H) = deg(\mathcal{G}_\epsilon, \mathcal{D}, 0)$$

where ϵ and λ are related by (6.3) and $\mathcal{D}$ and $\mathcal{E}$ are related by

$$\mathcal{D} = H + \frac{1}{\lambda}\mathcal{E}. \tag{6.6}$$

For convenience, let us define the function

$$g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R},\ g(u) = \frac{1}{|u|^{q-1}},$$

so $\nabla g(u) = -(q-1)\frac{u}{|u|^{q+1}}$. From now on, $H : S^1 \to \mathbb{R}^n$ shall be a periodic second primitive of $-h$, which is unique up to translations.

6.2 The repulsive case

In this section we shall improve Theorem 4.1.1 for the repulsive case. In first place, let us prove the existence of an extra solution by a direct degree argument:

Theorem 6.2.1. *In the conditions of Theorem 4.1.1, the repulsive case of problem* (6.1) *admits at least* $r+1$ *solutions.*

It is worth noticing, however, that Theorem 6.2.1 shall be improved as well at the end of this section by studying the winding number of H with respect to the connected components of $\mathbb{R}^2 \setminus Im(H)$.

We will make use of the following two lemmas, which shall provide us *a priori* bounds for the solutions. Later on, these bounds will be used also in the proofs of Theorems 6.5.1 and 6.5.2 for higher dimensions, so the results will be stated in $\mathbb{R}^n$. We remark that nothing of this can be extended to the attractive case.

Lemma 1. *Given* $\lambda_* > 0$ *there exist constants* $R, r > 0$ *such that*

$$r \le u(t) \le R \quad \forall t \in I$$

for any $u : I \to \mathbb{R}^n$ *solution of* (6.1) *with* $\lambda \le \lambda_*$.

Proof. Let prove first that solutions are uniformly bounded away from the origin. Let u be a solution. We define the energy and compute its derivative

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{(q-1)|u(t)|^{q-1}}$$

$$E'(t) = \langle u', u'' \rangle - \frac{\langle u', u \rangle}{|u|^{q+1}} = \langle u', \lambda h \rangle$$

$$|E'(t)| \leq \lambda |u'||h| \leq \lambda \|h\|_\infty \sqrt{2E(t)} \leq C\lambda_* \sqrt{E(t)}. \tag{6.7}$$

Now multiply the equation 6.1 by u and integrate,

$$\int_0^1 \langle u, u'' \rangle - \int_0^1 \frac{1}{|u|^{q-1}} = \lambda \int_0^1 \langle h, u \rangle$$

$$-\|u'\|_2^2 - \int_0^1 \frac{1}{|u|^{q-1}} = \lambda \int_0^1 \langle h, u \rangle = \lambda \int_0^1 \langle h, u - \overline{u} \rangle$$

$$\frac{1}{2}\|u'\|_2^2 + \int_0^1 \frac{q-2}{q-1} \frac{1}{|u|^{q-1}} + \int_0^1 E = -\lambda \int_0^1 \langle h, u - \overline{u} \rangle$$

$$\int_0^1 E \leq \lambda \left| \int_0^1 \langle h, u - \overline{u} \rangle \right| \leq \lambda \|h\|_2 \|u - \overline{u}\|_2$$

$$\int_0^1 E \leq \lambda_* C \|u'\|_2 \leq \lambda_* C \sqrt{\int_0^1 E}. \tag{6.8}$$

From this inequality, a bound for $E(t_0)$ for some t_0 is obtained. Using (6.7), we get a bound for $E(t)$ for all t, and hence a bound for $\frac{1}{|u(t)|^{q-1}}$ depending only on λ_*.

Next, let us prove that solutions are uniformly bounded. By contradiction, suppose there exists a sequence $\{u_n\}$ such that $\|u_n\|_\infty \to \infty$. Then $\|u_n - \overline{u_n}\|_\infty \leq C\|u_n'\|_\infty \leq C\sqrt{\|E\|_\infty} \leq C$ which, in turn, implies that if n is large then the image of u_n lies in a half-space. This contradicts the fact that $\overline{g(u_n)} = 0$. □

Lemma 2. *Problem* (6.1) *has no solutions for* $\lambda = 0$.

Proof. Let $\lambda = 0$ and suppose u is a solution. Multiply equation (6.1) by u and integrate, then

$$-\int_0^1 |u'|^2 = \int_0^1 u''u = \int_0^1 \frac{1}{|u|^{q-1}},$$

a contradiction. □

Lemma 3. *Solutions of* (6.1) *are also uniformly bounded in* $C^2(I, \mathbb{R}^n)$.

Proof. We know from Lemma 1 that $\|u\|_\infty$ and $\|u'\|_\infty$ are bounded. From (6.1), it follows that $\|u''\|_\infty$ is bounded as well. □

Lemma 4. *Let* $\mathcal{F} : \mathcal{E} \subset C^2(I, \mathbb{R}^n) \to C^2(I, \mathbb{R}^n)$ *where* $\mathcal{F}$ *is the functional associated to* (6.1) *and* $\mathcal{E} = \{u \in C^2 : r < |u| < R, \|u'\| < C, \|u''\| < C\}$ *where* r*,* R *and* C *are the bounds obtained in the preceding lemmas.*

Then $deg(\mathcal{F}, \mathcal{E}, -\lambda_* H) = deg(\mathcal{G}_{\epsilon_*}, \mathcal{D}, 0) = 0$ *where* ϵ_* *and* $\mathcal{D}$ *are defined as in* (6.3) *and* (6.6).

Proof. It follows immediately from the continuation theorem. □

Using this fact, we are able to obtain an extra solution of (6.1).

Proof of Theorem 6.2.1: Following the notation and the proof of Theorem 4.1.1, set

$$\mathcal{A}_k := \{z \in C^2(I, \mathbb{R}^2) : Im(z) \subseteq \Omega_k^*, \|z'\|_\infty < C, \|z''\|_\infty < C\}$$

with $\Omega_k^* \subseteq \Omega_k$ such that $deg(\nabla g, \Omega_k^*, 0) = 1$, and take $\mathcal{E}$ as in the previous lemma.

We know that, for some ϵ_* small enough, $deg(\mathcal{G}_{\epsilon_*}, \mathcal{A}_k, 0) = 1$ and the problem has a solution in $\mathcal{A}_k$.

By formula (6.6), $\mathcal{D} = \{z / \frac{r}{\lambda_*} < |z - H| < \frac{R}{\lambda_*}\}$, so taking R large enough it is seen that $\mathcal{A}_k \subseteq \mathcal{D}$ for all k. By the previous lemma, $deg(\mathcal{G}_{\epsilon_*}, \mathcal{D}, 0) = 0$. Defining $\mathcal{B} = \mathcal{D} \setminus \overline{\bigcup_{k=1}^r \mathcal{A}_k}$ we obtain $deg(\mathcal{G}_{\epsilon_*}, \mathcal{B}, 0) = -r$, so there exists at least one more solution in $\mathcal{B}$, which is obviously different from the others. □

In the preceding proof, when $r > 1$ it is worth observing that, although the degree of $\mathcal{G}_{\epsilon_*}$ is equal to $-r$ we cannot assert the existence of r different extra solutions since we are not able to ensure that they are non-degenerate. But we are still able to distinguish solutions using properties that are invariant under continuation.

We recall the definition of the index given in chapter 4

Definition 5. *Let $\gamma : I \to \mathbb{R}^2$ be a continuous curve and let $x \in \mathbb{R}^2 \setminus \mathrm{Im}(\gamma)$. Let $j(x, \gamma) \in \mathbb{Z}$ be defined as the winding number of γ around x. The function $x \mapsto j(x, \gamma)$ is constant in each connected component of $\mathbb{R}^2 \setminus Im(\gamma)$. Thus, if $\Omega \subseteq \mathbb{R}^2$ is one of these components, we define the winding number $j(\Omega, \gamma)$ of γ around Ω.*

Theorem 6.2.2. *In the conditions of Theorem 6.2.1, let $\Omega_1, \ldots, \Omega_r$ be the connected components of $\mathbb{R}^2 \setminus Im(H)$ and let s be the cardinality of the set $\{j(\Omega_k, H)/k = 1, \ldots, r\}$. Then the repulsive case of (6.1) admits at least $r + s$ solutions.*

Proof. Let $\mathcal{E}$ be defined as before and consider the 'winding number function' $J : \mathcal{E} \to \mathbb{Z}$ defined by $J(x) = j(0, x)$, which determines in $\mathcal{E}$ the connected components $\mathcal{E}_i = \{u \in \mathcal{E} / J(x) = i\}$ for $i \in \mathbb{Z}$. Since $\partial\mathcal{E} = \bigcup_{i \in \mathbb{Z}} \partial\mathcal{E}_i$, we deduce from Lemmas 1 and 2 that $deg(\mathcal{F}, \mathcal{E}_i, -\lambda H) = 0$ for every i. Using again formula (6.6), we obtain the decomposition $\mathcal{D} = \bigcup_{i \in \mathbb{Z}} \mathcal{D}_i$ and $deg(\mathcal{G}_{\epsilon_*}, \mathcal{D}_i, 0) = 0$. Repeating the argument of the previous theorem for each k, $\mathcal{A}_k \subseteq \mathcal{D}_i$ where $i = j(\Omega_k, H)$, so if

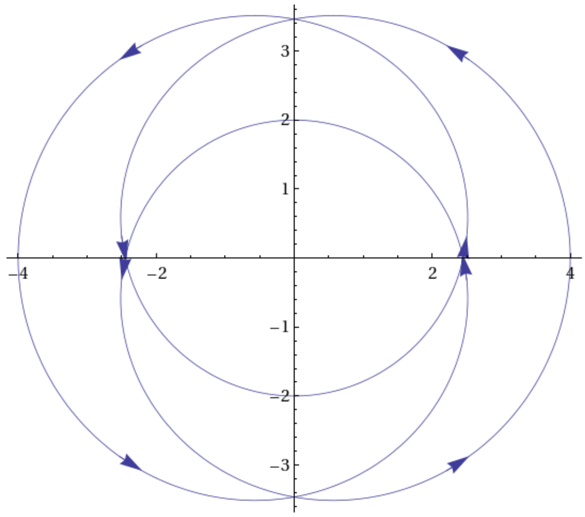
$$\mathcal{B}_i = \mathcal{D}_i \setminus \bigcup_{k/j(\Omega_k, H) = i} \mathcal{A}_k$$

then $deg(\mathcal{G}_{\epsilon_*}, \mathcal{B}_i, 0) = -\#\{k/j(\Omega_k, H) = i\}$ (which might be eventually 0, if there is no k such that $j(\Omega_k, H) = i$).

We conclude that there exists one solution for each $\mathcal{B}_{j(\Omega_k, H)}$.

□

As an example, in 3 we considered, using complex notation, $h(t) := e^{it} + 27e^{3it}$. The function $H(t) = e^{it} + 3e^{3it}$ is a parameterization of the epicycloid:



As observed, $\mathbb{R}^2 \setminus H(\mathbb{R})$ has five bounded connected components with corresponding indices $3, 2, 2, 1, 1$. Hence we obtained five periodic solutions. But according to theorem 6.2.2, in the repulsive case the number of solutions is at least 8.

In general, there is no way to guarantee that solutions of a given problem are non-degenerate without knowing them explicitly. But since our functional is smooth, this property can be achieved by arbitrarily small perturbations. It is in some sense a 'generic' property. This is the content of the next section.

6.3 Genericity

Some of the results of this section are proved in [44] in a more general situation. We include the proofs for the sake of completeness and clarity.

Let us define the spaces

$$C^i_{per} = \{x \in C^i(I, \mathbb{R}^n) : x^{(j)}(0) = x^{(j)}(1), 0 \leq j < i\}$$

so C^0_{per} is just $C^0 = C^0(I, \mathbb{R}^n)$. Also, if X is any of these spaces, we define

$$\tilde{X} = \{x \in X : \overline{x} = 0\}$$

We shall prove the following theorem:

Theorem 6.3.1. *There is a residual subset $\Sigma^2 \subseteq \tilde{C}^2_{per}$ with the following property: if $O \subseteq C^2_{per}$ is an open bounded set such that $deg(\mathcal{F}, O, y) = n$ for some $y \in \Sigma^2$, then the equation $\mathcal{F}(x) = y$ has at least n distinct solutions.*

As a corollary, we shall obtain:

Corollary 6.3.1. *In the conditions of Theorem 6.2.1, there exists a residual subset $\Sigma^0 \subset \tilde{C}^0_{per}$ such that* (6.1) *has at least $2r$ solutions for $\lambda h \in \Sigma^0$ and λ large enough.*

It is interesting to compare this result with the fact, mentioned in [7], that (6.1) has no solutions when $\|\lambda h\|_0 \leq \eta$ for some $\eta > 0$ depending only on q. In particular, the image of the operator $\mathcal{F}$ is not dense.

In order to prove our results, let us recall the Sard-Smale theorem stated in Chapter 5.

Theorem 6.3.2. *(Sard-Smale) Let $f : X \to Y$ be a Fredholm map between Banach manifolds. i.e, f is C^1 and $df(x) : T_xX \to T_{f(x)}Y$ is a Fredholm operator for all x. Then the set of regular values is a residual set in Y.*

The main difficulty for applying the Sard-Smale theorem to our situation is the fact that we need to solve an equation of the form $\hat{F}(x) = -\lambda H$ with $H \in \tilde{C}^0$, but $C^0 \setminus \tilde{C}^0$ is already a residual set. No information can be obtained from the theorem applied to $\hat{F} : C^2_{per} \to C^0$. Thus, we need to study a suitable restriction of the functional.

We shall use the notation $\hat{F}$ for functions defined in the ambient spaces C^2_{per}, C^0 and F for their restrictions to subspaces or submanifolds.

Consider the operators

$$N : \mathcal{U}^i \subseteq C^i \to C^i, \;\; N(u)(t) = n(u(t)) := \frac{u(t)}{|u(t)|^{q+1}},$$

$$Q : C^i \to \mathbb{R}^n, \;\; Q(u) = \overline{u}$$

where $\mathcal{U}^i = \{x \in C^i : x(t) \neq 0, \;\; \forall t\}$.

Lemma 1. *$N : \mathcal{U}^2 \to C^0$ is C^1. In particular, as Q is linear and continuous, then QN is C^1.*

Proof. Take $x \in \mathcal{U}^2$ and let $\eta := d(\mathrm{Im}(x), 0)/2$. If $v \in T_x\mathcal{U}^2 = C^2$ satisfies $\|v\|_\infty \leq \eta$, then

$$n(x(t) + v(t)) = n(x(t)) + dn(x(t))(v(t)) + R(v(t)), \quad R(x) = o(x).$$

In fact, we have $|R(x)| \leq C|x|^2$ with C depending only on η. Then

$$N(x + v) = N(x) + dN(x)(v) + R \circ v, \quad \|R \circ v\|_\infty = |o(\|v\|_\infty)| \leq o(\|v\|_{C^2})$$

and the proof follows.

□

Lemma 2. *The operator $QN : C^2_{per} \to \mathbb{R}^n$ has 0 as a regular value. In particular, the set $M := \{x \in C^2_{per} : QN(x) = 0\}$ is a differentiable manifold.*

Proof. For each $w \in \mathbb{R}^n$ take $v(t) = dn(x(t))^{-1}(w)$, then

$$d(QN)(x)(v) = Q(dN(x)(v)) = \int_0^1 dn(x(t))(v(t))dt = w$$

and hence $d(QN)(x)$ is an epimorphism for every x. □

Let us consider the operator $\hat{D} : C^2_{per} \to C^0_{per}$ defined by $\hat{D}(x) := x''$, so its restriction $D : \tilde{C}^2_{per} \to \tilde{C}^0_{per}$ is an isomorphism of Banach spaces. Let $\hat{F} : \mathcal{U}^2 \cap C^2_{per} \to C^0_{per}$ be given by

$$\hat{F}(x) = D(x) - N(x)$$

and consider its restriction $F : M \to \tilde{C}^0_{per}$.

Lemma 3. *The operator F is a Fredholm map of index 0.*

Proof. Clearly $d\hat{F}(x) = D - dN(x)$ is the sum of a Fredholm map of index 0 and a compact operator. Thus, $d\hat{F}(x)$ is a Fredholm linear operator.

The fact that $dF(x) : T_xM \to \tilde{C}^0_{per}$ is Fredholm of index 0 follows from the following general argument on vector spaces:

Let $\hat{L} : X \to Y$ be any linear Fredholm operator and let $L : V \to W$ be its restriction to spaces of finite codimension. Consider the following commutative diagram of Banach spaces

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "L"] & X \arrow[r] \arrow[d, "\hat{L}"] & X/V \arrow[r] \arrow[d, "\iota"] & 0 \\ 0 \arrow[r] & W \arrow[r] & Y \arrow[r] & Y/W \arrow[r] & 0 \end{tikzcd}$$

where ι is canonical.

The long exact sequence given by the snake lemma has only finite dimensional spaces, which implies that $ind(L) - ind(\hat{L}) + ind(\iota) = 0$.

In our particular case, $X/V = Y/W = \mathbb{R}^n$ and $Qd_x\hat{F} = d_xQN$, so $\iota = id$. $\square$

The following lemma is an immediate consequence of the above argument:

Lemma 4. *For $x \in M$, d_xF is an isomorphism if and only if $d_x\hat{F}$ is an isomorphism.*

Combining the previous results, we obtain:

Lemma 5. *The set of regular values of F is a residual set $\Sigma^0 \subseteq \tilde{C}^0_{per}$. Moreover:*

1. *For $y \in \tilde{C}^0_{per}$ $F^{-1}(y) = \hat{F}^{-1}(y)$ and regular values of F are also regular values of $\hat{F}$.*

2. *For each $y \in \Sigma^0$ and $x \in F^{-1}(y)$, $\hat{F}$ is a local diffeomorphism between neighborhoods of $x \in C^2_{per}$ and $y \in C^0_{per}$.*

Next, let us compose $\hat{F}$ with the isomorphism $id \oplus D^{-1}$ so we get

$$\mathcal{F} : C^2_{per} \to C^2_{per}, \mathcal{F}(x) = QNx \oplus D^{-1}PNx + Px.$$

This is the functional associated to equation (6.1). It is clear that u is a solution of (6.1) if and only if $\mathcal{F}(u) = -\lambda H$.

Now we are in conditions to prove the main theorem of this section:

Proof of Theorem 6.3.1: Take $y \in \Sigma^0 \subseteq \tilde{C}^0_{per}$. For every $x \in \mathcal{F}^{-1}(y)$, the function $\mathcal{F}$ is a diffeomorphism between neighborhoods of $x \in C^2_{per}$ and $D^{-1}(y) \in C^2_{per}$. Taking O_x a small neighborhood of x, from the product formula for the Leray-Schauder degree we obtain:

$$deg(\mathcal{F}, O_x, D^{-1}(y)) = \pm 1.$$

Finally, take $\Sigma^2 = D^{-1}(\Sigma^0) \subseteq \tilde{C}^2_{per}$, which is residual since D is an isomorphism. The result now follows by excision and additivity (properties VIII and VII from the introduction). $\square$

6.4 Some examples

Proposition 1. *For the repulsive case, if* $\mathrm{Im}(H)$ *is contained in a line then the equation 6.2 has no solution for any* $\epsilon > 0$.

Proof. Let us take coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. We may assume that the y coordinate of H is zero, that is $H(t) = (H_1(t), 0)$.

Let $z(t) = (x(t), y(t))$ be a solution. Multiply y'' by y and integrate, then

$$-\int_0^1 |y'(t)|^2 dt = \int_0^1 \langle y(t), y''(t)\rangle dt = \int_0^1 \langle y(t), \frac{\epsilon y(t)}{|z(t) - H(t)|^{q+1}}\rangle dt \geq 0,$$

so $y \equiv 0$.

Now, as z lies in the same line as H, we have $H_1 > x$ or $H_1 < x$ for all t so either x'' is positive or negative. This contradicts the fact that z is periodic.

□

Example 2. *In Theorem 4.1.1 the existence of connected components of* $\mathbb{R}^2 \setminus \mathrm{Im}(H)$ *is not necessary to deduce the existence of solutions.*

Indeed, take $H_l(t) = e^{il\sin(t)}$. The curve $H_l \subseteq \mathbb{R}^2$ is degenerate for $l < \pi$, in the sense that $\mathbb{R}^2 \setminus \mathrm{Im}(H_l)$ has no bounded connected components.

For $l = \pi$, we may construct as in 4.4 an open set Ω^* with $deg(F, \Omega^*, 0) = 1$, where F is the function defined in (6.4). Using the continuity of the Brouwer degree, we deduce that $deg(F, \Omega, 0) = 1$ for some $l < \pi$ close to π. This provides a periodic solution of (6.1), although $\mathbb{R}^2 \setminus \mathrm{Im}(H_l)$ has no bounded connected components.

Theorem 6.4.1. *Assume that* $\mathrm{Im}(H)$ *is not contained in a line. Then for* λ *large there exists a solution of the repulsive problem for some reparameterization of* H.

Proof. Take $s_\epsilon : I \to I$ such that s_ϵ is C^1 and increasing for $\epsilon > 0$, and s_0 is piecewise constant, $s_0([0, a]) = t_0, s_0((a, 1]) = t_1$. Define $H_\epsilon(t) = H(s_\epsilon(t))$. It is easy to see that F has one non-degenerate zero x_0 of index -1. Choosing a, t_0, t_1 appropriately we can ensure that x_0 is not in $\mathrm{Im}(H)$ (here we use the fact that $\mathrm{Im}(H)$ is not contained in a line) and we take a small neighborhood U of x_0 not touching H. We obtain a zero $x_1 \in U$ of F for small values of ϵ as a non-degenerate zero of index -1, which is isolated by U. This point is continued to a small solution of (6.1) for λ large enough and such that

$$deg(\mathcal{F}_\lambda, \mathcal{U}, 0) = -1$$

where

$$\mathcal{U} = \{x \in C^2_{per} / \mathrm{Im}(x) \subseteq U\}.$$

□

6.5 Higher dimensions

Theorem 4.1.1 and the extensions Theorem 6.2.1 and Theorem 6.3.1 can be carried out in dimension $n > 2$ without any modification of the proofs, as long as we can find open sets $\Omega \in \mathbb{R}^n \setminus \mathrm{Im}(H)$ such that the degree over Ω of the map $F : \mathbb{R}^n \setminus \mathrm{Im}(H) \to \mathbb{R}^n$ defined in the introduction is well defined and different from zero.

In this section we shall construct open sets with this property. We make use of singular homology theory with coefficients in a field to obtain information about the degree of F.

6.5.1 Dimension 3

For convenience, let us define the function $G : \mathbb{R}^n \setminus \mathrm{Im}(H) \to \mathbb{R}$ given by

$$G(x) := \int_0^1 g(x - H(t))dt = \int_0^1 \frac{1}{|x - H(t)|^{q-1}} dt.$$

Theorem 6.5.1. *If $\mathbb{R}^3 \setminus \mathrm{Im}(H)$ is not simply connected then there exists an open set $\Omega \subseteq \mathbb{R}^3 \setminus \mathrm{Im}(H)$ such that $deg(\nabla G, \Omega, 0) \neq 0$. Moreover, if $r := \dim(H_1(\mathbb{R}^3 \setminus \mathrm{Im}(H)))$ then $r \neq 0$ and $deg(\nabla G, \Omega, 0) \geq r$.*

As a consequence we obtain

Corollary 6.5.1. *If $\mathbb{R}^3 \setminus \mathrm{Im}(H)$ is not simply connected then equation 6.1 admits a periodic solution for λ large enough.*

Remark 1. *The number r counts the self-intersections of H. In the case that $\mathrm{Im}(H)$ is contained in a plane P, it is exactly the number of connected components of $P \setminus \mathrm{Im}(H)$. In this way, we recover Theorem 4.1.1, although not in its full generality.*

Also, it is worth noticing that the fundamental group of $\mathbb{R}^3 \setminus \mathrm{Im}(H)$ distinguishes whether H is or not a non-trivial knot, but the homology does not. In fact, using Alexander duality (Lemma 7), one can show that in most cases r depends on the image of H and not on how this image is embedded in $\mathbb{R}^3$.

Our proof of Theorem 6.5.1 will require several lemmas; all of them shall be stated in $\mathbb{R}^n$.

Lemma 2. *The function G is C^∞ smooth in $\mathbb{R}^n \setminus \mathrm{Im}(H)$.*

Lemma 3.

- $G(x) \to \infty$ *when* $x \to x_0 \in H$.
- $G(x) \to 0$ *when* $|x| \to \infty$.
- $G(x) < g(d(x, H))$

Lemma 4. *The function G is sub-harmonic for $q > n - 1$ and harmonic for $q = n - 1$. In particular, it has no local maxima in the interior of the domain of definition. In consequence, if $U \subseteq \mathbb{R}^n$ is an open and bounded set such that $H \cap \overline{U} = \emptyset$ then it attains its maximum at the boundary.*

Proof. It follows directly from the fact that g is sub-harmonic for $q > n-1$ and harmonic for $q = n - 1$, and that $\Delta G = \int_0^1 \Delta g(x - H(t))dt$. □

Lemma 5. *If B is a large ball centered at the origin then $-\nabla G$ is homotopic to the identity in ∂B by a homotopy of non vanishing vector fields.*

Proof. As

$$dG(x)(x) = \int_0^1 dg(x - H(t))(x)dt$$

and

$$dg(x - H(t))(x) = -(q - 1)|x - H(t)|^{-(q+1)}(\langle x, x\rangle - \langle H(t), x\rangle),$$

then for $|x| > \|H\|_\infty$ the value of $\langle \nabla G(x), x\rangle$ is negative.

□

Now some general lemmas

Lemma 6. *For $n \geq 3$ the set $\mathbb{R}^n \setminus \mathrm{Im}(H)$ is arcwise-connected.*

Proof. It follows as an application of transversality:

If a smooth curve γ joins two points not in $\mathrm{Im}(H)$, it can be homotoped with fixed end-points in $\mathbb{R}^3$ to be transversal to H. Since the dimensions of the images of the differentials of two curves sum at most 2, transversality in this case means γ and H are disjoint. $\square$

Lemma 7. *(Alexander duality, see [62, p. 296, thm 16])*

Let U be an open set in S^n with smooth boundary. If k_* and e_* denote the reduced Betti numbers of U and $S^n \setminus U$ respectively then

$$k_q = e_{n-1-q}.$$

For convenience, if $U \subset \mathbb{R}^n$ is open and bounded and $\phi : \partial U \to \mathbb{R}^n \setminus \{0\}$ is continuous, we shall use the notation $deg(\phi, \partial U, 0) := deg(\hat{\phi}, U, 0)$ where $\hat{\phi} : \overline{U} \to \mathbb{R}^n$ is any continuous extension of ϕ.

Lemma 8. *(Hopf, see [25, Satz VI])*

Let U be an open bounded set in $\mathbb{R}^n$ with smooth boundary.
Let $n : \partial U \to \mathbb{R}^n$ be the outer-pointing unit normal vector field. Then $deg(n, \partial U, 0) = \chi(U)$, where χ denotes the Euler characteristic.

Proof of Theorem 6.5.1: We will construct an open set U with the following properties:

- ∂U is smooth.
- $U \supseteq \mathrm{Im}(H)$.
- G is constant and $\nabla G \neq 0$ in ∂U.
- $\chi(U) \leq 1 - r$ where $r := dim(H^1(\mathbb{R}^3 \setminus \mathrm{Im}(H)))$.

Once we have this set U, we may notice that $-\nabla G$ is orthogonal to ∂U, so clearly $deg(-\nabla G, \partial U, 0) = deg(n, \partial U, 0)$ where n is the outer-pointing unit normal vector field.

Then using Lemma 8 it follows that $deg(-\nabla G, \partial U, 0) = \chi(U) \leq 1 - r$. Next, take a large ball B given by Lemma 5 and observe that $deg(-\nabla G, \partial B, 0) = 1$. Finally, as $\mathrm{Im}(H) \subseteq U$, it follows that G is well defined in $\Omega = B \setminus U$ and $deg(-\nabla G, \Omega, 0) \geq r$.

Now we may construct U as follows. For convenience, we shall regard $\mathbb{R}^3$ as embedded in S^3, and call $N \in S^3$ the north pole. We remark that the function G extends continuously to N by setting $G(N) = 0$.

It follows from the hypothesis that there exist smooth curves $\gamma_1 \ldots \gamma_r \in \mathbb{R}^3 \setminus \mathrm{Im}(H)$ generating $H_1(\mathbb{R}^3 \setminus \mathrm{Im}(H))$ as a basis. By Lemma 6, each γ_i is connected to N by another curve δ_i. Since G is continuous in $\bigcup \gamma_i \cup \delta_i$, then it is bounded there by a number α_0. By Sard's lemma, there exists a regular value $\alpha > \alpha_0$.

Next, take V the connected component of $\{G < \alpha\}$ that contains N and let $U = \overline{V}^c$. U and V are manifolds with common boundary in S^3. Moreover, U has a finite number of connected components that are disjoint manifolds with boundary. Obviously $\mathrm{Im}(H) \subseteq U$.

We claim that U is in fact connected. Indeed, let U' be a connected component of U. Then $U' \cap \mathrm{Im}(H) \neq \emptyset$: otherwise, from Lemma 4 we deduce that $G|_{U'}$ attains its maximum at some $x_0 \in \partial U' \subseteq \partial U$ so $G(x_0) = \alpha$. But also $G \leq \alpha$ in V, so x_0 is a local maximum of

G in $\mathbb{R}^3 \setminus \mathrm{Im}(H)$, a contradiction since α is a regular value. As $\mathrm{Im}(H) \subseteq U$ is connected, we conclude that U is connected.

Since $G = \alpha$ in ∂U and $G < \alpha$ in $\bigcup \gamma_i \cup \delta_i$, it follows that $\gamma_i \subseteq V$ and that U is bounded.

Notice that there is a homomorphism induced by the inclusion $H_1(V) \to H_1(\mathbb{R}^3 \setminus \mathrm{Im}(H))$, which sends $[\gamma_i]_{H_1(V)}$ to the generators, so it is surjective and hence $e_1 = \dim(H_1(V)) \geq r$ (using the notation of Lemma 7).

Using the Alexander duality (Lemma 7), from the fact that V is connected we obtain $k_2 = e_0 = 0$. Also, $k_0 = 0$ because U is connected, and $k_1 = e_1 \geq r$. The Euler characteristic of U is $\chi(U) = 1 + k_0 - k_1 + k_2 \leq 1 - r$, and this completes the proof. □

6.5.2 Dimension $n > 3$

In this section we shall prove, for $n > 3$, the existence of a set U as in the proof of Theorem 6.5.1.

Theorem 6.5.2. *If H is an embedding and $q > 2$ then there exists an open set $U \subseteq \mathbb{R}^n$ such that $H \subseteq U$ and $deg(\nabla G, U, 0) = 0$.*

As a consequence, we may construct an open set Ω such that

$$deg(-\nabla G, \Omega, 0) = 1$$

thus proving that equation 6.1 has a periodic solution for λ large enough.

Lemma 9. *If H is an embedding and $q > 1$ then the critical points of G do not accumulate in H.*

Proof. Assume, by contradiction, that there is a sequence x_n of critical points of G accumulating somewhere in H. Without loss of generality we may assume $x_n \to x_0 \in \mathrm{Im}(H)$. Fix $t_n \in I$ such that the distance from x_n to $\mathrm{Im}(H)$ is realized in $H(t_n)$. Let $v_n = x_n - H(t_n)$ and $\lambda_n = |v_n|^{-1}$. Again without loss of generality, we may assume $\lambda_n v_n \to y \in \mathbb{R}^n$ and $t_n \to t_0$. By periodicity we may also assume $t_0 = 0$. Let us compute $dG(x_n)$:

$$0 = dG(x_n) = \int_0^1 dg(x_n - H(t))dt = \lambda_n^q \int_0^1 dg(\lambda_n x_n - \lambda_n H(t))dt$$

$$= \lambda_n^{q-1} \int_0^{\lambda_n} dg(\lambda_n x_n - \lambda_n H(\frac{s}{\lambda_n} + t_n))ds$$

$$= \lambda_n^{q-1} \int_0^{\lambda_n} dg(\lambda_n (x_n - H(t_n)) - \lambda_n (H(\frac{s}{\lambda_n} + t_n) - H(t_n))ds.$$

It follows from the assumptions and the regularity of H that

$$\lambda_n(x_n - H(t_n)) \to y \qquad \text{and} \qquad \lambda_n(H(\frac{s}{\lambda_n} + t_n) - H(t_n)) \to sH'(t_0)$$

for every s. In order to establish the convergence of the integral, let us estimate $\lambda_n x_n - \lambda_n H(\frac{s}{\lambda_n} + t_n)$.

Consider the continuous function

$$\gamma(t,s) := \begin{cases} \frac{|H(t+s)-H(t)|}{|s|} \text{ if } t \neq s \\ |H'(t)| \text{ if } t = s \end{cases}$$

As H is an embedding, $\gamma(s,t) > 0$ for all s and t then by compactness $|H(t+s)-H(t)| \geq \epsilon|s|$ for all s, t and some small $\epsilon > 0$.

Using this we obtain

$$\left|\lambda_n x_n - \lambda_n H\left(\frac{s}{\lambda_n} + t_n\right)\right| \geq \lambda_n \left(\left|H\left(\frac{s}{\lambda_n} + t_n\right) - H(t_n)\right| - |v_n|\right) \geq \epsilon|s| - 1.$$

As $H(t_n)$ is the nearest point to x_n, the left hand side is also bounded from below by $\lambda_n|v_n| = 1$ so

$$|dg(\lambda_n x_n - \lambda_n H(\frac{s}{\lambda_n} + t_n))| \leq \max\{1, (\epsilon|s| - 1)^{-q}\}(q-1)$$

which is integrable in $\mathbb{R}$ for $q > 1$.

By dominated convergence we conclude that $\lambda_n^{-(q-1)} dG(x_n)$ converges to $\int_0^\infty dg(y - sH'(t_0))ds$. Notice that $\langle H'(t_n), \lambda_n v_n\rangle = 0$ and, taking limits, $\langle H'(t_0), y\rangle = 0$ so the integral can be explicitly calculated and is different from zero, a contradiction. $\square$

Proof of Theorem 6.5.2: Using Lemma 3 we can take $\alpha > 0$ large enough so that $U := \{G > \alpha\}$ is close to $\mathrm{Im}(H)$. By Lemma 9 we can ensure that there are no critical points of G in U.

Applying the Morse deformation lemma to G at level $+\infty$ we deduce that $\mathrm{Im}(H)$ is a deformation retract of U so $deg(-\nabla G, \partial U, 0) = \chi(U) = \chi(H) = \chi(S^1) = 0$.

$\square$

Remark 10. *As the set of embeddings of S^1 in $\mathbb{R}^n$ for $n \geq 3$ is open in the C^1 topology and dense in the C^∞ topology, Theorem 6.5.2 is a result about the generic situation.*

6.6 On Morse functions and knots

In this section we prove that, generically, the function G has some differential structure that allows us to obtain more solutions when H is a knotted curve in $\mathbb{R}^3$. The results involving knot theory will require G to be a Morse function and in order to emphasize the dependence on H, we shall use the notation $G_H(x) := \int_0^1 g(x - H(t))dt$. So firstly we shall prove the following

Theorem 6.6.1. *There exists a residual set $\Sigma \subseteq C^2_{per}(I, \mathbb{R}^n)$ such that if $H \in \Sigma$ then G_H is a Morse function.*

To this end let us consider the parametric version of the Sard-Smale's theorem. Firstly, we need the following

Definition 1. *Let $f : X \to Y \supset W$ be a smooth function between Banach manifolds and W a submanifold. We shall say that f is transversal to W and write $f \pitchfork W$ if for each $x \in f^{-1}(W)$ we have that the composition*

$$T_x X \to T_{f(x)} Y \to T_{f(x)} Y / T_{f(x)} W$$

is a submersion (namely, it is surjective and its kernel is complemented).

Theorem 6.6.2. *(see [9])*

Let $f : X \times B \to Y \supset W$ be a smooth function between Banach manifolds and W a submanifold. Assume that X is finite dimensional and W is finite codimensional.

If $f \pitchfork W$ then $f_b \pitchfork W$ for b in a residual subset of B, where f_b is the function $x \mapsto f(x, b)$.

We write $E = \mathbb{R}^n$ the Euclidean space and consider the function

$$DG : C^2_{per}(I, \mathbb{R}^n) \times E \to E^*$$

$$DG(H, x) = \int_0^1 dg(x - H(t))dt$$

where, as before, $g(x) = \frac{1}{|x|^{q-1}}$.
Proof of Theorem 6.6.1: The theorem is consequence of the following two lemmas:

Lemma 2. *The function DG is transversal to $0 \in E^*$.*

Proof. Notice that $d^2g(x) \in L(E, E^*)$ is always invertible. For given $\gamma \in E^*, H \in C^2, x \in E$ take $\hat{H}(t) = (d^2g(x - H(t)))^{-1}(\gamma)$, $\hat{H} \in C^2(S^1, \mathbb{R}^n)$. Then $d(DG)(H, x)(\hat{H}, 0) = -2\pi\gamma$. Since $Ker(d(DG))$ is finite codimensional, it splits. Then DG is transversal to 0 (notice that $DG^{-1}(0)$ is a differentiable Banach manifold). □

Lemma 3. *DG_H is transversal to 0 if and only if G_H is a Morse function.*

Proof. Note that $dG_H(x) = DG(H, x)$ so DG_H is transversal to $0 \in E^*$ if an only if dG_H is transversal to 0, if and only if for each critical point x of G_H, $d^2_x G_H = d_x(dG_H) \in L(E, E^*)$ is invertible. This G_H is a Morse function. □

□

Now Morse functions may be used as follows:

Lemma 4. *Let $f : U \to R$ be a Morse function and let x_0 be a critical point of index λ. Then there exists a neighborhood V that isolates x_0 as a critical point, and such that*

$$deg(\nabla f, V, 0) = (-1)^\lambda.$$

Proof. Using the Morse lemma we obtain a chart $(\hat{V}, x_1, \ldots, x_n)$ such that $f(x_1, \ldots, x_n) = -\sum_{i<\lambda} x_i^2 + \sum_{i \geq \lambda} x_i^2$. Take a square inside $\hat{V}$ of the form

$$V := \{(x_1, \ldots, x_n) : \|(x_1, \ldots, x_\lambda)\| \leq \epsilon, \|(x_{\lambda+1}, \ldots, x_n)\| \leq \epsilon\}.$$

This is the required neighborhood. □

As a consequence, when $H \in \Sigma$ we obtain at least one solution of (6.1) for each critical point of G_H.

Remark 5. *In the present case, in which G is a Morse function, the use of Leray-Schauder degree may be avoided by simply invoking the implicit function theorem for Banach spaces.*

The following theorems give us relations between the number of critical points of G and some knot invariants of H (see [3] for a general overview on knot theory). Although its statement is contained in that of Theorem 6.6.4, we shall give an independent proof because it is much simpler, it uses different tools and part of it shall be used later.

Theorem 6.6.3. *Assume H is a non-trivial knot embedded in $\mathbb{R}^3$ and that G is a Morse function. Then G has at least 3 critical points in $\mathbb{R}^3$.*

Proof. Changing the function G near $\mathrm{Im}(H)$ we may assume that $-G$ is a Morse function in S^3 with a global minimum $m \in \mathrm{Im}(H)$ and an index 1 critical point in $p \in \mathrm{Im}(H)$. Consider the Morse decomposition of S^3 through the function $-G$. Call $M^c = \{x / -G(x) \leq c\}$ and $M^{-\infty} = \mathrm{Im}(H) = S^1$.

We have a cell complex given by the Morse decomposition of S^3.

$$0 \to H_3(X_3, X_2) \to H_2(X_2, X_1) \to H_1(X_1, X_0) \to H_0(X_0) \to 0$$

$$0 \to \mathbb{Z}^{m_3} \oplus N.\mathbb{Z} \to \mathbb{Z}^{m_2} \to \mathbb{Z}^{m_1} \oplus H.\mathbb{Z} \to \mathbb{Z}^{m_0} \oplus m.\mathbb{Z} \to 0$$

with homology $H_*(S^3) = \{\mathbb{Z}, 0, 0, \mathbb{Z}\}$.

Here m_i is the number of critical points of $-G$ of index i in $\mathbb{R}^3 \setminus \mathrm{Im}(H)$ and N is the north pole, where we have a global maximum. Since G is subharmonic we have $m_0 = 0$.

Computing the Euler characteristic we have

$$\chi(S^3) = \chi(\mathrm{Im}(H)) + m_0 - m_1 + m_2 - (m_3 + 1)$$

$$0 = 0 - m_1 + m_2 - (m_3 + 1)$$

$$m_2 - m_1 = m_3 + 1 \geq 1. \tag{6.9}$$

We see that the only way to have just 1 critical point in $\mathbb{R}^3$ is $m_2 = 1, m_1 = 0, m_3 = 0$. All other possibilities give 3 or more. Then the cell complex reduces to

$$0 \to N.\mathbb{Z} \xrightarrow{0} e.\mathbb{Z} \to H.\mathbb{Z} \xrightarrow{0} m.\mathbb{Z} \to 0$$

where we think of H as the curve generated by the index 1 critical point p.

We know that $d(N) = 0$ because $H^3(S^3) = \mathbb{Z}$, and $d(H) = m - m = 0$.

Now consider the attaching map $f : S^1 = \partial e \to X_1$ corresponding to the unique 2-cell e.

Since $d : e.\mathbb{Z} \to H.\mathbb{Z}$ is an isomorphism we must have $d(e) = \pm H$.

Now we may consider $f : D^2 \to S^3$ as the inclusion of the 2-cell. Composing with the isotopy generated by the negative gradient, we may assume that $G(f(S^1))$ is uniformly large so $\mathrm{Im}(f)$ is uniformly close to $\mathrm{Im}(H)$ and thus lies in a tubular neighborhood of $\mathrm{Im}(H)$. It is clear that f is homotopic to H or H^{-1} inside that tubular neighborhood.

It follows that $f(S^1)$ is a satellite knot whose companion is H. Since f is homotopically nontrivial in the tubular neighbourhood we see that $f(S^1)$ is a nontrivial companion of a non trivial knot and is therefore itself non trivial. This may be deduced for example from the genus formula for satellite knots. But this is a contradiction, since $f(S^1)$ is the boundary of an embedded 2-cell in $\mathbb{R}^3$. □

Remark 6. *A lower bound for the number of solutions may be also obtained by considering the presentation of the knot group given by the Morse decomposition of the knot complement. Namely, a Morse function in $S^3 \setminus \mathrm{Im}(H)$ gives a presentation of $\pi_1(S^3 \setminus \mathrm{Im}(H))$ with one generator for each critical point of index 1 and one relation for each critical point of index 2. Thus an obvious lower bound for the number of critical points would be the minimal numbers of generators (and relations) of a presentation of the group.*

A knot invariant that seems to be closely related to the minimal number of critical points of G is the following:

Definition 7. *Let $K \subseteq S^3$ be a knot. A tunnel is an embedded arc with endpoints in K. The tunnel number of K, $t(K)$ is the minimal number of tunnels t_i such that if $N = N(K, t_i)$ is a tubular neighbourhood of the knots and the tunnels, then $S^3 \setminus N$ is a handlebody. We call this set of tunnels a tunnel decomposition.*

The only knot with tunnel number zero is the trivial knot, so Theorem 6.6.3 is a particular case of our next result.

Theorem 6.6.4. *Assume H is a knot embedded in $\mathbb{R}^3$ and that G is a Morse function. Then G has at least $2t(H) + 1$ critical points and consequently equation (6.1) has at least $2t(H) + 1$ periodic solutions for λ large enough.*

Remark 8. *If a knot has tunnel number one, it will be a one-relator knot and therefore prime [51]. Then for any composite knot we will have at least 5 critical points. Also, it is worth noting that a decomposition with t tunnels gives a presentation of $\pi_1(S^3 \setminus \mathrm{Im}(H))$ with t relations so the present theorem provides a better bound for the number of solutions than the one stated in the previous remark.*

<u>*Proof of Theorem 6.6.4*</u>: By the Kupka-Smale theorem [10, pp 159, thm 6.6], we may perturb G (preserving the critical points and its indices), in order to obtain a Morse-Smale function.

For each critical point p of $-G$ of index 1, denote $\gamma_p : \mathbb{R} \to S^3$ the embedding of the unstable manifold.

Clearly γ_p is connected with $\mathrm{Im}(H)$ because G is Morse-Smale. Now take a tubular neighborhood U of $\mathrm{Im}(H) \cup \mathrm{Im}(\gamma_{p_1}) \cup \ldots \cup \mathrm{Im}(\gamma_{p_k})$. We will show that $S^3 \setminus U$ is a handlebody then k, the number of γ curves, is greater than or equal to $t(H)$.

For each critical point q of $-G$ of index 2, denote $\delta_q : \mathbb{R} \to S^3$ the embedding of the stable manifold. Again, δ_q connects with $N \in S^3$ because G is Morse-Smale. Obviously, $\gamma_{p_i}, \delta_{q_j}$ are disjoint.

We shall construct a tubular neighborhood V of $\{N\} \cup \mathrm{Im}(\delta_{q_1}) \cup \ldots \cup \mathrm{Im}(\delta_{q_s})$. V is a handlebody and we will show that $S^3 \setminus U$ retracts to V.

Consider the positive gradient flow of $-G$ (i.e. $-\nabla G$) denoted by $\phi^t(x)$. Take a point $x \in S^3 \setminus U$. If the orbit of x converges to a critical point q then it belongs to the unstable manifold of $-G$ at q. We know that x cannot belong to any γ_{p_i} so q is a critical point of $-G$ of index 2 or $q = N$. We deduce that $q \in V$.

Orbits always converge to critical points, then every point in $(S^3 \setminus U) \setminus V$ must eventually enter at V. If we manage to construct V so that its boundary is transversal to the flow, then, by the inverse function theorem, the arrival time at V is a continuous function $t(x)$. This allows us to construct the required deformation as

$$\psi(t, x) = \phi^{\min\{t, t(x)\}}(x).$$

Finally, using formula (6.9) we obtain at least $t(H) + 1$ critical points of index 2.

Now we shall construct V. With this purpose, take a critical point q of index 2 and a Morse like system of coordinates x_i in a neighborhood of q of the form $U = [-1, 1] \times D^2$ where $\{\pm 1\} \times D^2$ is the exit set. We know that δ_q converges to the attractor N.

Consider

$$\alpha^{\pm} : \mathbb{R}_+ \times D^2 \to S^3$$

$$\alpha^{\pm}(t, x) = \phi^t(\pm 1, x)$$

Let us show that $\alpha^{\pm}$ is an embedding.

Clearly $t \mapsto \phi^t(x)$ is injective because f is strictly decreasing in the integral curves. Moreover, points leaving U can never return since f restricted to the exit set is less or equal to f restricted to the entrance set. We conclude that the orbits of the points in $\{\pm 1\} \times D^2$ are all different and then $\alpha^{\pm}$ is injective. It is clear, also, that the image is open and α is an embedding.

Notice that α sends $\mathbb{R}_+ \times \{x\}$ to integral lines. Now we may easily construct a neighborhood V_q of $\mathbb{R}_+ \times \{0\}$ with boundary transversal to the horizontal flow $t, x \mapsto t + s, x$. Call $F = V_q \cap \{0\} \times D^2$. It follows that $\alpha^+(V_q) \cup ([-1, 1] \times F) \cup \alpha^-(V_q)$ is a neighborhood of δ_q. Taking the union of these neighborhoods and an attracting neighborhood of N we obtain the set V with the desired properties. □

6.7 Links and the restricted n-body problem

Consider the equation

$$\begin{cases} z''(t) = \sum_{i=1}^{n-1} \nabla g(z - \lambda p_i(t)) + \lambda h(t) \\ z(0) = z(1), \quad z'(0) = z'(1) \end{cases} \tag{6.10}$$

for $z \in \mathbb{R}^3$, where $p_i : I \to \mathbb{R}^3$ are arbitrary periodic functions.

This equation describes the motion of a particle z under the force of the gravitational attraction of $n - 1$ bodies moving along large periodic trajectories $\lambda p_i(t)$, and under an arbitrary force $\lambda h(t)$ of comparable intensity. The letter n stands for the number of bodies and not for the dimension, which is now equal to 3.

As before, assume $\overline{h} = 0$ so we are able to make the change of variables

$$z = \lambda(x - H(t))$$

where H is a periodic second primitive of $-h$.

With the new variables, the equation is transformed into:

$$\begin{cases} x''(t) = \epsilon \sum_{i=1}^{n-1} \nabla g(x - (H(t) + p_i(t))) \\ x(0) = x(1), \quad x'(0) = x'(1). \end{cases} \tag{6.11}$$

Let $G_t : W \to \mathbb{R}$ be given by

$$G_t(x) = \sum g(x - (H(t) + p_i(t))),$$

then clearly

$$\nabla G_t(x) = \sum \nabla g(x - (H(t) + p_i(t)))$$

so the equation becomes

$$\begin{cases} x''(t) = \epsilon \nabla G_t(x) \\ x(0) = x(1), \quad x'(0) = x'(1). \end{cases} \tag{6.12}$$

Thus, solutions of (6.10) are related to the function $G(x) = \int_0^1 G_t(x)dt$ in the same manner as in the previous sections.

The set in which $G = \infty$ is the union of the curves $k_i = H + p_i$, so we define $K = \bigcup_{i=1}^{n-1} Im(k_i)$.

Theorem 6.5.1 may easily be generalized as follows:

Theorem 6.7.1. *Let $r = dim(H^1(\mathbb{R}^3 \setminus K))$. Then there exists an open set $\Omega \subseteq \mathbb{R}^3 \setminus K$ such that $deg(\nabla G, \Omega, 0) \geq r - n + 2$.*

Proof. The proof follows exactly as in the proof of Theorem 6.5.1 except that now the set U is not connected. From the discussion in the mentioned proof, it follows that every connected component of U touches K and thus U has at most $n-1$ connected components. We deduce that $\chi(U) \leq n - 1 - r$ and so completes the proof. □

Remark 1. *Notice that the trajectories of the curves k_i may have self-intersections and intersect each other. In the statement of the preceding theorem, the number $r - n + 2$ may be replaced by $r - c + 1$, where c is the number of connected components of K.*

Theorem 6.6.4 may be generalized as follows:

Theorem 6.7.2. *Assume K is a link embedded in $\mathbb{R}^3$ and that G is a Morse function. Then G has at least $2t(K) + 1$ critical points.*

Remark 2. *It is easy to see that G is a Morse function for 'generic' K.*

The proof requires no modification at all with respect to that of Theorem 6.6.3, but some explanation about formula (6.9) is needed. Indeed, now we have to represent each strand of the link K by one 0-cell and one 1-cell attached. The cell complex has to be replaced by

$$0 \to \mathbb{Z}^{m_3} \oplus N.\mathbb{Z} \to \mathbb{Z}^{m_2} \to \mathbb{Z}^{m_1} \oplus \mathbb{Z}^{n-1} \to \mathbb{Z}^{m_0} \oplus \mathbb{Z}^{n-1} \to 0$$

so formula (6.9) still holds.

We are already in conditions to establish the main results of this section

Theorem 6.7.3. *Define r as in Theorem 6.7.1. If $r \geq n-1$, then for λ large enough there exist at leasts one periodic solution of (6.10), and generically $r - n + 2$ distinct solutions.*

Theorem 6.7.4. *Assume that K is a link embedded in $\mathbb{R}^3$ and that G is a Morse function. Then equation (6.10) has at least $2t(K) + 1$ distinct solutions for λ large enough. In particular, it must have at least $2n - 3$ distinct solutions even if K is the unlink.*

Chapter 7

On resonant elliptic systems with rapidly rotating nonlinearities

7.1 Introduction

Finally we turn our attention to a partial differential equation which is a generalization of the Landesman-Lazer theorem. The problem becomes quite interesting because we shall see effects of the geometry of the domain of definition, apart from the geometric conditions for the non-linear part.

We consider the Neumann problem for the elliptic system

$$\begin{cases} \Delta u + g(u) = p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $p \in C(\overline{\Omega}, \mathbb{R}^n)$ has zero average, i.e.

$$\overline{p} := \frac{1}{\text{meas}(\Omega)} \int_\Omega p = 0.$$

This problem has been extensively studied. Due to its resonant structure, it is still an open problem to characterize the range of the semilinear operator $\Delta u + g(u)$, i.e. the set of all possible functions p such that (7.1) admits at least one weak solution. For a single equation it takes the form of the well-known Landesman-Lazer theorem stated at the introduction in 2.2.1.

This theorem has been generalized in several ways. On the one hand, analogous versions have been obtained for nonlinear operators of p-Laplacian or Φ-Laplacian type. On the other hand, the assumption on the existence of limits can be relaxed. For instance, it is easy to prove that the result is still valid under the weaker condition

$$g(-u)g(u) < 0 \quad \text{for } \ u \geq R \tag{7.2}$$

for some large enough R. From a topological point of view, condition (7.2) says two different things: firstly, that g does not vanish outside a compact set; secondly, that its Brouwer degree over the interval $(-R, R)$ is different from zero when R is large. Thus, one might believe that a natural extension of the preceding result for a system of n equations could be to require that

$$g(u) \neq 0 \quad \text{for } |u| \geq R \tag{7.3}$$

and

$$\deg(g, B_R(0), 0) \neq 0, \tag{7.4}$$

where 'deg' refers to the Brouwer degree of the function $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $B_R(0)$ is the open ball of radius R centered at the origin. For $N = 1$ this possible extension was analyzed by Ortega and Sánchez in [53], where they constructed an example showing that (7.3) and (7.4) are not sufficient to guarantee the existence of a solution. Specifically, for $n = 2$ they defined, in complex notation,

$$g_0(z) := \frac{z}{\sqrt{1 + |z|^2}} e^{i\mathrm{Re}(z)},$$

$$g(z) := g_0(z) - \gamma \quad \text{with } 0 < \gamma < 1, \tag{7.5}$$

and showed that problem

$$z'' + g(z) = \lambda \sin t$$

does not have a 2π-periodic solution when λ is large enough.

Already in the early seventies Nirenberg [50] proved the following generalization of the Landesman-Lazer result for systems:

Theorem 7.1.1 (Nirenberg)**.** *Let $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ be bounded. Assume that the radial limits*

$$g_v := \lim_{s \to +\infty} g(sv)$$

exist uniformly and $g_v \neq 0$ for every $v \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Furthermore, assume that (7.4) *holds for R sufficiently large. Then for each $p \in C(\overline{\Omega}, \mathbb{R}^n)$ with $\overline{p} = 0$ problem* (7.1) *admits at least one solution.*

As for a single equation, it is possible to replace the hypothesis on existence of limits at infinity by an interpretation of (7.2) for $n > 1$ which is more accurate than (7.3)-(7.4). This was done by Ruiz and Ward in [58]. The following result is adapted from their main theorem.

We write $B_r(v) := \{x \in \mathbb{R}^n : |x - v| < r\}$ and $\overline{B}_r(v)$ for its closure, and $\mathrm{co}(A)$ for the convex hull of a subset A of $\mathbb{R}^n$.

Theorem 7.1.2 (Ruiz-Ward)**.** *Assume that $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ is bounded and satisfies the following condition:*
For each $r > 0$ there exists $R > r$ such that

$$0 \notin \mathrm{co}(g(\overline{B}_r(v))) \qquad \text{if } v \in \mathbb{R}^n \text{ and } |v| = R. \tag{7.6}$$

Then, if (7.4) *holds, problem* (7.1) *admits at least one solution for each $p \in C(\overline{\Omega}, \mathbb{R}^n)$ with $\overline{p} = 0$.*

This result was established in [58] for a system of ordinary differential equations with periodic boundary conditions although, as the authors mention, its generalization to the Neumann problem (7.1) in a bounded smooth domain of higher dimension is straightforward. Theorem 7.1.2 still holds if g is unbounded but *sublinear*, that is,

$$\frac{g(u)}{|u|} \to 0 \qquad \text{as } |u| \to \infty. \tag{7.7}$$

In a recent work [8], the result has been extended also for singular g.

The role of condition (7.6) becomes clear when (7.1) is solved by Leray-Schauder degree methods. Indeed, the key step for proving Theorem 7.1.2 consists in showing that, for $0 < \lambda \leq 1$, problem

$$\begin{cases} \Delta u = \lambda(p - g(u)) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.8}$$

has no solution on $\partial\mathcal{U}$, where

$$\mathcal{U} := \{u \in C(\overline{\Omega}, \mathbb{R}^n) : \|u - \overline{u}\|_\infty < r, \ \ |\overline{u}| < R\}$$

for some suitable r and the corresponding R given by condition (7.6). An appropriate value of r is obtained after observing that, if u is satisfies (7.8), then

$$\|\nabla u\|_\infty \leq Q\|\Delta u\|_\infty \leq Q\left(\|p\|_\infty + \|g\|_\infty\right) \tag{7.9}$$

for some constant Q, independent of p and g (but depending on Ω). This yields the a priori bound $\|u - \overline{u}\|_\infty < r$ for r large enough. Next, if we assume that $|\overline{u}| = R$, we obtain a contradiction as follows: since the convex hull of $g(\overline{B}_r(\overline{u}))$ is compact, the geometric version of the Hahn-Banach theorem asserts that there exists a hyperplane H passing through the origin such that $g(\overline{B}_r(\overline{u})) \subset \mathbb{R}^n \setminus H$. As $\|u - \overline{u}\|_\infty < r$, we conclude that $g(u(x))$ remains on the same side of H for every $x \in \Omega$. This contradicts the fact that $\int_\Omega g(u(x))\, dx = \int_\Omega p(x)\, dx = 0$.

Condition (7.6) sheds some light on the counter-example (7.5) of Ortega and Sánchez where the 'pathological' g rotates rapidly. Condition (7.6) does not allow fast rotation, as it forces $g(\overline{B}_r(v))$ to remain at one side of a hyperplane for $v \in \partial B_R(0)$.

One may ask, in first place, if rotation has the same effect as shown in [53] for higher dimensions. We shall prove that, indeed, the example by Ortega and Sánchez may be extended as follows:

Proposition 1. *Let $\phi : \overline{\Omega} \to \mathbb{R}$ be a non-constant eigenfunction of $-\Delta$ with Neumann boundary condition and let $p_\lambda = (\lambda\phi, 0)$. Then, problem*

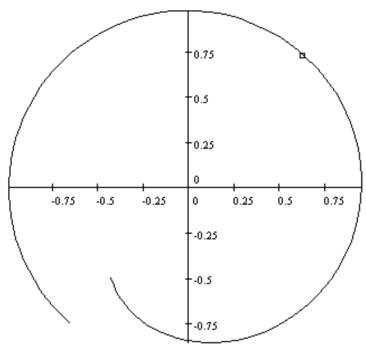
$$\begin{cases} \Delta u + g(u) = p_\lambda & \textit{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\Omega, \end{cases} \tag{7.10}$$

with g as in (7.5) *has no solution for λ large enough.*

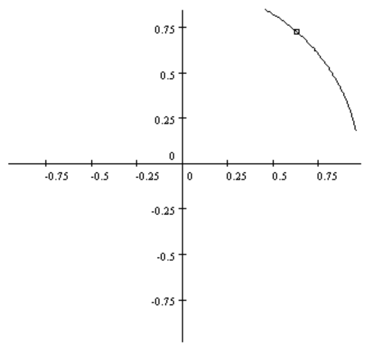
A closer look at the function (7.5) shows that the effect of rotation appears only when we consider the image of the whole ball $B_r(z)$ under the function g, whereas the image of a vertical strip

$$\mathcal{S}(z) := \{u \in B_r(z) : |\mathrm{Re}(u) - \mathrm{Re}(z)| < \delta\}$$

under g remains in the same half-plane for δ small enough.



$$g_0(4+t), \qquad t \in [-\pi, \pi]$$



$$g_0(4+it), \;\; t \in [-\pi, \pi]$$

This suggests replacing assumption (7.6) in Theorem 7.1.2 by a weaker one. We shall prove that $g(B_r(v))$ can be allowed to intersect all the hyperplanes passing through the origin, provided that, for some particular H_v, the function g maps some 'strip' in $B_r(v)$ sufficiently far away from H_v.

To make this statement precise, we need to introduce some notation. A *strip of width* 2δ in $B_r(v)$ is a set

$$\mathcal{S}(v) := \{u \in B_r(v) : |\langle u - v, \xi_v \rangle| < \delta\},$$

for some $\xi_v \in \mathbb{S}^{n-1}$. We consider the metric in Ω given by

$$d(x,y) := \inf\{\text{length}(\gamma) : \gamma \text{ is a smooth curve in } \Omega \text{ joining } x \text{ and } y\}.$$

The open ball of radius ρ for this metric will be denoted by $U_\rho(x)$, i.e.

$$U_\rho(x) := \{y \in \Omega : d(x,y) < \rho\}.$$

Further, we define

$$c(\rho) := \inf_{x\in\Omega} \text{meas}(U_\rho(x)).$$

Assume that (7.7) holds. For $\alpha > 1$ we choose $K > 0$ as follows: fix $\varepsilon \in [0, +\infty)$ such that

$$M_\varepsilon := \sup_{u\in\mathbb{R}^n} (|g(u)| - \varepsilon\,|u|) < \infty \tag{7.11}$$

and $Q\varepsilon\,\mathrm{diam}_d(\Omega)(1+\alpha) < 1$, where Q is the constant in (7.9) and $\mathrm{diam}_d(\Omega)$ is the diameter of Ω with respect to the metric d. Next, choose $K > 0$ such that

$$K > \frac{Q(\|p\|_\infty + M_\varepsilon)}{1 - Q\varepsilon\,\mathrm{diam}_d(\Omega)(1+\alpha)} > 0.$$

Our main result is the following:

Theorem 7.1.3. *Assume that $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfies (7.7). Let $p \in C(\overline{\Omega}, \mathbb{R}^n)$ with $\overline{p} = 0$, and $\alpha > 1$. Fix $K > 0$ as above and set $r := K\,\mathrm{diam}_d(\Omega)$. Assume there exists a domain $D \subset B_{\alpha r}(0)$ with the following properties:*
($\mathbf{D}_1$) *For every $v \in \partial D$ there exist a hyperplane H_v passing through the origin and a strip $\mathcal{S}(v)$ of width 2δ in $B_r(v)$ such that $g(\mathcal{S}(v)) \subset \mathbb{R}^n \setminus H_v$ and*

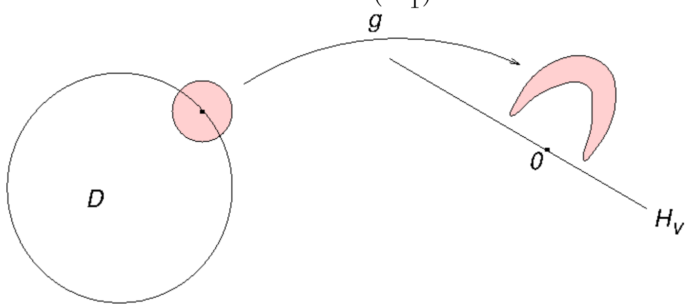
$$\mathrm{dist}(g(\mathcal{S}(v)), H_v) > \kappa\ \mathrm{dist}(g(u), H_v)$$

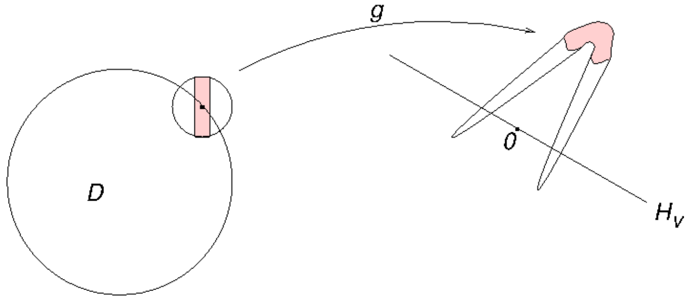
for every $u \in B_r(v)$ with $g(u) \in H_v^-$, where H_v^- denotes the closure of the connected component of $\mathbb{R}^n \setminus H_v$ not containing $g(\mathcal{S}(v))$, and $\kappa := \frac{\mathrm{meas}(\Omega)}{c(\delta/K)} - 1$.
($\mathbf{D}_2$) $\deg(g, D, 0) \neq 0$.
Then (7.1) admits at least one solution u such that $\overline{u} \in D$ and $\|u - \overline{u}\|_\infty < r$.

Here 'dist' denotes the euclidean distance in $\mathbb{R}^n$.

For a system of ordinary differential equations with periodic boundary conditions this result was recently established in [1]. Note that, for $N = 1$, $\mathrm{meas}(\Omega) = \mathrm{diam}_d(\Omega) = r/K$ and $c(\delta/K) = \delta/K$, so $\kappa = \frac{r}{\delta} - 1$ coincides with the constant given by Theorem 1.2 in [1], conveniently adapted for the Neumann conditions.

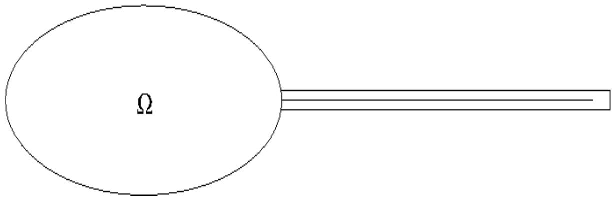
The following figures, taken from [1], illustrate the difference between condition (7.6) in Theorem 7.1.2 and condition ($\mathbf{D}_1$) in Theorem 7.1.3.





Condition (7.6) requires that the image under g of the whole ball $\overline{B}_r(v)$ lies on one side of a hyperplane H_v through the origin, whereas condition ($\mathbf{D}_1$) only requires the image of some strip $\mathcal{S}(v)$ to lie on one side of H_v but the image of the rest of the ball may cross the hyperplane, thus allowing for fast rotations of g. Note that ($\mathbf{D}_1$) is trivially satisfied for any κ if (7.6) holds. The effect of the constant κ only appears when g rotates fast enough, that is, when $g(\overline{B}_r(v))$ intersects H_v. Then, the distance of the image of the strip to H_v is not only restricted by the rotational effect of g, as in the ODE case considered in [1], but also by the geometry of Ω, as the following example shows.

Example 2. *Assume that g is bounded and that condition $(\mathbf{D}_1)$ holds for some domain Ω which contains the origin, some $p \in C(\overline{\Omega}, \mathbb{R}^n)$, some $D \subset \mathbb{R}^n$ and some $\delta > 0$. Let $T_{\theta,\eta} := \{(t, y) \in \mathbb{R} \times \mathbb{R}^{N-1} : t \in [0, \theta],\ |y| \leq \eta\}$ and let Ω_η be a bounded smooth domain in $\mathbb{R}^N$ such that $\Omega \cup T_{\theta,0} \subset \Omega_\eta \subset \Omega \cup T_{\theta+1,\eta}$ for $\eta > 0$ and some θ to be established.*



Observe that the best constant Q_η for the inequality (7.9) associated to the domain Ω_η is bounded from below by

$$Q_* := \sup_{u \in C_0^2(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_\infty}{\|\Delta u\|_\infty} \leq \sup_{u \in \mathcal{A}(\Omega_\eta)\setminus\{0\}} \frac{\|\nabla u\|_\infty}{\|\Delta u\|_\infty} =: Q_\eta$$

for every $\eta > 0$, where

$$\mathcal{A}(\Omega_\eta) := \{u \in C^1(\overline{\Omega}_\eta, \mathbb{R}^n) : \|\Delta u\|_\infty < \infty, \frac{\partial u}{\partial \nu}|_{\partial\Omega_\eta} = 0\}.$$

Let $p_\eta \in C(\overline{\Omega_\eta}, \mathbb{R}^n)$ be an extension of p. Since g is bounded, we may take $\varepsilon = 0$ in (7.11), and $K = K_\eta > Q_\eta(\|p_\eta\|_\infty + \|g\|_\infty)$. Then,

$$\frac{\delta}{K_\eta} < \frac{\delta}{Q_*(\|p\|_\infty + \|g\|_\infty)} := d_0$$

for all $\eta > 0$. Setting θ such that $\mathrm{dist}((\theta, 0), \Omega) > d_0$, we have that the open ball $U_{\delta/K_\eta}(\theta, 0)$ for the metric d in Ω_η satisfies $U_{\delta/K_\eta}(\theta, 0) \subset T_{\theta+1,\eta}$. Therefore,

$$c\,(\delta/K_\eta) := \inf_{x \in \Omega_\eta} \mathrm{meas}(U_{\delta/K_\eta}(x)) \leq \mathrm{meas}(T_{\theta+1,\eta}) \to 0 \quad \textit{as } \eta \to 0.$$

Thus,

$$\kappa_\eta := \frac{\mathrm{meas}(\Omega_\eta)}{c\,(\delta/K_\eta)} - 1 \to \infty \quad \textit{as } \eta \to 0.$$

So condition $(\mathbf{D}_1)$ will not hold for η sufficiently small.

7.2 The proof of the main result

For the sake of completeness, let us firstly prove the existence of the constant Q introduced in (7.9).

By standard regularity results (see e.g. [23, Thm. 2.3.3.2]), if $u \in C(\overline{\Omega}, \mathbb{R}^n)$ is a solution of (7.8) then $u \in \mathcal{A}(\Omega) \subset W^{2,s}(\Omega, \mathbb{R}^n)$ for any $s < \infty$, where $\mathcal{A}(\Omega)$ is defined as in the previous section. Next, suppose that, for a sequence $(u_k) \subset \mathcal{A}(\Omega)$, $\|\nabla u_k\|_\infty > k\|\Delta u_k\|_\infty$. Let $v_k := u_k/\|\nabla u_k\|_\infty$, then $\|\Delta v_k\|_\infty \to 0$ and hence $\|\Delta v_k\|_{L^2} \to 0$. This implies that $\|\nabla v_k\|_{L^2} \to 0$ and, consequently, that $\|v_k - \overline{v}_k\|_{H^1} \to 0$. Thus $\|v_k - \overline{v}_k\|_{H^2} \to 0$ which, in turn, implies that $\|v_k - \overline{v}_k\|_{W^{1,2^*}} \to 0$. Again, we conclude that $\|v_k - \overline{v}_k\|_{W^{2,2^*}} \to 0$ and by a standard bootstrapping argument we deduce that $\|v_k - \overline{v}_k\|_{W^{2,s}} \to 0$ for some $s > N$. By the Sobolev imbedding $W^{2,s}(\Omega, \mathbb{R}^n) \hookrightarrow C^1(\overline{\Omega}, \mathbb{R}^n)$, this implies $\|\nabla v_k\|_\infty \to 0$, a contradiction.

Proof of Theorem: Consider the set

$$\mathcal{U} = \{u \in C(\overline{\Omega}, \mathbb{R}^n) : \|u - \overline{u}\|_\infty < r,\ \overline{u} \in D\}.$$

By the classical continuation theorems [35], it suffices to prove that problem (7.8) has no solution on $\partial\mathcal{U}$ for $\lambda \in (0, 1]$.

Assume that u satisfies (7.8) for some $\lambda \in (0, 1]$. Then

$$\|\nabla u\|_\infty \le Q\|\Delta u\|_\infty \le Q(\|p\|_\infty + \varepsilon\|u\|_\infty + M_\varepsilon),$$

where $\varepsilon > 0$ is the number such that $Q\varepsilon\, \mathrm{diam}_d(\Omega)(1 + \alpha) < 1$ chosen to define K, and M_ε is given by (7.11). Thus,

$$\|\nabla u\|_\infty \le Q\|\Delta u\|_\infty \le Q\left(\|p\|_\infty + M_\varepsilon + \varepsilon\left[|\overline{u}| + \mathrm{diam}_d(\Omega)\|\nabla u\|_\infty\right]\right). \tag{7.12}$$

As $D \subset B_{\alpha r}(0)$, it follows that $|\overline{u}| < \alpha K \mathrm{diam}_d(\Omega)$. We claim that

$$\|\nabla u\|_\infty < K \qquad \text{and} \qquad \|u - \overline{u}\|_\infty < r. \tag{7.13}$$

Indeed, if $\|\nabla u\|_\infty \ge K$, inequality (7.12) would yield

$$K\left[1 - Q\varepsilon\, \mathrm{diam}_d(\Omega)(1 + \alpha)\right] \le Q(\|p\|_\infty + M_\varepsilon),$$

contradicting our choice of K. Thus, $\|\nabla u\|_\infty < K$, which implies $\|u - \overline{u}\|_\infty < r$. It remains to prove that $\overline{u} \notin \partial D$.

Taking w_v as the unit normal vector to H_v such that $\langle g(v), w_v\rangle > 0$, it is straightforward to check that condition $(\mathbf{D}_1)$ is equivalent to

$(\mathbf{D}_1')$ For each $v \in \partial D$ there exist a vector $w_v \in \mathbb{S}^{n-1}$ and a strip $\mathcal{S}(v)$ of width 2δ in $B_r(v)$ such that

$$\inf_{y \in \mathcal{S}(v)} \langle g(y), w_v\rangle + \left(\frac{\mathrm{meas}(\Omega)}{c(\delta/K)} - 1\right)\langle g(u), w_v\rangle > 0 \tag{7.14}$$

for every $u \in B_r(v)$ such that $\langle g(u), w_v\rangle \le 0$.

Next, arguing by contradiction, suppose that $\overline{u} \in \partial D$ and take $w_{\overline{u}} \in \mathbb{S}^{n-1}$ and the strip $\mathcal{S}(\overline{u}) = \{u \in B_r(\overline{u}) : |\langle u - \overline{u}, \xi_{\overline{u}}\rangle| < \delta\}$ with $\xi_{\overline{u}} \in \mathbb{S}^{n-1}$ such that (7.14) holds for $v = \overline{u}$. As u solves (7.8), we have that

$$0 = \int_\Omega \langle g(u(x)), w_{\overline{u}}\rangle\, dx = \int_\Omega \langle g(u(x)) - T w_{\overline{u}}, w_{\overline{u}}\rangle\, dx + T\ \mathrm{meas}(\Omega),$$

where

$$T := \inf_{x \in \Omega} \langle g(u(x)), w_{\overline{u}}\rangle.$$

Hence, $T \le 0$.

Define $\varphi(u) := \langle u, \xi_{\overline{u}} \rangle$. From the mean value theorem for vector integrals we deduce that $\overline{u} \in \mathrm{co}(\mathrm{Im}(u))$. Thus, $\varphi(\overline{u}) \in \varphi(\mathrm{Im}(u))$. Consequently, we may fix $\overline{x} \in \Omega$ such that $\varphi(u(\overline{x})) = \varphi(\overline{u})$, and from (7.13) we obtain

$$|\varphi(u(x)) - \varphi(\overline{u})| \leq |u(x) - u(\overline{x})| \leq K\, d(x, \overline{x}).$$

This implies that $u(x) \in \mathcal{S}(\overline{u})$ for $d(x,\overline{x}) < \frac{\delta}{K}$. Thus, if

$$A := \{x \in \Omega : u(x) \in \mathcal{S}(\overline{u})\},$$

then $U_{\delta/K}(\overline{x}) \subset A$, and hence $\mathrm{meas}(A) \geq c(\delta/K)$. Moreover, as $\overline{\Omega}$ is compact, we may choose $x_0 \in \overline{\Omega}$ such that $\langle g(u(x_0)), w_{\overline{u}} \rangle = T \leq 0$. Then,

$$\begin{aligned} 0 &\geq \int_A \langle g(u(x)) - T w_{\overline{u}}, w_{\overline{u}} \rangle \, dx + T \mathrm{meas}(\Omega) \\ &\geq c(\delta/K) \inf_{v \in \mathcal{S}(\overline{u})} \langle g(v), w_{\overline{u}} \rangle + T \left(\mathrm{meas}(\Omega) - c(\delta/K) \right) \\ &= c(\delta/K) \left[\inf_{v \in \mathcal{S}(\overline{u})} \langle g(u), w_{\overline{u}} \rangle + \left(\frac{\mathrm{meas}(\Omega)}{c(\delta/K)} - 1 \right) \langle g(u(x_0)), w_{\overline{u}} \rangle \right], \end{aligned}$$

which contradicts (7.14). □

7.3 The proof of the nonexistence result

The following lemma will be used to prove Proposition 1.

Lemma 1. *Let $U \subset \mathbb{R}^N$ be a smooth bounded domain, $\Gamma \in C^1(\mathbb{R}, \mathbb{R})$, $h_k \in C^1(\overline{U}, \mathbb{R})$, $\varphi, \omega_k \in C^2(\overline{U}, \mathbb{R})$, $A_k, \lambda_k \in \mathbb{R}$ and $\alpha > 1$ be such that*

$$|\nabla \varphi(x)| \geq \frac{1}{\alpha} \qquad \text{for all } x \in \overline{U},$$

$$\|\Gamma\|_{C^1}, \|h_k\|_{C^1}, \|\varphi\|_{C^2}, \|\omega_k\|_{C^1} \leq \alpha, \left\|\omega_k''\right\|_\infty \leq \alpha \lambda_k.$$

Assume that $\lambda_k \to +\infty$. Then

$$\lim_{k \to \infty} \int_U h_k(x) \Gamma'(\lambda_k \varphi(x) + \omega_k(x) + A_k)\, dx = 0.$$

Proof. We consider two cases.

Case 1: $N = 1$.

Let $U = (a,b)$. Since $|\varphi'(t)| \geq \frac{1}{\alpha}$ for all $t \in [a,b]$ and $\|\omega_k'\|_{C^0} \leq \alpha$, there exists $\lambda_* > 0$, independent of k, such that $\left|\varphi'(t) + \frac{1}{\lambda}\omega_k'(t)\right| \geq \frac{1}{2\alpha}$ for all $t \in [a,b]$ and $\lambda \in [\lambda_*, \infty)$. In particular, the function

$$f_k(t) := \frac{h_k(t)}{\varphi'(t) + \frac{1}{\lambda_k}\omega_k'(t)}$$

is well defined for $\lambda_k \in [\lambda_*, \infty)$. Since

$$f_k'(t) := \frac{h_k'(t)\left(\varphi'(t) + \frac{1}{\lambda_k}\omega_k'(t)\right) - h_k(t)\left(\varphi''(t) + \frac{1}{\lambda_k}\omega_k''(t)\right)}{\left(\varphi'(t) + \frac{1}{\lambda_k}\omega_k'(t)\right)^2},$$

we have that $\|f_k\|_\infty \le 2\alpha^2$, $\|f_k'\|_\infty \le 16\alpha^4$. Integrating by parts we obtain

$$\int_a^b h_k(t)\Gamma'(\lambda_k\varphi(t) + \omega_k(t) + A_k)\,dt = \frac{1}{\lambda_k}\int_a^b f_k(t)\frac{d}{dt}[\Gamma(\lambda_k\varphi(t) + \omega_k(t) + A_k)]\,dt$$

$$= \frac{1}{\lambda_k}\left[f_k(\cdot)\Gamma(\lambda_k\varphi(\cdot) + \omega_k(\cdot) + A_k)\Big|_a^b - \int_a^b f_k'(t)\Gamma(\lambda_k\varphi(t) + \omega_k(t) + A_k)\,dt\right].$$

Hence, if $\lambda_k \in [\lambda_*, \infty)$,

$$\left|\int_a^b h_k(t)\Gamma'(\lambda_k\varphi(t) + \omega_k(t) + A_k)\,dt\right| \le \frac{16\alpha^5(b-a+1)}{\lambda_k}.$$

As $\lambda_k \to \infty$, the result follows.

Case 2: $N > 1$.

Let $U_1, \dots, U_N$ be open subsets such that $\overline{U} = \cup_{i=1}^N \overline{U}_i$ and

$$\left|\frac{\partial\varphi}{\partial x_i}(x)\right| \ge \frac{1}{\sqrt{N}\alpha} \qquad \text{for all } x \in \overline{U}_i.$$

Write $x = (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, and set $J_{1,y} := \{t \in \mathbb{R} : (t, y) \in U_1\}$. Then, for each $y \in \mathbb{R}^{N-1}$, we may apply the case $N = 1$ to conclude that

$$\int_{J_{1,y}} h_k(t,y)\Gamma'(\lambda_k\varphi(t,y) + \omega_k(t,y) + A_k)\,dt \to 0.$$

Since U_1 is bounded and $\{h_k\Gamma'(\lambda_k\varphi + \omega_k + A_k)\}$ is uniformly bounded in U, Fubini's theorem and the dominated convergence theorem yield

$$\lim_{k\to\infty}\int_{U_1} h_k(x)\Gamma'(\lambda_k\varphi(x) + \omega_k(x) + A_k)\,dx = 0.$$

Similarly for $U_2, \dots, U_N$. Thus, the result follows. □

Proof of Proposition: Arguing by contradiction, assume there is a sequence $\lambda_k \to \infty$ such that problem (7.10) has a solution z_k. For convenience, from now on we shall write p_k instead of p_{λ_k}. Define $w_k = z_k + \frac{1}{\mu}p_k$, where μ is the eigenvalue associated to ϕ. Then

$$\Delta w_k = \Delta z_k + \frac{1}{\mu}\Delta p_k = p_k - g(z_k) - p_k,$$

that is to say,

$$\Delta w_k + g_0(w_k - \frac{1}{\mu}p_k) = \gamma. \tag{7.15}$$

Next, observe that

$$\int_\Omega \Delta w_k = \int_{\partial\Omega}\frac{\partial w_k}{\partial\nu} = \int_{\partial\Omega}\frac{\partial z_k}{\partial\nu} + \frac{1}{\mu}\int_{\partial\Omega}\frac{\partial p_k}{\partial\nu} = 0,$$

so integrating (7.15) yields

$$\int_\Omega g_0(w_k - \frac{1}{\mu}p_k) = \gamma\,\mathrm{meas}(\Omega). \tag{7.16}$$

Fix a positive $\varepsilon < \gamma\text{meas}(\Omega)$. Since the critical set $\mathcal{C}$ of ϕ is a compact subset of $\overline{\Omega}$ and has zero measure (see Lemma 2 below), we may fix a smooth domain U_ε with $\overline{U}_\varepsilon \subset \Omega$ such that $\overline{U}_\varepsilon \cap \mathcal{C} = \emptyset$ and

$$\left| \int_{\Omega\backslash U_\varepsilon} g_0(w_k - \frac{1}{\mu}p_k) \right| \leq \text{ meas}(\Omega\backslash U_\varepsilon) < \varepsilon.$$

Writing $w_k = x_k + iy_k$ we compute

$$\begin{aligned} g_0(w_k - \frac{1}{\mu}p_k) &= \frac{w_k - \frac{1}{\mu}p_k}{\sqrt{1 + |w_k - \frac{1}{\mu}p_k|^2}} e^{i\text{Re}(w_k - \frac{1}{\mu}p_k)} \\ &= \frac{(x_k - \frac{\lambda_k}{\mu}\phi)\cos(x_k - \frac{\lambda_k}{\mu}\phi) - y_k \sin(x_k - \frac{\lambda_k}{\mu}\phi)}{\sqrt{1 + |w_k - \frac{1}{\mu}p_k|^2}} \\ &\quad + i\frac{y_k \cos(x_k - \frac{\lambda_k}{\mu}\phi) + (x_k - \frac{\lambda_k}{\mu}\phi)\sin(x_k - \frac{\lambda_k}{\mu}\phi)}{\sqrt{1 + |w_k - \frac{1}{\mu}p_k|^2}}. \end{aligned}$$

Next we write the previous equality as $g_0 = g_0^1 + g_0^2 + i(g_0^3 + g_0^4)$ and apply Lemma 1 to each one of these four summands. For example, for the first one we set

$$h_k := \frac{x_k - \frac{\lambda_k}{\mu}\phi}{\sqrt{1 + |w_k - \frac{1}{\mu}p_k|^2}}, \quad \Gamma := \sin, \quad \varphi := \frac{1}{\mu}\phi$$

$$\omega_k := x_k - \overline{x_k}, \quad A_k := \overline{x_k}.$$

As $\|\Delta w_k\|_\infty \leq |\gamma| + 1$, a uniform bound (i. e. independent of k) for ω_k in $C^1(\overline{U_\varepsilon}, \mathbb{R})$ follows from the standard Sobolev estimates. However, Lemma 1 cannot be applied yet since $\|h_k\|_{C^1(\overline{U_\varepsilon}, \mathbb{R})}$ or $\|\omega_k/\lambda_k\|_{C^2(\overline{U_\varepsilon}, \mathbb{R})}$ might not be uniformly bounded. In order to overcome this difficulty, take a subsequence if necessary to define

$$\rho := \lim_{k\to\infty} \frac{\mu\overline{x_k}}{\lambda_k}.$$

Suppose firstly that $|\rho| < \infty$. As $\|w_k - \overline{w_k}\|_\infty$ is bounded, $\frac{\mu x_k}{\lambda_k}$ converges uniformly to ρ.

For each $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$|w_k - \frac{1}{\mu}p_k| \geq |\text{Re}(w_k - \frac{1}{\mu}p_k)| = \frac{\lambda_k}{\mu}\left| \frac{\mu x_k}{\lambda_k} - \phi \right| \geq c_\delta \lambda_k$$

on $\Omega_\delta := \phi^{-1}([\rho - \delta, \rho + \delta]^c)$. Moreover, as $\|\nabla w_k\|_\infty$ is bounded, it follows that $\|\nabla(w_k - \frac{1}{\mu}p_k)\|_\infty = O(\lambda_k)$. Thus, if we set

$$\theta(x, y) := \frac{x}{\sqrt{1 + x^2 + y^2}},$$

then $|\nabla\theta(w_k - \frac{1}{\mu}p_k)| \leq \frac{1}{\sqrt{1 + |w_k - \frac{1}{\mu}p_k|^2}} \leq \frac{1}{c_\delta\lambda_k}$ on Ω_δ. Using the chain rule, we conclude that $\theta(w_k - \frac{1}{\mu}p_k)$ is bounded in $C^1(\overline{\Omega}_\delta, \mathbb{R})$.

The same conclusion holds for

$$\chi(x,y) := \frac{y}{\sqrt{1+x^2+y^2}};$$

in particular, $\|\nabla g_0(w_k - \frac{1}{\mu}p_k)\|_\infty = O(\lambda_k)$. Thus, using (7.15), the interior Schauder estimates provide a uniform C^2 bound for $\frac{\omega_k}{\lambda_k}$ over $\overline{U_\varepsilon \cap \Omega_\delta}$.

As $\Omega = \phi^{-1}(\rho) \cup \bigcup_{k\in\mathbb{N}} \Omega_{1/k}$ and $\mathrm{meas}(\phi^{-1}(\rho)) = 0$, we may fix $\delta > 0$ such that $\mathrm{meas}(\Omega \setminus \Omega_\delta)$ is arbitrarily small.

Taking α large enough, Lemma 1 implies that

$$\lim_{k\to\infty} \int_{U_\varepsilon \cap \Omega_\delta} g_0^1(w_k - \frac{1}{\mu}p_k) = 0$$

and hence

$$\lim_{k\to\infty} \int_{U_\varepsilon} g_0^1(w_k - \frac{1}{\mu}p_k) = 0.$$

If we suppose, on the contrary, that $\rho = \pm\infty$, then it is immediately seen that for some $c > 0$

$$|w_k - \frac{1}{\mu}p_k| \geq c\lambda_k$$

on Ω, and the conclusion follows.

The procedure is similar for the other summands. We conclude that

$$\limsup_{k\to\infty} \left| \int_\Omega g_0(w_k - \frac{1}{\mu}p_k) \right| \leq \varepsilon,$$

which contradicts (7.16). □

Lemma 2. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be analytic with $\psi \not\equiv 0$. Then meas$\{\psi = 0\} = 0$.*

Proof. We proceed by induction. The case $n = 1$ is trivial. For each $x \in \mathbb{R}^{n-1}$ the functions $\psi^t(x) = \psi_x(t) = \psi(t,x)$ are analytic.

Fix t such that $\psi^t \not\equiv 0$. From the inductive hypothesis,

$$meas(\{x/\psi^t(x) = 0\}) = 0$$

and also

$$meas(\{x/\psi_x \equiv 0\}) = 0.$$

On the other hand, $meas(\{t/\psi_x(t) = 0\}) = 0$ for every x such that $\psi_x \not\equiv 0$. Thus, if we define

$$NZ = \{(t,x)/\psi(t,x) = 0 \text{ and } \psi_x \not\equiv 0\},$$

then by Fubini's Theorem we deduce that $meas(NZ) = 0$. We conclude that

$$\{\psi = 0\} = \mathbb{R} \times \{x/\psi_x \equiv 0\} \cup NZ$$

has zero measure. □

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