



UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Problemas elípticos con crecimiento no estándar y falta de compacidad

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área
Ciencias Matemáticas

Analía Concepción Silva

Director de tesis: Julián Fernández Bonder

Consejero de estudios : Julián Fernández Bonder

Buenos Aires, 29 de Noviembre de 2012

Para mis viejos.

Problemas elípticos con crecimiento no estándar y falta de compacidad

(Resumen)

En esta tesis estudiamos el teorema de inmersión de Sobolev $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ y el Teorema de Trazas de Sobolev $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ para espacios de exponente variable, en el caso en que las inclusiones no son compactas.

Para este propósito, primero extendemos el celebrado Principio de compacidad por concentración de P.L. Lions para el caso de exponente variable que describe con precisión los motivos por los cuales una sucesión es débil convergente pero no convergente en norma.

Como primera aplicación del principio de compacidad por concentración encontramos condiciones en términos de las constantes óptimas para las mencionadas inmersiones que garantizan la existencia de extremales para las mismas. Finalmente, damos condiciones locales en los exponentes $p(x)$, $q(x)$ y $r(x)$ para garantizar la existencia de extremales.

Como segunda aplicación estudiamos resultados de existencia y multiplicidad para ecuaciones elípticas con crecimiento crítico cuando el operador involucrado es el llamado $p(x)$ -laplaciano.

Palabras Claves: Espacios de exponente variable, principio de compacidad por concentración, exponente crítico, inmersiones de Sobolev .

Elliptic problems with non standard growth and lack of compactness

(Abstract)

In this Thesis we study the Sobolev immersion Theorem $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and the Sobolev trace Theorem $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ in the variable exponent setting in the critical case, i.e. when the immersions are not longer compact.

For this purpose, we first extend the celebrated concentration compactness principle of P.L. Lions to the variable exponent case which describe the mechanism why a sequence is weakly but not strongly convergent.

As a first application of the concentration compactness principle, we find conditions in terms of the optimal constants in the above mentioned immersions in order to guaranty the existence of extremals for the immersions. Finally, we give local conditions on the exponents $p(x)$, $q(x)$ and $r(x)$ to ensure the existence of such extremals.

As a second application we study existence and multiplicity results for solutions to critical elliptic equations when the elliptic operator is the so-called $p(x)$ -laplacian.

Key words: Variable exponent spaces, Concentration compactness principle, critical exponent, Sobolev embeddings

Agradecimientos

El doctorado fue para mí un ramo de rosas que casi no tuvo espinas. Y eso se lo debo a la gente asombrosa que me acompañaó todo este tiempo. Por eso, quiero aprovechar esta oportunidad para decir gracias.

Gracias a Julián por haber sido mi guía. Por hacer que la matemática parezca más fácil. Por darme ánimo y confiar en mí en todo momento. Por tenerme paciencia y dedicarme tiempo. Por su bondad y generosidad, que siempre estuvo presente. Pero por sobre todo, le quiero agradecer, porque fue tan maravillosa la experiencia que lo volvería a elegir.

Gracias a Nico por haber compartido con nosotros este problema que nos hizo tan felices. Por haberme enseñado geometría diferencial y a calcular asintóticos. Por la buena onda que tiene para trabajar. Pero, más que nada, por haber aceptado el desafío de dirigir mi post-doc. Merci beaucoup!!!

Gracias a Noemi, Sandra, Joana, Ricardo, Juan Pablo, Irene, Leandro, Ariel S., Ariel L., Pablo de Napoli y Enrique. Por los seminarios, los cafés y los congresos compartidos.

Gracias a los jurados por sus valiosas correcciones.

Gracias a Mari, Pati, Coti, Ro, Patri y Manu. Por los matecitos con cuartitos de alfajor. Por aguantarme en los buenos y en los malos días. Por hacer de la ofi un ambiente tan agradable.

Gracias a Caro, Vicky, Jony, Vero y Mer. Por su amistad sincera, por sus consejos y por pensar siempre en mi.

Gracias a Sigrid, Ema, Romi, Pau A., Nahu, Yani, Manu, Ana F., Juli G., Pablo T., Mazzi, Dani, Román, Pau K., Nico C., por las conversaciones de pasillo y las salidas compartidas.

Gracias a los chicos del cini por alegrarme los findes.

Gracias a Magui por su amistad incondicional, que me ayuda a crecer y a ser más feliz. Gracias por ser la hermana que elegí.

Gracias a mi familia por su apoyo, paciencia y amor.

Contents

Resumen	v
Abstract	vii
Agradecimientos	ix
Contents	xiii
1 Introducción	1
1.1 Resultados preliminares	1
1.2 Espacios de Lebesgue y de Sobolev con exponente variable	5
1.3 Motivaciones Físicas	7
1.4 Descripción de los resultados	9
1.5 Publicaciones incluidas	13
2 Introduction	14
2.1 Preliminary results	14
2.2 Variable exponent setting	18
2.3 Physical motivation	20
2.4 Description of the results	21
2.5 Included publications	26
3 Preliminaries	27
3.1 Variable exponent Sobolev spaces	27
3.2 Mountain pass theorem	30
3.3 A topological tool: the genus	31

3.4	The variational principle of Ekeland	32
4	The concentration–compactness principle for variable exponent spaces	34
4.1	The concentration–compactness principle for the Sobolev immersion	34
4.1.1	Preliminary Lemmas	36
4.1.2	Proof of the Concentration Compactness Principle	39
4.2	The concentration–compactness principle for the Sobolev trace immersion	41
5	Existence of extremals for Sobolev Embeddings	44
5.1	Compact case	44
5.2	Non-compact case 1: The Sobolev immersion Theorem	47
5.3	Global conditions for the validity of $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$	52
5.4	Continuity of the Sobolev constant with respect to p and q	53
5.5	Investigation on the validity of $\bar{S} = \inf_{x \in \mathcal{A}} K^{-1}(N, p(x))$	55
5.6	Local conditions for the validity of $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$	56
5.7	Non-compact case 2: The Sobolev trace Theorem	58
5.8	Global conditions for the validity of $T(p(\cdot), r(\cdot), \Omega) < \bar{T}$	63
5.9	Investigation on the validity of $\bar{T} = \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1}$	64
5.10	Local conditions for the validity of $T(p(\cdot), r(\cdot), \Omega) < \bar{T}$	66
6	Existence results for critical elliptic equations with compact perturbations	68
6.1	Superlinear–like compact perturbation	69
6.2	Sublinear–like compact perturbation	75
6.3	Multiplicity result	80
7	Existence results for critical elliptic equations via local conditions	90
7.1	Critical equation with Dirichlet boundary conditions	91
7.2	Critical equation with nonlinear boundary conditions	98
A	Asymptotic expansions	103
A.1	Asymptotic expansions for the Sobolev immersion constant	104
A.2	Asymptotic expansions for the Sobolev trace constant	109
	Bibliography	119

Index

123

1

Introducción

1.1 Resultados preliminares

El propósito de esta Tesis es el estudio de los Teoremas de inmersión de Sobolev en espacios de exponente variable con pérdida de compacidad.

Vamos a comenzar con un breve resúmen de resultados conocidos. Sea $\Omega \subset \mathbb{R}^N$ un dominio acotado y notamos por $\mathcal{M}(\Omega)$ el conjunto de funciones medibles en Ω con valores en la recta real extendida $[-\infty, +\infty]$. Los espacios de Lebesgue para exponente constante son definidos como:

$$L^p(\Omega) = \left\{ f \in \mathcal{M}(\Omega) : \int_{\Omega} |f|^p dx < \infty \right\}, \quad 1 \leq p < \infty, \quad \text{y} \quad L^\infty(\Omega) = \left\{ f \in \mathcal{M}(\Omega) : \sup_{\Omega} |f| < \infty \right\}.$$

Aquí, y a lo largo de esta Tesis, denotamos \sup como el supremo esencial con respecto a la medida de Lebesgue.

Estos espacios están equipados con las normas

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \text{y} \quad \|f\|_{L^\infty(\Omega)} = \|f\|_\infty = \text{ess sup}_{\Omega} |f|.$$

Los espacios de Sobolev son definidos como

$$W^{1,p}(\Omega) = \{ f \in L^p(\Omega) : \partial_i f \in L^p(\Omega) \text{ para todo } i = 1, \dots, N \}, \quad 1 \leq p \leq \infty,$$

donde $\partial_i f = \frac{\partial f}{\partial x_i}$ representa la i -ésima derivada parcial en el sentido de las distribuciones.

La norma de Sobolev es definida como

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{1,p} = (\|f\|_p^p + \|\nabla f\|_p^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \text{y} \quad \|f\|_{W^{1,\infty}(\Omega)} = \|f\|_{1,\infty} = \|f\|_\infty + \|\nabla f\|_\infty.$$

Sea $1 \leq p < N$ fijo. El teorema de inmersión de Sobolev dice que para Ω Lipschitz, uno tiene, para toda $f \in W^{1,p}(\Omega)$, la desigualdad, cf. [21]

$$\|f\|_q \leq C(p, q, \Omega) \|f\|_{1,p},$$

para $1 \leq q \leq p^*$, donde p^* es llamado *exponente crítico de Sobolev* dado por

$$p^* = \frac{Np}{N-p}. \quad (1.1)$$

Equivalentemente,

$$0 < \tilde{S}(p, q, \Omega) \triangleq \inf_{v \in W^{1,p}(\Omega)} \frac{\|v\|_{1,p}}{\|v\|_q}. \quad (1.2)$$

Esta constante $\tilde{S}(p, q, \Omega)$ es llamada la (mejor u óptima) constante de Sobolev y las funciones v que realizan el ínfimo (si existen) son llamadas *extremales*.

Una cuestión básica en análisis y en ecuaciones en derivadas parciales, es el cálculo explícito de las constantes óptimas de Sobolev y sus correspondientes extremales.

Vamos a remarcar que un extremal es una solución débil de la correspondiente ecuación de Euler–Lagrange

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda |u|^{q-2}u & \text{en } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = 0 & \text{en } \partial\Omega, \end{cases} \quad (1.3)$$

donde $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ es el operador p -laplaciano y $\partial u / \partial n$ es la derivada normal con respecto al vector normal unitario exterior n de Ω .

La constante λ es un multiplicador de Lagrange y depende de la normalización de u . Por ejemplo, si u es elegido tal que $\|u\|_q = 1$, entonces $\lambda = \tilde{S}(p, q, \Omega)^p$.

Cuando q es *subcrítico*, i.e. $1 \leq q < p^*$, la inmersión $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ es compacta, entonces el método directo del cálculo de variaciones permite probar de manera inmediata la existencia de extremales para (1.2) y, por ende, la existencia de una solución para (1.3).

Por otro lado, cuando $q = p^*$ es fácil de ver que la compacidad se pierde y entonces la existencia de un extremal para (1.2) o la existencia de una solución para la ecuación (1.3) es un problema no trivial.

Es común en la literatura, debido a muchas aplicaciones, considerar el subespacio de $W^{1,p}(\Omega)$ que consiste en todas las funciones que se anulan en la frontera de Ω . Este subespacio es definido como

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)},$$

donde $C_c^\infty(\Omega)$ son las funciones suaves con soporte compacto y la clausura es tomada en la norma $\|\cdot\|_{1,p}$.

En este espacio, se verifica la desigualdad de Poincaré

$$\|f\|_p \leq C(p, \Omega) \|\nabla f\|_p, \quad \text{para } f \in W_0^{1,p}(\Omega). \quad (1.4)$$

De (1.4) se sigue que $\|\nabla f\|_p$ define una norma en $W_0^{1,p}(\Omega)$ que es equivalente a $\|f\|_{1,p}$. Entonces, cuando uno trabaja con el espacio $W_0^{1,p}(\Omega)$, el teorema de inmersión de Sobolev puede ser rescrito como

$$0 < S(p, q, \Omega) = \inf_{v \in W_0^{1,p}(\Omega)} \frac{\|\nabla v\|_p}{\|v\|_q} \quad (1.5)$$

y la ecuación de Euler–Lagrange para extremales de (1.5) se convierte en

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

Otra vez, para el problema de existencia de extremales, el único caso no trivial es el crítico, $q = p^*$. Observemos que en el espacio $W_0^{1,p}(\Omega)$ para que valga el teorema de inmersión no es necesaria ninguna hipótesis de regularidad sobre el dominio Ω .

En el caso crítico se sabe que la constante $S(p, p^*, \Omega)$ no se alcanza para ningún abierto Ω acotado y que la ecuación de Euler–Lagrange (1.6) no tiene solución para Ω estrellado respecto de un punto.

Además, la constante $S(p, p^*, \Omega)$ es independiente de Ω . En efecto,

$$S(p, p^*, \Omega) = K(N, p)^{-1} := \inf_{v \in D^{1,p}(\mathbb{R}^N)} \frac{\|\nabla v\|_p}{\|v\|_{p^*}},$$

donde $D^{1,p}(\mathbb{R}^N)$ es el conjunto de funciones f en $L^{p^*}(\mathbb{R}^N)$ tal que $\partial_i f \in L^p(\mathbb{R}^N)$, $i = 1, \dots, N$.

Es también bien sabido que los extremales para $K(N, p)^{-1}$ forman una familia bi-paramétrica definida por

$$U_{\lambda, x_0}(x) = \lambda^{-\frac{N-p}{p}} U\left(\frac{x-x_0}{\lambda}\right),$$

con U dada por

$$U(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}}.$$

Ver [37, 53]. En particular esto permite calcular explícitamente el valor de $K(N, p)^{-1}$ (ver [53])

$$K(N, p) = \pi^{\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma(1 + \frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(1 - N - \frac{N}{p})}\right)^{\frac{1}{N}},$$

donde $\Gamma(x)$ es la función Gamma, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

Otra inmersión de Sobolev muy importante es el llamado *teorema de trazas de Sobolev*. Este teorema permite restringir una función de Sobolev al borde del dominio (que tiene medida de Lebesgue cero).

Para poder enunciar el teorema de trazas, necesitamos definir los espacios de Lebesgue en $\partial\Omega$. Suponemos que Ω es C^1 entonces $\partial\Omega$ es una variedad $(N-1)$ -dimensional C^1 inmersa en \mathbb{R}^N (Menos regularidad en $\partial\Omega$ es suficiente para que valga el teorema de trazas, pero la regularidad C^1 alcanza para nuestro propósito). En este caso la medida del borde concuerda con la medida $(N-1)$ -Hausdorff restringida a $\partial\Omega$. Denotamos esta medida por dS . Entonces, los espacios son definidos por

$$L^p(\partial\Omega) = \left\{f \in \mathcal{M}(\partial\Omega) : \int_{\partial\Omega} |f|^p dS < \infty\right\}, \quad 1 \leq p < \infty$$

y la obvia definición para $L^\infty(\partial\Omega)$. Las normas son definidas de manera usual y se notan como $\|f\|_{L^p(\partial\Omega)} = \|f\|_{p, \partial\Omega}$.

El Teorema de trazas de Sobolev dice que, para $1 \leq p < N$ y $1 \leq r \leq p_* = (N-1)p/(N-p)$, existe un operador lineal acotado $T : W^{1,p}(\Omega) \rightarrow L^r(\partial\Omega)$ tal que : $Tv = v|_{\partial\Omega}$ si $v \in W^{1,p}(\Omega) \cap C(\Omega)$ y $\|Tv\|_{L^r(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\partial\Omega)}$ o equivalentemente:

$$0 < T(p, r, \Omega) = \inf_{v \in W^{1,p}(\Omega)} \frac{\|v\|_{1,p}}{\|v\|_{r,\partial\Omega}}. \quad (1.7)$$

La ecuación de Euler–Lagrange para (1.7) es

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda |u|^{r-2}u & \text{on } \partial\Omega. \end{cases} \quad (1.8)$$

Otra vez, λ es un multiplicador de Lagrange que depende de la normalización de u . Si u es normalizada por $\|u\|_{r,\partial\Omega} = 1$ entonces $\lambda = T(p, r, \Omega)^p$.

Cuando r es subcrítico, i.e. $1 \leq r < p_*$, la inmersión $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ es compacta y, como en el caso previo, la existencia de extremales para (1.7) junto con la existencia de una solución no trivial para (1.8) es una consecuencia del método directo del Cálculo de Variaciones.

En el caso crítico, $r = p_*$, la inmersión no es compacta. Entonces el problema de existencia, otra vez, es no trivial.

El problema crítico de trazas presenta grandes diferencias con el problema de inmersión de Sobolev. En [26] se muestra que se tiene la desigualdad

$$T(p, p_*, \Omega) \leq \bar{K}(N, p)^{-1} = \inf_{v \in \bar{D}^{1,p}(\mathbb{R}_+^N)} \frac{\|\nabla v\|_{p,\mathbb{R}_+^N}}{\|v\|_{p_*,\partial\mathbb{R}_+^N}},$$

donde $\bar{D}^{1,p}(\mathbb{R}_+^N)$ es el conjunto de funciones medibles f tales que $\partial_i f \in L^p(\mathbb{R}_+^N)$, $i = 1, \dots, N$ y $f|_{\partial\mathbb{R}_+^N} \in L^{p_*}(\partial\mathbb{R}_+^N)$.

Más aún, en [26], se prueba que si se tiene la desigualdad estricta

$$T(p, p_*, \Omega) < \bar{K}(N, p)^{-1}, \quad (1.9)$$

entonces existen extremales para (1.7) y por ende una solución no trivial para (1.8).

Una condición global para Ω que implica (1.9) es

$$\frac{|\Omega|^{\frac{1}{p}}}{\mathcal{H}^{N-1}(\partial\Omega)^{\frac{1}{p_*}}} < \bar{K}(N, p)^{-1}. \quad (1.10)$$

Observar que la clase de conjuntos que verifican (1.10) es grande. En particular, para un dominio Ω dado, si denotamos $\Omega_t = t \cdot \Omega$ entonces Ω_t verifica (1.10) para todo $t > 0$ suficientemente pequeño.

Más interesante es encontrar condiciones locales en Ω que garanticen (1.9). Para $p = 2$ esto fue conseguido por Adimurthi y Yadava usando el hecho de que los extremales para $\bar{K}(N, 2)^{-1}$ son conocidos explícitamente desde el trabajo de Escobar [19]. De hecho, en [1], los autores

prueban que si el borde de Ω contiene un punto con curvatura media positiva, entonces se verifica (1.9).

Recientemente Nazaret, en [43], encontró extremales para $\bar{K}(N, p)^{-1}$ usando métodos de transporte de masa. Estos extremales son de la forma

$$V_{\lambda, y_0}(y, t) = \lambda^{-\frac{N-p}{p-1}} V\left(\frac{y-y_0}{\lambda}, \frac{t}{\lambda}\right), \quad y \in \mathbb{R}^{N-1}, t > 0$$

y

$$V(y, t) = r^{-\frac{N-p}{p-1}}, \quad r = \sqrt{(1+t)^2 + |y|^2}.$$

Del conocimiento explícito de los extremales uno puede calcular el valor de la constante $\bar{K}(N, p)$ (ver, por ejemplo, [27]). Se verifica que

$$\bar{K}(N, p) = \pi^{\frac{1-p}{2}} \left(\frac{p-1}{N-p}\right)^{p-1} \left(\frac{\Gamma(\frac{p(N-1)}{2(p-1)})}{\Gamma(\frac{N-1}{2(p-1)})}\right)^{\frac{p-1}{N-1}}.$$

Usando estos extremales, Fernández Bonder y Saintier en [27] extendieron [1] y probaron que (1.9) se verifica si $\partial\Omega$ contiene un punto de curvatura media positiva para $1 < p < (N+1)/2$. Ver también [44] para un resultado relacionado.

1.2 Espacios de Lebesgue y de Sobolev con exponente variable

Antes de esta Tesis muy pocos resultados eran conocidos sobre los teoremas de inmersión de Sobolev cuando uno reemplaza los espacios de Lebesgue y Sobolev usuales por sus contrapartes de exponente variable.

Vamos a comenzar con una breve descripción de los espacios de Lebesgue y Sobolev con exponente variable. Daremos una discusión más detallada en el Capítulo 3.

Notamos por $\mathcal{P}(\Omega)$ el conjunto de funciones medibles $p: \Omega \rightarrow [1, +\infty)$. Este es el conjunto de exponentes finitos.

Para cualquier $p \in \mathcal{P}(\Omega)$ definimos el Espacio de Lebesgue de exponente variable como

$$L^{p(x)}(\Omega) = \left\{ f \in \mathcal{M}(\Omega) : \int_{\Omega} |f|^{p(x)} dx < \infty \right\}.$$

Estos espacios poseen una norma (llamada *Norma de Luxemburg*) la cual es definida como

$$\|f\|_{L^{p(x)}(\Omega)} = \|f\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Con esta norma $L^{p(x)}(\Omega)$ resulta un espacio de Banach. Más aún, si $1 < \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) < \infty$, $L^{p(x)}(\Omega)$ resulta ser un espacio reflexivo cuyo dual viene dado por $L^{p'(x)}(\Omega)$, $1/p(x) + 1/p'(x) = 1$.

Se define el espacio de Sobolev de exponente variable como

$$W^{1,p(x)}(\Omega) = \{f \in L^{p(x)}(\Omega) : \partial_i f \in L^{p(x)}(\Omega), i = 1, \dots, N\}$$

y la norma en este espacio es definida como

$$\|f\|_{W^{1,p(x)}(\Omega)} = \|f\|_{1,p(x)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|f|^{p(x)} + |\nabla f|^{p(x)}}{\lambda^{p(x)}} dx \leq 1 \right\}.$$

Observemos que es posible definir la norma como $\|f\|_{p(x)} + \|\nabla f\|_{p(x)}$. Ambas normas resultan equivalentes, pero para nuestros propósitos es más conveniente trabajar con la primera.

Análogamente al caso de exponentes constantes, definimos el subespacio de funciones que se anulan en el borde como

$$W_0^{1,p(x)}(\Omega) = \overline{C_c^\infty(\Omega)},$$

la clausura siendo tomada en la norma $\|\cdot\|_{1,p(x)}$.

En orden de recuperar la desigualdad de Poincaré en este contexto, son necesarias algunas hipótesis en el exponente $p(x)$. Se sabe que la desigualdad de Poincaré es válida si, por ejemplo, el exponente $p(x)$ es log-Hölder continuo, cf. Capítulo 3.

Entonces, bajo la hipótesis de log-Hölder continuidad de $p(x)$ la norma $\|\nabla f\|_{p(x)}$ es equivalente a la norma $\|f\|_{1,p(x)}$ para $f \in W_0^{1,p(x)}(\Omega)$.

Luego, si $\sup_{\Omega} p(x) < N$, el Teorema de inmersión de Sobolev para exponente variable resulta

$$0 < S(p(\cdot), q(\cdot), \Omega) = \inf_{v \in W_0^{1,p(x)}(\Omega)} \frac{\|\nabla v\|_{p(x)}}{\|v\|_{q(x)}}, \quad (1.11)$$

para cualquier $q \in \mathcal{P}(\Omega)$ tal que

$$q(x) \leq p^*(x) = \frac{Np(x)}{N - p(x)}.$$

Para recuperar la compacidad de la inmersión $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, necesitamos que el exponente $q(x)$ sea *uniformemente subcrítico*, i.e.

$$\inf_{\Omega} (p^*(x) - q(x)) > 0. \quad (1.12)$$

Como en el caso de exponente constante, bajo la hipótesis (1.12) se sigue por el método directo del Cálculo de Variaciones la existencia de extremales para (1.11).

El principal objetivo de esta Tesis es estudiar la existencia de extremales para (1.11) cuando (1.12) es violada.

Para la desigualdad de trazas definimos los espacios de Lebesgue en $\partial\Omega$ como

$$L^{p(x)}(\partial\Omega) = \left\{ f \in \mathcal{M}(\partial\Omega) : \int_{\partial\Omega} |f|^{p(x)} dS < \infty \right\}$$

y su correspondiente norma (de Luxemburg) es

$$\|f\|_{L^{p(x)}(\partial\Omega)} = \|f\|_{p(x),\partial\Omega} = \inf \left\{ \lambda > 0: \int_{\partial\Omega} \left| \frac{f}{\lambda} \right|^{p(x)} dS \leq 1 \right\}.$$

Entonces, el teorema de trazas de Sobolev para exponentes variables dice:

$$0 < T(p(\cdot), r(\cdot), \Omega) = \inf_{v \in W^{1,p(x)}(\Omega)} \frac{\|v\|_{1,p(x)}}{\|v\|_{r(x),\partial\Omega}}, \quad (1.13)$$

para cualquier $r \in \mathcal{P}(\partial\Omega)$ tal que

$$r(x) \leq p_*(x) = \frac{(N-1)p(x)}{N-p(x)}.$$

Otra vez, para recuperar la compacidad de la inmersión $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$, necesitamos que el exponente $r(x)$ sea uniformemente subcrítico, i.e.

$$\inf_{\partial\Omega} (p_*(x) - r(x)) > 0. \quad (1.14)$$

Otro de los principales intereses de esta Tesis es el estudio de la existencia de extremales para (1.13) cuando (1.14) no se verifica.

Para una revisión más completa de los espacios de Lebesgue y Sobolev con exponente variable, ver el Capítulo 3.

1.3 Motivaciones Físicas

En esta sección revisamos uno de los más importantes problemas físicos que modelan los espacios de exponente variable. Esta es la descripción matemática de los fluidos electroreológicos.

El modelamiento de los fluidos electroreológicos está ampliamente desarrollado en el libro de Růžička [48].

Los fluidos electroreológicos tienen la habilidad especial de cambiar sus propiedades mecánicas dependiendo de manera dramática del campo eléctrico aplicado. Algunas aplicaciones de estos incluyen: amortiguadores, soportes de motor, embragues y telemedicina.

Un modelo interesante para estos fluidos fue estudiado por Růžička [48]. A saber,

$$\operatorname{div}(E + P) = 0 \quad (1.15)$$

$$\operatorname{curl}(E) = 0 \quad (1.16)$$

$$\operatorname{div} S + \nabla v \cdot v + \nabla \phi = f + (\nabla E)P \quad (1.17)$$

$$\operatorname{div} v = 0, \quad (1.18)$$

donde E es el campo eléctrico, P es la polarización, v la velocidad, S el tensor de estres extra, ϕ es la presión y f es la fuerza mecánica.

Suponemos que el tensor de stres extra S depende de manera no lineal del campo eléctrico E y de la componente simétrica del gradiente de velocidades D , $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^t)$.

Experimentalmente, se observa que S depende polinomialmente de D con el grado dependiendo de E . Suponemos que S tiene un crecimiento no estándar dada por

$$|S(D, E)| \leq L(1 + |D|^2)^{\frac{\rho(|E|^2)-1}{2}}.$$

Bajo las hipótesis de que la polarización P es constante, el sistema (1.15)–(1.18) se desacopla y podemos resolver primero (1.15)–(1.16) para E y entonces, si denotamos $p(x) = p(|E|^2)$ el sistema (1.17)–(1.18) se convierte en una ecuación no lineal con crecimiento no estándar.

La energía asociada naturalmente a este problema es

$$\int_{\Omega} |D(v)|^{p(x)} dx.$$

En el caso de la velocidad escalar u , la energía se convierte en

$$\int_{\Omega} |\nabla u|^{p(x)} dx,$$

o

$$\mathcal{J}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx. \quad (1.19)$$

El operador asociado al funcional \mathcal{J} se llama el $p(x)$ -laplaciano y está dado por

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u).$$

Cuando $p(x) \equiv p$ es constante, este operador es el bien conocido p -laplaciano. Además, cuando $p(x) \equiv 2$ se convierte en el usual operador de Laplace.

Este hecho provee una motivación para el estudio de la siguiente ecuación elíptica no lineal,

$$-\Delta_{p(x)} u = f(x, u) \quad \text{en } \Omega, \quad (1.20)$$

complementada con condiciones de borde (Dirichlet, Neumann, etc.). Por supuesto, son necesarias algunas hipótesis sobre el crecimiento de $f(x, t)$. Ver la próxima sección.

Otra aplicación interesante en donde el $p(x)$ -laplaciano juega un rol importante, es en el procesamiento de imágenes.

En efecto, Y. Chen, S. Levin y R. Rao en [11] propusieron el siguiente modelo para la restauración de imágenes:

$$E(u) = \int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} + f(|u(x) - I(x)|) dx \rightarrow \min,$$

donde $p(x)$ es una función que varía entre 1 y 2 y f es una función convexa. En su aplicación, ellos eligieron $p(x)$ cerca de 1 en los lugares donde presuponen que hay bordes y $p(x)$ cerca de 2 en los lugares donde presuponen que no hay bordes. De esta manera los autores pueden remover el ruido de la imagen preservando los bordes.

1.4 Descripción de los resultados

Como mencionamos antes, el principal objetivo de esta Tesis es el estudio de las inmersiones de Sobolev

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \quad \text{y} \quad W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$$

en el rango crítico, es decir en el caso donde $1 \leq q(x) \leq p^*(x)$, $1 \leq r(x) \leq p_*(x)$ con

$$\mathcal{A} = \{x \in \Omega: q(x) = p^*(x)\} \neq \emptyset \quad \text{y} \quad \mathcal{A}_T = \{x \in \partial\Omega: r(x) = p_*(x)\} \neq \emptyset.$$

Nuestro principal interés es el estudio de la falta de compacidad en la inmersión. Por otro lado, buscamos condiciones en Ω , $p(x)$, $q(x)$ y $r(x)$ que garanticen la existencia de extremales para las constantes de Sobolev.

$$S(p(\cdot), q(\cdot), \Omega) = \inf_{u \in W_0^{1,p(x)}(\Omega)} \frac{\|\nabla u\|_{p(x)}}{\|u\|_{q(x)}} \quad \text{y} \quad T(p(\cdot), r(\cdot), \Omega) = \inf_{u \in W^{1,p(x)}(\Omega)} \frac{\|u\|_{1,p(x)}}{\|u\|_{r(x),\partial\Omega}}.$$

En el caso de exponente constante, con Ω acotado, la falta de compacidad fue descrita por P.L. Lions con el llamado *Principio de compacidad por concentración* (PCC) [37]. Ver también los trabajos de H. Brezis y L. Nirenberg [9] y T. Aubin [5].

El PCC afirma que si una sucesión $f_n \in W_0^{1,p}(\Omega)$ converge débil a f , pero f_n no converge fuerte a f en $L^p(\Omega)$. Entonces, la falta de convergencia fuerte viene dada por la aparición de masas puntuales.

El primer resultado de esta Tesis es la extensión del PCC para el caso de exponente variable. Un aspecto importante de nuestra extensión es el hecho de que las masas puntuales están localizadas en el conjunto crítico \mathcal{A} (o \mathcal{A}_T en el caso de la inmersión de trazas).

Estos resultados están contenidos en el Capítulo 4. Resultados similares fueron obtenidos independientemente por Y. Fu en [32] pero nuestros resultados son más generales, dado que en [32] sólo es considerada la inmersión de Sobolev $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ y, más aún, en nuestro resultado obtenemos las constantes precisas en las estimaciones. Esto último juega un rol preponderante en las aplicaciones del resultado.

Respecto de la existencia de extremales, queremos recordar el resultado de compacidad de [42]. En ese trabajo, se muestra que si el conjunto crítico \mathcal{A} es *pequeño* y tenemos un control preciso de como el exponente $q(x)$ alcanza al crítico $p^*(x)$ en \mathcal{A} , entonces la inmersión $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ permanece compacta y, por ende, la existencia de extremales se sigue por el método directo del Cálculo de Variaciones.

En esta dirección, obtenemos al comienzo del Capítulo 5 un resultado similar para el problema de trazas aplicando las técnicas desarrolladas en [42].

Sin embargo, las condiciones de [42] son bastante restrictivas. Entonces, es deseable obtener un resultado más general que garantice la existencia de extremales.

Recordemos que para el caso de exponente constante no existen extremales en ningún dominio acotado para las constantes de Sobolev $S(p, p^*, \Omega)$ y que esa constante es independiente de Ω .

En el caso de exponentes constantes, algunos resultados positivos fueron obtenidos usando perturbaciones del problema original.

En efecto, en [9] y [5] los autores consideran el problema perturbado.

$$\lambda_1 = \inf_{v \in W_0^{1,2}(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 + h(x)|u|^2 dx}{\left(\int_{\Omega} |u|^{2^*} dx \right)^{\frac{2}{2^*}}},$$

y encuentran condiciones locales en h que aseguran la existencia de extremales para λ_1 .

Observemos que si $h(x) > 0$, se sabe que la obstrucción de Pohožaev implica que no existen extremales para λ_1 si Ω es estrellado con respecto a un punto, luego h debe ser negativa en algún lugar. De cualquier manera, se requiere que

$$\|u\|^2 := \int_{\Omega} |\nabla u|^2 + h(x)|u|^2 dx$$

defina una norma equivalente en $W_0^{1,2}(\Omega)$.

Nuestro primer resultado para la inmersión de Sobolev dice que es válida la siguiente desigualdad.

$$S(p(\cdot), q(\cdot), \Omega) \leq \inf_{x \in \mathcal{A}} \sup_{\varepsilon > 0} S(p(\cdot), q(\cdot), B_{\varepsilon}(x)) \leq \inf_{x \in \mathcal{A}} K(N, p(x))^{-1}$$

y, además si la desigualdad es estricta

$$S(p(\cdot), q(\cdot), \Omega) < \inf_{x \in \mathcal{A}} \sup_{\varepsilon > 0} S(p(\cdot), q(\cdot), B_{\varepsilon}(x)) \quad (1.21)$$

entonces existe un extremal para $S(p(\cdot), q(\cdot), \Omega)$. Después, encontramos condiciones suficientes para que valga esa desigualdad estricta.

Primero, con una estimación muy cruda, encontramos que si el conjunto subcrítico $\Omega \setminus \mathcal{A}$ contiene un bola suficientemente grande, entonces (1.21) vale y por ende existe un extremal para $S(p(\cdot), q(\cdot), \Omega)$.

Este último resultado no es completamente satisfactorio. Uno desearía encontrar condiciones locales en Ω , $p(x)$ y $q(x)$, en el espíritu de [9, 5], que impliquen la validez (1.21). En el Capítulo 5, damos ese tipo de condiciones.

Para la desigualdad de trazas, en el caso de exponentes constantes, se sabe (ver [1] para $p = 2$ y [26, 27, 44] para $p \neq 2$) que si el borde contiene a un punto de curvatura media positiva (por ejemplo, cualquier dominio acotado) entonces existe un extremal para $T(p, p^*, \Omega)$. Recordemos que en los trabajos antes mencionados, algunas restricciones en p son necesarias, i.e. $p < (N + 1)/2$.

En el caso de exponente variable, primero obtenemos un resultado general análogo al caso de la inmersión de Sobolev. Se verifica que

$$T(p(\cdot), r(\cdot), \Omega) \leq \inf_{x \in \mathcal{A}_T} \sup_{\varepsilon > 0} T(p(\cdot), r(\cdot), \Omega_{\varepsilon}, \Gamma_{\varepsilon}) \leq \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1},$$

donde $\Omega_\varepsilon = \Omega \cap B_\varepsilon(x)$, $\Gamma_\varepsilon = \Omega \cap \partial B_\varepsilon(x)$ y $T(p(\cdot), r(\cdot), \Omega, \Gamma)$ es la mejor constante de trazas de Sobolev para funciones que se anulan en $\Gamma \subset \partial\Omega$.

Además, probamos que si la primera desigualdad estricta, es decir si se verifica

$$T(p(\cdot), r(\cdot), \Omega) < \inf_{x \in \mathcal{A}_T} \sup_{\varepsilon > 0} T(p(\cdot), r(\cdot), \Omega_\varepsilon, \Gamma_\varepsilon), \quad (1.22)$$

entonces existe un extremal para $T(p(\cdot), r(\cdot), \Omega)$.

Después de eso, primero encontramos condiciones globales similares a (1.10) para que (1.22) sea estricta.

Al final del Capítulo 5, encontramos condiciones locales en $p(x)$, $r(x)$ y en la geometría de Ω que aseguran la existencia de extremales para $T(p(\cdot), r(\cdot), \Omega)$.

Como una aplicación de los resultados previos, el resto de la Tesis, se dedica a estudiar la existencia de soluciones para algunas ecuaciones con crecimiento crítico en el sentido de las inclusiones de Sobolev.

Más precisamente, analizamos el problema de existencia de soluciones de la ecuación

$$\begin{cases} -\Delta_{p(x)} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.23)$$

donde el crecimiento de f verifica

$$|f(x, t)| \sim |t|^{q(x)-1}$$

y $q(x)$ es crítico, i.e. $\mathcal{A} = \{x \in \bar{\Omega} : q(x) = p^*(x)\} \neq \emptyset$.

En los últimos años han aparecido una gran cantidad de resultados dedicados al estudio del problema de existencia para (1.23) con diferentes condiciones de contorno (Dirichlet, Neumann, flujo no lineal, etc). Ver, por ejemplo [10, 16, 22, 40, 41] y sus referencias.

En estos trabajos, y en la mayoría de los artículos que se encuentran en la literatura, sólo el caso subcrítico es considerado ($\mathcal{A} = \emptyset$). Observemos que dado que los exponentes se suponen continuos, $\mathcal{A} = \emptyset$ es equivalente a (1.12).

Al igual que en el caso de exponentes constantes, el problema de hallar soluciones de (1.23) resulta equivalente al de hallar puntos críticos del funcional asociado. Este funcional viene dado por

$$\mathcal{F}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_{\Omega} F(x, u) dx,$$

donde $F(x, t) = \int_0^t f(x, s) ds$.

Bajo condiciones muy generales para f , el funcional \mathcal{F} verifica las hipótesis geométricas del Teorema del Paso de la Montaña. En el caso subcrítico, la compacidad de la inmersión de Sobolev implica la condición de the Palais–Smale para cualquier nivel de energía $c \in \mathbb{R}$ y entonces, se obtiene la existencia de un punto crítico para \mathcal{F} , y por ende, la existencia de una solución para (1.23). Para un breve resumen del Teorema de paso de la montaña y algunas técnicas usuales del cálculo de variaciones, ver el Capítulo 3.

Cuando la subcriticalidad es violada, i.e. $\mathcal{A} \neq \emptyset$, si bien las hipótesis geométricas del Teorema del Paso de la Montaña siguen verificándose, la falta de compacidad en la inmersión hace que la condición de Palais–Smale ya no pueda verificarse. En consecuencia existen sólo unos pocos resultados de existencia de soluciones para (1.23) que describiremos brevemente abajo.

En [42], como discutimos en la sección previa, los autores dan condiciones muy restrictivas para que la inclusión de Sobolev se mantenga compacta, y las técnicas usuales pueden ser aplicadas para encontrar un solución no trivial de (1.23). Cuando falta compacidad en la inmersión, en el mismo trabajo los autores prueban que si el conjunto subcrítico $\Omega \setminus \mathcal{A}$ contiene una bola suficientemente grande, entonces (1.23) tiene una solución no trivial no negativa.

El estudio de (1.23) planteado en todo \mathbb{R}^N es analizado en [4, 22]. En estos trabajos, los autores estudian el problema en el caso donde $p(x)$, $q(x)$ y f son funciones radiales y dan condiciones que garantian la existencia de una solución radial no trivial.

En esta Tesis, obtenemos dos tipos de resultados de existencia para (1.23). El primero se obtiene perturbando subcríticamente la no linealidad f . En efecto, consideremos

$$\begin{cases} -\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.24)$$

donde $|g(t)| \sim |t|^{s(x)-1}$ y el exponente $s(x)$ es subcrítico.

En el caso de exponentes constantes, el problema de existencia para (1.24) fue analizado en [33]. Nosotros seguimos las mismas ideas en nuestro caso.

Para estos problemas encontramos que si $p(x) < s(x) < q(x)$ y $\lambda(x)$ es grande en el conjunto crítico \mathcal{A} entonces existe una solución del problema (1.24). Además, cuando $s(x) < p(x) < q(x)$ y $\lambda(x)$ es uniformemente pequeño, entonces existen infinitas soluciones del problema (1.24) bajo la suposición de que g es impar.

Sin suponer condiciones de paridad en g podemos obtener un resultado de multiplicidad para (1.24). Para hacer esto, necesitamos que $\sup p(x) < \inf s(x) \leq \sup s(x) < \inf q(x)$ y que $\inf \lambda(x)$ sea suficientemente grande y obtenemos tres soluciones no triviales, una positiva, una negativa y otra que cambia de signo. Esto extiende un trabajo previo [12], donde fue tratado el mismo problema para exponentes constantes. Ver también el trabajo de Struwe [52] donde el caso subcrítico para exponentes constantes fue analizado.

Estos resultados están contenidos en el Capítulo 6. Resultados similares (algo más restrictivos que los nuestros) fueron obtenidos independientemente por [32].

Finalmente, el último Capítulo de la Tesis se ocupa del problema de existencia para (1.23) sin perturbaciones subcríticas. Para este problema podemos mostrar que el funcional asociado \mathcal{F} verifica la condición de Palais–Smale para niveles de energía c por debajo de un cierto nivel de energía crítico c^* (que podemos calcular explícitamente en términos de la constante de Sobolev).

Entonces, el problema de existencia se reduce a encontrar una sucesión de Palais–Smale con nivel de energía por debajo de c^* . En el espíritu de [9, 5], etc. encontramos condiciones locales en $p(x)$ y $q(x)$ que implican la existencia de esa sucesión.

Esto se logra por medio de un análisis asintótico fino obtenido al concentrar el extremal de $K(N, p)^{-1}$ alrededor de algún punto crítico de $p(x)$ y $q(x)$.

Usando el mismo tipo de argumentos, el problema con condiciones de flujo no lineal en la frontera

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial n} = |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases}$$

puede ser tratado pero en este caso el análisis asintótico es aún más delicado debido a que entra en juego la geometría de $\partial\Omega$.

Finalmente, al final de la Tesis incluimos un apéndice donde son calculadas las expansiones asintóticas necesarias para el Capítulo 7. Elegimos dejar estos cálculos para el apéndice debido a que las mismas son excesivamente técnicas y largas.

1.5 Publicaciones incluidas

Los resultados de esta Tesis han aparecido publicados como artículos de investigación. Estos resultados pueden ser leídos como contribuciones individuales unidos por un tema común y la mayoría de los mismos ya han sido publicados. La Tesis contiene los siguientes artículos:

- [28] J. Fernández Bonder, N. Saintier, A. Silva. *On the Sobolev trace Theorem for variable exponent spaces in the critical range*. Preprint.
- [29] J. Fernández Bonder, N. Saintier, A. Silva. *Existence of solution to a critical equation with variable exponent*. Ann. Acad. Sci. Fenn. Math., 37 (2012), 579–594.
- [30] J. Fernández Bonder, N. Saintier, A. Silva. *On the Sobolev embedding theorem for variable exponent spaces in the critical range*. J. Differential Equations, 253 (2012), no. 5, 1604–1620.
- [31] J. Fernández Bonder, A. Silva. *The concentration-compactness principle for variable exponent spaces and applications*. Electron. J. Differential Equations, 2010 (2010), no. 141, 1–18.
- [51] A. Silva. *Multiple solutions for the $p(x)$ -laplace operator with critical growth*. Advanced Nonlinear Studies, 11 (2011), 63–75.

2

Introduction

2.1 Preliminary results

The purpose of this Thesis is the study of the Sobolev immersion theorems in variable exponent spaces with lack of compactness.

Let us begin with a brief account of the known results. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and denote by $\mathcal{M}(\Omega)$ the set of measurable function on Ω with values in the extended real line $[-\infty, +\infty]$. The Lebesgue spaces for constant exponents are defined as

$$L^p(\Omega) = \left\{ f \in \mathcal{M}(\Omega) : \int_{\Omega} |f|^p dx < \infty \right\}, \quad 1 \leq p < \infty, \quad \text{and} \quad L^\infty(\Omega) = \left\{ f \in \mathcal{M}(\Omega) : \sup_{\Omega} |f| < \infty \right\}.$$

Here and in this Thesis, by sup we mean de essential supremum with respect to the Lebesgue measure.

This spaces are equipped with the norms

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} = \|f\|_\infty = \text{ess sup}_{\Omega} |f|.$$

The Sobolev spaces are defined as

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) : \partial_i f \in L^p(\Omega), \quad i = 1, \dots, N\}, \quad 1 \leq p \leq \infty,$$

where $\partial_i f = \frac{\partial f}{\partial x_i}$ stands for the distributional partial derivative.

The Sobolev norm is defined as

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{1,p} = (\|f\|_p^p + \|\nabla f\|_p^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \text{and} \quad \|f\|_{W^{1,\infty}(\Omega)} = \|f\|_{1,\infty} = \|f\|_\infty + \|\nabla f\|_\infty.$$

Let $1 \leq p < N$ be fixed. The Sobolev immersion Theorem, says that for Ω Lipschitz, one has, for any $f \in W^{1,p}(\Omega)$, the inequality, cf. [21]

$$\|f\|_q \leq C(p, q, \Omega) \|f\|_{1,p}$$

for $1 \leq q \leq p^*$, where p^* is the so-called *critical Sobolev exponent* and is given by

$$p^* = \frac{Np}{N-p}. \quad (2.1)$$

Equivalently, there holds

$$0 < \tilde{S}(p, q, \Omega) = \inf_{v \in W^{1,p}(\Omega)} \frac{\|v\|_{1,p}}{\|v\|_q}. \quad (2.2)$$

These constants $\tilde{S}(p, q, \Omega)$ are called the (best or optimal) Sobolev constants and the functions v that realize the above infimum (if they exist) are called *extremals*.

One basic question in analysis and partial differential equations is the (explicit) computation of the optimal Sobolev constants and of their corresponding extremals.

Let us recall that an extremal is a (weak) solution of the corresponding Euler–Lagrange equation

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda|u|^{q-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.3)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the well-known p -Laplace operator and $\partial u / \partial n$ is the normal derivative with respect to the outer unit normal vector n to Ω .

The constant λ is a Lagrange multiplier and depends on the normalization of u . For instance if u is chosen so that $\|u\|_q = 1$, then $\lambda = \tilde{S}(p, q, \Omega)^p$.

When q is *subcritical*, i.e. $1 \leq q < p^*$, the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, and so the direct method of the Calculus of Variations immediately yields the existence of an extremal for (2.2), and therefore the existence of a weak solution to (2.3).

On the other hand, when $q = p^*$ it is easy to see that the compactness fails and so the existence of an extremal for (2.2) or the existence of a weak solution to (2.3) is a nontrivial matter.

It is customary in the literature, due to many applications, to consider the subspace of $W^{1,p}(\Omega)$ consisting in those functions having zero boundary values. This subspace is defined as

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)},$$

where $C_c^\infty(\Omega)$ stands for the smooth functions with compact support and the closure is taken in the $\|\cdot\|_{1,p}$ -norm.

In this space, the well-known Poincaré inequality holds

$$\|f\|_p \leq C(p, \Omega) \|\nabla f\|_p, \quad \text{for } f \in W_0^{1,p}(\Omega). \quad (2.4)$$

From (2.4) it follows that $\|\nabla f\|_p$ defines a norm in $W_0^{1,p}(\Omega)$ which is equivalent to $\|f\|_{1,p}$. So, when one works within the space $W_0^{1,p}(\Omega)$, the Sobolev immersion Theorem can be restated as

$$0 < S(p, q, \Omega) = \inf_{v \in W_0^{1,p}(\Omega)} \frac{\|\nabla v\|_p}{\|v\|_q} \quad (2.5)$$

and the Euler-Lagrange equation for the extremals of (2.5) becomes

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.6)$$

Again, for the existence of extremals problem, the only nontrivial case is the critical one, $q = p^*$. Let us observe that in the space $W_0^{1,p}(\Omega)$ no regularity hypothesis is needed on Ω for the Sobolev immersion Theorem to hold.

In the critical case it is known that $S(p, p^*, \Omega)$ is not attained for any Ω bounded and the Euler–Lagrange equation (2.6) does not have a solution for Ω bounded and starshaped with respect to some point.

Moreover, the constant $S(p, p^*, \Omega)$ is independent of Ω . It holds

$$S(p, p^*, \Omega) = K(N, p)^{-1} := \inf_{v \in D^{1,p}(\mathbb{R}^N)} \frac{\|\nabla v\|_p}{\|v\|_{p^*}}$$

where $D^{1,p}(\mathbb{R}^N)$ is the set of functions f in $L^{p^*}(\mathbb{R}^N)$ such that $\partial_i f \in L^p(\mathbb{R}^N)$, $i = 1, \dots, N$.

It is known, see [37, 53], that the extremals for $K(N, p)^{-1}$ form a two-parameter family given by

$$U_{\lambda, x_0}(x) = \lambda^{-\frac{N-p}{p}} U\left(\frac{x-x_0}{\lambda}\right),$$

with U given by

$$U(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}}.$$

In particular, see [53], this allows one to compute explicitly the value of $K(N, p)^{-1}$,

$$K(N, p) = \pi^{\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma(1 + \frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(1 - N - \frac{N}{p})}\right)^{\frac{1}{N}},$$

where $\Gamma(x)$ is the Gamma function, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

Another very important Sobolev immersion is the *Sobolev trace Theorem*. This theorem allows one to restrict a Sobolev function to the boundary of the domain (that has Lebesgue measure zero).

In order to state the trace Theorem we need to define the Lebesgue spaces on $\partial\Omega$. We assume that Ω is C^1 so $\partial\Omega$ is a $(N-1)$ -dimensional C^1 immersed manifold on \mathbb{R}^N (less regularity on $\partial\Omega$ is enough for the trace Theorem to hold, but the C^1 regularity is enough for our purposes). Therefore the boundary measure agrees with the $(N-1)$ -Hausdorff measure restricted to $\partial\Omega$. We denote this measure by dS . So, the spaces are defined as

$$L^p(\partial\Omega) = \left\{f \in \mathcal{M}(\partial\Omega) : \int_{\partial\Omega} |f|^p dS < \infty\right\}, \quad 1 \leq p < \infty$$

and an obvious definition for $L^\infty(\partial\Omega)$.

The norms are defined in the usual manner and are denoted by $\|f\|_{L^p(\partial\Omega)} = \|f\|_{p,\partial\Omega}$.

The Sobolev trace Theorem states that, for $1 \leq p < N$ and $1 \leq r \leq p_* = (N-1)p/(N-p)$, there exists a bounded linear operator $T : W^{1,p}(\Omega) \rightarrow L^r(\partial\Omega)$ such that $Tv = v|_{\partial\Omega}$ si $v \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $\|Tv\|_{L^r(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\partial\Omega)}$ or equivalently

$$0 < T(p, r, \Omega) = \inf_{v \in W^{1,p}(\Omega)} \frac{\|v\|_{1,p}}{\|v\|_{r,\partial\Omega}}. \quad (2.7)$$

The Euler-Lagrange equation for (2.7) is

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda |u|^{r-2}u & \text{on } \partial\Omega. \end{cases} \quad (2.8)$$

Again, λ is a Lagrange multiplier that depends on the normalization of u . If u is taken as $\|u\|_{r,\partial\Omega} = 1$ then $\lambda = T(p, r, \Omega)^p$.

If r is subcritical, i.e. $1 \leq r < p_*$, the immersion $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is compact and, as in the previous case, the existence of extremals for (2.7) and therefore the existence of a solution to (2.8) follows by the direct method of the Calculus of Variations.

In the critical case, $r = p_*$, the immersion is no longer compact. So the existence problem, again, becomes nontrivial.

The critical trace problem presents striking differences with respect to the critical Sobolev immersion problem.

It is known, see [26], that

$$T(p, p_*, \Omega) \leq \bar{K}(N, p)^{-1} = \inf_{v \in \bar{D}^{1,p}(\mathbb{R}_+^N)} \frac{\|\nabla v\|_p}{\|v\|_{p_*, \partial\mathbb{R}_+^N}},$$

where $\bar{D}^{1,p}(\mathbb{R}_+^N)$ is the set of measurable functions f such that $\partial_i f \in L^p(\mathbb{R}_+^N)$, $i = 1, \dots, N$ and $f|_{\partial\mathbb{R}_+^N} \in L^{p_*}(\partial\mathbb{R}_+^N)$.

Moreover, in [26] it is shown that if

$$T(p, p_*, \Omega) < \bar{K}(N, p)^{-1}, \quad (2.9)$$

then there exists an extremal for (2.7) and so, a solution to (2.8).

One trivial global condition on Ω that implies (2.9) is

$$\frac{|\Omega|^{\frac{1}{p}}}{\mathcal{H}^{N-1}(\partial\Omega)^{\frac{1}{p_*}}} < \bar{K}(N, p)^{-1}. \quad (2.10)$$

Observe that the family of sets that verify (2.10) is large. In particular, for any fixed set Ω if we denote $\Omega_t = t \cdot \Omega$ then Ω_t verifies (2.10) for any $t > 0$ small.

More interesting is to find local conditions on Ω that ensure (2.9). For $p = 2$ this was done by Adimurthi and Yadava by using the fact that the extremals for $\bar{K}(N, 2)^{-1}$ were explicitly known

since the work of Escobar [19]. In fact, in [1], the authors proved that if the boundary of Ω contains a point with positive mean curvature, then (2.9) holds true.

Recently Nazaret, in [43], found the extremals for $\bar{K}(N, p)^{-1}$ by means of mass transportation methods. These extremals are of the form

$$V_{\lambda, y_0}(y, t) = \lambda^{-\frac{N-p}{p-1}} V\left(\frac{y-y_0}{\lambda}, \frac{t}{\lambda}\right), \quad y \in \mathbb{R}^{N-1}, t > 0$$

and

$$V(y, t) = r^{-\frac{N-p}{p-1}}, \quad r = \sqrt{(1+t)^2 + |y|^2}.$$

From the explicit knowledge of the extremals one can compute the value of the constant $\bar{K}(N, p)$ (see, for example, [27]). It holds

$$\bar{K}(N, p) = \pi^{\frac{1-p}{2}} \left(\frac{p-1}{N-p}\right)^{p-1} \left(\frac{\Gamma(\frac{p(N-1)}{2(p-1)})}{\Gamma(\frac{N-1}{2(p-1)})}\right)^{\frac{p-1}{N-1}}.$$

Using these extremals, Fernández Bonder and Saintier in [27] extended [1] and prove that (2.9) holds true if $\partial\Omega$ contains a point of positive mean curvature for $1 < p < (N+1)/2$. See also [44] for a related result.

2.2 Variable exponent setting

Prior to this Thesis very little was known on the Sobolev immersion theorems when one replaces the usual Lebesgue and Sobolev spaces for their variable exponent counterparts.

Let us begin with a brief description of the Lebesgue and Sobolev spaces with variable exponent. A more detailed discussion is given in Chapter 3.

We denote by $\mathcal{P}(\Omega)$ the set of measurable functions $p: \Omega \rightarrow [1, +\infty)$. This is the set of finite exponents.

For any $p \in \mathcal{P}(\Omega)$ we define the variable exponent Lebesgue space as

$$L^{p(x)}(\Omega) = \left\{ f \in \mathcal{M}(\Omega) : \int_{\Omega} |f|^{p(x)} dx < \infty \right\}.$$

These spaces are endowed with a norm (so-called the *Luxemburg norm*) which is defined as

$$\|f\|_{L^{p(x)}(\Omega)} = \|f\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

With this norm, $L^{p(x)}(\Omega)$ becomes a Banach space and if $1 < \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) < \infty$, it is a reflexive space with dual given by $L^{p'(x)}(\Omega)$, $1/p(x) + 1/p'(x) = 1$.

The variable exponent Sobolev space is defined as

$$W^{1,p(x)}(\Omega) = \{ f \in L^{p(x)}(\Omega) : \partial_i f \in L^{p(x)}(\Omega), i = 1, \dots, N \}.$$

The norm within this space is defined as

$$\|f\|_{W^{1,p(x)}(\Omega)} = \|f\|_{1,p(x)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|f|^{p(x)} + |\nabla f|^{p(x)}}{\lambda^{p(x)}} dx \leq 1 \right\}.$$

Observe that we can define the norm as $\|f\|_{p(x)} + \|\nabla f\|_{p(x)}$. Both norms turn out to be equivalent, but is more convenient for our purposes to work with the first one.

Analogously to the constant exponent case, the subspace of the functions with zero boundary values is defined as

$$W_0^{1,p(x)}(\Omega) = \overline{C_c^\infty(\Omega)},$$

the closure being taken in the $\|\cdot\|_{1,p(x)}$ -norm.

In order to recover Poincaré inequality in this context, some hypotheses on the exponent $p(x)$ are needed. Up to date, Poincaré inequality is known to hold if $p(x)$ is log-Hölder continuous, cf. Chapter 3.

So, under log-Hölder continuity of $p(x)$ the norm $\|\nabla f\|_{p(x)}$ is an equivalent norm to $\|f\|_{1,p(x)}$ for $f \in W_0^{1,p(x)}(\Omega)$.

Thus, if $\sup_{\Omega} p(x) < N$, the Sobolev immersion Theorem for variable exponents reads

$$0 < S(p(\cdot), q(\cdot), \Omega) = \inf_{v \in W_0^{1,p(x)}(\Omega)} \frac{\|\nabla v\|_{p(x)}}{\|v\|_{q(x)}}, \quad (2.11)$$

for any $q \in \mathcal{P}(\Omega)$ such that

$$q(x) \leq p^*(x) = \frac{Np(x)}{N - p(x)}.$$

In order to recover the compactness of the immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, we require the exponent $q(x)$ to be *uniformly subcritical*, i.e.

$$\inf_{\Omega} (p^*(x) - q(x)) > 0. \quad (2.12)$$

As in the constant exponent case, under hypothesis (2.12) the existence of extremals for (2.11) follows immediately by direct minimization.

The main objective of the Thesis is to study the existence of extremals for (2.11) when (2.12) is violated.

As for the trace inequality, the Lebesgue spaces on $\partial\Omega$ are defined as

$$L^{p(x)}(\partial\Omega) = \left\{ f \in \mathcal{M}(\partial\Omega) : \int_{\partial\Omega} |f|^{p(x)} dS < \infty \right\}$$

and the corresponding (Luxemburg) norm is

$$\|f\|_{L^{p(x)}(\partial\Omega)} = \|f\|_{p(x),\partial\Omega} = \inf \left\{ \lambda > 0 : \int_{\partial\Omega} \left| \frac{f}{\lambda} \right|^{p(x)} dS \leq 1 \right\}.$$

Hence, the Sobolev trace Theorem for variable exponents reads

$$0 < T(p(\cdot), r(\cdot), \Omega) = \inf_{v \in W^{1,p(x)}(\Omega)} \frac{\|v\|_{1,p(x)}}{\|v\|_{r(x),\partial\Omega}}, \quad (2.13)$$

for any $r \in \mathcal{P}(\partial\Omega)$ such that

$$r(x) \leq p_*(x) = \frac{(N-1)p(x)}{N-p(x)}.$$

Again, in order to recover compactness of the immersion $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$, we require the exponent $r(x)$ to be uniformly subcritical, i.e.

$$\inf_{\partial\Omega} (p_*(x) - r(x)) > 0. \quad (2.14)$$

Another main concern of this Thesis is the study of the existence of extremals for (2.13) when (2.14) is violated.

For a more comprehensive review of Lebesgue and Sobolev spaces with variable exponents, see Chapter 3.

2.3 Physical motivation

In this section we review one of the most important physical problems where variable exponent spaces play a crucial role in modeling. This is the mathematical description of electrorheological fluids.

The modeling of electrorheological fluids is fully developed in the book of Růžička [48].

Electrorheological fluids have the special feature that their mechanical properties depend in a dramatic way on an applied electric field. Some applications of these include: vibration absorbers, engine mounts, earthquake-resistant buildings, clutches, actuators and telemedicine.

An interesting model for such fluids was studied by Růžička [48]. Namely,

$$\operatorname{div}(E + P) = 0 \quad (2.15)$$

$$\operatorname{curl}(E) = 0 \quad (2.16)$$

$$\operatorname{div} S + \nabla v \cdot v + \nabla \phi = f + (\nabla E)P \quad (2.17)$$

$$\operatorname{div} v = 0, \quad (2.18)$$

where E is the electric field, P the polarization, v the velocity, S the extra stress tensor, ϕ the pressure and f is the mechanical force.

The extra stress tensor S is assumed to depend in a nonlinear manner of the electric field E and of the symmetric velocity gradient D , $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^t)$.

Experimentally, it is observed that S depends polynomially on D with degree depending on E . It is assumed that S has the nonstandard growth given by

$$|S(D, E)| \leq L(1 + |D|^2)^{\frac{p(E^2)-1}{2}}.$$

Under the assumption that the polarization P is constant, the system (2.15)–(2.18) decouples and we can first solve (2.15)–(2.16) for E and so, if we denote $p(x) = p(|E|^2)$ the system (2.17)–(2.18) becomes a nonlinear differential equation with nonstandard growth.

The natural energy associate to this problem is

$$\int_{\Omega} |D(v)|^{p(x)} dx.$$

In case the velocity is a scalar u , this energy becomes

$$\int_{\Omega} |\nabla u|^{p(x)} dx,$$

or

$$\mathcal{J}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx. \quad (2.19)$$

The Euler-Lagrange equation associated to the functional (2.19) is the so-called $p(x)$ -laplacian given by

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u).$$

When $p(x) \equiv p$ is constant, this operator is the well-known p -Laplacian. Moreover, when $p(x) \equiv 2$ it becomes the usual Laplace operator.

This fact is a motivation for the study of the following nonlinear elliptic equation,

$$-\Delta_{p(x)} u = f(x, u) \quad \text{in } \Omega, \quad (2.20)$$

complemented with boundary conditions (Dirichlet, Neumann, etc.). Of course, some hypotheses on the source term $f(x, t)$ are needed. See the next section.

There is another interesting application where the $p(x)$ -laplacian plays an important role. This application comes from Image Processing.

In fact, Y. Chen, S. Levin and R. Rao in [11] proposed the following model for image restoration:

$$E(u) = \int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} + f(|u(x) - I(x)|) dx \rightarrow \min,$$

where $p(x)$ is a function varying between 1 and 2 and f is a convex function. In their application, they chose $p(x)$ close to 1 where there is likely to be edges and close to 2 where it is unlikely to be edges. In this way, the authors are allowed to remove the noise from the image preserving the boundaries.

2.4 Description of the results

As we mentioned before, the main objective of this Thesis is the study of the Sobolev immersions

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \quad \text{and} \quad W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$$

in the critical range, i.e. in the case where $1 \leq q(x) \leq p^*(x)$, $1 \leq r(x) \leq p_*(x)$ and

$$\mathcal{A} = \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset \quad \text{and} \quad \mathcal{A}_T = \{x \in \partial\Omega : r(x) = p_*(x)\} \neq \emptyset.$$

The main questions that we address is the study of the failure of compactness of the immersion. On the other, we look for conditions on Ω , $p(x)$, $q(x)$ and $r(x)$ that guaranty the existence of extremals for the Sobolev constants

$$S(p(\cdot), q(\cdot), \Omega) = \inf_{u \in W_0^{1,p(x)}(\Omega)} \frac{\|\nabla u\|_{p(x)}}{\|u\|_{q(x)}} \quad \text{and} \quad T(p(\cdot), r(\cdot), \Omega) = \inf_{u \in W^{1,p(x)}(\Omega)} \frac{\|u\|_{1,p(x)}}{\|u\|_{r(x),\partial\Omega}}.$$

In the constant exponent case, with Ω bounded, the failure of compactness was fully described by the so-called *Concentration–Compactness Principle* (CCP) by P.L. Lions in [37]. See also the works of H. Brezis and L. Nirenberg [9] and T. Aubin [5].

The CCP states that if a sequence $f_n \in W_0^{1,p}(\Omega)$ weakly converges to f , but f_n does not converges to f strongly in $L^p(\Omega)$. Then, the failure for this strong convergence comes from the appearance of point masses.

The first result in this Thesis is the extension of the CCP to the variable exponent case. An important feature of our extension is the fact that the point masses are located in the critical set \mathcal{A} (or \mathcal{A}_T in the case of the trace immersion).

These results are the content of Chapter 4. Let us mention that similar results were obtained independently by Y. Fu in [32] but our results are more general since in [32] only the case of the Sobolev immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ was considered and, moreover, in our results the precise constants entering the estimates are obtained. The precise knowledge of these constants play a decisive role in the applications of our result.

As for the existence of extremals, we recall the compactness result of [42]. In that work it is shown that if the critical set \mathcal{A} is *small* and we have a certain precise rate at which the exponent $q(x)$ reaches the critical one $p^*(x)$ at \mathcal{A} , then the immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ remains compact, and so the existence of extremals follows by direct minimization.

In this direction, we obtain at the beginning of Chapter 5 a similar result for the trace problem applying the same technique developed in [42].

Nevertheless, the conditions of [42] are rather restrictive. So, a more general result in order to obtain the existence of extremals is desirable.

Recall that in the constant exponent case extremals for the Sobolev constant $S(p, p^*, \Omega)$ do not exist in any bounded domain and that this constant is independent of Ω .

Still in the constant exponent case, some positive results were obtained for perturbations of the extremal problem.

In fact, in [9] and [5] the authors considered the perturbed problem

$$\lambda_1 = \inf_{v \in W_0^{1,2}(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 + h(x)|v|^2 dx}{\left(\int_{\Omega} |v|^{2^*} dx \right)^{\frac{2}{2^*}}},$$

and found local conditions on h that ensure the existence of an extremal for λ_1 .

Observe that if $h(x) > 0$ and Ω is starshaped with respect to some point, it is well known that the Pohozaev obstruction implies that no extremal for λ_1 exists, so h must be negative somewhere. Anyway, it is required that

$$\|u\|^2 := \int_{\Omega} |\nabla u|^2 + h(x)|u|^2 dx$$

define an equivalent norm in $W_0^{1,2}(\Omega)$.

Our first result for the Sobolev immersion says that the following inequality holds

$$S(p(\cdot), q(\cdot), \Omega) \leq \inf_{x \in \mathcal{A}} \sup_{\varepsilon > 0} S(p(\cdot), q(\cdot), B_\varepsilon(x)) \leq \inf_{x \in \mathcal{A}} K(N, p(x))^{-1}$$

and, moreover if the strict inequality holds

$$S(p(\cdot), q(\cdot), \Omega) < \inf_{x \in \mathcal{A}} \sup_{\varepsilon > 0} S(p(\cdot), q(\cdot), B_\varepsilon(x)) \quad (2.21)$$

then there exists an extremal for $S(p(\cdot), q(\cdot), \Omega)$. Then, we find sufficient conditions for the above strict inequality to hold.

First, with a very crude estimate, we find that if the subcritical set $\Omega \setminus \mathcal{A}$ contains a sufficiently large ball. Then, (2.21) holds and therefore an extremal for $S(p(\cdot), q(\cdot), \Omega)$ exists.

This latter result is not satisfactory. It would be desirable to find *local* conditions on $p(x)$ and $q(x)$, in the spirit of [9, 5], that imply the validity of (2.21). We find such conditions in Chapter 5.

As for the trace inequality, in the constant exponent case. It is known (see [1] for $p = 2$ and [26, 27, 44] for $p \neq 2$) that if the boundary of the domain contains a point of positive mean curvature (for instance, any bounded domain) then there exists an extremal for $T(p, p^*, \Omega)$. Recall that in the above mentioned works, some restrictions on p are in order, i.e. $p < (N + 1)/2$.

In the variable exponent case, we first obtain a general result analogous to the Sobolev immersion case. There holds

$$T(p(\cdot), r(\cdot), \Omega) \leq \inf_{x \in \mathcal{A}_T} \sup_{\varepsilon > 0} T(p(\cdot), r(\cdot), \Omega_\varepsilon, \Gamma_\varepsilon) \leq \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1},$$

where $\Omega_\varepsilon = \Omega \cap B_\varepsilon(x)$, $\Gamma_\varepsilon = \Omega \cap \partial B_\varepsilon(x)$ and $T(p(\cdot), r(\cdot), \Omega, \Gamma)$ stands for the best trace constant on functions that vanish on $\Gamma \subset \partial\Omega$.

Moreover, we prove that if the strict inequality holds

$$T(p(\cdot), r(\cdot), \Omega) < \inf_{x \in \mathcal{A}_T} \sup_{\varepsilon > 0} T(p(\cdot), r(\cdot), \Omega_\varepsilon, \Gamma_\varepsilon), \quad (2.22)$$

then there exists an extremal for $T(p(\cdot), r(\cdot), \Omega)$.

After that, we first find global conditions similar to (2.10) in order to (2.22) holds true.

Finally, at the end of Chapter 5, we find local conditions on $p(x)$, $r(x)$ and on the geometry of Ω that ensure the existence of extremals for $T(p(\cdot), r(\cdot), \Omega)$.

As an application of the previous results, the rest of the Thesis, is devoted to the study of the existence of solution to some partial differential equations with critical growth in the sense of the Sobolev embeddings.

To be precise, we analyze the existence problem for solutions of the equation

$$\begin{cases} -\Delta_{p(x)}u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.23)$$

where the source term f verifies

$$|f(x, t)| \sim |t|^{q(x)-1}$$

and $q(x)$ is critical, i.e. $\mathcal{A} = \{x \in \bar{\Omega}: q(x) = p^*(x)\} \neq \emptyset$.

In recent years a vast amount of literature that dealt with the existence problem for (2.23) with different boundary conditions (Dirichlet, Neumann, nonlinear, etc) has appeared. See, for instance [10, 16, 22, 40, 41] and references therein.

In these works, and in most of the papers found in the literature, only the subcritical case is considered ($\mathcal{A} = \emptyset$). Recall that as the exponents are assumed to be continuous, $\mathcal{A} = \emptyset$ is equivalent to (2.12).

In the subcritical case, the compactness of the Sobolev immersion immediately gives that, under reasonable assumptions on f , the functional associated to (2.23)

$$\mathcal{F}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_{\Omega} F(x, u) dx,$$

where $F(x, t) = \int_0^t f(x, s) ds$, verifies the Palais–Smale condition for any energy level $c > 0$ and so, by means of the Mountain Pass Theorem the existence of a critical point for \mathcal{F} , and therefore of a solution to (2.23), follows as in the constant exponent case. For a brief account on the Mountain-Pass Theorem and on some of the most common variational techniques in the calculus of variation, see Chapter 3.

When the subcriticality is violated, i.e. $\mathcal{A} \neq \emptyset$, there are only a handful of results on the existence of solutions to (2.23) that we briefly describe below.

In [42], as we discussed in the previous section, the authors give very restrictive conditions to ensure that the Sobolev immersion remains compact, and so the usual techniques can be applied to find a nontrivial solution to (2.23). When the immersion fails to be compact in the same work the authors prove that if the subcriticality set $\Omega \setminus \mathcal{A}$ contains a sufficiently large ball, then (2.23) has a nonnegative nontrivial solution.

The study of (2.23) posed in the whole \mathbb{R}^N is analyzed in [4, 22]. In those works the authors studied the problem in the case where $p(x)$, $q(x)$ and f are radial functions and give somewhat restrictive conditions to ensure the existence of a nontrivial radial solution.

In this Thesis, we obtain two type of existence results for (2.23). The first ones are obtained by perturbing the critical source f by a subcritical one. That is, we consider

$$\begin{cases} -\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.24)$$

where $|g(t)| \sim |t|^{s(x)-1}$ and $s(x)$ is subcritical.

In the constant exponent case, the existence problem for (2.24) was analyzed in [33]. We follow the same line of approach in our case.

For these problems we find that if $p(x) < s(x) < q(x)$ and $\lambda(x)$ is large in the critical set \mathcal{A} then there exists a solution to (2.24). Moreover, when $s(x) < p(x) < q(x)$ and $\lambda(x)$ is uniformly small, then there exists infinitely many solutions to (2.24) under the assumption that g is odd.

Without the oddness assumption on g we can still obtain a multiplicity result for (2.24). In order to do this, we need to assume that $\sup p(x) < \inf s(x) \leq \sup s(x) < \inf q(x)$ and that $\inf \lambda(x)$ large enough and we obtain three nontrivial solutions, one positive, one negative and the other one is sign changing. This extend previous work [12], where the same problem but with constant exponents was treated. See also the paper from Struwe [52] where the subcritical case in the constant exponent framework was analyzed.

These results are the content of Chapter 6. Let us mentioned that some related results (more restrictive than ours, though) were obtained independently in [32].

Finally, the last Chapter of the Thesis deals with the existence problem for (2.23) without subcritical perturbations. For this problem we can show that the associated functional \mathcal{F} verifies the Palais–Smale condition for energy levels c below some critical energy c^* (that can be computed explicitly in terms of the Sobolev constants).

So, the existence problem is reduced to find a Palais–Smale sequence with energy level below c^* . In the spirit of [9, 5], etc. we find local conditions on $p(x)$ and $q(x)$ that imply the existence of such sequence.

This is done by a refined asymptotic analysis obtained by concentrating the extremal for $K(N, p)^{-1}$ around some critical point of $p(x)$ and $q(x)$.

By using the same type of arguments, the nonlinear boundary condition case

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial n} = |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases}$$

can be treated but in this case the asymptotic analysis is even more delicate since the geometry of $\partial\Omega$ comes into play.

Finally, at the end of the Thesis we included an appendix where the asymptotic expansions needed in Chapter 7 are computed. We have chosen to leave this computations in a separated appendix since they are rather technical and long.

2.5 Included publications

The results in this thesis have appeared published as research articles. These results are readable as individuals contributions linked by a common theme and most of them have been already published. The Thesis contains the following papers:

- [28] J. Fernández Bonder, N. Saintier, A. Silva. *On the Sobolev trace Theorem for variable exponent spaces in the critical range*. Preprint.
- [29] J. Fernández Bonder, N. Saintier, A. Silva. *Existence of solution to a critical equation with variable exponent*. Ann. Acad. Sci. Fenn. Math., 37 (2012), 579–594.
- [30] J. Fernández Bonder, N. Saintier, A. Silva. *On the Sobolev embedding theorem for variable exponent spaces in the critical range*. J. Differential Equations, 253 (2012), no. 5, 1604–1620.
- [31] J. Fernández Bonder, A. Silva. *The concentration-compactness principle for variable exponent spaces and applications*. Electron. J. Differential Equations, 2010 (2010), no. 141, 1–18.
- [51] A. Silva. *Multiple solutions for the $p(x)$ -laplace operator with critical growth*. Advanced Nonlinear Studies, 11 (2011), 63–75.

3

Preliminaries

3.1 Variable exponent Sobolev spaces

In this chapter we review some preliminary results regarding Lebesgue and Sobolev spaces with variable exponent, they differ from classical L^p in that the exponent p is not constant but a function from Ω to $[1, \infty]$. All of these results and a comprehensive study of these spaces can be found in [14]. For the definition of the variable exponent spaces is necessary to introduce the kind of variable exponents that we are interested in.

Definition 3.1. Let (E, Σ, μ) be a σ -finite, complete measure space. We define $\mathcal{P}(E, \mu)$ to be the set of all μ measurable functions $p: E \rightarrow [1, \infty]$. The functions $p \in \mathcal{P}(E, \mu)$ are called variable exponents on Ω . We define $p^- := p_E^- := \inf_{y \in E} p(y)$ and $p^+ := p_E^+ := \sup_{y \in E} p(y)$. If $p^+ < \infty$ then we call p a bounded variable exponent.

For $p \in \mathcal{P}(E, \mu)$, we define $p' \in \mathcal{P}(E, \mu)$ by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, with the usual convention $\frac{1}{\infty} := 0$. The function $p' \in \mathcal{P}(E, \mu)$ is called the dual variable exponent of p .

Remark 3.2. The two most important cases that we consider in this Thesis are

- $(E, \Sigma, \mu) = (\Omega, \Sigma(\Omega), dx)$, where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\Sigma(\Omega)$ are the Lebesgue measurable subsets of Ω and dx is de Lebesgue measure in \mathbb{R}^N .
- $(E, \Sigma, \mu) = (\partial\Omega, \mathcal{B}(\partial\Omega), dS)$, where $\Omega \subset \mathbb{R}^N$ is an open set with C^2 boundary, $\mathcal{B}(\partial\Omega)$ are the Borel sets of $\partial\Omega$ and dS is the surface measure that agrees with the Hausdorff $(N - 1)$ -dimensional measure \mathcal{H}^{N-1} restricted to $\partial\Omega$.

Definition 3.3. Let $p \in \mathcal{P}(E, \mu)$ bounded and let $\rho_{p(x)}$ be de modular given by

$$\rho_{p(x)}(u) := \int_E |u|^{p(x)} d\mu.$$

So, the variable exponent Lebesgue space $L_\mu^{p(x)}(E)$ is defined by

$$L_\mu^{p(x)}(E) := \left\{ u \in \mathcal{M}_\mu(E) : \rho_{p(x)}(u) < \infty \right\},$$

where $\mathcal{M}_\mu(E) := \{u: E \rightarrow [-\infty, +\infty], \mu - \text{measurable}\}$.

This space is endowed with the norm, so-called *Luxemburg norm*,

$$\|u\|_{L_\mu^{p(x)}(E)} = \|u\|_{p(x),E} = \|u\|_{p(x),\mu} = \|u\|_{p(x)} := \inf \left\{ \lambda > 0: \rho_{p(x)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

The following proposition is proved in [36] and it will be most useful (see also [14], Chapter 2, Section 1).

Proposition 3.4. *Set $\rho(u) = \rho_{p(x)}(u)$. For $u \in L_\mu^{p(x)}(E)$ and $\{u_k\}_{k \in \mathbb{N}} \subset L_\mu^{p(x)}(E)$, we have*

$$u \neq 0 \Rightarrow \left(\|u\|_{p(x)} = \lambda \Leftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1 \right). \quad (3.1)$$

$$\|u\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1). \quad (3.2)$$

$$\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}. \quad (3.3)$$

$$\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}^{p^-}. \quad (3.4)$$

$$\lim_{k \rightarrow \infty} \|u_k\|_{p(x)} = 0 \Leftrightarrow \lim_{k \rightarrow \infty} \rho(u_k) = 0. \quad (3.5)$$

$$\lim_{k \rightarrow \infty} \|u_k\|_{p(x)} = \infty \Leftrightarrow \lim_{k \rightarrow \infty} \rho(u_k) = \infty. \quad (3.6)$$

When $(E, \Sigma, \mu) = (\Omega, \Sigma(\Omega), dx)$ as in Remark 3.2 we can define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega): \partial_i u \in L^{p(x)}(\Omega) \text{ for } i = 1, \dots, N\},$$

where $\partial_i u = \frac{\partial u}{\partial x_i}$ is the i^{th} -distributional partial derivative of u .

This space has a corresponding modular given by

$$\rho_{1,p(x)}(u) := \int_{\Omega} |u|^{p(x)} + |\nabla u|^{p(x)} dx$$

and so the corresponding norm for this space is

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{1,p(x)} := \inf \left\{ \lambda > 0: \rho_{1,p(x)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Analogously, the $W^{1,p(x)}(\Omega)$ norm can be defined as $\|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Both norms turn out to be equivalent but we use the first one for convenience.

In order to deal with the important case of zero boundary values, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the $W^{1,p(x)}(\Omega)$ norm.

From now on, we will focus on the case $(E, \Sigma, \mu) = (\Omega, \Sigma(\Omega), dx)$.

The spaces $L_\mu^{p(x)}(E)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces when $1 < p^- \leq p^+ < \infty$ if E is a locally compact metric space.

In order for the variable exponent Sobolev spaces to enjoy similar properties to the usual constant exponent counterparts, some regularity hypothesis is needed on the exponent $p(x)$. To this end, we state the following definition.

Definition 3.5. Let $p \in \mathcal{P}(\Omega)$. We say that $p(x)$ is log-Hölder continuous if there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{|\log|x - y||}, \quad \text{for } x, y \in \Omega, x \neq y.$$

Remark 3.6. Although this regularity assumption is not needed to define the Lebesgue or Sobolev spaces with variable exponent $p(x)$, it turns out to be very useful for these Sobolev spaces to enjoy all the usual properties like Sobolev embeddings, Poincaré inequality and so on. We will therefore assume it from now on for simplicity.

As usual, we denote the Sobolev conjugate exponent by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

The following result is proved in [36, 23] (see also [14], pp. 79, Lemma 3.2.20 (3.2.23)).

Proposition 3.7 (Hölder-type inequality). *Let $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$. Then the following inequality holds*

$$\|fg\|_{s(x)} \leq \left(\left(\frac{s}{p} \right)^+ + \left(\frac{s}{q} \right)^+ \right) \|f\|_{p(x)} \|g\|_{q(x)},$$

where

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

The Sobolev embedding Theorem is also proved in [23], Theorem 2.3.

Theorem 3.8 (Sobolev embedding). *Let $q \in \mathcal{P}(\Omega)$ be such that $q(x) \leq p^*(x) < \infty$ for all $x \in \Omega$. Then there is a continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Moreover, if $\inf_{\Omega}(p^* - q) > 0$ then, the embedding is compact.

As in the constant exponent spaces, Poincaré inequality holds true (see [14], pp. 249, Theorem 8.2.4)

Proposition 3.9 (Poincaré inequality). *Let $p \in \mathcal{P}(\Omega)$ be a log-Hölder exponent. Then there exists a constant $C > 0$, $C = C(\Omega, p(x))$, such that*

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Remark 3.10. From Poincaré inequality it follows immediately that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Remark 3.11. Is at this point where the log-Hölder continuity is used. Without any hypotheses on $p(x)$, Poincaré inequality fails. Let us mentioned the work of Harjulehto et al. [35] where Poincaré inequality is proved under weaker assumptions on $p(x)$, but a different definition of the space $W_0^{1,p(x)}(\Omega)$ is used.

The Sobolev trace Theorem is proved in [23]. When the exponent is critical, It requires more regularity on the exponent $p(x)$ (Lipschitz regularity is enough). This regularity can be relaxed when the exponent is strictly subcritical. It holds,

Theorem 3.12. *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary and let $p \in \mathcal{P}(\Omega)$ be Lipschitz such that $p \in W^{1,\gamma}(\Omega)$ with $1 \leq p_- \leq p^+ < N < \gamma$. Then there is a continuous boundary trace embedding $W^{1,p(x)}(\Omega) \subset L^{p_*(x)}(\partial\Omega)$, where*

$$p_*(x) = \frac{(N-1)p(x)}{N-p(x)}.$$

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Suppose that $p \in C^0(\bar{\Omega})$ and $1 < p^- \leq p^+ < N$. If $r \in \mathcal{P}(\partial\Omega)$ and there exists a positive constant ε such that*

$$r(x) + \varepsilon \leq p_*(x) \quad \text{for } x \in \partial\Omega$$

then the boundary trace embedding $W^{1,p(x)}(\Omega) \rightarrow L^{r(x)}(\partial\Omega)$ is compact.

Corollary 3.14. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Suppose that $p \in C^0(\bar{\Omega})$ and $1 < p_- \leq p_+ < N$. If $r \in C^0(\partial\Omega)$ satisfies the condition*

$$1 \leq r(x) < p_*(x) \quad x \in \partial\Omega$$

then there is a compact boundary trace embedding $W^{1,p(x)}(\Omega) \rightarrow L^{r(x)}(\partial\Omega)$

For much more on these spaces, we refer to [14].

3.2 Mountain pass theorem

In this section we review some well known results on the existence of critical points for functionals $\mathcal{F} : E \rightarrow \mathbb{R}$ where E is a (real) Banach space. The main tool here is the celebrated Mountain pass Theorem.

We begin with some basic definitions.

Definition 3.15. We say that $\mathcal{F} : E \rightarrow \mathbb{R}$ is (Fréchet) differentiable at $u_0 \in E$ if there exists $f \in E'$ such that

$$\mathcal{F}(u) = \mathcal{F}(u_0) + \langle f, u - u_0 \rangle + o(\|u - u_0\|), \quad \text{for every } u \in E,$$

where E' is the dual space to E and $\langle \cdot, \cdot \rangle$ is the duality product.

We denote $f = \mathcal{F}'(u)$

Definition 3.16. We say that $\mathcal{F} \in C^1 = C^1(E) = C^1(E, \mathbb{R})$ if \mathcal{F} is differentiable in E and $\mathcal{F}' : E \rightarrow E'$ is continuous in the strong topologies.

Definition 3.17. We say that $u \in E$ is a critical point of \mathcal{F} , if $\mathcal{F}'(u) = 0$.

Let us introduce the sets

$$\mathcal{F}^c := \{u \in E : \mathcal{F}(u) \leq c\} \quad \text{and} \quad K_c := \{u \in E : \mathcal{F}(u) = c \text{ and } \mathcal{F}'(u) = 0\}.$$

Definition 3.18. We say that $c \in \mathbb{R}$ is a critical value of \mathcal{F} if $K_c \neq \emptyset$

Definition 3.19. We say that $\{u_n\}_{n \in \mathbb{N}} \subset E$ is a Palais–Smale sequence of level c if

$$\mathcal{F}(u_n) \rightarrow c \quad \text{and} \quad \mathcal{F}'(u_n) \rightarrow 0 \text{ in } E'.$$

We say that \mathcal{F} satisfies the Palais–Smale condition of level c if every Palais–Smale sequence of level c contains a strongly convergent subsequence.

Now, we introduce the well known Mountain pass theorem. The proof of this Theorem can be found, for instance, in [21].

Theorem 3.20 (Mountain pass theorem). *Let $\mathcal{F} \in C^1$ be such that:*

- $\mathcal{F}(0) = 0$.
- There exist $r, a > 0$ such that if $u \in E$ with $\|u\| = r$ then $\mathcal{F}(u) \geq a$.
- There exists $v \in E$ with $\|v\| > r$ such that $\mathcal{F}(v) \leq 0$.

Then, if we denote

$$c := \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} \mathcal{F}(g(t)),$$

where $\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = v\}$ and \mathcal{F} verifies the Palais–Smale condition of level c , then c is a critical value of \mathcal{F} .

3.3 A topological tool: the genus

In this section we define a topological tool, called the *genus* that is commonly used in the literature in order to obtain the existence of multiple critical points for even functionals.

We begin with the definition and some basic properties. For an excellent reference on this subject and much more interesting results we refer to [47].

Definition 3.21. Let E be a real Banach space. We define \mathcal{K} to be the class of closed symmetric subsets of E , i.e.

$$\mathcal{K} = \{A \subset E - \{0\} : A \text{ is closed and } A = -A\}.$$

For any $A \in \mathcal{K}$ we define the *genus* of A , denoted by $\gamma(A)$, as the smallest $n \in \mathbb{N}$ such that there exists $\varphi \in C(A, \mathbb{R}^n - \{0\})$ odd.

If no such $n \in \mathbb{N}$ exists, then we define $\gamma(A) = \infty$. Finally, we define $\gamma(\emptyset) = 0$.

Now, we give some properties of γ .

Lemma 3.22. 1. *If there exists $f \in C(A, f(A))$ odd, then $\gamma(A) \leq \gamma(f(A))$*

2. *If $A \subset B$ then $\gamma(A) \leq \gamma(B)$*

3. *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$*

4. *If A is compact, then $\gamma(A) < \infty$ and there exist a $\delta > 0$ such that $\gamma(N_\delta(A)) = \gamma(A)$, where $N_\delta(A) = \bigcup_{x \in A} B_\delta(x)$.*

Now, we can introduce an important result. The following Theorem will give us a multiplicity result for even functionals. This theorem is proved in [33].

Theorem 3.23. *Let $\mathcal{F} \in C^1(E, \mathbb{R})$ be an even functional, bounded below. Assume that for $\varepsilon > 0$ small, there holds $\gamma(\mathcal{F}^{-\varepsilon}) \geq k$. Let us denote*

$$\mathcal{K}_k = \{A \subset \mathcal{K} : \gamma(A) \geq k\}.$$

Then

$$c_k = \inf_{A \in \mathcal{K}_k} \sup_{u \in A} \mathcal{F}(u)$$

is a negative critical value of \mathcal{F} and, moreover, if $c = c_k = \dots = c_{k+r}$, then $\gamma(K_c) \geq r + 1$.

Remark 3.24. Recall that if $\gamma(A) > 1$ then A has infinite pairs of distinct points. Therefore, Theorem 3.23 gives the existence of an infinite number of critical points of \mathcal{F} .

3.4 The variational principle of Ekeland

This section follows Appendix C in the Lecture Notes of Peral, [45]. The original result is due to Ekeland and can be found in [18].

The basic idea of Ekeland's variational principle is as follows: Suppose that \mathcal{F} is a real function defined in a metric space (X, d) which is lower semicontinuous and bounded below.

The principle ensures the construction of a minimizing sequence $\{x_\varepsilon\}_{\varepsilon > 0} \subset X$ for \mathcal{F} with some kind of control, more precisely, that verifies

$$\inf_{x \in X} \{\mathcal{F}(x)\} + \varepsilon > \mathcal{F}(x_\varepsilon)$$

and

$$\mathcal{F}(y) \geq \mathcal{F}(x_\varepsilon) - \varepsilon d(x_\varepsilon, y),$$

that is, the graph of \mathcal{F} stays above a *cone* centered at x_ε .

Theorem 3.25. *Let (X, d) be a metric space and we let $\mathcal{F} : X \rightarrow (-\infty, \infty]$ be a lower semi-continuous function such that $\mathcal{F}(x) \geq \beta$ for all $x \in X$. Let $\varepsilon > 0$ and $u \in X$ be such such that:*

$$\mathcal{F}(u) \leq \inf_X \mathcal{F} + \varepsilon.$$

Then there exist $v \in X$ such that:

- $\mathcal{F}(u) \geq \mathcal{F}(v)$,
- $d(u, v) \leq 1$ and
- $\mathcal{F}(w) \geq \mathcal{F}(v) - \varepsilon d(v, w)$ for every $w \in X$.

The following corollary is extremely useful in finding critical points for C^1 functionals.

Corollary 3.26. *Let E be a real Banach Space and $\mathcal{F} \in C^1(E, \mathbb{R})$ be bounded below. Then for every $\varepsilon > 0$ and for every $u \in E$ such that*

$$\mathcal{F}(u) \leq \inf_E \mathcal{F} + \varepsilon,$$

there exists $v \in E$ such that

- $\mathcal{F}(v) \leq \mathcal{F}(u)$,
- $\|u - v\| \leq \varepsilon^{\frac{1}{2}}$ and
- $\|\mathcal{F}'(v)\| \leq \varepsilon^{\frac{1}{2}}$.

To finish this section, we state two corollaries that generalize Corollary 3.26 to the case of differentiable manifolds in Banach spaces.

Corollary 3.27. *Let M be differential manifold in a Banach space E and $\mathcal{F} \in C^1(M, \mathbb{R})$ be bounded below. Then for every $\varepsilon > 0$ and for every $u \in M$ such that*

$$\mathcal{F}(u) \leq \inf_M \mathcal{F} + \varepsilon,$$

there exists $v \in M$ such that

- $\mathcal{F}(v) \leq \mathcal{F}(u)$,
- $\|u - v\| \leq \varepsilon^{\frac{1}{2}}$ and
- $\|\mathcal{F}'(v)\|_{T_v M} \leq \varepsilon^{\frac{1}{2}}$.

where $T_u M \subset E'$ is the tangent space to M at u .

Corollary 3.28. *Let M and \mathcal{F} be as in Corollary 3.27. Then given any minimizing sequence $\{u_k\}_{k \in \mathbb{N}} \subset M$ for Φ , there exists another minimizing sequence $\{v_k\}_{k \in \mathbb{N}} \subset M$ such that:*

- $\mathcal{F}(v_k) \leq \mathcal{F}(u_k)$,
- $\|u_k - v_k\| \rightarrow 0$ when $k \rightarrow \infty$ and
- $\|\mathcal{F}'(v_k)\|_{T_{v_k} M} \rightarrow 0$ when $k \rightarrow \infty$.

4

The concentration–compactness principle for variable exponent spaces

One of the main goals while working in a non compact setting is the need to understand the reason for a sequence to be weakly, but not strongly convergent and, ultimately, to fully describe the possible behaviors of such sequences.

In the case of Sobolev spaces with constant exponents this was achieved by P.L. Lions in the seminal paper [37] and is now called the *concentration–compactness principle*. Roughly speaking, it says that in a bounded domain, if a sequence is weakly convergent in $W_0^{1,p}(\Omega)$ but not strongly convergent in $L^{p^*}(\Omega)$, then this lack of compactness comes from the appearance of point masses where the sequence concentrate.

This principle has been proved to be a fundamental tool when dealing with nonlinear elliptic equations with critical growth (in the sense of the Sobolev embeddings). Just to cite a few, see [2, 3, 6, 17, 25, 33] but there is an impressive list of references on this.

The objective of this chapter is to extend the concentration–compactness principle of P.L. Lions to the variable exponent setting.

The method of the proof follows the lines of the ones in the original work of P.L. Lions and the main novelty in our result is the fact that we do not require the exponent $q(x)$ to be critical everywhere. Moreover, we show that the delta masses are concentrated in the set where $q(x)$ is critical.

4.1 The concentration–compactness principle for the Sobolev immersion

In this section we analyze the failure of compactness in the immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ when $\mathcal{A} = \{x \in \Omega: q(x) = p^*(x)\} \neq \emptyset$. More precisely, we prove,

Theorem 4.1. *Let $p, q \in \mathcal{P}(\Omega)$ be such that $q(x)$ is continuous and $p(x)$ is log-Hölder continu-*

ous. Assume that $q(x) \leq p^*(x)$ in Ω . Let $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a weakly convergent sequence with weak limit u , and such that:

$$|\nabla u_j|^{p(x)} \rightharpoonup \mu \quad \text{and} \quad |u_j|^{q(x)} \rightharpoonup \nu \quad \text{weakly-* in the sense of measures.}$$

Assume, moreover that $\mathcal{A} = \{x \in \Omega : q(x) = p^*(x)\}$ is nonempty. Then, for some finite index set I , we have:

$$\nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i} \quad \nu_i > 0 \quad (4.1)$$

$$\mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i} \quad \mu_i > 0 \quad (4.2)$$

$$\bar{S}_{x_i} \nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)} \quad \forall i \in I. \quad (4.3)$$

where $\{x_i\}_{i \in I} \subset \mathcal{A}$ and \bar{S}_x is the localized Sobolev constant defined by

$$\bar{S}_x = \sup_{\varepsilon > 0} S(p(\cdot), q(\cdot), B_\varepsilon(x)) = \lim_{\varepsilon \rightarrow 0^+} S(p(\cdot), q(\cdot), B_\varepsilon(x)). \quad (4.4)$$

Let $\{u_j\}_{j \in \mathbb{N}}$ be a bounded sequence in $W_0^{1,p(x)}(\Omega)$ and let $q \in C(\bar{\Omega})$ be such that $q \leq p^*$ with $\mathcal{A} = \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset$. Then there exists a subsequence that we still denote by $\{u_j\}_{j \in \mathbb{N}}$, such that

- $u_j \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$,
- $u_j \rightarrow u$ strongly in $L^{s(x)}(\Omega)$ for every $s \in \mathcal{P}(\Omega)$ such that $\inf_{x \in \Omega} (p^*(x) - s(x)) > 0$,
- $|u_j|^{q(x)} \rightharpoonup \nu$ weakly-* in the sense of measures,
- $|\nabla u_j|^{p(x)} \rightharpoonup \mu$ weakly-* in the sense of measures.

Take $\phi \in C^\infty(\bar{\Omega})$ and from Theorem 3.8, we obtain

$$S(p(\cdot), q(\cdot), \Omega) \|\phi u_j\|_{q(x)} \leq \|\nabla(\phi u_j)\|_{p(x)}. \quad (4.5)$$

Observe that if $\phi \in C_c^\infty(\Omega)$ then the constant $S(p(\cdot), q(\cdot), \Omega)$ can be replaced by $S(p(\cdot), q(\cdot), U)$ with $U \subset \Omega$ any open set containing $\text{supp}(\phi)$.

On the other hand,

$$|\|\nabla(\phi u_j)\|_{p(x)} - \|\phi \nabla u_j\|_{p(x)}| \leq \|u_j \nabla \phi\|_{p(x)}.$$

We first assume that $u = 0$. Then, we observe that the right side of the inequality converges to 0. In fact, we can assume that $\rho_{p(x)}(u) < 1$, then

$$\begin{aligned} \|u_j \nabla \phi\|_{p(x)} &\leq (\|\nabla \phi\|_\infty + 1)^{p^+} \|u_j\|_{p(x)} \\ &\leq (\|\nabla \phi\|_\infty + 1)^{p^+} \rho_{p(x)}(u_j)^{1/p^-} \rightarrow 0 \end{aligned}$$

Finally, if we take the limit for $j \rightarrow \infty$ in (4.5), we arrive at

$$S(p(\cdot), q(\cdot), \Omega) \|\phi\|_{q(x), \nu} \leq \|\phi\|_{p(x), \mu}. \quad (4.6)$$

The last inequality is a ‘‘Reverse Hölder inequality’’ for ϕ with measures μ and ν .

4.1.1 Preliminary Lemmas

Now we need a lemma that plays a key role in the proof of Theorem 4.1.

Lemma 4.2. *Let μ, ν be two non-negative and bounded measures on $\bar{\Omega}$, such that for $p, q \in \mathcal{P}(\Omega)$ bounded exponents with $p^+ < q^-$, there exists some constant $C > 0$ such that*

$$\|\phi\|_{q(x),\nu} \leq C \|\phi\|_{p(x),\mu}, \quad (4.7)$$

for every $\phi \in C_c^\infty(\Omega)$. Then, there exists a finite index set I , points $\{x_i\}_{i \in I} \subset \bar{\Omega}$ and scalars $\{\nu_i\}_{i \in I} \subset (0, \infty)$, such that

$$\nu = \sum_{i \in I} \nu_i \delta_{x_i},$$

where δ_{x_i} is the Dirac's delta measure supported on x_i .

For the proof of Lemma 4.2 we need a couple of preliminary results. The next Lemma is due to P.L. Lions in [37], but we include it here for the sake of completeness.

Lemma 4.3. *Let ν be a non-negative bounded Borel measure on $\bar{\Omega}$. Assume that there exists $\delta > 0$ such that for every Borel set A we have that, $\nu(A) = 0$ or $\nu(A) \geq \delta$. Then, there exist a finite index set I , points $\{x_i\}_{i \in I} \subset \bar{\Omega}$ and scalars $\{\nu_i\}_{i \in I} \in (0, \infty)$ such that*

$$\nu = \sum_{i \in I} \nu_i \delta_{x_i}.$$

Proof. Let A such that $\nu(A) \geq \delta$ we want to prove that there exist x_0 such that $\nu(\{x_0\}) \geq \delta$. Let $\{Q\}$ be a covering of A by cubes. Then, there exists a cube $Q_1 \in \{Q\}$ such that $\nu(Q_1) \geq \delta$.

Making a dyadic decomposition of the cube Q_1 , we obtain a decreasing sequence of cubes $\{Q_k\}_{k \in \mathbb{N}}$ such that $\nu(Q_k) \geq \delta$ for every $k \in \mathbb{N}$.

Let $\{x_0\} = \bigcap_{k \in \mathbb{N}} Q_k$, then $\nu(\{x_0\}) = \lim_{k \rightarrow \infty} \nu(Q_k) \geq \delta$.

Let $\{x_i\}_{i \in I} \subset \bar{\Omega}$ be the set of points such that $\nu(x_i) \geq \delta$. Since ν is bounded, it is easy to see that I is a finite set.

The preceding argument easily shows that the measure is supported on $\{x_i\}_{i \in I}$, and if we denote by $\nu_i := \nu(\{x_i\})$, we obtain the desired result. \square

Lemma 4.4. *Let ν be a non-negative, bounded Borel measure such that for some $p, q \in \mathcal{P}(\Omega)$ bounded exponents with $p^+ < q^-$ it holds that*

$$\|\phi\|_{q(x),\nu} \leq C \|\phi\|_{p(x),\nu}, \quad (4.8)$$

for every $\phi \in C_c^\infty(\Omega)$. Then there exists $\delta > 0$ such that for all Borel set $A \subset \bar{\Omega}$, $\nu(A) = 0$ or $\nu(A) \geq \delta$.

Proof. We can assume that $\nu(A) < 1$, then

$$\int_{\Omega} \left(\frac{\chi_A(x)}{\nu(A)^{\frac{1}{p^+}}} \right)^{p(x)} d\nu \leq \int_{\Omega} \left(\frac{\chi_A(x)}{\nu(A)^{\frac{1}{p(x)}}} \right)^{p(x)} d\nu = 1,$$

therefore $\|\chi_A\|_{p(x),\nu} \leq \nu(A)^{\frac{1}{p^+}}$. On the other hand,

$$\int_{\Omega} \left(\frac{\chi_A(x)}{\nu(A)^{\frac{1}{q^-}}} \right)^{q(x)} d\nu \geq \int_{\Omega} \frac{\chi_A(x)}{\nu(A)} d\nu = 1,$$

so $\nu(A)^{\frac{1}{q^-}} \leq \|\chi_A\|_{q(x),\nu}$. Now, from (4.8) we conclude that

$$\nu(A)^{\frac{1}{q^-}} \leq C\nu(A)^{\frac{1}{p^+}},$$

from where it follows that, since $p^+ < q^-$, $\nu(A) = 0$ or

$$\nu(A) \geq \left(\frac{1}{C} \right)^{\frac{p^+ q^-}{q^- - p^+}} > 0.$$

This completes the proof. □

Now we are ready to prove Lemma 4.2

Proof of Lemma 4.2. By the reverse Hölder inequality (4.7), the measure ν is absolutely continuous with respect to μ . As consequence there exists $f \in L^1_{\mu}(\Omega)$, $f \geq 0$, such that $\nu = \mu \llcorner f$. Also by (4.7) we have,

$$\min \left\{ \nu(A)^{\frac{1}{q^-}}, \nu(A)^{\frac{1}{q^+}} \right\} \leq C \max \left\{ \mu(A)^{\frac{1}{p^-}}, \mu(A)^{\frac{1}{p^+}} \right\}$$

for any Borel set $A \subset \Omega$. In particular, $f \in L^{\infty}_{\mu}(\Omega)$. On the other hand the Lebesgue decomposition of μ with respect to ν gives us

$$\mu = \nu \llcorner g + \sigma, \text{ where } g \in L^1_{\nu}(\Omega), g \geq 0$$

and σ is a bounded positive measure, singular with respect to ν .

Let $\psi \in C_c^{\infty}(\Omega)$, now consider (4.7) applied to the test function

$$\phi = g^{\frac{1}{q(x)-p(x)}} \chi_{\{g \leq n\}} \psi.$$

We obtain

$$\begin{aligned} \|g^{\frac{1}{q(x)-p(x)}} \chi_{\{g \leq n\}} \psi\|_{q(x),\nu} &\leq C \|g^{\frac{1}{q(x)-p(x)}} \chi_{\{g \leq n\}} \psi\|_{p(x),\mu} \\ &= \|g^{\frac{1}{q(x)-p(x)}} \chi_{\{g \leq n\}} \psi\|_{p(x),g d\nu + d\sigma} \\ &\leq \|g^{\frac{q(x)}{p(x)(q(x)-p(x))}} \chi_{\{g \leq n\}} \psi\|_{p(x),\nu} + \|g^{\frac{1}{q(x)-p(x)}} \chi_{\{g \leq n\}} \psi\|_{p(x),\sigma}. \end{aligned}$$

Since $\sigma \perp \nu$, we have

$$\|g^{\frac{1}{q(x)-p(x)}} \chi_{\{g \leq n\}} \psi\|_{q(x), \nu} \leq C \|g^{\frac{q(x)}{p(x)(q(x)-p(x))}} \chi_{\{g \leq n\}} \psi\|_{p(x), \nu},$$

hence if we denote $d\nu_n = g^{\frac{q(x)}{(q(x)-p(x))}} \chi_{g \leq n} d\nu$ the following reverse Hölder inequality holds

$$\|\psi\|_{q(x), \nu_n} \leq \|\psi\|_{p(x), \nu_n}.$$

Now, by Lemma 4.3 and Lemma 4.4, there exists $\{x_i^n\}_{i \in I^n}$ and $K_i^n > 0$ such that $\nu_n = \sum_{i \in I^n} K_i^n \delta_{x_i^n}$. On the other hand, $\nu_n \nearrow g^{\frac{r(x)}{r(x)-p(x)}} \nu$. Then, we have

$$g^{\frac{r(x)}{r(x)-p(x)}} \nu = \sum_{i \in I} K_i \delta_{x_i},$$

where $K_i = g^{\frac{r(x_i)}{r(x_i)-p(x_i)}} (x_i) \nu(\{x_i\})$. This finishes the proof. \square

The following Lemma is the extension to variable exponents of the well-known Brezis-Lieb Lemma (see [8]). The proof is analogous to that of [8].

Lemma 4.5. *Let $f_n \rightarrow f$ a.e and $f_n \rightharpoonup f$ in $L^{p(x)}(\Omega)$ then*

$$\lim_{n \rightarrow \infty} \left(\int_{\Omega} |f_n|^{p(x)} dx - \int_{\Omega} |f - f_n|^{p(x)} dx \right) = \int_{\Omega} |f|^{p(x)} dx.$$

Proof. First, it is easy to see that given $\varepsilon > 0$, there exists $C_\varepsilon = C_\varepsilon(p^-, p^+) > 0$ such that for every $a, b \in \mathbb{R}$,

$$\|a + b\|^{p(x)} - \|a\|^{p(x)} \leq \varepsilon \|a\|^{p(x)} + C_\varepsilon \|b\|^{p(x)}.$$

We define

$$W_{\varepsilon, n}(x) = (|f_n(x)|^{p(x)} - |f(x) - f_n(x)|^{p(x)} - |f(x)|^{p(x)} - \varepsilon |f_n(x)|^{p(x)})_+$$

and note that $W_{\varepsilon, n}(x) \rightarrow 0$ as $n \rightarrow \infty$ a.e. On the other hand,

$$\begin{aligned} |f_n(x)|^{p(x)} - |f(x) - f_n(x)|^{p(x)} - |f(x)|^{p(x)} &\leq |f_n(x)|^{p(x)} - |f(x) - f_n(x)|^{p(x)} + |f(x)|^{p(x)} \\ &\leq \varepsilon |f_n(x)|^{p(x)} + C_\varepsilon |f(x)|^{p(x)} + |f(x)|^{p(x)} \end{aligned}$$

i.e.

$$|f_n(x)|^{p(x)} - |f(x) - f_n(x)|^{p(x)} - |f(x)|^{p(x)} - \varepsilon |f_n(x)|^{p(x)} \leq (C_\varepsilon + 1) |f(x)|^{p(x)},$$

therefore

$$0 \leq W_{\varepsilon, n}(x) \leq (C_\varepsilon + 1) |f(x)|^{p(x)}$$

By the dominated convergence Theorem, we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega} W_{\varepsilon, n}(x) dx = 0.$$

On the other hand,

$$\| |f_n(x)|^{p(x)} - |f(x) - f_n(x)|^{p(x)} - |f(x)|^{p(x)} \| \leq W_{\varepsilon,n}(x) + \varepsilon |f_n(x)|^{p(x)}$$

Then, if we denote

$$I_n = \int_{\Omega} |f_n(x)|^{p(x)} - |f(x) - f_n(x)|^{p(x)} - |f(x)|^{p(x)} dx,$$

we get

$$I_n \leq \int_{\Omega} W_{\varepsilon,n}(x) dx + \varepsilon \rho_{p(x)}(f_n) \leq \int_{\Omega} W_{\varepsilon,n}(x) dx + \varepsilon \sup_{n \in \mathbb{N}} \rho_{p(x)}(f_n) = \int_{\Omega} W_{\varepsilon,n}(x) dx + \varepsilon C,$$

for some constant $C > 0$. Hence, we can conclude that $\limsup I_n \leq \varepsilon C$, for every $\varepsilon > 0$. \square

4.1.2 Proof of the Concentration Compactness Principle

Now we are in position to prove Theorem 4.1.

Proof of Theorem 4.1. Given any $\phi \in C^\infty(\Omega)$ we write $v_j = u_j - u$ and by lemma 4.5, we have

$$\lim_{j \rightarrow \infty} \left(\int_{\Omega} |\phi|^{q(x)} |u_j|^{q(x)} - \int_{\Omega} |\phi|^{q(x)} |v_j|^{q(x)} dx \right) = \int_{\Omega} |\phi|^{q(x)} |u|^{q(x)} dx.$$

On the other hand, by the reverse Hölder inequality (4.6) and Lemma 4.2, taking limits we obtain the representation

$$v = |u|^{q(x)} dx + \sum_{i \in I} v_i \delta_{x_i}. \quad (4.9)$$

Let us now show that the points x_j actually belong to the *critical set* \mathcal{A} .

In fact, assume by contradiction that $x_1 \in \Omega \setminus \mathcal{A}$. Let $B = B(x_1, r) \subset \subset \Omega - \mathcal{A}$. Then $q(x) < p^*(x) - \delta$ for some $\delta > 0$ in \bar{B} and, by Proposition 3.8, The embedding $W^{1,p(x)}(B) \hookrightarrow L^{q(x)}(B)$ is compact. Therefore, $u_j \rightarrow u$ strongly in $L^{q(x)}(B)$ and so $|u_j|^{q(x)} \rightarrow |u|^{q(x)}$ strongly in $L^1(B)$. This is a contradiction to our assumption that $x_1 \in B$.

Now we proceed with the proof.

Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \phi \leq 1$, $\phi(0) = 1$ and $\text{supp}(\phi) \subset B_1(0)$. Now, for each $i \in I$ and $\varepsilon > 0$, we denote $\phi_{\varepsilon,i}(x) := \phi((x - x_i)/\varepsilon)$.

Since $\text{supp}(\phi_{\varepsilon,i} u_n) \subset B_\varepsilon(x_i)$, by (4.6) and the subsequent remark, we obtain

$$S(p(\cdot), q(\cdot), B_\varepsilon(x_i)) \|\phi_{\varepsilon,i}\|_{L_v^{q(x)}(B_\varepsilon(x_i))} \leq \|\phi_{\varepsilon,i}\|_{L_\mu^{p(x)}(B_\varepsilon(x_i))}.$$

By (4.9), we have

$$\begin{aligned} \rho_{q(x), \nu}(\phi_{i_0, \varepsilon}) &:= \int_{B_\varepsilon(x_{i_0})} |\phi_{i_0, \varepsilon}|^{q(x)} d\nu \\ &= \int_{B_\varepsilon(x_{i_0})} |\phi_{i_0, \varepsilon}|^{q(x)} |\mu|^{q(x)} dx + \sum_{i \in I} \nu_i \phi_{i_0, \varepsilon}(x_i)^{q(x_i)} \\ &\geq \nu_{i_0}. \end{aligned}$$

From now on, we will denote

$$\begin{aligned} q_{i, \varepsilon}^+ &:= \sup_{B_\varepsilon(x_i)} q(x), & q_{i, \varepsilon}^- &:= \inf_{B_\varepsilon(x_i)} q(x), \\ p_{i, \varepsilon}^+ &:= \sup_{B_\varepsilon(x_i)} p(x), & p_{i, \varepsilon}^- &:= \inf_{B_\varepsilon(x_i)} p(x). \end{aligned}$$

If $\rho_\nu(\phi_{i_0, \varepsilon}) < 1$ then

$$\|\phi_{i_0, \varepsilon}\|_{L_\nu^{q(x)}(B_\varepsilon(x_{i_0}))} \geq \rho_\nu(\phi_{i_0, \varepsilon})^{1/q_{i, \varepsilon}^-} \geq \nu_{i_0}^{1/q_{i, \varepsilon}^-}.$$

Analogously, if $\rho_\nu(\phi_{i_0, \varepsilon}) \geq 1$ then

$$\|\phi_{i_0, \varepsilon}\|_{L_\nu^{q(x)}(B_\varepsilon(x_{i_0}))} \geq \nu_{i_0}^{1/q_{i, \varepsilon}^+}.$$

Therefore,

$$\min \left\{ \nu_i^{\frac{1}{q_{i, \varepsilon}^+}}, \nu_i^{\frac{1}{q_{i, \varepsilon}^-}} \right\} S(p(\cdot), q(\cdot), B_\varepsilon(x_i)) \leq \|\phi_{i, \varepsilon}\|_{L_\mu^{p(x)}(B_\varepsilon(x_i))}.$$

On the other hand,

$$\int_{B_\varepsilon(x_i)} |\phi_{i, \varepsilon}|^{p(x)} d\mu \leq \mu(B_\varepsilon(x_i)),$$

hence

$$\begin{aligned} \|\phi_{i, \varepsilon}\|_{L^{p(x)}(B_\varepsilon(x_i))} &\leq \max \left\{ \rho_\mu(\phi_{i, \varepsilon})^{\frac{1}{p_{i, \varepsilon}^+}}, \rho_\mu(\phi_{i, \varepsilon})^{\frac{1}{p_{i, \varepsilon}^-}} \right\} \\ &\leq \max \left\{ \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i, \varepsilon}^+}}, \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i, \varepsilon}^-}} \right\}, \end{aligned}$$

so we obtain,

$$S(p(\cdot), q(\cdot), B_\varepsilon(x_i)) \min \left\{ \nu_i^{\frac{1}{q_{i, \varepsilon}^+}}, \nu_i^{\frac{1}{q_{i, \varepsilon}^-}} \right\} \leq \max \left\{ \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i, \varepsilon}^+}}, \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i, \varepsilon}^-}} \right\}.$$

As p and q are continuous functions and as $q(x_i) = p^*(x_i)$, letting $\varepsilon \rightarrow 0$, we get

$$\left(\lim_{\varepsilon \rightarrow 0} S(p(\cdot), q(\cdot), B_\varepsilon(x_i)) \right) \nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)},$$

where $\mu_i := \lim_{\varepsilon \rightarrow 0} \mu(B_\varepsilon(x_i))$.

Finally, we show that $\mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i}$.

In fact, we have that $\mu \geq \tilde{\mu} := \sum_{i \in I} \mu_i \delta_{x_i}$. On the other hand, since $u_j \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$ then $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^{p(x)}(U)$ for all $U \subset \Omega$. By the weakly lower semicontinuity of norm we obtain that $d\mu \geq |\nabla u|^{p(x)} dx$ and, as $|\nabla u|^{p(x)} dx$ is orthogonal to $\tilde{\mu}$, we conclude the desired result.

This finishes the proof. \square

4.2 The concentration–compactness principle for the Sobolev trace immersion

This section is devoted to the extension of the CCP to the trace immersion.

In order to state the Theorem correctly, let us first introduce some notation. For $\Omega \subset \mathbb{R}^N$, let $\Gamma \subset \partial\Omega$ be closed, $\Gamma \neq \partial\Omega$ (possibly empty) and define

$$W_\Gamma^{1,p(x)}(\Omega) := \overline{\{\phi \in C^\infty(\bar{\Omega}) : \phi \text{ vanishes in a neighbourhood of } \Gamma\}},$$

where the closure is taken in the $\|\cdot\|_{1,p(x)}$ -norm. This is the set of functions in $W^{1,p(x)}(\Omega)$ that has zero boundary values on Γ . Obviously, $W_\emptyset^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega)$. In general $W_\Gamma^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega)$ if and only if the $p(x)$ -capacity of Γ is 0, see [35].

Let $r \in \mathcal{P}(\partial\Omega)$ be a continuous critical exponent in the sense that

$$\mathcal{A}_T := \{x \in \partial\Omega : r(x) = p_*(x)\} \neq \emptyset.$$

We define the Sobolev trace constant in $W_\Gamma^{1,p(x)}(\Omega)$ as

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) := \inf_{v \in W_\Gamma^{1,p(x)}(\Omega)} \frac{\|v\|_{1,p(x)}}{\|v\|_{r(x),\partial\Omega}} = \inf_{v \in W_\Gamma^{1,p(x)}(\Omega)} \frac{\|v\|_{1,p(x)}}{\|v\|_{r(x),\partial\Omega \setminus \Gamma}}$$

More precisely, we prove

Theorem 4.6. *Let $\{u_n\}_{n \in \mathbb{N}} \subset W^{1,p(x)}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. Then there exists a finite set I , positive numbers $\{\mu_i\}_{i \in I}$ and $\{v_i\}_{i \in I}$ and points $\{x_i\}_{i \in I} \subset \mathcal{A}_T \subset \partial\Omega$ such that*

$$|u_n|^{r(x)} dS \rightharpoonup \nu = |u|^{r(x)} dS + \sum_{i \in I} v_i \delta_{x_i} \quad \text{weakly-* in the sense of measures,} \quad (4.10)$$

$$|\nabla u_n|^{p(x)} dx \rightharpoonup \mu \geq |\nabla u|^{p(x)} dx + \sum_{i \in I} \mu_i \delta_{x_i} \quad \text{weakly-* in the sense of measures,} \quad (4.11)$$

$$\bar{T}_{x_i} v_i^{\frac{1}{r(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}, \quad (4.12)$$

where $\bar{T}_{x_i} = \sup_{\varepsilon > 0} T(p(\cdot), q(\cdot), \Omega_{\varepsilon,i}, \Gamma_{\varepsilon,i})$ is the localized Sobolev trace constant where

$$\Omega_{\varepsilon,i} = \Omega \cap B_\varepsilon(x_i) \quad \text{and} \quad \Gamma_{\varepsilon,i} := \partial B_\varepsilon(x_i) \cap \Omega.$$

Proof. The proof is very similar to the one for the Sobolev immersion Theorem, Theorem 4.1, so we only make a sketch stressing the differences between the two cases.

As in Theorem 4.1 it is enough to consider the case where $u_n \rightharpoonup 0$ weakly in $W^{1,p(x)}(\Omega)$.

Arguing as in (4.6), we obtain

$$T(p(\cdot), r(\cdot), \Omega) \|\phi\|_{L_v^{r(x)}(\partial\Omega)} \leq \|\phi\|_{L_\mu^{p(x)}(\Omega)}, \quad (4.13)$$

for every $\phi \in C^\infty(\bar{\Omega})$. Observe that if $\phi \in C_c^\infty(\mathbb{R}^N)$ and $U \subset \mathbb{R}^N$ is any open set that contains the support of ϕ , the constant in (4.13) can be replaced by $T(p(\cdot), q(\cdot), \Omega \cap U, \partial U \cap \Omega)$.

Now, the exact same proof of Theorem 4.1 can be applied to obtain that (4.10) and (4.11) hold. Again, exactly as in Theorem 4.1 it follows that the points $\{x_i\}_{i \in I}$ belong to the *critical set* \mathcal{A}_T .

It remains to see (4.12). Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \phi \leq 1$, $\phi(0) = 1$ and $\text{supp}(\phi) \subset B_1(0)$. Now, for each $i \in I$ and $\varepsilon > 0$, we denote $\phi_{\varepsilon,i}(x) := \phi((x - x_i)/\varepsilon)$.

From (4.13) and the subsequent remark we obtain

$$T(p(\cdot), r(\cdot), \Omega_{\varepsilon,i}, \Gamma_{\varepsilon,i}) \|\phi_{\varepsilon,i}\|_{L_v^{r(x)}(\partial\Omega \cap B_\varepsilon(x_i))} \leq \|\phi_{\varepsilon,i}\|_{L_\mu^{p(x)}(\Omega \cap B_\varepsilon(x_i))}.$$

By (4.10), we have

$$\begin{aligned} \rho_v(\phi_{i_0,\varepsilon}) &:= \int_{\partial\Omega \cap B_\varepsilon(x_{i_0})} |\phi_{i_0,\varepsilon}|^{r(x)} dv \\ &= \int_{\partial\Omega \cap B_\varepsilon(x_{i_0})} |\phi_{i_0,\varepsilon}|^{r(x)} |u|^{r(x)} dS + \sum_{i \in I} v_i \phi_{i_0,\varepsilon}(x_i)^{r(x_i)} \\ &\geq v_{i_0}. \end{aligned}$$

From now on, we will denote

$$\begin{aligned} r_{i,\varepsilon}^+ &:= \sup_{\partial\Omega \cap B_\varepsilon(x_i)} r(x), & r_{i,\varepsilon}^- &:= \inf_{\partial\Omega \cap B_\varepsilon(x_i)} r(x), \\ p_{i,\varepsilon}^+ &:= \sup_{\Omega \cap B_\varepsilon(x_i)} p(x), & p_{i,\varepsilon}^- &:= \inf_{\Omega \cap B_\varepsilon(x_i)} p(x). \end{aligned}$$

If $\rho_v(\phi_{i_0,\varepsilon}) < 1$ then

$$\|\phi_{i_0,\varepsilon}\|_{L_v^{r(x)}(\partial\Omega \cap B_\varepsilon(x_{i_0}))} \geq \rho_v(\phi_{i_0,\varepsilon})^{1/r_{i_0,\varepsilon}^-} \geq v_{i_0}^{1/r_{i_0,\varepsilon}^-}.$$

Analogously, if $\rho_v(\phi_{i_0,\varepsilon}) > 1$ then

$$\|\phi_{i_0,\varepsilon}\|_{L_v^{r(x)}(\partial\Omega \cap B_\varepsilon(x_{i_0}))} \geq v_{i_0}^{1/r_{i_0,\varepsilon}^+}.$$

Therefore,

$$T(p(\cdot), r(\cdot), \Omega_{\varepsilon,i}, \Gamma_{\varepsilon,i}) \min \left\{ \nu_i^{\frac{1}{r_{i,\varepsilon}^+}}, \nu_i^{\frac{1}{r_{i,\varepsilon}^-}} \right\} \leq \|\phi_{i,\varepsilon}\|_{L_\mu^{p(x)}(\Omega \cap B_\varepsilon(x_i))}.$$

On the other hand,

$$\int_{\Omega \cap B_\varepsilon(x_i)} |\phi_{i,\varepsilon}|^{p(x)} d\mu \leq \mu(\Omega \cap B_\varepsilon(x_i))$$

hence

$$\begin{aligned} \|\phi_{i,\varepsilon}\|_{L^{p(x)}(\Omega \cap B_\varepsilon(x_i))} &\leq \max \left\{ \rho_\mu(\phi_{i,\varepsilon})^{\frac{1}{p_{i,\varepsilon}^+}}, \rho_\mu(\phi_{i,\varepsilon})^{\frac{1}{p_{i,\varepsilon}^-}} \right\} \\ &\leq \max \left\{ \mu(\Omega \cap B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^+}}, \mu(\Omega \cap B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^-}} \right\}, \end{aligned}$$

so we obtain,

$$T(p(\cdot), r(\cdot), \Omega_{\varepsilon,i}, \Gamma_{\varepsilon,i}) \min \left\{ \nu_i^{\frac{1}{r_{i,\varepsilon}^+}}, \nu_i^{\frac{1}{r_{i,\varepsilon}^-}} \right\} \leq \max \left\{ \mu(\Omega \cap B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^+}}, \mu(\Omega \cap B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^-}} \right\}.$$

As p and r are continuous functions and as $r(x_i) = p_*(x_i)$, letting $\varepsilon \rightarrow 0$, we get

$$\bar{T}_{x_i} \nu_i^{1/p_*(x_i)} \leq \mu_i^{1/p(x_i)},$$

where $\mu_i := \lim_{\varepsilon \rightarrow 0} \mu(\Omega \cap B_\varepsilon(x_i))$.

The proof is now complete. □

5

Existence of extremals for Sobolev Embeddings

In this chapter we study the existence problem for extremals of the Sobolev immersion Theorem for variable exponents $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and for the Sobolev trace Theorem $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$. As we discuss in the introduction, the only nontrivial case is when the exponents q and r are critical, i.e.

$$\mathcal{A} = \{x \in \Omega: q(x) = p^*(x)\} \neq \emptyset \quad \text{and} \quad \mathcal{A}_T = \{x \in \partial\Omega: r(x) = p_*(x)\} \neq \emptyset.$$

Recall that by extremals we mean functions where the Sobolev constant are attained, i.e. the existence of $u \in W_0^{1,p(x)}(\Omega)$ and $v \in W^{1,p(x)}(\Omega)$ such that

$$S(p(\cdot), q(\cdot), \Omega) = \frac{\|\nabla u\|_{p(x)}}{\|u\|_{q(x)}} \quad \text{and} \quad T(p(\cdot), q(\cdot), \Omega) = \frac{\|v\|_{1,p(x)}}{\|v\|_{r(x),\partial\Omega}}. \quad (5.1)$$

Also recall that the critical exponents are defined as

$$p^*(x) = \frac{Np(x)}{N-p(x)} \quad \text{and} \quad p_*(x) = \frac{(N-1)p(x)}{N-p(x)}.$$

5.1 Compact case

We begin this chapter by finding conditions that implies that though the exponents q and r are critical, the immersions $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ remains compact. Therefore, in these cases, the existence of extremals for (5.1) follows directly by minimization.

Roughly speaking, these conditions require that the critical set is *small* and that one has a strict control on how the exponent q and r reaches the critical one when one is approaching the critical set. For the Sobolev immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, this result was obtained in [42]. Following the same ideas we can prove a similar result for the trace immersion.

First we need to introduce some notations. For any compact set $K \subset \mathbb{R}^N$, we denote by $\delta_K(x)$ the distance of x to K and we define $K(r) := \{x \in \mathbb{R}^N: \delta_K(x) \leq r\}$ for $r > 0$.

For a compact set K in \mathbb{R}^N and $s \in [0, N]$, we say that $(N - s)$ -dimensional upper Minkowski content of K is finite if there exists a constant $C > 0$ such that

$$|K(r)| \leq Cr^s, \quad \text{for every } r > 0.$$

More precisely, the result obtained in [42] is the following:

Theorem 5.1 ([42], Theorem 3.4). *Let $\varphi: [r_0^{-1}, \infty) \rightarrow (0, \infty)$ be a continuous function such that: $\varphi(r)/\ln r$ is nonincreasing in $[r_0^{-1}, \infty)$ for some $r_0 \in (0, e^{-1})$ and $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$. Let K be a compact set in \mathbb{R}^N whose $(N - s)$ -dimensional upper Minkowski content is finite for some s with $0 < s \leq N$.*

Let $p, q \in \mathcal{P}(\Omega)$ be such that $p^+ < N$ and $q(x) \leq p^(x)$. Assume that $q(x)$ is subcritical outside a neighborhood of K , i.e. $\inf_{\Omega \setminus K(r_0)}(p^*(x) - q(x)) > 0$. Moreover, assume that $q(x)$ reaches $p^*(x)$ in K at the following rate*

$$q(x) \leq p^*(x) - \frac{\varphi(\frac{1}{\delta_K(x)})}{\ln(\frac{1}{\delta_K(x)})} \quad \text{for almost every } x \in K(r_0) \cap \Omega.$$

Then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

When the critical set consists of a single point, we immediately obtain the following corollary. We will use the notation $\ln^n(t) = \ln(\ln^{n-1}(t))$.

Corollary 5.2 ([42], Corollary 3.5). *Let $p, q \in \mathcal{P}(\Omega)$ be such that $p^+ < N$ and $q(x) \leq p^*(x)$. Suppose that there exist $x_0 \in \Omega$, $C > 0$, $n \in \mathbb{N}$, $r_0 > 0$ such that $\inf_{\Omega \setminus B_{r_0}(x_0)}(p^*(x) - q(x)) > 0$ and $q(x) \leq p^*(x) - c \frac{\ln^n(\frac{1}{|x-x_0|})}{\ln(\frac{1}{|x-x_0|})}$ for almost every $x \in B_{r_0}(x_0)$. Then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Following the same ideas of [42] we can obtain an analogous result for the trace immersion.

First, we define the upper Minkowsky content for sets contained in $\partial\Omega$. We say that a compact set $K \subset \partial\Omega$ has finite $(N - 1 - s)$ -boundary dimensional upper Minkowsky content if there exists a constant $C > 0$ such that

$$\mathcal{H}^{N-1}(K(r) \cap \partial\Omega) \leq Cr^s, \quad \text{for all } r > 0.$$

Theorem 5.3. *Let $\varphi: [r_0^{-1}, \infty) \rightarrow (0, \infty)$ be a continuous function such that: $\varphi(r)/\ln r$ is non-increasing in $[r_0^{-1}, \infty)$ for some $r_0 \in (0, e^{-1})$ and $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$. Let $\partial\Omega$ lipschitz and $K \subset \partial\Omega$ be a compact set whose $(N - 1 - s)$ -boundary dimensional upper Minkowski content is finite for some s with $0 < s \leq N - 1$.*

Let $p \in \mathcal{P}(\Omega)$ and $q \in \mathcal{P}(\partial\Omega)$ be such that $p^+ < N$ and $r(x) \leq p_(x)$. Assume that $r(x)$ is subcritical outside a neighborhood of K , i.e. $\inf_{\partial\Omega \setminus K(r_0)}(p_*(x) - r(x)) > 0$. Moreover, assume that $r(x)$ reaches $p_*(x)$ in K at the following rate*

$$r(x) \leq p_*(x) - \frac{\varphi(\frac{1}{\delta_K(x)})}{\ln(\frac{1}{\delta_K(x)})} \quad \text{for Hausdorff } (n-1)\text{-almost every } x \in K(r_0) \cap \partial\Omega.$$

Then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ is compact.

Proof. Let us prove that

$$\lim_{\varepsilon \rightarrow 0^+} \sup \left\{ \int_{K(\varepsilon) \cap \partial\Omega} |v(x)|^{r(x)} dS : v \in W^{1,p(x)}(\Omega) \text{ and } \|v\|_{W^{1,p(x)}(\Omega)} \leq 1 \right\} = 0. \quad (5.2)$$

First, we take β such that $0 < \beta < s/p_*^+$ and $\varepsilon > 0$ such that $\varepsilon^{-1} > r_0^{-1}$ and $\varphi(\frac{1}{\varepsilon}) \geq 1$. For each $n \in \mathbb{N}$ we consider $\eta_n = \varepsilon^{-\beta n}$. We choose $x \in (K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \partial\Omega$, then, we have

$$\eta_n^{r(x)-p_*(x)} \leq \eta_n^{-\frac{\varphi(\frac{1}{\delta K(x)})}{\ln(\frac{1}{\delta K(x)})}} \leq \eta_n^{-\frac{\varphi(\frac{1}{\varepsilon^{n+1}})}{\ln(\frac{1}{\varepsilon^{n+1}})}} = \varepsilon^{-\frac{\beta n}{n+1} \varphi(\frac{1}{\varepsilon^{n+1}})} = A_n$$

On the other hand, we know that $\mathcal{H}(K(r) \cap \partial\Omega) \leq Cr^s$ and we can estimate the following term

$$\int_{(K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \partial\Omega} \eta_n^{r(x)} dS \leq \eta_n^{p_*^+} \int_{K(\varepsilon^n) \cap \partial\Omega} dS \leq C \varepsilon^{n(s-\beta p_*^+)}$$

Now, we have

$$\begin{aligned} & \int_{(K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \partial\Omega} |v(x)|^{r(x)} dS \\ & \leq \int_{(K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \partial\Omega} |v(x)|^{r(x)} \left(\frac{|v(x)|}{\eta_n} \right)^{p_*(x)-r(x)} dS + \int_{(K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \partial\Omega} \eta_n^{r(x)} dS \\ & \leq A_n \int_{(K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \partial\Omega} |v(x)|^{p_*(x)} dS + C \varepsilon^{n(s-\beta p_*^+)} \end{aligned}$$

for each $n_0 \in \mathbb{N}$, we obtain

$$\begin{aligned} \int_{K(\varepsilon^{n_0}) \cap \partial\Omega} |v(x)|^{r(x)} dS &= \sum_{n=n_0}^{\infty} \int_{(K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \partial\Omega} |v(x)|^{r(x)} dS \\ &\leq (\sup_{n \geq n_0} A_n) \int_{K(\varepsilon^{n_0}) \cap \partial\Omega} |v(x)|^{p_*(x)} dS + C \sum_{n=n_0}^{\infty} \varepsilon^{n(s-\beta p_*^+)} \end{aligned}$$

Using that $\|v\|_{p_*, \partial\Omega} \leq C \|v\|_{1,p}$ and that $(s - \beta p_*^+) > 0$, we can conclude (5.2).

Finally, let $\{v_n\}_{n \in \mathbb{N}} \subset W^{1,p(x)}(\Omega)$ and $v \in W^{1,p(x)}(\Omega)$ be such that

$$v_n \rightharpoonup v \quad \text{weakly in } W^{1,p(x)}(\Omega).$$

Then,

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{weakly in } L^{r(x)}(\partial\Omega), \\ v_n &\rightarrow v \quad \text{strongly in } L^{s(x)}(\partial\Omega) \text{ for every } s \text{ such that } \inf_{\partial\Omega} (p_*(x) - s(x)) > 0, \end{aligned}$$

therefore $v_n \rightarrow v$ in $L^{r(x)}(\partial\Omega \setminus K(\varepsilon))$ for each $\varepsilon > 0$ small. Hence,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\partial\Omega} |v_n(x) - v(x)|^{r(x)} dS &= \limsup_{n \rightarrow \infty} \left(\int_{K(\varepsilon) \cap \partial\Omega} |v_n(x) - v(x)|^{r(x)} dS \right. \\ &\quad \left. + \int_{\partial\Omega \setminus K(\varepsilon)} |v_n(x) - v(x)|^{r(x)} dS \right) \\ &\leq \sup_{n \in \mathbb{N}} \int_{K(\varepsilon) \cap \partial\Omega} |v_n(x) - v(x)|^{r(x)} dS \end{aligned}$$

So, by (5.2), we conclude the desired result. \square

Now it is straightforward to derive, analogous to Corollary 5.2,

Corollary 5.4. *Let $p \in \mathcal{P}(\Omega)$ be such that $p^+ < N$ and let $r \in \mathcal{P}(\partial\Omega)$. Suppose that there exist $x_0 \in \Omega$, $C > 0$, $n \in \mathbb{N}$, $r_0 > 0$ such that $\inf_{\partial\Omega \setminus B_{r_0}(x_0)} (p_*(x) - r(x)) > 0$ and $r(x) \leq p_*(x) - c \frac{\ln^n(\frac{1}{|x-x_0|})}{\ln(\frac{1}{|x-x_0|})}$ for Hausdorff $(n-1)$ -almost every $x \in \partial\Omega \cap B_{r_0}(x_0)$. Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ is compact.*

5.2 Non-compact case 1: The Sobolev immersion Theorem

In the general case $\mathcal{A} \neq \emptyset$, up to our knowledge, there are no results regarding the existence or not of extremals for the Sobolev immersion Theorem. In this chapter we first prove a general result that implies the existence of extremals in the noncompact case.

This general result says that if the Sobolev constant is smaller than the smallest localized Sobolev constant on the critical set \mathcal{A} , then the existence of extremals follows.

Consequently, in the next sections of this chapter, we give conditions that ensure that this strict inequality holds. We give both global and local conditions.

Global conditions are easily obtained by making some rough estimates on the Sobolev constant. In our case, this global condition says that if the subcritical set, $\Omega \setminus \mathcal{A}$, contains a sufficiently large ball, then the strict inequality holds, and so the existence of extremals follows.

Local conditions are much harder and requires of a fine asymptotic analysis of the Sobolev constant when the Rayleigh quotient is evaluated in a precise function concentrating around some critical point.

In order to state our main results, let us introduce some notation.

- The Rayleigh quotient will be denoted by

$$\mathcal{Q}_{p,q,\Omega}(v) := \frac{\|\nabla v\|_{p(x)}}{\|v\|_{q(x)}}. \quad (5.3)$$

- The localized Sobolev constant by

$$\bar{S}_x = \sup_{\varepsilon > 0} S(p(\cdot), q(\cdot), B_\varepsilon(x)) = \lim_{\varepsilon \rightarrow 0^+} S(p(\cdot), q(\cdot), B_\varepsilon(x)), \quad x \in \mathcal{A}.$$

- The critical constant by

$$\bar{S} = \inf_{x \in \mathcal{A}} \bar{S}_x. \quad (5.4)$$

With these notations, our main results can be stated as

Theorem 5.5. *Assume that $p, q \in \mathcal{P}(\Omega)$ are continuous functions with modulus of continuity $\rho(t)$ such that*

$$\rho(t) \log(1/t) \rightarrow 0 \quad \text{as } t \rightarrow 0^+.$$

Assume, moreover, that the criticality set \mathcal{A} is nonempty and $p^+ < q^-$.

Then, for every domain Ω it holds

$$S(p(\cdot), q(\cdot), \Omega) \leq \bar{S} \leq \inf_{x \in \mathcal{A}} K^{-1}(N, p(x)),$$

Theorem 5.6. *Under the same assumptions of the previous Theorem, if the strict inequality holds*

$$S(p(\cdot), q(\cdot), \Omega) < \bar{S},$$

then there exists an extremal for the immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

The proof of Theorem 5.6 heavily relies on the Concentration–Compactness Theorem for variable exponents that we prove in Chapter 4.

The other key ingredient in the proof is the adaptation of a convexity argument due to P.L. Lions, F. Pacella and M. Tricarico [38] in order to show that a minimizing sequence either concentrates at a single point or is strongly convergent.

First we prove a uniform upper bound for $S(p(\cdot), q(\cdot), \Omega)$ depending only on $p(x)$ with $x \in \mathcal{A}$.

Lemma 5.7. *With the assumptions of Theorem 5.5, it holds that*

$$S(p(\cdot), q(\cdot), \Omega) \leq \inf_{x \in \mathcal{A}} K^{-1}(N, p(x)).$$

Proof. First, we observe that our regularity assumptions on p and q implies that

$$\begin{aligned} q(x_0 + \lambda x) &= q(x_0) + \rho_1(\lambda, x) = p^*(x_0) + \rho_1(\lambda, x), \\ p(x_0 + \lambda x) &= p(x_0) + \rho_2(\lambda, x), \end{aligned}$$

where ρ_1 and ρ_2 are modulus of continuity such that $\lim_{\lambda \rightarrow 0^+} \lambda^{\rho_k(\lambda, x)} = 1$ uniformly in Ω ($k = 1, 2$).

Now, let $\phi \in C_c^\infty(\Omega)$, and define ϕ_λ to be the rescaled function around $x_0 \in \mathcal{A}$ as $\phi_\lambda = \lambda^{\frac{-n}{p^*(x_0)}} \phi\left(\frac{x-x_0}{\lambda}\right)$. Then we have

$$1 = \int_{\Omega} \left(\frac{\phi_\lambda(x)}{\|\phi_\lambda\|_{q(x)}} \right)^{q(x)} dx = \int_{\Omega_\lambda} \lambda^{\frac{-N(p^*(x_0)+\rho_1(\lambda, x_0+\lambda y))}{p^*(x_0)} + N} \left(\frac{\phi(y)}{\|\phi_\lambda\|_{q(x)}} \right)^{q(x_0)+\rho_1(\lambda, x_0+\lambda y)} dy.$$

where $\Omega_\lambda := \{y : \lambda y + x_0 \in \Omega\}$. Since

$$\lambda^{\frac{-N\rho_1(\lambda, x_0+\lambda y)}{p^*(x_0)}} \left(\frac{\phi(y)}{\|\phi_\lambda\|_{q(x)}} \right)^{\rho_1(\lambda, x_0+\lambda y)} \rightarrow 1 \text{ when } \lambda \rightarrow 0+ \text{ in } \{\phi > 0\} \subset \Omega,$$

By dominated convergence, we get

$$1 = \frac{\int_{\mathbb{R}^N} |\phi(y)|^{q(x_0)} dy}{\lim_{\lambda \rightarrow 0} \|\phi_\lambda\|_{q(x)}^{q(x_0)}}.$$

Analogously,

$$\begin{aligned} 1 &= \int_{\Omega} \left(\frac{|\nabla \phi_\lambda(x)|}{\|\nabla \phi_\lambda\|_{p(x)}} \right)^{p(x)} dx \\ &= \int_{\Omega_\lambda} \lambda^{\frac{-N(p(x_0)+\rho_2(\lambda x_0+\lambda y))}{p^*(x_0)} + N} \left(\frac{\frac{1}{\lambda} |\nabla \phi(y)|}{\|\nabla \phi_\lambda(y)\|_{p(x)}} \right)^{p(x_0)+\rho_2(\lambda x_0+\lambda y)} dx \\ &= \int_{\Omega_\lambda} \lambda^{\frac{-N(p(x_0)+\rho_2(\lambda x_0+\lambda y))}{p^*(x_0)} + N - p(x_0) - \rho_2(\lambda x_0+\lambda y)} \left(\frac{|\nabla \phi(y)|}{\|\nabla \phi_\lambda(y)\|_{p(x)}} \right)^{p(x_0)+\rho_2(\lambda x_0+\lambda y)} dx. \end{aligned}$$

Again,

$$\lambda^{\frac{-N\rho_2(\lambda x_0+\lambda y)}{p^*(x_0)} - \rho_2(\lambda x_0+\lambda y)} \left(\frac{|\nabla \phi(y)|}{\|\nabla \phi_\lambda(y)\|_{p(x)}} \right)^{\rho_2(\lambda x_0+\lambda y)} \rightarrow 1 \text{ when } \lambda \rightarrow 0+ \text{ in } \{|\nabla \phi| > 0\} \subset \Omega,$$

so we arrive at

$$1 = \frac{\int_{\mathbb{R}^N} |\nabla \phi(y)|^{p(x_0)} dy}{\lim_{\lambda \rightarrow 0+} \|\nabla \phi_\lambda\|_{p(x)}^{p(x_0)}}.$$

Now, by definition of $S(p(\cdot), q(\cdot), \Omega)$,

$$S(p(\cdot), q(\cdot), \Omega) \leq \frac{\|\nabla \phi_\lambda\|_{p(x)}}{\|\phi_\lambda\|_{q(x)}}$$

and taking limit $\lambda \rightarrow 0+$, we obtain

$$S(p(\cdot), q(\cdot), \Omega) \leq \frac{\|\nabla \phi\|_{p(x_0), \mathbb{R}^N}}{\|\phi\|_{q(x_0), \mathbb{R}^N}}$$

for every $\phi \in C_c^\infty(\Omega)$. Then,

$$S(p(\cdot), q(\cdot), \Omega) \leq K^{-1}(N, p(x_0)),$$

so,

$$S(p(\cdot), q(\cdot), \Omega) \leq \inf_{x \in \mathcal{A}} K^{-1}(N, p(x))$$

as we wanted to show. \square

Now, the proof of Theorem 5.5 follows easily as a simple corollary of Lemma 5.7.

Proof of Theorem 5.5. Applying Lemma 5.7 to the case $\Omega = B_\varepsilon(x_0)$ for $x_0 \in \mathcal{A}$ we get that

$$S(p(\cdot), q(\cdot), B_\varepsilon(x_0)) \leq K^{-1}(N, p(x_0))$$

for every $\varepsilon > 0$. So

$$\bar{S}_{x_0} \leq K^{-1}(N, p(x_0)).$$

Now, for the first inequality, we just observe that the Sobolev constant is nondecreasing with respect to inclusion, so

$$S(p(\cdot), q(\cdot), \Omega) \leq S(p(\cdot), q(\cdot), B_\varepsilon(x_0))$$

for every ball $B_\varepsilon(x_0) \subset \Omega$.

So the result follows. \square

Now we focus on our second theorem. We begin by adapting a convexity argument used in [38] to the variable exponent case.

Theorem 5.8. *Assume that $p^+ < q^-$, with the assumptions of theorem 5.5. Let $\{u_n\}_{n \in \mathbb{N}}$ be a minimizing sequence for $S(p(\cdot), q(\cdot), \Omega)$. Then the following alternative holds*

- $\{u_n\}_{n \in \mathbb{N}}$ has a strongly convergence subsequence in $L^{q(x)}(\Omega)$ or
- $\{u_n\}_{n \in \mathbb{N}}$ has a subsequence such that $|u_n|^{q(x)} \rightharpoonup \delta_{x_0}$ weakly in the sense of measures and $|\nabla u_n|^{p(x)} \rightharpoonup \bar{S}_{x_0}^{p(x_0)} \delta_{x_0}$ weakly in the sense of measures, for some $x_0 \in \mathcal{A}$.

Proof. Let $\{u_n\}_{n \in \mathbb{N}}$ be a normalized minimizing sequence, that is,

$$S(p(\cdot), q(\cdot), \Omega) = \lim_{n \rightarrow \infty} \|\nabla u_n\|_{p(x)}$$

and

$$\|u_n\|_{q(x)} = 1.$$

Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$, by the Concentration–Compactness Theorem (Theorem 4.6), we have that, for a subsequence that we still denote by $\{u_n\}_{n \in \mathbb{N}}$,

$$\begin{aligned} |u_n|^{q(x)} &\rightharpoonup \nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i}, && \text{weakly-* in the sense of measures,} \\ |\nabla u_n|^{p(x)} &\rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i}, && \text{weakly-* in the sense of measures,} \end{aligned}$$

where $u \in W_0^{1,p(x)}(\Omega)$, I is a finite set, $x_i \in \mathcal{A}$ and $\bar{S}_{x_i}^{-1} \mu_i^{1/p(x_i)} \geq \nu_i^{1/p^*(x_i)}$

Hence, using Theorem 5.5,

$$\begin{aligned}
1 &= \lim_{n \rightarrow \infty} \int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{\|\nabla u_n\|_{p(x)}^{p(x)}} dx \\
&\geq \int_{\Omega} |S(p(\cdot), q(\cdot), \Omega)^{-1} \nabla u|^{p(x)} dx + \sum_{i \in I} S(p(\cdot), q(\cdot), \Omega)^{-p(x_i)} \mu_i \\
&\geq \int_{\Omega} |S(p(\cdot), q(\cdot), \Omega)^{-1} \nabla u|^{p(x)} dx + \sum_{i \in I} \bar{S}_{x_i}^{-p(x_i)} \mu_i \\
&\geq \min\{(S(p(\cdot), q(\cdot), \Omega)^{-1} \|\nabla u\|_{p(x)})^{p^+}, (S(p(\cdot), q(\cdot), \Omega)^{-1} \|\nabla u\|_{p(x)})^{p^-}\} + \sum_{i \in I} v_i^{\frac{p(x_i)}{p^*(x_i)}} \\
&\geq \min\{\|u\|_{q(x)}^{p^+}, \|u\|_{q(x)}^{p^-}\} + \sum_{i \in I} v_i^{\frac{p(x_i)}{p^*(x_i)}}
\end{aligned}$$

where in the last inequality we have used the definition of $S(p(\cdot), q(\cdot), \Omega)$.

Now, as $\|u_n\|_{q(x)} = 1$ and $u_n \rightharpoonup u$ weakly in $L^{q(x)}(\Omega)$, it follows that $\|u\|_{q(x)} \leq 1$, hence

$$\min\{\|u\|_{q(x)}^{p^+}, \|u\|_{q(x)}^{p^-}\} = \|u\|_{q(x)}^{p^+} \geq \rho_{q(x)}(u)^{\frac{p^+}{q^-}}.$$

So we find that

$$\rho_{q(x)}(u)^{\frac{p^+}{q^-}} + \sum_{i \in I} v_i^{\frac{p(x_i)}{p^*(x_i)}} \leq 1. \quad (5.5)$$

On the other hand, as u_n is normalized, we get that

$$1 = \rho_{q(x)}(u) + \sum_{i \in I} v_i. \quad (5.6)$$

Since $p^+ < q^-$, by (5.5) and (5.6), we can conclude that either $\rho_{q(x)}(u) = 1$ and the set I is empty, or $u = 0$ and the set I contains a single point.

If the first case occurs, then $1 = \|u_n\|_{q(x)} = \rho_{q(x)}(u_n) = \rho_{q(x)}(u) = \|u\|_{q(x)}$ and, as $L^{q(x)}(\Omega)$ is a strictly convex Banach space, it follows that $u_n \rightarrow u$ strongly in $L^{q(x)}(\Omega)$.

If the second case occurs it easily follows that $v_0 = 1$ and $\mu_0 = \bar{S}_{x_0}^{p(x_0)}$. \square

With the aid of this result, we are now ready to prove Theorem 5.6.

Proof of Theorem 5.6. Let $\{u_n\}_{n \in \mathbb{N}}$ be a minimizing sequence for $S(p(\cdot), q(\cdot), \Omega)$.

If $\{u_n\}_{n \in \mathbb{N}}$ has a strongly convergence subsequence in $L^{q(x)}(\Omega)$, then the result holds.

Assume that this is not the case. Then, by the previous Theorem, there exists $x_0 \in \mathcal{A}$ such that $|u_n|^{q(x)} \rightharpoonup \delta_{x_0}$ weakly in the sense of measures and $|\nabla u_n|^{p(x)} \rightharpoonup \bar{S}_{x_0}^{p(x_0)} \delta_{x_0}$ weakly in the sense of measures

So, for $\varepsilon > 0$, we have

$$\int_{\Omega} \left(\frac{|\nabla u_n|}{\bar{S}_{x_0} - \varepsilon} \right)^{p(x)} dx \rightarrow \frac{\bar{S}_{x_0}^{p(x_0)}}{(\bar{S}_{x_0} - \varepsilon)^{p(x_0)}} > 1.$$

Then, there exists n_0 such that for all $n \geq n_0$, we know that:

$$\|\nabla u_n\|_{p(x)} > \bar{S}_{x_0} - \varepsilon.$$

Taking limit, we obtain

$$S(p(\cdot), q(\cdot), \Omega) \geq \bar{S}_{x_0} - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, the result follows. \square

5.3 Global conditions for the validity of $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$

In this section we provide with an example of a domain Ω and exponents p, q where the condition $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$ is satisfied.

The condition is the existence of a large ball where the exponent q is subcritical. Up to our knowledge it is not known if $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$ can hold when $q \equiv p^*$ on Ω .

This example somewhat relates to the one analyzed in [42]. More precisely, we can show

Theorem 5.9. *Assume that $B_R \subset \Omega \setminus \mathcal{A}$ where B_R is a ball of radius R . Moreover, assume that $q_{B_R}^+ < (p^*)_{B_R}^-$.*

Then, if R is large enough, we have that $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$.

Proof. Assume that Ω contains a subcritical ball B_R . Take $u \in C_c^\infty(B_1)$ such that $|u|, |\nabla u| \leq 1$, and consider $u_R(x) = u(x/R)$. We take R big enough to have

$$R^{N-p^+} \int_{B_1} |\nabla u|^{p^+} dx > 1, \quad R^N \int_{B_1} |u|^{q^+} > 1$$

and

$$\frac{\|\nabla u\|_{p_{B_R}^-}}{\|u\|_{q_{B_R}^+}} R^{N(1/(p_{B_R}^-)^* - 1/q_{B_R}^+)} < S.$$

Then we claim that

$$\frac{\|\nabla u_R\|_{p(x), B_R}}{\|u_R\|_{q(x), B_R}} < S.$$

We first note that

$$\int_{B_R} |\nabla u_R|^{p(x)} dx = \int_{B_1} R^{N-p(Rx)} |\nabla u|^{p(Rx)}(x) dx \geq R^{N-p^+} \int_{B_1} |\nabla u|^{p^+} dx > 1$$

so that, by Proposition 3.4,

$$\|\nabla u_R\|_{p(x), B_R} \leq \left(\int_{B_R} |\nabla u_R|^{p(x)} dx \right)^{1/p_{B_R}^-} \leq R^{\frac{N-p_{B_R}^-}{p_{B_R}^-}} \left(\int_{B_1} |\nabla u|^{p_{B_R}^-} dx \right)^{1/p_{B_R}^-}.$$

In the same way

$$\int_{B_R} |u_R|^{q(x)} dx = R^N \int_{B_1} |u|^{q(Rx)} dx \geq R^N \int_{B_1} |u|^{q^+} > 1$$

so that

$$\|u_R\|_{q(x), B_R} \geq \left(\int_{B_R} |u_R|^{q(x)} dx \right)^{1/q_{B_R}^+} \geq R^{N/q_{B_R}^+} \|u\|_{q_R^+},$$

from which we deduce our claim. This finishes the proof. \square

5.4 Continuity of the Sobolev constant with respect to p and q

In this section, we prove the continuity of the Sobolev constant $S(p(\cdot), q(\cdot), \Omega)$ with respect to p and q in the $L^\infty(\Omega)$ topology for monotone sequences.

This result will be useful in order to provide local conditions that imply $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$, but we have chosen to put it in a separate section since we believe that is of independent interest.

We first prove an easy Lemma on the continuity of the Rayleigh quotient.

Lemma 5.10. *Let $p, p_n, q, q_n \in \mathcal{P}(\Omega)$ be such that $p_n \rightarrow p$ and $q_n \rightarrow q$ in $L^\infty(\Omega)$. Then, for every $v \in C_c^\infty(\Omega)$, $\mathcal{Q}_{p_n, q_n, \Omega}(v) \rightarrow \mathcal{Q}_{p, q, \Omega}(v)$.*

Proof. We only need to prove that

$$\|\nabla v\|_{p_n(x)} \rightarrow \|\nabla v\|_{p(x)} \quad \text{and} \quad \|v\|_{q_n(x)} \rightarrow \|v\|_{q(x)}.$$

For that, we have

$$\int_{\Omega} \left(\frac{|v|}{\|v\|_{q(x)} + \delta} \right)^{q_n(x)} dx \rightarrow \int_{\Omega} \left(\frac{|v|}{\|v\|_{q(x)} + \delta} \right)^{q(x)} dx < 1,$$

so, there exist n_0 such that for every $n \geq n_0$,

$$\int_{\Omega} \left(\frac{|v|}{\|v\|_{q(x)} + \delta} \right)^{q_n(x)} dx < 1.$$

Therefore $\|v\|_{q_n(x)} \leq \|v\|_{q(x)} + \delta$. Analogously, we obtain $\|v\|_{q(x)} - \delta \leq \|v\|_{q_n(x)}$. In conclusion, for every $\delta > 0$ we get

$$\|v\|_{q(x)} - \delta \leq \liminf \|v\|_{q_n(x)} \leq \limsup \|v\|_{q_n(x)} \leq \|v\|_{q(x)} + \delta.$$

In a complete analogous fashion, we get

$$\|\nabla v\|_{p(x)} - \delta \leq \liminf \|\nabla v\|_{p_n(x)} \leq \limsup \|\nabla v\|_{p_n(x)} \leq \|\nabla v\|_{p(x)} + \delta.$$

This finishes the proof. \square

Now we prove the main result of the section.

Theorem 5.11. *Let $p, p_n, q, q_n \in \mathcal{P}(\Omega)$ be such that $p_n \rightarrow p$ and $q_n \rightarrow q$ in $L^\infty(\Omega)$. Assume, moreover, that $p_n \geq p$ and that $q_n \leq q$. Then $S(p_n(\cdot), q_n(\cdot), \Omega) \rightarrow S(p(\cdot), q(\cdot), \Omega)$.*

Proof. Given $\delta > 0$ we pick $u \in C_c^\infty(\Omega)$ such that $\mathcal{Q}_{p,q,\Omega}(u) \leq S(p(\cdot), q(\cdot), \Omega) + \delta$. Since, by Lemma 5.10, $\lim_{n \rightarrow \infty} \mathcal{Q}_{p_n, q_n, \Omega}(u) = \mathcal{Q}_{p,q,\Omega}(u)$, we obtain, using u as a test-function to estimate $S(p_n(\cdot), q_n(\cdot), \Omega)$, that

$$\begin{aligned} \limsup_{n \rightarrow \infty} S(p_n(\cdot), q_n(\cdot), \Omega) &\leq \limsup_{n \rightarrow \infty} \mathcal{Q}_{p_n, q_n, \Omega}(u) \\ &= \mathcal{Q}_{p,q,\Omega}(u) \\ &\leq S(p(\cdot), q(\cdot), \Omega) + \delta, \end{aligned}$$

for any $\delta > 0$. It follows that

$$\limsup_{n \rightarrow \infty} S(p_n(\cdot), q_n(\cdot), \Omega) \leq S(p(\cdot), q(\cdot), \Omega).$$

We now claim that there holds

$$\liminf_{n \rightarrow \infty} S(p_n(\cdot), q_n(\cdot), \Omega) \geq S(p(\cdot), q(\cdot), \Omega).$$

The claim will follow if we prove that for any $u \in C_c^\infty(\Omega)$,

$$\|\nabla u\|_{L^{p_n(x)}(\Omega)} \geq (1 + o(1)) \|\nabla u\|_{L^{p(x)}(\Omega)}, \quad (5.7)$$

and

$$\|u\|_{L^{q_n(x)}(\Omega)} \leq (1 + o(1)) \|u\|_{L^{q(x)}(\Omega)}, \quad (5.8)$$

where $o(1)$ is uniform in u . Since $p_n \geq p$ we can use Hölder inequality (Theorem 3.7), with $\frac{1}{p} = \frac{1}{p_n} + \frac{1}{s_n}$ to obtain

$$\begin{aligned} \|\nabla u\|_{p(x)} &\leq \left((p/p_n)^+ + (p/s_n)^+ \right) \|\nabla u\|_{p_n(x)} \|1\|_{s_n(x)} \\ &\leq (1 + o(1)) \|\nabla u\|_{p_n(x)} \max\{|\Omega|^{(1/s_n)^+}, |\Omega|^{(1/s_n)^-}\} \\ &= (1 + o(1)) \|\nabla u\|_{p_n(x)}, \end{aligned}$$

where the $o(1)$ are uniform in u . Equation (5.7) follows. We prove (5.8) in the same way considering $\frac{1}{q} = \frac{1}{q_n} + \frac{1}{t_n}$ and writing that

$$\begin{aligned} \|v\|_{q_n(x)} &\leq \left((q_n/q)^+ + (q_n/t_n)^+ \right) \|v\|_{q(x)} \|1\|_{t_n(x)} \\ &= (1 + o(1)) \|v\|_{q(x)}. \end{aligned}$$

The proof is now complete. □

5.5 Investigation on the validity of $\bar{S} = \inf_{x \in \mathcal{A}} K^{-1}(N, p(x))$

Another key result that is needed in order to provide local conditions that assure $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$ is the validity of $\bar{S} = \inf_{x \in \mathcal{A}} K^{-1}(N, p(x))$ since for $K^{-1}(N, p(x))$ we have the knowledge of the extremals.

So in this section we investigate whether the equality

$$\bar{S} = \inf_{x \in \mathcal{A}} K^{-1}(N, p(x)) \quad (5.9)$$

holds or not.

We show that, under certain assumptions on $p(x_0)$ and $q(x_0)$, $x_0 \in \mathcal{A}$ the equality

$$\bar{S}_{x_0} = K^{-1}(N, p(x_0)) \quad (5.10)$$

is valid.

As far as we know, it is an open problem to determine whether the equality holds true or not in general.

The aim of this section is to prove the following Theorem.

Theorem 5.12. *Let $p, q \in \mathcal{P}(\Omega)$ be continuous and assume that $p(\cdot)$ has a local minimum and $q(\cdot)$ has a local maximum at $x_0 \in \mathcal{A}$. Then*

$$\bar{S}_{x_0} = \lim_{\varepsilon \rightarrow 0} S(p(\cdot), q(\cdot), B_\varepsilon(x_0)) = K^{-1}(N, p(x_0)).$$

This Theorem is a direct consequence of Theorem 5.11 and the following result:

Proposition 5.13. *Assume $0 \in \mathcal{A}$ and denote by $p = p(0)$, $B_\varepsilon = B_\varepsilon(0)$.*

For any $u \in C_c^\infty(B_\varepsilon)$, there holds

$$\|u\|_{q(x), B_\varepsilon} = \varepsilon^{N/p^*} (1 + o(1)) \|u_\varepsilon\|_{q_\varepsilon(x), B_1}$$

and

$$\|\nabla u\|_{p(x), B_\varepsilon} = \varepsilon^{N/p^*} (1 + o(1)) \|\nabla u_\varepsilon\|_{p_\varepsilon(x), B_1},$$

where $o(1)$ is uniform in u , $p_\varepsilon(x) := p(\varepsilon x)$, $q_\varepsilon(x) := q(\varepsilon x)$ and $u_\varepsilon(x) := u(\varepsilon x)$.

Assuming Proposition 5.13 we can prove Theorem 5.12.

Proof of Theorem 5.12. We have

$$Q(p(\cdot), q(\cdot), B_\varepsilon)(u) = (1 + o(1)) Q(p_\varepsilon(\cdot), q_\varepsilon(\cdot), B_1)(u_\varepsilon),$$

where the $o(1)$ is uniform in u , so that, noticing that the map $u \in C_c^\infty(B_\varepsilon) \mapsto u_\varepsilon \in C_c^\infty(B_1)$ is bijective, using Theorem 5.11,

$$\begin{aligned} S(p(\cdot), q(\cdot), B_\varepsilon) &= (1 + o(1)) S(p_\varepsilon(\cdot), q_\varepsilon(\cdot), B_1) = (1 + o(1)) S(p(0), q(0), B_1) \\ &= (1 + o(1)) S(p(0), p(0)^*, B_1) \end{aligned}$$

which proves Theorem 5.12. \square

It remains to prove Proposition 5.13.

Proof of Proposition 5.13. Given $u \in C_c^\infty(B_\varepsilon)$ we have

$$\|u\|_{q(x), B_\varepsilon} = \inf\{\lambda > 0: I_q^{\lambda, \varepsilon}(u) \leq 1\},$$

where

$$I_q^{\lambda, \varepsilon}(u) := \int_{B_\varepsilon} \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx = \int_{B_1} \left| \frac{u_\varepsilon(x)}{\lambda \varepsilon^{-\frac{N}{q_\varepsilon(x)}}} \right|^{q_\varepsilon(x)} dx.$$

Writing that

$$\varepsilon^{-\frac{N}{q_\varepsilon(x)}} = \exp\{-N \ln \varepsilon (q(0) + O(\varepsilon))^{-1}\} = \varepsilon^{-N/p^*} (1 + o(1)),$$

where the $O(\varepsilon)$ and the $o(1)$ are uniform in x and u , we obtain

$$\begin{aligned} \|u\|_{q(x), B_\varepsilon} &= \inf\{\lambda > 0: I_q^{\lambda, \varepsilon}(u) \leq 1\} \\ &= \varepsilon^{N/p^*} (1 + o(1)) \inf\{\tilde{\lambda} > 0: I_{q_\varepsilon}^{\tilde{\lambda}, 1}(u_\varepsilon) \leq 1\}, \end{aligned} \quad (5.11)$$

from which we deduce the result. The proof of the result for the gradient term is similar: we have

$$\|\nabla u\|_{p(x), B_\varepsilon} = \inf\{\lambda > 0: I_p^{\lambda, \varepsilon}(\nabla u) \leq 1\},$$

and

$$I_p^{\lambda, \varepsilon}(\nabla u) = \int_{B_\varepsilon} \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} dx = \int_{B_1} \left| \frac{\nabla u_\varepsilon(x)}{\lambda \varepsilon^{1-\frac{N}{p_\varepsilon(x)}}} \right|^{p_\varepsilon(x)} dx,$$

and we can end the proof as before. \square

5.6 Local conditions for the validity of $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$

In this section, applying the results of the preceding ones, we find local conditions that ensures the validity of $S(p(\cdot), q(\cdot), \Omega) < \bar{S}$. The main assumptions are the existence of a point $x_0 \in \mathcal{A}$ such that $\bar{S} = \bar{S}_{x_0}$ and that x_0 is a local minimum of p and a local maximum of q . Under these assumptions, it holds that $\bar{S} = K^{-1}(N, p(x_0))$.

Therefore, a natural candidate to evaluate the Rayleigh quotient $Q_{p, q, \Omega}$ is a rescaled extremal for $K^{-1}(N, p(x_0))$. Then, a detailed asymptotic analysis with respect to the scaling parameter ε allows us to show that if the rescaled function u_ε is concentrated enough then $Q_{p, q, \Omega}(u_\varepsilon) < K^{-1}(N, p(x_0))$ and we obtain the desired result.

So, in this section we assume that $0 \in \mathcal{A}$, we write $p = p(0)$ and observe that $q = q(0) = p^*$ and we also assume that 0 is a local maximum of q and a local minimum of p and so

$$\bar{S} = \lim_{\varepsilon \rightarrow 0} S(p(\cdot), q(\cdot), B_\varepsilon(0)) = K^{-1}(N, p).$$

In order to estimate $S(p(\cdot), q(\cdot), \Omega)$ we compute the Rayleigh quotient $Q_{p, q, \Omega}$ given in (5.3) in the test function u_ε constructed in Proposition A.1 in the Appendix.

Hence, by Proposition A.1 and Remark A.2, we obtain

$$\int_{\Omega} \left(\frac{u_{\varepsilon}}{\lambda} \right)^{q(x)} dx = \frac{A_0}{\lambda^{p^*}} + \frac{A_1}{\lambda^{p^*}} \varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2 \ln \varepsilon), \quad (5.12)$$

with

$$A_0 = \int_{\mathbb{R}^N} U^{p^*} dx, \quad A_1 = -\frac{1}{2p^*} \Delta q(0) \int_{\mathbb{R}^N} |x|^2 U^{p^*} dx$$

and

$$\int_{\Omega} \left(\frac{|\nabla u_{\varepsilon}|}{\lambda} \right)^{p(x)} dx = \frac{B_0}{\lambda^p} + \frac{B_1}{\lambda^p} \varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2 \ln \varepsilon), \quad (5.13)$$

with

$$B_0 = \int_{\mathbb{R}^N} |\nabla U|^p dx, \quad B_1 = -\frac{1}{2p} \Delta p(0) \int_{\mathbb{R}^N} |x|^2 |\nabla U|^p dx.$$

From (5.12) and (5.13) we obtain

$$\|u_{\varepsilon}\|_{q(x)} = \|U\|_{p^*, \mathbb{R}^N} + C_1 o(1)$$

and

$$\|\nabla u_{\varepsilon}\|_{p(x)} = \|\nabla U\|_{p, \mathbb{R}^N} - C_2 o(1),$$

with $C_1, C_2 > 0$ and $o(1) > 0$. So

$$\begin{aligned} S(p(\cdot), q(\cdot), \Omega) &\leq Q_{p,q,\Omega}(u_{\varepsilon}) \\ &= \frac{\|\nabla U\|_{p, \mathbb{R}^N} - C_2 o(1)}{\|U\|_{p^*, \mathbb{R}^N} + C_1 o(1)} \\ &= K^{-1}(N, p) - C o(1) < K^{-1}(N, p) \end{aligned}$$

if either $\Delta p(0) > 0$ or $\Delta q(0) < 0$.

So we have proved the following theorem

Theorem 5.14. *Let $p, q \in \mathcal{P}(\Omega)$ be C^2 and that $p^+ < q^-$. Assume that there exists $x_0 \in \mathcal{A}$ such that $\bar{S} = \sup_{x \in \mathcal{A}} \bar{S}_x$ and that x_0 is a local minimum of $p(x)$ and a local maximum of $q(x)$. Moreover, assume that either $\Delta p(0) > 0$ or $\Delta q(0) < 0$.*

Finally, assume that $p(x_0) < \sqrt{N}$ if $N \geq 5$, $p(x_0) < 2$ if $N = 4$ and $p(x_0) < 3/2$ if $N = 3$. Then the strict inequality holds

$$S(p(\cdot), q(\cdot), \Omega) < \bar{S}$$

and therefore, there exists an extremal for $S(p(\cdot), q(\cdot), \Omega)$.

5.7 Non-compact case 2: The Sobolev trace Theorem

In this section we parallel the results for the Sobolev immersion Theorem, Section 5.2, to the Sobolev trace Theorem.

In that spirit, the result that we obtain says that if the Sobolev trace constant is strictly smaller than the smallest localized Sobolev trace constant in the critical set \mathcal{A}_T , then there exists an extremal for the trace inequality.

Then, the objective will be to find conditions on $p(x)$, $q(x)$ and Ω in order to ensure that strict inequality. Again, we find global and local conditions.

As in the previous case, global conditions are easily obtained and they say that if the surface measure of the boundary is larger than the volume of the domain, then the strict inequality holds and therefore an extremal for the trace inequality exists.

Once again, local conditions are more difficult to find. In this case, the geometry of the domain comes into play.

Let us begin by recalling some notation introduced in Section 4.2.

Let $\Omega \subset \mathbb{R}^N$ be smooth (C^2 is enough) and let $\Gamma \subset \partial\Omega$ be a (possibly empty) closed set.

We consider

$$W_\Gamma^{1,p(x)}(\Omega) := \overline{\{\phi \in C^\infty(\bar{\Omega}) : \phi \text{ vanishes in a neighborhood of } \Gamma\}},$$

the closure being taken in $\|\cdot\|_{1,p(x)}$ -norm.

The critical Sobolev trace constant is defined as

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) := \inf_{v \in W_\Gamma^{1,p(x)}(\Omega)} \frac{\|v\|_{1,p(x)}}{\|v\|_{r(x),\partial\Omega}}, \quad (5.14)$$

with $r \in \mathcal{P}(\partial\Omega)$ critical, i.e. $\mathcal{A}_T = \{x \in \partial\Omega \setminus \Gamma : r(x) = p_*(x)\} \neq \emptyset$.

The localized Sobolev trace constant is defined, for $x \in \mathcal{A}_T$, as

$$\bar{T}_x = \sup_{\varepsilon > 0} T(p(\cdot), r(\cdot), \Omega_\varepsilon, \Gamma_\varepsilon) = \lim_{\varepsilon \rightarrow 0} T(p(\cdot), r(\cdot), \Omega_\varepsilon, \Gamma_\varepsilon), \quad (5.15)$$

where $\Omega_\varepsilon = \Omega \cap B_\varepsilon(x)$ and $\Gamma_\varepsilon = \partial B_\varepsilon(x) \cap \bar{\Omega}$ and the smallest localized Sobolev trace constant is denoted by

$$\bar{T} := \inf_{x \in \mathcal{A}_T} \bar{T}_x. \quad (5.16)$$

Finally, the optimal Sobolev trace constant for constant exponents in $\mathbb{R}_+^N = \mathbb{R}^{N-1} \times (0, \infty)$ is denoted by

$$\bar{K}(N, p)^{-1} = \inf_{f \in \bar{D}^{1,p}(\mathbb{R}_+^N)} \frac{\|\nabla f\|_{p, \mathbb{R}_+^N}}{\|f\|_{p^*, \partial\mathbb{R}_+^N}},$$

where $\bar{D}^{1,p}(\mathbb{R}_+^N)$ is the set of functions f such that $\partial_i f \in L^p(\mathbb{R}_+^N)$ and $f|_{\partial\mathbb{R}_+^N} \in L^{p^*}(\partial\mathbb{R}_+^N)$.

We begin with a lemma that gives a bound for the constant $T(p(\cdot), r(\cdot), \Omega, \Gamma)$. This is the analogous to Lemma 5.7.

Lemma 5.15. *Assume that the exponents $p \in \mathcal{P}(\Omega)$ and $r \in \mathcal{P}(\partial\Omega)$ are continuous functions with modulus of continuity ρ such that*

$$\ln(\lambda)\rho(\lambda) \rightarrow 0 \quad \text{as } \lambda \rightarrow 0+.$$

Then, it holds that

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) \leq \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1}.$$

Proof. The proof uses the same rescaling argument as in Lemma 5.7 but we have to be more careful with the boundary term.

Let $x_0 \in \mathcal{A}_T$. Without loss of generality, we can assume that $x_0 = 0$ and that there exists $r > 0$ such that

$$\Omega_r := B_r \cap \Omega = \{x \in B_r : x_N > \psi(x')\}, \quad B_r \cap \partial\Omega = \{x \in B_r : x_N = \psi(x')\},$$

where $x = (x', x_N)$, $x' \in \mathbb{R}^{N-1}$, $x_N \in \mathbb{R}$, B_r is the ball centered at the origin of radius r and $\psi: \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ is of class C^2 with $\psi(0) = 0$ and $\nabla\psi(0) = 0$.

First, we observe that our regularity assumptions on p and r imply that

$$\begin{aligned} r(\lambda x) &= r(0) + \rho_1(\lambda, x) = p_*(0) + \rho_1(\lambda, x), \\ p(\lambda x) &= p(0) + \rho_2(\lambda, x), \end{aligned}$$

with ρ_1 and ρ_2 modulus of continuity such that $\lim_{\lambda \rightarrow 0+} \lambda^{\rho_1(\lambda, x)} = 1$ uniformly in $\bar{\Omega}_r$. From now on, for simplicity, we write $p = p(0)$ and $p_* = p_*(0)$.

Now, let $\phi \in C_c^\infty(\mathbb{R}^N)$, and define ϕ_λ to be the rescaled function around $0 \in \mathcal{A}_T$ as $\phi_\lambda = \lambda^{\frac{-(N-1)}{p_*}} \phi(\frac{x}{\lambda})$ and observe that, since Γ is closed and $0 \notin \Gamma$, $\phi_\lambda \in W_\Gamma^{1, p(x)}(\Omega)$ for λ small. Then we have

$$\begin{aligned} \int_{\partial\Omega} \phi_\lambda(x)^{r(x)} dS &= \int_{\partial\Omega_\lambda} \lambda^{\frac{-(N-1)\rho_1(\lambda, y)}{p_*}} \phi(y)^{p_* + \rho_1(\lambda, y)} dS \\ &= \int_{\mathbb{R}^{N-1}} \lambda^{\frac{-(N-1)\rho_1(\lambda, y)}{p_*}} \phi(y', \psi_\lambda(y'))^{p_* + \rho_1(\lambda, y')} \sqrt{1 + |\nabla\psi_\lambda(y')|^2} dy', \end{aligned}$$

where $\Omega_\lambda = \frac{1}{\lambda} \cdot \Omega$ and $\psi_\lambda(y') = \frac{1}{\lambda} \psi(\lambda y')$.

Since $\psi(0) = 0$ and $\nabla\psi(0) = 0$ we have that $\psi_\lambda(y') = O(\lambda)$ and $|\nabla\psi_\lambda(y')| = O(\lambda)$ uniformly in y' for $y' \in \text{supp}(\phi)$ which is compact. Moreover, our assumption on ρ_1 imply that

$$\lambda^{\frac{-(N-1)\rho_1(\lambda, y)}{p_*}} \phi(y)^{\rho_1(\lambda, y)} \rightarrow 1 \quad \text{when } \lambda \rightarrow 0+$$

uniformly in y .

Therefore, we get

$$\rho_{r(x), \partial\Omega}(\phi_\lambda) = \int_{\partial\Omega} \phi_\lambda(x)^{r(x)} dS \rightarrow \int_{\mathbb{R}^{N-1}} |\phi(y', 0)|^{p_*} dy', \quad \text{as } \lambda \rightarrow 0+. \quad (5.17)$$

In particular, (5.17) imply that $\|\phi_\lambda\|_{r(x),\partial\Omega}$ is bounded away from 0 and ∞ . Moreover, arguing as before, we find

$$\begin{aligned} 1 &= \int_{\partial\Omega} \left(\frac{\phi_\lambda(x)}{\|\phi_\lambda\|_{r(x),\partial\Omega}} \right)^{r(x)} dS \\ &= \int_{\mathbb{R}^{N-1}} \lambda^{-\frac{(N-1)\rho_1(\lambda,y)}{p^*}} \left(\frac{\phi(y', \psi_\lambda(y'))}{\|\phi_\lambda\|_{r(x),\partial\Omega}} \right)^{p^*+\rho_1(\lambda,y')} \sqrt{1 + |\nabla\psi_\lambda(y')|^2} dy', \end{aligned}$$

so

$$\lim_{\lambda \rightarrow 0^+} \|\phi_\lambda\|_{r(x),\partial\Omega} = \|\phi\|_{p^*,\partial\mathbb{R}_+^N}.$$

For the gradient term, we have

$$\begin{aligned} \int_{\Omega} |\nabla\phi_\lambda|^{p(x)} dx &= \int_{\Omega} \lambda^{-\frac{N}{p}p(x)} |\nabla\phi(\frac{x}{\lambda})|^{p(x)} dx \\ &= \int_{\Omega_\lambda} \lambda^{-\frac{N}{p}\rho_2(\lambda,y)} |\nabla\phi(y)|^{p+\rho_2(\lambda,y)} dy. \end{aligned}$$

Now, observing that $\Omega_\lambda \rightarrow \mathbb{R}_+^N$ and from our hypothesis on ρ_2 , we arrive at

$$\rho_{p(x),\Omega}(\phi_\lambda) = \int_{\Omega} |\nabla\phi_\lambda(x)|^{p(x)} dx \rightarrow \int_{\mathbb{R}_+^N} |\nabla\phi(y)|^p dy \quad \text{as } \lambda \rightarrow 0^+.$$

Similar computations show that

$$\rho_{p(x),\Omega}(\phi_\lambda) = \int_{\Omega} |\phi_\lambda(x)|^{p(x)} dx = O(\lambda^p),$$

so

$$\rho_{1,p(x),\Omega}(\phi_\lambda) = \int_{\Omega} |\nabla\phi_\lambda(x)|^{p(x)} + |\phi_\lambda(x)|^{p(x)} dx \rightarrow \int_{\mathbb{R}_+^N} |\nabla\phi(y)|^p dy \quad \text{as } \lambda \rightarrow 0^+.$$

Arguing as in the boundary term, we conclude that

$$\lim_{\lambda \rightarrow 0^+} \|\phi_\lambda\|_{1,p(x),\Omega} = \|\nabla\phi\|_{p,\mathbb{R}_+^N}.$$

Now, by the definition of $T(p(\cdot), r(\cdot), \Omega, \Gamma)$, it follows that

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) \leq \frac{\|\phi_\lambda\|_{1,p(x)}}{\|\phi_\lambda\|_{r(x),\partial\Omega}}$$

and taking the limit $\lambda \rightarrow 0^+$, we obtain

$$T(p(\cdot), q(\cdot), \Omega, \Gamma) \leq \frac{\|\nabla\phi\|_{p,\mathbb{R}_+^N}}{\|\phi\|_{p^*,\partial\mathbb{R}_+^N}}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Then,

$$T(p(\cdot), q(\cdot), \Omega, \Gamma) \leq \bar{K}(N, p)^{-1}$$

and so, since $x_0 = 0$ is arbitrary,

$$T(p(\cdot), q(\cdot), \Omega, \Gamma) \leq \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1},$$

as we wanted to show. \square

Now we prove a Lemma that gives us some monotonicity of the constants $T(p(\cdot), q(\cdot), \Omega, \Gamma)$ with respect to Ω and $\Gamma \subset \partial\Omega$.

Lemma 5.16. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be two C^2 domains, and let $\Gamma_i \subset \partial\Omega_i$, $i = 1, 2$ be closed.*

If $\Omega_2 \subset \Omega_1$, $(\partial\Omega_2 \cap \Omega_1) \subset \Gamma_2$ and $(\Gamma_1 \cap \partial\Omega_2) \subset \Gamma_2$, then

$$T(p(\cdot), q(\cdot), \Omega_1, \Gamma_1) \leq T(p(\cdot), q(\cdot), \Omega_2, \Gamma_2).$$

Proof. The proof is a simple consequence that if $v \in W_{\Gamma_2}^{1,p(x)}(\Omega_2)$, then extending v by 0 to $\Omega_1 \setminus \Omega_2$ gives that $v \in W_{\Gamma_1}^{1,p(x)}(\Omega_1)$. \square

Remark 5.17. Lemma 5.16 will be used in the following situation: For $\Omega \subset \mathbb{R}^N$ and $\Gamma \subset \partial\Omega$ closed, we take $x_0 \in \partial\Omega \setminus \Gamma$ and $r > 0$ such that $(B_r(x_0) \cap \partial\Omega) \cap \Gamma = \emptyset$.

Then, if we call $\Omega_r := \Omega \cap B_r(x_0)$, $\Gamma_r = \partial B_r(x_0) \cap \bar{\Omega}$, we obtain

$$T(p(\cdot), q(\cdot), \Omega, \Gamma) \leq T(p(\cdot), q(\cdot), \Omega_r, \Gamma_r).$$

As a consequence of Lemma 5.15 and Lemma 5.16 we easily obtain the following Theorem that is the trace analog of Theorem 5.5.

Theorem 5.18. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain and $\Gamma \subset \partial\Omega$ be closed. Let $p \in \mathcal{P}(\Omega)$ and $r \in \mathcal{P}(\partial\Omega)$ be continuous functions with modulus of continuity ρ such that*

$$\ln(\lambda)\rho(\lambda) \rightarrow 0 \quad \text{as } \lambda \rightarrow 0+.$$

Then, it holds that

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) \leq \bar{T} \leq \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1}.$$

Now, in the spirit of Theorem 5.8, we use the convexity method of [38] to prove that a minimizing sequence either is strongly convergent or concentrates around a single point.

Theorem 5.19. *Under the same assumption of Theorem 5.18. Let $\{u_n\}_{n \in \mathbb{N}} \subset W_{\Gamma}^{1,p(x)}(\Omega)$ be a minimizing sequence for $T(p(\cdot), r(\cdot), \Omega, \Gamma)$. Then the following alternative holds:*

- $\{u_n\}_{n \in \mathbb{N}}$ has a strongly convergence subsequence in $L^{r(x)}(\partial\Omega)$ or
- $\{u_n\}_{n \in \mathbb{N}}$ has a subsequence such that $|u_n|^{r(x)} dS \rightarrow \delta_{x_0}$ weakly in the sense of measures and $|\nabla u_n|^{p(x)} dx \rightarrow \bar{T}_{x_0}^{p(x)} \delta_{x_0}$ weakly in the sense of measures, for some $x_0 \in \mathcal{A}_T$ and $u_n \rightarrow 0$ strongly in $L^{p(x)}(\Omega)$.

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset W_{\Gamma}^{1,p(x)}(\Omega)$ be a normalized minimizing sequence for $T(p(\cdot), r(\cdot), \Omega, \Gamma)$, i.e.

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) = \lim_{n \rightarrow \infty} \|u_n\|_{1,p(x)}$$

and

$$\|u_n\|_{r(x), \partial\Omega} = 1.$$

For simplicity, we denote by $T = T(p(\cdot), r(\cdot), \Omega, \Gamma)$. The concentration compactness principle for the trace immersion, Theorem 4.6, together with the estimate given in Theorem 5.18 gives

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \int_{\Omega} \frac{|\nabla u_n|^{p(x)} + |u_n|^{p(x)}}{\|u_n\|_{1,p(x)}^{p(x)}} dx \\ &\geq \int_{\Omega} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{T^{p(x)}} dx + \sum_{i \in I} T^{-p(x_i)} \mu_i \\ &\geq \min\{(T^{-1}\|u\|_{1,p(x)})^{p^+}, (T^{-1}\|u\|_{1,p(x)})^{p^-}\} + \sum_{i \in I} \bar{T}_{x_i}^{-p(x_i)} \mu_i \\ &\geq \min\{\|u\|_{r(x), \partial\Omega}^{p^+}, \|u\|_{r(x), \partial\Omega}^{p^-}\} + \sum_{i \in I} v_i^{\frac{p(x_i)}{p^*(x_i)}} \\ &\geq \|u\|_{r(x), \partial\Omega}^{p^+} + \sum_{i \in I} v_i^{\frac{p(x_i)}{p^*(x_i)}} \\ &\geq \rho_{r(x), \partial\Omega}(u)^{\frac{p^+}{r^-}} + \sum_{i \in I} v_i^{\frac{p(x_i)}{p^*(x_i)}}. \end{aligned}$$

On the other hand, since $\{u_n\}_{n \in \mathbb{N}}$ is normalized in $L^{r(x)}(\partial\Omega)$, we get

$$1 = \int_{\partial\Omega} |u|^{r(x)} dS + \sum_{i \in I} v_i$$

So, since $p^+ < r^-$, we can conclude that either u is a minimizer of the corresponding problem and the set I is empty, or $v = 0$ and the set I contains a single point.

If the second case occur, it is easily seen that the second alternative holds. \square

With the aid of Theorem 5.19 we can now prove the main result of the section.

Theorem 5.20. *Let Ω be a bounded domain in \mathbb{R}^N with $\partial\Omega \in C^1$. Let $\Gamma \subset \partial\Omega$ be closed. Let $p \in \mathcal{P}(\Omega)$ and $r \in \mathcal{P}(\partial\Omega)$ be exponents that satisfy the regularity assumptions of Theorem 5.18. Assume, moreover, that $p^+ < r^-$.*

Then, if the following strict inequality holds

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) < \bar{T} \tag{5.18}$$

Then the infimum (5.14) is attained.

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset W_{\Gamma}^{1,p(x)}(\Omega)$ be a minimizing sequence for (5.14) normalized in $L^{r(x)}(\partial\Omega)$.

If $\{u_n\}_{n \in \mathbb{N}}$ has a strongly convergence subsequence in $L^{r(x)}(\partial\Omega)$, then the result holds.

Assume that this is not the case. Then, by Theorem 5.19, there exists $x_0 \in \mathcal{A}_T$ such that $|u_n|^{r(x)} dS \rightharpoonup \delta_{x_0}$ and $|\nabla u_n|^{p(x)} dx \rightharpoonup \bar{T}_{x_0}^{p(x_0)} \delta_{x_0}$ weakly in the sense of measures.

So for $\varepsilon > 0$, we have,

$$\int_{\Omega} \frac{|\nabla u_n|^{p(x)} + |u_n|^{p(x)}}{(\bar{T}_{x_0} - \varepsilon)^{p(x)}} dx \rightarrow \frac{\bar{T}_{x_0}^{p(x_0)}}{(\bar{T}_{x_0} - \varepsilon)^{p(x_0)}} > 1.$$

Then, there exists n_0 such that for all $n \geq n_0$, we know that

$$\|u_n\|_{1,p(x)} > \bar{T}_{x_0} - \varepsilon.$$

Taking limit, we obtain

$$T(p(\cdot), r(\cdot), \Omega, \Gamma) \geq \bar{T}_{x_0} - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, the result follows. \square

5.8 Global conditions for the validity of $T(p(\cdot), r(\cdot), \Omega) < \bar{T}$

Now we want to show an example of when the condition (5.18) is guaranteed. We assume that $\Gamma = \emptyset$ and using $v = 1$ as a test function we can estimate

$$T(p(\cdot), r(\cdot), \Omega) \leq \frac{\|1\|_{1,p(x)}}{\|1\|_{r(x),\partial\Omega}}.$$

It is easy to see that

$$\|1\|_{1,p(x)} = \|1\|_{p(x)} \leq \max \left\{ |\Omega|^{\frac{1}{p^+}}, |\Omega|^{\frac{1}{p^-}} \right\}$$

and

$$\|1\|_{r(x),\partial\Omega} \geq \min \{ |\partial\Omega|^{\frac{1}{r^+}}, |\partial\Omega|^{\frac{1}{r^-}} \}.$$

So, if Ω satisfies

$$\frac{\max \left\{ |\Omega|^{\frac{1}{p^+}}, |\Omega|^{\frac{1}{p^-}} \right\}}{\min \{ |\partial\Omega|^{\frac{1}{r^+}}, |\partial\Omega|^{\frac{1}{r^-}} \}} < \bar{T}, \quad (5.19)$$

then by Theorem 5.20 there exists an extremal for $T(p(\cdot), r(\cdot), \Omega)$.

Observe that the family of sets that verify (5.19) is large. In fact, for any open set Ω with C^1 boundary, if we denote $\Omega_t = t \cdot \Omega$ we have

$$\frac{\max \left\{ |\Omega_t|^{\frac{1}{p^+}}, |\Omega_t|^{\frac{1}{p^-}} \right\}}{\min \{ |\partial\Omega_t|^{\frac{1}{r^+}}, |\partial\Omega_t|^{\frac{1}{r^-}} \}} = \frac{t^{\frac{N}{p^+}} |\Omega|^{\frac{1}{p^+}}}{t^{\frac{N-1}{r^+}} |\partial\Omega|^{\frac{1}{r^+}}}$$

Now, the hypothesis $\frac{p^+}{r^-} < 1$ imply that $\frac{N}{p^+} - \frac{N-1}{r^-} \geq \frac{1}{p^+} > 0$, so we can conclude that:

$$T(p(\cdot), r(\cdot), \Omega_t) < \bar{T},$$

if $t > 0$ is small enough.

5.9 Investigation on the validity of $\bar{T} = \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1}$

In this section we find conditions on the exponents $p(x)$ and $r(x)$ that ensure the validity of $\bar{T} = \inf_{x \in \mathcal{A}_T} \bar{K}(N, p(x))^{-1}$. This fact, as in the Sobolev immersion case, plays a crucial role in order to find local condition for the existence of extremals for $T(p(\cdot), r(\cdot), \Omega, \Gamma)$.

We begin with a Lemma that is a refinement of the asymptotic expansions found in the proof of Lemma 5.15 since we obtain uniform convergence for bounded sets of $W^{1,p(x)}(\Omega)$. Though this lemma can be proved for variable exponents, we choose to prove it in the constant exponent case since this will be enough for our purposes and simplifies the arguments.

In order to prove the Lemma, we use the so-called *Fermi coordinates* in a neighborhood of some point $x_0 \in \partial\Omega$. Roughly speaking the Fermi coordinates around $x_0 \in \partial\Omega$ is to describe $x \in \Omega$ by (y, t) with $y \in \mathbb{R}^{N-1}$, $t > 0$ and y are the coordinates in a local chart of $\partial\Omega$ and t is the distance to the boundary along the inward unit normal vector. See Definition A.6 and Lemma A.7 in the Appendix for a description of the Fermi coordinates for C^2 open sets in \mathbb{R}^N .

Lemma 5.21. *Let $1 < p < N$ be a constant exponent and let u be a smooth function on $\bar{\Omega}$. Then, there holds*

$$\begin{aligned} \|u\|_{p^*, B_\varepsilon(x_0) \cap \partial\Omega} &= \varepsilon^{\frac{N-1}{p^*}} (1 + o(1)) \|\tilde{u}_\varepsilon\|_{p^*, V \cap \partial\mathbb{R}_+^N}, \\ \|u\|_{p, B_\varepsilon(x_0) \cap \Omega} &= \varepsilon^{\frac{N}{p}} (1 + o(1)) \|\tilde{u}_\varepsilon\|_{p, V \cap \mathbb{R}_+^N}, \\ \|\nabla u\|_{p, B_\varepsilon(x_0) \cap \Omega} &= \varepsilon^{\frac{N-p}{p}} (1 + o(1)) \|\nabla \tilde{u}_\varepsilon\|_{p, V \cap \mathbb{R}_+^N}, \end{aligned}$$

where V is the unit ball transformed under the Fermi coordinates, $o(1)$ is uniform in u for u bounded in $W^{1,p}(\Omega)$, $\tilde{u}_\varepsilon(y) = \tilde{u}(\varepsilon y)$ and \tilde{u} is u read in Fermi coordinates.

Proof. If we denote by $\Phi(y, t)$ the change of variables from Fermi coordinates to Euclidian coordinates, then, from Lemma A.7 we have

$$J\Phi = 1 + O(\varepsilon) \quad \text{in } B_\varepsilon(x_0) \cap \Omega,$$

where $J\Phi$ is the Jacobian of Φ ,

$$J_{\partial\Omega}\Phi = 1 + O(\varepsilon) \quad \text{in } B_\varepsilon(x_0) \cap \partial\Omega,$$

where $J_{\partial\Omega}\Phi$ is the tangential Jacobian of Φ and

$$|\nabla \tilde{u}_\varepsilon| = (1 + O(\varepsilon)) |\nabla u|$$

with $O(\varepsilon)$ uniform in u .

For a more comprehensive study of the Fermi coordinates see [19] and the book [34].

Now, we simply compute

$$\begin{aligned} \int_{B_\varepsilon(x_0) \cap \partial\Omega} |u|^{p^*} dS &= \int_{(\varepsilon \cdot V) \cap \partial\mathbb{R}_+^N} |\tilde{u}(y, 0)|^{p^*} (1 + O(\varepsilon)) dy \\ &= \varepsilon^{N-1} (1 + O(\varepsilon)) \int_{V \cap \partial\mathbb{R}_+^N} |\tilde{u}_\varepsilon(y, 0)|^{p^*} dy. \end{aligned}$$

In the same way,

$$\begin{aligned} \int_{B_\varepsilon(x_0) \cap \partial\Omega} |\nabla u|^p dx &= \int_{(\varepsilon \cdot V) \cap \partial\mathbb{R}_+^N} |\nabla \tilde{u}(y)|^p (1 + O(\varepsilon)) dy \\ &= \varepsilon^{N-p} (1 + O(\varepsilon)) \int_{V \cap \partial\mathbb{R}_+^N} |\nabla \tilde{u}_\varepsilon(y)|^p dy \end{aligned}$$

and

$$\begin{aligned} \int_{B_\varepsilon(x_0) \cap \partial\Omega} |u|^p dx &= \int_{(\varepsilon \cdot V) \cap \partial\mathbb{R}_+^N} |\tilde{u}(y)|^p (1 + O(\varepsilon)) dy \\ &= \varepsilon^N (1 + O(\varepsilon)) \int_{V \cap \partial\mathbb{R}_+^N} |\tilde{u}_\varepsilon(y)|^p dy. \end{aligned}$$

This completes the proof. \square

Now we can prove the main result of the section

Theorem 5.22. *Let $p \in \mathcal{P}(\Omega)$ and $r \in \mathcal{P}(\partial\Omega)$ be as in Theorem 5.18. Assume that $x_0 \in \mathcal{A}_T$ is a local minimum of $p(x)$ and a local maximum of $r(x)$. Then*

$$\bar{T}_{x_0} = \bar{K}(N, p(x_0))^{-1}.$$

Proof. From the proof of Theorem 5.18, it follows that $\bar{T}_{x_0} \leq \bar{K}(N, p(x_0))^{-1}$.

Let us see that if x_0 is a local minimum of $p(x)$ and a local maximum of $r(x)$ then the reverse inequality holds. Let us call $p = p(x_0)$ and then $p_* = r(x_0)$.

Since $p(x) \geq p$, by Young's inequality with $\frac{1}{p} = \frac{1}{p(x)} + \frac{1}{s(x)}$ we obtain

$$\int_{\Omega_\varepsilon} |u|^p + |\nabla u|^p dx \leq \frac{p}{p_\varepsilon^-} \int_{\Omega_\varepsilon} |u|^{p(x)} + |\nabla u|^{p(x)} dx + 2 \frac{p}{s_\varepsilon^-} |B_\varepsilon|$$

where $p_\varepsilon^- = \sup_{\Omega_\varepsilon} p(x)$.

It then follows that

$$\|\lambda^{-1} u\|_{1,p,\Omega_\varepsilon}^p \leq (1 + o(1)) \rho_{1,p(x),\Omega_\varepsilon}(\lambda^{-1} u) + O(\varepsilon^n).$$

So, for any $\delta > 0$, taking $\lambda = \|u\|_{1,p(x),\Omega_\varepsilon} + \delta$ we obtain

$$\|u\|_{1,p,\Omega_\varepsilon} \leq \|u\|_{1,p(x),\Omega_\varepsilon} + \delta, \quad (5.20)$$

if ε is small, depending only on δ .

Arguing in much the same way, we obtain

$$\|u\|_{r(x),\partial\Omega_\varepsilon} \leq \|u\|_{p_*,\partial\Omega_\varepsilon} + \delta, \quad (5.21)$$

for ε is small, depending only on δ .

Now, by (5.20) and (5.21) it follows that

$$\bar{Q}(p(\cdot), r(\cdot), \Omega_\varepsilon)(u) = \frac{\|u\|_{1,p(x),\Omega_\varepsilon}}{\|u\|_{r(x),\partial\Omega_\varepsilon}} \geq \frac{\|u\|_{1,p,\Omega_\varepsilon}}{\|u\|_{p^*,\partial\Omega_\varepsilon}} + O(\delta).$$

Finally, by Lemma 5.21, we get

$$\bar{Q}(p(\cdot), r(\cdot), \Omega_\varepsilon)(u) \geq \frac{\|\nabla \tilde{u}_\varepsilon\|_{p,V \cap \mathbb{R}_+^N}}{\|\tilde{u}_\varepsilon\|_{p^*,V \cap \partial \mathbb{R}_+^N}} + o(1) + O(\delta) \geq \bar{K}(N, p)^{-1} + o(1) + O(\delta).$$

So, taking infimum in $u \in W_{\Gamma_\varepsilon}^{1,p(x)}(\Omega_\varepsilon)$, $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$ we obtain the desired result. \square

5.10 Local conditions for the validity of $T(p(\cdot), r(\cdot), \Omega) < \bar{T}$

In this section we find local conditions to ensure the validity of $T(p(\cdot), r(\cdot), \Omega, \Gamma) < \bar{T}$, and so the existence of an extremal for $T(p(\cdot), r(\cdot), \Omega, \Gamma)$.

We assume, to begin with, that there exists a point $x_0 \in \mathcal{A}_T$ such that $\bar{T} = \bar{T}_{x_0}$. Moreover, this critical point x_0 is assume to be a local minimum of $p(x)$ and a local maximum of $q(x)$, so by Theorem 5.22, we obtain that $\bar{T} = \bar{T}_{x_0} = \bar{K}(N, p(x_0))^{-1}$.

The idea, then, is similar to the one used in Section 5.6. We estimate $T(p(\cdot), r(\cdot), \Omega, \Gamma)$ evaluating the corresponding Rayleigh quotient $\bar{Q}(p(\cdot), q(\cdot), \Omega)$ in a properly rescaled function of the extremal for $\bar{K}(N, p(x_0))^{-1}$.

A fine asymptotic analysis of the Rayleigh quotient with respect to the scaling parameter will yield the desired result.

Hence the main result of the section reads

Theorem 5.23. *Let $p \in \mathcal{P}(\Omega)$ and $r \in \mathcal{P}(\partial\Omega)$ be C^2 and that $p^+ < r^-$. Assume that there exists $x_0 \in \mathcal{A}_T$ such that $\bar{T} = \bar{T}_{x_0}$ and that x_0 is a local minimum of $p(x)$ and a local maximum of $r(x)$. Moreover, assume that either $\partial_t p(x_0) > 0$ or $H(x_0) > 0$.*

Then the strict inequality holds

$$T(p(\cdot), q(\cdot), \Omega, \Gamma) < \bar{T}$$

and therefore, there exists an extremal for $T(p(\cdot), q(\cdot), \Omega, \Gamma)$.

Proof. The proof is an immediate consequence of Propositions A.8, A.9 and A.10 of the Appendix. In fact, without loss of generality we can assume that $x_0 = 0$, denote $p(0) = p$ and

assume first that $\partial_t p(0) > 0$. Then

$$\begin{aligned} T(p(\cdot), q(\cdot), \Omega, \Gamma) &\leq \bar{Q}(p(\cdot), q(\cdot), \Omega)(u_\varepsilon) = \frac{\bar{D}_0^{\frac{1}{p}} \left(1 + \frac{\bar{D}_1}{p\bar{D}_0} \varepsilon \ln \varepsilon + o(\varepsilon \ln \varepsilon) \right)}{\bar{A}_0^{\frac{1}{p^*}} \left(1 + \frac{\bar{A}_1}{p^*\bar{A}_0} \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) \right)} \\ &= \bar{K}(N, p)^{-1} \frac{1 + \frac{\bar{D}_1}{p\bar{D}_0} \varepsilon \ln \varepsilon + o(\varepsilon \ln \varepsilon)}{1 + \frac{\bar{A}_1}{p^*\bar{A}_0} \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon)}. \end{aligned}$$

The proof will be finished if we show that

$$\frac{1 + \frac{\bar{D}_1}{p\bar{D}_0} \varepsilon \ln \varepsilon + o(\varepsilon \ln \varepsilon)}{1 + \frac{\bar{A}_1}{p^*\bar{A}_0} \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon)} < 1,$$

or, equivalently,

$$\frac{\bar{D}_1}{p\bar{D}_0} + o(1) < \frac{\bar{A}_1}{p^*\bar{A}_0} \varepsilon + o(\varepsilon).$$

But this former inequality holds, since $\bar{D}_1 < 0$ and $\bar{D}_0 > 0$.

The case where $\partial_t p(0) = 0$ and $H > 0$ is analogous. □

6

Existence results for critical elliptic equations with compact perturbations

In this chapter we begin our analysis of critical elliptic equations when the (nonlinear) elliptic operator is the $p(x)$ -Laplacian. That is, we will study the existence problem for the elliptic equation

$$-\Delta_{p(x)}u = f(x, u), \quad \text{in } \Omega, \quad (6.1)$$

complemented with some boundary conditions (in this chapter will be Dirichlet, but see next chapter).

By *critical* we mean that the source term $f(x, t)$ has critical growth in the sense of the Sobolev embedding, i.e.

$$c \leq \liminf_{t \rightarrow \infty} \frac{f(x, t)}{t^{p^*(x)-1}} \leq \limsup_{t \rightarrow \infty} \frac{f(x, t)}{t^{p^*(x)-1}} \leq C, \quad (6.2)$$

for some constants $0 < c \leq C < \infty$.

Observe that this problem is *variational* in the sense that weak solutions to (6.1) are critical points of the associated functional

$$\mathcal{F}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_{\Omega} F(x, u) dx,$$

where $F(x, t) = \int_0^t f(x, s) ds$ and that, from (6.2), \mathcal{F} is a well defined functional in the Sobolev space $W^{1,p(x)}(\Omega)$.

The main tool that we employ in order to find critical points for \mathcal{F} is the *Mountain Pass Theorem* that was described in Chapter 3, Theorem 3.20. The Mountain pass theorem has two types of hypotheses, geometrical and topological. Under fairly general conditions on $f(x, t)$ it can be checked that the geometrical hypotheses are satisfied. The topological hypotheses, i.e. the *Palais–Smale condition*, Definition 3.19, requires the mapping

$$u \mapsto \int_{\Omega} F(x, u) dx$$

to be compact. When $F(x, t)$ is subcritical, the compactness of the Sobolev embedding yields the desired result, but when one is working in the critical range this point becomes a delicate matter.

It is well known since the work of Pohožaev [46] that in the constant exponent case $p(x) = p$ and for the pure power critical source $f(x, t) = |t|^{p^*-2}t$ solutions to (6.1) with Dirichlet boundary conditions do not exist, if the domain Ω is starshaped with respect to some point.

In order to obtain existence of solutions to the critical equation in the constant exponent case, one need to perform some sort of perturbations. In this direction, we point out the seminal works of [5, 9] where the authors perturbed the equation studied by Pohožaev and obtain local conditions that ensure the existence of solutions.

Later on, we like to mention the work of García-Azorero and Peral [33] where the authors considered problem (6.2) with constant exponent $p(x) = p$ and a source of the form $f(x, t) = |t|^{p^*-2}t + \lambda|t|^{q-2}t$ with q subcritical. Then by choosing λ large, or small (depending on the q being smaller or larger than p), the authors obtained the existence of nontrivial solutions.

In all the above mentioned works, the general approach to the problem is similar. It is proved that the functional \mathcal{F} satisfies the Palais–Smale condition below a certain energy level c^* (that can be computed explicitly in terms of the Sobolev constants) and then prove that if the perturbation is chosen properly, then there exists a Palais–Smale sequence with energy below c^* .

Let us also point out that related existence results for problems like (6.1) with critical growth were obtained independently by Fu in [32]. In that paper the author considered a source of the form $f(x, u) = a(x)|u|^{p^*(x)-2}u + \lambda g(x, u)$ where $g(x, t) \sim |t|^{r(x)-2}t$ with $r(x)$ subcritical. Under these assumptions the author obtains existence of a nontrivial weak solution for positive and small enough $a(x)$ and for every $\lambda > 0$. No multiplicity results were obtained.

In this chapter, we follow the approach of [33] and extend those results to the variable exponent case.

6.1 Superlinear–like compact perturbation

In this section, we study the existence problem for the following elliptic equation

$$\begin{cases} -\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \quad (6.3)$$

where $r(x) < p^*(x) - \varepsilon$, $q(x) \leq p^*(x)$ and $\mathcal{A} = \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset$.

We define $A_\delta := \bigcup_{x \in \mathcal{A}} (B_\delta(x) \cap \Omega) = \{x \in \Omega : \text{dist}(x, \mathcal{A}) < \delta\}$ the δ -tubular neighborhood of \mathcal{A} .

In this case, the associated functional reads

$$\mathcal{F}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} - \frac{|u|^{q(x)}}{q(x)} - \lambda(x) \frac{|u|^{r(x)}}{r(x)} dx.$$

We assume in this section that the compact perturbation term $\lambda(x)|u|^{r(x)-2}u$ is superlinear-like, i.e. $p^+ \leq r^-$.

For this problem we can prove

Theorem 6.1. *Assume that $p, q, r \in \mathcal{P}(\Omega)$ are continuous and that $p(x)$ and $q(x)$ verify the hypotheses of Theorem 4.1. Assume moreover that $p^+ < r^- \leq r(x) < q(x) \leq p^*(x)$ with $\mathcal{A} := \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset$.*

Then, there exists $\lambda_0 > 0$ depending only on p, q, r, N and Ω , such that if

$$\inf_{x \in A_\delta} \lambda(x) > \lambda_0 \quad \text{for some } \delta > 0,$$

problem (6.3) has at least one nontrivial solution in $W_0^{1,p(x)}(\Omega)$.

We begin by proving the Palais-Smale condition for the functional \mathcal{F} , below certain level of energy.

Lemma 6.2. *Assume that $r \leq q$. Let $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ a Palais-Smale sequence then $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.*

Proof. By definition

$$\mathcal{F}(u_j) \rightarrow c \quad \text{and} \quad \mathcal{F}'(u_j) \rightarrow 0.$$

Now, we have

$$c + 1 \geq \mathcal{F}(u_j) = \mathcal{F}(u_j) - \frac{1}{r^-} \langle \mathcal{F}'(u_j), u_j \rangle + \frac{1}{r^-} \langle \mathcal{F}'(u_j), u_j \rangle,$$

where

$$\langle \mathcal{F}'(u_j), u_j \rangle = \int_{\Omega} |\nabla u_j|^{p(x)} - |u_j|^{q(x)} - |u_j|^{r(x)} dx.$$

Then, if $r(x) \leq q(x)$ we conclude

$$c + 1 \geq \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \int_{\Omega} |\nabla u_j|^{p(x)} dx - \frac{1}{r^-} |\langle \mathcal{F}'(u_j), u_j \rangle|.$$

We can assume that $\|\nabla u_j\|_{p(x), \Omega} \geq 1$, if not the sequence is bounded. As $\|\mathcal{F}'(u_j)\|_{W^{-1,p'}(\Omega)}$ is bounded we have that

$$c + 1 \geq \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \|\nabla u_j\|_{p(x), \Omega}^{p^-} - \frac{C}{r^-} \|\nabla u_j\|_{p(x), \Omega}.$$

We deduce that u_j is bounded.

This finishes the proof. □

From the fact that $\{u_j\}_{j \in \mathbb{N}}$ is a Palais-Smale sequence it follows, by Lemma 6.2, that $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Hence, by Theorem 4.1, we have

$$|u_j|^{q(x)} \rightharpoonup \nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i} \quad \nu_i > 0, \quad (6.4)$$

$$|\nabla u_j|^{p(x)} \rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i} \quad \mu_i > 0, \quad (6.5)$$

$$\bar{S} \nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)}. \quad (6.6)$$

Note that if $I = \emptyset$, we have that $\|u_j\|_{q(x), \Omega} \rightarrow \|u\|_{q(x), \Omega}$ and we know that $L^{q(x)}(\Omega)$ is uniformly convex then $u_j \rightarrow u$ strongly in $L^{q(x)}(\Omega)$. We know that $\{x_i\}_{i \in I} \subset \mathcal{A}$.

Let us show that if $c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}}\right) \bar{S}^N$ and $\{u_j\}_{j \in \mathbb{N}}$ is a Palais-Smale sequence, with energy level c , then $I = \emptyset$.

In fact, suppose that $I \neq \emptyset$. Then let $\phi \in C_0^\infty(\mathbb{R}^N)$ with support in the unit ball of \mathbb{R}^N . Consider the rescaled functions $\phi_{i,\varepsilon}(x) = \phi\left(\frac{x-x_i}{\varepsilon}\right)$.

As $\mathcal{F}'(u_j) \rightarrow 0$ in $(W_0^{1,p(x)}(\Omega))'$, we obtain that

$$\lim_{j \rightarrow \infty} \langle \mathcal{F}'(u_j), \phi_{i,\varepsilon} u_j \rangle = 0.$$

On the other hand,

$$\langle \mathcal{F}'(u_j), \phi_{i,\varepsilon} u_j \rangle = \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla(\phi_{i,\varepsilon} u_j) - \lambda(x) |u_j|^{r(x)} \phi_{i,\varepsilon} - |u_j|^{q(x)} \phi_{i,\varepsilon} dx$$

Then, passing to the limit as $j \rightarrow \infty$, we get

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \left(\int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla(\phi_{i,\varepsilon} u_j) dx \right) \\ &\quad + \int_{\Omega} \phi_{i,\varepsilon} d\mu - \int_{\Omega} \phi_{i,\varepsilon} d\nu - \int_{\Omega} \lambda(x) |u|^{r(x)} \phi_{i,\varepsilon} dx. \end{aligned}$$

By Hölder inequality, it is easy to check that

$$\lim_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla(\phi_{i,\varepsilon} u_j) dx = 0.$$

On the other hand,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{i,\varepsilon} d\mu = \mu_i \phi(0), \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{i,\varepsilon} d\nu = \nu_i \phi(0).$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \lambda(x) |u|^{r(x)} \phi_{i,\varepsilon} dx = 0.$$

So, we conclude that $(\mu_i - \nu_i)\phi(0) = 0$, i.e., $\mu_i = \nu_i$. Then,

$$\bar{S}^N \nu_i^{1/p^*(x_i)} \leq \nu_i^{1/p(x_i)},$$

so it is clear that $\nu_i = 0$ or $\bar{S}^N \leq \nu_i$.

On the other hand, as $r^- > p^+$,

$$\begin{aligned} c &= \lim_{j \rightarrow \infty} \mathcal{F}(u_j) = \lim_{j \rightarrow \infty} \mathcal{F}(u_j) - \frac{1}{p^+} \langle \mathcal{F}'(u_j), u_j \rangle \\ &= \lim_{j \rightarrow \infty} \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\nabla u_j|^{p(x)} dx + \int_{\Omega} \left(\frac{1}{p^+} - \frac{1}{q(x)} \right) |u_j|^{q(x)} dx \\ &\quad + \lambda \int_{\Omega} \left(\frac{1}{p^+} - \frac{1}{r(x)} \right) |u_j|^{r(x)} dx \\ &\geq \lim_{j \rightarrow \infty} \int_{\Omega} \left(\frac{1}{p^+} - \frac{1}{q(x)} \right) |u_j|^{q(x)} dx \\ &\geq \lim_{j \rightarrow \infty} \int_{\mathcal{A}_\delta} \left(\frac{1}{p^+} - \frac{1}{q(x)} \right) |u_j|^{q(x)} dx \\ &\geq \lim_{j \rightarrow \infty} \int_{\mathcal{A}_\delta} \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) |u_j|^{q(x)} dx \end{aligned}$$

But

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\mathcal{A}_\delta} \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) |u_j|^{q(x)} dx &= \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) \left(\int_{\mathcal{A}_\delta} |u|^{q(x)} dx + \sum_{j \in I} \nu_j \right) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) \nu_i \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) \bar{S}^N. \end{aligned}$$

As $\delta > 0$ is arbitrary, and q is continuous, we get

$$c \geq \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-} \right) \bar{S}^N.$$

Therefore, if

$$c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-} \right) \bar{S}^N,$$

the index set I is empty.

Now we are ready to prove the Palais-Smale condition below level c .

Theorem 6.3. *Let $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence, with energy level c . If $c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-} \right) \bar{S}^N$, then there exist $u \in W_0^{1,p(x)}(\Omega)$ and $\{u_{j_k}\}_{k \in \mathbb{N}} \subset \{u_j\}_{j \in \mathbb{N}}$ a subsequence such that $u_{j_k} \rightarrow u$ strongly in $W_0^{1,p(x)}(\Omega)$.*

Proof. We have that $\{u_j\}_{j \in \mathbb{N}}$ is bounded. Then, for a subsequence that we still denote $\{u_j\}_{j \in \mathbb{N}}$, $u_j \rightarrow u$ strongly in $L^{q(x)}(\Omega)$. We define $\mathcal{F}'(u_j) := \phi_j$. By the Palais-Smale condition, with energy level c , we have $\phi_j \rightarrow 0$ in $(W_0^{1,p(x)}(\Omega))'$.

By definition $\langle \mathcal{F}'(u_j), z \rangle = \langle \phi_j, z \rangle$ for all $z \in W_0^{1,p(x)}(\Omega)$, i.e.,

$$\int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla z \, dx - \int_{\Omega} |u_j|^{q(x)-2} u_j z \, dx - \int_{\Omega} \lambda(x) |u_j|^{r(x)-2} u_j z \, dx = \langle \phi_j, z \rangle.$$

Then, u_j is a weak solution of the following equation.

$$\begin{cases} -\Delta_{p(x)} u_j = |u_j|^{q(x)-2} u_j + \lambda(x) |u_j|^{r(x)-2} u_j + \phi_j =: f_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.7)$$

We define $T: (W_0^{1,p(x)}(\Omega))' \rightarrow W_0^{1,p(x)}(\Omega)$, $T(f) := u$ where u is the weak solution of the following equation.

$$\begin{cases} -\Delta_{p(x)} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.8)$$

Then it is easily seen, analogous to the constant case, that T is a continuous invertible operator.

To finish the proof, it is sufficient to show that f_j converges in $(W_0^{1,p(x)}(\Omega))'$. We only need to prove that $|u_j|^{q(x)-2} u_j \rightarrow |u|^{q(x)-2} u$ strongly in $(W_0^{1,p(x)}(\Omega))'$.

In fact,

$$\begin{aligned} \langle |u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u, \psi \rangle &= \int_{\Omega} (|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u) \psi \, dx \\ &\leq \|\psi\|_{q(x), \Omega} \|(|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u)\|_{q'(x), \Omega}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|(|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u)\|_{(W_0^{1,p(x)}(\Omega))'} &= \sup_{\substack{\psi \in W_0^{1,p(x)}(\Omega) \\ \|\psi\|_{W_0^{1,p(x)}(\Omega)} = 1}} \int_{\Omega} (|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u) \psi \, dx \\ &\leq \|(|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u)\|_{q'(x), \Omega} \end{aligned}$$

and now, by 4.5 we know that

$$\begin{aligned} \int_{\Omega} (|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u)^{q'(x)} \, dx &= \int_{\Omega} |u_j|^{q(x)-2} u_j^{q'(x)} \, dx - \int_{\Omega} |u|^{q(x)-2} u^{q'(x)} \, dx + o(1) \\ &= \int_{\Omega} |u_j|^{q(x)} \, dx - \int_{\Omega} |u|^{q(x)} \, dx + o(1) \\ &= \int_{\Omega} |u_j - u|^{q(x)} \, dx + o(1) \end{aligned}$$

this last term goes to zero as $j \rightarrow \infty$ because $u_j \rightarrow u$ strongly in $L^{q(x)}(\Omega)$.

The proof is finished. □

We are now in position to prove Theorem 6.1.

Proof of Theorem 6.1. In view of the previous result, we seek for critical values below level c . For that purpose, we want to use the Mountain Pass Theorem. Hence we have to check the following condition:

1. There exist constants $R, r > 0$ such that when $\|u\|_{1,p(x),\Omega} = R$, then $\mathcal{F}(u) > r$.
2. There exist $v_0 \in W^{1,p(x)}(\Omega)$ such that $\mathcal{F}(v_0) < r$.

Let us first check (1). We suppose that $\|\nabla u\|_{p(x),\Omega} \leq 1$ and $\|u\|_{p(x),\Omega} \leq 1$. The other cases can be treated similarly.

By Poincaré inequality (Proposition 4.5) we have,

$$\begin{aligned} & \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} - \frac{|u|^{q(x)}}{q(x)} - \lambda(x) \frac{|u|^{r(x)}}{r(x)} dx \\ & \geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - \frac{1}{q^-} \int_{\Omega} |u|^{q(x)} dx - \frac{\|\lambda\|_{\infty}}{r^-} \int_{\Omega} |u|^{r(x)} dx \\ & \geq \frac{1}{p^+} \|\nabla u\|_{p(x),\Omega}^{p^+} - \frac{1}{q^-} \|u\|_{q(x),\Omega}^{q^-} - \frac{\|\lambda\|_{\infty}}{r^-} \|u\|_{r(x),\Omega}^{r^-} \\ & \geq \frac{1}{p^+} \|\nabla u\|_{p(x),\Omega}^{p^+} - \frac{1}{q^-} \|\nabla u\|_{p(x),\Omega}^{q^-} - \frac{\|\lambda\|_{\infty}}{r^-} \|\nabla u\|_{p(x),\Omega}^{r^-}. \end{aligned}$$

Let $g(t) = \frac{1}{p^+} t^{p^+} - \frac{1}{q^-} t^{q^-} - \frac{\|\lambda\|_{\infty}}{r^-} t^{r^-}$, then it is easy to check that $g(R) > r$ for some $R, r > 0$. This proves (1).

Now (2) is immediate as for a fixed $w \in W_0^{1,p(x)}(\Omega)$ we have

$$\lim_{t \rightarrow \infty} \mathcal{F}(tw) = -\infty.$$

Now the candidate for critical value according to the Mountain Pass Theorem is

$$c = \inf_{g \in C} \sup_{t \in [0,1]} \mathcal{F}(g(t)),$$

where $C = \{g : [0, 1] \rightarrow W_0^{1,p(x)}(\Omega) : g \text{ continuous and } g(0) = 0, g(1) = v_0\}$.

We will show that, if $\inf_{x \in \mathcal{A}_\delta} \lambda(x)$ is big enough for some $\delta > 0$ then $c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-}\right) \bar{S}^n$ and so the local Palais-Smale condition (Theorem 6.3) can be applied.

We fix $w \in W_0^{1,p(x)}(\Omega)$. Then, if $t < 1$ we have

$$\begin{aligned} \mathcal{F}(tw) &\leq \int_{\Omega} t^{p(x)} \frac{|\nabla w|^{p(x)}}{p^-} - t^{q(x)} \frac{|w|^{q(x)}}{q^+} - \lambda(x) t^{r(x)} \frac{|w|^{r(x)}}{r^+} dx \\ &\leq \frac{t^{p^-}}{p^-} \int_{\Omega} |\nabla w|^{p(x)} dx - \frac{t^{r^+}}{r^+} \int_{\Omega} \lambda(x) |w|^{r(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} \int_{\Omega} |\nabla w|^{p(x)} dx - \frac{t^{r^+}}{r^+} \int_{\mathcal{A}_\delta} \lambda(x) |w|^{r(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} \int_{\Omega} |\nabla w|^{p(x)} dx - \frac{t^{r^+}}{r^+} \int_{\mathcal{A}_\delta} (\inf_{x \in \mathcal{A}_\delta} \lambda(x)) |w|^{r(x)} dx \end{aligned}$$

We define $g(t) := \frac{t^{p^-}}{p^-} a_1 - (\inf_{x \in \mathcal{A}_\delta} \lambda(x)) \frac{t^{r^+}}{r^+} a_3$, where a_1 and a_2 are given by $a_1 = \|\nabla w\|_{1,\Omega}^{p(x)}$ and $a_3 = \| |w|^{r(x)} \|_{1,\mathcal{A}_\delta}$.

The maximum of g is attained at $t_\lambda = \left(\frac{a_1}{(\inf_{x \in \mathcal{A}_\delta} \lambda(x)) a_3} \right)^{\frac{1}{r^+ - p^-}}$. So, we conclude that there exists $\lambda_0 > 0$ such that if $(\inf_{x \in \mathcal{A}_\delta} \lambda(x)) \geq \lambda_0$ then

$$\mathcal{F}(tw) < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}} \right) \bar{S}^N$$

This finishes the proof. \square

Remark 6.4. Observe that if $\lambda(x)$ is continuous it suffices to assume that $\lambda(x)$ is large in the criticality set \mathcal{A} .

6.2 Sublinear-like compact perturbation

In this section we study the existence problem for (6.3) but now the compact perturbation is assumed to be sublinear-like, i.e. $r^+ < p_-$.

With this hypothesis we prove

Theorem 6.5. *Let $p, q, r \in \mathcal{P}(\Omega)$ be continuous such that $p(x)$ and $q(x)$ verify the hypotheses of Theorem 4.1. Moreover, assume that $r^+ \leq p^- \leq p^+ \leq q^- \leq p^*(x)$ with $\mathcal{A} := \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset$.*

Then, there exists a constant $\lambda_1 > 0$ depending only on p, q, r, N and Ω such that if $\lambda(x)$ verifies $0 < \inf_{x \in \Omega} \lambda(x) \leq \|\lambda\|_{\infty, \Omega} < \lambda_1$, then there exists infinitely many solutions to (6.3) in $W_0^{1,p(x)}(\Omega)$.

In order to prove Theorem 6.5, we begin by checking the Palais–Smale condition for this case.

Lemma 6.6. *Let $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais–Smale sequence for \mathcal{F} then $\{u_j\}_{j \in \mathbb{N}}$ is bounded.*

Proof. Let $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence, that is

$$\mathcal{F}(u_j) \rightarrow c \quad \text{and} \quad \mathcal{F}'(u_j) \rightarrow 0.$$

Therefore there exists a sequence $\varepsilon_j \rightarrow 0$ such that

$$|\mathcal{F}'(u_j)w| \leq \varepsilon_j \|\nabla w\|_{p(x),\Omega} \text{ for all } w \in W_0^{1,p(x)}(\Omega).$$

Now we have,

$$\begin{aligned} c + 1 &\geq \mathcal{F}(u_j) - \frac{1}{q^-} \mathcal{F}'(u_j)u_j + \frac{1}{q^-} \mathcal{F}'(u_j)u_j \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |\nabla u_j|^{p(x)} dx + \int_{\Omega} \left(\frac{\lambda(x)}{q^-} - \frac{\lambda(x)}{r^-} \right) |u_j|^{r(x)} dx + \frac{1}{q^-} \mathcal{F}'(u_j)u_j \end{aligned}$$

We can assume that $\|\nabla u_j\|_{p(x),\Omega} > 1$. Then we have, by Proposition 3.4 and by Poincaré inequality,

$$\begin{aligned} c + 1 &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \|\nabla u_j\|_{p(x),\Omega}^{p^-} + \|\lambda\|_{\infty} \left(\frac{1}{q^-} - \frac{1}{r^-} \right) \|u_j\|_{r(x),\Omega}^{r^+} - \frac{1}{q^-} \|\nabla u_j\|_{p(x),\Omega} \varepsilon_j \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \|\nabla u_j\|_{p(x),\Omega}^{p^-} + \|\lambda\|_{\infty} \left(\frac{1}{q^-} - \frac{1}{r^-} \right) C \|\nabla u_j\|_{p(x),\Omega}^{r^+} - \frac{1}{q^-} \|\nabla u_j\|_{p(x),\Omega} \end{aligned}$$

from where it follows that $\|\nabla u_j\|_{p(x),\Omega}$ is bounded (recall that $p^+ \leq q^-$ and $r^+ < p^-$). \square

Let $\{u_j\}_{j \in \mathbb{N}}$ be a Palais-Smale sequence for \mathcal{F} . Therefore, by the previous Lemma, it follows that $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Then, by Theorem 4.1 we can assume that there exist two measures μ, ν and a function $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\begin{aligned} u_j &\rightharpoonup u && \text{weakly in } W_0^{1,p(x)}(\Omega), \\ |\nabla u_j|^{p(x)} &\rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i} && \text{weakly in the sense of measures,} \\ |u_j|^{q(x)} &\rightharpoonup \nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i} && \text{weakly in the sense of measures,} \\ \bar{S} \nu_i^{1/p^*(x_i)} &\leq \mu_i^{1/p(x_i)}. \end{aligned}$$

As before, assume that $I \neq \emptyset$. Now the proof follows exactly as in the previous case, until we get to

$$c \geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |u|^{q(x)} dx + \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \bar{S}^N + \|\lambda\|_{\infty, \Omega} \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \int_{\Omega} |u|^{r(x)} dx.$$

Applying now Hölder inequality, we find

$$c \geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |u|^{q(x)} dx + \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \bar{S}^N \\ + \|\lambda\|_{\infty, \Omega} \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \|u^{r(x)}\|_{q(x)/r(x), \Omega} |\Omega|^{\frac{q^+}{q^- - p^+}}.$$

If $\|u^{r(x)}\|_{q(x)/r(x), \Omega} \geq 1$, we have

$$c \geq c_1 \|u^{r(x)}\|_{q(x)/r(x), \Omega}^{(q/r)^-} + c_3 - \|\lambda\|_{\infty, \Omega} c_2 \|u^{r(x)}\|_{q(x)/r(x), \Omega},$$

so, if $f_1(x) := c_1 x^{(q/r)^-} - \|\lambda\|_{\infty, \Omega} c_2 x$, this function reaches its absolute minimum at $x_0 = \left(\frac{\|\lambda\|_{\infty, \Omega} c_2}{c_1 (q/r)^-} \right)^{\frac{1}{(q/r)^- - 1}}$.

On the other hand, if $\|u^{r(x)}\|_{q(x)/r(x), \Omega} < 1$, then

$$c \geq c_1 \|u^{r(x)}\|_{q(x)/r(x), \Omega}^{(q/r)^+} + c_3 - \|\lambda\|_{\infty, \Omega} c_2 \|u\|_{q(x)/r(x), \Omega},$$

so, if $f_2(x) = c_1 x^{(q/r)^+} - \|\lambda\|_{\infty, \Omega} c_2 x$, this function reaches its absolute minimum at $x_0 = \left(\frac{\|\lambda\|_{\infty, \Omega} c_2}{c_1 (q/r)^+} \right)^{\frac{1}{(q/r)^+ - 1}}$.

Then, we obtain

$$c \geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \bar{S}^N + K \min \left\{ \|\lambda\|_{\infty, \Omega}^{\frac{(q/r)^-}{(q/r)^- - 1}}, \|\lambda\|_{\infty, \Omega}^{\frac{(q/r)^+}{(q/r)^+ - 1}} \right\},$$

which contradicts our hypothesis.

Therefore $I = \emptyset$ and so $u_j \rightarrow u$ strongly in $L^{q(x)}(\Omega)$.

With these preliminaries the Palais-Smale condition can now be easily checked.

Lemma 6.7. *Let $(u_j) \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence for \mathcal{F} , with energy level c . There exists a constant K depending only on p, q, r and Ω such that, if $c < \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \bar{S}^N + K \min \left\{ \|\lambda\|_{\infty, \Omega}^{\frac{(q/r)^-}{(q/r)^- - 1}}, \|\lambda\|_{\infty, \Omega}^{\frac{(q/r)^+}{(q/r)^+ - 1}} \right\}$, then there exists a subsequence $\{u_{j_k}\}_{k \in \mathbb{N}} \subset \{u_j\}_{j \in \mathbb{N}}$ that converges strongly in $W_0^{1,p(x)}(\Omega)$.*

Proof. At this point, the proof follows by the continuity of the solution operator as in Theorem 6.3. \square

Assume now that $\|\nabla u\|_{p(x), \Omega} \leq 1$. Then, applying Poincaré inequality, we have

$$\mathcal{F}(u) \geq \frac{1}{p^+} \|\nabla u\|_{p(x), \Omega}^{p^+} - \frac{1}{q^-} \|u\|_{q(x), \Omega}^{q^-} - \frac{\|\lambda\|_{\infty, \Omega}}{r^-} \|u\|_{r(x), \Omega}^{r^-} \\ \geq \frac{1}{p^+} \|\nabla u\|_{p(x), \Omega}^{p^+} - \frac{C}{q^-} \|\nabla u\|_{p(x), \Omega}^{q^-} - \frac{\|\lambda\|_{\infty, \Omega} C}{r^-} \|\nabla u\|_{p(x), \Omega}^{r^-} =: J_1(\|\nabla u\|_{p(x), \Omega}),$$

where $J_1(x) = \frac{1}{p^+}x^{p^+} - \frac{C}{q^-}x^{q^-} - \frac{\|\lambda\|_{\infty,\Omega}C}{r^-}x^{r^-}$. We recall that $p^+ \leq q^-$ and $r^- < r^+ < p^- < p^+$.

As J_1 attains a local, but not a global, minimum (J_1 is not bounded below), we have to perform some sort of truncation. To this end let x_0, x_1 be such that $m < x_0 < M < x_1$ where m is the local minimum and M is the local maximum of J_1 and $J_1(x_1) > J_1(m)$. For these values x_0 and x_1 we can choose a smooth function $\tau_1(x)$ such that $\tau_1(x) = 1$ if $x \leq x_0$, $\tau_1(x) = 0$ if $x \geq x_1$ and $0 \leq \tau_1(x) \leq 1$.

If $\|\nabla u\|_{p(x),\Omega} > 1$, we argue similarly and obtain

$$\mathcal{F}(u) \geq \frac{1}{p^+}\|\nabla u\|_{p(x),\Omega}^{p^+} - \frac{C}{q^-}\|\nabla u\|_{p(x),\Omega}^{q^+} - \frac{\|\lambda\|_{\infty,\Omega}C}{r^-}\|\nabla u\|_{p(x),\Omega}^{r^+} =: J_2(\|\nabla u\|_{p(x),\Omega})$$

where $J_2(x) = \frac{1}{p^+}x^{p^+} - \frac{C}{q^-}x^{q^+} - \frac{\|\lambda\|_{\infty,\Omega}C}{r^-}x^{r^+}$. As in the previous case, J_2 attains a local but not a global minimum. So let x_0, x_1 be such that $m < x_0 < M < x_1$ where m is the local minimum of J_2 and M is the local maximum of J_2 and $J_2(x_1) > J_2(m)$. For these values x_0 and x_1 we can choose a smooth function $\tau_2(x)$ with the same properties as τ_1 . Finally, we define

$$\tau(x) = \begin{cases} \tau_1(x) & \text{if } x \leq 1 \\ \tau_2(x) & \text{if } x > 1. \end{cases}$$

Next, let $\varphi(u) = \tau(\|\nabla u\|_{p(x),\Omega})$ and define the truncated functional as follows

$$\tilde{\mathcal{F}}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} \varphi(u) dx - \int_{\Omega} \frac{\lambda(x)}{r(x)} |u|^{r(x)} dx$$

Now we state a Lemma that contains the main properties of $\tilde{\mathcal{F}}$.

Lemma 6.8. $\tilde{\mathcal{F}}$ is C^1 , if $\tilde{\mathcal{F}}(u) \leq 0$ then $\|\nabla u\|_{p(x),\Omega} < x_0$ and $\mathcal{F}(v) = \tilde{\mathcal{F}}(v)$ for every v close enough to u . Moreover there exists $\lambda_1 > 0$ such that if $0 < \|\lambda\|_{\infty,\Omega} < \lambda_1$ then $\tilde{\mathcal{F}}$ satisfies a local Palais-Smale condition for $c \leq 0$.

Proof. We only have to check the local Palais-Smale condition. Observe that every Palais-Smale sequence for $\tilde{\mathcal{F}}$ with energy level $c \leq 0$ must be bounded, therefore by Lemma 6.7 if λ verifies $0 < \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \bar{S}^N + K \min \left\{ \|\lambda\|_{\infty,\Omega}^{\frac{(q/r)^-}{(q/r)^- - 1}}, \|\lambda\|_{\infty,\Omega}^{\frac{(q/r)^+}{(q/r)^+ - 1}} \right\}$, then there exists a convergent subsequence \square

The following Lemma gives the final ingredients needed in the proof.

Lemma 6.9. For every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that

$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \geq n$$

where $\tilde{\mathcal{F}}^{-\varepsilon} = \{u \in W_0^{1,p(x)}(\Omega) : \tilde{\mathcal{F}}(u) \leq -\varepsilon\}$ and γ is the Krasnoselskii genus.

Proof. Let $E_n \subset W_0^{1,p(x)}(\Omega)$ be a n -dimensional subspace. Hence we have, for $u \in E_n$ such that $\|\nabla u\|_{p(x),\Omega} = 1$,

$$\begin{aligned}\tilde{\mathcal{F}}(tu) &= \int_{\Omega} \frac{|\nabla(tu)|^{p(x)}}{p(x)} dx - \int_{\Omega} \frac{|tu|^{q(x)}}{q(x)} \varphi(tu) dx - \int_{\Omega} \frac{\lambda(x)}{r(x)} |tu|^{r(x)} dx \\ &\leq \int_{\Omega} \frac{|\nabla(tu)|^{p(x)}}{p^-} dx - \int_{\Omega} \frac{|tu|^{q(x)}}{q^+} \varphi(tu) dx - \int_{\Omega} \frac{\lambda(x)}{r^+} |tu|^{r(x)} dx.\end{aligned}$$

If $t < 1$, then

$$\begin{aligned}\tilde{\mathcal{F}}(tu) &\leq \int_{\Omega} \frac{t^{p^-} |\nabla u|^{p(x)}}{p^-} dx - \int_{\Omega} \frac{t^{q^+} |u|^{q(x)}}{q^+} dx - \int_{\Omega} \frac{\inf_{x \in \Omega} \lambda(x)}{r^+} t^{r^+} |u|^{r(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} - \frac{t^{q^+}}{q^+} a_n - \inf_{x \in \Omega} \lambda(x) \frac{t^{r^+}}{r^+} b_n,\end{aligned}$$

where

$$a_n = \inf \left\{ \int_{\Omega} |u|^{q(x)} dx : u \in E_n, \|\nabla u\|_{p(x),\Omega} = 1 \right\}$$

and

$$b_n = \inf \left\{ \int_{\Omega} |u|^{r(x)} dx : u \in E_n, \|\nabla u\|_{p(x),\Omega} = 1 \right\}.$$

Now, we have

$$\tilde{\mathcal{F}}(tu) \leq \frac{t^{p^-}}{p^-} - \frac{t^{q^+}}{q^+} a_n - \inf_{x \in \Omega} \lambda(x) \frac{t^{r^+}}{r^+} b_n \leq \frac{t^{p^-}}{p^-} - \inf_{x \in \Omega} \lambda(x) \frac{t^{r^+}}{r^+} b_n$$

Observe that $a_n > 0$ and $b_n > 0$ because E_n is finite dimensional. As $r^+ < p^-$ and $t < 1$ we obtain that there exists positive constants ρ and ε such that

$$\tilde{\mathcal{F}}(\rho u) < -\varepsilon \quad \text{for } u \in E_n, \|\nabla u\|_{p(x),\Omega} = 1.$$

Therefore, if we set $S_{\rho,n} = \{u \in E_n : \|u\| = \rho\}$, we have that $S_{\rho,n} \subset \tilde{\mathcal{F}}^{-\varepsilon}$. Hence by monotonicity of the genus

$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \geq \gamma(S_{\rho,n}) = n$$

as we wanted to show. \square

The proof of Theorem 6.5 now follows exactly as in that of [33] using Lemma 6.9 and Theorem 3.23.

Proof of Theorem 6.5. We denote $J^{-\varepsilon} = \{u \in W_0^{1,p(x)}(\Omega) : J(u) \leq -\varepsilon\}$. By Lemma 6.9, for every $k \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $\gamma(J^{-\varepsilon}) \geq k$.

Since J is continuous and even, $J^{-\varepsilon} \in \Sigma_k$; then $C_k \leq -\varepsilon < 0$, for every $k \in \mathbb{N}$. But J is bounded from below; hence, $c_k > -\infty \forall k$.

Let us assume that $c = c_k = \dots = c_{k+r}$. So, applying Theorem 3.23 we obtain $\gamma(K_c) \geq r$ and by Remark 3.24, the proof is finished. \square

6.3 Multiplicity result

In the preceding section we obtained a multiplicity result for (6.3) (with a sublinear-like perturbation). That multiplicity result relied heavily on the oddness of the source $f(x, t)$. The oddness of f implies that the associated functional \mathcal{F} is even and therefore there exists a whole machinery for even functionals that can be applied in order to obtain the existence of multiple solutions.

Without the oddness assumption, multiplicity results are in general harder to obtain and this is the goal of this section.

So, here, we consider (6.1) complemented with Dirichlet boundary conditions, with a source term given by

$$f(x, t) = |t|^{q(x)-2}t + \lambda g(x, t),$$

where g is subcritical, but g is not assumed to be odd. i.e., we consider

$$\begin{cases} -\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (6.9)$$

where Ω is a bounded smooth domain in \mathbb{R}^N , $q(x)$ is critical in the sense that $\mathcal{A} = \{q(x) = p^*(x)\} \neq \emptyset$, λ is a positive parameter and the perturbation g is subcritical with some precise assumptions that we state below.

This problem with constant exponents was analyzed in [12]. In that paper, the authors prove the existence of at least three nontrivial solutions for (6.9), one positive, one negative and one that changes sign, under adequate assumptions on the source term g and the parameter λ .

The method in the proof used in [12] consists of restricting the functional associated to (6.9) to three different Banach manifolds, one consisting of positive functions, one consisting of negative functions and the third one consisting of sign-changing functions, all of them under a normalization condition. Then, by means of a suitable version of the Mountain Pass Theorem due to Schwartz [50] and the concentration-compactness principle of P.L. Lions [37] the authors can prove the existence of a critical point of each restricted functional and, finally, the authors were able to prove that critical points of each restricted functional are critical points of the unrestricted one.

This method was introduced by M. Struwe [52] where the subcritical case (in the sense of the Sobolev embeddings) for the p -Laplacian was treated. A related result for the p -Laplacian under nonlinear boundary condition can be found in [24]. Also, a similar problem in the case of the $p(x)$ -Laplacian, but with subcritical nonlinearities was analyzed in [39].

In all the above mentioned works, the main feature on the perturbation g is that no oddness condition is imposed.

The precise assumptions on the perturbation g are as follows:

- (G1) $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $g(x, 0) = 0$ for every $x \in \Omega$.

(G2) There exist constants $c_1 > 1/(q^- - 1)$, $c_2 \in (p^+, q^-)$, $0 < c_3 < c_4$, such that for any $u \in L^{q(x)}(\Omega)$ and $p^- \leq p^+ < r^- \leq r^+ < q^- \leq q^+$.

$$\begin{aligned} c_3 \rho_{r(x)}(u) &\leq c_2 \int_{\Omega} G(x, u) dx \leq \int_{\Omega} g(x, u) u dx \\ &\leq c_1 \int_{\Omega} g_u(x, u) u^2 dx \leq c_4 \rho_{r(x)}(u) \end{aligned}$$

Remark 6.10. Observe that this set of hypotheses on the nonlinear term g are similar than the ones considered by [12].

Remark 6.11. We exhibit now one example of nonlinearities that fulfills all of our hypotheses. $g(x, u) = |u|^{r(x)-2}u + (u_+)^{s(x)-1}$, if $s(x) < r(x)$, $q^- - 1 > s^- > p^+$. Hypotheses (G1)–(G2) are clearly satisfied.

More precisely, the main result said that:

Theorem 6.12. *Let $p, q, r \in \mathcal{P}(\Omega)$ be such that p and q verify the hypotheses of Theorem 4.1. Under assumptions (G1)–(G2), there exists $\lambda^* > 0$ depending only on N, p, q and the constant c_3 in (G2), such that for every $\lambda > \lambda^*$, there exist three different, nontrivial, solutions to problem (6.9). Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

The proof uses the same approach as in [52]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that \mathcal{F} has a critical point in K_i . These sets will be subsets of C^1 -manifolds $M_i \subset W^{1,p(x)}(\Omega)$ that will be constructed by imposing a sign restriction and a normalizing condition.

In fact, let

$$\begin{aligned} \mathcal{J}(v) &= \int_{\Omega} |\nabla v|^{p(x)} - |v|^{q(x)} dx, \\ M_1 &= \left\{ u \in W_0^{1,p(x)}(\Omega) : \int_{\Omega} u_+ dx > 0 \text{ and } \mathcal{J}(u_+) = \int_{\Omega} \lambda g(x, u) u_+ dx \right\}, \\ M_2 &= \left\{ u \in W_0^{1,p(x)}(\Omega) : \int_{\Omega} u_- dx > 0 \text{ and } \mathcal{J}(u_-) = - \int_{\Omega} \lambda g(x, u) u_- dx \right\}, \\ M_3 &= M_1 \cap M_2. \end{aligned}$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of u . We define

$$\begin{aligned} K_1 &= \{u \in M_1 \mid u \geq 0\}, \\ K_2 &= \{u \in M_2 \mid u \leq 0\}, \\ K_3 &= M_3. \end{aligned}$$

First, we need a Lemma to show that these sets are nonempty and, moreover, give some properties that will be useful in the proof of the result.

Lemma 6.13. *For every $w_0 \in W_0^{1,p(x)}(\Omega)$, $w_0 > 0$ ($w_0 < 0$), there exists $t_\lambda > 0$ such that $t_\lambda w_0 \in M_1$ ($\in M_2$). Moreover, $\lim_{\lambda \rightarrow \infty} t_\lambda = 0$. As a consequence, given $w_0, w_1 \in W_0^{1,p(x)}(\Omega)$, $w_0 > 0$, $w_1 < 0$, with disjoint supports, there exists $\bar{t}_\lambda, \underline{t}_\lambda > 0$ such that $\bar{t}_\lambda w_0 + \underline{t}_\lambda w_1 \in M_3$. Moreover $\bar{t}_\lambda, \underline{t}_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$.*

Proof. We prove the lemma for M_1 , the other cases being similar. For $w \in W_0^{1,p(x)}(\Omega)$, $w \geq 0$, we consider the functional

$$\varphi_1(w) = \int_{\Omega} |\nabla w|^{p(x)} - |w|^{q(x)} - \lambda g(x, w)w \, dx.$$

Given $w_0 > 0$, in order to prove the lemma, we must show that $\varphi_1(t_\lambda w_0) = 0$ for some $t_\lambda > 0$. Using hypothesis (G2), if $t < 1$, we have that:

$$\varphi_1(tw_0) \geq At^{p^+} - Bt^{q^-} - \lambda c_4 C t^{r^-}$$

and

$$\varphi_1(tw_0) \leq At^{p^-} - Bt^{q^+} - \lambda c_3 C t^{r^+},$$

where the coefficients A , B and C are given by:

$$A = \int_{\Omega} |\nabla w_0|^{p(x)} \, dx, \quad B = \int_{\Omega} |w_0|^{q(x)} \, dx, \quad C = \int_{\Omega} |w_0|^{r(x)} \, dx.$$

Since $p^- \leq p^+ < r^- \leq r^+ < q^- \leq q^+$ it follows that $\varphi_1(tw_0)$ is positive for t small enough, and negative for t big enough. Hence, by Bolzano's theorem, there exists some $t = t_\lambda$ such that $\varphi_1(t_\lambda u) = 0$. (This t_λ need not to be unique, but this does not matter for our purposes).

In order to give an upper bound for t_λ , it is enough to find some t_1 , such that $\varphi_1(t_1 w_0) < 0$. We observe that:

$$\varphi_1(tw_0) \leq \max\{At^{p^-} - \lambda c_3 C t^{r^+}; At^{p^+} - \lambda c_3 C t^{r^-}\}$$

so it is enough to choose t_1 such that $\max\{At_1^{p^-} - \lambda c_3 C t_1^{r^+}; At_1^{p^+} - \lambda c_3 C t_1^{r^-}\} = 0$, i.e.,

$$t_1 = \left(\frac{A}{c_3 \lambda C} \right)^{1/(r^+ - p^-)} \quad \text{or} \quad t_1 = \left(\frac{A}{c_3 \lambda C} \right)^{1/(r^- - p^+)}.$$

Hence, again by Bolzano's theorem, we can choose $t_\lambda \in [0, t_1]$, which implies that $t_\lambda \rightarrow 0$ as $\lambda \rightarrow +\infty$. \square

Lemma 6.14. *There exists $C_1, C_2 > 0$ depending on $p(x)$ and on c_2 such that, for every $u \in K_i$, $i = 1, 2, 3$, we have*

$$\int_{\Omega} |\nabla u|^{p(x)} \, dx = \left(\lambda \int_{\Omega} g(x, u)u \, dx + \int_{\Omega} |u|^{q(x)} \, dx \right) \leq C_1 \mathcal{F}(u) \leq C_2 \left(\int_{\Omega} |\nabla u|^{p(x)} \, dx \right).$$

Proof. The equality is clear since $u \in K_i$. Now, by (G2), $G(x, u) \geq 0$; so

$$\begin{aligned}\mathcal{F}(u) &= \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} - \frac{1}{q(x)} |u|^{q(x)} - \lambda G(x, u) dx \\ &\leq \frac{1}{p^-} \int_{\Omega} |\nabla u|^{p(x)} dx.\end{aligned}$$

To prove the final inequality, we proceed as follows. Using the norming condition of K_i and hypothesis (G2):

$$\begin{aligned}\mathcal{F}(u) &= \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} - \frac{1}{q(x)} |u|^{q(x)} - \lambda G(x, u) dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |u|^{q(x)} dx + \lambda \int_{\Omega} \left(\frac{1}{p^+} g(x, u) u - G(x, u) \right) dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |u|^{q(x)} dx + \left(\frac{1}{p^+} - \frac{1}{c_2} \right) \lambda \int_{\Omega} g(x, u) u dx.\end{aligned}$$

(Recall that $q^- > p^+$). This finishes the proof. \square

Lemma 6.15. *There exists $c > 0$ such that*

$$\|\nabla u_+\|_{p(x), \Omega} \geq c \quad \forall u \in K_1, \quad (6.10)$$

$$\|\nabla u_-\|_{p(x), \Omega} \geq c \quad \forall u \in K_2, \quad (6.11)$$

$$\|\nabla u_+\|_{p(x), \Omega}, \|\nabla u_-\|_{p(x), \Omega} \geq c \quad \forall u \in K_3. \quad (6.12)$$

Proof. Suppose that $\|\nabla u_{\pm}\|_{p(x), \Omega} < 1$. By the definition of K_i , by (G2) and the Poincaré inequality we have that

$$\begin{aligned}\|\nabla u_{\pm}\|_{p(x), \Omega}^{p^+} &\leq \rho_{p(x)}(\nabla u_{\pm}) = \int_{\Omega} \lambda g(x, u) u_{\pm} + |u_{\pm}|^{q(x)} dx \\ &\leq C \rho_{r(x)}(u_{\pm}) + \rho_{q(x)}(u_{\pm}) \leq C \|u_{\pm}\|_{r(x), \Omega}^{r^-} + \|u_{\pm}\|_{q(x), \Omega}^{q^-} \\ &\leq c_1 \|\nabla u_{\pm}\|_{p(x), \Omega}^{r^-} + c_2 \|\nabla u_{\pm}\|_{p(x), \Omega}^{q^-}.\end{aligned}$$

As $p^+ < r^- < q^-$, this finishes the proof. \square

The following lemma describes the properties of the manifolds M_i .

Lemma 6.16. *M_i is a C^1 sub-manifold of $W_0^{1, p(x)}(\Omega)$ of co-dimension 1 ($i = 1, 2$), 2 ($i = 3$) respectively. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$W_0^{1, p(x)}(\Omega) = T_u M_i \oplus \text{span}\{u_+, u_-\},$$

where $T_u M$ is the tangent space at u of the Banach manifold M . Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i .

Proof. Let us denote

$$\begin{aligned}\bar{M}_1 &= \left\{ u \in W_0^{1,p(x)}(\Omega) : \int_{\Omega} u_+ dx > 0 \right\}, \\ \bar{M}_2 &= \left\{ u \in W_0^{1,p(x)}(\Omega) : \int_{\Omega} u_- dx > 0 \right\}, \\ \bar{M}_3 &= \bar{M}_1 \cap \bar{M}_2.\end{aligned}$$

Observe that $M_i \subset \bar{M}_i$. The set \bar{M}_i is open in $W_0^{1,p(x)}(\Omega)$. Therefore it is enough to prove that M_i is a C^1 sub-manifold of \bar{M}_i . In order to do this, we will construct a C^1 function $\varphi_i : \bar{M}_i \rightarrow \mathbb{R}^d$ with $d = 1$ ($i = 1, 2$), $d = 2$ ($i = 3$) respectively and M_i will be the inverse image of a regular value of φ_i .

In fact, we define: For $u \in \bar{M}_1$,

$$\varphi_1(u) = \int_{\Omega} |\nabla u_+|^{p(x)} - |u_+|^{q(x)} - \lambda g(x, u) u_+ dx.$$

For $u \in \bar{M}_2$,

$$\varphi_2(u) = \int_{\Omega} |\nabla u_-|^{p(x)} - |u_-|^{q(x)} - \lambda g(x, u) u_- dx.$$

For $u \in \bar{M}_3$,

$$\varphi_3(u) = (\varphi_1(u), \varphi_2(u)).$$

Obviously, we have $M_i = \varphi_i^{-1}(0)$. From standard arguments (see [15], or the appendix of [47]), φ_i is of class C^1 . Therefore, we only need to show that 0 is a regular value for φ_i . To this end we compute, for $u \in M_1$,

$$\begin{aligned}\langle \nabla \varphi_1(u), u_+ \rangle &\leq p^+ \rho_p(\nabla u_+) - q^- \rho_q(u_+) - \lambda \int_{\Omega} g(x, u) u_+ - g_u(x, u) u_+^2 dx \\ &\leq q^- (\rho_p(\nabla u_+) - \rho_q(u_+)) - \lambda \int_{\Omega} g(x, u) u_+ - g_u(x, u) u_+^2 dx \\ &\leq (q^- \lambda - \lambda) \int_{\Omega} g(x, u) u_+ dx - \int_{\Omega} g_u(x, u) u_+^2 dx.\end{aligned}$$

By (G2) the last term is bounded by

$$\begin{aligned}(q^- \lambda - \lambda - \frac{\lambda}{c_1}) \int_{\Omega} g(x, u) u_+ dx &= \left(q^- - 1 - \frac{1}{c_1} \right) (\rho_{p(x)}(\nabla u_+) - \rho_{q(x)}(u_+)) \\ &\leq \left(q^- - 1 - \frac{1}{c_1} \right) \rho_{p(x)}(\nabla u_+).\end{aligned}$$

Recall that $c_1 < 1/(q^- - 1)$. Now, the last term is strictly negative by Lemma 6.15. Therefore, M_1 is a C^1 sub-manifold of $W_0^{1,p(x)}(\Omega)$. The exact same argument applies to M_2 . Since trivially

$$\langle \nabla \varphi_1(u), u_- \rangle = \langle \nabla \varphi_2(u), u_+ \rangle = 0$$

for $u \in M_3$, the same conclusion holds for M_3 .

To see that K_i is complete, let u_k be a Cauchy sequence in K_i , then $u_k \rightarrow u$ in $W_0^{1,p(x)}(\Omega)$. Moreover, $(u_k)_\pm \rightarrow u_\pm$ in $W_0^{1,p(x)}(\Omega)$. Now it is easy to see, by Lemma 6.15 and by continuity that $u \in K_i$.

Finally, by the first part of the proof we have the decomposition

$$T_u W^{1,p(x)}(\Omega) = T_u M_i \oplus \text{span}\{u_+\}$$

where $M_1 = \{u : \varphi_1(u) = 0\}$ and $T_u M_1 = \{v : \langle \nabla \varphi_1(u), v \rangle = 0\}$. Now let $v \in T_u W_0^{1,p(x)}(\Omega)$ be a unit tangential vector, then $v = v_1 + v_2$ where $v_2 = \alpha u_+$ and $v_1 = v - v_2$. Let us take α as

$$\alpha = \frac{\langle \nabla \varphi_1(u), v \rangle}{\langle \nabla \varphi_1(u), u_+ \rangle}.$$

With this choice, we have that $v_1 \in T_u M_1$. Now

$$\langle \varphi_1(u), v_1 \rangle = 0.$$

The very same argument to show that $T_u W^{1,p(x)}(\Omega) = T_u M_2 \oplus \langle u_- \rangle$ and $T_u W^{1,p(x)}(\Omega) = T_u M_3 \oplus \langle u_+, u_- \rangle$. From these formulas and from the estimates given in the first part of the proof, the uniform continuity of the projections onto $T_u M_i$ follows. \square

Now, we need to check the Palais-Smale condition for the functional \mathcal{F} restricted to the manifold M_i . We begin by proving the Palais-Smale condition for the functional \mathcal{F} unrestricted, below certain level of energy.

Lemma 6.17. *Assume that $r \leq q$. Let $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence. Then $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.*

Proof. By definition

$$\mathcal{F}(u_j) \rightarrow c \quad \text{and} \quad \mathcal{F}'(u_j) \rightarrow 0.$$

Now, we have

$$c + 1 \geq \mathcal{F}(u_j) = \mathcal{F}(u_j) - \frac{1}{c_2} \langle \mathcal{F}'(u_j), u_j \rangle + \frac{1}{c_2} \langle \mathcal{F}'(u_j), u_j \rangle,$$

where

$$\langle \mathcal{F}'(u_j), u_j \rangle = \int_{\Omega} |\nabla u_j|^{p(x)} - |u_j|^{q(x)} - \lambda f(x, u_j) u_j \, dx.$$

Then, if $c_2 < q^-$ we conclude

$$c + 1 \geq \left(\frac{1}{p^+} - \frac{1}{c_2} \right) \int_{\Omega} |\nabla u_j|^{p(x)} \, dx - \frac{1}{c_2} |\langle \mathcal{F}'(u_j), u_j \rangle|.$$

We can assume that $\|\nabla u_j\|_{p(x), \Omega} \geq 1$. As $\|\mathcal{F}'(u_j)\|$ is bounded we have that

$$c + 1 \geq \left(\frac{1}{p^+} - \frac{1}{c_2} \right) \|\nabla u_j\|_{p(x), \Omega}^{p^-} - \frac{C}{c_2} \|\nabla u_j\|_{p(x), \Omega}.$$

We deduce that u_j is bounded. This finishes the proof. \square

From the fact that $\{u_j\}_{j \in \mathbb{N}}$ is a Palais-Smale sequence it follows, by Lemma 6.17, that $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Hence, by Theorem 4.1, we have

$$\begin{aligned} |u_j|^{q(x)} &\rightharpoonup \nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i} \quad \nu_i > 0, \\ |\nabla u_j|^{p(x)} &\rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i} \quad \mu_i > 0, \\ \bar{S}_{x_i} \nu_i^{1/p^*(x_i)} &\leq \mu_i^{1/p(x_i)} \end{aligned}$$

where $\{x_i\}_{i \in I} \subset \mathcal{A}$.

Note that if $I = \emptyset$ then $u_j \rightarrow u$ strongly in $L^{q(x)}(\Omega)$. We define $q_{\mathcal{A}}^- := \inf_{\mathcal{A}} q(x)$.

Let us show that if $c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-}\right) \bar{S}^N$ and $\{u_j\}_{j \in \mathbb{N}}$ is a Palais-Smale sequence, with energy level c , then $I = \emptyset$. In fact, suppose that $I \neq \emptyset$. Then let $\phi \in C_0^\infty(\mathbb{R}^N)$ with support in the unit ball of \mathbb{R}^N . Consider the rescaled functions $\phi_{i,\varepsilon}(x) = \phi\left(\frac{x-x_i}{\varepsilon}\right)$, $x_i \in \mathcal{A}$.

As $\mathcal{F}'(u_j) \rightarrow 0$ in $(W_0^{1,p(x)}(\Omega))'$, we obtain that

$$\lim_{j \rightarrow \infty} \langle \mathcal{F}'(u_j), \phi_{i,\varepsilon} u_j \rangle = 0.$$

On the other hand,

$$\langle \mathcal{F}'(u_j), \phi_{i,\varepsilon} u_j \rangle = \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla(\phi_{i,\varepsilon} u_j) - \lambda f(x, u_j) u_j \phi_{i,\varepsilon} - |u_j|^{q(x)} \phi_{i,\varepsilon} dx.$$

Then, passing to the limit as $j \rightarrow \infty$, we get

$$0 = \lim_{j \rightarrow \infty} \left(\int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla(\phi_{i,\varepsilon}) u_j dx \right) + \int_{\Omega} \phi_{i,\varepsilon} d\mu - \int_{\Omega} \phi_{i,\varepsilon} d\nu - \int_{\Omega} \lambda f(x, u) u \phi_{i,\varepsilon} dx.$$

By Hölder inequality, it is easy to check that

$$\lim_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla(\phi_{i,\varepsilon}) u_j dx = 0.$$

On the other hand,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{i,\varepsilon} d\mu = \mu_i \phi(0), \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_{i,\varepsilon} d\nu = \nu_i \phi(0)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \lambda f(x, u) u \phi_{i,\varepsilon} dx = 0.$$

So, we conclude that $(\mu_i - \nu_i) \phi(0) = 0$, i.e., $\mu_i = \nu_i$. Then,

$$\bar{S}_{x_i} \nu_i^{1/p^*(x_i)} \leq \nu_i^{1/p(x_i)},$$

so it is clear that $\nu_i = 0$ or $\bar{S}^N \leq \nu_i$. On the other hand, we consider the δ -tubular neighborhood of \mathcal{A} , namely

$$\mathcal{A}_\delta := \bigcup_{x \in \mathcal{A}} (B_\delta(x) \cap \Omega).$$

So, as $c_2 > p^+$,

$$\begin{aligned}
c &= \lim_{j \rightarrow \infty} \mathcal{F}(u_j) = \lim_{j \rightarrow \infty} \mathcal{F}(u_j) - \frac{1}{p^+} \langle \mathcal{F}'(u_j), u_j \rangle \\
&= \lim_{j \rightarrow \infty} \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\nabla u_j|^{p(x)} dx + \int_{\Omega} \left(\frac{1}{p^+} - \frac{1}{q(x)} \right) |u_j|^{q(x)} dx \\
&\quad - \lambda \int_{\Omega} F(x, u_j) dx + \frac{\lambda}{p^+} \int_{\Omega} f(x, u_j) u_j dx \\
&\geq \lim_{j \rightarrow \infty} \int_{\Omega} \left(\frac{1}{p^+} - \frac{1}{q(x)} \right) |u_j|^{q(x)} dx \\
&\geq \lim_{j \rightarrow \infty} \int_{\mathcal{A}_\delta} \left(\frac{1}{p^+} - \frac{1}{q(x)} \right) |u_j|^{q(x)} dx \\
&\geq \lim_{j \rightarrow \infty} \int_{\mathcal{A}_\delta} \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) |u_j|^{q(x)} dx.
\end{aligned}$$

But

$$\begin{aligned}
\lim_{j \rightarrow \infty} \int_{\mathcal{A}_\delta} \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) |u_j|^{q(x)} dx &= \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) \left(\int_{\mathcal{A}_\delta} |u|^{q(x)} dx + \sum_{i \in I} v_i \right) \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}_\delta}^-} \right) v_i \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-} \right) \bar{S}^N.
\end{aligned}$$

As $\delta > 0$ is arbitrary, and q is continuous, we get

$$c \geq \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-} \right) \bar{S}^N.$$

Therefore, if

$$c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-} \right) \bar{S}^N,$$

the index set I is empty.

Now we are ready to prove the Palais-Smale condition below level c .

Lemma 6.18. *Let $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence, with energy level c . If $c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-} \right) \bar{S}^N$, then there exist $u \in W_0^{1,p(x)}(\Omega)$ and $\{u_{j_k}\}_{k \in \mathbb{N}} \subset \{u_j\}_{j \in \mathbb{N}}$ a subsequence such that $u_{j_k} \rightarrow u$ strongly in $W_0^{1,p(x)}(\Omega)$.*

Proof. We have that $\{u_j\}_{j \in \mathbb{N}}$ is bounded. Then, for a subsequence that we still denote $\{u_j\}_{j \in \mathbb{N}}$, $u_j \rightarrow u$ strongly in $L^{q(x)}(\Omega)$. We define $\mathcal{F}'(u_j) := \phi_j$. By the Palais-Smale condition, with energy level c , we have $\phi_j \rightarrow 0$ in $(W_0^{1,p(x)}(\Omega))'$.

By definition $\langle \mathcal{F}'(u_j), z \rangle = \langle \phi_j, z \rangle$ for all $z \in W_0^{1,p(x)}(\Omega)$, i.e.,

$$\int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla z \, dx - \int_{\Omega} |u_j|^{q(x)-2} u_j z \, dx - \int_{\Omega} \lambda g(x, u_j) z \, dx = \langle \phi_j, z \rangle.$$

Then, u_j is a weak solution of the following equation.

$$\begin{cases} -\Delta_{p(x)} u_j = |u_j|^{q(x)-2} u_j + \lambda g(x, u_j) + \phi_j =: f_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.13)$$

We define $T: (W_0^{1,p(x)}(\Omega))' \rightarrow W_0^{1,p(x)}(\Omega)$, $T(f) := u$ where u is the weak solution of the following equation.

$$\begin{cases} -\Delta_{p(x)} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.14)$$

Then T is a continuous invertible operator.

It is sufficient to show that f_j converges in $(W_0^{1,p(x)}(\Omega))'$. We only need to prove that $|u_j|^{q(x)-2} u_j \rightarrow |u|^{q(x)-2} u$ strongly in $(W_0^{1,p(x)}(\Omega))'$. In fact,

$$\begin{aligned} \langle |u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u, \psi \rangle &= \int_{\Omega} (|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u) \psi \, dx \\ &\leq \|\psi\|_{q(x), \Omega} \|(|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u)\|_{q'(x), \Omega}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|(|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u)\|_{(W_0^{1,p(x)}(\Omega))'} &= \sup_{\psi \in W_0^{1,p(x)}(\Omega), \|\nabla \psi\|_{p(x), \Omega} = 1} \int_{\Omega} (|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u) \psi \, dx \\ &\leq \|(|u_j|^{q(x)-2} u_j - |u|^{q(x)-2} u)\|_{q'(x), \Omega} \end{aligned}$$

and now, by the Dominated Convergence Theorem this last term goes to zero as $j \rightarrow \infty$. \square

Now, we can prove the Palais-Smale condition for the restricted functional.

Lemma 6.19. *The functional $\mathcal{F}|_{K_i}$ satisfies the Palais-Smale condition for energy level c for every $c < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}}\right) \bar{S}^N$.*

Proof. Let $\{u_k\} \subset K_i$ be a Palais-Smale sequence, that is $\mathcal{F}(u_k)$ is uniformly bounded and $\mathcal{F}'|_{K_i}(u_k) \rightarrow 0$ strongly. We need to show that there exists a subsequence u_{k_j} that converges strongly in K_i .

Let $v_j \in W_0^{1,p(x)}(\Omega)$ be a unit vector such that

$$\langle \mathcal{F}'(u_j), v_j \rangle = \|\mathcal{F}'(u_j)\|_{(W_0^{1,p(x)}(\Omega))'}.$$

Now, by Lemma 6.16, $v_j = w_j + z_j$ with $w_j \in T_{u_j}M_i$ and $z_j \in \text{span}\{(u_j)_+, (u_j)_-\}$.

Since $\mathcal{F}(u_j)$ is uniformly bounded, by Lemma 6.14, u_j is uniformly bounded in $W_0^{1,p(x)}(\Omega)$ and hence, by Lemma 6.16, w_j is uniformly bounded in $W_0^{1,p(x)}(\Omega)$. Moreover, by the definition of the sets K_i it follows that $\langle \mathcal{F}'(u_j), (u_j)^\pm \rangle = 0$. Therefore

$$\|\mathcal{F}'(u_j)\|_{(W_0^{1,p(x)}(\Omega))'} = \langle \mathcal{F}'(u_j), v_j \rangle = \langle \mathcal{F}'|_{K_i}(u_j), w_j \rangle = o(1),$$

Since w_j is uniformly bounded and $\mathcal{F}'|_{K_i}(u_k) \rightarrow 0$ strongly. Now the result follows from Lemma 6.18. \square

The following lemma now follows easily.

Lemma 6.20. *Let $u \in K_i$ be a critical point of the restricted functional $\mathcal{F}|_{K_i}$. Then u is also a critical point of the unrestricted functional \mathcal{F} and hence a weak solution to (6.9).*

Proof. To prove the Theorem, we need to check that the functional $\mathcal{F}|_{K_i}$ verifies the hypotheses of Ekeland's Variational Principle, Corollary 3.28. But this follows directly by the construction of the manifolds K_i .

Then, by Corollary 3.28, there exist $v_k \in K_i$ such that

$$\mathcal{F}(v_k) \rightarrow c_i := \inf_{K_i} \mathcal{F} \quad \text{and} \quad (\mathcal{F}|_{K_i})'(v_k) \rightarrow 0.$$

We have to check that if we choose λ large, we have that $c_i < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}}\right)\bar{\mathcal{S}}^N$. This follows easily from Lemma 6.13. For instance, for c_1 , we have that choose $w_0 \geq 0$, if $t_\lambda < 1$

$$c_1 \leq \mathcal{F}(t_\lambda w_0) \leq \frac{1}{p^-} t_\lambda^{p^+} \int_{\Omega} |\nabla w_0|^{p(x)} dx$$

Hence $c_1 \rightarrow 0$ as $\lambda \rightarrow \infty$. Moreover, it follows from the estimate of t_λ in Lemma 6.13, that $c_i < \left(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}}\right)\bar{\mathcal{S}}^N$ for $\lambda > \lambda^*(p, q, n, c_3)$. The other cases are similar.

From Lemma 6.19, it follows that v_k has a convergent subsequence, that we still call v_k . Therefore \mathcal{F} has a critical point in K_i , $i = 1, 2, 3$ and, by construction, one of them is positive, other is negative and the last one changes sign. \square

7

Existence results for critical elliptic equations via local conditions

In this final chapter, we continue with our study of existence of solutions for critical equations, in the sense of the Sobolev embeddings, when the elliptic operator is the $p(x)$ -laplacian.

As we discuss in the former chapter, it is not difficult to prove that the associated functional verifies the geometrical assumptions of the Mountain Pass Theorem and also that the Palais–Smale condition is verified below some critical energy level c^* . Therefore, the main difficulty is to exhibit some Palais–Smale sequence with energy below the critical level c^* .

In Chapter 6 this was done by perturbing the equation with a compact term and then showing that if the perturbation is chosen appropriately then the desired result holds.

In this Chapter, we follow a different line of approach, more closely to the one in the works of [5, 9]. That is, we treat the unperturbed problem and try to find *local conditions* on the exponents and on the coefficients of the equation in order to obtain the result.

To be precise, we consider two types of problems, with Dirichlet boundary conditions and with a nonlinear boundary conditions. i.e. we consider the equation

$$\begin{cases} -\Delta_{p(x)}u + h(x)|u|^{p(x)-2}u = |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (7.1)$$

and

$$\begin{cases} -\Delta_{p(x)}u + h(x)|u|^{p(x)-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial n} = |u|^{r(x)-2}u & \text{on } \partial\Omega, \end{cases} \quad (7.2)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p, q \in \mathcal{P}(\Omega)$, $r \in \mathcal{P}(\partial\Omega)$, $1 < p^- \leq p^+ < N$ and q and r are critical in the sense that $\mathcal{A} \neq \emptyset$, $\mathcal{A}_T \neq \emptyset$ where, as usual \mathcal{A} and \mathcal{A}_T are the critical sets of q and r respectively.

So, the rest of the chapter is divided into two sections. The first one is devoted to the study of problem (7.1) and the second one to (7.2).

7.1 Critical equation with Dirichlet boundary conditions

As we mentioned in the introduction of this chapter, in order to study (7.1) by means of variational methods, we analyze the functional $\mathcal{F} : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$ which is defined as

$$\mathcal{F}(u) := \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + h(x)|u|^{p(x)}) dx - \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx. \quad (7.3)$$

This functional is naturally associated to (7.1) in the sense that weak solutions of (7.1) are critical points of \mathcal{F} .

We need to assume that the smooth function h is such that the functional

$$\mathcal{J}(u) := \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + h(x)|u|^{p(x)}) dx \quad (7.4)$$

is coercive in the sense that the norm

$$\|u\| := \inf \left\{ \lambda > 0 \int_{\Omega} \left| \frac{\nabla u + h(x)u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

is equivalent to the usual norm of $W_0^{1,p(x)}(\Omega)$.

Under the assumption that $p^+ < q^-$, it is easy to show, similar to the previous chapter, that \mathcal{F} satisfies the geometric assumptions of the Mountain–Pass Theorem. Hence if we assume moreover that the exponent $q(x)$ is subcritical then \mathcal{F} satisfies the Palais–Smale condition, and the existence of a nontrivial solution to (7.1) follows easily.

When $q(x)$ is critical, again as in Chapter 6, we can prove that the functional \mathcal{F} verifies the Palais–Smale condition below some critical value.

Theorem 7.1. *Assume that $p^+ < q^-$. Then the functional \mathcal{F} satisfies the Palais–Smale condition at level $c \in (0, \frac{1}{N} \bar{S}^N)$ where \bar{S} is given in (5.4).*

Proof. The scheme of the proof is classical (see e.g. [49]) and relies on the concentration–compactness principle proved in Chapter 4, Theorem 4.1.

Let $\{u_k\}_{k \in \mathbb{N}} \subset W^{1,p(x)}(\Omega)$ be a Palais–Smale sequence for \mathcal{F} . Recalling that the functional \mathcal{J} defined by (7.4) is assumed to be coercive, it then follows that $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $W^{1,p(x)}(\Omega)$. In fact, for k large, we have that

$$\begin{aligned} c + 1 + o(1) \|\nabla u_k\|_{p(x)} &\geq \mathcal{J}(u_k) - \frac{1}{q^-} \langle \mathcal{F}'(u_k), u_k \rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |\nabla u_k|^{p(x)} + h(x)|u_k|^{p(x)} dx - \int_{\Omega} \left(\frac{1}{q(x)} - \frac{1}{q^-} \right) |u_k|^{q(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |\nabla u_k|^{p(x)} + h(x)|u_k|^{p(x)} dx. \end{aligned}$$

from where the claim follows recalling that $p^+ < q^-$. We may thus assume that $u_k \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$. We claim that u turns out to be a weak solution to (7.1). The proof of this fact follows closely the one in [49] and this argument is taken from [13, 20], where the constant exponent case is treated.

In fact, since $\{u_k\}_{k \in \mathbb{N}}$ is a Palais–Smale sequence, we have that

$$\langle \mathcal{F}'(u_k), v \rangle = \int_{\Omega} |\nabla u_k|^{p(x)-2} \nabla u_k \nabla v \, dx + \int_{\Omega} h |u_k|^{p(x)-2} u_k v \, dx - \int_{\Omega} |u_k|^{q(x)-2} u_k v \, dx = o(1),$$

for any $v \in C_c^\infty(\Omega)$. Without loss of generality, we can assume that $u_k \rightarrow u$ a.e. in Ω and in $L^{p(x)}(\Omega)$ because $\{u_k\}$ is a Palais Smale sequence, so it is bounded. It is easy to see, from standard integration theory, that

$$\int_{\Omega} h |u_k|^{p(x)-2} u_k v \, dx \rightarrow \int_{\Omega} h |u|^{p(x)-2} u v \, dx \quad \text{and} \quad \int_{\Omega} |u_k|^{q(x)-2} u_k v \, dx \rightarrow \int_{\Omega} |u|^{q(x)-2} u v \, dx,$$

so the claim will follow if we show that

$$\int_{\Omega} |\nabla u_k|^{p(x)-2} \nabla u_k \nabla v \, dx \rightarrow \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx.$$

This is a consequence of the monotonicity of the $p(x)$ -Laplacian. We can assume that there exist $\xi \in (L^{p'(x)}(\Omega))^N$ such that

$$|\nabla u_k|^{p(x)-2} \nabla u_k \rightharpoonup \xi \quad \text{weakly in } L^{p'(x)}(\Omega).$$

The idea is to show that $\nabla u_k \rightarrow \nabla u$ a.e. in Ω , then this will imply that $\xi = |\nabla u|^{p(x)-2} \nabla u$ and thus, the claim.

Let $\delta > 0$ then, by Egoroff's Theorem, there exists $E_\delta \subset \Omega$ such that $|\Omega \setminus E_\delta| < \delta$ and $u_k \rightarrow u$ uniformly in E_δ . As a consequence, given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $|u_k(x) - u(x)| < \varepsilon/2$ for $x \in E_\delta$ and for any $k \geq k_0$.

Define the truncation β_ε as

$$\beta_\varepsilon(t) = \begin{cases} -\varepsilon & \text{if } t \leq -\varepsilon \\ t & \text{if } -\varepsilon < t < \varepsilon \\ \varepsilon & \text{if } t \geq \varepsilon. \end{cases}$$

Now we make use of the following well known monotonicity inequality

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq 0 \tag{7.5}$$

which is valid for any $x, y \in \mathbb{R}^N$ and $p \geq 1$ and we obtain

$$(|\nabla u_k|^{p(x)-2} \nabla u_k - |\nabla u|^{p(x)-2} \nabla u) \nabla \beta_\varepsilon(u_k - u) \geq 0,$$

since $\nabla \beta_\varepsilon(u_k - u) = \nabla u_k - \nabla u$ in E_δ and $\nabla \beta_\varepsilon(u_k - u) = 0$ in $\Omega \setminus E_\delta$. Therefore, we obtain

$$\int_{E_\delta} (|\nabla u_k|^{p(x)-2} \nabla u_k - |\nabla u|^{p(x)-2} \nabla u) (\nabla u_k - \nabla u) \, dx \leq \int_{\Omega} (|\nabla u_k|^{p(x)-2} \nabla u_k - |\nabla u|^{p(x)-2} \nabla u) \nabla \beta_\varepsilon(u_k - u) \, dx.$$

Now, observe that $\beta_\varepsilon(u_k - u) \rightharpoonup 0$ weakly in $W_0^{1,p(x)}(\Omega)$ and so

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \beta_\varepsilon(u_k - u) dx \rightarrow 0.$$

Now, for k sufficiently large, we obtain that

$$\int_{\Omega} |\nabla u_k|^{p(x)-2} \nabla u_k \nabla \beta_\varepsilon(u_k - u) dx \leq C\varepsilon$$

for some constant $C > 0$. In fact, since $\beta_\varepsilon(u_k - u)$ is bounded in $W_0^{1,p(x)}(\Omega)$,

$$\langle \mathcal{F}'(u_k), \beta_\varepsilon(u_k - u) \rangle = o(1),$$

so that

$$\int_{\Omega} |\nabla u_k|^{p(x)-2} \nabla u_k \nabla \beta_\varepsilon(u_k - u) dx = o(1) + I_1 + I_2,$$

where

$$|I_1| = \left| \int_{\Omega} |u_k|^{q(x)-2} u_k \beta_\varepsilon(u_k - u) dx \right| \leq \varepsilon \int_{\Omega} |u_k|^{q(x)-1} dx \leq C\varepsilon$$

and

$$|I_2| = \left| \int_{\Omega} h |u_k|^{p(x)-2} u_k \beta_\varepsilon(u_k - u) dx \right| \leq \varepsilon \|h\|_{\infty} \int_{\Omega} |u_k|^{p(x)-1} dx \leq C\varepsilon.$$

As a consequence, we get that

$$0 \leq \limsup_{k \rightarrow \infty} \int_{E_\delta} (|\nabla u_k|^{p(x)-2} \nabla u_k - |\nabla u|^{p(x)-2} \nabla u)(\nabla u_k - \nabla u) dx \leq C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $(|\nabla u_k|^{p(x)-2} \nabla u_k - |\nabla u|^{p(x)-2} \nabla u)(\nabla u_k - \nabla u) \rightarrow 0$ strongly in $L^1(E_\delta)$ and thus, up to a subsequence, also a.e. in E_δ . By a standard diagonal argument, we can assume that $(|\nabla u_k|^{p(x)-2} \nabla u_k - |\nabla u|^{p(x)-2} \nabla u)(\nabla u_k - \nabla u) \rightarrow 0$ a.e. in E_δ for every $\delta > 0$ and so the convergence holds a.e. in Ω .

Finally, it is easy to see that $(|x_k|^{p-2} x_k - |x|^{p-2} x)(x_k - x) \rightarrow 0$ for $x_k, x \in \mathbb{R}^N$ and $p > 1$ imply that $x_k \rightarrow x$, so we get that $\nabla u_k \rightarrow \nabla u$ a.e. in Ω . This concludes the proof of the claim.

By Theorem 4.1 it holds that

$$\begin{aligned} |u_k|^{q(x)} &\rightharpoonup \nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i} \quad \text{weakly in the sense of measures,} \\ |\nabla u_k|^{p(x)} &\rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i} \quad \text{weakly in the sense of measures,} \\ \bar{S}_{x_i} \nu_i^{1/p^*(x_i)} &\leq \mu_i^{1/p(x_i)}, \end{aligned}$$

where I is a countable set, $\{\nu_i\}_{i \in I}$ and $\{\mu_i\}_{i \in I}$ are positive numbers and the points $\{x_i\}_{i \in I}$ belong to the critical set \mathcal{A} .

It is not difficult to check that $v_k := u_k - u$ is Palais–Smale sequence for $\tilde{\mathcal{F}}(v) := \mathcal{F}(v) - \int_{\Omega} \frac{1}{p(x)} h|v|^{p(x)}$. Now, by Lemma 4.5 we get

$$\begin{aligned} \mathcal{F}(u_k) - \mathcal{F}(u) &= \int_{\Omega} \frac{1}{p(x)} [|\nabla v_k|^{p(x)} + h|v_k|^{p(x)}] dx - \int_{\Omega} \frac{1}{q(x)} |v_k|^{q(x)} dx + o(1) \\ &= \tilde{\mathcal{F}}(v_k) + \int_{\Omega} \frac{1}{p(x)} h|v_k|^{p(x)} dx + o(1) \\ &= \tilde{\mathcal{F}}(v_k) + o(1). \end{aligned}$$

Since u is a weak solution of (7.1), and since $p^+ < q^-$,

$$\begin{aligned} \mathcal{F}(u) &\geq \frac{1}{p^+} \int_{\Omega} (|\nabla u|^{p(x)} + h(x)|u|^{p(x)}) dx - \frac{1}{q^-} \int_{\Omega} |u|^{q(x)} dx \\ &= \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\Omega} |u|^{q(x)} dx \\ &\geq 0. \end{aligned}$$

Therefore,

$$\mathcal{F}(u_k) \geq \tilde{\mathcal{F}}(v_k) + o(1).$$

Let $\phi \in C_c^\infty(\Omega)$. As $\tilde{\mathcal{F}}'(v_k) \rightarrow 0$, we have

$$\begin{aligned} o(1) &= \langle \tilde{\mathcal{F}}'(v_k), v_k \phi \rangle \\ &= \int_{\Omega} |\nabla v_k|^{p(x)} \phi dx - \int_{\Omega} |v_k|^{q(x)} \phi dx + \int_{\Omega} |\nabla v_k|^{p(x)-2} \nabla v_k \nabla \phi v_k dx \\ &= A - B + C. \end{aligned}$$

Since $v_k \rightarrow 0$ weakly in $W_0^{1,p(x)}(\Omega)$ it is easy to see that $C \rightarrow 0$ as $k \rightarrow \infty$. By means of Lemma 4.5 it follows that

$$A \rightarrow \int_{\Omega} \phi d\tilde{\mu} \quad \text{and} \quad B \rightarrow \int_{\Omega} \phi d\tilde{\nu},$$

where $\tilde{\mu} = \mu - |\nabla u|^{p(x)}$ and $\tilde{\nu} = \nu - |u|^{q(x)}$. So we conclude that $\tilde{\mu} = \tilde{\nu}$. In particular $\nu_i \geq \mu_i$ ($i \in I$) from where we obtain that $\nu_i \geq \bar{S}^N$. Hence

$$\begin{aligned} c &= \lim_{k \rightarrow \infty} \mathcal{F}(u_k) \geq \lim_{k \rightarrow \infty} \tilde{\mathcal{F}}(v_k) = \int \frac{1}{p(x)} d\tilde{\mu} - \int \frac{1}{q(x)} d\tilde{\nu} \\ &= \int \left(\frac{1}{p(x)} - \frac{1}{q(x)} \right) d\tilde{\nu} = \sum_{i \in I} \left(\frac{1}{p(x_i)} - \frac{1}{p^*(x_i)} \right) \nu_i \\ &\geq \#(I) \frac{1}{N} \bar{S}^N. \end{aligned}$$

We deduce that if $c < \frac{1}{N} \bar{S}^N$ then I must be empty. This implies that $u_k \rightarrow u$ strongly in $W^{1,p(x)}(\Omega)$. \square

As a corollary, we can apply the Mountain–Pass Theorem to obtain the following necessary existence condition:

Theorem 7.2. *If there exists $v \in W_0^{1,p(x)}(\Omega)$ such that*

$$\sup_{t>0} \mathcal{F}(tv) < \frac{1}{N} \bar{S}^N, \quad (7.6)$$

then (7.1) has a non-trivial nonnegative solution.

Proof. The proof is an immediate consequence of the Mountain–Pass Theorem, Theorem 7.1 and assumption (7.6).

In fact, it suffices to verify that \mathcal{F} has the Mountain–Pass geometry and that $\mathcal{F}(tu) < 0$ for some $t > 0$. Concerning the latter condition notice that for $t > 1$,

$$\begin{aligned} \mathcal{F}(tu) &= \int_{\Omega} \frac{t^{p(x)}}{p(x)} (|\nabla u|^{p(x)} + h(x)|u|^{p(x)}) dx - \int_{\Omega} \frac{t^{q(x)}}{q(x)} |\nabla u|^{q(x)} dx \\ &\leq t^{p^+} \mathcal{J}(u) - t^{q^-} \int_{\Omega} \frac{1}{q(x)} |\nabla u|^{q(x)} dx, \end{aligned}$$

which tends to $-\infty$ as $t \rightarrow +\infty$ since $q^- > p^+$.

It remains to see that \mathcal{F} has the Mountain–Pass geometry. But $\mathcal{F}(0) = 0$ and, if $\|\nabla v\|_{p(x),\Omega} = r$ small enough, then

$$\int_{\Omega} |\nabla v|^{p(x)} + h|v|^{p(x)} dx \geq c_1 \|\nabla v\|_{p(x),\Omega}^{p^+}$$

and

$$\|v\|_{q(x),\Omega} \leq C \|\nabla v\|_{p(x),\Omega} = Cr < 1,$$

so

$$\int_{\Omega} |v|^{q(x)} dx \leq c_2 \|\nabla v\|_{p(x),\Omega}^{q^-}.$$

Therefore

$$\mathcal{F}(v) \geq \frac{c_1}{p^+} r^{p^+} - \frac{c_2}{q^-} r^{q^-} > 0,$$

since $p^+ < q^-$. This completes the proof. \square

Eventually the following result provide a sufficient local condition for (7.6) to hold:

Theorem 7.3. *Assume that there exists a point $x_0 \in \mathcal{A}$ such that $\bar{S}_{x_0} = \bar{S}$ and that x_0 is a local minimum of p and a local maximum of q . In particular*

$$-\Delta p(x_0) \leq 0 \leq -\Delta q(x_0). \quad (7.7)$$

Assume moreover that p, q are C^2 in a neighborhood of x_0 , and that $h(x_0) < 0$ if $1 < p(x_0) < 2$ ($N \geq 4$), or if $2 \leq p(x_0) < \sqrt{N}$ ($N \geq 5$), that at least one of the two inequalities in (7.7) is strict, but $h(x_0)$ is arbitrary. Under these assumptions (7.6) holds. In particular (7.1) has a non-trivial nonnegative solution.

Remark 7.4. In the constant exponent case, the well known Pohozaev obstruction [46] affirms that if $h \geq 0$ and Ω is starshaped then there are no (positive) solutions to (7.1). Our result shows that for variable p and q and $p(x) \geq 2$ this does not need to be the case, showing a striking difference between the constant exponent case and the variable exponent one.

Proof. Let $x_0 \in \mathcal{A}$ be such that

$$\bar{S} := \inf_{x \in \mathcal{A}} \lim_{\varepsilon \rightarrow 0} S(p(\cdot), q(\cdot), B_\varepsilon(x)) = \lim_{\varepsilon \rightarrow 0} S(p(\cdot), q(\cdot), B_\varepsilon(x_0)) = \bar{S}_{x_0}.$$

For ease of notation we assume that $x_0 = 0$, write $p = p(0)$ and observe that $q = q(0) = p^*$. From Theorem 5.12, we have that if 0 is a local maximum of q and a local minimum of p , then

$$\bar{S} = \bar{S}_0 = K(N, p)^{-1}.$$

Let U be an extremal for the constant $K(N, p)$. It follows that U verifies

$$-\Delta_p U = \frac{K(N, p)^{-p}}{\|U\|_{p^*, \mathbb{R}^N}^{p^*-p}} U^{p^*-1} = C U^{p^*-1}.$$

Then $W = C^{\frac{1}{p^*-p}} U = \frac{K(N, p)^{-\frac{N-p}{p}}}{\|U\|_{p^*}} U$ solves $-\Delta_p W = W^{p^*-1}$ and satisfy

$$\|\nabla W\|_{p, \mathbb{R}^N} = K(N, p)^{-N/p}.$$

Consider the test function

$$w_\varepsilon(x) = \varepsilon^{-\frac{N-p}{p}} W\left(\frac{x}{\varepsilon}\right) \eta(x) = C^{\frac{1}{p^*-p}} u_\varepsilon(x).$$

From Proposition A.1, we obtain the following asymptotic expansions

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{t^{q(x)}}{q(x)} w_\varepsilon^{q(x)} dx &= \frac{t^{p^*}}{p^*} K(N, p)^{-N} + \frac{t^{p^*}}{p^*} A \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon), \\ \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} |\nabla w_\varepsilon|^{p(x)} dx &= \frac{t^p}{p} K(N, p)^{-N} + \frac{t^p}{p} B \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon), \\ \int_{\mathbb{R}^N} h(x) \frac{t^{p(x)}}{p(x)} |w_\varepsilon|^{p(x)} dx &= h(0) \frac{t^p}{p} C \varepsilon^p + o(\varepsilon^p), \end{aligned} \quad (7.8)$$

with

$$\begin{aligned} A &= -\frac{\Delta q(0)}{2p^*} K(N, p)^{-N} \|U\|_{p^*}^{-p^*} \int_{\mathbb{R}^N} |x|^2 U^{p^*} dx, \\ B &= -\frac{\Delta p(0)}{2p} K(N, p)^{p-N} \|U\|_{p^*}^{-p} \int_{\mathbb{R}^N} |x|^2 |\nabla U|^p dx, \\ C &= K(N, p)^{p-N} \|U\|_{p^*}^{-p} \|U\|_p^p. \end{aligned}$$

Using w_ε as a test-function in (7.6) we can see that there exists $t_0 > 1$ such that $\mathcal{F}(tw_\varepsilon) < 0$ for $t > t_0$. Now if $p < 2$, we can write

$$f_\varepsilon(t) := \mathcal{F}(tw_\varepsilon) = f_0(t) + \varepsilon^p f_1(t) + o(\varepsilon^p)$$

C^1 -uniformly in $t \in [0, t_0]$, with

$$f_0(t) = K(N, p)^{-N} \left(\frac{t^p}{p} - \frac{t^{p^*}}{p^*} \right), \quad \text{and} \quad f_1(t) = \frac{1}{p} t^p h(0) C.$$

Notice that f_0 reaches its maximum in $[0, t_0]$ at $t = 1$. Moreover it is a nondegenerate maximum since $f_0''(1) = (p - p^*)K(N, p)^{-N} \neq 0$. It follows that f_ε reaches a maximum at $t_\varepsilon = 1 + a\varepsilon^p + o(\varepsilon^p)$ for $a = -\frac{f_1'(1)}{f_0''(1)}$. Hence

$$\sup_{t>0} \mathcal{F}(tw_\varepsilon) = \mathcal{F}(t_\varepsilon w_\varepsilon) = \frac{1}{N} K(N, p)^{-N} + f_1(1)\varepsilon^p + o(\varepsilon^p)$$

Then if $h(0) < 0$ we get $\sup_{t>0} \mathcal{F}(tw_\varepsilon) < \frac{1}{N} K(N, p)^{-N}$.

We now assume that $p \geq 2$. Then

$$f_\varepsilon(t) = \mathcal{F}(tw_\varepsilon) = f_0(t) + f_2(t)\varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon),$$

C^1 -uniformly in $t \in [0, t_0]$, with

$$f_2(t) = \frac{t^{p^*}}{p^*} A - \frac{t^p}{p} B.$$

As before f_ε reaches its maximum at $t_\varepsilon = 1 + a\varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon)$ with $a = -\frac{f_2'(1)}{f_0''(1)}$. Hence

$$\begin{aligned} \sup_{t>0} \mathcal{F}(tw_\varepsilon) &= \mathcal{F}(t_\varepsilon w_\varepsilon) = f_0(1) + f_2(1)\varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) \\ &= \frac{1}{N} K(N, p)^{-N} + f_2(1)\varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon). \end{aligned}$$

We thus need $f_2(1) < 0$ i.e.

$$-\Delta p(0) < -\Delta q(0)(p/p^*)^2 D(N, p), \quad \text{where} \quad D(N, p) := \frac{\int_{\mathbb{R}^N} |\nabla U|^p dx \int_{\mathbb{R}^N} |x|^2 U^{p^*} dx}{\int_{\mathbb{R}^N} U^{p^*} dx \int_{\mathbb{R}^N} |x|^2 |\nabla U|^p dx}. \quad (7.9)$$

Since 0 is a local maximum of q and a local minimum of p we already know that (7.7) holds. Then if one of the two inequalities in (7.7) is strict we see that (7.9) holds.

This ends the proof of Theorem 7.3. \square

Remark 7.5. As a final remark, we notice that we can compute $D(N, p)$ exactly. To do this let

$$I_p^q := \int_0^\infty t^{q-1} (1+t)^{-p} dt = B(q, p-q) = \frac{\Gamma(q)\Gamma(p-q)}{\Gamma(p)}, \quad (7.10)$$

where $B(x, y) := \int_0^\infty t^{x-1}(1+t)^{-x-y} dt$ is the Beta function. This formula can be found, for instance, in [7]. Passing to spherical coordinates and then performing the change of variable $t = r^{\frac{p}{p-1}}$, $dr = \frac{p-1}{p} t^{-\frac{1}{p}} dt$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} U^{p^*} dx &= \omega_{N-1} \frac{p-1}{p} I_N^{N \frac{p-1}{p}}, \\ \int_{\mathbb{R}^N} |x|^2 U^{p^*} dx &= \omega_{N-1} \frac{p-1}{p} I_N^{N \frac{p-1}{p} - \frac{2}{p} + 2}, \\ \int_{\mathbb{R}^N} |\nabla U|^p dx &= \omega_{N-1} \frac{p-1}{p} \left(\frac{N-p}{p-1} \right)^p I_N^{N \frac{p-1}{p} + 1}, \\ \int_{\mathbb{R}^N} |x|^2 |\nabla U|^p dx &= \omega_{N-1} \frac{p-1}{p} \left(\frac{N-p}{p-1} \right)^p I_N^{N \frac{p-1}{p} - \frac{2}{p} + 3}. \end{aligned}$$

Then

$$D(N, p) = \frac{I_N^{\frac{N(p-1)}{p} + 1} I_N^{\frac{N(p-1)}{p} - \frac{2}{p} + 2}}{I_N^{\frac{N(p-1)}{p}} I_N^{\frac{N(p-1)}{p} - \frac{2}{p} + 3}} = \frac{N}{N-p} \frac{(N-p) - 2(p-1)}{N+2},$$

where we used that

$$I_p^{q+1} = \frac{q}{p-q-1} I_p^q$$

which follows from (7.10) and the formula $\Gamma(z+1) = z\Gamma(z)$.

7.2 Critical equation with nonlinear boundary conditions

In order to study (7.2) by means of variational methods, we need to consider the functional $\mathcal{F}: W^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mathcal{F}(u) := \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + h(x)|u|^{p(x)}) dx - \int_{\partial\Omega} \frac{1}{r(x)} |u|^{r(x)} dS. \quad (7.11)$$

Again, $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (7.2) if and only if u is a critical point of \mathcal{F} .

We need to assume that the smooth function h is such that the functional \mathcal{F} given in (7.4) is coercive in $W^{1,p(x)}(\Omega)$.

Our first result provides a condition for the functional \mathcal{F} defined by (7.11) to satisfy the Palais–Smale condition.

Theorem 7.6. *The functional \mathcal{F} satisfies the Palais–Smale condition at level*

$$0 < c < \inf_{x \in \mathcal{A}_T} \left(\frac{1}{p(x)} - \frac{1}{p_*(x)} \right) \bar{T}_x^{\frac{p(x)p_*(x)}{p_*(x)-p(x)}}.$$

Proof. Let $\{u_k\}_{k \in \mathbb{N}} \subset W^{1,p(x)}(\Omega)$ be a Palais–Smale sequence for \mathcal{F} .

Recalling that the functional \mathcal{F} defined by (7.4) is assumed to be coercive, it then follows, arguing as in the proof of Theorem 7.1 that $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $W^{1,p(x)}(\Omega)$.

We may thus assume that $u_k \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. Again, arguing as in the proof of Theorem 7.1 it follows that u is a weak solution to (7.2).

By the Concentration Compactness Principle for variable exponents, Theorem 4.6, it holds that

$$\begin{aligned} |u_k|^{r(x)} dS &\rightharpoonup \nu = |u|^{r(x)} dS + \sum_{i \in I} \nu_i \delta_{x_i} \quad \text{weakly in the sense of measures,} \\ |\nabla u_k|^{p(x)} dx &\rightharpoonup \mu \geq |\nabla u|^{p(x)} dx + \sum_{i \in I} \mu_i \delta_{x_i} \quad \text{weakly in the sense of measures,} \\ \bar{T}_{x_i} \nu_i^{1/p^*(x_i)} &\leq \mu_i^{1/p(x_i)}, \end{aligned}$$

where I is a countable set, $\{\nu_i\}_{i \in I}$ and $\{\mu_i\}_{i \in I}$ are positive numbers and the points $\{x_i\}_{i \in I}$ belong to the critical set $\mathcal{A}_T \subset \partial\Omega$.

The same argument used in the proof of Theorem 7.1 leads us to conclude that $\nu_i \geq \mu_i$ ($i \in I$) from where we obtain that $\nu_i \geq \bar{T}_{x_i}^{\frac{p(x_i)p_*(x_i)}{p_*(x_i)-p(x_i)}}$. Hence

$$\begin{aligned} c &= \lim_{k \rightarrow \infty} \mathcal{F}(u_k) \geq \int \frac{1}{p(x)} d\mu - \int \frac{1}{r(x)} d\nu \geq \sum_{i \in I} \left(\frac{1}{p(x_i)} - \frac{1}{p_*(x_i)} \right) \nu_i \\ &\geq \#(I) \inf_{x \in \mathcal{A}_T} \left(\frac{1}{p(x)} - \frac{1}{p_*(x)} \right) \bar{T}_x^{\frac{p(x)p_*(x)}{p_*(x)-p(x)}}. \end{aligned}$$

We deduce that if $c < \inf_{x \in \mathcal{A}_T} \left(\frac{1}{p(x)} - \frac{1}{p_*(x)} \right) \bar{T}_x^{\frac{p(x)p_*(x)}{p_*(x)-p(x)}}$ then I must be empty implying that $u_k \rightarrow u$ strongly in $W^{1,p(\cdot)}(\Omega)$. \square

As a corollary, we can apply the Mountain–Pass Theorem to obtain the following necessary existence condition:

Theorem 7.7. *If there exists $v \in W^{1,p(x)}(\Omega)$ such that*

$$\sup_{t>0} \mathcal{F}(tv) < \inf_{x \in \mathcal{A}_T} \left(\frac{1}{p(x)} - \frac{1}{p_*(x)} \right) \bar{T}_x^{\frac{p(x)p_*(x)}{p_*(x)-p(x)}} \quad (7.12)$$

then (7.2) has a non-trivial nonnegative solution.

Proof. The proof is an immediate consequence of the Mountain–Pass Theorem, Theorem 7.6 and assumption (7.12).

In fact, it suffices to verify that \mathcal{F} has the Mountain–Pass geometry and that $\mathcal{F}(tu) < 0$ for some $t > 0$. But these facts follows in exactly the same manner as in Theorem 7.2 using the fact that $p^+ < r^-$. \square

In order to find local conditions that imply (7.12) we need to recall the *Fermi coordinates* that are described in the Appendix in Definition A.6 and Lemma A.7.

Briefly speaking, the Fermi coordinates describe a neighborhood of a point $x_0 \in \partial\Omega$ with variables (y, t) where $y \in \mathbb{R}^{N-1}$ are the coordinates in a local chart of $\partial\Omega$ such that $y = 0$ corresponds to x_0 and $t > 0$ is the distance along the unit inward normal vector.

Eventually the following result provides a sufficient local condition for (7.12) to hold:

Theorem 7.8. *Assume that there exists a point $x_0 \in \mathcal{A}_T$ such that $\bar{T} = \bar{T}_{x_0}$ and such that x_0 is a local minimum of $p(x)$ and a local maximum of $r(x)$ and $p(x_0) < \min\{\sqrt{N}, \frac{N^2}{3N-2}\}$. Moreover assume that one of the following conditions hold*

1. $\frac{\partial p}{\partial t}(x_0) > 0$,
2. $\frac{\partial p}{\partial t}(x_0) = 0$ and $H(x_0) > 0$ or
3. $\frac{\partial p}{\partial t}(x_0) = 0$, $H(x_0) = 0$, $1 < p(x_0) < 2$ and $h(x_0) < 0$ or
4. $\frac{\partial p}{\partial t}(x_0) = 0$, $H(x_0) = 0$, $p(x_0) \geq 2$ and $\Delta p(x_0) > 0$ or $\Delta_y r(x_0) < 0$.

Then there exists a nontrivial solution to (7.2). Here, by $\frac{\partial}{\partial t}$ and Δ_y we refer to derivatives with respect to the Fermi coordinates.

Proof. We assume, without loss of generality that $x_0 = 0$ and denote $p = p(0)$. Observe that $r(0) = p_*$.

The idea, similar to Theorem 7.2, is to evaluate (7.12) with $v = Cv_\varepsilon$ where v_ε is the function described in Section A.2 that is constructed by rescaling and truncating properly the extremal for $\bar{K}(N, p)^{-1}$ in Fermi coordinates.

In fact, we consider $z_\varepsilon = Cv_\varepsilon$ with $C = \bar{K}(N, p)^{-\frac{p}{p_*-p}} \|V\|_{p_*, \partial\mathbb{R}_+^N}^{-1}$. This constant C is chosen so that if $Z(y, t) = CV(y, t)$ where V is the extremal for $\bar{K}(N, p)^{-1}$, then

$$\int_{\mathbb{R}_+^N} |\nabla Z|^p dy dt = \int_{\partial\mathbb{R}_+^N} |Z|^{p_*} dy = \bar{K}(N, p)^{-\frac{pp_*}{p_*-p}}.$$

We first consider the case where $\partial_t p(0) > 0$. In fact, from Propositions A.8, A.9 and A.10, we have

$$\begin{aligned} f_\varepsilon(s) &= \mathcal{F}(sz_\varepsilon) = \bar{D}_0 + \bar{D}_1 \varepsilon \ln \varepsilon - \bar{A}_0 + o(\varepsilon \ln \varepsilon) \\ &= f_0(s) + \varepsilon \ln \varepsilon f_1(s) + O(\varepsilon) \end{aligned}$$

C^1 - uniformly in $s \in [0, s_0]$, with

$$f_0(s) = \bar{K}(N, p)^{-\frac{pp_*}{p_*-p}} \left(\frac{s^p}{p} - \frac{s^{p_*}}{p_*} \right)$$

and

$$f_1(s) = -\frac{N}{p} \frac{s^p}{p} \partial_t p(0) \int_{\mathbb{R}_+^N} t |\nabla Z|^p dydt$$

Notice that f_0 reaches its maximum in $[0, s_0]$ at $s = 1$. Moreover, it is a nodedegenerate maximum since $f_0''(1) = (p - p_*) \bar{K}(N, p)^{-\frac{p_* p}{p_* - p}} \neq 0$. It follows that f_ε reaches a maximum at $s_\varepsilon = 1 + a\varepsilon \ln \varepsilon + O(\varepsilon)$ for $a = -\frac{f_1'(1)}{f_0''(1)}$. Hence

$$\sup_{s>0} \mathcal{F}(sz_\varepsilon) = \mathcal{F}(s_\varepsilon z_\varepsilon) = \left(\frac{1}{p} - \frac{1}{p_*} \right) \bar{K}(N, p)^{-\frac{p_* p}{p_* - p}} + f_1(1) \varepsilon \ln \varepsilon + O(\varepsilon)$$

If $\partial_t p(0) > 0$ then $f_1(1) < 0$ and the result follows.

Assume now that $\partial_t p(0) = 0$ and $H(0) > 0$. Then we have

$$\begin{aligned} \mathcal{F}(sz_\varepsilon) &= \bar{D}_0 + \bar{D}_2 \varepsilon + o(\varepsilon) - \bar{A}_0 \\ &= f_0(s) + f_2(s) \varepsilon + o(\varepsilon) \end{aligned}$$

C^1 - uniformly in $[0, s_0]$, with

$$f_2(s) = -H(0) \frac{s^p}{p} \int_{\mathbb{R}_+^N} t |\nabla Z|^p dydt + \frac{H(0)}{N-1} s^p \int_{\mathbb{R}_+^N} \frac{t|y|^2}{r^2} |\nabla Z|^p dydt$$

As before f_ε reaches its maximum at $s_\varepsilon = 1 + a\varepsilon + o(\varepsilon)$ with $a = \frac{f_2'(1)}{f_0''(1)}$. So,

$$\sup_{s>0} \mathcal{F}(sz_\varepsilon) = \mathcal{F}(s_\varepsilon z_\varepsilon) = \left(\frac{1}{p} - \frac{1}{p_*} \right) \bar{K}(N, p)^{-\frac{p_* p}{p_* - p}} + f_2(1) \varepsilon + o(\varepsilon)$$

So, we need that $f_2(1) < 0$, i.e.

$$-H(0) \frac{1}{p} \int_{\mathbb{R}_+^N} t |\nabla Z|^p dydt + \frac{H(0)}{N-1} \int_{\mathbb{R}_+^N} \frac{t|y|^2}{r^2} |\nabla Z|^p dydt < 0.$$

But,

$$\begin{aligned} -\frac{1}{p} \int_{\mathbb{R}_+^N} t |\nabla Z|^p dydt + \frac{1}{N-1} \int_{\mathbb{R}_+^N} \frac{t|y|^2}{r^2} |\nabla Z|^p dydt &\leq \\ \left(-\frac{1}{p} + \frac{1}{N-1} \right) \int_{\mathbb{R}_+^N} t |\nabla Z|^p dydt &< 0 \end{aligned}$$

if $p < N - 1$. So, since $H(0) > 0$, the result follows.

Now suppose that $\partial_t p(0) = 0$ and $H(0) = 0$. Then

$$\mathcal{F}(sz_\varepsilon) = \bar{D}_0 + \bar{D}_4 \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) + \bar{C}_0 \varepsilon^p + o(\varepsilon^p) - \bar{A}_0 - \bar{A}_1 \varepsilon^2 \ln \varepsilon.$$

If $1 < p < 2$

$$\mathcal{F}(sz_\varepsilon) = (\bar{D}_0 - \bar{A}_0) + \bar{C}_0 \varepsilon^p + o(\varepsilon^p) = f_0(s) + f_3(s) \varepsilon^p + o(\varepsilon^p)$$

with

$$f_3(s) = h(0) \frac{s^p}{p} \int_{\mathbb{R}_+^N} |\nabla Z|^p dy dt.$$

As before f_ε reaches its maximum at $s_\varepsilon = 1 + a\varepsilon^p + o(\varepsilon^p)$ with $a = \frac{f_3'(1)}{f_3''(1)}$. Then,

$$\sup_{s>0} \mathcal{F}(sz_\varepsilon) = \mathcal{F}(s_\varepsilon z_\varepsilon) = \left(\frac{1}{p} - \frac{1}{p^*} \right) \bar{K}(N, p)^{-\frac{p^*p}{p^*-p}} + f_3(1)\varepsilon^p + o(\varepsilon^p)$$

So, we need that $f_3(1) < 0$. But, this is equivalent to $h(0) < 0$.

If $p \geq 2$, we have

$$\mathcal{F}(sz_\varepsilon) = (\bar{D}_0 - \bar{A}_0) + (\bar{D}_4 - \bar{A}_1)\varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) = f_0(s) + f_4(s)\varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon),$$

with

$$\begin{aligned} f_4(s) = & -\frac{s^p}{p} \frac{N}{2p} \left(\partial_{tt} p(0) \int_{\mathbb{R}_+^N} t^2 |\nabla Z|^p dy dt + \Delta_y p(0) \int_{\mathbb{R}_+^N} |y|^2 |\nabla Z|^p dy dz \right) \\ & + \frac{s^{p^*}}{p^*} \frac{1}{2p^*} \Delta_y r(0) \int_{\partial \mathbb{R}_+^N} |y|^2 Z^{p^*} dy. \end{aligned}$$

As before, we need that $f_4(1) < 0$. Since 0 is a local minimum of $p(x)$ and a local maximum of $r(x)$ and $\partial_t p(0) = 0$ it easily follows that $f_4(1) \leq 0$. And if one of the following inequalities

$$\Delta_y r(0) \leq 0 \leq \Delta p(0)$$

is strict, then $f_4(1) < 0$ and the result follows. \square

A

Asymptotic expansions

The goal of this Appendix is to gather all of the asymptotic expansions needed in the course of the proofs of this Thesis, mainly in Chapter 7 (although in Chapter 5 some of the results in the Appendix are already used).

The computations of the asymptotic expansions are based on Taylor expansions of the test functions with respect to a (small) concentration parameter $\varepsilon > 0$. Though elementary, the computations are lengthy and delicate.

The Appendix is divided into two sections. The first one is devoted to asymptotic expansions of concentrations of extremals for $K(N, p)^{-1}$ and the second one to asymptotic expansions of extremals for $\bar{K}(N, p)^{-1}$.

First, recall some basic definitions.

$$K(N, p)^{-1} = \inf_{f \in D(N, p)} \frac{\|\nabla f\|_{p, \mathbb{R}^N}}{\|f\|_{p^*, \mathbb{R}^N}},$$

where $D(N, p) = \{f \in L^p(\mathbb{R}^N) : \partial_i f \in L^p(\mathbb{R}^N)\}$.

$$\bar{K}(N, p)^{-1} = \inf_{f \in \bar{D}(N, p)} \frac{\|\nabla f\|_{p, \mathbb{R}_+^N}}{\|f\|_{p^*, \partial \mathbb{R}_+^N}},$$

where $\bar{D}(N, p) = \{f \in \mathcal{M}(\mathbb{R}_+^N) : \partial_i f \in L^p(\mathbb{R}_+^N) \text{ and } f|_{\partial \mathbb{R}_+^N} \in L^p(\partial \mathbb{R}_+^N)\}$.

Both constants are attained at some extremals. These extremals are completely characterized. In fact an extremal for $K(N, p)^{-1}$ is given by

$$U(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{N}{p}}$$

and any other extremal for $K(N, p)^{-1}$ is obtained by a dilation and translation of U , namely

$$U_{\lambda, x_0}(x) = \lambda^{-\frac{N}{p}} U\left(\frac{x-x_0}{\lambda}\right).$$

As for $\bar{K}(N, p)^{-1}$ an extremal is given by

$$V(x', t) = (|x'|^2 + (1+t)^2)^{-\frac{N-p}{2(p-1)}} \quad x' \in \mathbb{R}^{N-1}, t > 0$$

and any other extremal for $\bar{K}(N, p)^{-1}$ is given by a dilation and translation of V , namely

$$V_{\lambda, x'_0}(x', t) = \lambda^{-\frac{N-p}{p-1}} V\left(\frac{x'-x'_0}{\lambda}, \frac{t}{\lambda}\right).$$

A.1 Asymptotic expansions for the Sobolev immersion constant

In this section we perform the asymptotic expansions on appropriate test functions constructed by concentrating the extremals U given in the introduction of the Appendix and truncated by a cut-off function.

Assume that $0 \in \Omega$ and given $\delta > 0$ small we take a cut-off function $\eta \in C_c^\infty(B_{2\delta}, [0, 1])$ such that $\eta \equiv 1$ in B_δ . We then consider the test-function

$$u_\varepsilon(x) = U_{\varepsilon, 0}(x)\eta(x).$$

For this test function we have:

Proposition A.1. *Let $p, q \in \mathcal{P}(\mathbb{R}^N)$ be C^2 and assume that 0 is a critical point of p and q . We have*

- If $p \leq \frac{N}{2}$,

$$\int_{\mathbb{R}^N} f(x) u_\varepsilon^{q(x)} dx = A_0 + A_1 \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) \quad (\text{A.1})$$

with

$$A_0 = f(0) \int_{\mathbb{R}^N} U^{p^*} dx, \quad A_1 = -\frac{N-p}{p} \frac{f(0)}{2} \int_{\mathbb{R}^N} U^{p^*} (D^2 q(0)x, x) dx$$

- If $p < \min\{\sqrt{N}, \frac{N+2}{3}\}$,

$$\int_{\mathbb{R}^N} f(x) |\nabla u_\varepsilon|^{p(x)} dx = B_0 + B_1 \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) \quad (\text{A.2})$$

with

$$B_0 = f(0) \int_{\mathbb{R}^N} |\nabla U|^p dx, \quad B_1 = -\frac{N}{p} \frac{f(0)}{2} \int_{\mathbb{R}^N} |\nabla U|^p (D^2 p(0)x, x) dx$$

- If $p < \sqrt{N}$,

$$\int_{\mathbb{R}^N} f(x) |u_\varepsilon|^{p(x)} dx = C_0 \varepsilon^p + o(\varepsilon^p) \quad \text{with} \quad C_0 = f(0) \int_{\mathbb{R}^N} U^p dx. \quad (\text{A.3})$$

Remark A.2. Observe that if $g(x)$ is a radial function then

$$\int_{\mathbb{R}^N} g(x)(Ax, x) dx = \text{tr}(A) \int_{\mathbb{R}^N} g(x)x_1^2 dx = \frac{\text{tr}(A)}{N} \int_{\mathbb{R}^N} g(x)|x|^2 dx,$$

for any $A \in \mathbb{R}^{N \times N}$ (with adequate decaying assumptions at infinity on g). In fact this is a consequence of the fact that, for $i \neq j$,

$$\int_{\mathbb{R}^N} g(x)x_i x_j dx = 0.$$

With this observation, we easily conclude that

$$A_1 = -\frac{f(0)}{p^*} \Delta q(0) \int_{\mathbb{R}^N} U^{p^*} |x|^2 dx$$

and

$$B_1 = -\frac{f(0)}{2p} \Delta p(0) \int_{\mathbb{R}^N} |\nabla U|^p |x|^2 dx.$$

The proof of Proposition A.1 is divided into three steps. Each step provides the proof of one of the asymptotic expansions.

As 0 is a local minimum of $p(\cdot)$ we can assume that $p_{2\delta}^- := \min_{x \in B_{2\delta}} p(x) = p$.

Lemma A.3. *Under the assumptions of Proposition A.1, (A.1) holds true*

Proof. We first write

$$\int_{\mathbb{R}^N} f(x)u_\varepsilon(x)^{q(x)} dx = \int_{B_{2\delta} \setminus B_{\varepsilon^{1/p}}} f(x)u_\varepsilon^{q(x)} dx + \int_{B_{\varepsilon^{1/p}}} f(x)u_\varepsilon(x)^{q(x)} dx = I_1(\varepsilon) + I_2(\varepsilon).$$

Since $u_\varepsilon(x) \leq 1$ if $|x| \geq \varepsilon^{1/p}$, we have, letting $q_{2\delta}^- := \min_{B_{2\delta}} q$ that

$$\begin{aligned} I_1(\varepsilon) &\leq \|f\|_{\infty, B_{2\delta}} \int_{B_{2\delta} \setminus B_{\varepsilon^{1/p}}} u_\varepsilon(x)^{q_{2\delta}^-} dx \\ &\leq \|f\|_{\infty, B_{2\delta}} \varepsilon^{N - \frac{N-p}{p} q_{2\delta}^-} \int_{\mathbb{R}^N \setminus B_{\varepsilon^{-(p-1)/p}}} U(x)^{q_{2\delta}^-} dx, \end{aligned}$$

where the integral in the right hand side can be bounded by

$$C \int_{\varepsilon^{-(p-1)/p}}^{+\infty} (1 + r^{\frac{p}{p-1}})^{-\frac{N-p}{p} q_{2\delta}^-} r^{N-1} dr \leq C \int_{\varepsilon^{-(p-1)/p}}^{+\infty} r^{-1 + N - \frac{N-p}{p-1} q_{2\delta}^-} dr \leq C \varepsilon^{-N \frac{p-1}{p} + \frac{N-p}{p} q_{2\delta}^-}.$$

Hence $I_1(\varepsilon) \leq C \varepsilon^{N/p}$ so that

$$\begin{aligned} \int_{\mathbb{R}^N} f(x)u_\varepsilon(x)^{q(x)} dx &= \int_{B_{\varepsilon^{1/p}}} f(x)u_\varepsilon(x)^{q(x)} dx + O(\varepsilon^{N/p}) \\ &= \int_{B_{\varepsilon^{-(p-1)/p}}} f(\varepsilon x) \varepsilon^{N - q(\varepsilon x) \frac{N-p}{p}} U(x)^{q(\varepsilon x)} dx + O(\varepsilon^{N/p}). \end{aligned}$$

As $\nabla q(0) = 0$ we get

$$q(\varepsilon x) = q(0) + \frac{1}{2}\varepsilon^2(D^2q(0)x, x) + o(\varepsilon^2|x|^2),$$

with $q(0) = p(0)^* = p^*$, so

$$\begin{aligned} \int_{\mathbb{R}^N} f(x)u_\varepsilon(x)^{q(x)} dx &= A_0(\varepsilon) + A_1(\varepsilon)\varepsilon^2 \ln \varepsilon + \int_{B_{\varepsilon^{-(p-1)/p}}} o(\varepsilon^2 \ln \varepsilon)|x|^2 U(x)^{p^*} dx \\ &\quad + \varepsilon \int_{B_{\varepsilon^{-(p-1)/p}}} U(x)^{p^*} \nabla f(0) \cdot x dx + O(\varepsilon^{N/p}) \\ &= A_0(\varepsilon) + A_1(\varepsilon)\varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) + O(\varepsilon^{N/p}), \end{aligned}$$

where $A_0(\varepsilon)$ and $A_1(\varepsilon)$ are the same as A_0 and A_1 except that we integrate over $B_{\varepsilon^{-(p-1)/p}}$ instead of \mathbb{R}^N and we have used the fact that

$$\int_{B_{\varepsilon^{-(p-1)/p}}} U(x)^{p^*} \nabla f(0) \cdot x dx = 0,$$

since U is radially symmetric. We have

$$\begin{aligned} |A_0(\varepsilon) - A_0| &\leq C \int_{\mathbb{R}^N \setminus B_{\varepsilon^{-(p-1)/p}}} U(x)^{p^*} dx \\ &\leq C \int_{\varepsilon^{-(p-1)/p}}^{+\infty} (1 + r^{\frac{p}{p-1}})^{-N} r^{N-1} dr \\ &\leq C \int_{\varepsilon^{-(p-1)/p}}^{+\infty} r^{\frac{-Np}{p-1} + N-1} dr \\ &\leq C\varepsilon^{\frac{N}{p}}. \end{aligned}$$

If $p < (N + 2)/2$, we can estimate

$$\begin{aligned} |A_1(\varepsilon) - A_1| &\leq C \int_{\mathbb{R}^N \setminus B_{\varepsilon^{-(p-1)/p}}} |x|^2 U(x)^{p^*} dx \\ &\leq C \int_{\varepsilon^{-(p-1)/p}}^{+\infty} (1 + r^{\frac{p}{p-1}})^{-N} r^{N+1} dr \\ &\leq C\varepsilon^{\frac{N+2-2p}{p}}. \end{aligned}$$

We thus have

$$\int_{\mathbb{R}^N} f(x)u_\varepsilon(x)^{q(x)} dx - A_0 - A_1\varepsilon^2 \ln \varepsilon = O(\varepsilon^{N/p}) + o(\varepsilon^2 \ln \varepsilon),$$

which reduces to (A.1) if we assume that $p \leq N/2$. □

Lemma A.4. *Under the assumptions of Proposition A.1, (A.3) holds true*

Proof. As before,

$$\int_{\mathbb{R}^N} f(x)u_\varepsilon^{p(x)} dx = \int_{B_{\varepsilon^{1/p}}} f(x)u_\varepsilon^{p(x)} dx + \int_{B_{2\delta} \setminus B_{\varepsilon^{1/p}}} f(x)u_\varepsilon^{p(x)} dx$$

where, noticing that $p = p_{2\delta}^-$, the 2nd integral in the right hand side can be bounded by

$$\begin{aligned} \int_{B_{2\delta} \setminus B_{\varepsilon^{1/p}}} u_\varepsilon^p dx &\leq C\varepsilon^p \int_{\varepsilon^{1/p-1}}^\infty (1+r^{\frac{p}{p-1}})^{p-N} r^{N-1} dr \\ &\leq C\varepsilon^p \varepsilon^{\frac{N-p^2}{p}} \\ &= C\varepsilon^{\frac{N}{p}}, \end{aligned}$$

if $p^2 < N$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} f(x)u_\varepsilon^{p(x)} dx &= \int_{B_{\varepsilon^{1/p}}} f(x)u_\varepsilon^{p(x)} dx + O(\varepsilon^{\frac{N}{p}}) \\ &= \int_{B_{\varepsilon^{1/p-1}}} f(\varepsilon x)\varepsilon^{N-\frac{N-p}{p}p(\varepsilon x)} U(x)^{p(\varepsilon x)} dx + O(\varepsilon^{\frac{N}{p}}) \\ &= \varepsilon^p f(0) \int_{\mathbb{R}^N} U(x)^p dx + o(\varepsilon^p). \end{aligned}$$

□

Lemma A.5. *Under the assumptions of Proposition A.1, (A.2) holds true*

Proof. We first write

$$\int_{\mathbb{R}^N} f(x)|\nabla u_\varepsilon|^{p(x)} dx = \int_{\mathbb{R}^N} f(x)|\eta \nabla U_\varepsilon + U_\varepsilon \nabla \eta|^{p(x)} dx = \int_{\mathbb{R}^N} f(x)|\eta \nabla U_\varepsilon|^{p(x)} dx + R_\varepsilon,$$

where, using the inequality

$$\|a + b\|^q - \|a\|^q \leq C(\|b\|^q + \|b\|\|a\|^{q-1}),$$

(the constant C being uniform in q for q in a bounded interval of $[0, +\infty)$) we can estimate

$$|R_\varepsilon| \leq C \left[\int_{B_{2\delta} \setminus B_\delta} |\nabla \eta|^{p(x)} U_\varepsilon^{p(x)} dx + \int_{B_{2\delta} \setminus B_\delta} |\nabla \eta| U_\varepsilon(x) |\nabla U_\varepsilon|^{p(x)-1} dx \right] = C[I_1(\varepsilon) + I_2(\varepsilon)].$$

Since $U_\varepsilon \leq 1$ in $\mathbb{R}^N \setminus B_\delta$ for ε small, we can bound $I_1(\varepsilon)$ as before by

$$I_1(\varepsilon) \leq C \int_{B_{2\delta} \setminus B_\delta} U_\varepsilon^p dx \leq C\varepsilon^p \int_{\mathbb{R}^N \setminus B_{\delta/\varepsilon}} U^p dx \leq C\varepsilon^p \varepsilon^{\frac{N-p^2}{p-1}} = C\varepsilon^{\frac{N-p}{p-1}},$$

if $p^2 < N$. Since $|\nabla U_\varepsilon| \leq 1$ in $\mathbb{R}^N \setminus B_\delta$ for ε small, we also have

$$\begin{aligned} I_2(\varepsilon) &\leq C \int_{\mathbb{R}^N \setminus B_\delta} U_\varepsilon(x) |\nabla U_\varepsilon|^{p-1} dx \\ &\leq C \|U_\varepsilon\|_{p, \mathbb{R}^N \setminus B_\delta} \|\nabla U_\varepsilon\|_{p, \mathbb{R}^N \setminus B_\delta}^{p-1} \\ &\leq C \varepsilon^{\frac{N-p}{p(p-1)}} \|\nabla U_\varepsilon\|_{p, \mathbb{R}^N \setminus B_\delta}^{p-1}, \end{aligned}$$

with, since $|U'(r)| \sim r^{-\frac{N-1}{p-1}}$ as $r \sim +\infty$,

$$\int_{\mathbb{R}^N \setminus B_\delta} |\nabla U_\varepsilon|^p dx \leq C \int_{\delta/\varepsilon}^{+\infty} |U'(r)|^p r^{N-1} dr \leq C \varepsilon^{\frac{N-p}{p-1}}.$$

It follows that $I_2(\varepsilon) = O(\varepsilon^{\frac{N-p}{p-1}})$ and then $R_\varepsilon = O(\varepsilon^{\frac{N-p}{p-1}})$. Independently, since

$$|\nabla U_\varepsilon(x)| = \frac{N-p}{p-1} \varepsilon^{-N/p} \left(\frac{|x|}{\varepsilon} \right)^{\frac{1}{p-1}} \left(1 + \left(\frac{|x|}{\varepsilon} \right)^{\frac{p}{p-1}} \right)^{-N/p}$$

we have

$$|\nabla U_\varepsilon(x)| < 1 \quad \text{for } |x| > C_p \varepsilon^{\frac{N-p}{p(N-1)}}, \quad C_p = \left(\frac{N-p}{p-1} \right)^{\frac{p-1}{N-1}}. \quad (\text{A.4})$$

Taking some constant $C > C_p$, we thus write

$$\begin{aligned} \int_{\mathbb{R}^N} f(x) |\nabla u_\varepsilon|^{p(x)} dx &= \int_{B_{\frac{N-p}{C\varepsilon^{p(N-1)}}}} f(x) |\nabla U_\varepsilon|^{p(x)} dx \\ &\quad + \int_{\mathbb{R}^N \setminus B_{\frac{N-p}{C\varepsilon^{p(N-1)}}}} f(x) |\nabla U_\varepsilon|^{p(x)} dx + O(\varepsilon^{\frac{N-p}{p-1}}). \end{aligned}$$

Since $|\nabla U_\varepsilon(x)| < 1$ in $\mathbb{R}^N \setminus B_{\frac{N-p}{C\varepsilon^{p(N-1)}}}$, we can bound the second integral on the right hand side by

$$C \int_{\mathbb{R}^N \setminus B_{\frac{N-p}{C\varepsilon^{p(N-1)}}}} |\nabla U_\varepsilon|^p dx \leq C \int_{\varepsilon^{\frac{N(p-1)}{p(N-1)}}}^{+\infty} r^{\frac{p}{p-1}} \left(1 + r^{\frac{p}{p-1}} \right)^{-N} r^{N-1} dr \leq C \varepsilon^{\frac{N(N-p)}{p(N-1)}} = o(\varepsilon^{\frac{N-p}{p-1}}).$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^N} f(x) |\nabla u_\varepsilon|^{p(x)} dx &= \int_{B_{\frac{N-p}{C\varepsilon^{p(N-1)}}}} f(x) |\nabla U_\varepsilon|^{p(x)} dx + O(\varepsilon^{\frac{N-p}{p-1}}) \\ &= B_0(\varepsilon) + B_1(\varepsilon) \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) + O(\varepsilon^{\frac{N-p}{p-1}}) \end{aligned}$$

where $B_0(\varepsilon)$ and $B_1(\varepsilon)$ are the same as B_0, B_1 but integrating over $B_{\frac{N-p}{\varepsilon^{p(N-1)}}}$ instead of \mathbb{R}^N . Again, as in the computation of (A.1), the term involving $\nabla f(0)$ vanishes for symmetry reasons.

Since $|U'(r)|^p \sim r^{\frac{p(1-N)}{p-1}}$ as $r \sim +\infty$, we have

$$|B_0 - B_0(\varepsilon)| \leq C \int_{\mathbb{R}^N \setminus B_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} |\nabla U|^p dx \leq C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{+\infty} r^{\frac{p-N}{p-1}-1} dr \leq C\varepsilon^{\frac{N(N-p)}{p(N-1)}} = o(\varepsilon^{\frac{N-p}{p-1}}),$$

$$|B_1 - B_1(\varepsilon)| \leq C \int_{\mathbb{R}^N \setminus B_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} |x|^2 |\nabla U|^p dx \leq C\varepsilon^{\frac{N(N-3p+2)}{p(N-1)}} \quad \text{if } p < \frac{N+2}{3}.$$

Hence if $p < \frac{N+2}{3}$ we have

$$\int_{\mathbb{R}^N} f(x) |\nabla u_\varepsilon|^{p(x)} dx - B_0 - B_1 \varepsilon^2 \ln \varepsilon = o(\varepsilon^2 \ln \varepsilon).$$

□

Trivially, these lemmas imply Proposition A.1.

A.2 Asymptotic expansions for the Sobolev trace constant

In this section we provide the asymptotic expansions needed in order to deal with the Sobolev trace constant.

In order to construct the test functions in this case, we need to introduce the so-called *Fermi coordinates* around some point of $\partial\Omega$.

Definition A.6 (Fermi Coordinates). We consider the following change of variables around a point $x_0 \in \partial\Omega$.

We assume that $x_0 = 0$ and that $\partial\Omega$ has the following representation in a neighborhood of 0:

$$\partial\Omega \cap V = \{x \in V : x_n = \psi(x'), x' \in U \subset \mathbb{R}^{N-1}\}, \quad \Omega \cap V = \{x \in V : x_n > \psi(x'), x' \in U \subset \mathbb{R}^{N-1}\}.$$

The function $\psi : U \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ is assumed to be at least of class C^2 and that $\psi(0) = 0$, $\nabla\psi(0) = 0$.

The change of variables is then defined as $\Phi : U \times (0, \delta) \rightarrow \Omega \cap V$

$$\Phi(y, t) = (y, \psi(y)) + t\nu(y),$$

where $\nu(y)$ is the unit inward normal vector, i.e.

$$\nu(y) = \frac{(-\nabla\psi(y), 1)}{\sqrt{1 + |\nabla\psi(y)|^2}}.$$

It is well known that Φ defines a smooth diffeomorphism. In differential geometry, this is called the *Fermi coordinates* (see [19]).

Moreover, in [19] it is proved the following asymptotic expansions

Lemma A.7. *With the notation introduced in Definition A.6, the following asymptotic expansions hold*

$$J\Phi(y, t) = 1 - Ht + O(t^2 + |y|^2),$$

where H is the mean curvature of $\partial\Omega$.

Also, if we denote $v(y, t) = u(\Phi(y, t))$,

$$|\nabla u(x)|^2 = (\partial_t v)^2 + \sum_{i,j=1}^N (\delta^{ij} + 2h^{ij}t + O(t^2 + |y|^2)) \partial_{y_i} v \partial_{y_j} v,$$

where h^{ij} is the second fundamental form of $\partial\Omega$.

For a general construction of the Fermi coordinates in differential manifolds, we refer to the book [34].

Now, we are in position to construct the test functions needed in order to estimate the Sobolev trace constant. Assume that $0 \in \partial\Omega$. Then, the test-function in these coordinate is $(x = \Phi(y, t))$

$$v_\varepsilon(x) = \eta(y, t) V_{\varepsilon,0}(y, t),$$

where V is the extremal for $\bar{K}(N, p(0))^{-1}$ given in the introduction of the Appendix and $\eta \in C_c^\infty(B_{2\delta} \times [0, 2\delta], [0, 1])$ is a smooth cut-off function.

From now on, we assume that $p(x) \in \mathcal{P}(\Omega)$, $r(x) \in \mathcal{P}(\partial\Omega)$ are of class C^2 , $0 \in \partial\Omega$ and we deonte $p = p(0)$ and $r = r(0)$.

The goal of this section is to prove the following propositions.

Proposition A.8. *There holds*

$$\int_{\Omega} f(x) |v_\varepsilon|^{p(x)} dx = \bar{C}_0 \varepsilon^p + o(\varepsilon^p) \quad \text{with} \quad \bar{C}_0 = f(0) \int_{\mathbb{R}_+^N} V^p dx. \quad (\text{A.5})$$

Proposition A.9. *If $p < \frac{N-1}{2}$,*

$$\int_{\partial\Omega} f(x) |v_\varepsilon|^{r(x)} dS_x = \bar{A}_0 + \bar{A}_1 \varepsilon^2 \ln \varepsilon + o(\varepsilon^2 \ln \varepsilon) \quad (\text{A.6})$$

with

$$\bar{A}_0 = f(0) \int_{\mathbb{R}^{N-1}} V(y, 0)^{p^*} dy,$$

and

$$\begin{aligned} \bar{A}_1 &= -\frac{N-p}{2p} f(0) \int_{\mathbb{R}^{N-1}} (D^2 r(0)y, y) V(y, 0)^{p^*} dy \\ &= -\frac{1}{2p^*} f(0) \Delta r(0) \int_{\mathbb{R}^{N-1}} |y|^2 V(y, 0)^{p^*} dy. \end{aligned}$$

Proposition A.10. Assume that $p < N^2/(3N - 2)$. Then

$$\int_{\Omega} f(x)|\nabla v_{\varepsilon}(x)|^{p(x)} dx = \bar{D}_0 + \bar{D}_1 \varepsilon \ln \varepsilon + \bar{D}_2 \varepsilon + \bar{D}_3 (\varepsilon \ln \varepsilon)^2 + \bar{D}_4 \varepsilon^2 \ln \varepsilon + O(\varepsilon^2),$$

with

$$\bar{D}_0 = f(0) \int_{\mathbb{R}_+^N} |\nabla V|^p dy dt, \quad \bar{D}_1 = -\frac{N}{p} f(0) \partial_t p(0) \int_{\mathbb{R}_+^N} t |\nabla V|^p dy dt,$$

and, assuming that $\partial_t p(0) = 0$,

$$\begin{aligned} \bar{D}_2 &= (\partial_t f(0) - Hf(0)) \int_{\mathbb{R}_+^N} t |\nabla V|^p dy dt + p \bar{h} f(0) \int_{\mathbb{R}_+^N} \frac{t|y|^2}{r^2} |\nabla V|^p dy dt, \\ \bar{D}_3 &= 0 \\ \bar{D}_4 &= -\frac{N}{2p} f(0) \partial_{tt} p(0) \int_{\mathbb{R}_+^N} t^2 |\nabla V|^p dy dt - \frac{N}{2(N-1)p} f(0) \Delta_y p(0) \int_{\mathbb{R}_+^N} |y|^2 |\nabla V|^p dy dt \end{aligned}$$

Proof of Proposition A.8. We write

$$\int_{\Omega} f(x)|v_{\varepsilon}|^{p(x)} dx = \int_{\mathbb{R}_+^N} f(y, t)|v_{\varepsilon}(y, t)|^{p(y, t)} (1 + O(|y|^2 + |t|)) dy dt.$$

Now the result follows as in Lemma A.4. \square

Proof of Proposition A.9. We have

$$\int_{\partial\Omega} f v_{\varepsilon}^{r(x)} dS = \int_{\mathbb{R}^{N-1}} f(y, \psi(y)) v_{\varepsilon}(y, \psi(y))^{r(y, \psi(y))} (1 + O(|y|^2)) dy.$$

Now the proof follows from Lemma A.3. \square

To treat the gradient term, we need the following result:

Lemma A.11. Assume $p < N^2/(3N - 2)$ and that $p = p(y, t)$ has a local minimum at $(y, t) = (0, 0)$. Given a bounded $g \in C^2(\Omega)$ and real numbers a^{ij} , $1 \leq i, j \leq N - 1$, we have

$$\begin{aligned} \sum_{i, j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^N} g(y, t) \eta(y, t) |\nabla v_{\varepsilon}|^{p(y, t)-2} \partial_i v_{\varepsilon}(y, t) \partial_j v_{\varepsilon}(y, t) dy dt \\ = \bar{B}_0 + \bar{B}_1 \varepsilon \ln \varepsilon + \bar{B}_2 \varepsilon + \bar{B}_3 (\varepsilon \ln \varepsilon)^2 + \bar{B}_4 \varepsilon^2 \ln \varepsilon + O(\varepsilon^2) \end{aligned} \quad (\text{A.7})$$

where $\partial_i = \frac{\partial}{\partial y_i}$, and

$$\begin{aligned}\bar{B}_0 &= \bar{a}g(0) \int_{\mathbb{R}_+^N} |\nabla V(y, t)|^p \frac{|y|^2}{r^2} dydt, \quad \bar{B}_1 = -\frac{N}{p}g(0)\partial_t p(0)\bar{a} \int_{\mathbb{R}_+^N} |\nabla V(y, t)|^p \frac{|y|^2 t}{r^2} dydt \\ \bar{B}_2 &= \bar{a} \int_{\mathbb{R}_+^N} |\nabla V(y, t)|^p \frac{t|y|^2}{r^2} \{g(0)\partial_t p(0) \ln |\nabla V(y, t)| + \partial_t g(0)\} dydt \\ \bar{B}_3 &= \frac{N^2}{2p^2}g(0)\partial_t p(0)^2 \bar{a} \int_{\mathbb{R}_+^N} |\nabla V(y, t)|^p \frac{|y|^2 t^2}{r^2} dydt \\ \bar{B}_4 &= -\frac{N}{p}\bar{a} \int_{\mathbb{R}_+^N} |\nabla V(y, t)|^p \frac{|y|^2 t^2}{r^2} \left(-\frac{g(0)}{2}\partial_{tt} p(0) + \partial_t p(0)\partial_t g(0) + \partial_t p(0)^2 g(0) \ln |\nabla V(y, t)| \right) dydt \\ &\quad + \sum_{i=1}^{N-1} \frac{Ng(0)}{2p} a^{ii} \partial_{ii} p(0) \int_{\mathbb{R}_+^N} |\nabla V(y, t)|^p r^{-2} (y_1^4 - 3y_1^2 y_2^2) dydt \\ &\quad + \sum_{i,k=1}^{N-1} \frac{Ng(0)}{2p} (a^{ii} \partial_{kk} p(0) + 2a^{ik} \partial_{ik} p(0)) \int_{\mathbb{R}_+^N} |\nabla V(y, t)|^p r^{-2} y_1^2 y_2^2 dydt\end{aligned}$$

where $\bar{a} = \frac{1}{N-1} \sum_{i=1}^{N-1} a^{ii}$ and $r = r(y, t) = \sqrt{(1+t)^2 + |y|^2}$.

Proof. Notice that

$$|\nabla V_\varepsilon(y, t)| = \frac{N-p}{p-1} \varepsilon^{\frac{N-p}{p(N-1)}} ((\varepsilon+t)^2 + |y|^2)^{-\frac{N-1}{2(p-1)}}.$$

So, $|\nabla V_\varepsilon(y, t)| < 1$ if $|(y, t)| > C\varepsilon^{\frac{N-p}{p(N-1)}}$ where $C = \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{N-1}}$, and $\nabla = (\nabla_y, \partial_t)$. Moreover, since $p_{2\delta}^- = p := p(0, 0)$,

$$\begin{aligned}& \int_{B_{2\delta}^+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)-2} |\nabla_y V_\varepsilon|^2 dydt \leq \int_{B_{2\delta}^+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)} dydt \\ & \leq \int_{B_{2\delta}^+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^p dydt \leq C\varepsilon^{\frac{N-p}{p-1}} \int_{\mathbb{R}_+^N \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} \{(\varepsilon+t)^2 + |y|^2\}^{-\frac{p(N-1)}{2(p-1)}} dydt \\ & \leq C\varepsilon^{\frac{N-p}{p-1}} \int_{\mathbb{R}^N \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |(y, t)|^{-\frac{p(N-1)}{p-1}} dydt \leq C\varepsilon^{\frac{N-p}{p-1}} \int_{C\varepsilon^{\frac{N-p}{p(N-1)}}}^{+\infty} \rho^{N-1-\frac{p(N-1)}{p-1}} d\rho\end{aligned}$$

Then, we obtain

$$\int_{B_{2\delta}^+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)-2} |\nabla_y V_\varepsilon|^2 dydt \leq C\varepsilon^{\frac{N}{p^*}}.$$

Since $p \leq \frac{N^2}{3N-2}$, we get that $\frac{N}{p^*} \geq 2$, hence

$$\int_{B_{2\delta}^+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(x,t)-2} |\nabla_y V_\varepsilon|^2 dydt = O(\varepsilon^2).$$

Hence

$$\begin{aligned}
& a^{ij} \int_{\mathbb{R}_+^N} g(y, t) \eta(y, t) |\nabla V_\varepsilon|^{p(y, t)-2} \partial_i V_\varepsilon(y, t) \partial_j V_\varepsilon(y, t) dy dt \\
&= a^{ij} \int_{B^+_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} g(y, t) |\nabla V_\varepsilon|^{p(y, t)-2} \partial_i V_\varepsilon(y, t) \partial_j V_\varepsilon(y, t) dy dt + O(\varepsilon^2) \\
&= a^{ij} \int_{B^+_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} g(\varepsilon y, \varepsilon t) \varepsilon^{N(1-\frac{p(\varepsilon y, \varepsilon t)}{p})} |\nabla V|^{p(\varepsilon y, \varepsilon t)-2} \partial_i V \partial_j V dy dt + O(\varepsilon^2).
\end{aligned}$$

Letting

$$\phi_{ij} = |\nabla V|^{p-2} \partial_i V \partial_j V = |\nabla V(y, t)|^p \frac{y_i y_j}{r^2}, \quad \nabla = (\nabla_y, \partial_t),$$

we obtain

$$\begin{aligned}
& \sum_{i, j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^n} g(y, t) \eta(y, t) |\nabla V_\varepsilon|^{p(y, t)-2} \partial_i V_\varepsilon \partial_j V_\varepsilon dy dt \\
&= \bar{B}_0(\varepsilon) + \bar{B}_1(\varepsilon) \varepsilon \ln \varepsilon + \bar{B}_2(\varepsilon) \varepsilon + \bar{B}_3(\varepsilon) (\varepsilon \ln \varepsilon)^2 + \bar{B}_4(\varepsilon) \varepsilon^2 \ln \varepsilon + \varepsilon^2 R(\varepsilon)
\end{aligned}$$

with coefficients $\bar{B}_i(\varepsilon)$, $i = 0, \dots, 4$, defined as

$$\begin{aligned}
\bar{B}_0 &= \sum_{i, j=1}^{N-1} a^{ij} g(0) \int_{\mathbb{R}_+^N} \phi_{ij}(y, t) dy dt \\
\bar{B}_1 &= -\frac{N}{p} g(0) \partial_t p(0) \sum_{i, j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^N} t \phi_{ij}(y, t) dy dt \\
\bar{B}_2 &= \sum_{i, j=1}^{N-2} a^{ij} \int_{\mathbb{R}_+^N} \phi_{ij}(y, t) (g(0) t \partial_t p(0) \ln |\nabla V| + \nabla g(0)(y, t)) dy dt \\
\bar{B}_3 &= \frac{N^2}{2p^2} g(0) \partial_t p(0)^2 \sum_{i, j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^N} t^2 \phi_{ij}(y, t) dy dt \\
\bar{B}_4 &= -\frac{N}{p} \sum_{i, j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^N} \phi_{ij}(x, t) \left(\frac{g(0)}{2} (D^2 p(0)(y, t), (y, t)) + \partial_t p(0) t (\nabla g(0), (y, t)) \right. \\
&\quad \left. + \partial_t p(0)^2 g(0) t^2 \ln |\nabla V| \right) dy dt,
\end{aligned}$$

but with integral over $B^+_{C\varepsilon^{\frac{N-p}{p(N-1)}}}$ instead of \mathbb{R}_+^N , and the error term $R(\varepsilon)$ satisfies

$$\begin{aligned}
|R(\varepsilon)| &\leq C \int_{B^+_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} r^2 |\nabla V|^p \ln |\nabla V| (1 + r \varepsilon \ln \varepsilon) dy dt \\
&\leq C \int_{B^+_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} r^2 |\nabla V|^p \ln |\nabla V| dy dt.
\end{aligned}$$

Clearly, this last integral is bounded by

$$C \int_1^{+\infty} \rho^{1-\frac{N-p}{p-1}} \ln \rho \, d\rho$$

which is finite since $p < \frac{N+2}{3}$. Moreover

$$|\bar{B}_0 - \bar{B}_0(\varepsilon)| \leq C \int_{\mathbb{R}_+^N \setminus B^+_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} |\nabla V|^p \, dydt \leq C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{-1-\frac{N-p}{p-1}} \, dr \leq C \varepsilon^{\frac{N(N-p)}{p(N-1)}} \leq C \varepsilon^2$$

since $p \leq \frac{N^2}{3N-2}$. Also for $i = 1, 2$,

$$\begin{aligned} |\bar{B}_i - \bar{B}_i(\varepsilon)| &\leq C \int_{\mathbb{R}_+^N \setminus B^+_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} |(y, t)|(1 + \ln |\nabla V|)|\nabla V|^p \, dydt \\ &\leq C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}} \ln r \, dr \\ &\leq C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}+\alpha} \, dr \quad \text{for any } \alpha > 0 \\ &\leq C \varepsilon^{\frac{N(N-2p+1)}{p(N-1)}-\beta} \quad \text{for any } \beta > 0 \text{ and if } p < \frac{N^2 + N}{3N-1}, \\ &= o(\varepsilon). \end{aligned}$$

Eventually, for any $i = 3, 4$,

$$\begin{aligned} |\bar{B}_i - \bar{B}_i(\varepsilon)| &\leq C \int_{\mathbb{R}_+^N \setminus B^+_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} |(y, t)|^2(1 + \ln |\nabla V|)|\nabla V|^p \, dydt \\ &\leq C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}} \ln r \, dr \\ &\leq C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}+\alpha} \, dr \quad \text{for any } \alpha > 0 \\ &= o(1), \end{aligned}$$

since $p < \frac{n+2}{3}$.

Hence if $p < N^2/(3N-2)$,

$$\begin{aligned} \sum_{i,j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^N} g(y, t) \eta(y, t) |\nabla V_{\varepsilon}|^{p(y,t)-2} \partial_i V_{\varepsilon}(y, t) \partial_j U_{\varepsilon}(y, t) \, dydt \\ = \bar{B}_0 + \bar{B}_1 \varepsilon \ln \varepsilon + \bar{B}_2 \varepsilon + \bar{B}_3 ((\varepsilon \ln \varepsilon)^2 + \bar{B}_4 \varepsilon^2 \ln \varepsilon + O(\varepsilon^2)). \end{aligned}$$

Finally, using the radial symmetry in the y variable, we can simplify the expressions for the \bar{B}_i 's.

For \bar{B}_4 , notice that

$$\begin{aligned}
& \sum_{i,j=1}^{N-1} a^{ij} \partial_{kl} p(0) \int_{\mathbb{R}_+^N} |\nabla V|^p r^{-2} y_i y_j y^k y^l dy dt \\
&= \sum_{i=1}^{N-1} a^{ii} \partial_{ii} p(0) \int_{\mathbb{R}_+^N} |\nabla V|^p r^{-2} y_1^4 dy dt + \left(\sum_{i \neq k} a^{ii} \partial_{kk} p(0) + 2a^{ik} \partial_{ik} p(0) \right) \int_{\mathbb{R}_+^N} |\nabla V|^p r^{-2} y_1^2 y_2^2 dy dt \\
&= \sum_{i=1}^{N-1} a^{ii} \partial_{ii} p(0) \int_{\mathbb{R}_+^N} |\nabla V|^p r^{-2} (y_1^4 - 3y_1^2 y_2^2) dy dt \\
&\quad + \sum_{i,k=1}^{N-1} (a^{ii} \partial_{kk} p(0) + 2a^{ik} \partial_{ik} p(0)) \int_{\mathbb{R}_+^N} |\nabla V|^p r^{-2} y_1^2 y_2^2 dy dt
\end{aligned}$$

The other simplifications follow in the same manner. \square

Lemma A.12. Assume $p < N^2/(3N-2)$. There holds that

$$\int_{\mathbb{R}_+^N} f(y, t) \eta(y, t) |\nabla V_\varepsilon|^{p(y,t)} dy dt = \bar{C}_0 + \bar{C}_1 \varepsilon \ln \varepsilon + \bar{C}_2 \varepsilon + \bar{C}_3 (\varepsilon \ln \varepsilon)^2 + \bar{C}_4 \varepsilon^2 \ln \varepsilon + O(\varepsilon^2)$$

with

$$\begin{aligned}
\bar{C}_0 &= f(0) \int_{\mathbb{R}_+^N} |\nabla V|^p dy dt, \quad \bar{C}_1 = -\frac{N}{p} f(0) \partial_t p(0) \int_{\mathbb{R}_+^N} t |\nabla V|^p dy dt \\
\bar{C}_2 &= \int_{\mathbb{R}_+^N} t |\nabla V|^p (f(0) \partial_t p(0) \ln |\nabla V| + \partial_t f(0)) dy dt \\
\bar{C}_3 &= \frac{N^2}{2p^2} f(0) \partial_t p(0)^2 \int_{\mathbb{R}_+^N} t^2 |\nabla V|^p dy dt \\
\bar{C}_4 &= -\frac{N}{p} \int_{\mathbb{R}_+^N} t^2 |\nabla V|^p \left(\frac{f(0)}{2} \partial_{tt} p(0) + \partial_t p(0) \partial_t f(0) + \partial_t p(0)^2 f(0) \ln |\nabla V| \right) dy dt \\
&\quad - \frac{N}{2(N-1)p} f(0) \Delta_y p(0) \int_{\mathbb{R}_+^N} |y|^2 |\nabla V|^p dy dt, \quad \Delta_y = \sum_{i=1}^{n-1} \partial_{ii}
\end{aligned}$$

Proof. As before

$$\int_{\mathbb{R}_+^N \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)} dy dt \leq C \varepsilon^{\frac{N}{p^*}} = O(\varepsilon^2).$$

so that

$$\begin{aligned}
\int_{\mathbb{R}_+^N} f(y, t) \eta(y, t) |\nabla V_\varepsilon|^{p(y,t)} dy dt &= \int_{B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} f(y, t) |\nabla V_\varepsilon|^{p(y,t)} dy dt + O(\varepsilon^2) \\
&= \bar{C}_0(\varepsilon) + \bar{C}_1(\varepsilon) \varepsilon \ln \varepsilon + \bar{C}_2(\varepsilon) \varepsilon + \bar{C}_3(\varepsilon) (\varepsilon \ln \varepsilon)^2 + \bar{C}_4(\varepsilon) \varepsilon^2 \ln \varepsilon + O(\varepsilon^2)
\end{aligned}$$

where the constants $\bar{C}_i(\varepsilon)$ are the same as

$$\begin{aligned}\bar{C}_0 &= f(0) \int_{\mathbb{R}_+^N} |\nabla V|^p dydt \\ \bar{C}_1 &= -\frac{N}{p} f(0) \partial_t p(0) \int_{\mathbb{R}_+^N} t |\nabla V|^p dydt \\ \bar{C}_2 &= \int_{\mathbb{R}_+^N} t |\nabla V|^p (f(0) \partial_t p(0) \ln |\nabla V| + \partial_t f(0)) dydt \\ \bar{C}_3 &= \frac{N^2}{2p^2} f(0) \partial_t p(0)^2 \int_{\mathbb{R}_+^N} t^2 |\nabla V|^p dydt \\ \bar{C}_4 &= -\frac{N}{p} \int_{\mathbb{R}_+^N} |\nabla V|^p \left(\frac{f(0)}{2} (D^2 p(0)(y, t), (y, t)) + \partial_t p(0) \partial_t f(0) t^2 + \partial_t p(0)^2 f(0) t^2 \ln |\nabla V| \right) dydt\end{aligned}$$

but with integral over B^+ instead of \mathbb{R}_+^N . We can estimate $|\bar{C}_i(\varepsilon) - \bar{C}_i|$ as we estimated $|\bar{B}_i(\varepsilon) - \bar{B}_i|$ in the previous lemma.

Again, using the radial symmetry of V we can simplify the constants \bar{C}_i as in the previous lemma. \square

With the aid of the previous Lemmas, we can now prove Proposition A.10.

Proof of Proposition A.10. First, by Lemma A.7,

$$\int_{\Omega} f(x) |\nabla v_\varepsilon|^{p(x)} dx = \int_{\mathbb{R}_+^N} f(y, t) |\nabla v_\varepsilon|^{p(y, t)} (1 - Ht + O(t^2 + |y|^2)) dydt,$$

where we denote $f(y, t) = f(\Phi(y, t))$ and $p(y, t) = p(\Phi(y, t))$.

Recall that, by Lemma A.7,

$$|\nabla v_\varepsilon|^2 = (\partial_t v_\varepsilon)^2 + \sum_{i,j=1}^{N-1} (\delta^{ij} + 2h^{ij}t + O(t^2 + |y|^2)) \partial_i v_\varepsilon \partial_j v_\varepsilon, \quad \partial_i = \frac{\partial}{\partial y_i}.$$

Then

$$\begin{aligned}& \int_{\mathbb{R}_+^N} f(y, t) |\nabla v_\varepsilon|^{p(y, t)} (1 - Ht + O(t^2 + |y|^2)) dydt \\ &= \int_{\mathbb{R}_+^N} f(y, t) |\nabla(\eta V_\varepsilon)|^{p(y, t)} (1 - Ht + O(t^2 + |y|^2)) dydt \\ &= \int_{\mathbb{R}_+^N} f(y, t) \eta(y, t)^{p(y, t)} |\nabla V_\varepsilon|^{p(y, t)} (1 - Ht + O(t^2 + |y|^2)) dydt + R(\varepsilon),\end{aligned}$$

where

$$|R(\varepsilon)| \leq C \int_{\mathbb{R}_+^N \setminus B_\delta} |V_\varepsilon|^{p(y, t)} dydt \leq C \varepsilon^p \int_{\delta/\varepsilon}^\infty r^{-\frac{p(N-p)}{p-1} + N-1} dr = O(\varepsilon^2),$$

if $p \leq (n + 2)/3$. Hence

$$\begin{aligned} \int_{\Omega} f(x) |\nabla V_{\varepsilon}|^{p(x)} dx &= \int_{\mathbb{R}_+^N} f(y, t) \eta(y, t)^{p(y, t)} \left[(\partial_t U_{\varepsilon})^2 \right. \\ &\quad \left. + \sum_{i, j=1}^{N-1} (\delta^{ij} + 2h^{ij}t + O(t^2 + |y|^2)) \partial_i V_{\varepsilon} \partial_j V_{\varepsilon} \right]^{\frac{p(y, t)}{2}} \\ &\quad (1 - Ht + O(t^2 + |y|^2)) dy dt + O(\varepsilon^2) \end{aligned}$$

with

$$\begin{aligned} &\left[(\partial_t V_{\varepsilon})^2 + \sum_{i, j=1}^{N-1} (\delta^{ij} + 2h^{ij}t + O(t^2 + |y|^2)) \partial_i V_{\varepsilon} \partial_j V_{\varepsilon} \right]^{\frac{p(y, t)}{2}} \\ &= |\nabla V_{\varepsilon}|^{p(y, t)} \left[1 + \sum_{i, j=1}^{N-1} p(y, t) t h^{ij} |\nabla V_{\varepsilon}|^{-2} \partial_i V_{\varepsilon} \partial_j V_{\varepsilon} + O(t^2 + |y|^2) \right] \\ &= |\nabla V_{\varepsilon}|^{p(y, t)} + p(y, t) t h^{ij} |\nabla V_{\varepsilon}|^{p(y, t)-2} \partial_i V_{\varepsilon} \partial_j V_{\varepsilon} + |\nabla V_{\varepsilon}|^{p(y, t)} O(t^2 + |y|^2) \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega} f(x) |\nabla V_{\varepsilon}|^{p(x)} dx &= \int_{\mathbb{R}_+^N} f(y, t) \eta(y, t)^{p(y, t)} |\nabla V_{\varepsilon}|^{p(y, t)} dy dt \\ &\quad + \sum_{i, j=1}^{N-1} h^{ij} \int_{\mathbb{R}_+^N} t f(y, t) p(y, t) \eta(y, t)^{p(y, t)} |\nabla V_{\varepsilon}|^{p(y, t)-2} \partial_i V_{\varepsilon} \partial_j V_{\varepsilon} dy dt \\ &\quad - H \int_{\mathbb{R}_+^N} t f(y, t) \eta(y, t)^{p(y, t)} |\nabla V_{\varepsilon}|^{p(y, t)} dy dt \\ &\quad + O(\varepsilon^2) \end{aligned}$$

since

$$\begin{aligned} \int_{\mathbb{R}_+^N} |\nabla V_{\varepsilon}|^{p(y, t)} O(t^2 + |y|^2) dy dt &\leq C \int_{\mathbb{R}_+^N} |(y, t)|^2 |\nabla V_{\varepsilon}|^{p(y, t)} dy dt \\ &\leq C \varepsilon^2 \int_{\mathbb{R}_+^N} |(y, t)|^2 |\nabla V|^{p+O(\varepsilon)} dy dt \\ &= C \varepsilon^2 \int_{\mathbb{R}_+^N} |(y, t)|^2 |\nabla V|^p (1 + O(\varepsilon) \ln |\nabla V|) dy dt. \end{aligned}$$

As before this last integral is finite provided that $p < (N + 2)/3$.

The proof now follows applying Lemmas A.11 and A.12. \square

Bibliography

- [1] Adimurthi and S. L. Yadava. Positive solution for Neumann problem with critical nonlinearity on boundary. *Comm. Partial Differential Equations*, 16(11):1733–1760, 1991.
- [2] Claudianor O. Alves. Existence of positive solutions for a problem with lack of compactness involving the p -Laplacian. *Nonlinear Anal.*, 51(7):1187–1206, 2002.
- [3] Claudianor O. Alves and Yanheng Ding. Existence, multiplicity and concentration of positive solutions for a class of quasilinear problems. *Topol. Methods Nonlinear Anal.*, 29(2):265–278, 2007.
- [4] Claudianor O. Alves and Marco A. S. Souto. Existence of solutions for a class of problems in \mathbb{R}^N involving the $p(x)$ -Laplacian. In *Contributions to nonlinear analysis*, volume 66 of *Progr. Nonlinear Differential Equations Appl.*, pages 17–32. Birkhäuser, Basel, 2006.
- [5] Thierry Aubin. Problèmes isopérimétriques et espaces de Sobolev. *C. R. Acad. Sci. Paris Sér. A-B*, 280(5):Aii, A279–A281, 1975.
- [6] Abbas Bahri and Pierre-Louis Lions. On the existence of a positive solution of semilinear elliptic equations in unbounded domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(3):365–413, 1997.
- [7] Richard Beals and Roderick Wong. *Special functions*, volume 126 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. A graduate text.
- [8] Haïm Brézis and Elliott Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983.
- [9] Haïm Brézis and Louis Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.
- [10] Alberto Cabada and Rodrigo L. Pouso. Existence theory for functional p -Laplacian equations with variable exponents. *Nonlinear Anal.*, 52(2):557–572, 2003.
- [11] Yunmei Chen, Stacey Levine, and Murali Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406 (electronic), 2006.

- [12] Pablo L. De Nápoli, Julián Fernández Bonder, and Analía Silva. Multiple solutions for the p -Laplace operator with critical growth. *Nonlinear Anal.*, 71(12):6283–6289, 2009.
- [13] Françoise Demengel and Emmanuel Hebey. On some nonlinear equations involving the p -Laplacian with critical Sobolev growth. *Adv. Differential Equations*, 3(4):533–574, 1998.
- [14] Lars Diening, Petteri Harjulehto, Peter Hästö, and Michael Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.
- [15] G. Dinca, P. Jebelean, and J. Mawhin. Variational and topological methods for Dirichlet problems with p -Laplacian. *Port. Math. (N.S.)*, 58(3):339–378, 2001.
- [16] Teodora-Liliana Dinu. Nonlinear eigenvalue problems in Sobolev spaces with variable exponent. *J. Funct. Spaces Appl.*, 4(3):225–242, 2006.
- [17] Pavel Drábek and Yin Xi Huang. Multiplicity of positive solutions for some quasilinear elliptic equation in \mathbf{R}^N with critical Sobolev exponent. *J. Differential Equations*, 140(1):106–132, 1997.
- [18] I. Ekeland. On the variational principle. *J. Math. Anal. Appl.*, 47:324–353, 1974.
- [19] José F. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)*, 136(1):1–50, 1992.
- [20] Lawrence C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1990.
- [21] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [22] Xian-Ling Fan and Qi-Hu Zhang. Existence of solutions for $p(x)$ -Laplacian Dirichlet problem. *Nonlinear Anal.*, 52(8):1843–1852, 2003.
- [23] Xianling Fan and Dun Zhao. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 263(2):424–446, 2001.
- [24] Julián Fernández Bonder. Multiple solutions for the p -Laplace equation with nonlinear boundary conditions. *Electron. J. Differential Equations*, pages No. 37, 7 pp. (electronic), 2006.
- [25] Julián Fernández Bonder and Julio D. Rossi. Existence results for the p -Laplacian with nonlinear boundary conditions. *J. Math. Anal. Appl.*, 263(1):195–223, 2001.
- [26] Julián Fernández Bonder and Julio D. Rossi. On the existence of extremals for the Sobolev trace embedding theorem with critical exponent. *Bull. London Math. Soc.*, 37(1):119–125, 2005.

- [27] Julián Fernández Bonder and Nicolas Saintier. Estimates for the Sobolev trace constant with critical exponent and applications. *Ann. Mat. Pura Appl. (4)*, 187(4):683–704, 2008.
- [28] Julián Fernández Bonder, Nicolas Saintier, and Analía Silva. On the sobolev trace theorem for variable exponent spaces in the critical range. In preparation.
- [29] Julián Fernández Bonder, Nicolas Saintier, and Analia Silva. Existence of solution to a critical equation with variable exponent. *Ann. Acad. Sci. Fenn. Math.*, 37:579–594, 2012.
- [30] Julián Fernández Bonder, Nicolas Saintier, and Analia Silva. On the Sobolev embedding theorem for variable exponent spaces in the critical range. *J. Differential Equations*, 253(5):1604–1620, 2012.
- [31] Julián Fernández Bonder and Analía Silva. Concentration-compactness principle for variable exponent spaces and applications. *Electron. J. Differential Equations*, pages No. 141, 18, 2010.
- [32] Yongqiang Fu. The principle of concentration compactness in $L^{p(x)}$ spaces and its application. *Nonlinear Anal.*, 71(5-6):1876–1892, 2009.
- [33] J. García Azorero and I. Peral Alonso. Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Amer. Math. Soc.*, 323(2):877–895, 1991.
- [34] Alfred Gray. *Tubes*. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1990.
- [35] Petteri Harjulehto, Peter Hästö, Mika Koskenoja, and Susanna Varonen. The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values. *Potential Anal.*, 25(3):205–222, 2006.
- [36] Ondrej Kováčik and Jiří Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116)(4):592–618, 1991.
- [37] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.
- [38] P.-L. Lions, F. Pacella, and M. Tricarico. Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions. *Indiana Univ. Math. J.*, 37(2):301–324, 1988.
- [39] Duchao Liu. Existence of multiple solutions for a $p(x)$ -Laplace equation. *Electron. J. Differential Equations*, pages No. 33, 11, 2010.
- [40] Mihai Mihăilescu. Elliptic problems in variable exponent spaces. *Bull. Austral. Math. Soc.*, 74(2):197–206, 2006.

- [41] Mihai Mihăilescu and Vicențiu Rădulescu. On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. *Proc. Amer. Math. Soc.*, 135(9):2929–2937 (electronic), 2007.
- [42] Yoshihiro Mizuta, Takao Ohno, Tetsu Shimomura, and Naoki Shioji. Compact embeddings for Sobolev spaces of variable exponents and existence of solutions for nonlinear elliptic problems involving the $p(x)$ -Laplacian and its critical exponent. *Ann. Acad. Sci. Fenn. Math.*, 35(1):115–130, 2010.
- [43] Bruno Nazaret. Best constant in Sobolev trace inequalities on the half-space. *Nonlinear Anal.*, 65(10):1977–1985, 2006.
- [44] A. I. Nazarov and A. B. Reznikov. On the existence of an extremal function in critical Sobolev trace embedding theorem. *J. Funct. Anal.*, 258(11):3906–3921, 2010.
- [45] Ireneo Peral Alonso. *Multiplicity of Solutions for the p -Laplacian*. Second School of Nonlinear Functional Analysis and Applications to Differential Equations. International Center for Theoretical Physics, 1997.
- [46] S. I. Pohožaev. On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR*, 165:36–39, 1965.
- [47] Paul H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.
- [48] Michael Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
- [49] Nicolas Saintier. Asymptotic estimates and blow-up theory for critical equations involving the p -Laplacian. *Calc. Var. Partial Differential Equations*, 25(3):299–331, 2006.
- [50] Jacob T. Schwartz. Generalizing the Lusternik-Schnirelman theory of critical points. *Comm. Pure Appl. Math.*, 17:307–315, 1964.
- [51] Analia Silva. Multiple solutions for the $p(x)$ -Laplace operator with critical growth. *Adv. Nonlinear Stud.*, 11(1):63–75, 2011.
- [52] Michael Struwe. Three nontrivial solutions of anticoercive boundary value problems for the pseudo-Laplace operator. *J. Reine Angew. Math.*, 325:68–74, 1981.
- [53] Giorgio Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.

Index

- $K(N, p)^{-1}$, 16
- L^p space, 14
- L^p - norm, 14
- $L^{p(x)}$ space, 18
- $L^{p(x)}$ -norm, 18
- $S(p(\cdot), q(\cdot), \Omega)$, 19
- $T(p(\cdot), r(\cdot), \Omega, \Gamma)$, 58
- $T(p(\cdot), r(\cdot), \Omega)$, 20
- $W^{1,p(x)}$ space, 18
- $W^{1,p(x)}$ -norm, 19
- $W_0^{1,p(x)}$ space, 19
- $W^{1,p}$ space, 14
- $W^{1,p}$ - norm, 14
- $W_0^{1,p}$ space, 15
- $\mathcal{P}(E, \mu)$, 27
- $\bar{K}(N, p)^{-1}$, 17
- \bar{S}_x , 35
- \bar{T} , 58
- \bar{T}_{x_i} , 41
- \bar{T}_x , 58
- \bar{S} , 48
- $\rho_{1,p(x)}$, 28
- $\rho_{p(x)}$, 27
- p -Laplacian, 15
- $p(x)$ -Laplacian, 21

- Best constant in Sobolev Immersion, 15
- Best constant in trace Immersion, 17
- boundary upper Minkowsky content, 45
- Brezis-Lieb lemma, 38

- Critical point, 31
- Critical set, 22
- Critical Sobolev exponent, 15

- Critical Sobolev exponent for variable exponent, 19
- Critical trace exponent, 17
- Critical value, 31

- dS, 27

- Electrorheological fluids, 20

- Fermi Coordinates, 109
- Fréchet differentiable, 30

- Genus, 31

- Hölder-type inequality, 29

- Log-Hölder continuous, 29

- Mountain pass theorem, 31

- Palais Smale, 31
- Poincaré inequality, 15, 29

- Rayleigh quotient, 47
- Reverse Hölder inequality, 35

- Sobolev Extremals, 15
- Sobolev immersion, 14
- Sobolev immersion for variable exponent spaces, 29

- The concentration–compactness principle for the Sobolev immersion, 34
- The concentration–compactness principle for the Sobolev trace immersion, 41
- The variational principle of Ekeland, 33
- Trace Extremals, 18
- Trace immersion, 30

Uniformly subcritical, 19, 20
upper Minkowski content, 45