



UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Geometría Riemanniana de grupos de operadores y espacios homogéneos

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

Alberto Manuel López Galván

Director de tesis: Gabriel Larotonda.
Consejero de estudios: Esteban Andruchow.

Lugar de trabajo: Instituto Argentino de Matemática. "Alberto Calderón".
CONICET.

Buenos Aires, 14 de marzo de 2016.

Título: Geometría Riemanniana de grupos de operadores y espacios homogéneos

Resumen: El objetivo de esta tesis es el estudio de la geometría de diferentes grupos de operadores, los cuales son perturbaciones de la identidad por un operador Hilbert-Schmidt. A través de este trabajo dotaremos a los espacios tangentes con distintas métricas Riemannianas y estudiaremos sus problemas métricos. La nueva métrica introducida aquí es la *métrica polar* que es definida usando la descomposición polar de los operadores inversibles. Compararemos esta métrica con las métricas clásicas invariantes a izquierda de los grupos de Lie. Además nos centraremos en algunos espacios homogéneos y analizaremos que métricas pueden ser definidas y que propiedades tienen.^{1 2}

¹MSC 2010: Primary 47D03; Secondary 58B20, 53C22.

²Palabras clave: Variedades Riemannianas, Grupos de Lie Banach, grupos autoadjuntos, espacios homogéneos, geodésicas, distancia geodésica, completitud.

Title: Riemannian geometry of operator groups and Homogeneous spaces

Abstract: The aim of this thesis is the geometric study of different groups of operators which are a perturbation of the identity by a Hilbert-Schmidt operator. Throughout this work we will endow the tangent spaces with different Riemannian metrics and we will study their metric problems. The new metric introduced here is the *polar metric*, which is defined using the classical polar decomposition of invertible operators. We will compare this new metric with the classical left-invariant metric of Lie groups. Moreover we will focus in some homogeneous spaces given by the action of these operator groups and we analyse which metrics can be defined and study their properties. ³ ⁴

³MSC 2010: Primary 47D03; Secondary 58B20, 53C22.

⁴Keywords: Riemannian-Hilbert manifolds, Banach-Lie general linear group, self-adjoint group, homogeneous spaces, geodesics, geodesic distance, completeness.

Agradecimientos

Quería agradecer a Gabriel y a Esteban por haberme dado la oportunidad para hacer el doctorado y por su ayuda incondicional en todos estos años. También quería agradecer a la Dra. Alejandra Maetripieri, al Dr. Carlos Olmos y al Dr. Daniel Belțiță por haber aceptado ser jurados de mi tesis.

Introduction

Precedents

In classical finite dimensional Riemannian theory it is well known the fact that given two points there is a minimal geodesic curve that joins them. In this case the completeness of the geodesic curves is equivalent to the completeness of the metric space with the geodesic distance; this is the Hopf-Rinow theorem. In the infinite dimensional case this is no longer true. In [23] and [2], McAlpin and Atkin showed in two examples how this theorem can fail. One of the natural questions then regards the completeness of the metric space induced by the geodesic distance.

In the 90's, Corach, Porta and Recht started to study the geometry of positive invertible elements (denoted by G) in C^* -algebras. There they endowed the tangent bundle with a Finsler structure given by the norm of the C^* -algebra; given a positive element a and $X \in T_a G$ the Finsler structure is given by $\|X\|_a = \|a^{-1/2} X a^{-1/2}\|$. The tangent bundle carries a canonical connection determined by the transport equation, with covariant derivative defined by $D_X Y = X(Y) - 1/2(Xa^{-1}Y + Ya^{-1}X)$. Moreover they proved that the geodesics given by this connection are short for the given endpoints. The geometry of the positive invertible unitized Hilbert-Schmidt operators with the above metric was studied in [18]; there the author obtained general geometric results about: Riemannian connection, geodesics, sectional curvature, convexity of geodesic distance and completeness. Another facts obtained there are decomposition theorems and the structure of self-adjoint operators groups.

Another work that it is relevant in this context has been developed in [3], there the authors studied left invariant metrics induced by the p -norms of the trace in the matrix algebra of the general lineal group. In particular the Riemannian geodesics, corresponding to the case $p = 2$, are characterized as the product of two one-parameter groups. It is also shown that geodesics are one-parameter groups if and only if the initial velocity is a normal matrix.

The homogeneous spaces for a group of operators have become a cen-

VIII

tral topic in the study of infinite dimensional geometry. One of the most known examples of these homogeneous spaces is the Hilbert-Schmidt restricted Grassmannian $Gr_{res}(p)$ (also known in the literature as the Sato Grassmannian). The connected component of an infinite projection p coincides with the unitary orbit $\{upu^* : u \in U_2(\mathcal{H})\}$ where $U_2(\mathcal{H})$ denote the Hilbert-Schmidt unitary group. The geometry of this orbit was studied in [4]. In that work the authors endowing each tangent space with the trace inner product and show that the geodesics given by the Levi-Civita connection of this metric have minimal length among all piecewise smooth curves in the orbit joining the same endpoints. Moreover they proved the completeness of the geodesic distance using the completeness of the Hilbert-Schmidt unitary group.

Another important Grassmannian is the Lagrangian Grassmannian; in finite dimension, it was introduced by V.I. Arnold in 1967 [1]. These notions have been generalized to infinite dimensional Hilbert spaces (see [11]) and have found several applications to Algebraic Topology, Differential Geometry and Physics. In [3] E. Andruchow and G. Larotonda introduced a linear connection in the Lagrangian Grassmannian and focused on the geodesic structure of this manifold. There they proved that any two Lagrangian subspaces can be joined by a minimal geodesic. The case of the Fredholm Lagrangian Grassmannian of an infinite dimensional symplectic Hilbert space \mathcal{H} , modelled on the space of compact operators, was studied by J. C. C. Eidam and P. Piccione in [10]. The reader can see also the paper by A. Abbondandolo and P. Majer for the general theory of infinite dimensional Grassmannians, and the book by G. Segal and A. Pressley for further references on the subject [26].

Main results

In this thesis we will introduce a new Riemannian metric into the group of invertible Hilbert-Schmidt operators. It will be defined through the existence of a unique polar decomposition in the invertible group of operators, this new metric will be name the *polar metric*. The geodesic curves of this metric will be computed and we will show that they are minimal between two given points. Moreover we will study the completeness of the geodesic distance and will compare this metric with the induced with the classical left-invariant metric.

Another group that we will work with is the group of symplectic operators which are a perturbation of the identity by a Hilbert-Schmidt operator. This subgroup of the symplectic group was introduced in Pierre de la Harpe's

classical book of Banach-Lie groups [13]. This group has many applications in quantum theory with infinitely many degrees of freedom, i.e. in canonical quantum field theory, string theory, statistical quantum physics and soliton theory. We will endow the tangent spaces with different Riemannian metrics; since the polar decomposition is stable into the group we can endow the symplectic group with the *polar metric* and compare the length of curves using the minimal curves of the unitary group and the positive invertible operators. Moreover we will study the geometry of the symplectic group with the left-invariant metric; its connection, geodesics and completeness.

In Chapter 3 we will study homogeneous spaces for the symplectic group, more precisely we will focus in a restricted version of the Lagrangian Grassmannian which we named the Hilbert-Schmidt Lagrangian Grassmannian. It is defined as the set of Lagrangian subspaces L such that there exists a Hilbert-Schmidt symplectic operator g such that $L = g(L_0)$ for a fixed Lagrangian subspace L_0 . Here we will focus on the geometric study and we will discuss which metric can be defined in each tangent space and which geometric properties it verifies. In particular we will find the geodesic curves of this structure and we will describe it in terms of exponentials of operators, moreover we will study the completeness of the geodesic distance.

In Chapter 4 we will extend some results of the Hilbert-Schmidt symplectic group into a more general class of Riemannian operator groups, the self-adjoint operator groups. The most important statement here will be the completeness of the geodesic distance with the left-invariant metric. Moreover we will study the completeness with the *polar metric* and will find the geodesic curves.

In the next items we summarize the most important statement in this thesis.

We denote by \mathcal{P} the *polar metric* and by \mathcal{I} the classical left-invariant metric of Lie-groups and $d_{\mathcal{P}}, d_{\mathcal{I}}$ denote the respective geodesic distances. The invertible group Hilbert-Schmidt perturbations of the identity is denoted by $\mathrm{GL}_2(\mathcal{H})$ and the Hilbert-Schmidt symplectic group by $\mathrm{Sp}_2(\mathcal{H})$, their Lie algebras will be denoted by $\mathcal{B}_2(\mathcal{H})$ and $\mathfrak{sp}_2(\mathcal{H})$ respectively. The cone of positive invertible Hilbert-Schmidt operators is denoted by $\mathrm{GL}_2^+(\mathcal{H})$ and its Riemannian metric is denoted by \mathfrak{p} . The Lagrangian Grassmannian is denoted by $\Lambda(\mathcal{H})$ and the Hilbert-Schmidt Lagrangian Grassmannian by \mathcal{O}_{L_0} . The quotient norm of homogeneous spaces will be denoted by \mathcal{Q} .

- **Theorem 1:** Let $p, q \in \mathrm{GL}_2(\mathcal{H})$, suppose that $u_p|p|$ and $u_q|q|$ are their polar decompositions. If we choose $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ such that $u_q = u_p e^z$

with $\|z\| \leq \pi$; then the curve

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \subset \mathrm{GL}_2(\mathcal{H})$$

has minimal length among all curves joining p to q , measured with the *polar metric* \mathcal{P} given by $\mathcal{P}((u, |g|), v) := (\mathcal{I}(u, x)^2 + \mathfrak{p}(|g|, y)^2)^{1/2}$ where $v = (x, y) \in T_u \mathrm{U}_2(\mathcal{H}_J) \times T_{|g|} \mathrm{GL}_2^+(\mathcal{H})$.

- **Theorem 2:** The metric space $(\mathrm{GL}_2(\mathcal{H}), d_{\mathcal{P}})$ is complete.
- **Theorem 3:** $\mathrm{Sp}_2(\mathcal{H})$ is a totally geodesic submanifold of $\mathrm{GL}_2(\mathcal{H})$ when we consider the polar metric.
- **Theorem 4:** $\mathrm{Sp}_2(\mathcal{H})$ is a totally geodesic submanifold of $\mathrm{GL}_2(\mathcal{H})$ when we consider the left invariant metric.
- **Theorem 5:** Let $p, q \in \mathrm{Sp}_2(\mathcal{H})$, suppose that $u_p |p|$ and $u_q |q|$ are their polar decompositions, if we choose $z \in \mathfrak{sp}_2(\mathcal{H})_{ah}$ such that $u_q = u_p e^z$ with $\|z\| \leq \pi$, then the curve

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \subset \mathrm{Sp}_2(\mathcal{H})$$

has minimal length among all curves joining p to q , measured with the induced *polar metric* of $\mathrm{GL}_2(\mathcal{H})$.

- **Theorem 6:** The metric spaces $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{P}})$ and $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{I}})$ are complete.
- **Theorem 7:** Let $\xi : [0, 1] \rightarrow \mathcal{O}_{L_0}$ be a geodesic curve of the Riemannian connection induced by the quotient metric \mathcal{Q} with initial position $\xi(0) = L$ and initial velocity $\dot{\xi}(0) = w \in T_{\xi(0)} \mathcal{O}_{L_0} = \mathcal{B}_2(L)_h$. Then

$$\xi(t) = e^{t(v^* - v)} e^{-tv^*}(L)$$

where v is a preimage of $-w$ by the differential of the action of $\mathrm{Sp}_2(\mathcal{H})$.

- **Theorem 8:** If (L_n) is a sequence in \mathcal{O}_{L_0} , $L \in \mathcal{O}_{L_0}$ and $d_{\mathcal{Q}}$ the geodesic distance with the quotient metric then

1. The metric space $(\mathcal{O}_{L_0}, d_{\mathcal{Q}})$ is complete.

2. The distance $d_{\mathcal{Q}}$ defines the given topology on \mathcal{O}_{L_0} . Equivalently,

$$L_n \xrightarrow{\mathcal{O}_{L_0}} L \iff L_n \xrightarrow{d_{\mathcal{Q}}} L.$$

- **Theorem 9** If G is a self-adjoint Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$ then it is totally geodesic with the polar and left invariant metrics.

- **Theorem 10:** Let G be a closed subgroup of $GL_n(\mathbb{C})$ then the metric space induced by the geodesic distance with the left invariant p -norms are complete.
- **Theorem 11:** Let G be a closed, self-adjoint Banach-Lie subgroup of $GL_2(\mathcal{H})$ then
 1. $(G, d_{\mathcal{P}})$ is complete.
 2. $(G, d_{\mathcal{I}})$ is complete.

The above results have been published in research articles [20],[21] and submitted in [22], for which I am the sole author.

Introducción

Antecedentes

En teoría Riemanniana de dimensión finita se sabe que dados dos puntos existe una curva geodesica minimal que los une. En este caso la completitud de las geodésicas es equivalente con la completitud del espacio métrico inducido por la distancia geodésica; esto es el Teorema de Hopf-Rinow. En el caso de dimensión infinita esto no es cierto. En [23] y [2], McAlpin y Atkin probaron en dos ejemplos como este teorema puede fallar. Por esta razón es interesante el estudio de los problemas métricos en cada caso. Una pregunta natural en esta dirección es acerca de la completitud del espacio métrico inducido por la distancia geodésica.

En los 90's, Corach, Porta y Recht comenzaron con el estudio de la geometría de operadores positivos inversibles in C^* -algebras. Ahí construyeron una métrica de Finsler en el fibrado tangente usando la norma de la C^* -algebra; dado un elemento positivo a y $X \in T_aG$ la estructura Finsler es $\|X\|_a = \|a^{-1/2}Xa^{-1/2}\|$. El fibrado tangente lleva una conexión canonica determinada por la ecuación de transporte, la derivada covariante es definida por $D_X Y = X(Y) - 1/2(Xa^{-1}Y + Ya^{-1}X)$. Además probaron que las geodésicas de esta conexión son minimales entre dos puntos. La geometría de los operadores positivos de Hilbert-Schmidt fué estudiada en [18] donde el autor obtuvo aspectos geometricos sobre: conexión Riemanniana, geodésicas, curvatura seccional, convexidad de la distancia geodésica y completitud. Otros hechos abordados ahí fueron los teoremas de Descomposición y los grupos de operadores autoadjuntos.

Otro trabajo que es relevante en este contexto ha sido desarrollado en [3], ahí los autores estudiaron métricas invariantes a izquierda inducidas por las normas p en el álgebra del grupo lineal. En particular han sido caracterizadas las geodésicas correspondientes al caso $p = 2$ y se describieron como producto de grupos a un parámetro. También se prueba que las geodésicas son grupos a un parámetro si y solo si la velocidad inicial es una matriz normal.

Los espacios homogéneos para grupos de operadores han recibido una im-

portancia central en el estudio de la geometría de dimensión infinita. Uno de los más conocidos de estos espacios homogéneos es la Grasmanniana restringida de Hilbert-Schmidt $G_{res}(p)$ (también conocida como la Grasmanniana de Sato). La componente conexa de un projector de rango infinito coincide con la órbita unitaria $\{upu^* : u \in U_2(\mathcal{H})\}$ donde $U_2(\mathcal{H})$ representa a los operadores unitarios de Hilbert-Schmidt. La geometría de esta órbita fue estudiada en [4]. En este trabajo los autores dotaron a los espacios tangentes con el producto interno dado por la traza y probaron que las geodésicas dadas por la conexión de Levi-Civita tienen longitud mínima entre todas las curvas suaves que unen los mismos puntos finales. Además probaron la completitud de la distancia geodésica usando la completitud del grupo de unitarios de Hilbert-Schmidt.

Otra Grasmanniana de importancia es la Grasmanniana Lagrangiana; en dimensión finita esta es introducida por V.I. Arnold en 1967 [1]. Estas nociones se han generalizado a espacios de Hilbert en dimensiones infinitas (véase [11]) y han encontrado varias aplicaciones a la topología algebraica, geometría diferencial y Física. En [3] E. Andruchow y G. Larotonda introdujeron una conexión lineal en la Grasmanniana Lagrangiana y se centraron en la estructura geodésica de esta conexión. Allí se demostró que dos subespacios Lagrangianos se pueden unir por una geodésica mínima. El caso de la Grasmanniana Lagrangiana de Fredholm de dimensión infinita en un espacio de Hilbert simpléctico \mathcal{H} , modelado en el espacio de los operadores compactos, fue estudiado por J.C.C. Eidam y P. Piccione en [10]. El lector puede ver también el artículo de A. Abbondandolo y P. Majer para la teoría general de Grasmannianas en dimensión infinita, y el libro de G. Segal y A. Pressley para más referencias sobre el tema [26].

Resultados principales

En esta tesis vamos a introducir una nueva métrica Riemanniana en el grupo de operadores inversibles de Hilbert-Schmidt. Esta se define a través de la existencia de la unicidad de la descomposición polar en el grupo de operadores inversibles, esta nueva métrica será llamada, la *métrica polar*. Se calcularán las curvas geodésicas de esta métrica y se mostrará que son mínimas entre dos puntos dados. Además se estudiará la completitud de la distancia geodésica y se comparará con la inducida por la métrica invariante a izquierda.

Otro grupo que trabajaremos es el grupo de operadores simplécticos que son una perturbación de la identidad por un operador de Hilbert-Schmidt. Este subgrupo del grupo simpléctico se introdujo en el clásico libro de Pierre de la Harpe de grupos de Banach-Lie [13]. Este grupo tiene muchas aplica-

ciones en la teoría cuántica con infinitos de grados de libertad, es decir, en la teoría canónica cuántica de campos, la teoría de cuerdas, la física cuántica y la teoría estadística soliton. En este trabajo vamos a dotar a los espacios tangentes con diferentes métricas Riemannianas. Como la descomposición polar es estable en el grupo podemos dotar al grupo simpléctico con la *métrica polar* y comparar la longitud de las curvas utilizando las curvas mínimas del grupo unitario y de los operadores invertibles positivos. Por otro lado vamos a estudiar la geometría del grupo simpléctico con la métrica invariante a izquierda; conexión, geodésicas y completitud.

En el Capítulo 3 estudiaremos los espacios homogéneos para el grupo simpléctico, más precisamente nos centraremos en una versión restringida de la Grasmanniana Lagrangiana; la cual llamaremos, Grasmanniana Lagrangiana de Hilbert Schmidt. Esta se define como el conjunto de subespacios Lagrangianos L tales que existe un operador g simpléctico de Hilbert-Schmidt tal que $L = g(L_0)$ para L_0 un subespacio Lagrangiano fijo. Aquí nos centraremos en el estudio geométrico y discutiremos que métricas se pueden definir en cada espacio tangente y que propiedades geométricas se verifican. En particular, encontraremos las curvas geodésicas de estas estructuras y las describiremos en términos de exponenciales de operadores, por otra parte se estudiará la completitud de la distancia geodésica.

En el Capítulo 4 ampliaremos algunos resultados del grupo simpléctico de Hilbert-Schmidt a una clase más general de grupos de operadores Riemannianos, los grupos de operadores autoadjuntos. El hecho más importante aquí será la completitud de la distancia geodésica con la métrica invariante a izquierda. Además se estudiará la completitud con la *métrica polar* y se calcularán las curvas geodésicas.

En los próximos items resumimos los hechos más relevantes en esta tesis.

Denotamos por \mathcal{P} a la *métrica polar* y por \mathcal{I} a la métrica invariante a izquierda de grupos de Lie y por $d_{\mathcal{P}}, d_{\mathcal{I}}$ las respectivas distancias geodésicas. El grupo de operadores inversibles que son perturbaciones de la identidad por un operador de Hilbert-Schmidt es notado por $\mathrm{GL}_2(\mathcal{H})$ y el grupo simpléctico de Hilbert-Schmidt $\mathrm{Sp}_2(\mathcal{H})$ sus álgebras de Lie son $\mathcal{B}_2(\mathcal{H}), \mathfrak{sp}_2(\mathcal{H})$ respectivamente. El cono de operadores positivos de Hilbert-Schmidt es notado por $\mathrm{GL}_2^+(\mathcal{H})$ y su métrica Riemanniana es notada por \mathfrak{p} . La Grasmanniana Lagrangiana es notada por $\Lambda(\mathcal{H})$ y la Grasmanniana Lagrangiana de Hilbert-Schmidt por \mathcal{O}_{L_0} . La métrica cociente de espacios homogéneos será notada por \mathcal{Q} .

- **Teorema 1:** Sean $p, q \in \mathrm{GL}_2(\mathcal{H})$, supongamos que $u_p|p|$ y $u_q|q|$ son sus

descomposiciones polares. Si elegimos $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ tal que $u_q = u_p e^z$ con $\|z\| \leq \pi$, luego la curva

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \subset \mathrm{GL}_2(\mathcal{H})$$

tiene longitud mínima entre todas las que unen p con q , medidas con la métrica polar \mathcal{P} dada por $\mathcal{P}((u, |g|), v) := (\mathcal{I}(u, x)^2 + \mathfrak{p}(|g|, y)^2)^{1/2}$ donde $v = (x, y) \in T_u \mathrm{U}_2(\mathcal{H}_J) \times T_{|g|} \mathrm{GL}_2^+(\mathcal{H})$.

- **Teorema 2:** El espacio métrico $(\mathrm{GL}_2(\mathcal{H}), d_{\mathcal{P}})$ es completo.
- **Teorema 3:** $\mathrm{Sp}_2(\mathcal{H})$ es una subvariedad total geodésica de $\mathrm{GL}_2(\mathcal{H})$ cuando consideramos la métrica polar.
- **Teorema 4:** $\mathrm{Sp}_2(\mathcal{H})$ es una subvariedad total geodésica de $\mathrm{GL}_2(\mathcal{H})$ cuando consideramos la métrica invariante a izquierda.
- **Teorema 5:** Sean $p, q \in \mathrm{Sp}_2(\mathcal{H})$, supongamos que $u_p |p|$ y $u_q |q|$ son sus descomposiciones polares, si elegimos $z \in \mathfrak{sp}_2(\mathcal{H})_{ah}$ tal que $u_q = u_p e^z$ con $\|z\| \leq \pi$, luego la curva

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \subset \mathrm{Sp}_2(\mathcal{H})$$

tiene longitud mínima entre todas las que unen p con q , medidas con la métrica polar inducida de $\mathrm{GL}_2(\mathcal{H})$.

- **Teorema 6:** Los espacios métricos $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{P}})$ y $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{I}})$ son completos.
- **Teorema 7:** Sea $\xi : [0, 1] \rightarrow \mathcal{O}_{L_0}$ una curva geodésica dada por la conexión inducida por la métrica cociente \mathcal{Q} con posición inicial $\xi(0) = L$ y velocidad inicial $\dot{\xi}(0) = w \in T_{\xi(0)} \mathcal{O}_{L_0} = \mathcal{B}_2(L)_h$. Luego

$$\xi(t) = e^{t(v^* - v)} e^{-tv^*}(L)$$

donde v es una preimagen de $-w$ vía la diferencial de la acción.

- **Teorema 8:** Si (L_n) es una sucesión en \mathcal{O}_{L_0} , $L \in \mathcal{O}_{L_0}$ y $d_{\mathcal{Q}}$ denota la distancia geodésica con la métrica cociente luego

1. El espacio métrico $(\mathcal{O}_{L_0}, d_{\mathcal{Q}})$ es completo.

2. La distancia $d_{\mathcal{Q}}$ define la topología en \mathcal{O}_{L_0} . Equivalentemente,

$$L_n \xrightarrow{\mathcal{O}_{L_0}} L \iff L_n \xrightarrow{d_{\mathcal{Q}}} L.$$

- **Teorema 9:** Si G es un subgrupo de Lie autoadjunto de $GL_2(\mathcal{H})$ luego es total geodésico con la métrica polar y con la métrica invariante a izquierda.
- **Teorema 10:** Sea G un subgrupo cerrado de $GL_n(\mathbb{C})$ luego los espacios métricos inducidos por la distancia geodésica con las métricas invariantes a izquierda con las normas p son completos.
- **Teorema 11:** Sea G un subgrupo de Lie, cerrado y autoadjunto de $GL_2(\mathcal{H})$ luego
 1. $(G, d_{\mathcal{P}})$ es completo.
 2. $(G, d_{\mathcal{I}})$ es completo.

Los resultados anteriores han sido publicados en [20],[21] y presentados en [22], en donde soy el autor.

Contents

Introduction	VII
Main results	VIII
1 Preliminaries	1
1.1 Linear Operators in Hilbert Spaces	1
1.2 Manifolds	4
1.2.1 Sprays and connections	5
1.3 Banach-Lie groups	6
1.3.1 Homogeneous manifolds	7
1.4 Riemannian Geometry	8
2 Riemannian metrics in operator groups	11
2.1 The Hilbert-Schmidt general linear group	11
2.2 The symplectic group	17
2.2.1 The Hilbert-Schmidt symplectic group	20
2.3 Riemannian metrics in $\mathrm{Sp}_2(\mathcal{H})$	22
2.3.1 Riemannian metrics in $\mathrm{Sp}_2^+(\mathcal{H})$	23
2.3.2 $\mathrm{Sp}_2^+(\mathcal{H})$ as submanifold of the ambient space	25
2.4 Polar Riemannian structure in $\mathrm{Sp}_2(\mathcal{H})$	27
2.5 The metric space $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{I}})$	29
3 An homogeneous space of $\mathrm{Sp}_2(\mathcal{H})$	33
3.1 Manifold structure of \mathcal{O}_{L_0}	34
3.2 Metric structure in \mathcal{O}_{L_0}	41
3.2.1 The ambient metric	41
3.2.2 The geodesic distance	42
3.2.3 The quotient metric	44
4 The self-adjoint groups	49
4.1 Riemannian geometry	49
4.1.1 Riemannian geometry with the left invariant metric	49
4.1.2 Riemannian geometry with the polar metric	50

4.2	Completeness of the geodesic distance	52
4.2.1	Completeness in finite dimension with p-norms	52
4.2.2	Completeness in the infinite dimensional case	53

Chapter 1

Preliminaries

En este capítulo fijaremos la notación que utilizaremos y recordaremos algunos hechos de operadores lineales, variedades, grupos de Lie y geometría Riemanniana.

In this chapter we will fix the notation that we will use throughout and recall some facts of Linear Operator, Banach manifolds, Lie groups and Riemannian geometry.

1.1 Linear Operators in Hilbert Spaces

We start this section giving some definitions and results about linear operators between Hilbert spaces. The Hilbert spaces will be denoted by \mathcal{H} . In general we use complex a Hilbert space, but in some cases of operator groups we will use a real Hilbert space. We denote the inner product by $\langle \cdot, \cdot \rangle$ and the induced norm by $\|\xi\| = \langle \xi, \xi \rangle^{1/2}$, $\xi \in \mathcal{H}$. A linear map (operator) $x : \mathcal{H} \rightarrow \mathcal{H}$ is said to be bounded if there is a number K such that $\|x\xi\| \leq K\|\xi\|$, $\forall \xi \in \mathcal{H}$. The infimum of all such K is called the uniform or spectral norm of x , written $\|x\|$. Boundedness of an operator is equivalent to continuity. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators acting on \mathcal{H} . To every bounded operator $x \in \mathcal{B}(\mathcal{H})$ there is another $x^* \in \mathcal{B}(\mathcal{H})$, called the adjoint of x , which is defined by the formula

$$\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle, \quad \forall \xi, \eta \in \mathcal{H}.$$

Then the uniform norm can be calculated by

$$\|x\| = \sup_{\|\xi\|=1} \|x\xi\| = \sup_{\|\xi\|\leq 1, \|\eta\|\leq 1} |\langle x\xi, \eta \rangle| = \|x^*\| = \|x^*x\|^{1/2}.$$

Definition 1.1.1. An operator $x \in \mathcal{B}(\mathcal{H})$ is called Hermitic if $x = x^*$. Analogously it is called anti-Hermitic if $x = -x^*$. An operator $x \in \mathcal{B}(\mathcal{H})$ is called positive ($x \geq 0$) if $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$. An operator $u \in \mathcal{B}(\mathcal{H})$ is called unitary if $uu^* = u^*u = 1$. We denote by $U(\mathcal{H})$ the set of all unitary operators.

If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is any subset of operators, we use the subscript h (resp. ah) to denote the subset of Hermitian (resp. anti-Hermitian) elements of it, i.e. $\mathcal{A}_h = \{x \in \mathcal{A} : x^* = x\}$ and $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$.

We denote by $GL(\mathcal{H})$ the general linear group of all invertible operators on \mathcal{H} and by $GL(\mathcal{H})^+$ the subset of all positive invertible operators. Let $|x| = (x^*x)^{1/2}$ be the modulus of x . It is known that every invertible operator g has an unique representation

$$g = u|g|,$$

where $u \in U(\mathcal{H})$. Such decomposition is called a polar decomposition of g . An operator $x \in \mathcal{B}(\mathcal{H})$ is said to be compact if $x(B_{\mathcal{H}})$ has compact closure in \mathcal{H} , where $B_{\mathcal{H}} = \{\xi \in \mathcal{H} : |\xi| = 1\}$ is the unit ball. The set of all compact operators will be denoted by $\mathcal{K}(\mathcal{H})$. The spectrum of any operator x will be denoted by $\sigma(x)$. The compact operators have many properties that we will use. We mention some of these:

1. A compact operator x is compact if and only if there exists a sequence (x_n) of finite range operators such that $\|x - x_n\| \rightarrow 0$.
2. x is compact if and only if x^* is compact.
3. $0 \in \sigma(x)$, and $\sigma(x) - \{0\}$ consists of eigenvalues of finite multiplicity (i.e. the dimension of the λ -eigenspace $\ker(x - \lambda)$ has finite dimension).
4. $\sigma(x) - \{0\}$ is either empty, finite or a sequence converging to 0.
5. If x is compact and normal with spectrum $\sigma(x) = \{0, \lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$ then by the spectral Theorem

$$x = \sum_{n=1}^{\infty} \lambda_n p_n$$

where p_n denotes the orthogonal projection to $\ker(x - \lambda_n)$.

Theorem 1.1.2. (Canonical form for compact operator) Let $x \in \mathcal{K}(\mathcal{H})$. Then x has the norm convergent expansion,

$$x = \sum_{n=1}^{\infty} s_n(x) \langle \phi_n, \cdot \rangle \psi_n$$

(where the sum may be finite or infinite), each $s_n(x) \geq 0$, decreasingly ordered with $s_n(x) \rightarrow 0$ and ϕ_n, ψ_n are orthonormal sets (not necessarily complete). Moreover, the $s_n(x)$ are uniquely determined. The $s_n := s_n(x)$ are eigenvalues of $|x| = (x^*x)^{1/2}$ counted with multiplicity and are called singular values of x .

If $x \in \mathcal{K}(\mathcal{H})$ we denote by $\{s_n(x)\}$ the sequence of singular value of x (decreasingly ordered). For $1 \leq p \leq \infty$, let

$$\|x\|_p := \left(\sum_{n=1}^{\infty} s_n(x)^p \right)^{1/p}$$

and

$$\mathcal{B}_p(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : \|x\|_p < \infty\}$$

called the p -Schatten class of $\mathcal{B}(\mathcal{H})$.

If x is any operator then the sum $Tr(x) := \sum_{n=1}^{\infty} \langle x\xi_n, \xi_n \rangle$ has the same value (finite or infinite) for any orthonormal basis $\{\xi_n\}$ of \mathcal{H} . This number is called the trace of x and it has the following properties:

- $Tr(\lambda x + \beta y) = \lambda Tr(x) + \beta Tr(y)$.
- $Tr(uxu^*) = Tr(x)$ for all $u \in U(\mathcal{H})$.
- If $0 \leq x \leq y$, then $Tr(x) \leq Tr(y)$.
- $Tr(xy) = Tr(yx)$.

Remark 1.1.3. $x \in \mathcal{B}_p(\mathcal{H})$ if and only if $\|x\|_p = Tr(|x|^p)^{1/p} < \infty$.

Now, let us recall some properties of the classes $\mathcal{B}_p(\mathcal{H})$, for a proof see the book [27] Chapter 3.

Theorem 1.1.4. Let $1 \leq p < \infty$,

1. $\mathcal{B}_p(\mathcal{H})$ is a $*$ -ideal of $\mathcal{B}(\mathcal{H})$.
2. $\|x\|_p = \|uxv\|_p$, for all $x \in \mathcal{B}_p(\mathcal{H})$ and $u, v \in U(\mathcal{H})$. That is, the unitary invariance property.
3. $\|x\| \leq \|x\|_p = \|x^*\|_p$, for all $x \in \mathcal{B}_p(\mathcal{H})$.
4. $\|xyz\| \leq \|x\| \|y\|_p \|z\|$, $x, z \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}_p(\mathcal{H})$.

When $p = 2$, the elements of $\mathcal{B}_2(\mathcal{H})$ are called Hilbert-Schmidt operators, they form a Hilbert space with the 2-norm. The inner product is given by

$$\langle x, y \rangle = Tr(y^*x).$$

1.2 Manifolds

In this Thesis we focus in smooth Riemannian manifolds modelled in spaces of infinite dimension called Riemann Hilbert manifolds. We refer to Lang's book [17] for the basic differential geometry of this type of manifolds.

Definition 1.2.1. *Let X be a set. An atlas of class C^r on X is a collection of pairs (called charts) (U_i, ϕ_i) satisfying the following conditions:*

1. *Each U_i is a subset of X and the U_i cover X .*
2. *Each ϕ_i is a bijection of U_i onto an open subset $\phi_i(U_i)$ of some Banach space E_i and for any i, j , $\phi_i(U_i \cap U_j)$ is open in E_i .*
3. *The map*

$$\phi_j \phi_i : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$$

is a C^r -isomorphism for each pair of indices i, j .

One can then show that there is a unique topology on X such that each U_i is open and each ϕ_i is a homeomorphism. If the Banach spaces E_i are Hilbert spaces, the above structure is called a Hilbert manifold on X . The definition of smooth function is analogous to the finite dimensional case. If we have a smooth function $f : X \rightarrow Y$ between manifolds we denote its differential at a point $x \in X$ by

$$d_x f : T_x X \rightarrow T_{f(x)} Y$$

However, if $I \subset \mathbb{R}$ is an interval and a curve $\gamma : I \rightarrow X$, its differential in $(t, 1) \in T_t I = I \times \mathbb{R}$ will be denoted by $\dot{\gamma}(t)$, that is $\dot{\gamma}(t) = d_t \gamma(1)$.

A smooth map $f : X \rightarrow Y$ is called a submersion at a point $x \in X$ if it satisfies:

- $\ker(d_x f)$ is a complemented subspace of $T_x X$, i. e. there exists a closed subspace \mathcal{F} such that $T_x X = \ker(d_x f) \oplus \mathcal{F}$.
- The map $d_x f : T_x X \rightarrow T_{f(x)} Y$ is surjective.

We say that f is a submersion if it is a submersion at every point.

Proposition 1.2.2. *Let $f : X \rightarrow Y$ a smooth map. Then f is a submersion at $x \in X$ if and only if f admits local sections, i.e, there exists a neighbourhood U of $f(x) \in Y$ and a smooth map $s : U \rightarrow X$ such that $f \circ s = id_U$.*

Let X be a manifold and $Y \subseteq X$ be a subset; we say that a chart $\phi : U \rightarrow \phi(U)$ is a submanifold chart for Y if $\phi(U \cap Y)$ is equal to the intersection of $\phi(U)$ with a closed vector subspace S . Then we say that:

$$\phi|_{U \cap Y} : U \cap Y \rightarrow \phi(U) \cap S$$

is the chart in Y induced by ϕ . The subset Y is said to be an embedded submanifold of X if for all $x \in X$ there exists a submanifold chart for Y whose domain contains x . The inclusion $i : Y \hookrightarrow X$ will be an embedding of Y in X , i.e., a differentiable immersion which is a homeomorphism onto its image endowed with the relative topology.

1.2.1 Sprays and connections

A second-order vector field on a manifold X is a vector field $F : TX \rightarrow TTX$ on TX satisfying $d\pi \circ F = id_{TX}$, where $\pi : TX \rightarrow X$ is the natural projection map. Let $t \in \mathbb{R}$ and let $sTX : TX \rightarrow TX$, $V \mapsto tV$ denote the multiplication by t in each tangent space. A second order vector field is called a spray if $F(tV) = d(sTX)(tF(V))$ for all $t \in \mathbb{R}$ and $V = (x, v) \in TX$. In a local chart (U, ϕ) , using the identification $TU \cong U \times E$ and $TTU \cong (U \times E) \times (E \times E)$, a spray can be written as

$$F(x, v) = (x, v, v, f(x, v))$$

where $f : U \times E \rightarrow E$ is a smooth map that verifies that $f(x, \cdot)$ is a quadratic map for each $x \in U$. Using the polarization formula we have the bilinear form,

$$\Gamma_x(v, w) = 1/2 \{F_x(v + w) - F_x(v) - F_x(w)\}, \text{ for } x \in U, v, w \in T_x X \quad (2.1)$$

associated to the spray. We also have the covariant derivative of the spray

$$D_t \eta = \dot{\eta} - \Gamma(\eta, \dot{\alpha}) \quad (2.2)$$

where $\alpha : (-\epsilon, \epsilon) \rightarrow X$ is any smooth curve and η is a tangent field along α . A smooth curve $\alpha : I \rightarrow X$, is called a geodesic of the spray F if it verifies the equation

$$\ddot{\alpha} = F(\ddot{\alpha}) \quad (2.3)$$

Let \mathcal{D} the set of vectors $V \in TX$ such that the solution α of the above equation is defined at least on the interval $[0, 1]$; we define the exponential map by

$$Exp(V) = \alpha_v(1)$$

where $V = (x, v)$ and α_v is a solution of equation (2.3) with initial velocity v . We denote by Exp_x, \mathcal{D}_x the restriction of the map Exp to the tangent space $T_x X$. Thus,

$$Exp_x : \mathcal{D}_x \subset T_x X \rightarrow X.$$

1.3 Banach-Lie groups

A Banach-Lie group G is a smooth Banach manifold which is also endowed with a group structure such that the map $G \times G \rightarrow G$ defined by $(x, y) \mapsto xy^{-1}$ is smooth. The tangent space $T_e G$ at the identity $e \in G$ is its Banach-Lie algebra and it is denoted by \mathfrak{g} . If we denote by L_g the differential map of the left action of G on itself and by R_g the differential map of the right action, then the tangent space at $g \in G$ is

$$T_g G = L_g \mathfrak{g} \cong R_g \mathfrak{g}.$$

A 1-parameter subgroup of G is a group homomorphism $\gamma : \mathbb{R} \rightarrow G$. For each $v \in \mathfrak{g}$ there exists a unique 1-parameter subgroup such that $\dot{\gamma}_v(0) = v$. This allows us to define the exponential map

$$\exp_G : \mathfrak{g} \rightarrow G, \quad \exp_G(v) = \gamma_v(1).$$

See the Chapter 2 Section 2 in the book [5] for further information about the exponential map.

Definition 1.3.1. *A subgroup H of a Banach-Lie group G is a Banach-Lie subgroup if:*

1. *H is a Banach-Lie group and its topology coincides which inherits from G .*
2. *The map $i : H \hookrightarrow G$ is an immersion with closed range.*
3. *There exists a closed subspace \mathcal{F} such that $d_e i(\mathfrak{h}) \oplus \mathcal{F} = \mathfrak{g}$.*

Because of condition 2 we always identify \mathfrak{h} (the Banach-Lie algebra of H) with $\text{Ran}(d_e i)$, so that we think of \mathfrak{h} as a closed subalgebra of the Banach-Lie algebra \mathfrak{g} . In this way $d_e i$ is just the inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$.

The following statement supplies a very useful characterization of Banach-Lie subgroups. It can be found in the Chapter 4, Proposition 4.4 in the book [5].

Theorem 1.3.2. *Assume that G is a Banach-Lie group with the Lie algebra \mathfrak{g} , H is a closed subgroup of G and denote*

$$\mathfrak{h} = \{x \in \mathfrak{g} : \exp_G(tx) \in H, \forall t \in \mathbb{R}\}$$

Then \mathfrak{h} is a closed Lie subalgebra of \mathfrak{g} and there exist on H an uniquely determined topology τ and a manifold structure making H into a Banach-Lie group such that $L(H) = \mathfrak{h}$, the inclusion map $H \hookrightarrow G$ is smooth and $d_e i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is an inclusion.

Corollary 1.3.3. *In the setting of the above theorem, if we assume that there exist an open neighborhood V of $0 \in \mathfrak{g}$ and an open neighborhood U of $1 \in G$ such that \exp_G induces a diffeomorphism of V onto U and $\exp_G(V \cap \mathfrak{h}) = U \cap H$. Then the topology τ coincides with the topology inherited by H from G .*

Remark 1.3.4. *If G is a finite dimensional Lie group and H is a closed subgroup, then H is a Banach-Lie subgroup of G . See the book [5] Chapter 4, Remark 4.6 for the proof.*

Definition 1.3.5. *We say that a subspace $\mathfrak{m} \subset \mathfrak{g}$ is a Lie triple system if $[[x, y], z] \in \mathfrak{m}$ for any $x, y, z \in \mathfrak{m}$.*

1.3.1 Homogeneous manifolds

Let G be a Banach-Lie group and X a smooth manifold. A smooth action of G on X is a smooth map $\pi : G \times X \rightarrow X$, $(g, x) \mapsto g.x$ such that $(g_1 g_2).x = g_1.(g_2.x)$ and $e.x = x$ for all $g_1, g_2 \in G$ and $x \in X$. Given an action, the orbit of $x \in X$ is the set $\mathcal{O}_x = \{g.x : g \in G\}$. We denote by π_x the smooth map given by

$$\pi_x : G \rightarrow X, \quad \pi_x(g) = g.x.$$

The subgroup given by $G_x = \{g \in G : g.x = x\}$ is called the isotropy group at $x \in X$. It is not difficult to see that there is a bijection between \mathcal{O}_x and G/G_x .

A smooth action is called transitive if for all $x_1, x_2 \in X$ there exists $g \in G$ such that $g.x_1 = x_2$, i.e. $\mathcal{O}_x = X$. Let π be a smooth transitive action, we say that X is a homogeneous space if there exists $x \in X$ such that π_x is a submersion at $e \in G$. It is not difficult to see that if π_x is a submersion at $e \in G$ then it is a submersion for all $g \in G$. If the orbit is a homogeneous space then, since $G_x = \pi_x^{-1}(x)$ and by the inverse function Theorem (see the book [17] Chapter 1, Corollary 5.5), the isotropy group results in a Banach-Lie subgroup of G with Banach-Lie algebra $\ker d_e \pi_x$. Moreover, we have a diffeomorphism between G/G_x and \mathcal{O}_x .

1.4 Riemannian Geometry

A Riemannian metric (or Riemannian structure) on a smooth manifold X is a correspondence which associates to each point x of X an inner product $(\langle \cdot, \cdot \rangle_x)$ on the tangent space $T_x X$, which varies smoothly. A manifold with a Riemannian metric will be called a Riemannian manifold. In other words a Riemannian structure is a smooth section $h : X \rightarrow \mathcal{Bil}(TX)$ such that $h(x)(v, w) = \langle v, w \rangle_x$ with image in positive definite forms. We denote by $\mathfrak{b}(x, v) = \langle v, v \rangle_x$ the metric in each tangent space. The length of a smooth curve α measured with the metric \mathfrak{b} will be denoted by

$$L_{\mathfrak{b}}(\alpha) = \int_0^1 \mathfrak{b}(\alpha(t), \dot{\alpha}(t)) dt.$$

We define the geodesic distance between two points $x, y \in X$ as the infimum of the length of all piecewise smooth curves in X joining x to y ,

$$d_{\mathfrak{b}}(x, y) = \inf \{L_{\mathfrak{b}}(\alpha) : \alpha \subset X, \alpha(0) = x, \alpha(1) = y\}.$$

Thus, X is a metric space with respect to the distance $d_{\mathfrak{b}}$.

Let $f : X \rightarrow Y$ be an immersion, if Y has a Riemannian structure, f induces a Riemannian structure on X by defining $\langle v, u \rangle_x = \langle d_x f(v), d_x f(u) \rangle$ for all $v, u \in T_x X$. This metric is then called the metric induced by f , and f is an isometric immersion.

We say that a Riemannian metric on a Lie group G is left invariant if $\langle v, u \rangle_y = \langle d_y L_x(v), d_y L_x(u) \rangle_{L_x(y)}$ for all $x, y \in G$, $v, u \in T_y G$. Analogously, we can define a right invariant metric. We can always introduce a left invariant Riemannian metric on a Lie group G taking any arbitrary inner product $\langle \cdot, \cdot \rangle_e$ on its Lie algebra and define

$$\langle v, u \rangle_x = \langle d_x L_{x^{-1}}(v), d_x L_{x^{-1}}(u) \rangle_e, \quad \forall u, v \in T_x G, x \in G. \quad (4.4)$$

In an analogous manner we can build a right invariant metric using the right multiplication.

If we have two Riemannian manifolds X and Y , then we can consider the cartesian product $X \times Y$ with the manifold product structure. Let $pr_1 : X \times Y \rightarrow X$ and $pr_2 : X \times Y \rightarrow Y$ be the projections. Then, we can introduce a Riemannian metric on $X \times Y$ as follows:

$$\langle v, u \rangle_{(x,y)} = \langle dpr_1 \cdot v, dpr_1 \cdot u \rangle_x + \langle dpr_2 \cdot v, dpr_2 \cdot u \rangle_y, \quad (4.5)$$

for all $(x, y) \in X \times Y$, $v, u \in T_{(x,y)} X \times Y$. Thus, we can give a Riemannian structure on the product manifold $X \times Y$.

The following two theorems characterize the second-order vector field on a Riemannian manifold X . For more details see the book [17], Chapter 4.

Theorem 1.4.1. *Let X be a Riemannian manifold. There exists a unique covariant derivative D such that for all vector fields η, β, μ , we have*

$$D_\eta \langle \beta, \mu \rangle = \langle D_\eta \beta, \mu \rangle + \langle \beta, D_\eta \mu \rangle. \quad (4.6)$$

This covariant derivative is called The Levi-Civita derivative and the above equation means the compatibility with the metric.

Theorem 1.4.2. *Let X be a Riemannian manifold. There exists a unique spray F on X satisfying the following two equivalent conditions:*

1. *In a chart,*

$$\langle F_x(v), h(x)z \rangle = -\langle d_x h(v), z \rangle + 1/2 \langle d_x h(v), v \rangle$$

for all $z, v \in T_x X$. This spray is called the metric spray.

2. *The covariant derivative associated to the spray is the Levi-Civita derivative.*

Proposition 1.4.3. *Let X, Y be Riemannian manifolds. If we consider the product Riemannian structure on $X \times Y$ (4.5) then the Levi-Civita connection is given by (∇^X, ∇^Y) where ∇^X and ∇^Y denote the Levi-Civita connection of X and Y .*

Proof. We will suppose that the tangent space at any $(x, y) \in X \times Y$ is given by $T_x X \times T_y Y$. We want to prove that the covariant derivative is given by $(\nabla_{W_1}^X V_1, \nabla_{W_2}^Y V_2)$ where V_1, W_1 and V_2, W_2 are fields on TX and TY respectively. It is clearly symmetric and verifies all the formal identities of a connection therefore the proof that it is the Levi-Civita connection relies on the compatibility condition between the connection and the metric. Indeed, let $V(t) = (V_1(t), V_2(t))$ and $W(t) = (W_1(t), W_2(t))$ fields along a curve $\gamma = (\gamma_1, \gamma_2) \subset X \times Y$ then using the compatibility on each factor we have,

$$\begin{aligned} \frac{d}{dt} \left(\langle (V_1, V_2), (W_1, W_2) \rangle_{(\gamma_1, \gamma_2)} \right) &= \frac{d}{dt} \left(\langle V_1, W_1 \rangle_{\gamma_1} \right) + \frac{d}{dt} \left(\langle V_2, W_2 \rangle_{\gamma_2} \right) \\ &= \langle D_t V_1, W_1 \rangle_{\gamma_1} + \langle V_1, D_t W_1 \rangle_{\gamma_2} + \langle D_t V_2, W_2 \rangle_{\gamma_2} + \langle V_2, D_t W_2 \rangle_{\gamma_2} = \\ &= \langle (D_t V_1, D_t V_2), (W_1, W_2) \rangle_{(\gamma_1, \gamma_2)} + \langle (V_1, V_2), (D_t W_1, D_t W_2) \rangle_{(\gamma_1, \gamma_2)} \end{aligned}$$

□

Let $B_{d_b}(x, r)$ be the open ball with respect to the geodesic distance centered in x of radio r .

Theorem 1.4.4. *Let $x \in X$, there exists $c > 0$ such that for all $r < c$ the map Exp_x gives a differential isomorphism*

$$Exp_x : U \subset T_x X \rightarrow B_{d_b}(x, r)$$

where U is an open neighbourhood of $0 \in T_x X$.

In a Riemannian manifold X we have another notion of completeness. A Riemannian manifold is geodesically complete if the maximal interval of definition of every geodesic in X is all of \mathbb{R} . Let $\alpha : [0, 1] \rightarrow X$ be a geodesic. We say that α is a minimal geodesic if $L_b(\alpha) \leq L_b(\gamma)$ for every path joining $\alpha(0)$ and $\alpha(1)$.

Theorem 1.4.5. *Let X be a Riemannian manifold, then:*

1. *Every geodesic is locally minimal.*
2. *If a smooth curve α verifies that $L_b(\alpha) \leq L_b(\gamma)$ for every path joining $\alpha(0)$ and $\alpha(1)$, then it is a geodesic.*

Let us considerate the following conditions:

1. As a metric space under d_b , X is complete.
2. All geodesic in X are defined in \mathbb{R} .
3. For every $x \in X$, the exponential Exp_x is defined on all of $T_x X$.

Theorem 1.4.6. *Each of the above conditions implies the next, i.e $1 \Rightarrow 2 \Rightarrow 3$.*

In the finite dimensional case the above conditions are equivalent. That is the Hopf-Rinow theorem. See the book [16], Chapter 1, Theorem 10.3.

Theorem 1.4.7. *(Hopf-Rinow) Assume that X is connected, geodesically complete and finite dimensional. Then any two point in X can be joined by a minimal geodesic.*

Definition 1.4.8. *A submanifold $Y \subset X$ is said to be totally geodesic if any geodesic of the manifold Y (with respect to the metric induced on Y by the metric of the ambient manifold X) is at the same time a geodesic of the ambient manifold X .*

Proposition 1.4.9. *Let $Y \subset X$ be a Riemannian submanifold of the Riemannian manifold X and denote by ∇_1, ∇_2 the respective covariants derivative of them, then the following statements are equivalent:*

1. *Y is a totally geodesic submanifold of X .*
2. *$(\nabla_2)_\xi \eta \in TY$ for every vector fields $\xi, \eta \in TY$.*
3. *$\nabla_1 = \nabla_2|_{TY}$.*

Chapter 2

Riemannian metrics in operator groups

En este Capítulo estudiaremos posibles métricas Riemannianas en distintos grupos de operadores, más precisamente estudiamos grupos de operadores invertibles que son perturbaciones de la identidad por un operador de Hilbert-Schmidt. Estos grupos de Banach fueron introducidos por Pierre de la Harpe en su libro de Grupos de Lie-Banach [13]. Nos centraremos en el grupo lineal y en el grupo simpléctico. En ambos casos estudiaremos estructuras Riemannianas y sus propiedades geométricas.

In this chapter we will study possible Riemannian metrics in different operator groups, more precisely we study groups of invertible linear operator which are a perturbation of the identity by a Hilbert-Schmidt operator. These Banach-Lie groups were introduced by Pierre de la Harpe in his book of Banach-Lie groups [13]. We will focus in the general linear group and in the symplectic group. In both cases we will study Riemannian structures and we will study its geometric properties.

2.1 The Hilbert-Schmidt general linear group

The Hilbert-Schmidt general linear group is denoted by

$$\mathrm{GL}_2(\mathcal{H}) = \{g \in \mathrm{GL}(\mathcal{H}) : g - 1 \in \mathcal{B}_2(\mathcal{H})\}.$$

This group has a differentiable structure when endowed with the metric $\|g_1 - g_2\|_2$ (note that $g_1 - g_2 \in \mathcal{B}_2(\mathcal{H})$); it is a Banach-Lie group with Banach-Lie

algebra $\mathcal{B}_2(\mathcal{H})$. The exponential map is given by the classical exponential

$$\exp(x) = e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

Using the left action of $\mathrm{GL}_2(\mathcal{H})$ on itself, the tangent space at $g \in \mathrm{GL}_2(\mathcal{H})$ is

$$T_g \mathrm{GL}_2(\mathcal{H}) = g \cdot \mathcal{B}_2(\mathcal{H}).$$

The classical unitary subgroup is denoted by

$$\mathrm{U}_2(\mathcal{H}) = \{g \in \mathrm{U}(\mathcal{H}) : g - 1 \in \mathcal{B}_2(\mathcal{H})\}.$$

It is not difficult to see, using Theorem 1.3.2, that $\mathrm{U}_2(\mathcal{H})$ is a Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$ with Banach-Lie algebra $\mathcal{B}_2(\mathcal{H})_{ah}$. Given $u \in \mathrm{U}_2(\mathcal{H})$ its tangent space is $T_u \mathrm{U}_2(\mathcal{H}) = u \mathcal{B}_2(\mathcal{H})_{ah}$.

We introduce the left invariant metric (4.4) for $v \in T_g \mathrm{GL}_2(\mathcal{H})$ by

$$\mathcal{I}(g, v) = \|g^{-1}v\|_2. \quad (1.1)$$

This metric comes from the inner product

$$\langle v, w \rangle_g = \langle g^{-1}v, g^{-1}w \rangle = \mathrm{Tr}((gg^*)^{-1}vw^*).$$

In the followings steps we recall the metric spray of $\mathrm{GL}_2(\mathcal{H})$ with the left invariant metric. For the metric expression $g \mapsto I_g$ where $I_g v = (gg^*)^{-1}v$ we obtain the metric spray

$$F_g(v) = vg^{-1}v + gv^*I_g v - vv^*(g^*)^{-1}.$$

Using the polarization formula (2.1) we obtain the bilinear form associated to the spray, that is for $g \in \mathrm{GL}_2(\mathcal{H})$ and $v = gx, w = gy \in T_g \mathrm{GL}_2(\mathcal{H})$,

$$2g^{-1}\Gamma_g(gx, gy) = xy + yx + x^*y + y^*x - xy^* - yx^*.$$

The covariant derivative of the spray is $D_t \eta = \dot{\eta} - \Gamma(\eta, \dot{\alpha})$ where $\alpha : (-\epsilon, \epsilon) \rightarrow \mathrm{GL}_2(\mathcal{H})$ is any smooth curve and η is a tangent field along α . Let $g_0 \in \mathrm{GL}_2(\mathcal{H})$ and $v_0 \in \mathcal{B}_2(\mathcal{H})$, then the unique geodesic of the Levi-Civita connection induced by the trace inner-product metric, with initial position g_0 and initial speed $g_0 v_0$, is given by

$$\alpha(t) = g_0 e^{tv_0^*} e^{t(v_0 - v_0^*)}.$$

In this context, the Riemannian exponential map is given, for fixed $g \in \mathrm{GL}_2(\mathcal{H})$, by the expression

$$\mathrm{Exp}_g(v) = g e^{v^*} e^{v - v^*},$$

and the exponential flow is certainly a smooth map from $\mathbb{R} \times \mathcal{B}_2(\mathcal{H})$ to $\mathrm{GL}_2(\mathcal{H})$. For more details see [3].

The following proposition allows us an easier interpretation of the covariant derivative in terms of the fields at the identity.

Proposition 2.1.1. *If η is a field along a curve α we define $\beta = \alpha^{-1}\dot{\alpha}$ and $\mu = \alpha^{-1}\eta$, the fields at the identity, then the covariant derivative can be expressed by*

$$\alpha^{-1}D_t\eta = \dot{\mu} + 1/2\{[\beta, \mu] + [\beta, \mu^*] + [\mu, \beta^*]\}.$$

Proof. From the covariant derivative formula, we have

$$\alpha^{-1}D_t\eta = \alpha^{-1}\dot{\eta} - \alpha^{-1}\Gamma(\alpha\mu, \alpha\beta).$$

If we write $\eta = \alpha\mu$ and $\dot{\alpha} = \alpha\beta$, using the product rule to differentiate η we obtain

$$\alpha^{-1}\dot{\eta} = \alpha^{-1}\dot{\alpha}\mu + \dot{\mu} = \beta\mu + \dot{\mu}.$$

□

Let $\mathrm{GL}_2^+(\mathcal{H}) := \mathrm{GL}(\mathcal{H})^+ \cap \mathrm{GL}_2(\mathcal{H})$ be the subset of positive invertible operators. It is known that $\mathrm{GL}_2^+(\mathcal{H})$ is a submanifold of the open set $\Delta = \{\beta + X \in \mathbb{C} \oplus \mathcal{B}_2(\mathcal{H}) : \beta + X > 0\}$. For $p \in \mathrm{GL}_2^+(\mathcal{H})$, we identify the tangent space $T_p\mathrm{GL}_2^+(\mathcal{H})$ with $\mathcal{B}_2(\mathcal{H})_h$ and endow this manifold with a complete Riemannian metric by means of the formula

$$\mathfrak{p}(p, x) = \|p^{-1/2}xp^{-1/2}\|_2 \quad (1.2)$$

for $p \in \mathrm{GL}_2^+(\mathcal{H})$ and $x \in T_p\mathrm{GL}_2^+(\mathcal{H})$. Using the compatibility condition (4.6) between the connection and the metric it is not difficult to see that the Levi-Civita connection is given by

$$\nabla_\eta\mu_p = \eta(\mu)_p - 1/2(\eta_p p^{-1}\mu_p + \mu_p p^{-1}\eta_p) \quad (1.3)$$

where η, μ are tangent fields and $\eta(\mu)$ denotes derivation of the vector field μ in the direction of η .

Euler's equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ for the covariant derivative introduced by the Riemannian connection reads $\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}$, and it is not hard to see that the unique solution of this equation with $\gamma(0) = p$ and $\gamma(1) = q$ is given by the smooth curve

$$\gamma_{pq}(t) = p^{1/2}(p^{-1/2}qp^{-1/2})^t p^{1/2}.$$

The exponential map of $\mathrm{GL}_2^+(\mathcal{H})$ is given by

$$\mathrm{Exp}_p : T_p\mathrm{GL}_2^+(\mathcal{H}) \rightarrow \mathrm{GL}_2^+(\mathcal{H}), \quad \mathrm{Exp}_p(v) = p^{1/2} \exp(p^{-1/2}vp^{-1/2})p^{1/2}.$$

In [18] G. Larotonda obtained general geometric results about $\mathrm{GL}_2^+(\mathcal{H})$ with the above metric: Riemannian connection, geodesic, sectional curvature, convexity of geodesic distance and completeness.

Since the modulus operator of $g \in \mathrm{GL}_2(\mathcal{H})$ can be written in terms of the exponential operator, that is $|g| = \exp(\frac{1}{2} \ln(g^*g))$, the polar decomposition induces a diffeomorphism into the product manifold $\mathrm{U}_2(\mathcal{H}) \times \mathrm{GL}_2^+(\mathcal{H})$. This fact was noted in Prop.14 (iv) on page 98 of the book [13]. We denote it by

$$\begin{aligned} \mathrm{GL}_2(\mathcal{H}) &\xrightarrow{\varphi} \mathrm{U}_2(\mathcal{H}) \times \mathrm{GL}_2^+(\mathcal{H}) \\ g &\longmapsto (u, |g|). \end{aligned} \quad (1.4)$$

If we put the left invariant metric on $\mathrm{U}_2(\mathcal{H})$ (i.e. the metric \mathcal{I} induced by the ambient manifold $\mathrm{GL}_2(\mathcal{H})$) and the positive metric (1.2) on $\mathrm{GL}_2^+(\mathcal{H})$, then we can endow the product manifold $\mathrm{U}_2(\mathcal{H}) \times \mathrm{GL}_2^+(\mathcal{H})$ with the usual product metric (4.5). Thus, if $v = (x, y) \in T_u \mathrm{U}_2(\mathcal{H}) \times T_{|g|} \mathrm{GL}_2^+(\mathcal{H})$ we denote the product metric by,

$$\begin{aligned} \mathcal{P}((u, |g|), v) &:= \left(\mathcal{I}(u, x)^2 + \mathfrak{p}(|g|, y)^2 \right)^{1/2} \\ &= \left(\|x\|_2^2 + \| |g|^{-1/2} y |g|^{-1/2} \|_2^2 \right)^{1/2}. \end{aligned} \quad (1.5)$$

The map φ is in particular an immersion, from this we can define a new Riemannian metric in the group in the following way: if $v, w \in T_g \mathrm{GL}_2(\mathcal{H})$ we put

$$\langle v, w \rangle_g := \langle d\varphi_g(v), d\varphi_g(w) \rangle_{(u, |g|)}.$$

It is clear that φ is an isometric map with the above metric and if α is any curve in the group $\mathrm{GL}_2(\mathcal{H})$ we can measure its length as $L_{\mathcal{P}}(\varphi \circ \alpha)$.

Proposition 2.1.2. *The Levi-Civita connection of the polar metric is given by*

$$\nabla_{\mu}^{\mathcal{P}} \eta_{(u, |g|)} = \left(\frac{1}{2} [\eta_1, \mu_1] , \eta_2(\mu_2)_{|g|} - \frac{1}{2} [\eta_2|g|^{-1} \mu_2|g| + \mu_2|g|^{-1} \eta_2|g|] \right)$$

where $\eta = (\eta_1, \eta_2)$ and $\mu = (\mu_1, \mu_2)$ are the fields in $T_u \mathrm{U}_2(\mathcal{H}) \times T_{|g|} \mathrm{GL}_2^+(\mathcal{H})$.

Proof. It is a direct consequence of Proposition 1.4.3 using the Levi-Civita connection of the $\mathrm{GL}_2^+(\mathcal{H})$ and $\mathrm{U}_2(\mathcal{H})$. \square

Theorem 2.1.3. *Let $g \in \mathrm{GL}_2(\mathcal{H})$ with polar decomposition $u|g|$ and suppose that $u = e^x$ with $x \in \mathcal{B}_2(\mathcal{H})_{ah}$ and $\|x\| \leq \pi$, then the curve $\alpha(t) = e^{tx}|g|^t \subset \mathrm{GL}_2(\mathcal{H})$ has minimal length among all curves joining 1 to g , if we endow $\mathrm{GL}_2(\mathcal{H})$ with the polar Riemannian metric (1.5).*

Proof. By the polar decomposition, $\varphi \circ \alpha(t) = (e^{tx}, |g|^t)$ and its length is

$$L_{\mathcal{P}}(\varphi \circ \alpha) = \int_0^1 \mathcal{P}((e^{tx}, |g|^t), (xe^{tx}, \ln |g||g|^t)) dt = (\|x\|_2^2 + \|\ln |g|\|_2^2)^{1/2}.$$

Let β be another curve that joins the same endpoints and suppose that $\beta = \beta_1\beta_2$ is its polar decomposition where $\beta_1 \subset \mathbf{U}_2(\mathcal{H})$ and $\beta_2 \subset \mathbf{GL}_2^+(\mathcal{H})$, then

$$L_{\mathcal{P}}(\varphi \circ \beta) = \int_0^1 \mathcal{P}((\beta_1, \beta_2), (\dot{\beta}_1, \dot{\beta}_2)) dt = \int_0^1 (\mathcal{I}(\beta_1, \dot{\beta}_1)^2 + \mathbf{p}(\beta_2, \dot{\beta}_2)^2)^{1/2} dt.$$

Using the Minkowski inequality (see inequality 201 of [12]) we have,

$$\begin{aligned} \int_0^1 (\mathcal{I}(\beta_1, \dot{\beta}_1)^2 + \mathbf{p}(\beta_2, \dot{\beta}_2)^2)^{1/2} dt &\geq \left(\left\{ \int_0^1 \mathcal{I}(\beta_1, \dot{\beta}_1) \right\}^2 + \left\{ \int_0^1 \mathbf{p}(\beta_2, \dot{\beta}_2) \right\}^2 \right)^{1/2} \\ &= \left(L_{\mathcal{I}}(\beta_1)^2 + L_{\mathbf{p}}(\beta_2)^2 \right)^{1/2}. \end{aligned} \quad (1.6)$$

It is known that the geodesic curve e^{tx} has minimal length among all smooth curves in $\mathbf{U}_2(\mathcal{H})$ joining the same endpoints (see [4]); using this fact and since the curve $|g|^t$ has minimal length with the positive metric \mathbf{p} (see [18]) we have,

$$L_{\mathcal{I}}(\beta_1) \geq L_{\mathcal{I}}(e^{tx}) = \|x\|_2 \quad \text{and} \quad L_{\mathbf{p}}(\beta_2) \geq L_{\mathbf{p}}(e^{t \ln |g|}) = \|\ln |g|\|_2$$

then it is clear that $L_{\mathcal{P}}(\varphi \circ \beta) \geq L_{\mathcal{P}}(\varphi \circ \alpha)$. \square

Remark 2.1.4. Let $p, q \in \mathbf{GL}_2(\mathcal{H})$, suppose that $u_p|p|$ and $u_q|q|$ are their polar decompositions. From the surjectivity of the exponential map we can choose $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ such that $u_q = u_p e^z$ with $\|z\| \leq \pi$. Then the curve

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \subset \mathbf{GL}_2(\mathcal{H})$$

has minimal length among all curves joining p to q .

The above fact shows that the curve $\alpha_{p,q}$ is a geodesic of the Levi-Civita connection of the polar metric. Its length is

$$\left(\|z\|_2^2 + \|\ln |p|^{-1/2} |q| |p|^{-1/2}\|_2^2 \right)^{1/2}.$$

From this, the geodesic distance is

$$d_{\mathcal{P}}(p, q) = (d_{\mathcal{I}}(u_p, u_q)^2 + d_{\mathbf{p}}(|p|, |q|)^2)^{1/2}.$$

Proposition 2.1.5. *The metric space $(\mathrm{GL}_2(\mathcal{H}), d_{\mathcal{P}})$ is complete.*

Proof. Let $(x_n) \subset \mathrm{GL}_2(\mathcal{H})$ be a Cauchy sequence with $d_{\mathcal{P}}$, if $x_n = u_{x_n}|x_n|$ is its polar decomposition, we have that

$$d_{\mathcal{I}}(u_{x_n}, u_{x_m}) \leq d_{\mathcal{P}}(x_n, x_m) = (d_{\mathcal{I}}(u_{x_n}, u_{x_m})^2 + d_{\mathfrak{p}}(|x_n|, |x_m|)^2)^{1/2}$$

then the unitary part is a Cauchy sequence in $(\mathrm{U}_2(\mathcal{H}), d_{\mathcal{I}})$ and by [4] it is $d_{\mathcal{I}}$ convergent to an element $u \in \mathrm{U}_2(\mathcal{H})$. Analogously the positive part is a Cauchy sequence in $(\mathrm{GL}_2^+(\mathcal{H}), d_{\mathfrak{p}})$ then it is convergent to an element $g \in \mathrm{GL}_2^+(\mathcal{H})$ (see [18]). If we put $x := ug \in \mathrm{GL}_2(\mathcal{H})$ then,

$$d_{\mathcal{P}}(x_n, x) = (d_{\mathcal{I}}(u_{x_n}, u)^2 + d_{\mathfrak{p}}(|x_n|, g)^2)^{1/2} \rightarrow 0.$$

□

In the next proposition we will compare the geodesic distance measured with the polar metric versus the left invariant metric.

Proposition 2.1.6. *Given $p, q \in \mathrm{GL}_2(\mathcal{H})$, if we denote $v := |p|^{-1/2}|q||p|^{-1/2}$ we can estimate the geodesic distance $d_{\mathcal{I}}$ by the geodesic distance $d_{\mathcal{P}}$ as,*

$$d_{\mathcal{I}}(p, q) \leq c(p, q)d_{\mathcal{P}}(p, q)$$

where

$$c(p, q)^2 = 2 \max \left\{ e^{4\|\ln(v)\|} (\|p\|\|p^{-1}\|)^2, \|p\|\|p^{-1}\| \right\}.$$

Proof. If we differentiate $\alpha_{p,q}$ we have,

$$\dot{\alpha}_{p,q} = u_p z e^{tz} |p|^{1/2} e^{t \ln(v)} |p|^{1/2} + u_p e^{tz} |p|^{1/2} \ln(v) e^{t \ln(v)} |p|^{1/2}$$

and the inverse of the curve $\alpha_{p,q}$ is

$$\alpha_{p,q}^{-1} = |p|^{-1/2} e^{-t \ln(v)} |p|^{-1/2} e^{-tz} u_p^{-1}.$$

After some simplifications we can write

$$\alpha_{p,q}^{-1} \dot{\alpha}_{p,q} = |p|^{-1/2} e^{-t \ln(v)} |p|^{-1/2} z |p|^{1/2} e^{t \ln(v)} |p|^{1/2} + |p|^{-1/2} \ln(v) |p|^{1/2}.$$

Let $x := |p|^{1/2} e^{t \ln(v)} |p|^{1/2}$, taking the norm and using the parallelogram rule we have,

$$\begin{aligned} \|\alpha_{p,q}^{-1} \dot{\alpha}_{p,q}\|_2^2 &= \|x^{-1} z x + |p|^{-1/2} \ln(v) |p|^{1/2}\|_2^2 \\ &\leq 2(\|x^{-1} z x\|_2^2 + \||p|^{-1/2} \ln(v) |p|^{1/2}\|_2^2) \\ &\leq 2(\|x^{-1}\|_2^2 \|z\|_2^2 + \||p|^{-1/2}\|_2^2 \|\ln(v)\|_2^2 \||p|^{1/2}\|_2^2). \end{aligned} \quad (1.7)$$

We can estimate $\|x\|^2$ and $\|x^{-1}\|^2$ by

$$\|x\|^2 \leq \| |p|^{1/2} \|^4 e^{2\|\ln(v)\|} = \|p\|^2 e^{2\|\ln(v)\|}$$

and

$$\|x^{-1}\|^2 \leq \| |p|^{-1/2} \|^4 e^{2\|\ln(v)\|} = \|p^{-1}\|^2 e^{2\|\ln(v)\|}.$$

If we define

$$c(p, q)^2 = 2 \max \left\{ e^{4\|\ln(v)\|} (\|p\| \|p^{-1}\|)^2, \|p\| \|p^{-1}\| \right\}.$$

from (1.7) and taking square roots we have,

$$\|\alpha_{p,q}^{-1} \dot{\alpha}_{p,q}\|_2 \leq c(p, q) (\|z\|_2^2 + \|\ln(v)\|_2^2)^{1/2} = c(p, q) d_{\mathcal{P}}(p, q),$$

then

$$d_{\mathcal{I}}(p, q) \leq L_{\mathcal{I}}(\alpha_{p,q}) \leq c(p, q) d_{\mathcal{P}}(p, q).$$

□

2.2 The symplectic group

In this section we will consider \mathcal{H} as a real Hilbert space. We fix a complex structure; that is a linear isometry $J \in \mathcal{B}(\mathcal{H})$ such that,

$$J^2 = -1 \text{ and } J^* = -J.$$

The symplectic form w is given by $w(\xi, \eta) = \langle J\xi, \eta \rangle$. We denote by $\text{Sp}(\mathcal{H})$ the subgroup of invertible operators which preserve the symplectic form, that is $g \in \text{Sp}(\mathcal{H})$ if $w(g\xi, g\eta) = w(\xi, \eta)$ for all $\xi, \eta \in \mathcal{H}$. Algebraically we can describe this subgroup as,

$$\text{Sp}(\mathcal{H}) = \{g \in \text{GL}(\mathcal{H}) : g^* J g = J\}.$$

Denote by \mathcal{H}_J the Hilbert space \mathcal{H} with the action of the complex field \mathbb{C} given by J , that is; if $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ and $\xi \in \mathcal{H}$ we can define the action as $\lambda\xi := \lambda_1\xi + \lambda_2 J\xi$ and the complex inner product as $\langle \xi, \eta \rangle_{\mathbb{C}} = \langle \xi, \eta \rangle - iw(\xi, \eta)$.

Denote by $\mathcal{B}(\mathcal{H}_J)$ the space of bounded complex linear operators in \mathcal{H}_J . A straightforward computation shows that $\mathcal{B}(\mathcal{H}_J)$ consists of the elements of $\mathcal{B}(\mathcal{H})$ which commute with J .

One of the most important properties of this operator group is the stability of the adjoint operation.

Proposition 2.2.1. *If $g \in \text{Sp}(\mathcal{H})$ then $g^* \in \text{Sp}(\mathcal{H})$.*

Proof. The proof is a short computation using the definition, indeed if $g \in \text{Sp}(\mathcal{H})$ then $g^*J = Jg^{-1}$ and times by gJ we obtain $gJg^*J = -1$ then $gJg^* = J$. \square

Proposition 2.2.2. *The symplectic group is a closed subgroup of $\text{GL}(\mathcal{H})$.*

Proof. Let $(g_n) \subset \text{Sp}(\mathcal{H})$ be a convergent sequence $g_n \rightarrow g$, it is clear that g verifies the relation $g^*Jg = J$, so the only fact to prove is that g is an invertible operator. Since $g_n^* \in \text{Sp}(\mathcal{H})$ then we have $g_nJg_n^* = J$, thus this relation is transferred through the limit to g . We can now define the inverse of g as $g^{-1} := -Jg^*J$, it verifies:

$$g^{-1}g = -Jg^*Jg = 1 \quad \text{and} \quad gg^{-1} = g(-Jg^*J) = 1.$$

\square

Let us denote $\mathfrak{sp}(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : xJ = -Jx^*\}$, it is clear that $\mathfrak{sp}(\mathcal{H})$ is a closed subalgebra of $\mathcal{B}(\mathcal{H})$. If we compute the exponential on $\mathfrak{sp}(\mathcal{H})$, its image belongs in $\text{Sp}(\mathcal{H})$. Indeed, if x verifies $xJ = -Jx^*$ then $e^xJ = Je^{-x^*} = J(e^{x^*})^{-1}$ and thus $e^xJe^{x^*} = J$. Therefore, we have

$$\exp : \mathfrak{sp}(\mathcal{H}) \rightarrow \text{Sp}(\mathcal{H})$$

and if we derive the equality $e^{tv^*}Je^{tv} = J$, ($t \in \mathbb{R}$) we get $vJ = -Jv^*$. So,

$$\mathfrak{sp}(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : e^{tv} \in \text{Sp}(\mathcal{H}) \forall t \in \mathbb{R}\}.$$

Therefore by Theorem 1.3.2 the symplectic group is a Banach-Lie group with Banach-Lie algebra $\mathfrak{sp}(\mathcal{H})$.

Theorem 2.2.3. *$\text{Sp}(\mathcal{H})$ is a Banach-Lie subgroup of $\text{GL}(\mathcal{H})$.*

Proof. We will give a constructive proof. An alternative proof can be obtained using the fact that $\text{Sp}(\mathcal{H})$ is an algebraic subgroup of $\text{GL}(\mathcal{H})$ (see the book [5] Chapter 4 Theorem 4.13 or the paper [14] Proposition 2). We start proving that the topology τ (given by the Banach-Lie structure) coincides with the topology inherited from $\text{GL}(\mathcal{H})$. It is a straightforward computation using the logarithmic series, indeed if $g \in \text{Sp}(\mathcal{H})$ meets $\|g - 1\| < r$ ($r < 1$) the exponential is a diffeomorphism and then its inverse is given by the logarithmic series $x = \log(g) = \sum_{n=1}^{\infty} (-1)^n \frac{(1-g)^n}{n} \in \mathcal{B}(\mathcal{H})$, then since $\|g - 1\| = \|(g - 1)J\| = \|J((g^*)^{-1} - 1)\| = \|(g^*)^{-1} - 1\|$ we have

$$\begin{aligned} xJ &= \sum_{n=1}^{\infty} (-1)^n \frac{(1-g)^n}{n} J = J \sum_{n=1}^{\infty} (-1)^n \frac{(1-(g^*)^{-1})^n}{n} \\ &= J \log((g^*)^{-1}) = -Jx^*. \end{aligned}$$

Therefore, if we take the neighborhood $V := \{g \in \mathrm{GL}(\mathcal{H}) : \|g - 1\| < 1\}$ of the identity and if $U := \log(V)$, by the above computation it is clear that $\exp(U \cap \mathfrak{sp}(\mathcal{H})) = \mathrm{Sp}(\mathcal{H}) \cap V$ and then by Corollary 1.3.3 the topology τ coincides with the topology inherited from $\mathrm{GL}(\mathcal{H})$. Now we have to prove that the Banach-Lie algebra $\mathfrak{sp}(\mathcal{H})$ is complemented into $\mathcal{B}(\mathcal{H})$. To this end we consider the linear map

$$\Pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}), \quad \Pi(x) = 1/2(x + Jx^*J)$$

A simple computation shows that it is an idempotent map, and moreover its range is $\mathfrak{sp}(\mathcal{H})$. Indeed, if $x \in \mathfrak{sp}(\mathcal{H})$ then $\Pi(x) = 1/2(x + Jx^*J) = 1/2(x - J^2x) = x$ and on the other hand,

$$\Pi(x)J = 1/2(xJ - Jx^*) = 1/2J(-JxJ - x^*) = -J\Pi(x)^*.$$

Therefore, $\mathcal{B}(\mathcal{H}) = \mathfrak{sp}(\mathcal{H}) \oplus \ker \Pi$. \square

Note that the Banach-Lie algebra $\mathfrak{sp}(\mathcal{H})$ is closed under adjoints. Therefore we have a Cartan decomposition

$$\mathfrak{sp}(\mathcal{H}) = \mathfrak{sp}(\mathcal{H})_h \oplus \mathfrak{sp}(\mathcal{H})_{ah}. \quad (2.8)$$

Let

$$\mathrm{U}(\mathcal{H}_J) = \{g \in \mathrm{U}(\mathcal{H}) : gJ = Jg\} \quad \text{and} \quad \mathrm{Sp}^+(\mathcal{H}) = \{g \in \mathrm{Sp}(\mathcal{H}) : g > 0\}$$

be the intersection of $\mathrm{Sp}(\mathcal{H})$ with the set of unitary operators and with the positive definite operator respectively. Since the exponential map $\exp : \mathcal{B}(\mathcal{H}_J)_{ah} \rightarrow \mathrm{U}(\mathcal{H}_J)$ is surjective (see [4]), it is clear that $\exp(\mathfrak{sp}(\mathcal{H})_{ah}) = \mathrm{U}(\mathcal{H}_J)$.

Proposition 2.2.4. *$\mathrm{U}(\mathcal{H}_J)$ is a Banach-Lie subgroup of $\mathrm{Sp}(\mathcal{H})$.*

Proof. Let U be a neighborhood of 0 in $\mathfrak{sp}(\mathcal{H})$ such that the exponential map is a diffeomorphism, we can assume that $U = \{x \in \mathfrak{sp}(\mathcal{H}) : \|x\| < r\}$ for a suitable $r > 0$. It is clear that we always have

$$\exp(\mathfrak{sp}(\mathcal{H})_{ah} \cap U) \subseteq \mathrm{U}(\mathcal{H}_J) \cap \exp(U).$$

Conversely, suppose that $g \in \mathrm{U}(\mathcal{H}_J) \cap \exp(U)$ then $g = e^y$ for some $y \in U$; hence $1 = gg^* = e^y e^{y^*}$ and then $e^y = e^{-y^*}$. Since $-y^*$ also belongs in U and the exponential is one to one, we have that $y = -y^*$ and thus $y \in \mathfrak{sp}(\mathcal{H})_{ah}$. Then we have $\exp(\mathfrak{sp}(\mathcal{H})_{ah} \cap U) = \mathrm{U}(\mathcal{H}_J) \cap \exp(U)$ and by equation (2.8) we conclude the statement. \square

Now, we want to show that if $g \in \mathrm{Sp}(\mathcal{H})$ then both factors in the polar decomposition are in $\mathrm{Sp}(\mathcal{H})$.

Proposition 2.2.5. *The exponential map $\exp : \mathfrak{sp}(\mathcal{H})_h \rightarrow \mathrm{Sp}^+(\mathcal{H})$ is a diffeomorphism.*

Proof. Since clearly $\exp(\mathfrak{sp}(\mathcal{H})_h) \subset \mathrm{Sp}^+(\mathcal{H})$, then it suffices to show that the map $\exp : \mathfrak{sp}(\mathcal{H})_h \rightarrow \mathrm{Sp}^+(\mathcal{H})$ is onto. Given $g \in \mathrm{Sp}^+(\mathcal{H})$ there exists a unique symmetric endomorphism with $g = e^x$, then $g^{-1} = e^{-x} = Je^xJ^{-1} = \sum_{n=0}^{\infty} \frac{Jx^nJ^{-1}}{n!} = \sum_{n=0}^{\infty} \frac{(JxJ^{-1})^n}{n!} = e^{-JxJ}$, therefore $xJ = -Jx$ and $x \in \mathfrak{sp}(\mathcal{H})_h$. \square

Corollary 2.2.6. *If $u|g|$ is the polar decomposition of an element $g \in \mathrm{Sp}(\mathcal{H})$ then its unitary part u and its positive part $|g|$ belong to $\mathrm{Sp}(\mathcal{H})$ and therefore the map*

$$\begin{aligned} \mathrm{Sp}(\mathcal{H}) &\rightarrow \mathrm{U}(\mathcal{H}_J) \times \mathrm{Sp}^+(\mathcal{H}) \\ g &\mapsto (u, |g|) \end{aligned}$$

is a diffeomorphism.

Proof. Since $gg^* \in \mathrm{Sp}^+(\mathcal{H})$ then there exist $x \in \mathfrak{sp}(\mathcal{H})_h$ such that $e^x = gg^*$ and then it is clear that $|g| = (gg^*)^{1/2} = e^{x/2} \in \mathrm{Sp}^+(\mathcal{H})$. \square

2.2.1 The Hilbert-Schmidt symplectic group

Now, we consider the Hilbert-Schmidt symplectic group given by

$$\mathrm{Sp}_2(\mathcal{H}) = \{g \in \mathrm{Sp}(\mathcal{H}) : g - 1 \in \mathcal{B}_2(\mathcal{H})\}.$$

Given $g_1, g_2 \in \mathrm{Sp}_2(\mathcal{H})$, it is obvious that $g_1 - g_2$ belongs in $\mathcal{B}_2(\mathcal{H})$; hence we can endow the Hilbert-Schmidt symplectic group with the metric $\|g_1 - g_2\|_2$.

Proposition 2.2.7. *$\mathrm{Sp}_2(\mathcal{H})$ is a closed subgroup of $\mathrm{GL}_2(\mathcal{H})$ or equivalently the metric space $(\mathrm{Sp}_2(\mathcal{H}), \|\cdot\|_2)$ is complete.*

Proof. Let $(x_n) \subset \mathrm{Sp}_2(\mathcal{H})$ be a Cauchy sequence, then $x_n - 1$ is a Cauchy sequence in $\mathcal{B}_2(\mathcal{H})$. From this, we can take $x \in \mathcal{B}_2(\mathcal{H})$ such that $x_n \rightarrow 1 + x := x_0$ in $\|\cdot\|_2$. It is clear that x_0 verifies the algebraic relation $x_0^*Jx_0 = J$; to complete the proof we will see that x_0 is invertible. Indeed, from $x_n^* \in \mathrm{Sp}_2(\mathcal{H})$ we have $x_nJx_n^* = J$, then this relation is transferred through the limit to x_0 . We can now define the inverse of x_0 as $x_0^{-1} := -Jx_0^*J$, it verifies:

$$x_0^{-1}x_0 = -Jx_0^*Jx_0 = 1 \quad \text{and} \quad x_0x_0^{-1} = x_0(-Jx_0^*J) = 1.$$

\square

Now we will show that $\mathrm{Sp}_2(\mathcal{H})$ is a Banach-Lie group. Let us denote

$$\mathfrak{sp}_2(\mathcal{H}) = \{x \in \mathcal{B}_2(\mathcal{H}) : xJ = -Jx^*\}.$$

It is clear that $\mathfrak{sp}_2(\mathcal{H})$ is a Banach-Lie subalgebra of $\mathcal{B}_2(\mathcal{H})$ and since $\exp : \mathcal{B}_2(\mathcal{H}) \rightarrow 1 + \mathcal{B}_2(\mathcal{H})$ it is clear that

$$\mathfrak{sp}_2(\mathcal{H}) = \{x \in \mathcal{B}_2(\mathcal{H}) : e^{tx} \in \mathrm{Sp}_2(\mathcal{H}) \forall t \in \mathbb{R}\}.$$

Theorem 2.2.8. $\mathrm{Sp}_2(\mathcal{H})$ is a Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$.

Proof. The proof is similar to Theorem 2.2.3. Since the exponential is a local diffeomorphism between

$$U = \{x = \log(g) : \|g - 1\|_2 < 1\} \xrightarrow{\exp} V = \{g \in 1 + \mathcal{B}_2(\mathcal{H}) : \|g - 1\|_2 < 1\}$$

then $\exp(U \cap \mathfrak{sp}_2(\mathcal{H})) = \mathrm{Sp}_2(\mathcal{H}) \cap V$. Moreover $\ker \Pi|_{\mathcal{B}_2(\mathcal{H})} \oplus \mathfrak{sp}_2(\mathcal{H}) = \mathcal{B}_2(\mathcal{H})$. \square

Since the Banach-Lie algebra $\mathfrak{sp}_2(\mathcal{H})$ is closed under adjoints, here we have a Cartan decomposition as in the case of the full symplectic group,

$$\mathfrak{sp}_2(\mathcal{H}) = \mathfrak{sp}_2(\mathcal{H})_h \oplus \mathfrak{sp}_2(\mathcal{H})_{ah}. \quad (2.9)$$

As before we denote by

$$\mathrm{U}_2(\mathcal{H}_J) = \{g \in \mathrm{U}_2(\mathcal{H}) : gJ = Jg\} \text{ and } \mathrm{Sp}_2^+(\mathcal{H}) = \{g \in \mathrm{Sp}_2(\mathcal{H}) : g > 0\}$$

its unitary part and positive part respectively.

Proposition 2.2.9. The exponential map $\exp : \mathfrak{sp}_2(\mathcal{H})_h \rightarrow \mathrm{Sp}_2^+(\mathcal{H})$ is a diffeomorphism.

Proof. We know that $\exp : \mathfrak{sp}(\mathcal{H})_h \rightarrow \mathrm{Sp}^+(\mathcal{H})$ is onto, so if we have any point $g \in \mathrm{Sp}_2^+(\mathcal{H})$ there exist a unique $x \in \mathfrak{sp}(\mathcal{H})_h$ such that $g = e^x$, we have to prove that x belongs to $\mathcal{B}_2(\mathcal{H})$. Indeed, since $g \in \mathrm{Sp}_2^+(\mathcal{H})$ it has a spectral decomposition $g = P_0 + \sum_{k \geq 1} (1 + \alpha_k) P_k$, where α_k are the non zero eigenvalues of $g - 1 \in \mathcal{B}_2(\mathcal{H})_h$. Since $g = e^x$ then $\alpha_k = e^{t_k} - 1$ where $t_k \in \sigma(x) \subset \mathbb{R}$. Since the quotient $\frac{t_k^2}{(e^{t_k} - 1)^2} \rightarrow 1$ then the sequence (t_k) is square summable. Let $y = \sum_{k=1}^{\infty} t_k P_k$ thus $y \in \mathcal{B}_2(\mathcal{H})_h$ and $e^y = g = e^x$, therefore by the injectivity of the exponential on the symmetric operators we have $x = y \in \mathfrak{sp}_2(\mathcal{H})_h$. \square

Since the exponential map $\exp : \mathcal{B}_2(\mathcal{H}_J)_{ah} \rightarrow \mathrm{U}_2(\mathcal{H}_J)$ is surjective (see [4]), it is clear that $\exp(\mathfrak{sp}_2(\mathcal{H})_{ah}) = \mathrm{U}_2(\mathcal{H}_J)$.

Proposition 2.2.10. *The unitary subgroup $U_2(\mathcal{H}_J)$ is a Lie-subgroup of $Sp_2(\mathcal{H})$.*

Proof. Let U be a neighborhood of 0 in $\mathfrak{sp}_2(\mathcal{H})$ such that the exponential map is a diffeomorphism, we can assume that $U = \{x \in \mathfrak{sp}_2(\mathcal{H}) : \|x\|_2 < r\}$ for a suitable $r > 0$. It is clear that we always have

$$\exp(\mathfrak{sp}_2(\mathcal{H})_{ah} \cap U) \subseteq U_2(\mathcal{H}_J) \cap \exp(U).$$

Conversely, suppose that $g \in U_2(\mathcal{H}_J) \cap \exp(U)$ then $g = e^y$ for some $y \in U$; hence $1 = gg^* = e^y e^{y^*}$ and then $e^y = e^{-y^*}$. Since $-y^*$ also belongs in U and the exponential is one to one, we have that $y = -y^*$ and thus $y \in \mathfrak{sp}_2(\mathcal{H})_{ah}$. Then we have $\exp(\mathfrak{sp}_2(\mathcal{H})_{ah} \cap U) = U_2(\mathcal{H}_J) \cap \exp(U)$ and this implies that $U_2(\mathcal{H}_J)$ is a Lie-subgroup of $Sp_2(\mathcal{H})$. \square

2.3 Riemannian metrics in $Sp_2(\mathcal{H})$

Since $Sp_2(\mathcal{H})$ is a Banach-Lie subgroup of $GL_2(\mathcal{H})$ we can endow it with the left-invariant metric of the ambient manifold $GL_2(\mathcal{H})$. So, if $g \in Sp_2(\mathcal{H})$ and $v \in (TSp_2(\mathcal{H}))_g = g \cdot \mathfrak{sp}_2(\mathcal{H})$ the left-invariant metric is

$$\mathcal{I}(g, v) := \|g^{-1}v\|_2.$$

Proposition 2.3.1. *$Sp_2(\mathcal{H})$ is a totally geodesic submanifold of $GL_2(\mathcal{H})$. Equivalently, if $\alpha \subset Sp_2(\mathcal{H})$ is a curve and η a field along α then*

$$D_t \eta \in (TSp_2(\mathcal{H}))_\alpha = \alpha \cdot \mathfrak{sp}_2(\mathcal{H}).$$

Proof. Let $\beta = \alpha^{-1}\dot{\alpha}$ and $\mu = \alpha^{-1}\eta$ be the fields moved to $\mathfrak{sp}_2(\mathcal{H})$, we will show that $\alpha^{-1}D_t \eta \in \mathfrak{sp}_2(\mathcal{H})$. Indeed μ verifies $\mu J = -J\mu^*$, if we derive, we obtain $\dot{\mu}J = -J\dot{\mu}^*$ and $\dot{\mu}$ is a Hilbert-Schmidt operator that lies in $\mathfrak{sp}_2(\mathcal{H})$. The brackets $[\beta, \mu], [\beta, \mu^*], [\mu, \beta^*]$ are all in $\mathfrak{sp}_2(\mathcal{H})$ since it is a Banach-Lie algebra, then using the above proposition

$$\alpha^{-1}D_t \eta = \dot{\mu} + 1/2\{[\beta, \mu] + [\beta, \mu^*] + [\mu, \beta^*]\} \in \mathfrak{sp}_2(\mathcal{H}).$$

\square

In particular the geodesics of $Sp_2(\mathcal{H})$ are the same than those of $GL_2(\mathcal{H})$; if $g_0 \in Sp_2(\mathcal{H})$ and $g_0 v_0 \in g_0 \cdot \mathfrak{sp}_2(\mathcal{H})$ are the initial position and the initial velocity then

$$\alpha(t) = g_0 e^{tv_0^*} e^{t(v_0 - v_0^*)} \in Sp_2(\mathcal{H})$$

satisfies $D_t \dot{\alpha} = 0$. In this context the Riemannian exponential for $g \in Sp_2(\mathcal{H})$ is

$$Exp_g(v) = g e^{v^*} e^{v - v^*}$$

with $v \in \mathfrak{sp}_2(\mathcal{H})$.

2.3.1 Riemannian metrics in $\mathrm{Sp}_2^+(\mathcal{H})$

From the stability of the adjoint operation in $\mathrm{Sp}_2(\mathcal{H})$ we can restrict the natural action of the invertible group to the set of positive invertible operators.

Lemma 2.3.2. *The natural action $l : \mathrm{Sp}_2(\mathcal{H}) \times \mathrm{Sp}_2^+(\mathcal{H}) \longrightarrow \mathrm{Sp}_2^+(\mathcal{H})$ given by*

$$(g, a) \longmapsto gag^*$$

is well defined and transitive.

Proof. Since the group is closed under the adjoint, the map $(g, a) \longmapsto gag^*$ is well defined and it is clear that $gag^* \in \mathrm{Sp}_2(\mathcal{H})$ and it is positive. If $X, Y \in \mathrm{Sp}_2^+(\mathcal{H})$, we can assume that $X = e^x$, $Y = e^y$ where $x, y \in \mathfrak{sp}_2(\mathcal{H})_h$; then if we consider the operator $g = e^{x/2}e^{-y/2} \in \mathrm{Sp}_2(\mathcal{H})$ it verifies that $X = gYg^*$. \square

Now we endow the closed submanifold $\mathrm{Sp}_2^+(\mathcal{H})$ with the induced metric (1.2) of $\mathrm{GL}_2^+(\mathcal{H})$; if $a \in \mathrm{Sp}_2^+(\mathcal{H})$ and

$$x \in T_a\mathrm{Sp}_2^+(\mathcal{H}) = \{a^{1/2} \ln(a^{-1/2}qa^{-1/2})a^{1/2} : q \in \mathrm{Sp}_2^+(\mathcal{H})\}$$

we put the metric of positive operators given by

$$\mathfrak{p}(a, x) := \|a^{-1/2}xa^{-1/2}\|_2.$$

Remark 2.3.3. *The above metric is invariant for the action of the group $\mathrm{Sp}_2(\mathcal{H})$, that is: if $x \in T_a\mathrm{Sp}_2^+(\mathcal{H})$ then*

$$\mathfrak{p}(gag^*, gxg^*) = \mathfrak{p}(a, x).$$

Since $\mathfrak{sp}_2(\mathcal{H})_h$ is a Lie triple system and $\exp(\mathfrak{sp}_2(\mathcal{H})_h) = \mathrm{Sp}_2^+(\mathcal{H}) \subset \mathrm{GL}_2^+(\mathcal{H})$ then it is a geodesically convex submanifold and therefore totally geodesic, see for instance Corollary 3.13 and Proposition 3.6 in [19].

Corollary 2.3.4. *The covariant derivative in $\mathrm{Sp}_2^+(\mathcal{H})$ with the induced positive metric is given by*

$$\nabla_\eta \mu_p = \eta(\mu)_p - \frac{1}{2}(\eta_p p^{-1} \mu_p + \mu_p p^{-1} \eta_p) \quad (3.10)$$

where η, μ are tangent fields on $T\mathrm{Sp}_2^+(\mathcal{H})$ and $\eta(\mu)$ denotes derivation of the vector field μ in the direction of η .

Proof. Since $\mathrm{Sp}_2^+(\mathcal{H})$ is totally geodesic by Theorem 1.4.9 we have that the covariant derivative coincides with the covariant derivative of the ambient submanifold $\mathrm{GL}_2^+(\mathcal{H})$. \square

Euler's equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ for the covariant derivative introduced by the Riemannian connection reads $\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}$, the unique solution of this equation with $\gamma(0) = p$ and $\gamma(1) = q$ is given by the smooth curve

$$\gamma_{pq}(t) = p^{1/2}(p^{-1/2}qp^{-1/2})^t p^{1/2}.$$

The curve $\gamma_{pq}(t) = p^{1/2}(p^{-1/2}qp^{-1/2})^t p^{1/2} = p^{1/2}e^{t(\ln(p^{-1/2}qp^{-1/2}))}p^{1/2} \subset \mathrm{Sp}_2^+(\mathcal{H})$ joins p to q and its length is

$$L_{\mathfrak{p}}(\gamma_{pq}) = \|\ln(p^{-1/2}qp^{-1/2})\|_2.$$

This curve is minimal among all curves in $\mathrm{Sp}_2^+(\mathcal{H})$ that join p to q . We will give a short proof of this fact, the key is the following inequality.

Remark 2.3.5. (See [15]) *If $d\exp_x$ denotes the differential of exponential at x of the usual exponential map, then*

$$\mathfrak{p}(e^x, d\exp_x(y)) = \|e^{-x/2}d\exp_x(y)e^{-x/2}\|_2 \geq \|y\|_2. \quad (3.11)$$

for any $x, y \in \mathcal{B}_2(\mathcal{H})_h$.

Theorem 2.3.6. *Let $p, q \in \mathrm{Sp}_2^+(\mathcal{H})$ then $\gamma_{pq} \subset \mathrm{Sp}_2^+(\mathcal{H})$ has minimal length among all curves that joins p to q .*

Proof. We can suppose that $p = 1$, then $\gamma_{1q}(t) = e^{tx}$ where $x = \ln(q)$ and its length is $\|x\|_2 = \|\ln(q)\|_2$. If α is another curve that joins the same points, then it can be written as $\alpha(t) = e^{\beta(t)}$ where $\beta(t) = \ln(\alpha(t)) \in \mathfrak{sp}_2(\mathcal{H})_h$. Using the above remark we have

$$L_{\mathfrak{p}}(\gamma_{1q}) = \|x - 0\|_2 = \left\| \int_0^1 \dot{\beta}(t) dt \right\|_2 \leq \int_0^1 \|\dot{\beta}(t)\|_2 dt$$

and also

$$\begin{aligned} \mathfrak{p}(\alpha, \dot{\alpha}) &= \mathfrak{p}(e^{\beta(t)}, d\exp_{\beta(t)}(\dot{\beta}(t))) \\ &= \|e^{-\beta(t)/2}d\exp_{\beta(t)}(\dot{\beta}(t))e^{-\beta(t)/2}\|_2 \geq \|\dot{\beta}(t)\|_2. \end{aligned}$$

□

It can be shown that the metric space $(\mathrm{Sp}_2^+(\mathcal{H}), d_{\mathfrak{p}})$ is complete. This fact was proved in [8] or [18] in another context; in this context we also can derive from (3.11) the known inequality

$$d_{\mathfrak{p}}(p, q) \geq \|\log p - \log q\|_2$$

for $p, q \in \mathrm{Sp}_2^+(\mathcal{H})$; the proof of completeness can be adapted easily, therefore we omit it.

2.3.2 $\mathrm{Sp}_2^+(\mathcal{H})$ as submanifold of the ambient space

Here we will think $\mathrm{Sp}_2^+(\mathcal{H})$ as a submanifold of the real Hilbert space $H_{\mathbb{R}} := \mathbb{R} \oplus \mathcal{B}_2(\mathcal{H})_h$ with the natural inner product

$$\langle \lambda + a, \mu + b \rangle = \lambda\mu + \mathrm{Tr}(b^*a).$$

From the action given by Lemma 2.3.2 we can define for each $a \in \mathrm{Sp}_2^+(\mathcal{H})$ the map

$$\pi_a : \mathrm{Sp}_2(\mathcal{H}) \rightarrow \mathrm{Sp}_2^+(\mathcal{H}), \quad \pi_a(g) = gag^*.$$

Observe that, since the action is transitive this map is onto and as in the case of the full space of positive invertible operators $\mathcal{B}(\mathcal{H})^+$ (see [9]), we have that $\sigma_a(b) = b^{1/2}a^{-1/2}$ defines a global smooth section of π_a . Note that this map is well defined and its image belongs clearly to $\mathrm{Sp}_2(\mathcal{H})$.

If g is any element in $\mathrm{Sp}_2^+(\mathcal{H})$, we can consider the real linear map

$$\Pi_g : H_{\mathbb{R}} \longrightarrow H_{\mathbb{R}}, \quad x \longmapsto \frac{1}{2}(x + gJxJg).$$

This map is well defined and a short computation shows that the range belongs to $\mathcal{B}_2(\mathcal{H})_h$.

Lemma 2.3.7. *The map Π_g is idempotent and its range is $g^{1/2}\mathfrak{sp}_2(\mathcal{H})_hg^{1/2}$. Moreover, its adjoint map for the trace inner product is $\Pi_{g^{-1}}$. If $g = 1$ this map is the orthogonal projection onto $\mathfrak{sp}_2(\mathcal{H})_h$.*

Proof. First we prove that Π_g is an idempotent map. Indeed, using the fact that $gJg = J$,

$$\begin{aligned} \Pi_g^2(x) &= \Pi_g\left(\frac{1}{2}(x + gJxJg)\right) = \frac{1}{4}(x + gJxJg + gJ(x + gJxJg)Jg) = \\ &= \frac{1}{4}(x + 2gJxJg + (gJg)JxJ(gJg)) = \Pi_g(x). \end{aligned}$$

Now we will prove that $\mathrm{Ran}(\Pi_g) = g^{1/2}\mathfrak{sp}_2(\mathcal{H})_hg^{1/2}$. Indeed, let $g^{1/2}xg^{1/2}$ with $x \in \mathfrak{sp}_2(\mathcal{H})_h$, then using that $g^{1/2}Jg^{1/2} = J$ (that is $g^{1/2} \in \mathrm{Sp}_2^+(\mathcal{H})$) and the relation of x with J we have

$$\Pi_g(g^{1/2}xg^{1/2}) = \frac{1}{2}(g^{1/2}xg^{1/2} + g^{1/2}g^{1/2}Jg^{1/2}xg^{1/2}Jg) = g^{1/2}xg^{1/2}.$$

Finally, note that the range is contained in $g^{1/2}\mathfrak{sp}_2(\mathcal{H})_hg^{1/2}$;

$$\frac{1}{2}(x + gJxJg) = g^{1/2}\frac{1}{2}\left(g^{-1/2}xg^{-1/2} + g^{1/2}JxJg^{1/2}\right)g^{1/2}.$$

To conclude we must show that the expression in the bracket anti-commutes with J , here we will use that $J^2 = -1$ and the relation $g^{1/2}J = Jg^{-1/2}$:

$$\begin{aligned} (g^{-1/2}xg^{-1/2} + g^{1/2}JxJg^{1/2})J &= -g^{-1/2}JJxJg^{1/2} - Jg^{-1/2}xg^{-1/2} = \\ &= -J(g^{1/2}JxJg^{1/2} + g^{-1/2}xg^{-1/2}). \end{aligned}$$

Now we will show that $\Pi_g^* = \Pi_{g^{-1}}$; first note that if $x, y \in H_{\mathbb{R}}$ by the invariant and cyclic properties of the trace we have

$$\begin{aligned} \text{Tr}(ygJxJg) &= \text{Tr}(-JygJxJgJ) = \text{Tr}(JygJxg^{-1}) = \text{Tr}(g^{-1}JygJx) \\ &= \text{Tr}(g^{-1}JyJg^{-1}x). \end{aligned}$$

Then the inner product is

$$\begin{aligned} \langle \Pi_g(x), y \rangle &= \text{Tr}\left(y\left(\frac{1}{2}(x + gJxJg)\right)\right) = \frac{1}{2}\text{Tr}(yx + ygJxJg) = \\ &= \frac{1}{2}(\text{Tr}(yx) + \text{Tr}(g^{-1}JyJg^{-1}x)). \end{aligned}$$

On the other hand, we have

$$\langle x, \Pi_{g^{-1}}(y) \rangle = \text{Tr}\left(\frac{1}{2}(y + g^{-1}JyJg^{-1})x\right) = \frac{1}{2}(\text{Tr}(yx) + \text{Tr}(g^{-1}JyJg^{-1}x)).$$

□

It is natural to consider a Hilbert-Riemann metric in $\text{Sp}_2^+(\mathcal{H})$, which consists of endowing each tangent space with the trace inner product. Therefore the Levi-Civita connection of this metric is given by differentiating in the ambient space $H_{\mathbb{R}}$ and projecting onto $T\text{Sp}_2^+(\mathcal{H})$. For this, we define the positive ambient metric as;

$$\mathfrak{p}_{amb}(g, x) := \|x\|_2$$

where $x \in T_g\text{Sp}_2^+(\mathcal{H})$. Using the formula of the projector over its range and Lemma 2.3.7, we can calculate the orthogonal projection onto $T_g\text{Sp}_2^+(\mathcal{H})$; that is

$$E_{T_g\text{Sp}_2^+(\mathcal{H})} = \Pi_g(\Pi_g + \Pi_g^* - 1)^{-1} = (\Pi_g + \Pi_g^* - 1)^{-1}\Pi_g^* = (\Pi_g + \Pi_{g^{-1}} - 1)^{-1}\Pi_{g^{-1}}.$$

Then, if γ is a smooth curve in $\text{Sp}_2^+(\mathcal{H})$ and $\mathcal{X}(t)$ is a smooth tangent field along γ the covariant derivative is

$$\frac{D}{dt}\mathcal{X}(t) = E_{\gamma(t)}(\dot{\mathcal{X}}(t)).$$

Proposition 2.3.8. *A curve α is a geodesic of the Levi-Civita connection if and only if it satisfies the differential equation*

$$\alpha\ddot{\alpha} + J\ddot{\alpha}J = 0.$$

Proof. Using the last expression of the orthogonal projection E , we have

$$\frac{D}{dt}\dot{\alpha}(t) = 0 \Leftrightarrow \Pi_{\alpha^{-1}(t)}(\ddot{\alpha}(t)) = 0 \Leftrightarrow \ddot{\alpha} + \alpha^{-1}J\ddot{\alpha}J\alpha^{-1} = 0.$$

□

2.4 Polar Riemannian structure in $\mathrm{Sp}_2(\mathcal{H})$

Since the polar decomposition is stable in the group we can restrict the map (1.4) on $\mathrm{Sp}_2(\mathcal{H})$. So, we endow the product manifold $\mathrm{U}_2(\mathcal{H}_J) \times \mathrm{Sp}_2^+(\mathcal{H})$ with the usual product metric, that is: if $v = (x, y) \in T_u\mathrm{U}_2(\mathcal{H}_J) \times T_{|g|}\mathrm{Sp}_2^+(\mathcal{H})$ we put

$$\begin{aligned} \mathcal{P}((u, |g|), v) &:= \left(\mathcal{I}(u, x)^2 + \mathfrak{p}(|g|, y)^2 \right)^{1/2} \\ &= \left(\|x\|_2^2 + \||g|^{-1/2}y|g|^{-1/2}\|_2^2 \right)^{1/2}, \end{aligned}$$

then we can define the polar Riemannian metric in the Hilbert-Schmidt symplectic group: if $v, w \in (T\mathrm{Sp}_2(\mathcal{H}))_g$ we put

$$\langle v, w \rangle_g := \langle d\varphi_g(v), d\varphi_g(w) \rangle_{(u, |g|)}.$$

In other words this polar metric is simply the induced polar metric (1.5) of the ambient manifold $\mathrm{GL}_2(\mathcal{H})$.

Proposition 2.4.1. *$\mathrm{Sp}_2(\mathcal{H})$ is a totally geodesic submanifold of $\mathrm{GL}_2(\mathcal{H})$ when we consider the polar metric.*

Proof. Since the manifolds $\mathrm{Sp}_2^+(\mathcal{H})$ and $\mathrm{U}_2(\mathcal{H}_J)$ are totally geodesic (Corollary 2.3.4) we have that the Levi-Civita derivative on the ambient manifold $\mathrm{U}_2(\mathcal{H}) \times \mathrm{GL}_2^+(\mathcal{H})$ restricts to $\mathrm{U}_2(\mathcal{H}_J) \times \mathrm{Sp}_2^+(\mathcal{H})$. □

Theorem 2.4.2. *Let $g \in \mathrm{Sp}_2(\mathcal{H})$ with polar decomposition $u|g|$ and suppose that $u = e^x$ with $x \in \mathfrak{sp}_2(\mathcal{H})_{ah}$ and $\|x\| \leq \pi$, then the curve $\alpha(t) = e^{tx}|g|^t \subset \mathrm{Sp}_2(\mathcal{H})$ has minimal length among all curves joining 1 to g .*

Proof. The proof is an analogous computation as that of Theorem 2.4.2. Indeed, the polar length of α is $(\|x\|_2^2 + \|\ln |g|\|_2^2)^{1/2}$, let β be another curve that joins the same endpoints and let β_1, β_2 be its polar decomposition. By the stability of polar decomposition we have $\beta_1 \subset \mathbf{U}_2(\mathcal{H}_J)$ and $\beta_2 \subset \mathbf{Sp}_2^+(\mathcal{H})$ therefore using Theorem 2.3.6 and the minimality of exponential in the unitary group we have,

$$L_{\mathcal{I}}(\beta_1) \geq L_{\mathcal{I}}(e^{tx}) = \|x\|_2 \quad \text{and} \quad L_{\mathfrak{p}}(\beta_2) \geq L_{\mathfrak{p}}(e^{t \ln(|g|)}) = \|\ln |g|\|_2$$

then it is clear that $L_{\mathcal{P}}(\varphi \circ \beta) \geq L_{\mathcal{P}}(\varphi \circ \alpha)$. □

Remark 2.4.3. Let $p, q \in \mathbf{Sp}_2(\mathcal{H})$, suppose that $u_p|p|$ and $u_q|q|$ are their polar decompositions, from the surjectivity of the exponential map we can choose $z \in \mathfrak{sp}_2(\mathcal{H})_{ah}$ such that $u_q = u_p e^z$ with $\|z\| \leq \pi$, then the curve

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \subset \mathbf{Sp}_2(\mathcal{H}) \quad (4.12)$$

has minimal length among all curves joining p to q .

Therefore, its length is

$$\left(\|z\|_2^2 + \|\ln |p|^{-1/2} |q| |p|^{-1/2}\|_2^2 \right)^{1/2},$$

and the geodesic distance is

$$d_{\mathcal{P}}(p, q) = \left(d_{\mathcal{I}}(u_p, u_q)^2 + d_{\mathfrak{p}}(|p|, |q|)^2 \right)^{1/2}.$$

Special case: normal speed. If the initial condition $v \in \mathfrak{sp}_2(\mathcal{H})$ is normal, then the geodesics starting at the identity map coincide with the geodesics from the polar metric. Indeed, if $v = x + y$ is the decomposition in $\mathfrak{sp}_2(\mathcal{H})_h \oplus \mathfrak{sp}_2(\mathcal{H})_{ah}$ and v is normal a straightforward computation shows that x commutes with y , thus we have

$$e^{tv^*} e^{t(v-v^*)} = e^{tv} = e^{tx} e^{ty}.$$

This equation shows that the geodesics are one-parameter groups when the initial speed is normal.

Proposition 2.4.4. The metric space $(\mathbf{Sp}_2(\mathcal{H}), d_{\mathcal{P}})$ is complete.

Proof. Let $(x_n) \subset \mathrm{Sp}_2(\mathcal{H})$ be a Cauchy sequence with $d_{\mathcal{P}}$, if $x_n = u_{x_n}|x_n|$ is its polar decomposition, we have that

$$d_{\mathcal{I}}(u_{x_n}, u_{x_m}) \leq d_{\mathcal{P}}(x_n, x_m) = (d_{\mathcal{I}}(u_{x_n}, u_{x_m})^2 + d_{\mathfrak{p}}(|x_n|, |x_m|)^2)^{1/2}$$

then the unitary part is a Cauchy sequence in $(\mathrm{U}_2(\mathcal{H}_J), d_{\mathcal{I}})$ and by [4] it is $d_{\mathcal{I}}$ convergent to an element $u \in \mathrm{U}_2(\mathcal{H}_J)$. Analogously the positive part is a Cauchy sequence in $(\mathrm{Sp}_2^+(\mathcal{H}), d_{\mathfrak{p}})$ then it is convergent to an element $g \in \mathrm{Sp}_2^+(\mathcal{H})$. If we put $x := ug \in \mathrm{Sp}_2(\mathcal{H})$ then,

$$d_{\mathcal{P}}(x_n, x) = (d_{\mathcal{I}}(u_{x_n}, u)^2 + d_{\mathfrak{p}}(|x_n|, g)^2)^{1/2} \rightarrow 0.$$

□

In the next steps we will compare the geodesic distance measured with the polar metric versus the left invariant metric. It is a computation analogous to that in Proposition 2.1.6.

Proposition 2.4.5. *Given $p, q \in \mathrm{Sp}_2(\mathcal{H})$, if we denote $v := |p|^{-1/2}|q||p|^{-1/2}$ we can estimate the geodesic distance $d_{\mathcal{I}}$ by the geodesic distance $d_{\mathcal{P}}$ as,*

$$d_{\mathcal{I}}(p, q) \leq c(p, q)d_{\mathcal{P}}(p, q)$$

where

$$c(p, q)^2 = 2 \max \left\{ e^{4\|\ln(v)\|} (\|p\|\|p^{-1}\|)^2, \|p\|\|p^{-1}\| \right\}.$$

Proof. Given two points p, q we can build the smooth curve $\alpha_{p,q} \subset \mathrm{Sp}_2(\mathcal{H})$ (4.12) that joins p to q ; therefore if we repeat the argument that we gave in Proposition 2.1.6 we get the same inequality.

□

2.5 The metric space $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{I}})$

In this section we will prove the main result of this chapter, that is the completeness of $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{I}})$, it will be deduced from the completeness of $(\mathrm{U}_2(\mathcal{H}_J), d_{\mathcal{I}})$ and from Proposition 2.4.5. The next lemma is essential for the proof.

Lemma 2.5.1. *If $(x_n) \subset \mathrm{Sp}_2(\mathcal{H})$ is a Cauchy sequence in $(\mathrm{Sp}_2(\mathcal{H}), d_{\mathcal{I}})$ then it is a Cauchy sequence in $(\mathrm{Sp}_2(\mathcal{H}), \|\cdot\|_2)$.*

Proof. First we take W, U geodesic neighborhoods of 0 and 1 respectively such that

$$\text{Exp}_1 : W \longrightarrow U := \text{Exp}_1(W) \subset \text{Sp}_2(\mathcal{H})$$

is a diffeomorphism. If (x_n) is $d_{\mathcal{I}}$ -Cauchy, given small ε there exist $n(\varepsilon)$ such that $d_{\mathcal{I}}(x_n^{-1}x_{n+p}, 1) = d_{\mathcal{I}}(x_{n+p}, x_n) < \varepsilon \forall p$. Then we can suppose that $x_n^{-1}x_{n+p} \in U$ for all p . Let $\alpha_p(t) = e^{tv_p^*}e^{t(v_p - v_p^*)} = \text{Exp}_1(tv_p)$ with $v_p \in W$ be the minimal curve that joins 1 to $x_n^{-1}x_{n+p}$, then

$$d_{\mathcal{I}}(x_n^{-1}x_{n+p}, 1) = L_{\mathcal{I}}(\alpha_p) = \|v_p\|_2 < \varepsilon.$$

We have

$$\begin{aligned} \|x_n^{-1}x_{n+p} - 1\|_2 &\leq \int_0^1 \|\dot{\alpha}_p(t)\|_2 dt \leq \int_0^1 \|\alpha_p(t)\| \|\alpha_p^{-1}\dot{\alpha}_p(t)\|_2 dt, \\ \|\alpha_p(t)\| &= \|e^{tv_p^*}e^{t(v_p - v_p^*)}\| \leq e^{3\|v_p\|_2} \leq e^{3\varepsilon}. \end{aligned}$$

From this,

$$\|x_n^{-1}x_{n+p} - 1\|_2 \leq e^{3\varepsilon}\varepsilon, \text{ for all } p.$$

This fact shows that the sequence is bounded in the uniform norm; indeed if we take ε_0 such that the sequence belongs in the geodesic neighborhood U , then there exists n_0 (fixed) such that $\|x_{n_0}^{-1}x_{n_0+p} - 1\|_2 \leq e^{3\varepsilon_0}\varepsilon_0$, for all p . Then if $m = n_0 + p > n_0$, we have

$$\| \|x_{n_0}\| - \|x_m\| \| \leq \|x_{n_0} - x_m\|_2 \leq \|x_{n_0}\| \|x_{n_0}^{-1}x_{n_0+p} - 1\|_2 \leq \|x_{n_0}\| e^{3\varepsilon_0}\varepsilon_0;$$

then

$$\|x_{m=n_0+p}\| \leq \| \|x_m\| - \|x_{n_0}\| \| + \|x_{n_0}\| \leq \|x_{n_0}\| (1 + e^{3\varepsilon_0}\varepsilon_0) \forall p.$$

To complete the proof, if n is large, we have

$$\|x_{n+p} - x_n\|_2 = \|x_n(x_n^{-1}x_{n+p} - 1)\|_2 \leq \|x_n\| e^{3\varepsilon}\varepsilon \leq K e^{3\varepsilon}\varepsilon \quad \forall p.$$

□

Now we are in a position to obtain our main result in this chapter.

Theorem 2.5.2. *The metric space $(\text{Sp}_2(\mathcal{H}), d_{\mathcal{I}})$ is complete.*

Proof. Let $(x_n) \subset \text{Sp}_2(\mathcal{H})$ be a $d_{\mathcal{I}}$ -Cauchy sequence, by the above lemma it is $\|\cdot\|_2$ -Cauchy; then from Proposition 2.2.7 there exists $x \in \text{Sp}_2(\mathcal{H})$ such that $x_n \xrightarrow{\|\cdot\|_2} x$. Now we will show that $x_n \xrightarrow{d_{\mathcal{I}}} x$; indeed from the continuity of the module we have that $|x_n|$ converges to $|x|$ in $\|\cdot\|_2$ and its unitary part $u_{x_n} =$

$x_n|x_n|^{-1}$ converges to $u_x = x|x|^{-1}$. The sequence $|x|^{-1/2}|x_n||x|^{-1/2}$ converges to 1 and then the geodesic distance $d_{\mathfrak{p}}(|x_n|, |x|) = \|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\|_2 \rightarrow 0$. By the equivalence of metrics in $U_2(\mathcal{H}_J)$ (see [4] for a proof) we have

$$\sqrt{1 - \frac{\pi^2}{12}} d_{\mathcal{I}}(u_{x_n}, u_x) \leq \|u_{x_n} - u_x\|_2 \leq d_{\mathcal{I}}(u_{x_n}, u_x)$$

and then

$$d_{\mathcal{P}}(x_n, x) = (d_{\mathcal{I}}(u_{x_n}, u_x)^2 + d_{\mathfrak{p}}(|x_n|, |x|)^2)^{1/2} \rightarrow 0.$$

From Proposition 2.4.5 we have $d_{\mathcal{I}}(x, x_n) \leq c(x, x_n)d_{\mathcal{P}}(x, x_n)$; now we will see that $c(x, x_n)$ is uniformly bounded. Indeed, for n large we can assume that $\|\ln(v_n)\| \leq 1$ where $v_n = |x|^{-1/2}|x_n||x|^{-1/2}$ as we denoted in Proposition 2.4.5, then we have

$$\begin{aligned} c(x, x_n)^2 &= 2 \max\{ e^{4\|\ln(v_n)\|} (\|x\|\|x^{-1}\|)^2, \|x\|\|x^{-1}\| \} \\ &\leq 2 \max\{ e^4 (\|x\|\|x^{-1}\|)^2, \|x\|\|x^{-1}\| \} \end{aligned}$$

and it is clearly uniformly bounded thus it is clear that $d_{\mathcal{I}}(x, x_n) \rightarrow 0$.

□

Chapter 3

An homogeneous space of $\mathrm{Sp}_2(\mathcal{H})$

A lo largo de este capítulo presentaremos un espacio homogéneo del grupo simpléctico; La Grasmanniana Lagrangiana de Hilbert Schmidt. Estudiaremos su estructura diferencial y sus posibles métricas.

Throughout this chapter we introduce an homogeneous space of the symplectic group; the Hilbert-Schmidt Lagrangian Grassmannian. We will study its smooth structure and its possible metrics.

The Lagrangian Grassmannian $\Lambda(\mathcal{H})$ is the set of closed linear subspaces $L \subset \mathcal{H}$ such that $J(L) = L^\perp$. Clearly $\mathrm{Sp}(\mathcal{H})$ acts on $\Lambda(\mathcal{H})$ by means of $g.L = g(L)$. Indeed, it is sufficient prove that $J(g(L)) \subset g(L)^\perp$. If $\eta, \xi \in L$ then

$$\langle J(g(\eta)), g(\xi) \rangle = \langle g^* Jg(\eta), \xi \rangle = \langle J(\eta), \xi \rangle = 0.$$

Since the action of the unitary group $U(\mathcal{H}_J)$ is transitive on $\Lambda(\mathcal{H})$ (see [28] Theorem 3.5), it is clear that the action of $\mathrm{Sp}(\mathcal{H})$ is also transitive on $\Lambda(\mathcal{H})$, so we can think of $\Lambda(\mathcal{H})$ as an orbit for a fixed $L_0 \in \Lambda(\mathcal{H})$, i.e

$$\Lambda(\mathcal{H}) = \{g(L_0) : g \in \mathrm{Sp}(\mathcal{H})\}.$$

We denote by $P_L \in \mathcal{B}(\mathcal{H})$ the orthogonal projection onto L . It is customary to parametrize closed subspaces via orthogonal projections, $L \leftrightarrow P_L$, in order to carry on geometric or analytic computations. We shall also consider here an alternative description of the Lagrangian subspaces using projections and symmetries. That is, L is a Lagrangian subspace if and only if $P_L J + J P_L = J$, see [11] for a proof. Another description of this equation using symmetries is $\epsilon_L J = -J \epsilon_L$, where $\epsilon_L = 2P_L - 1$ is the symmetric orthogonal transformation which acts as the identity in L and minus the identity in L^\perp .

The isotropy subgroup at L is

$$\mathrm{Sp}(\mathcal{H})_L = \{g \in \mathrm{Sp}(\mathcal{H}) : g(L) = L\}.$$

It is obvious that this subgroup is a closed subgroup of $\mathrm{Sp}(\mathcal{H})$. In the infinite dimensional setting we know that this does not guarantee a nice submanifold structure; in Proposition 3.1.8 we will prove that $\mathrm{Sp}(\mathcal{H})_L$ is a Banach-Lie subgroup of $\mathrm{Sp}(\mathcal{H})$.

We can restrict the natural action of the symplectic group in $\Lambda(\mathcal{H})$ to the Hilbert-Schmidt symplectic group and it will also be smooth. As before, we can consider the isotropy group at L

$$\mathrm{Sp}_2(\mathcal{H})_L = \{g \in \mathrm{Sp}_2(\mathcal{H}) : g(L) = L\}.$$

We will also prove in Proposition 3.1.8 that this is a Banach-Lie subgroup of $\mathrm{Sp}_2(\mathcal{H})$, with the topology induced by the metric $\|g_1 - g_2\|_2$.

If T is any operator we denote by Gr_T its graph, i.e. the subset $Gr_T = \{v + Tv : v \in \mathrm{Dom}(T)\} \subset \mathcal{H} \oplus \mathcal{H}$. Fix a Lagrangian subspace $L_0 \subset \mathcal{H}$, we consider the subset of $\Lambda(\mathcal{H})$

$$\mathcal{O}_{L_0} = \{g(L_0) : g \in \mathrm{Sp}_2(\mathcal{H})\} \subseteq \Lambda(\mathcal{H}).$$

We will see that this set is strictly contained in $\Lambda(\mathcal{H})$ and thus the action of $\mathrm{Sp}_2(\mathcal{H})$ on the Lagrangian Grassmannian is not transitive. Perhaps a more natural approach would be to consider the set of pairs (L_1, L_2) of Lagrangians such that $L_2 = g(L_1)$ for some $g \in \mathrm{Sp}_2(\mathcal{H})$. However the orbit approach makes the presentation of the metrics simple. The purpose of this chapter is the geometric study of this orbit; its manifold structure and relevant metrics.

3.1 Manifold structure of \mathcal{O}_{L_0}

We start by proving that the subset \mathcal{O}_{L_0} is strictly contained in $\Lambda(\mathcal{H})$, to do it we need the following lemma.

Lemma 3.1.1. *Let $g \in \mathrm{Sp}_2(\mathcal{H})$ then $P_{g(L_0)} - P_{L_0} \in \mathcal{B}_2(\mathcal{H})$.*

Proof. To prove it, we use the formula of the orthogonal projector over the range of an operator Q given by

$$P_{R(Q)} = QQ^*(1 - (Q - Q^*)^2)^{1/2}. \quad (1.1)$$

This formula can be obtained using a block matrix representation. If we denote by Q the idempotent associated with $g(L_0)$, i.e. $Q := gP_{L_0}g^{-1}$ and if we suppose that $g = 1 + k$ and $g^{-1} = 1 + k'$ where $k, k' \in \mathcal{B}_2(\mathcal{H})$ we have

$$QQ^* = (1 + k)P_{L_0}(1 + k')(1 + k'^*)P_{L_0}(1 + k^*)$$

$$\begin{aligned}
&= \underbrace{(P_{L_0} + P_{L_0}k' + kP_{L_0} + kP_{L_0}k')}_Q \underbrace{(P_{L_0} + P_{L_0}k^* + k'^*P_{L_0} + k'^*P_{L_0}k^*)}_{Q^*} \\
&= P_{L_0} + \underbrace{P_{L_0}k^* + P_{L_0}k'^*P_{L_0} + \dots}_{\in \mathcal{B}_2(\mathcal{H})} = P_{L_0} + T \in P_{L_0} + \mathcal{B}_2(\mathcal{H}).
\end{aligned}$$

It is clear that $Q - Q^* \in \mathcal{B}_2(\mathcal{H})$, then $(Q - Q^*)^2 \in \mathcal{B}_1(\mathcal{H})$. From the spectral theorem we have,

$$1 - (Q - Q^*)^2 = 1 + \sum_i \lambda_i P_i = P_0 + \sum_i (\lambda_i + 1) P_i$$

where $(\lambda_i) \in \ell^1$ and P_0 is the projection to the kernel. Taking square roots, we have

$$\begin{aligned}
(1 - (Q - Q^*)^2)^{1/2} &= P_0 + \sum_i (\lambda_i + 1)^{1/2} P_i \\
&= P_0 + \sum_i [(\lambda_i + 1)^{1/2} - 1] P_i + \sum_i 1 P_i \\
&= 1 + \sum_i [(\lambda_i + 1)^{1/2} - 1] P_i = 1 + T' \in 1 + \mathcal{B}_2(\mathcal{H})
\end{aligned}$$

where $((\lambda_i + 1)^{1/2} - 1) \in \ell^2$, because $(\lambda_i) \in \ell^1$ and $\lim_{x \rightarrow 0} \frac{((x + 1)^{1/2} - 1)^2}{x} = 0$. Then by the formula (1.1) we have

$$P_{g(L_0)} = (P_{L_0} + T)(1 + T') \in P_{L_0} + \mathcal{B}_2(\mathcal{H}).$$

□

Corollary 3.1.2. *The inclusion $\mathcal{O}_{L_0} \subset \Lambda(\mathcal{H})$ is strict.*

Proof. Suppose that $\Lambda(\mathcal{H}) = \mathcal{O}_{L_0}$, since L_0^\perp is Lagrangian, there exists $g \in \text{Sp}_2(\mathcal{H})$ such that $L_0^\perp = g(L_0)$, then using its orthogonal projector and the above lemma we have,

$$1 - P_{L_0} = P_{L_0^\perp} = P_{g(L_0)} = P_{L_0} + T$$

for some $T \in \mathcal{B}_2(\mathcal{H})$. Therefore, $2P_{L_0} - 1 = -T \in \mathcal{B}_2(\mathcal{H})$ and this is a contradiction because $2P_{L_0} - 1$ is a unitary operator.

□

To build a manifold structure on \mathcal{O}_{L_0} , we will consider the charts of $\Lambda(\mathcal{H})$ given by the parametrization of Lagrangian subspaces as graphs of functions and we will adapt this charts to our set. This charts were used in [6] to describe the manifold structure of $\Lambda(\mathcal{H})$; in the followings steps we recall this charts and we fix the notation.

Given $L \in \Lambda(\mathcal{H})$, we have the Lagrangian decomposition $\mathcal{H} = L \oplus L^\perp$ and we denote by

$$\Omega(L^\perp) = \{W \in \Lambda(\mathcal{H}) : \mathcal{H} = W \oplus L^\perp\}.$$

In [11] it was proved that these sets are open in $\Lambda(\mathcal{H})$. We consider the map $\phi_L : \Omega(L^\perp) \rightarrow \mathcal{B}(L)_s$ given by

$$W = Gr_T \longmapsto J|_{L^\perp} T$$

where $T : L \rightarrow L^\perp$ is the linear operator whose graph is W , more precisely

$$T = \pi_1|_W \circ (\pi_0|_W)^{-1}$$

where π_0, π_1 are the orthogonal projections to L and L^\perp .

Remark 3.1.3. *The map ϕ_L is onto: Let $\psi \in \mathcal{B}(L)_s$, we consider the operator $T := -J|_L \psi$ (T maps L into L^\perp) and $W := Gr_T$. Since ψ is a symmetric operator, W is a Lagrangian subspace and $\mathcal{H} = Gr_T \oplus L^\perp$. Then $W \in \Omega(L^\perp)$ and it is a preimage of ψ .*

The maps $\{\phi_L\}_{L \in \Lambda(\mathcal{H})}$ constitute a smooth atlas for $\Lambda(\mathcal{H})$, so that $\Lambda(\mathcal{H})$ becomes a smooth Banach manifold (see [25]). For every $W \in \Lambda(\mathcal{H})$ we can identify the tangent space $T_W \Lambda(\mathcal{H})$ with the Banach space $\mathcal{B}(W)_s$, this identification was used in [6] and [25]. For $W \in \Omega(L^\perp)$, the differential $d\phi_L$ of the chart at W is given by

$$d_W \phi_L(H) = \eta^* H \eta \tag{1.2}$$

for all $H \in \mathcal{B}(W)_h$, where $\eta : L \rightarrow W$ is the isomorphism given by the restriction to L of the projection $W \oplus L^\perp \rightarrow W$. It is easy to see that the inverse $d_\psi \phi_L^{-1}$ of this map at a point $\psi = \phi_L(W)$ is given by

$$\mathcal{B}(L)_h \xrightarrow{d_\psi \phi_L^{-1}} \mathcal{B}(W)_h$$

$$H \longmapsto (\eta^{-1})^* H \eta^{-1}.$$

Since the symplectic group acts smoothly we can consider for fixed $L \in \Lambda(\mathcal{H})$ the smooth map $\pi_L : \mathrm{Sp}(\mathcal{H}) \rightarrow \Lambda(\mathcal{H})$ given by $g \mapsto g(L)$. Its differential map at a point $g \in \mathrm{Sp}(\mathcal{H})$ is given by

$$T_g \mathrm{Sp}(\mathcal{H}) = \mathfrak{sp}(\mathcal{H})g \ni Xg \mapsto P_{g(L)} JX|_{g(L)} \in \mathcal{B}(g(L))_h,$$

see [6] and [25] for a proof. Throughout, we will denote by $d_1\pi_L$ the differential at the identity. If $L \in \mathcal{O}_{L_0}$ we can restrict the map π_L to the subgroup $\mathrm{Sp}_2(\mathcal{H})$ obtaining a surjective map onto \mathcal{O}_{L_0} ,

$$\pi_L|_{\mathrm{Sp}_2(\mathcal{H})} : \mathrm{Sp}_2(\mathcal{H}) \rightarrow \mathcal{O}_{L_0}.$$

Theorem 3.1.4. *The set \mathcal{O}_{L_0} is a submanifold of $\Lambda(\mathcal{H})$ and the natural map $i : \mathcal{O}_{L_0} \hookrightarrow \Lambda(\mathcal{H})$ is an embedding.*

Proof. We will adapt the above local chart ϕ_L to our set. Let $L = g(L_0) \in \mathcal{O}_{L_0}$, first we see that $\phi_L(\Omega(L^\perp) \cap \mathcal{O}_{L_0}) \subset \mathcal{B}_2(L)_h$. Indeed, if W belongs to $\Omega(L^\perp) \cap \mathcal{O}_{L_0}$ then we can write $W = Gr_T = h(L_0)$ for some $h \in \mathrm{Sp}_2(\mathcal{H})$ and since $L_0 = g^{-1}(L)$ we have that $W = hg^{-1}(L)$ and it is obvious that we can write now $W = \tilde{g}(L)$ with $\tilde{g} \in \mathrm{Sp}_2(\mathcal{H})$. If we write $\tilde{g} = 1 + k$ where $k \in \mathcal{B}_2(\mathcal{H})$ then the orthogonal projection π_1 restricted to W can be written as

$$\pi_1|_W(w) = \pi_1(\tilde{g}l) = \pi_1(l + kl) = \pi_1(kl) = \pi_1(k(\tilde{g}^{-1}w))$$

where $W \ni w = \tilde{g}(l)$ and $l \in L$. Thus we have

$$\pi_1|_W = \pi_1 \circ k \circ \tilde{g}^{-1}|_W \in \mathcal{B}_2(W, L^\perp).$$

Then it is clear that $\phi_L(W) = J|_{L^\perp}T \in \mathcal{B}_2(L)_h$. Now we have the restricted chart

$$\phi_L|_{\Omega(L^\perp) \cap \mathcal{O}_{L_0}} : \Omega(L^\perp) \cap \mathcal{O}_{L_0} \longrightarrow \mathcal{B}_2(L)_h.$$

To conclude we will see that this restricted map is also onto. Let $\psi \in \mathcal{B}_2(L)_h$ and as we did in Remark 3.1.3 we consider the operator $T := -J|_L\psi$, then the only fact to prove is that

$$Gr_T = \{v + (-J|_L\psi)v : v \in L\} \in \mathcal{O}_{L_0}.$$

To prove it we define $f := 1 - J|_L\psi P_L \in 1 + \mathcal{B}_2(\mathcal{H})$; it is invertible with inverse given by $1 + J|_L\psi P_L$ and it is clear that $Gr_T = f(L)$. Now we have to show that f is symplectic. Indeed, let $\xi, \eta \in \mathcal{H}$ then

$$\begin{aligned} w((1 - J|_L\psi P_L)\xi, (1 - J|_L\psi P_L)\eta) = \\ w(\xi, \eta) + w(\xi, -J|_L\psi P_L\eta) + w(-J|_L\psi P_L\xi, \eta) + \underbrace{w(J|_L\psi P_L\xi, J|_L\psi P_L\eta)}_{=0} \end{aligned}$$

and since J is an isometry we have

$$\begin{aligned} w(\xi, -J|_L\psi P_L\eta) + w(-J|_L\psi P_L\xi, \eta) &= \langle J\xi, -J|_L\psi P_L\eta \rangle + \langle J(-J|_L\psi P_L)\xi, \eta \rangle \\ &= -\langle \xi, \psi P_L\eta \rangle + \langle \psi P_L\xi, \eta \rangle. \end{aligned}$$

If $\xi = \xi_0 + \xi_0^\perp$ and $\eta = \eta_0 + \eta_0^\perp$ are the respective decompositions in $L \oplus L^\perp$, then by the symmetry of ψ the above equality results in

$$\begin{aligned} -\langle \xi, \psi P_L \eta \rangle + \langle \psi P_L \xi, \eta \rangle &= \langle \xi_0 + \xi_0^\perp, \psi \eta_0 \rangle + \langle \psi \xi_0, \eta_0 + \eta_0^\perp \rangle \\ &= -\langle \xi_0, \psi \eta_0 \rangle + \langle \psi \xi_0, \eta_0 \rangle = 0. \end{aligned}$$

Then

$$w((1 - J|_L \psi P_L)\xi, (1 - J|_L \psi P_L)\eta) = w(\xi, \eta)$$

and $f \in \mathrm{Sp}_2(\mathcal{H})$. Since $L = g(L_0)$ we have

$$Gr_T = f(L) = fg(L_0) \in \mathcal{O}_{L_0}.$$

□

As in the case of the full Lagrangian Grassmannian, for every $L \in \mathcal{O}_{L_0}$ we can identify the tangent space $T_L \mathcal{O}_{L_0}$ with the Hilbert space $\mathcal{B}_2(L)_h$.

Since the differential of the inclusion map is an inclusion map, it is clear that the differential of the adapted charts is the restriction of the differential of full charts given by equation (1.2). So, if $W \in \Omega(L^\perp) \cap \mathcal{O}_{L_0}$ then the differential of the adapted chart is given by $d_W \phi_L|_{\Omega(L^\perp) \cap \mathcal{O}_{L_0}}(H) = \eta^* H \eta$ where $H \in \mathcal{B}_2(W)_h$ and its inverse is

$$\begin{aligned} \mathcal{B}_2(L)_h &\xrightarrow{d_\psi \phi_L^{-1}|_{\Omega(L^\perp) \cap \mathcal{O}_{L_0}}} \mathcal{B}_2(W)_h = T_W \mathcal{O}_{L_0} \\ H &\longmapsto (\eta^{-1})^* H \eta^{-1}. \end{aligned} \tag{1.3}$$

Remark 3.1.5. *The differential of the map $\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$ at a point $g \in \mathrm{Sp}_2(\mathcal{H})$ is the restriction of the differential map $d_g \pi_L$ at $T_g \mathrm{Sp}_2(\mathcal{H})$ i.e.*

$$d_g \pi_L|_{\mathrm{Sp}_2(\mathcal{H})} : T_g \mathrm{Sp}_2(\mathcal{H}) = \mathfrak{sp}_2(\mathcal{H})g \ni Xg \mapsto P_{g(L)} JX|_{g(L)} \in \mathcal{B}_2(g(L))_h.$$

Indeed, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}(\mathcal{H}) & \xrightarrow{\pi_L} & \Lambda(\mathcal{H}) \\ i_2 \uparrow & & \uparrow i_1 \\ \mathrm{Sp}_2(\mathcal{H}) & \xrightarrow{\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}} & \mathcal{O}_{L_0} \end{array}$$

If we differentiate at a point $g \in \mathrm{Sp}_2(\mathcal{H})$ the equation $\pi_L \circ i_2 = i_1 \circ \pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$, and use that the differential of the inclusion maps i_1 and i_2 at $h(L_0)$ and at h respectively are inclusions, we have $d_g \pi_L|_{\mathrm{Sp}_2(\mathcal{H})}(Xg) = d_g \pi_L(Xg)$ for every $X \in \mathfrak{sp}_2(\mathcal{H})$.

In the followings steps we will obtain the main result of this section, the Lie subgroup structure of the isotropy group. To do it we will use the above submanifold structure constructed on \mathcal{O}_{L_0} . If M and N are smooth Banach manifolds a smooth map $f : M \rightarrow N$ is a submersion if the tangent map $d_x f$ is onto and its kernel is a complemented subspace of $T_x M$ for all $x \in M$. This fact is equivalent to the existence of smooth local sections (see [17]). The next proposition is essential for the proof.

Proposition 3.1.6. *The map $\pi_{L_0} : \mathrm{Sp}(\mathcal{H}) \rightarrow \Lambda(\mathcal{H})$ and its restriction $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})} : \mathrm{Sp}_2(\mathcal{H}) \rightarrow \mathcal{O}_{L_0}$ are smooth submersions when we consider in $\Lambda(\mathcal{H})$ (resp. in \mathcal{O}_{L_0}) the above manifold structure.*

Proof. First we will prove that the map $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})} : \mathrm{Sp}_2(\mathcal{H}) \rightarrow \mathcal{O}_{L_0}$ has local cross sections on a neighborhood of L_0 , the proof is adapted from [3]. Using the symmetry over $R(Q)$ we have

$$\epsilon_{R(Q)} = 2P_{R(Q)} - 1 \in \epsilon_{L_0} + \mathcal{B}_2(\mathcal{H}). \quad (1.4)$$

For $L \in \mathcal{O}_{L_0}$ close to L_0 , we consider the element $g_L = 1/2(1 + \epsilon_L \epsilon_{L_0})$; it is invertible (in fact, it can be shown that it is invertible if $\|\epsilon_L - \epsilon_{L_0}\| < 2$) and it commutes with J , so it belongs to $GL(\mathcal{H}_J)$. From equation (1.4) we have

$$\epsilon_L \epsilon_{L_0} \in (\epsilon_{L_0} + \mathcal{B}_2(\mathcal{H})) \epsilon_{L_0} \in 1 + \mathcal{B}_2(\mathcal{H})$$

and then it is clear that $g_L \in 1 + \mathcal{B}_2(\mathcal{H}_J)$. Thus g_L is complex and invertible in a neighbourhood of ϵ_{L_0} . Note that

$$g_L \epsilon_{L_0} = 1/2(\epsilon_{L_0} + \epsilon_L) = \epsilon_L g_L$$

and also that $g^* g$ commutes with ϵ_{L_0} . If $|x| = (x^* x)^{1/2}$ denotes the modulus and $g_L = u_L |g_L|$ is the polar decomposition, then $u_L = g_L (g_L^* g_L)^{-1/2} \in U(\mathcal{H}_J) \subset \mathrm{Sp}(\mathcal{H})$. We define the local cross section for L close to L_0 as

$$\sigma(L) = u_L.$$

Now we have to prove that $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})}(\sigma(L)) = L$. If we identify the subspace with the symmetry this is equivalent to prove that $\epsilon_{\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})}(\sigma(L))} = \epsilon_L$. Indeed,

$$\epsilon_{\pi_{L_0}(u_L)} = u_L \epsilon_{L_0} u_L^* = g_L (g_L^* g_L)^{-1/2} \epsilon_{L_0} (g_L^* g_L)^{-1/2} g_L^* = g_L \epsilon_{L_0} g_L^{-1} = \epsilon_L.$$

Let us prove that it takes values in $\mathrm{Sp}_2(\mathcal{H})$. Since $\mathbb{C}1 + \mathcal{B}_2(\mathcal{H}_J)$ is a $*$ -Banach algebra and $g_L \in GL_2(\mathcal{H}_J)$ by the Riesz functional calculus we have that $u_L = g_L |g_L|^{-1} \in \mathbb{C}1 + \mathcal{B}_2(\mathcal{H}_J)$. Thus $u_L = \beta 1 + b$ with $b \in \mathcal{B}_2(\mathcal{H}_J)$. On

the other hand, note that $g_L^*g_L$ is a positive operator which lies in the C^* -algebra $\mathbb{C}1 + \mathcal{K}(\mathcal{H}_J)$. Therefore its square root is of the form $r1 + k$ with $r \geq 0$ and k compact. Then

$$g_L^*g_L = (r1 + k)^2 = r^2 \cdot 1 + k'$$

and since $g_L^*g_L \in GL_2(\mathcal{H}_J)$ we have

$$r^2 1 + k' = 1 + b'$$

with $b' \in \mathcal{B}_2(\mathcal{H}_J)$. Since $\mathbb{C}1$ and $\mathcal{K}(\mathcal{H}_J)$ are linearly independent, it follows that $r = 1$. Then it is clear that $u_L \in U_2(\mathcal{H}_J) \subset \mathrm{Sp}_2(\mathcal{H})$ and σ is well defined. To conclude the proof we now show that the local section σ is smooth. If L lies in a small neighborhood of L_0 we have

$$L = \phi_{L_0}^{-1}(\psi) = Gr_{-J|_L\psi} = (1 - J|_{L_0}\psi P_{L_0})(L_0) = g(L_0) \in \Omega(L_0^\perp) \cap \mathcal{O}_{L_0}.$$

The idempotent of range L is

$$Q := gP_{L_0}g^{-1} = (1 - J|_{L_0}\psi P_{L_0})P_{L_0}(1 + J|_{L_0}\psi P_{L_0}) = P_{L_0} - J|_{L_0}\psi P_{L_0}$$

and it is smooth as a function of ψ . Since the formula of the orthogonal projector (1.1) is smooth, the local expression of σ will also be smooth. Indeed, the symmetry in the chart will be

$$\epsilon_L = 2P_{R(gP_{L_0}g^{-1})} - 1 = 2QQ^*(1 - (Q - Q^*)^2)^{1/2} - 1$$

and it is clearly smooth as a function of ψ , because Q and the operations involved (product, involution, square root) are smooth. Then it is clear that the invertible element g_L and its unitary part u_L are smooth too. Finally the local expression $\sigma \circ \phi_{L_0}^{-1}$ is smooth as a function of ψ . Since the full Lagrangian Grassmannian can be expressed as an orbit for a fixed L_0 , the proof of smoothness of the local section of π_{L_0} is analogous to that of the restricted map $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})}$. \square

Corollary 3.1.7. *If L is any subspace in the full Lagrangian Grassmannian or in \mathcal{O}_{L_0} then the map $\pi_L : \mathrm{Sp}(\mathcal{H}) \rightarrow \Lambda(\mathcal{H})$ and its restriction $\pi_L|_{\mathrm{Sp}_2(\mathcal{H})} : \mathrm{Sp}_2(\mathcal{H}) \rightarrow \mathcal{O}_{L_0}$ have local cross sections on a neighborhood of L .*

Proof. The above map σ can be translated using the action to any $L = g(L_0)$. That is,

$$\sigma_L(h(L_0)) = g\sigma(g^{-1}h(L_0))g^{-1}$$

where $h(L_0)$ lies on a neighborhood of L . \square

Theorem 3.1.8. *The isotropy groups $\mathrm{Sp}(\mathcal{H})_L$ and $\mathrm{Sp}_2(\mathcal{H})_L$ of the symplectic group and of the restricted symplectic group are Lie subgroups of them with their respective topology. Their Lie algebras are*

$$\mathfrak{sp}(\mathcal{H})_L = \{x \in \mathfrak{sp}(\mathcal{H}) : x(L) \subseteq L\}$$

$$\mathfrak{sp}_2(\mathcal{H})_L = \{x \in \mathfrak{sp}_2(\mathcal{H}) : x(L) \subseteq L\}.$$

Proof. Since the maps $d_1\pi_L$ and $d_1\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$ are submersions then by the inverse function theorem, we have that the isotropy groups are Lie subgroups and their Lie algebras are $\ker d_1\pi_L$ and $\ker d_1\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$ respectively. A short computation shows us that $\ker d_1\pi_L = \{x \in \mathfrak{sp}(\mathcal{H}) : x(L) \subseteq L\}$ and $\ker d_1\pi_L|_{\mathrm{Sp}_2(\mathcal{H})} = \{x \in \mathfrak{sp}_2(\mathcal{H}) : x(L) \subseteq L\}$. Indeed, if $P_L JX|_L = 0$ then $JX|_L \in L^\perp$ and thus $-X|_L \in J(L^\perp) = L$. \square

Remark 3.1.9. *The Lie algebra $\mathfrak{sp}_2(\mathcal{H})_L$ consists of all operators $x \in \mathfrak{sp}_2(\mathcal{H})$ that are L invariant, so we can give another characterization of this algebra using the orthogonal projection P_L . That is,*

$$\mathfrak{sp}_2(\mathcal{H})_L = \{x \in \mathfrak{sp}_2(\mathcal{H}) : xP_L = P_LxP_L\}. \quad (1.5)$$

In block matrix form, this operators correspond to the upper triangular elements of $\mathfrak{sp}_2(\mathcal{H})$.

3.2 Metric structure in \mathcal{O}_{L_0}

In this section we will introduce a Riemannian structure in \mathcal{O}_{L_0} using the Hilbert-Schmidt inner product. We will prove that this Riemannian structure coincides with the Riemannian structure given by the quotient norm. We also study the completeness of the geodesic distance and moreover we will find the corresponding geodesic curves.

3.2.1 The ambient metric

Given $v, w \in T_W\mathcal{O}_{L_0} = \mathcal{B}_2(W)_h$, we define the inner product

$$\langle v, w \rangle_W := \mathrm{tr}_W(w^*v) = \sum_{i=1}^{\infty} \langle w^*ve_i, e_i \rangle$$

where $\{e_i\}$ is an orthonormal basis of the subspace W . The ambient metric for $v \in T_W\mathcal{O}_{L_0} = \mathcal{B}_2(W)_h$ is

$$\mathcal{A}(W, v) := \mathrm{tr}_W(v^*v)^{1/2}. \quad (2.6)$$

Using the orthogonal projection over W , it can be expressed by $\|vP_W\|_2^2$. Indeed, if $\{e_i\}$ is an orthonormal basis for \mathcal{H} then

$$\begin{aligned} \|vP_W\|_2^2 &= \sum_i \langle vP_W e_i, vP_W e_i \rangle = \sum_i \langle v^* v P_W e_i, P_W e_i \rangle \\ &= \sum_i \langle v^* v P_W e_i, e_i \rangle = \text{tr}(v^* v P_W) = \text{tr}_W(v^* v). \end{aligned} \quad (2.7)$$

To each point $W \in \mathcal{O}_{L_0}$, we associate the inner product $\langle \cdot, \cdot \rangle_W$ on the tangent space $T_W \mathcal{O}_{L_0}$. This correspondence allows us to introduce a Riemannian structure on the manifold \mathcal{O}_{L_0} . The fact to prove here is that the metric varies differentiably.

Proposition 3.2.1. *The Riemannian structure is well defined.*

Proof. Let $L \in \mathcal{O}_{L_0}$ and consider a neighborhood $U := \Omega(L^\perp) \cap \mathcal{O}_{L_0}$ of it. For any $W \in U$, we can write it in the local chart $W = \phi_L^{-1} \psi = \text{Gr}_{(-J|_L \psi)}$. Let $\eta_W : L \rightarrow W$ be the restriction of the orthogonal projection $W \oplus L^\perp \xrightarrow{\pi} W$, then its local expression is

$$\begin{aligned} \eta_W(v) &= \pi(v) = \pi((v - J|_L \psi(v)) + J|_L \psi(v)) \\ &= (1 - J|_L \psi)(v) \quad \text{for all } v \in L, \end{aligned}$$

and then it can be expressed by compression of the operator $1 - J|_L \psi P_L$ into the subspace L i.e. $\eta_W = (1 - J|_L \psi P_L)|_L$. If we write the local expression of the metric using the classical differential structure of the tangent bundle with the differential of the chart ϕ_L^{-1} given in the formula (1.3), for every $v \in TU$ we have

$$\mathcal{A}(W, v) = \|d_\psi \phi_L^{-1}(H) P_W\|_2 = \|(\eta_W^{-1})^* H \eta_W^{-1} P_W\|_2, \quad (2.8)$$

where $\psi \in \phi_L(U)$ and $H \in \mathcal{B}_2(L)_h$ is the preimage of v . Since the projector $P_W = P_{\text{Gr}_{(-J|_L \psi)}}$ is smooth and the local expression of η_W is also smooth as a function of ψ and by the smoothness of the operations involved (inverse, involution, product, trace) the formula (2.8) is smooth. \square

3.2.2 The geodesic distance

The length of a smooth curve measured with the ambient metric will be denoted by

$$L_{\mathcal{A}}(\gamma) = \int_0^1 \mathcal{A}(\gamma(t), \dot{\gamma}(t)) dt.$$

Given two Lagrangian subspaces S and T in \mathcal{O}_{L_0} , we denote by $d_{\mathcal{A}}$ the geodesic distance using the ambient metric,

$$d_{\mathcal{A}}(S, T) = \inf\{L_{\mathcal{A}}(\gamma) : \gamma \text{ joins } S \text{ and } T \text{ in } \mathcal{O}_{L_0}\}.$$

If $(L_n) \subset \mathcal{O}_{L_0}$ is any sequence we will denote by $L_n \xrightarrow{\mathcal{O}_{L_0}} L$ the convergence to some subspace $L \in \mathcal{O}_{L_0}$ in the topology given by the smooth structure of \mathcal{O}_{L_0} (Theorem 3.1.4).

There is a naturally defined Hilbert space inner product on the tangent space at 1 of the group $\mathrm{Sp}_2(\mathcal{H})$, which is identified with the space of Hilbert-Schmidt operators on \mathcal{H} , and this inner product is employed to define a left-invariant and a right-invariant Riemannian structure on the group.

Given a smooth curve α in $\mathrm{Sp}_2(\mathcal{H})$ we can measure its length with the left or right invariant metric, depending on which identification of tangent spaces we use in the group. In chapter 2 we used the left invariant metric. The length of a curve using this metric is $L_{\mathcal{L}}(\alpha) = \int_0^1 \|\alpha^{-1}\dot{\alpha}\|_2$. Here, we will use the right identification of the tangent spaces, so we have to introduce the right invariant metric. Although formally equivalent this choice will make some computations easier. Then the length of α is $L_{\mathcal{R}}(\alpha) = \int_0^1 \|\dot{\alpha}\alpha^{-1}\|_2$.

Remark 3.2.2. *Let G be a Banach-Lie group, if $d_{\mathcal{L}}$ and $d_{\mathcal{R}}$ denote the geodesic distance with the left and right invariant metrics respectively then,*

$$d_{\mathcal{L}}(x^{-1}, y^{-1}) = d_{\mathcal{R}}(x, y) \quad \forall x, y \in G.$$

Indeed, since the geodesic distances are left and right invariant respectively, the only fact left to prove is the equality $d_{\mathcal{L}}(x^{-1}, 1) = d_{\mathcal{R}}(x, 1)$ for all $x \in G$. Then, if α is any curve that joins 1 to x^{-1} , the curve $\beta(t) = \alpha(t)^{-1}$ joins 1 to x ; if we differentiate we have $\dot{\beta}(t)\beta(t)^{-1} = -\alpha(t)^{-1}\dot{\alpha}(t)$ and then the right length of β coincides with the left length of α .

If $\xi : [0, 1] \rightarrow \mathcal{O}_{L_0}$ is a curve with $\xi(0) = L$ then a lifting of ξ is a map $\phi : [0, 1] \rightarrow \mathrm{Sp}_2(\mathcal{H})$ with $\phi(0) = 1$ and $\phi(t)(L) = \xi(t)$, for all $t \in [0, 1]$. The next lemma is an adaptation of Lemma 25 in [6].

Lemma 3.2.3. *Every smooth curve $\xi : [0, 1] \rightarrow \mathcal{O}_{L_0}$ with $\xi(0) = L$ admits an isometric lifting, if we consider the right invariant metric in $\mathrm{Sp}_2(\mathcal{H})$.*

Proof. For each $t \in [0, 1]$, set $X(t) = -J\dot{\xi}(t)P_{\xi(t)} \in \mathfrak{sp}_2(\mathcal{H})$ and consider the solution of the ODE

$$\begin{cases} \dot{\phi}(t) = X(t)\phi(t) \\ \phi(0) = 1 \end{cases} \quad (2.9)$$

A simple computation using Remark 3.1.5 shows that both $t \mapsto \phi(t)(L)$ and $\xi(t)$ are integral curves of the vector field $\nu(t)(L) = P_L JX(t)|_L \in T_L \mathcal{O}_{L_0} = \mathcal{B}_2(L)_h$ both starting at L , therefore the two curves coincide. Now, it is easy to see that the solution of the differential equation (2.9) is an isometric lifting of ξ . Indeed, if we take norms in the equation we have,

$$\|\dot{\phi}(t)\phi^{-1}(t)\|_2 = \|\dot{\phi}(t)P_{\xi(t)}\|_2 = \|\dot{\xi}(t)P_{\xi(t)}\|_2 = \mathcal{A}(\xi(t), \dot{\xi}(t)).$$

□

The geodesic curves given by the left invariant metric in the group $\mathrm{Sp}_2(\mathcal{H})$ were calculated in Chapter 2 Proposition 2.3.1. This fact can be used to find the geodesics of the Levi-Civita connection induced by the ambient metric \mathcal{A} .

Theorem 3.2.4. *Let $\xi : [0, 1] \rightarrow \mathcal{O}_{L_0}$ be a geodesic curve of the Riemannian connection induced by the ambient metric \mathcal{A} with initial position $\xi(0) = L$ and initial velocity $\dot{\xi}(0) = w \in T_{\xi(0)}\mathcal{O}_{L_0} = \mathcal{B}_2(L)_h$. Then*

$$\xi(t) = e^{t(v^*-v)}e^{-tv^*}(L)$$

where $v \in \mathfrak{sp}_2(\mathcal{H})$ is a preimage of $-w$ by $d_1\pi_L$.

Proof. Since ξ is a geodesic curve, it is locally minimizing. Using Lemma 3.2.3 there exists an isometric lifting $\phi \subset \mathrm{Sp}_2(\mathcal{H})$ with initial condition $\phi(0) = 1$. By the isometric property ϕ results locally minimizing with the right invariant metric and then ϕ^{-1} results locally minimizing with the left invariant metric. Hence the curve $\phi^{-1} \subset \mathrm{Sp}_2(\mathcal{H})$ is a geodesic and it is $\phi^{-1}(t) = e^{tv^*}e^{t(v-v^*)}$ for some $v \in \mathfrak{sp}_2(\mathcal{H})$. Then it is clear that $\phi(t) = e^{t(v^*-v)}e^{-tv^*}$ and $\xi(t) = e^{t(v^*-v)}e^{-tv^*}(L)$. The only fact left to prove is that v is a lift of $-w$. Indeed, since $\dot{\xi}(t) = d_{e^{t(v^*-v)}e^{-tv^*}}\pi_L((v^* - v)e^{t(v^*-v)}e^{-tv^*} - e^{t(v^*-v)}e^{-tv^*}v^*)$, then $w = \dot{\xi}(0) = d_1\pi_L(-v) = -d_1\pi_L(v)$. □

3.2.3 The quotient metric

Since the action of the Hilbert-Lie group $\mathrm{Sp}_2(\mathcal{H})$ on the Grassmannian \mathcal{O}_{L_0} is smooth and transitive, we identify $\mathcal{O}_{L_0} \simeq \mathrm{Sp}_2(\mathcal{H})/\mathrm{Sp}_2(\mathcal{H})_{L_0}$ as manifolds. Then it is only natural to consider on our Grassmannian the quotient Riemannian metric. If $W \in \mathcal{O}_{L_0}$ and $v \in T_W\mathcal{O}_{L_0}$, we put

$$\mathcal{Q}(W, v) = \inf\{\|z\|_2 : z \in \mathfrak{sp}_2(\mathcal{H}), d_1\pi_W(z) = v\}.$$

This metric will be called the quotient metric of \mathcal{O}_{L_0} , because it is the quotient metric in the Banach space

$$T_W \mathcal{O}_{L_0} \simeq \mathfrak{sp}_2(\mathcal{H}) / \mathfrak{sp}_2(\mathcal{H})_W.$$

Indeed, since $\mathfrak{sp}_2(\mathcal{H})_W = \ker d_1 \pi_W$, if $z \in \mathfrak{sp}_2(\mathcal{H})$ with $d_1 \pi_W(z) = v$ then

$$\mathcal{Q}(W, v) = \inf \{ \|z - y\|_2 : y \in \mathfrak{sp}_2(\mathcal{H})_W \}.$$

If Q_L denotes the orthogonal projection onto $\mathfrak{sp}_2(\mathcal{H})_W$ then each $z \in \mathfrak{sp}_2(\mathcal{H})$ can be uniquely decomposed as

$$z = z - Q_L(z) + Q_L(z) = z_0 + Q_L(z)$$

hence

$$\|z - y\|_2^2 = \|z_0 + Q_L(z) - y\|_2^2 = \|z_0\|_2^2 + \|Q_L(z) - y\|_2^2 \geq \|z_0\|_2^2$$

for any $y \in \mathfrak{sp}_2(\mathcal{H})_W$ which shows that

$$\mathcal{Q}(W, v) = \|z_0\|_2 \tag{2.10}$$

where z_0 is the unique vector in $\mathfrak{sp}_2(\mathcal{H})_W^\perp$ such that $d_1 \pi_W(z_0) = v$.

We denote the length for a piecewise smooth curve in \mathcal{O}_{L_0} , measured with the quotient norm introduced above as $L_{\mathcal{Q}}(\gamma)$.

Theorem 3.2.5. *The quotient metric and the ambient metric are equal.*

Proof. The proof is a straightforward computation using the definition of the metrics; indeed let $W \in \mathcal{O}_{L_0}$ and $v \in T_W \mathcal{O}_{L_0}$, by formula (2.10) we have $\mathcal{Q}(W, v) = \|z_0\|_2$ where z_0 is the unique vector in $\mathfrak{sp}_2(\mathcal{H})_W^\perp$ such that $d_1 \pi_W(z_0) = v$. Since z_0 belongs to $\mathfrak{sp}_2(\mathcal{H})_W^\perp$, using the decomposition $W \oplus W^\perp$, we can write

$$z_0 = z_0 P_W - P_W z_0 P_W = (1 - P_W) z_0 P_W$$

and then since P_W is a Lagrangian projector we have $J z_0 = (J - J P_W) z_0 P_W = P_W J z_0 P_W$. Therefore using the definition of the ambient metric (2.6) we have,

$$\begin{aligned} \mathcal{A}(W, v) &= \|v P_W\|_2 = \|d_1 \pi_W(z_0) P_W\|_2 = \|P_W J z_0|_W P_W\|_2 \\ &= \|J z_0\|_2 = \|z_0\|_2 = \mathcal{Q}(W, v). \end{aligned}$$

□

Now, it is obvious that the geometry of these Riemannian metrics is the same, in particular the geodesics and the geodesic distance.

To prove the main theorem in this chapter we will use some facts that we obtained in Chapter 2. The key is to use the completeness of the metric space $(\mathrm{Sp}_2(\mathcal{H}), \|\cdot\|_2)$ and the lifting property given in Lemma 3.2.3.

Theorem 3.2.6. *If (L_n) is a sequence in \mathcal{O}_{L_0} and $L \in \mathcal{O}_{L_0}$ then*

1. $L_n \xrightarrow{\mathcal{O}_{L_0}} L \implies L_n \xrightarrow{d_{\mathcal{Q}}} L.$
2. *The metric space $(\mathcal{O}_{L_0}, d_{\mathcal{Q}})$ is complete.*
3. *The distance $d_{\mathcal{Q}}$ defines the given topology on \mathcal{O}_{L_0} . Equivalently, $L_n \xrightarrow{\mathcal{O}_{L_0}} L \iff L_n \xrightarrow{d_{\mathcal{Q}}} L.$*

Proof. Since $d_{\mathcal{A}}(S, T) = d_{\mathcal{Q}}(S, T)$ for all $S, T \in \mathcal{O}_{L_0}$, we can prove the three items with $d_{\mathcal{A}}$ to simplify the computations.

1. The map π_L has local continuous sections, let n_0 be such that $L_n \in U \subset \mathcal{O}_{L_0} \forall n \geq n_0$ (U a neighbourhood of L) and such that $\sigma_L : U \rightarrow \mathrm{Sp}_2(\mathcal{H})$ is a section for π_L . By continuity we have $\sigma_L(L_n) \xrightarrow{\|\cdot\|_2} \sigma_L(L) = 1$ if $n \geq n_0$. Since $\sigma_L(L_n)$ is close to 1, there is $z_n \in \mathfrak{sp}_2(\mathcal{H})$ such that $\sigma_L(L_n) = e^{z_n}$ and since $\|e^{z_n} - 1\|_2 = \|\sigma_L(L_n) - 1\|_2 \rightarrow 0$ we also have $\|z_n\|_2 \rightarrow 0$. Let $\gamma_n(t) = e^{tz_n}(L) \subset \mathcal{O}_{L_0}$ be a curve that joins L and L_n ; using the equality (2.7) its length is $L_{\mathcal{A}}(\gamma_n) = \int_0^1 \mathcal{A}(\gamma_n(t), \dot{\gamma}_n(t)) dt = \int_0^1 \|\dot{\gamma}_n(t) P_{\gamma_n(t)}\|_2 dt$. Since $\gamma_n(t) = \pi_L \circ e^{tz_n}$ using the chain rule and Proposition 3.1.5 we have

$$\dot{\gamma}_n(t) = d_{e^{tz_n}} \pi_L(z_n e^{tz_n}) = P_{e^{tz_n}(L)} J z_n |_{e^{tz_n}(L)},$$

then taking norms and using the symmetric property of the 2-norm ($\|xyz\|_2 \leq \|x\| \|y\| \|z\|$) we have

$$\|\dot{\gamma}_n(t) P_{\gamma_n(t)}\|_2 = \|P_{e^{tz_n}(L)} J z_n P_{e^{tz_n}(L)}\|_2 \leq \|z_n\|_2.$$

Then it is clear that $d_{\mathcal{A}}(L_n, L) \leq L_{\mathcal{A}}(\gamma_n) \rightarrow 0$.

2. Let (L_n) be a $d_{\mathcal{A}}$ -Cauchy sequence in \mathcal{O}_{L_0} and fix $\varepsilon > 0$. Then there exists n_0 such that $d_{\mathcal{A}}(L_n, L_m) \leq \varepsilon$ if $n, m \geq n_0$. For the fixed Lagrangian L_{n_0} , we have the map

$$\pi = \pi_{L_{n_0}} : \mathrm{Sp}_2(\mathcal{H}) \rightarrow \mathcal{O}_{L_0}, \quad \pi(g) = g(L_{n_0}).$$

If $n, m \geq n_0$ we can take a curve $\gamma_{n,m} \subset \mathcal{O}_{L_0}$ that joins L_n to L_m (for $t = 0$ and $t = 1$ respectively) such that

$$L_{\mathcal{A}}(\gamma_{n,m}) \leq d_{\mathcal{A}}(L_n, L_m) + \varepsilon.$$

Then by Lemma 3.2.3, the curves $\gamma_{n_0,m}$ are lifted, via π , to curves ϕ_m of $\mathrm{Sp}_2(\mathcal{H})$ with $\phi_m(0) = 1$ and $L_{\mathcal{R}}(\phi_m) = L_{\mathcal{A}}(\gamma_{n_0,m})$. Denote by $g_m = \phi_m(1) \in \mathrm{Sp}_2(\mathcal{H})$ the end point. Then

$$\varepsilon + d_{\mathcal{A}}(L_{n_0}, L_m) \geq L_{\mathcal{A}}(\gamma_{n_0,m}) = L_{\mathcal{R}}(\phi_m) \geq d_{\mathcal{R}}(1, g_m).$$

For each $n, m \geq n_0$ we have,

$$\begin{aligned} d_{\mathcal{R}}(g_n, g_m) &\leq d_{\mathcal{R}}(1, g_m) + d_{\mathcal{R}}(1, g_n) \\ &\leq 2\varepsilon + d_{\mathcal{A}}(L_{n_0}, L_m) + d_{\mathcal{A}}(L_{n_0}, L_n) \leq 4\varepsilon. \end{aligned}$$

Thus the sequence $(g_m) \subset \mathrm{Sp}_2(\mathcal{H})$ is $d_{\mathcal{R}}$ -Cauchy and then by Remark 3.2.2 we have that (g_m^{-1}) is $d_{\mathcal{L}}$ -Cauchy. Using Lemma 2.5.1 of Chapter 2 we have that the sequence (g_m^{-1}) is a Cauchy sequence in $(\mathrm{Sp}_2(\mathcal{H}), \|\cdot\|_2)$ and then since this metric space is closed, there exists $x \in \mathrm{Sp}_2(\mathcal{H})$ such that $g_m^{-1} \xrightarrow{\|\cdot\|_2} x$. By continuity we have $\pi(g_m) \xrightarrow{\mathcal{O}_{L_0}} \pi(x^{-1})$ and since ϕ_m is a lift of $\gamma_{n_0,m}$ we also have $\pi(g_m) = g_m(L_{n_0}) = \phi_m(1)(L_{n_0}) = \gamma_{n_0,m}(1) = L_m$, so $L_m \xrightarrow{\mathcal{O}_{L_0}} \pi(x^{-1})$. Thus using the first item of this theorem we have $d_{\mathcal{A}}(L_m, \pi(x^{-1})) \rightarrow 0$.

3. Suppose that $L_n \xrightarrow{d_{\mathcal{A}}} L$, then it is a $d_{\mathcal{A}}$ -Cauchy sequence. If we repeat the argument that we used above, there exists $x \in \mathrm{Sp}_2(\mathcal{H})$ such that $L_n \xrightarrow{\mathcal{O}_{L_0}} \pi(x^{-1})$. By the first point it is $d_{\mathcal{A}}$ convergent and therefore $L_n \xrightarrow{\mathcal{O}_{L_0}} L$.

□

Chapter 4

Riemannian metrics in self-adjoint groups

En este capítulo extenderemos algunos resultados de la geometría del grupo simpléctico de Hilbert-Schmidt que obtuvimos en el capítulo 2 a una clase mucho más amplia de grupos de operadores Riemannianos, los grupos de operadores autoadjuntos.

In this chapter we will extend some results of the geometry of the Hilbert-Schmidt symplectic group that we obtained in Chapter 2 to a more general class of Riemannian operator groups, the self-adjoint operator groups.

4.1 Riemannian geometry of a self-adjoint subgroup

The following definition is related to those of Sections 3 and 7 of Chapter IV in [16].

Definition 4.1.1. *Let G be a connected abstract subgroup of $\mathrm{GL}_2(\mathcal{H})$. We say that G is a self-adjoint subgroup if $g^* \in G$ whenever $g \in G$ (for short, we write $G^* = G$). Note that a connected Banach-Lie group G is self-adjoint if and only if $\mathfrak{g}^* = \mathfrak{g}$, where \mathfrak{g} denotes the Banach-Lie algebra of G .*

4.1.1 Riemannian geometry with the left invariant metric

Throughout this chapter G will denote a closed, connected self-adjoint Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$, moreover we will denote by $\mathfrak{g} \subset \mathcal{B}_2(\mathcal{H})$ its closed

Banach-Lie algebra. Using the left action on itself, the tangent space at $g \in G$ is

$$T_g G = g \cdot \mathfrak{g}.$$

We endow G with the induced left invariant metric of $\mathrm{GL}_2(\mathcal{H})$, so for $v \in T_g G$ we have

$$\mathcal{I}(g, v) = \|g^{-1}v\|_2. \quad (1.1)$$

Proposition 4.1.2. *Let $G \subset \mathrm{GL}_2(\mathcal{H})$ with the left invariant metric (1.1), then G is totally geodesic submanifold. In other words the Levi-Civita covariant derivative is given by*

$$\alpha^{-1}D_t\eta = \dot{\mu} + 1/2\{[\beta, \mu] + [\beta, \mu^*] + [\mu, \beta^*]\} \quad (1.2)$$

where $\alpha : (-\epsilon, \epsilon) \rightarrow G$ is any smooth curve, η is a tangent field along α and $\beta = \alpha^{-1}\dot{\alpha}$, $\mu = \alpha^{-1}\eta \in \mathfrak{g}$ are the fields at the identity.

Proof. The proof is similar to the proof of Proposition 2.3.1. Let $\beta = \alpha^{-1}\dot{\alpha}$ and $\mu = \alpha^{-1}\eta$ be the fields at \mathfrak{g} , we will show that $\alpha^{-1}D_t\eta \in \mathfrak{g}$. Indeed, since $\mu \in \mathfrak{g}$ then it is clear that $\dot{\mu}$ belongs to \mathfrak{g} , because it is a limit of operators that belong to the closed algebra \mathfrak{g} . Since G is self-adjoint, we have $\mathfrak{g}^* = \mathfrak{g}$, then μ^* and β^* belong to \mathfrak{g} and the brackets $[\beta, \mu]$, $[\beta, \mu^*]$, $[\mu, \beta^*]$ are all in \mathfrak{g} because it is a closed Banach-Lie algebra. Thus we have $\alpha^{-1}D_t\eta \in \mathfrak{g}$. \square

This shows that the Riemannian connection given by the left invariant metric in the group G matches the one of $\mathrm{GL}_2(\mathcal{H})$. In particular, the geodesics of G are the same than those of $\mathrm{GL}_2(\mathcal{H})$; if $g_0 \in G$ and $g_0v_0 \in g_0 \cdot \mathfrak{g}$ are the initial position and the initial velocity then

$$\alpha(t) = g_0 e^{tv_0^*} e^{t(v_0 - v_0^*)} \in G$$

satisfies $D_t\dot{\alpha} = 0$. Therefore, the Riemannian exponential for $g \in G$ is

$$\mathrm{Exp}_g(v) = g e^{v^*} e^{v - v^*}$$

with $v \in \mathfrak{g}$.

4.1.2 Riemannian geometry with the polar metric

The next theorem summarizes the most important properties of self-adjoint Banach-Lie groups. It was proved by G. Larotonda in [18].

Theorem 4.1.3. *Let $G = \langle \exp(\mathfrak{g}) \rangle$ be a connected, self-adjoint Banach-Lie group with Banach-Lie algebra $\mathfrak{g} \subset \mathcal{B}_2(\mathcal{H})$. Let P be the analytic map $g \mapsto g^*g$, $P : G \rightarrow G$. Let $\mathfrak{k} = \ker(d_1P)$, $\mathfrak{m} = \text{Ran}(d_1P)$. Let $M_G = \exp(\mathfrak{m})$ and $K = G \cap U_2(\mathcal{H}) = P^{-1}(1)$. Then*

1. *The set \mathfrak{m} is a closed Lie triple system. We have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. In particular \mathfrak{k} is a Banach-Lie subalgebra of \mathfrak{g} .*

2. *$P(G) = M_G$ and M_G is a geodesically convex submanifold of $\text{GL}_2^+(\mathcal{H})$.*

3. *For any $g = u_g|g|$ (polar decomposition), we have $|g| \in M_G$ and $u_g \in K$.*

4. *Let $g \in G$, $p \in M_G$, $I_g(p) = gpg^*$. Then $I_g \in I(M_G)$ (the group of isometries of M_G). If $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{\frac{1}{2}}$, then $I_g(p) = q$, namely G acts isometrically and transitively on M_G .*

5. *Let $u \in K$ and $x \in \mathfrak{m}$ (resp. \mathfrak{m}^\perp). Then $I_u(x) = uxu^* \in \mathfrak{m}$ (resp. \mathfrak{m}^\perp). If $p, q \in M_G$ then I_p maps T_pM_G (resp. $T_pM_G^\perp$) isometrically onto $T_{I_p(q)}M_G$ (resp. $T_{I_p(q)}M_G^\perp$).*

6. *The group K is a Banach-Lie subgroup of G with Lie algebra \mathfrak{k} .*

7. *$G \simeq K \times M_G$ as Hilbert manifolds. In particular K is connected and $G/K \simeq M_G$.*

Since $M_G = \exp(\mathfrak{m})$ is closed and a geodesically convex submanifold of $\text{GL}_2^+(\mathcal{H})$, then for any $p = e^x \in M_G$,

$$T_pM_G = \text{Exp}_p^{-1}(M_G) = \{p^{1/2} \ln(p^{-1/2}qp^{-1/2})p^{1/2} : q \in M_G\}.$$

For this reason it is clear, as in the case of the full space $\text{GL}_2^+(\mathcal{H})$, that given any $p, q \in M_G$ the curve

$$\gamma_{pq}(t) = p^{1/2}(p^{-1/2}qp^{-1/2})^t p^{1/2} \subset M_G$$

has minimal length among all curves in M_G that join p to q . Its length is measured with the metric (1.2) and it is $\|\ln(p^{-1/2}qp^{-1/2})\|_2$. Moreover the metric space (M_G, d_p) is complete.

The diffeomorphism given in point seven of Theorem 4.1.3 is the restriction of the map (1.4) on G . So, we can endow G with the polar Riemannian metric (1.5) using the product manifold $K \times M_G$.

If we consider a curve $\alpha \subset K$ and η is a tangent field along α then the covariant derivative (1.2) given by the left invariant metric is reduced to

$$\alpha^{-1}D_t\eta = \dot{\mu} + 1/2[\beta, \mu] \in \mathfrak{k}.$$

Therefore K is a totally geodesic manifold of the unitary group $U_2(\mathcal{H})$ and we have the following proposition that extends Proposition 2.4.1 to the self-adjoint groups.

Proposition 4.1.4. *G is a totally geodesic submanifold of $GL_2(\mathcal{H})$ when we consider the induced polar metric.*

Proof. Since the manifolds M_G and K are totally geodesic then we have that the Levi-Civita derivative on the ambient manifold $U_2(\mathcal{H}) \times GL_2^+(\mathcal{H})$ restricts onto $K \times M_G$. \square

Since the Levi-Civita derivative is given by the product, it is not difficult to see that given any initial velocity $v \in \mathfrak{g}$, if $v = x + y$ is the decomposition into $\mathfrak{k} \oplus \mathfrak{m}$, then the geodesics of the polar metric starting at the identity are

$$\alpha(t) = e^{tx}e^{ty}.$$

Proposition 4.1.5. *The geodesics of the left invariant metric coincide with the geodesics of the polar metric if the initial velocity $v \in \mathfrak{g}$ is normal.*

Proof. Let $v = x + y \in \mathfrak{k} \oplus \mathfrak{m}$ be the decomposition into its hermitian and anti-hermitian part, since v is normal a straightforward computation shows that x commutes with y , thus we have

$$e^{tv^*}e^{t(v-v^*)} = e^{tv} = e^{tx}e^{ty}.$$

This equation shows that the geodesics are one-parameter groups. \square

4.2 Completeness of the geodesic distance

4.2.1 Completeness in finite dimension with p-norms

Let $GL_n(\mathbb{C})$ be the general linear group in finite dimension. Let $\mathcal{M}_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. Since $GL_n(\mathbb{C})$ is open in the space $\mathcal{M}_n(\mathbb{C})$, we can identify the tangent space of $GL_n(\mathbb{C})$ at any point with $\mathcal{M}_n(\mathbb{C})$. In this algebra we consider the classical p -norms, if $x \in \mathcal{M}_n(\mathbb{C})$ and τ denote the real part of the trace we put,

$$\|x\|_p^p = \tau((x^*x)^{p/2}) \text{ for any } p \geq 1.$$

Since the dimension of $\mathrm{GL}_n(\mathbb{C})$ is finite, it is known that any closed subgroup G has a structure of Banach-Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$. In this context we denote the left invariant metric for any self-adjoint closed subgroup as $\mathcal{I}_p(g, v) = \|g^{-1}v\|_p$ for $g \in G$ and $v \in T_g G$.

Theorem 4.2.1. *The metric space $(G, d_{\mathcal{I}_p})$ is complete.*

Proof. If $p = 2$, by Hopf-Rinow's theorem, the space $(G, d_{\mathcal{I}_2})$ is complete since the manifold G is geodesically complete with the 2-norm (Proposition 4.1.2). Now, we claim that $d_{\mathcal{I}_p}$ is equivalent to $d_{\mathcal{I}_2}$ for any $p \geq 2$. Indeed, at each tangent space of G , the p -norm is equivalent with the 2-norm with constants which depend only on the dimension of $\mathcal{M}_n(\mathbb{C})$. Examining the length functionals, it follows that the metrics are equivalent, with the same constants. \square

4.2.2 Completeness in the infinite dimensional case

Since the map $g \mapsto g|g|^{-1} = u_g$ is continuous, it is clear that if $p, q \in G$ are close to each other its unitary parts (u_p, u_q) are close too. So, if $g \in G$ is close to the identity 1, then its unitary part u_g is close to 1 too and since the polar decomposition is in the group (Theorem 4.1.3) we can assume that u_g lies in $K \cap U$ where U is a neighbourhood of the identity in G . Since K is a Banach-Lie subgroup of G , if we reduce the neighbourhood U , we can assume that $u_g = \exp(z)$ where z belongs to a neighbourhood of 0 in the Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Now, if $p, q \in G$ are close to each other and $u_p|p|, u_q|q|$ are their polar decompositions, then the unitary element $u_p^{-1}u_q \in K$ is close to 1 and we can choose an element $z \in \mathfrak{k} \subset \mathfrak{g}$ such that $u_p^{-1}u_q = e^z$. So, we can build the following smooth curve in G ;

$$\alpha_{p,q}(t) = u_p e^{tz} |p|^{1/2} (|p|^{-1/2} |q| |p|^{-1/2})^t |p|^{1/2} \in G \quad (2.3)$$

that joins p to q . This curve will be used to obtain the completeness with both metrics.

Theorem 4.2.2. *Let G be a closed, connected self-adjoint Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$, then the metric space $(G, d_{\mathcal{P}})$ is complete.*

Proof. Let (x_n) be a $d_{\mathcal{P}}$ -Cauchy sequence, let $x_n = u_{x_n}|x_n|$ be its polar decomposition. First we will prove that $u_{x_n} \subset K$ and $|x_n| \subset M_G$ are $d_{\mathcal{I}}$ and $d_{\mathcal{P}}$ -Cauchy sequences respectively. Indeed, given $\varepsilon = 1/n$ there exist curves $\beta_n \subset G$ such that $\beta_n(0) = x_n$, $\beta_n(1) = x_m$ and $d_{\mathcal{P}}(x_n, x_m) + 1/n > L_{\mathcal{P}}(\beta_n)$. If $\beta_{1n} \subset K$ and $\beta_{2n} \subset M_G$ denote the unitary and positive part of β_n , then

since $x_n = \beta_n(0) = \beta_{1n}(0)\beta_{2n}(0) = u_{x_n}|x_n|$, it is clear that β_{1n} joins u_{x_n} to u_{x_m} and β_{2n} joins $|x_n|$ to $|x_m|$. Using the inequality (1.6) we have,

$$d_{\mathcal{P}}(x_n, x_m) + 1/n > L_{\mathcal{P}}(\beta_n) \geq L_{\mathcal{I}}(\beta_{1n}) \geq d_{\mathcal{I}}(u_{x_n}, u_{x_m})$$

for all n, m , then it is clear that $d_{\mathcal{I}}(u_{x_n}, u_{x_m}) \rightarrow 0$ when $n, m \rightarrow \infty$. An analogous computation using β_{2n} shows that $d_{\mathcal{P}}(|x_n|, |x_m|) \rightarrow 0$ when $n, m \rightarrow \infty$. Since (u_{x_n}) is a unitary sequence it is known that $\|u_{x_n} - u_{x_m}\|_2 \leq d_{\mathcal{I}}(u_{x_n}, u_{x_m})$ and therefore $(u_{x_n}) \subset K$ is a 2-norm Cauchy sequence. Since K is closed we can take $u \in K$ such that $u_{x_n} \xrightarrow{\|\cdot\|_2} u$. Then if n is large, we can suppose that $u^{-1}u_{x_n}$ is close to 1, therefore there exists a sequence $(z_n) \subseteq \mathfrak{k}$ such that $u^{-1}u_{x_n} = e^{z_n}$ and $z_n \xrightarrow{\|\cdot\|_2} 0$. On the other hand, there exists $g \in M_G$ such that $d_{\mathcal{P}}(|x_n|, g) = \|\ln(g^{-1/2}|x_n|g^{-1/2})\|_2 \rightarrow 0$. It is clear that $ug \in G$, then if n is large we can consider the curve $\alpha_{ug, x_n}(t) = ue^{tz_n}g^{1/2}(g^{-1/2}|x_n|g^{-1/2})^t g^{1/2}$ (2.3) that joins x_n and ug , therefore we have

$$d_{\mathcal{P}}(x_n, ug) \leq L_{\mathcal{P}}(\alpha_{ug, x_n}) = (\|z_n\|_2^2 + \|\ln(g^{-1/2}|x_n|g^{-1/2})\|_2^2)^{1/2} \rightarrow 0.$$

□

The following proposition is a generalization of Proposition 2.1.6 to G .

Proposition 4.2.3. *Suppose $p, q \in G$ are close to each other and let $v := |p|^{-1/2}|q||p|^{-1/2}$ then we can estimate the geodesic distance $d_{\mathcal{I}}$ by*

$$d_{\mathcal{I}}(p, q) \leq c(p, q) (\|z\|_2^2 + \|\ln(v)\|_2^2)^{1/2}$$

where

$$c(p, q)^2 = 2 \max \left\{ e^{4\|\ln(v)\|} (\|p\|\|p^{-1}\|)^2, \|p\|\|p^{-1}\| \right\}.$$

Proof. The proof is similar to Proposition 2.1.6. Since $p, q \in G$ are close, then we can build the smooth curve $\alpha_{p, q} \subset G$ (2.3) that joins p to q ; so we can repeat the argument that we gave in Proposition 2.1.6. Then in this case we have,

$$\|\alpha_{p, q}^{-1} \dot{\alpha}_{p, q}\|_2 \leq c(p, q) (\|z\|_2^2 + \|\ln(v)\|_2^2)^{1/2}.$$

and then $d_{\mathcal{I}}(p, q) \leq L_{\mathcal{I}}(\alpha_{p, q}) \leq c(p, q) (\|z\|_2^2 + \|\ln(v)\|_2^2)^{1/2}$. □

Lemma 4.2.4. *If $(x_n) \subset G$ is a Cauchy sequence in $(G, d_{\mathcal{I}})$ then it is a Cauchy sequence in $(G, \|\cdot\|_2)$.*

Proof. Since the geodesics of the Riemannian connection are of the form $\alpha(t) = \text{Exp}_g(tv) = ge^{tv^*}e^{t(v-v^*)}$, the proof of this lemma can be adapted easily from Lemma 2.5.1 in Chapter 2. □

Now we are in position to obtain our final result of the thesis.

Theorem 4.2.5. *Let G be a closed, connected self-adjoint Banach-Lie subgroup of $\mathrm{GL}_2(\mathcal{H})$, then the metric space $(G, d_{\mathcal{I}})$ is complete.*

Proof. Let $(x_n) \subset G$ be a $d_{\mathcal{I}}$ -Cauchy sequence, by the above lemma it is $\|\cdot\|_2$ -Cauchy; then since G is closed there exists $x \in G$ such that $x_n \xrightarrow{\|\cdot\|_2} x$. Now we will show that $x_n \xrightarrow{d_{\mathcal{I}}} x$; indeed from the continuity of the module we have that $|x_n|$ converges to $|x|$ in $\|\cdot\|_2$ and its unitary part $u_{x_n} = x_n|x_n|^{-1}$ converges to $u_x = x|x|^{-1}$. The sequence $|x|^{-1/2}|x_n||x|^{-1/2}$ converges to 1 and then $\|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\|_2 \rightarrow 0$. Since x_n converges to x , we can assume that x_n is close to x if $n \geq n_0$, therefore we can use Proposition 4.2.3 to estimate the geodesic distance. Then we have

$$d_{\mathcal{I}}(x, x_n) \leq c(x, x_n) (\|z_n\|_2^2 + \|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\|_2^2)^{1/2}$$

where $z_n \in \mathfrak{k} \subset \mathfrak{g}$ is such that $u_x^{-1}u_{x_n} = e^{z_n}$ (since $u_x^{-1}u_{x_n}$ is close to 1 and K is a Banach-Lie subgroup of G). We also have $\|z_n\|_2 \rightarrow 0$. Now we will see that $c(x, x_n)$ is uniformly bounded. Indeed, since $\|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\| \leq \|\ln(|x|^{-1/2}|x_n||x|^{-1/2})\|_2 \rightarrow 0$, then for n large we can assume that $\|\ln(v_n)\| \leq 1$ where $v_n = |x|^{-1/2}|x_n||x|^{-1/2}$ as we denoted in Proposition 4.2.3. Finally we have

$$\begin{aligned} c(x, x_n)^2 &= 2 \max\{ e^{4\|\ln(v_n)\|} (\|x\|\|x^{-1}\|)^2, \|x\|\|x^{-1}\| \} \\ &\leq 2 \max\{ e^4 (\|x\|\|x^{-1}\|)^2, \|x\|\|x^{-1}\| \} \end{aligned}$$

is clearly uniformly bounded and then it is clear that $d_{\mathcal{I}}(x, x_n) \rightarrow 0$. \square

Bibliography

- [1] V.I. Arnold. On a characteristic class entering into conditions of quantization, *Funkcional. Anal. i Priložen.* 1 (1967) 1-14 (in Russian).
- [2] C.J. Atkin. The Hopf-Rinow theorem is false in infinite dimensions. *Bull. London Math. Soc.* 7 (1975), 261-266.
- [3] E. Andruchow, G. Larotonda, L. Recht, A. Varela. The left invariant metric in the general linear group (2014). *Journal of Geometry and Physics*, volume 86 (2014), 241-257.
- [4] E. Andruchow, G. Larotonda. Hopf-Rinow Theorem in the Sato Grassmannian. *J. Funct. Anal.* 255 (2008) no.7, 1692-1712.
- [5] D. Beltiță. Smooth homogeneous structures in operator theory. Chapman and Hall/CRC. Monographs and Surveys in Pure and Applied Mathematics, 137. Chapman and Hall/CRC, Boca Raton, FL, 2006.
- [6] L. Biliotti, R. Exel, P. Piccione and D. V. Tausk. On The Singularities of the exponential map in infinite dimensional Riemannian Manifolds. *Math. Ann.* 336 (2) (2006) 247-267.
- [7] G. Corach, H. Porta, L. Recht. Geodesics and operator means in the space of positive operators. *Internat. J. Math.* 4 (1993), 193-202.
- [8] G. Corach, H. Porta, L. Recht. Convexity of the geodesic distance on the space of positive operators. *Illions J.Math.* 38 (1994) no. 1, 87-94.
- [9] G. Corach. Sobre la geometría del conjunto de operadores positivos en espacios de Hilbert, *Anal. Acad. Nac. Cs. Ex. Fis. y Nat.*, Buenos Aires, Argentina, 50 (1998), 109-118.
- [10] J.C.C. Eidam, P. Piccione. The essential Lagrangian-Grassmannian and the homotopy type of the Fredholm Lagrangian-Grassmannian, *Topol-ogy Appl.* 153 (2006), no. 15, 2782-2787.

- [11] K. Furutani. Fredholm-Lagrangian-Grassmannian and the Maslov index. *J. Geom. Phys.* 51 (3) (2004) 269-331.
- [12] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, London, 1934.
- [13] Pierre de la Harpe. *Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space*. Springer-Verlag. Berlin. Heidelberg. NewYork 1972.
- [14] L.A. Harris, W. Kaup, Linear algebraic groups in infinite dimensions, *Illinois J. Math.* 21 (1977), no. 3, 666-674.
- [15] F. Hiai, H. Kosaki. Comparison of various means of operators. *J. Funct. Anal.* 163 (1999), no 2, 300-323.
- [16] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. Academic Press. New York, 1978.
- [17] S. Lang. *Differentiable and Riemannian manifolds*. Third edition. Graduate Texts in Mathematics, 160. Springer-Verlag, New York, 1995.
- [18] G. Larotonda. Nonpositive Curvature: A Geometric Approach to Hilbert-Schmidt Operators. *Differential Geom. Appl.* 25 (2007) no. 6, 679-700.
- [19] G. Larotonda. *Geodesic Convexity Symmetric Spaces and Hilbert-Schmidt Operators*. Thesis.
- [20] M. López Galván. Riemannian metrics on an infinite dimensional symplectic group. *J. Math. Anal. Appl.* 428 (2015), pp. 1070-1084.
- [21] M. López Galván. The Hilbert-Schmidt Lagrangian Grassmannian. *J. Geom. Phys.* 99 (2016), pp. 174-183.
- [22] M. López Galván. Riemannian metrics on infinite dimensional self-adjoint operator groups. *Journal of Lie Theory* 26 (2016), no. 3, 717-728.
- [23] J. McAlpin. *Infinite dimensional manifolds and Morse theory*. Thesis, Columbia University, 1965.
- [24] G.D. Mostow. Some new decompositions theorems for semi-simple groups. *Mem. Amer. Math. Soc.* 14 (1955), 31-54.

- [25] P. Piccione and D. Victor Tausk. A Students Guide to Symplectic Spaces, Grassmannians and Maslov Index. Publicações Matemáticas do IMPA. Rio de Janeiro, Instituto Nacional de Matemática Pura e Aplicada (IMPA). xiv, 301 p. (2008).
- [26] A. Pressley, G. Segal. Loop groups. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.
- [27] Barry Simon. Trace Ideals and their Applications. Mathematical Surveys and Monographs. Volume 120.
- [28] R. C. Swanson. Linear Symplectic Structures on Banach Spaces. Rocky Mountain Journal of Mathematics. Volume 10, number 2, Spring 1980.