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Descomposiciones atómicas.
Algunos problemas de localidad en el espacio de fases.

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires
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José Luis Romero

Directora de tesis: Ursula Molter.
Consejera de estudios: Ursula Molter.
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Resumen: Considerar un espacio funcional como un espacio de coórbita significa reformular las propiedades que lo definen como condiciones de tamaño impuestas sobre alguna transformación adecuada. El espacio de fases es el conjunto de grados de libertad subyacentes en esa descripción. En esta tesis se estudia la relación entre ciertas operaciones en un espacio funcional y su descripción en el espacio de fases. Más precisamente, se prueba que algunas construcciones sobre el espacio de fases en efecto hacen lo que se espera que hagan.

Palabras clave: Descomposición atómica, Espacio de coórbitas, Espacio de fases, Wavelets, Gabor, Análisis de tiempo-frecuencia, Localización, Multiplicadores.

Atomic decompositions.

Some locality problems in phase-space.

Abstract: Considering a functional space as a coorbit space amounts to formulating the properties that define it as size conditions imposed on a certain transform. The phase-space is the underlying set of degrees of freedom in such a description. In this thesis we study the relation between certain operations on a functional spaces and their description in phase-space. More precesily, we show that certain constructions on phase-space indeed yield what they are expected to.

Keywords: Atomic decomposition, Coorbit space, Phase-space, Wavelets, Gabor, Time-frequency analysis, Localization, Multipliers.

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Introducción

Una descomposición atómica de un espacio funcional $\boldsymbol{X}$ es una familia de *átomos* $\mathcal{A} \subseteq \boldsymbol{X}$ y una operación que descompone cada elemento de $\boldsymbol{X}$ como superposición de átomos. Por ejemplo, si $\boldsymbol{X}$ es un espacio de Banach y el conjunto de los átomos consiste de una sucesión de vectores $\mathcal{A} = \{x_k : k \in \mathbb{N}\}$, la operación de descomposición podría estar dada por una familia de funcionales lineales $\{f_k : k \in \mathbb{N}\} \subseteq \boldsymbol{X}'$. La descomposición de un elemento $x \in \boldsymbol{X}$ como superposición de átomos se logra mediante una serie convergente,

$$x = \sum_k f_k(x) x_k.$$

Más generalmente, el conjunto de los átomos podría estar parametrizado por un espacio de medida, $\mathcal{A} = \{f_w : w \in \Omega\}$. La operación de descomposición es en este caso una transformación lineal $x \mapsto T(x)$ que envía un elemento $x \in \boldsymbol{X}$ a una función medible sobre Ω. Cada elemento $x \in \boldsymbol{X}$ se representa en términos de átomos como,

$$x = \int_\Omega T(x)(w) f_w dw. \tag{1}$$

Este modelo incluye al ejemplo previo como el caso donde $\boldsymbol{X}$ es un conjunto numerable provisto de la medida de contar, pero también permite resoluciones de la identidad "continuas".

La fórmula reproductiva de Calderón es un ejemplo de una resolución continua de la identidad. Sea $\psi : \mathbb{R}^d \to \mathbb{R}$ una función suave, radial, con varios momentos nulos y cuya transformada de Fourier satisface,

$$\int_0^{+\infty} \hat{\psi}(tw) \frac{dt}{t} = 1, \qquad (w \neq 0).$$

En estas condiciones, para cada $f \in L^2(\mathbb{R}^d)$,

$$f(x) = \int_0^{+\infty} (f * \psi_t * \psi_t)(x) \frac{dt}{t}, \tag{2}$$

donde $\psi_t(x) := t^{-d}\psi(x/t)$. (La integral debe ser interpretada en el sentido débil, no puntualmente). Si tomamos como colección de átomos el conjunto de todas las traslaciones y dilataciones de ψ,

$$\mathcal{A} = \left\{ \psi_{t,x} = t^{d/2}\psi_t(\cdot - x) \,\middle|\, x \in \mathbb{R}^d, t \in (0, +\infty) \right\},$$

y definimos la transformación W como,

$$W(f)(x,t) := (f * \psi_t)(x) = t^{-d/2} \int_{\mathbb{R}^d} f(y)\psi\left(\frac{x-y}{t}\right)dx, \tag{3}$$

la fórmula de la Ecuación (2) resulta ser,

$$f = \int_0^{+\infty} \int_{\mathbb{R}^d} W(f)(x,t)\psi_{t,x} dx \frac{dt}{t^{d+1}}. \tag{4}$$

La transformación de la Ecuación (3) se conoce como la transformada wavelet continua (con ventana ψ) y envía $L^2(\mathbb{R}^d)$ isométricamente dentro de $L^2(\mathbb{R}^d \times (0,+\infty), dxdt/t^{d+1})$. La fórmula de la Ecuación (4) recupera explícitamente una función $f \in L^2(\mathbb{R}^d)$ a partir de su transformada wavelet, presentando a f como una superposición de versiones trasladadas y reescaladas de ψ.

La expansión en términos de átomos en la Ecuación (4) es válida no sólo para $f \in L^2(\mathbb{R}^d)$ sino también para una amplia gama de espacios funcionales que incluye a los espacios de Lebesgue $L^p, (1 < p < +\infty)$, los espacios de Sobolev y, más generalmente, a toda la clase de espacios de Besov y Triebel-Lizorkin (véase la Sección 1.12). Más aún, la norma de una función f en cada uno de esos espacios es equivalente a la norma de $W(f)$ en un espacio de Lebesgue con pesos adecuado. Esto significa que las propiedades de suavidad que definen aquellos espacios pueden ser reformuladas como condiciones de tamaño y suavidad mediante la transformada wavelet.

El mismo tipo de análisis puede llevarse a cabo usando un rango discreto de traslaciones y dilataciones. El conjunto de átomos,

$$\mathcal{A} = \left\{ \psi_{k,j} = 2^{-j/2}\psi(2^{-j}\cdot - k) \,\middle|\, k \in \mathbb{Z}^d, j \in \mathbb{Z} \right\},$$

se llama un sistema wavelet. Para una función ψ cuidadosamente elegida, $\mathcal{A}$ resulta una base ortonormal de $L^2(\mathbb{R}^d)$, proporcionando entonces el desarrollo,

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left\langle f, \psi_{k,j} \right\rangle \psi_{k,j},$$

para toda $f \in L^2(\mathbb{R}^d)$. Más aún, ese desarrollo se extiende a la misma gama de espacios que en el caso continuo.

El análisis de tiempo-frecuencia da más ejemplos de descomposiciones atómicas. Las traslaciones de tiempo-frecuencia de una función $\varphi : \mathbb{R}^d \to \mathbb{C}$ están dadas por,

$$\pi(x,w)\varphi(y) := e^{2\pi i\langle x,w\rangle}\varphi(y-x). \tag{5}$$

Como antes, podemos considerar el conjunto de átomos,

$$\mathcal{A} = \left\{ \pi(x,w)\varphi \,\middle|\, (x,w) \in \mathbb{R}^d \times \mathbb{R}^d \right\}.$$

La transformada de Fourier de corto alcance (short-time Fourier transform) con ventana $\varphi \in L^2(\mathbb{R}^d)$ está dada por,

$$V(f)(x,w) := \int_{\mathbb{R}^d} f(y)\overline{\pi(x,w)\varphi(y)}dy.$$

Para una función “ventana“ φ adecuada, la transformada de Fourier de corto alcance envía $L^2(\mathbb{R}^d)$ isométricamente dentro de $L^2(\mathbb{R}^{2d})$. Más aún, cada $f \in L^2(\mathbb{R}^d)$ puede descomponerse como,

$$f = \int_{\mathbb{R}^d\times\mathbb{R}^d} V(f)(x,w)\pi(x,w)\varphi dxdw. \tag{6}$$

Así, cada f puede presentarse como una superposición de tralaciones en tiempo y frecuencia del átomo φ. El coeficiente correspondiente al átomo $\pi(x,w)\varphi$ es el número $V(f)(x,w)$. Esta cantidad es exactamente la transformada de Fourier de $f\overline{\varphi(\cdot - x)}$ evaluada en w. Luego, si la función ventana φ es suave y está bien concentrada espacialmente, el número $\pi(x,w)\varphi$ representa la influencia de la frecuencia w en f, cerca de x.

La fórmula de la Ecuación (6) puede extenderse también a otros espacios funcionales conocidos como *espacios de modulación* (véase la Sección 1.11). Estos espacios se definen mediante condiciones de concentración en tiempo y frecuencia. El espacio de modulación $M^p(\mathbb{R}^d)$, por ejemplo, es el conjunto de todas las distribuciones f tales que $V(f) \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$.

Al igual que en el caso de las descomposiciones tiempo-escala, hay una correspondiente teoría discreta. Un sistema de Gabor es un conjunto de átomos de la forma,

$$\mathcal{A} = \{ \varphi_{k,j} = \pi(\alpha k, \beta j)\varphi \,\big|\, k, j \in \mathbb{Z}^d \}, \tag{7}$$

donde $\alpha, \beta > 0$. Para una función ventana φ y parámetros α, β adecuados, cada $f \in L^2(\mathbb{R}^d)$ admite el desarrollo,

$$f = \sum_{k,j\in\mathbb{Z}^d} \left\langle f, \varphi_{k,j} \right\rangle \varphi_{k,j}. \tag{8}$$

Este desarrollo se extiende también a los espacios de modulación. Sin embargo, es sabido que si φ está bien concentrada en tiempo y frecuencia, entonces el sistema de la Ecuación (7) no puede ser una base. Luego, el desarrollo de la Ecuación (8) es necesariamente redundante.

Ambos ejemplos se encuadran en el marco de la llamada *teoría de coórbitas*. Los espacios de coórbitas son espacios funcionales definidos imponiendo condiciones de tamaño a una cierta transformación. Más precisamente, considerar un espacio funcional $\boldsymbol{X}$ como un espacio de coórbita consiste en dar una transformación $T : \boldsymbol{X} \to \boldsymbol{E}$ que incluye a $\boldsymbol{X}$ como un sumando directo de otro espacio funcional $\boldsymbol{E}$ que es *sólido*. Esto significa que la pertenencia a $\boldsymbol{E}$ está determinada por condiciones de tamaño (para una definición precisa ver la Sección 1.4).

El espacio $\boldsymbol{E}$ consiste en funciones sobre un conjunto $\mathcal{G}$ que frecuentemente es un grupo localmente compacto.

Cuando un espacio funcional $\boldsymbol{X}$ se identifica como un espacio de coórbitas, las propiedades de cada elemento $f \in \boldsymbol{X}$ quedan reformuladas como propiedades de decaimiento e integrabilidad de la función $T(f) \in \boldsymbol{E}$, que se suele llamar la *representación en el espacio de fases* de f. Los elementos de $\boldsymbol{X}$ pueden ser resintetizados a partir de su representación en el espacio de fases por medio de un operador $U : \boldsymbol{E} \to \boldsymbol{X}$ que es una inversa a izquierda de T (i.e. $f = UT(f)$).

La teoría de coórbitas de Feichtinger y Gröchenig estudia el caso en el que T surge de los coeficientes de representación asociados a la acción unitaria de un grupo localmente compacto (véase la Sección 1.10). Más precisamente, si π es una representación unitaria de un grupo localmente compacto $\mathcal{G}$ sobre un espacio de Hilbert $\mathbb{H}$ y $h \in \mathbb{H}$ es un vector adecuado, la transformada wavelet abstracta de un vector $f \in \mathbb{H}$ se define como,

$$V_h f(x) := \langle f, \pi(x)h \rangle, \quad (x \in \mathcal{G}).$$

Los espacios de coórbitas se definen imponiendo condiciones de tamaño y decaimiento a la transformada wavelet asociada a π. En este contexto, existe una versión de la fórmula de la Ecuación (1), con $\boldsymbol{X} = \mathbb{H}$, $\Omega = \mathcal{G}$, $T = V_h$, y el conjunto de átomos dado por la órbita de h, $\{\pi(x)h \,|\, x \in \mathcal{G}\}$. El ejemplo del análisis tiempo-escala se obtiene tomando π como la representación del grupo afín sobre $L^2(\mathbb{R}^d)$, dada por los operadores de traslación y dilatación, mientras que el caso del análisis de tiempo-frecuencia se obtiene a partir de la acción del grupo de Heisenberg sobre $L^2(\mathbb{R}^d)$ mediante las traslaciones de tiempo-frecuencia.

Uno de los resultados centrales de Feichtinger y Gröchenig es el hecho de que los espacios de coórbitas asociados a una representación de un grupo admiten una descomposición donde el conjunto de átomos se obtiene haciendo actuar cualquier subconjunto ”suficientemente denso“ de $\mathcal{G}$ sobre un vector admisible (véase la Sección 1.10 para más detalles). En los ejemplos de las descomposiciones tiempo-escala y tiempo-frecuencia, esto da los sistemas wavelet y de Gabor antes mencionados.

Aunque la mayoría de los ejemplos de lo que se entiende comúnmente por espacios de coórbitas están incluidos en el caso de las representaciones de un grupo, es ciertamente posible considerar espacios de coórbitas sin una acción de grupo subyacente. Un marco para un espacio de Hilbert $\mathbb{H}$ es un conjunto de vectores $\{f_k : k \in I\}$ que brinda un desarrollo

$$f = \sum_k c_k f_k, \tag{9}$$

donde los coeficientes $c \equiv \{c_k : k \in I\}$ dependen linealmente de $f \in \mathbb{H}$ y $\|f\|_{\mathbb{H}} \approx \|c\|_{\ell^2}$ (véase la Sección 1.8). Luego, los marcos son conjuntos de átomos para una descomposición atómica de un espacios de Hilbert. Los espacios de coórbitas asociados a un marco se definen imponiendo condiciones de sumabilidad a los coeficientes de la Ecuación (9) (véase la Sección 1.13).

Consideremos de nuevo el escenario abstracto de un espacio funcional $\boldsymbol{X}$ y una descomposición atómica instrumentada por un conjunto de átomos $\{f_w : w \in \Omega\}$, indexados por un espacio de medida Ω, y una transformación T que envía elementos de $\boldsymbol{X}$ a funciones medibles definidas sobre Ω, brindando la expansión,

$$x = \int_\Omega T(x)(w) f_w dw, \qquad (x \in \boldsymbol{X}). \tag{10}$$

El espíritu de la teoría de coórbitas es que el conjunto de átomos $\{f_w : w \in \Omega\}$ y la transformación T deben descomponer no sólo un espacio funcional $\boldsymbol{X}$, sino que deben servir de descripción simultánea para toda una gama de espacios. Los diferentes espacios queda caracterizados imponiendo diferentes normas sólidas a la transformación T. En el contexto abstracto los espacios de coórbitas se construyen así, mientras que en ejemplos concretos, espacios ya definidos tienen que ser identificados como espacios de coórbita de una cierta transformación.

El poder de una tal descripción de una familia de espacios funcionales está en el hecho de que las propiedades que definen cada espacio $\boldsymbol{X}$ quedan reformuladas como condiciones de tamaño mediante la transformación T. En estas condiciones es tentador tratar de describir diferentes operaciones sobre elementos $x \in \boldsymbol{X}$ como operaciones sobre $T(x)$. Esto se conoce como un enfoque de *espacio de fases*.

El término espacio de fases proviene de la física y se puede definir vagamente como un espacio donde cada estado de un sistema está representado por un único punto. Las funciones sobre el espacio de fases representan entonces una cantidad medible de un sistema. En el contexto de las descomposiciones atómicas, el término ”espacio de fases“ se usa mayormente en forma imprecisa, sin definir cuidadosamente su significado. En ese contexto, el rango de la transformada wavelet $T(X)$ juega el rol de la familia de funciones sobre el espacio de fases, mientras que el espacio de fases en sí mismo se entiende como el conjunto de ”grados de libertad“ subyacente para esa familia de funciones. Si por casualidad $T(X)$ resulta ser un álgebra C^* conmutativa, entonces el espacio de fases podría ser definido rigurosamente como su espectro, pero este es raramente el caso. Una definición formal del espacio de fases yace en el dominio de la geometría no conmutativa. En el caso del análisis de tiempo-frecuencia, la terminología del espacio de fases corresponde de hecho a conceptos relacionados en la mecánica cuántica. Más aún, sus relaciones con la geometría no conmutativa están siendo estudiadas rigurosamente (véase [85]).

En esta tesis se estudian dos problemas en el espacio de fases relacionados con descompociones atómicas y espacios de coórbitas. El primer problema que estudiamos es el de la *cirugía de marcos*. Dadas varias descomposiciones atómicas para un mismo espacio de coórbitas, construimos una nueva descomposición atómica para el mismo espacio, pegando porciones arbitrarias de las descomposiciones originales, siempre que la superposición entre estas porciones sea suficientemente grande. Las nociones de ”porción“ y ”superposición“ se consideran con respecto al espacio de fases. Esta técnica puede ser útil para producir descomposiciones atómicas cuando la forma exacta de los átomos es importante. Por ejemplo, los átomos podrían ser autovectores de ciertos operadores o representar ciertos funcionales lineales. Esperamos que la técnica de cirugía de marcos sea en una herramienta útil para construir descomposiciones atómicas adaptadas a problemas concretos, aunque no exploramеos aquí su aplicación a otras ramas de la matemática, posponíendola para una contribución futura.

Cuando se aplica a marcos de Gabor, el procedimiento de la cirugía da un resultado general de existencia para el concepto recientemente introducido de ”quilted Gabor frames“ [32]. Dada una familia de marcos, $G^i \equiv \left\{\pi(\lambda)g^i : \lambda \in \Lambda_i\right\}$, $(i \in I)$ (donde $\pi(\lambda)$ denota la traslación de tiempo-frecuencia definida en la Ecuación (5)), y un cubrimiento de $\mathbb{R}^d \times \mathbb{R}^d$, $\mathcal{E} \equiv \{E_i\}_{i \in I}$, construimos un nuevo marco de Gabor,

$$\left\{\pi(\lambda)g^i : i \in I, \lambda \in \Lambda_i, d(\lambda, E_i) \leq r\right\},$$

seleccionando de cada marco G^i aquellos elementos asociados con nodos de tiempo frecuencia cercanos a E_i. El concepto de ”quilted Gabor frame“ fue formalmente introducido en [31, 32] con el objeto de construir diccionarios funcionales adaptados al procesamiento de señales musicales. En efecto, existen muchas herramientas concretas para construir marcos de Gabor usando una función como ventana y un reticulado como conjunto de nodos de tiempo-frecuencia. La elección de la ventana determina el balance deseado entre resolución en tiempo y en frecuencia. El objetivo de los ”quilted Gabor frames“ es potenciar estas construcciones, permitiendo que el balance de resolución tiempo-frecuencia varíe a través del plano tiempo-frecuencia, como lo requiere, por ejemplo, la descripción de diferentes instrumentos musicales. Existen numerosos resultados numéricos para este tipo de sistemas (véase por ejemplo [12, 77]). Los resultados de esta tesis dan condiciones suficientes para la validez de esa construcción.

La cirugía de marcos puede aplicarse también al problema del muestreo irregular, considerando como átomos los vectores que representan los funcionales de evaluación. Dada una clase de funciones y familias de conjuntos de muestreo $\mathcal{X}$ para los que se tiene la estimación,

$$\|f\|_{L^p} \approx \|(f(x))_{x \in \mathcal{X}}\|_{\ell^p},$$

construimos nuevos conjuntos para los que esta relación sigue siendo válida. Más aún, dadas fórmulas explícitas de reconstrucción para los conjuntos originales, obtenemos fórmulas de reconstrucción aproximadas para los nuevos conjuntos.

Otras aplicaciones incluyen la identificación de ciertas clases de multiplicadores de tiempo-frecuencia. Los multiplicadores de Gabor surgen de aplicar una máscara a los coeficientes de un desarrollo de Gabor. Luego cada uno de estos operadores tiene la forma,

$$T = \sum_{\lambda \in \Lambda} c_\lambda P_\lambda,$$

donde $c_\lambda \in \mathbb{C}$ y P_λ es un operador de rango uno (esecialmente un projector sobre el subespacio generado por un átomo de tiempo-frecuencia). Cada operador en una clase de multiplicadores de Gabor, puede identificarse por medio de su *símbolo inferior*, que consiste en la sucesión de productos internos Hilbert-Schmidt $\{\langle T, P_\lambda\rangle \mid \lambda \in \Lambda\}$. Usando la técnica de cirugía obtenemos condiciones suficientes para identificar una clase de multiplicadores de Gabor mediante un *símbolo inferior mixto*, construido usando diferentes tipos de operadores de rango uno P_λ para λ en distintas regiones del plano tiempo-frecuencia.

El segundo problema que estudiamos es el de caracterizar espacios de coórbitas mediante multiplicadores del espacio de fase. Sea $\boldsymbol{X}$ un espacio funcional considerado como espacio de coórbita mediante una transformación $T : \boldsymbol{X} \to \boldsymbol{E}$ que incluye a $\boldsymbol{X}$ como un subespacio complementado de un espacio de funciones sólido $\boldsymbol{E}$, sobre un grupo localmente compacto. Como $T(\boldsymbol{X})$ es complementado en $\boldsymbol{E}$, existe una retracción $U : \boldsymbol{E} \to \boldsymbol{X}$ que sirve de inversa a izquiera de T (i.e., $UT(x) = x$).

Con el propósito de retocar la propiedades de una función f que están exhibidas por su representación en el espacio de fases $T(f)$, podemos considerar operadores de la forma $M_m(f) = U(mT(f))$, que aplican una máscara m a $T(f)$. Vamos a llamar a estos operadores *multiplicadores del espacio de fases*. Por supuesto, la interpretación rigurosa de $M_m(f)$ es problemática dado que, en general, $TM_m(f) \neq mT(f)$. Cuando T es la transformada wavelet abstracta asociada a una representación unitaria de un grupo, estos operadores se conocen como *operadores de localización* o *multiplicadores wavelet* [71, 113, 83]. En el caso del análisis de tiempo-frecuencia, estos operadores se conocer como *operadores de localización en tiempo-frecuencia* o multiplicadores de la transformada de Fourier de corto alcance [24, 21, 22, 13].

Estudiamos el problema de caracterizar la norma de un espacio de coórbitas en términos de familias de multiplicadores del espacio de fases asociadas a una partición de la unidad en $\mathcal{G}$. Específicamente, supongamos que $\boldsymbol{X}$ es un espacio de Banach considerado como espacio de coórbita mediante una transformación $T : \boldsymbol{X} \to \boldsymbol{E}$ que tiene una inversa a izquierda $U : \boldsymbol{E} \to \boldsymbol{X}$. Sea $\{\theta_\gamma\}_\gamma$ una partición de la unidad sobre $\mathcal{G}$ y consideramos los respectivos multiplicadores del espacio de fase $M_\gamma(f) = U(\theta_\gamma T(f))$. De la partición de la unidad sólo se asume que satisface ciertas condiciones de localización espacial, pero fuera de eso es arbitraria. Probamos que $\|f\|_{\boldsymbol{X}}$ es equivante a la norma de la sucesión $\{\|M_\gamma(f)\|_{\boldsymbol{B}}\}_\gamma$ en una versión discreta del espacio $\boldsymbol{E}$, donde el espacio $\boldsymbol{B}$ puede elegirse dentro de una amplia clase de espacios funcionales. Más aún, probamos que la aplación $f \mapsto \{M_\gamma(f)\}_\gamma$ incluye a $\boldsymbol{X}$ como un sumando directo de un espacio de sucesiones $\boldsymbol{B}$-valuadas, obtenido mediante una discretización de $\boldsymbol{E}$. Esto cuantifica la relación entre un elemento $f \in \boldsymbol{X}$ y sus partes localizadas en el espacio de fases $\{M_\gamma(f)\}_\gamma$.

En el caso del análisis de tiempo-frecuencia, Dörfler y Gröchenig obtuvieron recientemente este tipo de caracterización de los espacios de modulación [34], usando ciertas técnicas de álgebras de rotación (toro no conmutativo) desarrolladas en [66] y [64], y teoría espectral de espacios de Hilbert (véase también [33]). En esta tesis se usa un enfoque diferente para obtener resultados en contextos donde las técnicas de [34] no son aplicables como las descomposiciones tiempo-escala y los espacios de Besov. Como resultado adicional de las nuevas técnicas, obtenemos una versión más fuerte del resultado principal de [34], donde se limita las particiones de la unidad admisibles a traslaciones de una misma función por un reticulado y el espacio $\boldsymbol{B}$ es L^2. Estas restricciones parecen ser esenciales para las técnicas usadas en [34].

Para ambos problemas usamos el mismo enfoque general. Consideramos un modelo para el espacio de fases que consiste en un espacio funcional sólido, llamado ”medio ambiente”, y un cierto subespacio complementado, llamado “espacio atómico”. Probamos todos los resultados en este contexto y luego obtenemos las aplicaciones usando como espacio atómico el rango de distintas transformaciones.

En cuanto a la organización de la tesis, el Capítulo 1 da una breve referencia sobre la teoría de coórbitas, las descomposicones atómicas y temas relacionados, haciendo que este trabajo sea mayormente autocontenido. En los capítulos 2 y 4 adaptamos algunas herramientas conocidas para hacerlas aplicables a nuestro contexto. El Capítulo 3 introduce el modelo para el espacio de fases y da algunos ejemplos. El Capítulo 5 desarrolla el método de la cirugía de marcos, mientras que en los capítulos 6 y 7 se estudia el problema de la caracterización de espacios de coórbitas mediante multiplicadores del espacio de fases. Luego, los Capítulos 5, 6 y 7 contienen los resultados principales de esta tesis. Cada capítulo comienza con una introducción a los temas y resultados en cuestión, seguida de una discusión sobre las técnicas involucradas. Al final de cada capítulo se presentan aplicaciones de los resultados a descomposiciones atómicas y espacios de coórbitas. El Capítulo 7 es ligeramente no autocontenido. Ciertos resultados ahí requieren conceptos más refinados sobre grupos topológicos. Estas sutilezas son, sin embargo, mayormente irrelevantes para las aplicaciones propuestas.

La mayoría de los resultados de esta tesis fueron presentados en artículos de investigación. Las estimaciones de la Sección 4.2 fueron publicadas en [93]. Los resultados del Capítulo 5 están contenidos en [95], mientras que los de los Capítulos 6 y 7 fueron presentados en [94].

Introduction

Loosely speaking, an atomic decomposition of a functional space $\boldsymbol{X}$ is a family of *atoms* $\mathcal{A} \subseteq \boldsymbol{X}$ and a procedure to decompose every element of $\boldsymbol{X}$ as a superposition of atoms. For example, if $\boldsymbol{X}$ is a Banach space and the set of atoms consist of a sequence of vectors $\mathcal{A} = \{x_k : k \in \mathbb{N}\}$, the decomposition procedure could be a sequence of linear functionals $\{f_k : k \in \mathbb{N}\} \subseteq \boldsymbol{X}'$. The decomposition of an element x as superposition of atoms is achieved by a norm convergent series,

$$x = \sum_k f_k(x) x_k.$$

More generally, the set of atoms could be parametrized by a measure space $\mathcal{A} = \{f_w : w \in \Omega\}$. The decomposition procedure is in that case a linear transform $x \mapsto T(x)$ that maps an element $x \in \boldsymbol{X}$ to a measurable function over Ω and every element $x \in \boldsymbol{X}$ is represented in terms of atoms as

$$x = \int_\Omega T(x)(w) f_w dw. \tag{11}$$

This model includes the previous example as the case where $\boldsymbol{X}$ is a countable set endowed with the counting measure, and also allows for "continuous" resolutions of the identity.

Calderón's reproducing formula is an example of a continuous resolution of the identity. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a radial smooth function with several vanishing moments and a Fourier transform satisfying,

$$\int_0^{+\infty} \hat{\psi}(tw) \frac{dt}{t} = 1, \qquad (w \neq 0).$$

Then for any $f \in L^2(\mathbb{R}^d)$,

$$f(x) = \int_0^{+\infty} (f * \psi_t * \psi_t)(x) \frac{dt}{t}, \tag{12}$$

where $\psi_t(x) := t^{-d} \psi(x/t)$. (The integral should not be interpreted in the pointwise sense, but weakly). If we let the set of atoms be the collection of all translations and dilations of ψ,

$$\mathcal{A} = \{ \psi_{t,x} = t^{d/2} \psi_t(\cdot - x) \,\big|\, x \in \mathbb{R}^d, t \in (0, +\infty) \},$$

and let the transform W be given by,

$$W(f)(x,t) := (f * \psi_t)(x) = t^{-d/2} \int_{\mathbb{R}^d} f(y)\psi\left(\frac{x-y}{t}\right)dx, \tag{13}$$

then the formula in Equation (12) takes the form,

$$f = \int_0^{+\infty} \int_{\mathbb{R}^d} W(f)(x,t)\psi_{t,x} dx \frac{dt}{t^{d+1}}. \tag{14}$$

The transform in Equation (13) is known as the wavelet transform (with window ψ) and maps $L^2(\mathbb{R}^d)$ isometrically into $L^2(\mathbb{R}^d \times (0,+\infty), dxdt/t^{d+1})$. The formula in Equation (14) explicitly recovers a function $f \in L^2(\mathbb{R}^d)$ from its wavelet transform by presenting f as a superposition of shifted and scaled versions of ψ.

The expansion in terms of affine atoms in Equation (14) is valid not only for $f \in L^2(\mathbb{R}^d)$ but also for a wide range of functional spaces that includes the Lebesgue spaces $L^p, (1 < p < +\infty)$, Sobolev spaces and more generally the whole class of Besov and Triebel-Lizorkin spaces (see Section 1.12). Moreover, the norm of a function f in each of those spaces is equivalent to the norm of $W(f)$ in an adequate weighted Lebesgue space. This means that the smoothness properties defining these spaces can be reformulated as size and decay conditions by means of the wavelet transform.

It is also possible to carry out the same analysis using only a discrete set of shifts and scales. The set of atoms,

$$\mathcal{A} = \{ \psi_{k,j} = 2^{-j/2}\psi(2^{-j} \cdot -k) \,\big|\, k \in \mathbb{Z}^d, j \in \mathbb{Z} \},$$

is called a wavelet system. For a very carefully chosen function ψ, $\mathcal{A}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, yielding the expansion,

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left\langle f, \psi_{k,j} \right\rangle \psi_{k,j},$$

for $f \in L^2(\mathbb{R}^d)$. Moreover that expansion extends to the same functional spaces that the continuous one.

Time-frequency analysis provides more examples of atomic decompositions. The time-frequency shifts of a function $\varphi : \mathbb{R}^d \to \mathbb{C}$ are given by,

$$\pi(x,w)\varphi(y) := e^{2\pi i \langle x,w \rangle}\varphi(y-x). \tag{15}$$

As before, we can consider as set of atoms,

$$\mathcal{A} = \{ \pi(x,w)\varphi \,\big|\, (x,w) \in \mathbb{R}^d \times \mathbb{R}^d \}.$$

The short-time Fourier transform with window $\varphi \in L^2(\mathbb{R}^d)$ is given by,

$$V(f)(x,w) := \int_{\mathbb{R}^d} f(y)\overline{\pi(x,w)\varphi(y)}dy.$$

For an adequate window function φ, the short-time Fourier transform maps $L^2(\mathbb{R}^d)$ isometrically into $L^2(\mathbb{R}^{2d})$. Moreover, every $f \in L^2(\mathbb{R}^d)$ can be decomposed as,

$$f = \int_{\mathbb{R}^d \times \mathbb{R}^d} V(f)(x, w)\pi(x, w)\varphi dxdw. \tag{16}$$

Thus, every f can be presented as a superposition of time-frequency shifts of φ. The coefficient corresponding to the atom $\pi(x, w)\varphi$ is the number $V(f)(x, w)$, which equals the Fourier transform of $f\overline{\varphi(\cdot - x)}$ at w. Consequently, if the window φ is smooth and well-concentrated, the number $\pi(x, w)\varphi$ represents the influence of the frequency w near the point x.

The formula in Equation (16) can be extended to other functional spaces known as *modulation spaces* (see Section 1.11), defined by time-frequency concentration conditions. The modulation space $M^p(\mathbb{R}^d)$, for example, is defined as the set of all distributions f such that $V(f) \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$.

As in the case of time-scale decompositions, there is a corresponding discrete theory. A Gabor system is a set of atoms of the form,

$$\mathcal{A} = \{ \varphi_{k,j} = \pi(\alpha k, \beta j)\varphi \,\big|\, k, j \in \mathbb{Z}^d \}, \tag{17}$$

where $\alpha, \beta > 0$. For an adequate window function φ and lattice parameters α, β, every $f \in L^2(\mathbb{R}^d)$ admits the expansion,

$$f = \sum_{k,j \in \mathbb{Z}^d} \left\langle f, \varphi_{k,j} \right\rangle \varphi_{k,j}, \tag{18}$$

and this expansion also extends to modulation spaces. However, it is known that if φ is well-concentrated in time and frequency, the system in Equation (17) cannot be a basis. Hence, the expansion in Equation (18) is necessarily redundant.

Both examples fit into the general framework of coorbit theory. Coorbit spaces are functional spaces defined by imposing size conditions to a certain transform. More precisely, considering a functional space $\boldsymbol{X}$ as a coorbit space consists of giving a transform $T : \boldsymbol{X} \to \boldsymbol{E}$ that embeds $\boldsymbol{X}$ into another functional space $\boldsymbol{E}$ that is *solid*. This means that membership in $\boldsymbol{E}$ is determined by size conditions (for a precise definition see Section 1.4). The space $\boldsymbol{E}$ consists of functions defined on a set $\mathcal{G}$ that is commonly taken to be a locally compact group.

When a functional space $\boldsymbol{X}$ is identified as a coorbit space, the properties of an element $f \in \boldsymbol{X}$ are reformulated in terms of decay and integrability conditions for the function $T(f) \in \boldsymbol{E}$, that is sometimes referred to as the *phase-space representation* of f. The elements of $\boldsymbol{X}$ can be resynthesized from their phase-space representation by means of an operator $U : \boldsymbol{E} \to \boldsymbol{X}$ that is a left-inverse for T (i.e. $f = UT(f)$).

The coorbit theory of Feichtinger and Gröchenig studies the case when T arises as the representation coefficients of a unitary action of a locally compact group (see Section 1.10). More precisely, if π is a unitary representation of locally compact group $\mathcal{G}$ on a Hilbert space

$\mathbb{H}$ and $h \in \mathbb{H}$ is an adequate vector, then the abstract wavelet transform of a vector $f \in \mathbb{H}$ is defined as,

$$V_h f(x) := \langle f, \pi(x)h \rangle, \quad (x \in \mathcal{G}).$$

Coorbit spaces are defined by imposing size and decay conditions to the wavelet transform associated with π. In this setting, a version of the formula in Equation (11) is valid, where $\boldsymbol{X} = \mathbb{H}$, $\Omega = \mathcal{G}$, $T = V_h$ and the atoms are given by the orbit of h, $\{\pi(x)h \mid x \in \mathcal{G}\}$. The example of time-scale analysis is obtained by letting π be the representation of the affine group on $L^2(\mathbb{R}^d)$ given by the translation and dilation operators, whereas the one of time-frequency analysis is obtained by letting the Heisenberg group act on $L^2(\mathbb{R}^d)$ by time-frequency shifts.

One of the central results of Feichtinger and Gröchenig is the fact that coorbit spaces associated with group representations admit an atomic decomposition whose atoms are produced by letting any "sufficiently dense" subset of the group act on an admissible vector (see Section 1.10 for details). In the examples of time-scale and time-frequency analysis this yields wavelet and Gabor expansions.

Even though most of the examples of what is commonly understood as a coorbit space are covered by the setting of group representations, it is certainly possible to consider coorbit theory without an underlying group action. A frame for a Hilbert space $\mathbb{H}$ is a set of vectors $\{f_k : k \in I\}$ that provides an expansion,

$$f = \sum_k c_k f_k, \tag{19}$$

where the coefficients $c \equiv \{c_k : k \in I\}$ depend linearly on $f \in \mathbb{H}$ and $\|f\|_{\mathbb{H}} \approx \|c\|_{\ell^2}$ (see Section 1.8). Thus, frames are sets of atoms for an atomic decomposition of a Hilbert space. Coorbit spaces associated with frames are defined by imposing decay and summability conditions to the coefficients in Equation (19) (see Section 1.13).

Let us consider again the abstract setting of a functional space $\boldsymbol{X}$ and an atomic decomposition implemented by a set of atoms $\{f_w : w \in \Omega\}$ indexed by a measure space Ω, and a transform T that maps elements of $\boldsymbol{X}$ into measurable functions over Ω, providing an expansion,

$$x = \int_\Omega T(x)(w) f_w dw, \qquad (x \in \boldsymbol{X}). \tag{20}$$

The spirit of coorbit theory is that the set of atoms $\{f_w : w \in \Omega\}$ and the coefficient mapping T should not only decompose one functional space $\boldsymbol{X}$, but serve as a simultaneous description of a whole family of spaces. Different spaces are characterized by imposing different solid norms on the range of the transform T. In the abstract setting coorbit spaces are constructed in this way, while in specific examples, already existing spaces have to be identified as coorbit spaces of a certain transform.

The power of such a description of a family of functional spaces lies in the fact that the properties defining each space $\boldsymbol{X}$ are reformulated as size conditions by means of the transform T. It is then tempting to describe operations on an elements $x \in \boldsymbol{X}$ as operations on $T(x)$. This is commonly called a *phase-space* approach.

The term phase-space comes from physics and is vaguely defined as a space where each possible state of a system is represented by a unique point. Functions on phase-space then represent measurable quantities of a system. In the context of atomic decompositions the term "phase-space" is mostly used loosely, without giving it a precise meaning. In that context, the range of the wavelet transform $T(X)$ plays the role of the family of all functions on phase-space, and phase-space itself is understood as the underlying set of "degrees of freedom" for that family of functions. If $T(X)$ happens to be a commutative C^*-algebra, then phase-space could be precisely defined as its spectrum, but this is hardly ever the case. A formal definition of phase-space lies then in the domain of non-commutative geometry. In the case of time-frequency analysis, the phase-space terminology actually corresponds to related concepts in quantum mechanics. Moreover its links with non-commutative geometry are currently being rigorously studied (see [85]).

In this thesis we study two phase-space problems related to atomic decompositions and coorbit spaces. The first problem that we study is the one of *frame surgery*. Given several atomic decompositions for a coorbit space, we construct a new atomic decomposition for the same space by piecing together arbitrary portions of the original atoms, provided that the overlaps between these portions are large enough. The notions of "portion" and "overlap" are considered with respect to phase-space. This technique could be useful to produce atomic decompositions when the exact form of the atoms is important. For example, the atoms can be eigenvectors of a certain family of operators or representatives of certain linear functionals. We hope that frame surgery will became a useful tool to construct atomic decompositions adapted to concrete problems, although we do not study here its applications to other areas of mathematics, postponing it to a future contribution.

When applied to Gabor frames, the surgery scheme yields a general existence result for the recently introduced concept of quilted Gabor frame [32]. Given a family of Gabor frames, $G^i \equiv \left\{\pi(\lambda)g^i : \lambda \in \Lambda_i\right\}$, $(i \in I)$ (where $\pi(\lambda)$ denotes the time-frequency shift defined in Equation (15)), and a covering of $\mathbb{R}^d \times \mathbb{R}^d$, $\mathcal{E} \equiv \{E_i\}_{i\in I}$, we construct a new Gabor frame,

$$\left\{\pi(\lambda)g^i : i \in I, \lambda \in \Lambda_i, d(\lambda, E_i) \leq r\right\},$$

by selecting from each frame G^i those elements associated with time-frequency nodes lying near E_i. The concept of quilted Gabor frame was formally introduced in [31, 32] with the aim of constructing functional dictionaries that are well-suited for the processing of musical signals. Indeed, there are several tools available to construct Gabor frames using a single window function and a lattice as set of time-frequency nodes. The choice of the window function determines the desired balance between time and frequency resolution. The objective of quilted Gabor frames was to empower these constructions by allowing the time-frequency resolution balance to vary over the time-frequency plane, as required, for example, for the description of different kinds of musical instruments. There are several numerical results for these kinds of systems (see for example [12, 77]). The results in this thesis give general sufficient conditions for the validity of these constructions.

Frame surgery can be also applied to the irregular sampling problem, by considering as atoms the vectors representing the evaluation functionals. Given a class of functions and a family of sampling sets $\mathcal{X}$ for which a sampling estimate,

$$\|f\|_{L^p} \approx \|(f(x))_{x\in\mathcal{X}}\|_{\ell^p}$$

is known to hold, we can construct new sets for which the sampling inequality still holds. Moreover, given explicit reconstruction formulas for the original sets, we get approximate reconstruction formulas for the new sets.

Further applications include identifying certain classes of time-frequency multipliers. Gabor multipliers are operators that arise from applying a mask to the coefficients associated with a Gabor frame expansion; hence each of these operators has the form

$$T = \sum_{\lambda\in\Lambda} c_\lambda P_\lambda,$$

where $c_\lambda \in \mathbb{C}$ and P_λ is a rank-one operator (essentially a projector onto the subspace generated by a time-frequency atom). Each operator in a given class of Gabor multipliers can be identified by its associated *lower symbol* which consists of the Hilbert-Schmidt inner products $\{ \langle T, P_\lambda\rangle \mid \lambda \in \Lambda \}$. Using the surgery scheme we get a sufficient condition to identify a class of Gabor multipliers by a *mixed lower symbol* constructed by using different types of rank-one operators P_λ for λ in different regions of the time-frequency plane.

The second problem we study is the one of characterizing coorbit spaces through phase-space multipliers. Let $\boldsymbol{X}$ be a functional space that is regarded as a coorbit space by means of a transform $T : \boldsymbol{X} \to \boldsymbol{E}$ that embeds $\boldsymbol{X}$ as a complemented subspace of a solid function space $\boldsymbol{E}$ over a locally-compact group $\mathcal{G}$. Since $T(\boldsymbol{X})$ is complemented in $\boldsymbol{E}$, there exists a retraction $U : \boldsymbol{E} \to \boldsymbol{X}$ serving as a left-inverse of T (i.e., $UT(x) = x$).

In an attempt to finely adjust the properties of a function f that are expressed by its phase-space representation $T(f)$, one can consider operators of the form $M_m(f) = U(mT(f))$, that apply a mask m to $T(f)$. We will call these operators *phase-space multipliers*. Of course, the rigorous interpretation of $M_m(f)$ is problematic since, in general, $TM_m(f) \neq mT(f)$. When T is the abstract wavelet transform (representation-coefficients function) associated with a unitary representation of a group, these operators are know as *localization operators* or *wavelet multipliers* [71, 113, 83]. In the case of time-frequency analysis these operators are known as time-frequency localization operators or multipliers of the short-time Fourier transform [24, 21, 22, 13].

We study the problem of characterizing the norm of a coorbit space in terms of families of phase-space multipliers associated with a partition of unity in $\mathcal{G}$. Specifically, suppose that $\boldsymbol{X}$ is a Banach space that is regarded as a coorbit space by means of a transform $T : \boldsymbol{X} \to \boldsymbol{E}$, having a left-inverse $U : \boldsymbol{E} \to \boldsymbol{X}$. Let $\left\{\theta_\gamma\right\}_\gamma$ be a partition of unity on $\mathcal{G}$ and consider the corresponding phase-space multipliers given by $M_\gamma(f) = U(\theta_\gamma T(f))$. The partition of unity is only assumed to satisfy certain spatial localization conditions but it is otherwise arbitrary. We prove that $\|f\|_{\boldsymbol{X}}$ is equivalent to the norm of the sequence $\left\{\|M_\gamma(f)\|_{\boldsymbol{B}}\right\}_\gamma$ in a discrete version of the space $\boldsymbol{E}$, where the space $\boldsymbol{B}$ can be chosen among a large class of function spaces. Moreover, we prove that the map $f \mapsto \left\{M_\gamma(f)\right\}_\gamma$ embeds $\boldsymbol{X}$ as a complemented subspace of a space of $\boldsymbol{B}$-valued sequences, obtained as a discretization of $\boldsymbol{E}$. This quantifies the relation between an element $f \in \boldsymbol{X}$ and the phase-space localized pieces $\left\{M_\gamma(f)\right\}_\gamma$.

For the case of time-frequency analysis, Dörfler and Gröchenig have recently obtained this kind of characterization of modulation spaces [34], using techniques from rotation algebras (non-commutative tori) developed in [66] and [64] and spectral theory for Hilbert spaces (see also [33]). We use a different approach to obtain consequences for settings where the techniques in [34] are not applicable, such as time-scale decompositions and Besov spaces. As a by-product we derive a stronger version of the main result in [34] where the admissible partitions of unity are restricted to be lattice shifts of a non-negative function and the space $\boldsymbol{B}$ is L^2. These restrictions seem to be essential for the techniques used there.

For both problems we take the same general approach. We consider a model for the phase-space consisting of a solid functional space - called the environment - and a certain complemented subspace - called the atomic subspace. We prove all our results in this setting and then obtain applications to coorbit spaces by letting the atomic subspace be the range of the wavelet transform.

The thesis is organized as follows. Chapter 1 gives a brief background on the theory of coorbit spaces, atomic decompositions and related topics, making this work mostly self-contained. In Chapters 2 and 4 we adapt and extend some known tools and results in order to make them applicable to our context. Chapter 3 introduces the model for phase-space and gives several examples. Chapter 5 develops the frame surgery scheme, whereas Chapters 6 and 7 study the problem of characterizing coorbit spaces through phase-space multipliers. Thus, Chapters 5, 6 and 7 contain the main results of the thesis. Each chapter begins with an introduction to the main topic or results, followed by a discussion of the techniques involved. At the end of each chapter the main results are illustrated by presenting applications to atomic decompositions and coorbit spaces. Chapter 7 is slightly non self-contained. Certain results there use more refined concepts of topological groups. However, those subtleties are not very relevant for the proposed applications.

Most of the results in this thesis have been presented in research articles. The estimates of Section 4.2 have been published in [93]. The results of Chapter 5 are contained in [95]. The content of Chapter 6 and 7 has been presented in [94].

Contents

Chapter 1

Preliminaries

1.1 Notation

The cardinality of a set A will be denoted by $\#A$. The characteristic function of the set A will be denoted by χ_A.

If $\boldsymbol{E}$ is a Banach space, $B(\boldsymbol{E})$ will denote the set of all the bounded operators on $\boldsymbol{E}$. If $T : \mathbb{H} \to \mathbb{H}$ is a bounded operator on a Hilbert space, the set

$$\operatorname{spec}(T) = \{\, \lambda \in \mathbb{C} \,\big|\, T - \lambda I \text{ is not invertible} \,\},$$

is called the spectrum of T.

If $T : \mathbb{H} \to \mathbb{K}$ is a bounded operator between Hilbert spaces and has closed range, we will denote by $T^\dagger$ its Moore-Penrose pseudo-inverse. We will only need pseudo-inverses of self-adjoint operators. These can be easily described in the following way. An operator with closed range T restricts to an isomorphism $T|\ker(T)^\perp : \ker(T)^\perp \to \operatorname{rng}(T)$, from the orthogonal complement of its kernel onto its range. If T is self-adjoint, $T^\dagger : \mathbb{K} \to \mathbb{H}$ equals the orthogonal projection onto $\operatorname{rng}(T)$ followed by the inverse of $T|\ker(T)^\perp$.

A Radon measure on a topological space is a Borel measure that is finite on compact sets, outer regular on all Borel sets and inner regular on open sets. Outer regularity for a set means that it can be approximated in measure from the outside by an open set whereas inner regularity means that it can be approximated from the inside by a compact set. All σ-finite Borel measure are outer and inner regular on all Borel sets.

The *Fourier transform* of an integrable function $f : \mathbb{R}^d \to \mathbb{C}$, is defined by,

$$\mathcal{F}(f)(w) := \hat{f}(w) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x w} dx.$$

The modulation and translation operators are defined by,

$$\begin{aligned} T_x f(y) &:= f(y - x), \qquad (x, y \in \mathbb{R}^d), \\ M_w f(y) &:= e^{2\pi i w y} f(y), \qquad (w, y \in \mathbb{R}^d). \end{aligned} \tag{1.1}$$

(In the last definition the product wy denotes the inner (dot) product).

The symbol $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class of functions on the Euclidean space $\mathbb{R}^d$, while $\mathcal{S}'(\mathbb{R}^d)$ denotes the class of tempered distributions.

A subset $\Lambda \subseteq \mathbb{R}^d$ is called a *lattice* if there exists an invertible matrix $A \in \mathbb{R}^{d\times d}$ such that $\Lambda = A\mathbb{Z}^d$. This is sometimes referred to as a full-rank lattice.

Given two non-negative functions $f, g : \boldsymbol{X} \to [0, +\infty)$, the statement,

$$f \lesssim g,$$

means that there exists a constant $C \geq 0$ such that $f(x) \leq Cg(x)$, for all $x \in \boldsymbol{X}$. If $f \lesssim g$ and $g \lesssim f$, we write $f \approx g$.

1.2 Locally compact groups

In this thesis we will often work with a locally compact group $\mathcal{G}$. We will always further assume that $\mathcal{G}$ is σ-compact, i.e., that $\mathcal{G}$ is a countable union of compact sets. This will simplify all measure-theoretic considerations and will be sufficient for all the applications. However this restriction is not essential. For more details on how to overcome the measure-theoretic technicalities arising in the non-σ-compact case see Folland's book [51].

A *left Haar measure* on a topological group $\mathcal{G}$ is a non-zero (non-negative) Radon measure μ such that $\mu(xA) = \mu(A)$, for all $x \in \mathcal{G}$ and Borel sets A. Any locally-compact group $\mathcal{G}$ has a left Haar measure and any two such measures are multiples of each other. We will let $|\cdot|$ stand for a left Haar measure on $\mathcal{G}$, and will moreover call it *the* left Haar measure, the particular normalization being immaterial. Integration will be always considered with respect to the left Haar measure. The symbol $\langle \cdot, \cdot \rangle$ will stand for the L^2 inner product, $\langle f, g \rangle := \int_{\mathcal{G}} f(x)\overline{g}(x)dx$, whenever defined and the identity element of $\mathcal{G}$ will by denoted by e.

The *modular function* of $\mathcal{G}$ is the unique function $\Delta : \mathcal{G} \to (0, +\infty)$ that satisfies,

$$|Ax| = \Delta(x)\,|A|\,, \tag{1.2}$$

for all Borel sets A. Its existence is granted by the essential uniqueness of the left Haar measure. It is easy to see that $\Delta : \mathcal{G} \to (0, +\infty)$ is a continuous group morphism. A group is called *unimodular* if $\Delta \equiv 1$, i.e., if the left Haar measure is also right-invariant.

For a function $f : \mathcal{G} \to \mathbb{C}$ and $x \in \mathcal{G}$, the left and right translates of f by x are defined by,

$$L_x f(y) = f(x^{-1}y),$$
$$R_x f(y) = f(yx).$$

We also define the involution $^\vee$ by,

$$f^\vee(x) = f(x^{-1}).$$

Using Equation (1.2) it is easy to see that,

$$\int_{\mathcal{G}} R_y f(x)dx = \int_{\mathcal{G}} f(xy)dx = \Delta(y^{-1}) \int_{\mathcal{G}} f(x)dx = \Delta(y)^{-1} \int_{\mathcal{G}} f(x)dx.$$

The modular function also comes up in the change of variables $x \mapsto x^{-1}$,

$$\int_{\mathcal{G}} f^{\vee}(x)dx = \int_{\mathcal{G}} f(x^{-1})dx = \int_{\mathcal{G}} f(x)\Delta(x^{-1})dx.$$

Given two functions $f, g : \mathcal{G} \to \mathbb{C}$, the convolution $f * g$ is formally defined by,

$$f * g(x) := \int_{\mathcal{G}} f(y)g(y^{-1}x)dy.$$

We will sometimes regard this definition, not in the pointwise sense, but as a vector-valued integral,

$$f * g := \int_{\mathcal{G}} f(y)L_y g dy.$$

We now mention an important class of groups. A locally compact group $\mathcal{G}$ is said to be an *IN group* if there exists V, a relatively compact neighborhood of the identity that is invariant under inner automorphisms; i.e., $xVx^{-1} = V$, for all $x \in \mathcal{G}$. The abbreviation IN stands for "invariant neighborhood".

Some of the constructions below require choosing a relatively compact neighborhood V, but are largely independent of it in the sense that different choices of V will yield equivalent objects. When working with an IN group $\mathcal{G}$ with a distinguished neighborhood V, we will further assume that V is invariant.

1.2.1 Sets

A set $V \subseteq \mathcal{G}$ is called *symmetric* if $V = V^{-1}$. A set $\Lambda \subseteq \mathcal{G}$ is called *relatively separated* if for some (or any) $V \subseteq \mathcal{G}$, relatively compact neighborhood of e, the quantity

$$\rho_V(\Lambda) := \sup_{x \in \mathcal{G}} \#(\Lambda \cap xV)$$

is finite, i.e. if the amount of elements of Λ that lie in any left translate of V is uniformly bounded. Equivalently, Λ is relatively separated if for any compact set $K \subseteq \mathcal{G}$,

$$\sup_{\lambda \in \Lambda} \#\{\, \lambda' \in \Lambda \,\big|\, \lambda K \cap \lambda' K \neq \emptyset \,\} < +\infty.$$

For technical reasons, we will sometimes need to allow for repeated elements in sets. A *set with multiplicity* is simply a map $\Lambda \ni \lambda \mapsto \lambda^* \in \mathcal{G}$. Any subset of $\mathcal{G}$ can be considered as a set with multiplicity by letting the underlying map be the inclusion. By a slight abuse of notation, it is usual to refer to a set with multiplicity by the domain of the underlying map. For sets with multiplicity, the relative separation is defined by,

$$\rho_V(\Lambda) := \sup_{x\in\mathcal{G}} \#\{\, \lambda \in \Lambda \,\big|\, \lambda^* \in xV \,\}.$$

Every statement and proof that we give for "ordinary" sets also works for sets with multiplicity. To see this, it suffices to read λ^* instead of λ whenever an element λ of a set with multiplicity is used as an element of $\mathcal{G}$ instead of as an index set.

When $\mathcal{G} = \mathbb{R}^d$ we will always use $V = [-1/2, 1/2]^d$ as the distinguished neighborhood. The corresponding quantity,

$$\rho(\Lambda) := \max\left\{ \#(\Lambda \cap ([-1/2, 1/2]^d + x)) \,\big|\, x \in \mathbb{R}^d \right\} \tag{1.3}$$

will be called the *relative separation* of the set Λ. This is somehow an abuse of language since for a very separated set, this quantity is small.

A subset $\Lambda \subseteq \mathcal{G}$ of a locally-compact group $\mathcal{G}$ is called V*-dense* (for V, a relatively compact neighborhood of e) if $\mathcal{G} = \bigcup_{\lambda\in\Lambda} \lambda V$. Λ is called *well-spread* if it is both relatively separated and V-dense for some V. In the case of the Euclidean space $\mathbb{R}^d$ we say that Λ is L-dense if it is $[-L, L]^d$-dense.

1.3 Weights

Let $\mathcal{G}$ be a locally compact group. A locally bounded function $w : \mathcal{G} \to (0, +\infty)$ will be called a *weight* on $\mathcal{G}$. We will say that a weight w is *admissible* if it satisfies following conditions,

$$w(x) = \Delta(x^{-1})w(x^{-1}), \tag{1.4}$$

$$w(xy) \le w(x)w(y). \tag{1.5}$$

The second condition is called *submultiplicativity*. When $\mathcal{G}$ is unimodular, the first condition simply says that w is *symmetric* (i.e., $w(x) = w(x^{-1})$).

A second weight v is called w-moderate if,

$$v(xyz) \lesssim w(x)v(y)w(z), \qquad \text{for all } x, y, z \in \mathcal{G}.$$

In the case that $\mathcal{G}$ is the Euclidean space $\mathbb{R}^d$, an admissible weight is thus a weight $w : \mathbb{R}^d \to (0, +\infty)$ such that,

$$w(x+y) \le w(x)w(y), \qquad \text{for all } x, y \in \mathbb{R}^d, \tag{1.6}$$

$$w(x) = w(-x), \qquad \text{for all } x \in \mathbb{R}^d.$$

As an example, the *polynomial weights*

$$w_t(x) := (1 + |x|)^t, \tag{1.7}$$

are admissible if $t \geq 0$.

A second weight v is *w-moderated* if it satisfies,

$$v(x + y) \leq Cv(x)w(y), \tag{1.8}$$

for some constant $C > 0$ and every $x, y \in \mathbb{R}^d$. If the constant in Equation (1.8) is 1, we say that v is *strictly moderated* by w. The polynomial weight w_t is strictly w_s-moderated if $s \geq 0$ and $|t| \leq s$.

A weight w on the euclidean space $\mathbb{R}^d$ is called *subexponential* if it has the form $w(x) := e^{\rho(|x|)}$, for some norm $|\cdot|$ on $\mathbb{R}^d$ and some continuous, concave function $\rho : [0, \infty] \to [0, \infty]$ such that $\rho(0) = 0$ and $\lim_{x\to+\infty} \frac{\rho(x)}{x} = 0$. Under these conditions, the weight w satisfies: $w(0) = 1$, $w(x) = w(-x)$ and is submultiplicative. The condition $\lim_{x\to+\infty} \frac{\rho(x)}{x} = 0$ is equivalent to the *Gelfand-Raikov-Shilov* condition for w,

$$\lim_{g\to\infty} w(nx)^{1/n} = 1, \qquad \text{for all } x \in \mathbb{R}^d.$$

An example of a subexponential a weight is,

$$w(x) = e^{|x|^\beta}(1 + |x|)^s(\log(e + |x|))^t, \tag{1.9}$$

for $0 \leq \beta < 1$, $s \geq 0$ and $t \geq 0$.

We now state for future reference some facts about polynomial weights. The first lemma says that polynomial weights are *subconvolutive* (see [37].)

Lemma 1.3.1. *If $t > d$, then,*

$$w_{-t} * w_{-t} \leq K w_{-t},$$

for some constant $K \lesssim \max\{1, 1/(t - d)\}$.

There is a corresponding statement for relatively separated index sets. The important point is that the bounds depend only on the relative separation of the sets involved (and this quantity is translation invariant).

Lemma 1.3.2. *Let $\Gamma \subseteq \mathbb{R}^d$ be a relatively separated set of points and let $t > d$. Then, the following estimates hold for a constant $K \lesssim \max\{1, 1/(t - d)\}$.*

(a) $\sum_{\gamma\in\Gamma} w_{-t}(\gamma) \leq K\rho(\Gamma)$,

(b) $\sum_{\gamma:|\gamma|>M} w_{-t}(\gamma) \leq K\rho(\Gamma)M^{-(t-d)}$,

(c) $\sum_{\gamma\in\Gamma} w_{-t}(\gamma)w_{-t}(x - \gamma) \leq K\rho(\Gamma)w_{-t}(x)$, *for all $x \in \mathbb{R}^d$.*

1.4 Function spaces

Let $\mathcal{G}$ be a locally-compact group. A *BF space* is a Banach space $\boldsymbol{E}$ consisting of functions on $\mathcal{G}$ that is continuously embedded into $L^1_{\text{loc}}(\mathcal{G})$, the space of locally integrable functions. A BF space $\boldsymbol{E}$ is called *solid* if for every $f \in \boldsymbol{E}$ and every measurable function $g : \mathcal{G} \to \mathbb{C}$ such that $|g(x)| \le |f(x)|$ a.e., it is true that $g \in \boldsymbol{E}$ and $\|g\|_{\boldsymbol{E}} \le \|f\|_{\boldsymbol{E}}$. If $\boldsymbol{E}$ is a solid BF space and w is a weight, we let $\boldsymbol{E}_w$ be the set of all functions $f \in L^1_{\text{loc}}(\mathcal{G})$ such that $fw \in \boldsymbol{E}$ and endow it with the norm $\|f\|_{\boldsymbol{E}_w} := \|fw\|_{\boldsymbol{E}}$. If w is an admissible weight, then $L^1_w(\mathcal{G})$ is a convolution algebra, $\|f\|_{L^1_w} = \|f^\vee\|_{L^1_w}$, and $\|L_x\|_{L^1_w \to L^1_w} \le w(x)$.

We say that a BF space $\boldsymbol{E}$ is *translation invariant* if it satisfies the following.

(i) $\boldsymbol{E}$ is closed under left and right translations (i.e. $L_x\boldsymbol{E} \subseteq \boldsymbol{E}$ and $R_x\boldsymbol{E} \subseteq \boldsymbol{E}$, for all $x \in \mathcal{G}$).

(ii) The relations,

$$L^1_u(\mathcal{G}) * \boldsymbol{E} \subseteq \boldsymbol{E} \text{ and } \boldsymbol{E} * L^1_v(\mathcal{G}) \subseteq \boldsymbol{E}, \tag{1.10}$$

hold, with the corresponding norm estimates, where $u(x) := \|L_x\|_{\boldsymbol{E}\to\boldsymbol{E}}$ and $v(x) := \Delta(x^{-1})\|R_{x^{-1}}\|_{\boldsymbol{E}\to\boldsymbol{E}}$.

Left (resp. right) translation invariant spaces are defined similarly, requiring only the conditions for left (resp. right) translations and convolutions.

Remark 1.4.1. *Observe that, for a BF space, if the translations leave $\boldsymbol{E}$ invariant, then they are bounded by the closed graph theorem.*

Remark 1.4.2. *In the definition of translation invariant space, the technical assumption (ii) follows from (i) if the set of continuous functions with compact support is dense on $\boldsymbol{E}$, or more generally if the maps $x \mapsto L_x$ and $x \mapsto R_x$ are strongly continuous.*

We say that $\boldsymbol{E}$ is isometrically left (resp. right) translation invariant if it is translation invariant and, in addition, left (resp. right) translations are isometries on $\boldsymbol{E}$.

Given a BF space $\boldsymbol{E}$, a set of functions $\{ f_\lambda \,|\, \lambda \in \Lambda \} \subseteq L^1_{\text{loc}}(\mathcal{G})$ - indexed by a relatively separated set Λ - is called a set of $\boldsymbol{E}$*-molecules* if there exists a function $g \in \boldsymbol{E}$ - called envelope - such that

$$|f_\lambda(x)| \le L_\lambda g(x), \qquad (x \in \mathcal{G}, \lambda \in \Lambda).$$

Given a solid, translation invariant, BF space $\boldsymbol{E}$, we say that a weight $w : \mathcal{G} \to (0, +\infty)$ is *admissible for $\boldsymbol{E}$* if w is admissible and, in addition, it satisfies,

$$w(x) \gtrsim \max\left\{u(x),\ u(x^{-1}),\ v(x),\ \Delta(x^{-1})v(x^{-1})\right\}, \tag{1.11}$$

where $u(x) := \|L_x\|_{\boldsymbol{E}\to\boldsymbol{E}}$ and $v(x) := \Delta(x^{-1})\|R_{x^{-1}}\|_{\boldsymbol{E}\to\boldsymbol{E}}$. Under these conditions, $w(x) \gtrsim 1$, $L^1_w * \boldsymbol{E} \subseteq \boldsymbol{E}$ and $\boldsymbol{E} * L^1_w \subseteq \boldsymbol{E}$, with the corresponding norm estimates.

If $\boldsymbol{E}$ is a solid BF space, we construct discrete versions of it as follows. Given a well-spread set $\Lambda \subseteq \mathcal{G}$ and a symmetric, relatively compact neighborhood of the identity V, we define the space,

$$\boldsymbol{E}^d = \boldsymbol{E}^d(\Lambda) := \{ c \in \mathbb{C}^\Lambda \,\Big|\, \sum_\lambda |c_\lambda| \chi_{\lambda V} \in \boldsymbol{E} \}$$

and endow it with the norm,

$$\|(c_\lambda)_{\lambda\in\lambda}\|_{\boldsymbol{E}^d} := \|\sum_\lambda |c_\lambda| \chi_{\lambda V}\|_{\boldsymbol{E}}.$$

The definition, of course, depends on Λ and V, but a large class of neighborhoods V and sets Λ produce equivalent spaces (see [44, Lemma 3.5] for a precise statement). In the sequel, we will mainly use the space $\boldsymbol{E}^d$ keeping V fixed and making an explicit choice for Λ.

The space $\boldsymbol{E}^d$ is an example of a *BK-space* on the index set Λ, i.e., a Banach space of sequences that is continuously embedded into the product $\mathbb{C}^\Lambda$. When $\boldsymbol{E} = L^p_w$, for an admissible weight w, the corresponding discrete space $\boldsymbol{E}^d(\Lambda)$ is $\ell^p_w(\Lambda)$, where the weight w is restricted to the set Λ. This is so because the admissibility of w implies that for $x \in \lambda V$, $w(x) \approx w(\lambda)$.

We will also need a vector-valued version of $\boldsymbol{E}^d$. Given another BF space $\boldsymbol{B}$ we let,

$$\boldsymbol{E}^d_{\boldsymbol{B}} = \boldsymbol{E}^d_{\boldsymbol{B}}(\Lambda) := \{ c \in \boldsymbol{B}^\Lambda \,\Big|\, \sum_\lambda \|c_\lambda\|_{\boldsymbol{B}} \chi_{\lambda V} \in \boldsymbol{E} \},$$

and endow it with a norm in a similar fashion.

1.5 Some examples

1.5.1 The affine group

The affine group is the set $\mathcal{G} = \mathbb{R}^d \times (0, +\infty)$ together with the operation $(x, s) \cdot (x', s') = (x + sx', ss')$. The inverse of an element (x, s) is therefore $(-x/s, 1/s)$. The measure with density $s^{-(d+1)} dxds$ with respect to the Lebesgue measure is a left Haar measure. The corresponding modular function is given by $\Delta(x, s) = s^{-d}$, so the affine group is not unimodular. The translation operators are given by,

$$L_{(y,t)} f(x, s) = f\left(\frac{x - y}{t}, \frac{s}{t}\right), \qquad R_{(y,t)} f(x, s) = f(x + sy, st). \tag{1.12}$$

The *weighted Lebesgue spaces* $L^{p,q}_\alpha$, $L^\alpha_{p,q}$ are given by the norms,

$$\|f\|_{L^{p,q}_\alpha} = \left(\int_0^{+\infty} \left(\int_{\mathbb{R}^d} |f(x, s)|^p dx \right)^{q/p} s^{-\alpha q} \frac{ds}{s^{d+1}} \right)^{1/q}, \qquad (1 \le p, q \le \infty, \alpha \in \mathbb{R}), \tag{1.13}$$

$$\|f\|_{L^\alpha_{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_0^{+\infty} |f(x, s)|^q s^{-\alpha q} \frac{ds}{s} \right)^{p/q} dx \right)^{1/p}, \qquad (1 \le p, q \le \infty, \alpha \in \mathbb{R}), \tag{1.14}$$

(with the usual modifications when p or q are $+\infty$). These spaces are examples of solid BF spaces. Using Equation (1.12), the spaces $L^{p,q}_\alpha$ are easily seen to be translation invariant. Moreover,

$$\|L_{(y,t)}f\|_{L^{p,q}_\alpha} = t^{d(1/p-1/q)-\alpha}\|f\|_{L^{p,q}_\alpha}, \quad \|R_{(y,t)}f\|_{L^{p,q}_\alpha} = t^{d/q+\alpha}\|f\|_{L^{p,q}_\alpha}. \tag{1.15}$$

In contrast, the spaces $L^\alpha_{p,q}$ are only left-translation invariant.

The set $V = [-1/2, 1/2]^d \times (1/2, 2)$ is a relatively compact neighborhood of the identity, while the set

$$\Lambda := \{ \left(k2^{-j}, 2^{-j}\right) \bigm| k \in \mathbb{Z}^d, j \in \mathbb{Z} \},$$

is an example of a relatively separated set. Identifying the point $\lambda = (k2^{-j}, 2^{-j})$ with the index (k, j) it is easy to see that the norms of the discrete spaces corresponding to $L^{p,q}_\alpha$ and $L^\alpha_{p,q}$ are given by,

$$\|c\|_{(L^{p,q}_\alpha)^d} \approx \left(\sum_j \left(\sum_k \left|c_{k,j}\right|^p\right)^{q/p} 2^{j(\alpha+d/q-d/p)q}\right)^{1/q}, \tag{1.16}$$

$$\|c\|_{(L^\alpha_{p,q})^d} \approx \left\|\left(\sum_{k,j} \left|c_{k,j}\right|^q \chi_{[-1/2,1/2]^d}(2^j\cdot - k) 2^{j\alpha q}\right)^{1/q}\right\|_{L^p(\mathbb{R}^d)}, \tag{1.17}$$

with the usual modifications when p or q is $+\infty$.

1.5.2 The Heisenberg group

The *polarized reduced* Heisenberg group is the set $\mathbb{H} := \mathbb{R}^d \times \mathbb{R}^d \times S^1$, together with the operation $(x, w, \lambda) \cdot (x', w', \lambda') := (x + x', y + y, \lambda\lambda' e^{-2\pi i w x'})$. Here, S^1 denotes the set of unimodular complex numbers. The Haar measure on $\mathbb{H}$ is the (product) Lebesgue measure. This measure is also right-invariant, so $\mathbb{H}$ is unimodular.

For $(x, w) \in \mathbb{R}^d \times \mathbb{R}^d$, the *time-frequency shift* $\pi(x, w)$ is defined as $\pi(x, w) := M_w T_x$ (cf. Section 1.1). Hence, for a function $f : \mathbb{R}^d \to \mathbb{C}$ the time-frequency shift acts on f by,

$$\pi(x, w) f(y) := e^{2\pi i w y} f(y - x). \tag{1.18}$$

The time-frequency shifts do not determine a representation of $\mathbb{R}^{2d}$ on $L^2(\mathbb{R}^d)$ because they do not commute. However, π can be extended to an action of the Heisenberg group by setting $\overline{\pi}(x, w, \lambda) = \lambda\pi(x, w)$.[1]

[1]Other conventions for the order of modulation and translation in the time-frequency shifts give rise to other (equivalent) versions of the Heisenberg group.

1.6 Wiener amalgam spaces

Amalgam spaces are defined by the global behavior of certain local properties of their elements. The norm of the amalgam space is built up out of two other spaces: the local and the global component. In concrete examples, the local properties defining the amalgam space can be integrability or smoothness conditions whereas the global properties are size conditions. The best known example is the one of L^p-ℓ^q amalgams [54, 74] on the Euclidean space $\mathbb{R}^d$, given by the norm,

$$\|f\|_{W(L^p,\ell^q)} = \left(\sum_{k\in\mathbb{Z}^d} \|f\|^q_{L^p([0,1]^d+k)} \right)^{1/q}, \tag{1.19}$$

with the usual modification for $q = +\infty$. Convolution relations among L^p-ℓ^q amalgams play an important role, for example, in the work of Wiener [112, 110, 111].

In [38], Feichtinger introduced a far-reaching generalization of these amalgam spaces by allowing the inner (local) norm to be non-solid. Instead, the space used as local component is required to have a sufficiently rich algebra of pointwise multipliers. (The solidity of a space means that L^∞ embeds into its algebra of pointwise multipliers; so the point of the term "sufficiently rich" is that that algebra should be large, but not necessarily as large as L^∞). This allowed him to derive abstract convolution relations for amalgam spaces that express at the same time the regularizing effect of convolution together with the preservation of integrability given by Young-type inequalities.

We first introduce amalgam spaces in the context of the Euclidean space and then in the general setting of locally compact groups. Of course, the first is an special case of the second, but certain matters pertain only to setting of the Euclidean space.

1.6.1 Amalgams on the Euclidean space

We now describe amalgam spaces in the context of the Euclidean space $\mathbb{R}^d$, following the treatment in [41]. See that article for an exposition on the application of amalgam spaces to sampling theory, signal processing, spline approximation and error analysis.

Let us denote by $\mathcal{D}(\mathbb{R}^d)$ the set of all C^∞, compactly supported, complex-valued functions on $\mathbb{R}^d$, by $C^0(\mathbb{R}^d)$ the set of all continuous functions vanishing at infinity and by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz class.

Let $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ be a *uniformly localizable, isometrically translation invariant* Banach space. That is, $\boldsymbol{B}$ satisfies the following axioms.

- $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ are continuous embeddings whose composition is the canonical embedding $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

- If $h \in \mathcal{D}(\mathbb{R}^d)$ and $f \in \boldsymbol{B}$, then $hf \in \boldsymbol{B}$ and there is a constant $C = C(h) > 0$ such that $\|h(\cdot - x)f\|_{\boldsymbol{B}} \le C\|f\|_{\boldsymbol{B}}$, for all $x \in \mathbb{R}^d$ and $f \in \boldsymbol{B}$.

- If $f \in \boldsymbol{B}$ and $x \in \mathbb{R}^d$, then $f(\cdot - x) \in \boldsymbol{B}$ and $\|f(\cdot - x)\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$.
- Complex conjugation defines an isometry on $\boldsymbol{B}$. That is, if $f \in \boldsymbol{B}$, then $\overline{f} \in \boldsymbol{B}$ and $\|f\|_{\boldsymbol{B}} = \|\overline{f}\|_{\boldsymbol{B}}$.

We consider the space of distributions that belong to $\boldsymbol{B}$ locally,

$$\boldsymbol{B}_{loc} := \{\, f \in \mathcal{S}'(\mathbb{R}^d) \,\big|\, hf \in \boldsymbol{B}, \text{ for all } h \in \mathcal{D}(\mathbb{R}^d) \,\}.$$

Given $f \in \boldsymbol{B}_{loc}$ and a non-zero window $\eta \in \mathcal{D}(\mathbb{R}^d)$, we consider the *control function*

$$K(f)(x) := \|f\eta(\cdot - x)\|, \quad (x \in \mathbb{R}^d).$$

Let $\boldsymbol{E}$ be a solid, translation invariant function space (cf. Section 1.4) such that the polynomial weight $w(x) := (1 + |x|)^\alpha$ is admissible for it, for some $\alpha \geq 0$. The Wiener amalgam space $W(\boldsymbol{B}, \boldsymbol{E})$ is defined by,

$$W(\boldsymbol{B}, \boldsymbol{E}) := \{\, f \in \boldsymbol{B}_{loc} \,\big|\, K(f) \in \boldsymbol{E} \,\},$$

and is given the norm $\|f\|_{W(\boldsymbol{B},\boldsymbol{E})} := \|K(f)\|_{\boldsymbol{E}}$. This definition of course depends on the window function η but the space $W(\boldsymbol{B}, \boldsymbol{E})$ is independent of this choice in the sense that different windows yield equivalent norms. The amalgam $W(\boldsymbol{B}, \boldsymbol{E})$ is a Banach space.

When $\boldsymbol{B} = L^p$ and $\boldsymbol{E} = L^q_w$, for a polynomially moderated weight w, and $1 \leq p \leq \infty$, then $W(L^p, L^q_w)$ coincides with the classical amalgam space $W(L^p, \ell^q_w)$ from Equation (1.19).

The most important result about amalgam spaces is the following abstract convolution relation.

Theorem 1.6.1. *Suppose that* $(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3)$ *and* $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ *are convolution triples*[2] *, i.e.,*

$$\|f * g\|_{\boldsymbol{B}_3} \leq C_1 \|f\|_{\boldsymbol{B}_1} \|g\|_{\boldsymbol{B}_2}, \qquad \text{for all, } f \in \boldsymbol{B}_1, g \in \boldsymbol{B}_2,$$
$$\|f * g\|_{\boldsymbol{E}_3} \leq C_2 \|f\|_{\boldsymbol{E}_1} \|g\|_{\boldsymbol{E}_2}, \qquad \text{for all, } f \in \boldsymbol{E}_1, g \in \boldsymbol{E}_2,$$

hold for some constants $C_1, C_2 > 0$*. Then,* $(W(\boldsymbol{B}_1, \boldsymbol{E}_1), W(\boldsymbol{B}_2, \boldsymbol{E}_2), W(\boldsymbol{B}_3, \boldsymbol{E}_3))$ *is also a convolution triple, i.e.,*

$$\|f * g\|_{W(\boldsymbol{B}_3,\boldsymbol{E}_3)} \leq C_3 \|f\|_{W(\boldsymbol{B}_1,\boldsymbol{E}_1)} \|g\|_{W(\boldsymbol{B}_2,\boldsymbol{E}_2)}, \qquad \text{for all, } f \in W(\boldsymbol{B}_1, \boldsymbol{E}_1), g \in W(\boldsymbol{B}_2, \boldsymbol{E}_2),$$

holds for some constants $C_3 > 0$.

A great number of operators in spline approximation, sampling and signal analysis can be modeled as distributional convolutions. Theorem 1.6.1 allows then to quantify this formalism. As an example we mention that the relation (cf. Equation (1.19)),

$$\Big\| \sum_{k \in \Lambda} c_k f(\cdot - k) \Big\|_{W(L^\infty, \ell^p)} \lesssim \rho(\Lambda) \|c\|_{\ell^p} \|f\|_{W(L^\infty, \ell^1)},$$

follows from Theorem 1.6.1 by letting $\boldsymbol{E}_1 = \boldsymbol{E}_3 = L^p$, $\boldsymbol{E}_2 = L^1$, $\boldsymbol{B}_2 = \boldsymbol{B}_3 = L^\infty$ and taking $\boldsymbol{B}_1$ to be the space of all complex-valued measures with the norm of absolute variation.

[2] Here we also assume that the spaces $\boldsymbol{B}_i$ and $\boldsymbol{E}_i$ satisfy the hypothesis introduced above.

1.6.2 Amalgams on general groups

We now present amalgam spaces in the general context, as introduced in [38]. The definition is more technical than in the case of the Euclidean space since the reservoir of tempered distributions is not available. Such technicalities will be mostly irrelevant for this thesis.

Let $\mathcal{G}$ be a locally compact group. As local component of an amalgam we will use a Banach space $\boldsymbol{B}$ such that,

- There exist an isometrically left-translation invariant Banach space $\boldsymbol{A}$ that is also a regular Banach algebra under pointwise multiplication (i.e. separating points from closed sets) and isometrically closed under pointwise conjugation.
- $\boldsymbol{B}$ is embedded into $(\boldsymbol{A} \cap \mathcal{K}(\mathcal{G}))'$, where $\mathcal{K}(\mathcal{G})$ is the space of compactly supported functions on $\mathcal{G}$, with the inductive limit topology.
- $\boldsymbol{B}$ is a pointwise $\boldsymbol{A}$-module, i.e., if $f \in \boldsymbol{A}$ and $g \in \boldsymbol{B}$, then $fg \in \boldsymbol{B}$ and $\|fg\|_{\boldsymbol{B}} \lesssim \|f\|_{\boldsymbol{A}}\|g\|_{\boldsymbol{B}}$.

Note that the third condition makes sense because of the second one. Indeed, since $\boldsymbol{A}$ is a pointwise algebra, $\boldsymbol{A}'$ has a natural pointwise module structure over $\boldsymbol{A}$. The embedding $\boldsymbol{B} \hookrightarrow (\boldsymbol{A} \cap \mathcal{K}(\mathcal{G}))'$ allows us to restrict that action to $\boldsymbol{B}$. Note also that, since $\boldsymbol{A}$ is isometrically translation invariant,

$$\|(L_x f)g\|_{\boldsymbol{B}} \lesssim \|f\|_{\boldsymbol{A}}\|g\|_{\boldsymbol{B}},$$

holds for all $f \in \boldsymbol{A}$ and $g \in \boldsymbol{B}$ and $x \in \mathcal{G}$.

The space $\boldsymbol{B}_{loc}$ is defined as the set of all elements $f \in \boldsymbol{A}'$ such that $hf \in \boldsymbol{B}$, for all $h \in \boldsymbol{A} \cap \mathcal{K}(\mathcal{G})$. This space can be shown to be independent of the particular choice of the algebra $\boldsymbol{A}$.

For the global component, we let $\boldsymbol{E}$ be a solid, translation invariant BF space (cf. Section 1.4). The left *Wiener amalgam space* (or space of Wiener-type) with local component $\boldsymbol{B}$ and global component $\boldsymbol{E}$ is defined by,

$$W(\boldsymbol{B}, \boldsymbol{E}) := \{ f \in \boldsymbol{B}_{loc} \,|\, K_{\boldsymbol{B}}(f) \in \boldsymbol{E} \},$$

where the control function $K_{\boldsymbol{B}}(f)$ is given by,

$$K_{\boldsymbol{B}}(f)(x) := \|f\|_{\boldsymbol{B}(xV)} := \inf \{ \|g\|_{\boldsymbol{B}} \,|\, g = f \text{ on } xV \}.$$

We endow $W(\boldsymbol{B}, \boldsymbol{E})$ with the norm $\|f\|_{W(\boldsymbol{B},\boldsymbol{E})} := \|K_{\boldsymbol{B}}(f)\|_{\boldsymbol{E}}$. This definition depends on the choice of the neighborhood V, but a different choice produces the same space with an equivalent norm. $W(\boldsymbol{B}, \boldsymbol{E})$ is a Banach spaces and it is also independent of the choice of the algebra $\boldsymbol{A}$. The right amalgam space $W_R(\boldsymbol{B}, \boldsymbol{E})$ is defined similarly, this time using the control function,

$$K_{\boldsymbol{B}}(f, R)(x) := \|f\|_{\boldsymbol{B}(Vx^{-1})} := \inf \{ \|g\|_{\boldsymbol{B}} \,|\, g = f \text{ on } Vx^{-1} \}.$$

Theorem 1.6.1 can be generalized to the abstract setting when $\mathcal{G}$ is an IN group. We say that a triple of Banach spaces $(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3)$ is a convolution triple if $(\boldsymbol{X}_1 \cap \boldsymbol{\mathcal{K}}) * (\boldsymbol{X}_2 \cap \boldsymbol{\mathcal{K}}) \subseteq (\boldsymbol{X}_3 \cap \boldsymbol{\mathcal{K}})$, and convolution extends to a bounded bilinear mapping $\boldsymbol{X}_1 \times \boldsymbol{X}_2 \to \boldsymbol{X}_3$. This definition makes sense when each space $\boldsymbol{X}_i$ is a Banach function space, but also when they consist of extremal distributions as in the case of the local components above.

Theorem 1.6.2 ([38])**.** *Let $\mathcal{G}$ be an IN group. Suppose that $(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3)$ and $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ are convolution triples.*[3] *Then, $(W(\boldsymbol{B}_1, \boldsymbol{E}_1), W(\boldsymbol{B}_2, \boldsymbol{E}_2), W(\boldsymbol{B}_3, \boldsymbol{E}_3))$ is also a convolution triple.*

In this thesis we will use amalgam spaces with non-solid local components only for the Euclidean space, where we are interested in smoothness matters. For abstract groups we will mainly use solid BF spaces as local components. In this case, all the technicalities above disappear since $\boldsymbol{L}^1_{\text{loc}}$ can be used as a reservoir in all the definitions. Moreover, when $\boldsymbol{B}$ is a solid BF space, the restriction norm $\|f\|_{\boldsymbol{B}(xV)}$, can be simply computed as $\|f\|_{\boldsymbol{B}(xV)} = \|f\chi_{xV}\|$.

Amalgam spaces with L^1 and L^∞ as local components will be a key technical tool in this thesis. The spaces $W(L^\infty, \boldsymbol{E})$ and $W_R(L^\infty, \boldsymbol{E})$ can be easily described in terms of certain maximum functions. For a locally bounded function $f : \mathcal{G} \to \mathbb{C}$ we define the left and right *local maximum* functions by,

$$f^\#(x) := \operatorname*{supess}_{y\in V} |f(xy)|\,,$$
$$f_\#(x) := \operatorname*{supess}_{y\in V} |f(yx)|\,.$$

The neighborhood V can be always assumed to be symmetric. Then, the maximum functions are related by $(f_\#)^\vee = (f^\vee)^\#$. Using these definitions we have,

$$\|f\|_{W(L^\infty,\boldsymbol{E})} = \|f^\#\|_{\boldsymbol{E}}, \tag{1.20}$$
$$\|f\|_{W_R(L^\infty,\boldsymbol{E})} = \|(f_\#)^\vee\|_{\boldsymbol{E}}. \tag{1.21}$$

In particular, $\|f\|_{W_R(L^\infty,\boldsymbol{E})} = \|f^\vee\|_{W(L^\infty,\boldsymbol{E})}$. Note also that, by the solidity of $\boldsymbol{E}$, both $W(L^\infty, \boldsymbol{E})$ and $W_R(L^\infty, \boldsymbol{E})$ are continuously embedded into $\boldsymbol{E}$. The space $W(C_0, \boldsymbol{E})$, constructed using C_0 as local component can be characterized as the subspace of $W(L^\infty, \boldsymbol{E})$ formed by the continuous functions. A similar statement holds for $W_R(C_0, \boldsymbol{E})$.

When $\boldsymbol{E} = L^1_w$ for an admissible weight w we can drop the involution in Equation (1.21) yielding,

$$\|f\|_{W_R(L^\infty,L^1_w)} := \|f_\#\|_{L^1_w}. \tag{1.22}$$

In addition, since $L_x(f^\#) = (L_x f)^\#$, the space $W(L^\infty, L^1_w)$ is invariant under left translations and the norm of the left translations is dominated by w. A similar statement holds for $W_R(L^\infty, L^1_w)$ and right translations.

[3] Here we also assume that the spaces $\boldsymbol{B}_i$ and $\boldsymbol{E}_i$ satisfy the hypothesis introduced above.

We finally note that if $\mathcal{G}$ is an IN group, the left and right local maximum functions coincide and therefore $W(L^\infty, \boldsymbol{E}) = W_R(L^\infty, \boldsymbol{E})$.

In [44], Feichtinger and Gröchenig used amalgam-space techniques to discretize continuous reproducing formulas for functional spaces. The condition that the $\mathcal{G}$ be an IN group in Theorem 1.6.2 is then too restrictive as it would exclude the important case of time-scale decompositions from coorbit theory. They developed then variants of Theorem 1.6.2 where left and right amalgam spaces are combined.

We now state some facts mainly taken from [44] and [45]. In the cases when we were unable to find an exact reference we sketch a proof.

Lemma 1.6.1. *Let $\boldsymbol{E}$ be a solid, translation invariant BF space and let w be an admissible weight for it. The following embeddings hold, together with the corresponding norm estimates.*

(a) $\boldsymbol{E} * W(L^\infty, L^1_w) \hookrightarrow W(L^\infty, \boldsymbol{E})$ *and* $\boldsymbol{E} * W(C_0, L^1_w) \hookrightarrow W(C_0, \boldsymbol{E})$.

(b) $\boldsymbol{E} \hookrightarrow W(L^1, L^\infty_{1/w})$. *In addition, if $\boldsymbol{E}$ is isometrically left-translation invariant, then* $\boldsymbol{E} \hookrightarrow W(L^1, L^\infty)$.

(c) $W(L^1, L^\infty).W(L^\infty, L^1) \hookrightarrow L^1$ *and* $W(L^1, L^\infty_{1/w}).W(L^\infty, L^1_w) \hookrightarrow L^1$.

(d) $W(L^1, L^\infty) * W_R(L^\infty, L^1_w) \hookrightarrow L^\infty$ *and* $W(L^1, L^\infty_{1/w}) * W_R(L^\infty, L^1_w) \hookrightarrow L^\infty_{1/w}$.

Proof. Part (a) is proved in [45, Theorem 7.1]. By [44, Lemma 3.9], $\boldsymbol{E} \hookrightarrow W(L^1, L^\infty_{(u^\vee)^{-1}})$, where $u(x) := \|L_x\|_{\boldsymbol{E}\to\boldsymbol{E}}$. The admissibility of w implies that $u^\vee \lesssim w$, so part (b) follows.

To prove (c) first observe that for any $f \in L^1(\mathcal{G})$, since $V = V^{-1}$,

$$\int_{\mathcal{G}}\int_{\mathcal{G}} |f(x)|\,(L_y\chi_V(x))dxdy = \int_{\mathcal{G}} |f(x)| \int_{\mathcal{G}} \chi_V(y^{-1}x)dydx = |V| \int_{\mathcal{G}} |f(x)|\,dx.$$

Using this observation, for $f \in W(L^1, L^\infty_{1/w})$ and $g \in W(L^\infty, L^1_w)$,

$$\begin{aligned}\int_{\mathcal{G}} |f(x)|\,|g(x)|\,dx &\approx \int_{\mathcal{G}}\int_{yV} |f(x)|\,|g(x)|\,dxdy\\ &\lesssim \int_{\mathcal{G}} \|f\|_{L^1(yV)}\|g\|_{L^\infty(yV)}dy \leq \|f\|_{W(L^1,L^\infty_{1/w})}\|g\|_{W(L^\infty,L^1_w)}.\end{aligned}$$

The unweighted case follows similarly. To prove (d) let $f \in W(L^1, L^\infty_{1/w})$ and $g \in W(L^\infty, L^1_w)$. For $x \in \mathcal{G}$ we can use (c) to get,

$$\begin{aligned}|f * g(x)| &\leq \int_{\mathcal{G}} |f(y)|\,\left|L_x g^\vee(y)\right|\,dy \leq \|f\|_{W(L^1,L^\infty_{1/w})}\|L_x g^\vee\|_{W(L^\infty,L^1_w)}\\ &\leq \|f\|_{W(L^1,L^\infty_{1/w})}\|g^\vee\|_{W(L^\infty,L^1_w)}w(x).\end{aligned}$$

Since $\|g^\vee\|_{W(L^\infty,L^1_w)} = \|g\|_{W_R(L^\infty,L^1_w)}$ the weighted inequality in (d) is proved. The unweighted one follows similarly, this time using the unweighted bound in (c) and the fact that $w \gtrsim 1$. □

Lemma 1.6.2. *Let $\boldsymbol{E}$ be a solid, translation invariant BF space, let w be an admissible weight for it and let $\Lambda \subseteq \mathcal{G}$ be a relatively separated set. Then the following holds.*

(a) For every $f \in W(C_0, \boldsymbol{E})$, the sequence $f(\Lambda) = (f(\lambda))_{\lambda \in \Lambda}$ belongs to $\boldsymbol{E}^d(\Lambda)$ and,

$$\|f(\Lambda)\|_{\boldsymbol{E}^d} \lesssim \|f\|_{W(C_0,\boldsymbol{E})}.$$

(b) For every $f \in \boldsymbol{E}$ and $g \in W_R(C_0, L^1_w)$, the sequence $(\langle f, L_\lambda g\rangle)_{\lambda \in \Lambda}$ belongs to $\boldsymbol{E}^d(\Lambda)$ and,

$$\|(\langle f, L_\lambda g\rangle)_\lambda\|_{E^d} \lesssim \|f\|_E \|g\|_{W_R(L^\infty, L^1_w)}.$$

(c) If $(c_\lambda)_\lambda \in \boldsymbol{E}^d(\Lambda)$ and $f \in W_R(L^\infty, L^1_w)$, then $\sum_\lambda c_\lambda L_\lambda f \in \boldsymbol{E}$ and

$$\|\sum_\lambda c_\lambda L_\lambda f\|_{\boldsymbol{E}} \lesssim \|(c_\lambda)_\lambda\|_{\boldsymbol{E}^d} \|f\|_{W_R(L^\infty, L^1_w)}.$$

The series converges absolutely at every point and, if the set of bounded compactly supported functions is dense in $\boldsymbol{E}$, it also converges unconditionally in the norm of $\boldsymbol{E}$.

(d) $\boldsymbol{E}^d(\Lambda) \hookrightarrow \ell^\infty_{1/w}(\Lambda)$.

All the implicit constants depend on the set Λ.

Proof. Part (a) follows easily from the definitions (see for example [44, Lemma 3.8]). For (b) observe that $\langle f, L_\lambda g\rangle = (f * g^\vee)(\lambda)$. Hence Lemma 1.6.1 and part (a) imply that

$$\|(\langle f, L_\lambda g\rangle)_\lambda\|_{E^d} \lesssim \|f * g^\vee\|_{W(C_0,\boldsymbol{E})} \lesssim \|f\|_{\boldsymbol{E}} \|g^\vee\|_{W(L^\infty, L^1_w)} = \|f\|_{\boldsymbol{E}} \|g\|_{W_R(L^\infty, L^1_w)}.$$

Part (c) is Proposition 5.2 of [44]. Lemma 3.5 in [44] gives the embedding $\boldsymbol{E}^d(\Lambda) \hookrightarrow \ell^\infty_{1/u}(\Lambda)$, where $u(x) := \|L_x\|_{\boldsymbol{E}\to\boldsymbol{E}}$. Since $u \lesssim w$, part (d) follows. □

Finally we state the following lemma that will be used to justify treating convolutions pointwise.

Lemma 1.6.3. *Let $\boldsymbol{E}$ be a solid, translation invariant BF space and let w be an admissible weight for it. The following embeddings hold, together with the corresponding norm estimates.*

(a) $W(L^\infty, \boldsymbol{E}) \hookrightarrow L^\infty_{1/w}$.

(b) $W(L^\infty, \boldsymbol{E}) * L^1_w \hookrightarrow C_{1/w}$, where $C_{1/w}$ denotes the subspace of $L^\infty_{1/w}$ formed by the continuous functions.

Proof. By Lemma 1.6.2, $\boldsymbol{E}^d \hookrightarrow \ell^\infty_{1/w}$. This implies that $W(L^\infty, \boldsymbol{E}) \hookrightarrow W(L^\infty, L^\infty_{1/w}) = L^\infty_{1/w}$ (see for example [44, Proposition 3.7]). This proves part (a). The embedding $W(L^\infty, \boldsymbol{E}).L^1_w \hookrightarrow L^\infty_{1/w}.L^1_w \hookrightarrow L^1$, implies that, $W(L^\infty, \boldsymbol{E}) * L^1_w = W(L^\infty, \boldsymbol{E}) * {L^1_w}^\vee \hookrightarrow L^\infty_{1/w}$. Now part (b) follows from the fact that the class of continuous, compactly supported functions is dense in L^1_w. □

1.7 Spectral invariance of some matrix algebras

Let $\boldsymbol{A} \subseteq \boldsymbol{B}$ be two Banach algebras. $\boldsymbol{A}$ is said to be *spectral* or *inverse closed* in $\boldsymbol{B}$ if for each element $a \in \boldsymbol{A}$, the spectrum of a as an element of $\boldsymbol{A}$ equals its spectrum as an element of $\boldsymbol{B}$. This means that if an element $a \in \boldsymbol{A}$ has an inverse $a^{-1} \in \boldsymbol{B}$, then $a^{-1} \in \boldsymbol{A}$. We will be mainly interested in the case where $\boldsymbol{B}$ is the algebra of bounded operators on a certain Hilbert space.

Wiener's $1/f$ lemma states [112] that if a function $f : [0,1] \to \mathbb{C}$ has an absolutely convergent Fourier series,

$$f(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k x},$$

and is never vanishing, then $1/f$ has also an absolutely convergent Fourier series,

$$1/f(x) = \sum_{k\in\mathbb{Z}} b_k e^{2\pi i k x}.$$

This says that the algebra $\mathcal{F}^{-1}(\ell^1(\mathbb{Z}))$ of functions having an absolutely convergent Fourier series is spectral in $C([0,1])$, the algebra of continuous functions on $[0,1]$. Wiener's lemma can be reformulated without using the Fourier transform as follows. Consider the inclusion,

$$\begin{aligned} \ell^1(\mathbb{Z}) &\hookrightarrow B(\ell^2(\mathbb{Z})) \\ a &\mapsto a * -, \end{aligned}$$

which maps a sequence a into the convolution operator with kernel a. Wiener's lemma states that this inclusion is spectral.

This second formulation makes sense for every locally compact group. The problem of determining what classes of groups $\mathcal{G}$ are such that the inclusion

$$\begin{aligned} L^1(\mathcal{G}) &\hookrightarrow B(L^2(\mathcal{G})) \\ f &\mapsto f * -, \end{aligned}$$

is spectral has been extensively studied [73, 75, 81, 82, 84, 50, 48].

A more general problem is the one of finding conditions for a subalgebra $\boldsymbol{A} \subseteq B(L^2(\mathcal{G}))$ to be spectral. In the next sections we cite a number of results in this direction. Weighted versions of Wiener's lemma follow by applying these results to convolution operators.

Before doing that we make the following remark that will be important for the applications to function spaces.

Remark 1.7.1. *Let $B(\mathbb{H})$ be the algebra of bounded operators on some Hilbert space $\mathbb{H}$ and let $\boldsymbol{A}$ be a spectral subalgebra. If $T \in \boldsymbol{A}$ is a self-adjoint operator with closed range, then $T^\dagger \in \boldsymbol{A}$.*

Indeed, $L^\dagger = f(L)$, where $f(z) = z^{-1}$, for $z \neq 0$ and $f(0) = 0$. The function f is holomorphic on the spectrum of L because, since the range of L is closed, 0 is an isolated point of its spectrum. Since the inclusion $\boldsymbol{A} \hookrightarrow B(\mathbb{H})$ is closed under inversion, it is also closed under holomorphic functional calculus.

1.7.1 The Jaffard algebra

Let $\Lambda \subseteq \mathbb{R}^d$ be a relatively separated set of points and $s > d$. The algebra $\mathcal{J}_f$ consists of all matrices $A \in \mathbb{C}^{\Lambda\times\Lambda}$ such that,

$$\left|A_{k,j}\right| \leq C(1 + |k - j|)^{-s},$$

for some constant $C \geq 0$. The least value for that constant defines the norm of A. The Jaffard algebra is embedded into $B(\ell^2(\Lambda))$ by Schur's test (interpolation). Jaffard proved in [76] that this inclusion is spectral.

1.7.2 Weighted Schur algebras

Let $\Lambda \subseteq \mathbb{R}^d$ be a relatively separated set of points and let w be a subexponential weight (cf. Section 1.3). Suppose in addition that $w(x) \gtrsim (1 + \|x\|)^{\delta}$, for some $\delta > 0$. An example of such a weight is given by,

$$w(x) = e^{|x|^{\beta}}(1 + |x|)^s(\log(e + |x|))^t,$$

if $0 \leq \beta < 1$, $s > 0$ and $t \geq 0$.

Let $\mathcal{A}_w$ be the class of all matrices $A \in \mathbb{C}^{\Lambda\times\Lambda}$ such that,

$$\|A\|_{\mathcal{A}_w} := \max\left\{\sup_k \sum_j \left|A_{k,j}\right| w(k - j), \sup_j \sum_k \left|A_{k,j}\right| w(k - j)\right\}$$

is finite. Gröchenig and Leinert proved in [67] that $\mathcal{A}_w$ is a spectral subalgebra of $B(\ell^2(\Lambda))$ (see also [104]). It is known that the algebra $\mathcal{A}_w$ may not be spectral without the assumption $w(x) \gtrsim (1 + \|x\|)^{\delta}$ ([105]).

1.7.3 The Baskakov-Sjöstrand algebra

For $s \geq 0$, let C_s be the class of all matrices $A \in \mathbb{C}^{\mathbb{Z}^d\times\mathbb{Z}^d}$ such that,

$$\|A\|_{C_s} := \sum_k \sup_j \left|A_{j,j+k}\right| (1 + |k - j|)^s,$$

is finite. In [8, 9] Baskakov proved that C_s is a spectral subalgebra of $B(\ell^2(\mathbb{Z}^d))$ (see also [102]).

1.7.4 Controlled inversion

The results that we have just cited mean that certain off-diagonal decay conditions of a matrix A are inherited by its inverse A^{-1}. However they are only asymptotic because they do not specify what qualities of the matrix A determine the off-diagonal decay of the inverse matrix A^{-1}. Let us formulate this more precisely. All the above spectral invariance results take the following form. There is a certain matrix algebra $\mathcal{A}$ that is embedded into $B(\ell^2)$. From the knowledge that $\|A\|_{\mathcal{A}} < \infty$ and that A has an inverse in $B(\ell^2)$ we conclude that $\|A^{-1}\|_{\mathcal{A}} < \infty$. We would like to moreover have bounds on $\|A^{-1}\|_{\mathcal{A}}$ depending on $\|A\|_{\mathcal{A}}$ and possibly some additional quality of A. An obvious candidate for such extra quality is $\|A^{-1}\|_{B(\ell^2)}$.

The problem of quantifying spectral invariance is very subtle. It has been extensively studied by Nikolsky in [90] (see also [36]). Let us just mention that for the unweighted Baskakov-Sjöstrand algebra C_0, having bounds on $\|A\|_{C_0}$ and $\|A^{-1}\|_{B(\ell^2)}$ is not sufficient to bound $\|A^{-1}\|_{C_0}$.

In [104] Qiyu Sun proved a spectral invariance result for matrix algebras, using a very general geometric model for the underlying set of indices. As a by-product of his technique he not only established the inverse closedness of certain new matrix algebras but he also obtained a qualitative control for some of the classical ones. In Chapter 4 we cite an application of his result to the Jaffard algebra. For another result in that direction see [10].

1.8 Bases and frames for Hilbert spaces

We now introduce several concepts related to atomic decompositions in Hilbert spaces. For a general reference on these topics and proofs of the corresponding facts we refer the reader to [35, 114, 17].

1.8.1 Riesz bases

An *orthonormal basis* for a separable $\mathbb{C}$-Hilbert space $\mathbb{H}$ is a (countable) set $\{e_k\}_{k\in I} \subseteq \mathbb{H}$ that is orthonormal (i.e., $\left\langle e_k, e_j \right\rangle = 1$, if $k = j$, and 0 otherwise) and is *complete*, i.e., it generates a dense linear subspace. In this case, every element $f \in \mathbb{H}$ has an expansion,

$$f = \sum_k \langle f, e_k \rangle e_k,$$

with unconditional convergence in the norm of $\mathbb{H}$. The unconditionality of the convergence means that the net of finite sums

$$\left(\sum_{k\in F} \langle f, e_k \rangle e_k \right)_{f\subseteq I \text{ finite}}, \tag{1.23}$$

converges to f, where the class of finite subsets of I is ordered by inclusion. More concretely, given $\varepsilon > 0$, there exists a finite set $F_0 \subseteq I$ such that for every finite set $F_0 \subseteq F \subseteq I$, $\|\sum_{k\in F} \langle f, e_k \rangle e_k - f\|_{\mathbb{H}} < \varepsilon$.

A *Riesz basis* $\{f_k\}_{k\in I}$ for $\mathbb{H}$ is the image of an orthonormal basis $\{e_k\}_{k\in I}$ under an invertible operator $T : \mathbb{H} \to \mathbb{H}$, i.e., $f_k = T(e_k)$. If we let $g_k := (T^{-1})^*(e_k)$, it follows that every $f \in \mathbb{H}$ admits the expansion,

$$f = \sum_k \langle f, f_k \rangle \, g_k, \tag{1.24}$$

and also,

$$f = \sum_k \langle f, g_k \rangle \, f_k. \tag{1.25}$$

Here, the convergence is also unconditional in the norm of $\mathbb{H}$. The set $\{g_k\}_{k\in I}$ is also a Riesz basis and it is called the *dual basis* of $\{f_k\}_{k\in I}$. It is characterized by the bi-orthogonality relation: $\left\langle f_k, g_j \right\rangle = 1$, if $k = j$, and 0 otherwise. For any $c \in \ell^2$, the series $\sum_k c_k f_k$ converges unconditionally in the norm of $\mathbb{H}$ and moreover $\|c\|_{\ell^2} \approx \|\sum_k c_k f_k\|_{\mathbb{H}}$. As a consequence, the coefficients in Equation (1.25) provide the only way to write f as $\sum_k c_k f_k$ with $c \in \ell^2$.

The property that $\|c\|_{\ell^2} \approx \|\sum_k c_k f_k\|_{\mathbb{H}}$ in fact characterizes Riesz bases.

Theorem 1.8.1. *Let $\mathbb{H}$ be a Hilbert space and let $\{f_k\}_{k\in I} \subseteq \mathbb{H}$. Then $\{f_k\}_k$ is a Riesz basis if and only if it generates a dense linear subspace of $\mathbb{H}$ and satisfies the inequalities,*

$$A\|c\|_{\ell^2} \leq \|\sum_k c_k f_k\|_{\mathbb{H}} \leq B\|c\|_{\ell^2}, \qquad c \text{ a finitely supported sequence},$$

for some constants $0 < A \leq B < +\infty$. The optimal constants satisfying these inequalities are called the Riesz basis constants.

1.8.2 Frames

A *frame* $\{f_k\}_{k\in I}$ for a Hilbert space $\mathbb{H}$ is the image of an orthonormal basis $\{e_k\}_{k\in I}$ under a surjective bounded operator $T : \mathbb{K} \to \mathbb{H}$. Here, $\mathbb{K}$ is another Hilbert space, so the cardinality of a frame may exceed the algebraic dimension of the space. This consideration, of course, is only important when $\mathbb{H}$ is finitely dimensional. If $U : \mathbb{H} \to \mathbb{K}$ denotes the canonical section (right inverse) of T, characterized by $U(\mathbb{H}) = \ker(T)^\perp$, and we let $g_k := U^*(e_k)$, it follows that every $f \in \mathbb{H}$ admits the expansions in Equations (1.24) and (1.25), with the same kind of convergence. The choice for the system $\{g_k\}_k$ yielding those expansions is however non-unique due to the fact that T admits many other sections. The system defined by $g_k := U^*(e_k)$ can be seen to also be a frame and is called the *canonical dual frame* of $\{f_k\}_k$. The sequence $\{\langle f, g_k \rangle\}_k$ is characterized by the property of having minimal ℓ^2-norm among all sequences $c \in \ell^2$ such that $f = \sum_k c_k f_k$.

Hence, frames provide possibly redundant basis-like expansions. In several cases where it is impossible to obtain bases with a desired structure it is however possible to construct frames. Similarly to Theorem 1.8.1, the property of being a frame can characterized in terms of an inequality.

Theorem 1.8.2. *Let $\mathbb{H}$ be a Hilbert space and let $\{f_k\}_{k\in I} \subseteq \mathbb{H}$. Then $\{f_k\}_k$ is a frame for $\mathbb{H}$ if and only if, there exists two constants $0 < A \leq B < +\infty$ such that,*

$$A^2\|f\|_{\mathbb{H}}^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq B^2\|f\|_{\mathbb{H}}^2, \qquad \text{for all } f \in \mathbb{H}. \tag{1.26}$$

The optimal constants satisfying that condition are called the frame bounds.

According to Theorem 1.6.1, if $\{f_k\}_k$ is a frame for $\mathbb{H}$, then the operator $S : \mathbb{H} \to \mathbb{H}$ given by,

$$S(f) := \sum_k \langle f, f_k\rangle f_k, \tag{1.27}$$

is well-defined and satisfies $AI_{\mathbb{H}} \leq S \leq BI_{\mathbb{H}}$, for some $0 < A \leq B < +\infty$. Therefore, S is invertible. S is called the *frame operator*. The canonical dual frame of $\{f_k\}_k$ is easily seen to be given by $g_k = S^{-1}(f_k)$. If $A = B = 1$ we say that $\{f_k\}_k$ is a *Parseval frame*. In this case $g_k = f_k$ and the frame $\{f_k\}_k$ provides an orthonormal-basis-like expansion.

In general, however, reconstructing an element f from the coefficients $\{\langle f, f_k\rangle\}_k$ requires inverting the operator S. This can be done by using *Neumann series*, yielding the following *frame algorithm*. Let $0 < \alpha < 2/B$ and set $\delta := \max\{|1 - \alpha A|, |1 - \alpha B|\}$. For $f \in \mathbb{H}$, define,

$$\begin{cases} f_0 & := 0, \\ f_{n+1} & := f_n + \alpha S(f - f_n), \qquad (n \in \mathbb{N}). \end{cases}$$

Then, $f_n \to_n f$ with a geometric rate of convergence,

$$\|f - f_n\|_{\mathbb{H}} \leq \delta^n \|f\|_{\mathbb{H}}.$$

The optimal value for the *relaxation parameter* α is $\frac{2}{A+B}$. Since finding sharp estimates on the frame bounds can be difficult, the numerical efficiency of the frame algorithm is seriously limited. In practice, this limitation is addressed by combining the frame algorithm with an acceleration method like the one of conjugate gradients.

Riesz bases can be characterized as non-redundant frames.

Theorem 1.8.3. *Let $\mathbb{H}$ be a Hilbert space and let $\{f_k\}_{k\in I} \subseteq \mathbb{H}$ be a frame for $\mathbb{H}$. Then the following conditions are equivalent.*

- $\{f_k\}_k$ *is a Riesz basis.*
- *Every element $f \in \mathbb{H}$ has a unique expansion $f = \sum_k c_k f_k$, with $c \in \ell^2$.*
- $\|\sum_k c_k f_k\|_{\mathbb{H}} \approx \|c\|_{\ell^2}, \quad (c \in \ell^2)$.

1.8.3 Bessel sequences and frame pairs

A family $\{f_k\}_{k\in I}$ in a Hilbert space $\mathbb{H}$ is called *Bessel* if it satisfies the inequality,

$$\sum_k |\langle f, f_k\rangle|^2 \leq B^2\|f\|^2_{\mathbb{H}}, \qquad (f \in \boldsymbol{H}), \tag{1.28}$$

for some constant B. This condition means that the *analysis operator*,

$$\begin{aligned} &\mathbb{H} \to \ell^2(I) \\ &f \mapsto C(f) := (\langle f, f_k\rangle)_k, \end{aligned} \tag{1.29}$$

is well-defined and bounded, with norm at most B. The formal adjoint of C is the *synthesis operator*,

$$\ell^2(I) \to \mathbb{H} \tag{1.30}$$

$$c \mapsto R(c) := \sum_k c_k f_k. \tag{1.31}$$

Consequently, the Bessel condition is equivalent to the estimate,

$$\|\sum_k c_k f_k\|_{\mathbb{H}} \leq B\|c\|_{\ell^2}, \qquad c \text{ a finitely supported sequence.} \tag{1.32}$$

Riesz bases and frames can be easily characterized in terms of these operators.

Theorem 1.8.4. *Let $\mathbb{H}$ be a Hilbert space and let $\{f_k\}_{k\in I} \subseteq \mathbb{H}$ be a Bessel sequence. Then the following holds.*

- *$\{f_k\}_{k\in I}$ is a frame if and only if the analysis map C bounded below. This happens if and only if R is surjective.*

- *$\{f_k\}_{k\in I}$ is a Riesz basis if and only if the analysis map C is invertible. This happens if and only if R is invertible.*

A pair of Bessel sequences $(\{f_k\}_k, \{g_k\}_k)$ is called a *frame pair* if they provide the expansion,

$$f = \sum_k \langle f, g_k\rangle\, f_k,$$

for every $f \in \mathbb{H}$. When $\{f_k\}_k$ is a frame, the canonical dual frame $\{g_k\}_k$ yields a frame pair $(\{f_k\}_k, \{g_k\}_k)$ but there may be many others choices for $\{g_k\}_k$. These are called *non-canonical duals frames*.

1.8.4 Bases and frames for subspaces

A *Riesz sequence* is a subset of a Hilbert space that is a Riesz basis of the closure of the linear subspace that it generates. Similarly, a *frame sequence* is a subset that is a frame for that subspace. Theorem 1.8.4 can be reformulated as follows.

Theorem 1.8.5. *Let $\mathbb{H}$ be a Hilbert space and let $\{f_k\}_{k\in I} \subseteq \mathbb{H}$ be a Bessel sequence. Then the following holds.*

- *$\{f_k\}_{k\in I}$ is a frame sequence if and only if C has closed range. This happens if and only if R has closed range.*
- *$\{f_k\}_{k\in I}$ is a Riesz sequence if and only if R is bounded below.*

Given a Hilbert space $\mathbb{H}$ and a closed subspace $\mathbb{K}$, an *outer frame* (or *exterior frame*) for $\mathbb{K}$ is a family $\{f_k\}_k \subseteq \mathbb{H}$ whose orthogonal projection onto $\mathbb{K}$, $\{P_{\mathbb{K}}(f_k)\}_k$, forms a frame for $\mathbb{K}$. The family $\{f_k\}_k \subseteq \mathbb{H}$ needs not to be Bessel. Using Theorem 1.8.2 it follows that $\{f_k\}_k \subseteq \mathbb{H}$ is an outer frame for $\mathbb{K}$ if and only if there exist two constants $0 < A \leq B < +\infty$ such that the frame inequality,

$$A^2\|f\|_{\mathbb{H}}^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq B^2\|f\|_{\mathbb{H}}^2,$$

holds for all $f \in \mathbb{K}$. This is so, because $\langle f, f_k\rangle = \langle f, P_{\mathbb{K}}(f_k)\rangle$ in the inequality above.

In the context of a Hilbert space $\mathbb{H}$ and a distinguished subspace $\mathbb{K}$, a frame pair for $\mathbb{K}$ is a pair o Bessel sequences $(\{f_k\}_k, \{g_k\}_k)$ such that $(\{f_k\}_k, \{P_{\mathbb{K}}(g_k)\}_k)$ is a frame pair in the sense of Section 1.8.3. This simply means that the expansion,

$$f = \sum_k \langle f, g_k\rangle f_k,$$

holds for every $f \in \mathbb{K}$, although the analysing atoms $\{g_k\}_k$ are not assumed to belong to $\mathbb{K}$.

1.8.5 The Gramian matrix

Given a Hilbert space $\mathbb{H}$, the *Gramian matrix* of a family $\{f_k\}_{k\in I} \subseteq \mathbb{H}$ is the matrix $G \in \mathbb{C}^{I\times I}$ given by,

$$G_{k,j} := \left\langle f_k, f_j\right\rangle.$$

Note that the Gramian matrix is Hermitian (i.e., $G^* = G$). If we let C and R denote the analysis and synthesis operators given by Equations (1.29) and (1.30), then G is the transpose of the matrix representing the operator $CR : \ell^2(I) \to \ell^2(I)$. This operator is well-defined and bounded if $\{f_k\}_k$ is a Bessel sequence. Conversely, it is easy to show that if G defines a bounded operator on ℓ^2, then the sequence $\{f_k\}_k$ is Bessel. The Gramian matrix is dual to the frame operator S defined by Equation (1.27) in the sense that $G^t = CR = CC^*$ and $S = RC = C^*C$. As shown in Theorem 1.8.2, the invertibility of the frame operator is equivalent to the frame property for $\{f_k\}_k$. In contrast, the Gramian matrix depends only on the smallest closed subspace of $\mathbb{H}$ that contains the family $\{f_k\}_k$. Hence, it can only be used to decide if the system $\{f_k\}_k$ is a frame or Riesz sequence, but cannot decide about the completeness of the family in the whole space $\mathbb{H}$.

Theorem 1.8.6. *Let $\mathbb{H}$ be a Hilbert space and let $\{f_k\}_{k\in I} \subseteq \mathbb{H}$ be a Bessel sequence. Then the following holds.*

- $\{f_k\}_k$ *is a Riesz sequence if and only if the Gramian matrix G in invertible. In this case,* $A := \inf\{x : x \in \operatorname{spec}(G)\}$ *and* $B := \sup\{x : x \in \operatorname{spec}(G)\}$ *are the Riesz basis bounds for* $\{f_k\}_k$.
- $\{f_k\}_k$ *is a frame sequence if and only if G has closed range. This happens if and only if* 0 *is an isolated point of the spectrum of G (or does not belong to it at all). In this case,* $A := \inf\{x : x \in \operatorname{spec}(G) \setminus \{0\}\}$ *and* $B := \sup\{x : x \in \operatorname{spec}(G)\}$ *are the frame bounds for* $\{f_k\}_k$.

The canonical dual frame can be calculated using the Gramian matrix (instead of the frame operator). If $\{f_k\}_k$ is a frame (or frame sequence), Theorem 1.8.6 implies that its Gramian matrix G has closed range and therefore admits a Moore-Penrose pseudo-inverse. The canonical dual frame $\{g_k\}_k$ is given by,

$$g_j = \sum_k G^\dagger_{j,k} f_k, \quad (j \in I). \tag{1.33}$$

1.9 Atomic decompositions of Banach spaces

We now introduce atomic decompositions for abstract Banach spaces. The concept of frame for a Hilbert space can be generalized in two directions. The elements of a Hilbert-space frame $\{f_k\}_k$ can be considered as *atoms* of the space yielding expansions $\sum_k c_k f_k$ or as linear functionals $f \mapsto \langle f, f_k\rangle$ providing abstract "samples" of a vector that can then be used to reconstruct it. It was shown in Section 1.8 that, as a consequence of the self-duality of Hilbert spaces, these two notions are equivalent: any admissible system of "analyzing vectors" (cf. Equation (1.26)) is also an adequate set of atoms and vice versa.

We have faced already in the context of Hilbert spaces the need to distinguish the two usages. The elements of an outer frame $\{f_k\}_k$ for a subspace $\mathbb{K}$ of a Hilbert space $\mathbb{H}$ can be considered as non-canonical representations of the functionals $\mathbb{K} \ni f \mapsto \langle f, f_k\rangle$ but not as atoms of $\mathbb{K}$ since they do not even belong to $\mathbb{K}$.

For general Banach spaces the two notions must certainly be distinguished. Atomic decompositions will be defined as pairs of analyzing functionals and atoms. We now present a definition that fits all the examples that we are interested in. Let $\boldsymbol{X}$ be a separable Banach space over the complex field. A *Fatou topology* $\mathcal{T}$ on $\boldsymbol{X}$ is a locally-convex Hausdorff topology that is coarser than the norm topology and such that, for every $x \in \boldsymbol{X}$,

$$\|x\|_{\boldsymbol{X}} = \sup \left\{ |f(x)| \;\middle|\; f \in (\boldsymbol{X}, \mathcal{T})', \|f\|_{\boldsymbol{X}'} \leq 1 \right\}.$$

Here, $(\boldsymbol{X}, \mathcal{T})'$ denotes the set of linear functionals on $\boldsymbol{X}$ that are continuous with respect to $\mathcal{T}$. Hence, a Fatou topology is a topology with enough continuous linear functionals so as to approximate the norm of each element. The terminology comes from the fact that this condition is equivalent to the validity of Fatou's lemma: if $x_n \to x$ in the $\mathcal{T}$-topology, then $\|x\|_{\boldsymbol{X}} \leq \liminf_n \|x_n\|_{\boldsymbol{X}}$ (see [98]).

Recall (cf. Section 1.4) that a BK space is a Banach sequence space $\boldsymbol{S}$ over a countable index set I that is continuously embedded into $\mathbb{C}^I$; i.e., each coordinate projection, $a \mapsto a_k$, defines a continuous linear functional. Recall also that a BK space $\boldsymbol{S}$ is called solid if $a \in \boldsymbol{S}$ and $|b| \leq |a|$, imply that $b \in \boldsymbol{S}$ and $\|b\|_{\boldsymbol{S}} \leq \|a\|_{\boldsymbol{S}}$.

An *atomic decomposition of* $\boldsymbol{X}$ consists of a solid BK space $\boldsymbol{S}$ of sequences over a countable index set I together with a Fatou topology $\mathcal{T}$ on $\boldsymbol{X}$, a set of vectors $\{x_k : k \in I\} \subseteq \boldsymbol{X}$ and functionals $\{f_k : k \in I\} \subseteq \boldsymbol{X}'$ such that,

(i) For every $c \in \boldsymbol{S}$, the series $R(c) := \sum_k c_k x_k$ converges unconditionally in the topology $\mathcal{T}$. Moreover, the synthesis operator, $R : \boldsymbol{S} \to \boldsymbol{X}$ is bounded.

(ii) For every $x \in \boldsymbol{X}$, the sequence $C(x) := (f_k(x))_k$ belongs to $\boldsymbol{S}$. Moreover, the analysis operator $C : \boldsymbol{X} \to \boldsymbol{S}$ is bounded.

(iii) $RC = I_{\boldsymbol{X}}$. That is, every $x \in \boldsymbol{X}$ admits the expansion, $x = \sum_k f_k(x) x_k$.

In concrete examples, when $\boldsymbol{X}$ is a space of functions, the topology $\mathcal{T}$ can be some weak topology like the one of distributions. When the set of finitely supported sequences is dense in $\boldsymbol{S}$, due to the solidity of $\boldsymbol{S}$, the family of "delta sequences" $\{\delta_k\}_k$ - given by, $\delta_k(j) = 1$, if $k = j$ and 0 otherwise - forms an unconditional basis of $\boldsymbol{S}$. As a consequence, the series defining the synthesis operator R converge unconditionally in the norm of $\boldsymbol{X}$. Therefore, in that case, the topology $\mathcal{T}$ can be taken to be the norm-topology of $\boldsymbol{X}$. The definition of atomic decomposition given above makes sense even if the sequence space $\boldsymbol{S}$ is not assumed to be solid, but we shall only work in the context of solid sequence spaces.

A *Banach frame* for a (separable) Banach space $\boldsymbol{X}$ consist of a solid BK space $\boldsymbol{S}$ of sequences over a countable index set I together with a set of vectors $\{x_k : k \in I\} \subseteq \boldsymbol{X}$ - called *atoms* and a bounded linear retraction $R : \boldsymbol{S} \to \boldsymbol{X}$ such that $R(\delta_k) = f_k$. The fact that R is a retraction means that there exists another operator $C : \boldsymbol{E} \to \boldsymbol{E}_d$, called the *coefficient operator*, such that $RC = I_{\boldsymbol{E}}$. Since $\boldsymbol{S} \hookrightarrow \mathbb{C}^I$, this operator is implemented by some family of linear functionals $\{g_k\}_k \subseteq \boldsymbol{E}'$ by means of the formula $C(f) = (\langle f, g_k \rangle)_k$. When the operator R is implemented by a series converging in a Fatou topology, each choice of a coefficient operator gives rise to an atomic decomposition in the sense of the definition above. As we remarked before, this is the case whenever the set of finitely-supported sequences is dense in $\boldsymbol{S}$.

It is common in the literature to define Banach frames in terms of the coefficient functionals $\{g_k\}_k$ rather than the atoms $\{f_k\}_k$. A family of linear functionals $\{g_k\}_{k\in\Lambda} \subseteq \boldsymbol{X}'$, together with a solid BK sequence space $\boldsymbol{S}$ is called a *Banach frame* for $\boldsymbol{X}$ if the coefficient operator $\boldsymbol{X} \ni f \mapsto (\langle f, g_k\rangle)_k \in \boldsymbol{S}$ is a bounded section, i.e., there exists a bounded operator $R : \boldsymbol{S} \to \boldsymbol{X}$ such that $RC = I_{\boldsymbol{X}}$. In the abstract setting there is no possible confusion between the two usages since the atoms and the coefficient functionals belong to different spaces. However, in concrete examples where $\boldsymbol{X}$ is a classical function space and $\boldsymbol{X}'$ is identified with another classical function space, these two usages can be ambiguous. To avoid confusions, every reference to a Banach frame will be followed by a clarification about its precise meaning.

1.10 Coorbit theory

We now briefly introduce coorbit theory (see [44]). Let π be a (strongly continuous) unitary representation of a locally compact group $\mathcal{G}$ on a Hilbert space $\mathbb{H}$. For a fixed $h \in \mathbb{H}$, the abstract wavelet transform is defined as,

$$V_h f(x) := \langle f, \pi(x)h\rangle, \quad (f \in \mathbb{H}, x \in \mathcal{G}).$$

Let w be an admissible weight on $\mathcal{G}$. A vector $h \in \mathbb{H}$ is called *admissible* if $V_h h \in W_R(L^\infty, L^1_w)(\mathcal{G})$ and the reproducing formula,

$$V_h f = V_h f * V_h h,$$

holds for all $f \in \mathbb{H}$. Here, $W_R(L^\infty, L^1_w)(\mathcal{G})$ is the Wiener amalgam space from Section 1.6. Moreover, h is assumend to be *cyclic*, i.e. the orbit of h, $\{\pi(x)h \,|\, x \in \mathcal{G}\}$ is assumed to be dense in $\mathbb{H}$. Admissible vectors are the main ingredient of coorbit theory. They are known to exist under several circumstances. For a study about the validity of the reproducing formula see [58]. In concrete examples one can often produce explicit admissible vectors.

Let h be an admissible vector. Since $V_h h(x^{-1}) = \overline{V_h h(x)}$, it follows that $V_h h$ also belongs to $W(L^\infty, L^1_w)$. As a consequence of the reproducing formula, $V_h : \mathbb{H} \to L^2(\mathcal{G})$ is an isometry and therefore has an inverse on its (closed) range.

Under these conditions the space $\mathbb{H}^1_w$ is defined by

$$\mathbb{H}^1_w := \{ f \in \mathbb{H} \,|\, V_h f \in L^1_w \},$$

and endowed with the norm $\|f\|_{\mathbb{H}^1_w} := \|V_h f\|_{L^1_w}$. The anti-dual of $\mathbb{H}^1_w$ (i.e. the space of continuous conjugate-linear functionals) is denoted by $(\mathbb{H}^1_w)^\urcorner$. The inner product $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$ extends to a sesquilinear form on $\mathbb{H}^1_w \times (\mathbb{H}^1_w)^\urcorner \to \mathbb{C}$. Since h is assumed to belong to $\mathbb{H}^1_w$, the abstract wavelet transform can be defined for $f \in (\mathbb{H}^1_w)^\urcorner$.

Coorbit spaces are defined by selecting from the reservoir $(\mathbb{H}^1_w)^\urcorner$ those elements that satisfy a certain criteria. Let $\boldsymbol{E}$ be a solid, translation invariant BF space on $\mathcal{G}$ such that w is admissible for it. The coorbit space is defined by

$$\mathrm{Co}\boldsymbol{E} := \{\, f \in (\mathbb{H}^1_w)^\urcorner \,\big|\, V_h f \in \boldsymbol{E}\,\},$$

and endowed with the norm $\|f\|_{\mathrm{Co}\boldsymbol{E}} := \|V_h f\|_{\boldsymbol{E}}$.

The results in [44, 61] provide atomic decompositions for coorbit spaces, where the *coherent states* $\{\pi(\lambda)h : \lambda \in \Lambda\}$ can play both the role of atoms and coefficient functionals.

Theorem 1.10.1. *Let w be an admissible weight and $h \in \boldsymbol{H}$ an admissible vector. Then, there exists U, a relatively compact neighborhood of the identity in $\mathcal{G}$, such that for any U-dense and relatively separated set $\Lambda \subseteq \mathcal{G}$, the following atomic decomposition of* Co$\boldsymbol{E}$ *holds simultaneously, for all BF spaces $\boldsymbol{E}$ for which the weight w is admissible.*

(i) For every $c \in \boldsymbol{E}^d(\Lambda)$, the function,

$$f = \sum_\lambda c_\lambda \pi(\lambda)h,$$

belongs to Co$\boldsymbol{E}$ *and $\|f\|_{\mathrm{Co}\boldsymbol{E}} \lesssim \|c\|_{\boldsymbol{E}^d}$. The series converges unconditionally in the weak* topology of $(\mathbb{H}^1_w)^\urcorner$ and, if the set of bounded, compactly supported functions is dense in $\boldsymbol{E}$, it also converges unconditionally in the norm of* Co$\boldsymbol{E}$*.*

(ii) There exists a bounded linear operator $C : \mathrm{Co}\boldsymbol{E} \to \boldsymbol{E}^d(\Lambda)$ such that

$$f = \sum_\lambda C(f)_\lambda \pi(\lambda)h.$$

The operator C is given by $C(f)_\lambda = \langle f, g_\lambda\rangle$, for a certain family $\{g_\lambda : \lambda \in \Lambda\} \subseteq \mathbb{H}^1_w$, that is independent of $\boldsymbol{E}$. In addition, the quantity $\|(\langle f, g_\lambda\rangle)_\lambda\|_{\boldsymbol{E}^d}$ is an equivalent norm on Co$\boldsymbol{E}$*.*

(iii) The quantity $\left\|(\langle f, \pi(\lambda)h\rangle)_\lambda\right\|_{\boldsymbol{E}^d}$ is an equivalent norm on Co$\boldsymbol{E}$*. Moreover, there exists a bounded operator $R : \boldsymbol{E}^d(\Lambda) \to \mathrm{Co}\boldsymbol{E}$ such that $R((\langle f, \pi(\lambda)h\rangle)_\lambda) = f$, for all $f \in \mathrm{Co}\boldsymbol{E}$.*

1.11 Modulation spaces

For $f, h \in L^2(\mathbb{R}^d)$, the *Short-Time Fourier Transform* (STFT) (or *windowed Fourier Transform*) is defined by,

$$\mathcal{V}_h f(x, \varsigma) = \int_{\mathbb{R}^d} f(y) e^{-2\pi i \varsigma y} \overline{h(y - x)} dy.$$

Recall that the translation and modulation operators are given by $T_x f(y) := f(y - x)$ and $M_\varsigma f(y) := e^{2\pi i \varsigma y} f(y)$, so that,

$$\mathcal{V}_h f(x, \varsigma) := \langle f, M_\varsigma T_x h \rangle . \tag{1.34}$$

If h is suitably normalized, $\mathcal{V}_h : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is an isometry. The adjoint (inverse) STFT is given by,

$$\mathcal{V}_h^* F(x) = \int_{\mathbb{R}^{2d}} F(y, \varsigma) M_\varsigma T_y h(x) dy d\varsigma.$$

For a general $h \in L^2(\mathbb{R}^d)$ the integral should be understood in the weak sense (not pointwise).

Let ϕ be any non-zero Schwartz class function. For example, ϕ can be the Gaussian function $\phi(x) := e^{-|x|^2}$. The definition in Equation (1.34) then extends to tempered distributions. Modulation spaces are defined by imposing integrability conditions of the STFT. Let v be a polynomially moderated weight on $\mathbb{R}^{2d}$. For $1 \leq p, q \leq +\infty$, the modulation space $M_v^{p,q}$ is defined as,

$$M_v^{p,q} = M_v^{p,q}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,\middle|\, \mathcal{V}_\phi f \in L_v^{p,q}(\mathbb{R}^{2d}) \right\}$$

where,

$$\|F\|_{L_v^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, \varsigma)|^p v(x, \varsigma)^p dx \right)^{q/p} d\varsigma \right)^{1/q},$$

with the usual modifications when p or q are $+\infty$. $M_v^{p,q}$ is of course given the norm $\|f\|_{M_v^{p,q}} = \|\mathcal{V}_h f\|_{L_v^{p,q}}$. When $p = q$ we write $M_v^p = M_v^{p,p}$.

Thus, modulation-space norms measure the time-frequency concentration of a distribution. The decay properties of a distribution f are roughly evidenced by the decay of $\mathcal{V}_\phi f(x, w)$ in the variable x, where the smoothness of f near point $x = x_0$ is related to the decay of $\mathcal{V}_\phi f(x_0, w)$ in the variable w. Here are some precise statements [62, Proposition 11.3.1].

Proposition 1.11.1. *Let $v_t(x) := (1 + |x|)^t$, $t \in \mathbb{R}$, denote the polynomial weights on $\mathbb{R}^{2d}$. Then the following holds.*

(a) If $|f(x)| \lesssim v_{-t}(x)$, for some $t > d$, then $\left|V_\phi f(x, w)\right| \lesssim v_{-t}(x)$. If $\left|\hat{f}(w)\right| \lesssim v_{-t}(w)$, for some $t > d$, then $\left|V_\phi f(x, w)\right| \lesssim v_{-t}(w)$.

(b) If $v(x, w) = v(x)$, then $M_v^2 = L_v^2$.

(c) If $v(x, w) = v(w)$, then $M_v^2 = \mathcal{F}(L_v^2)$. In particular if $v(x, w) = v_t(w)$, then M_v^2 is a Bessel potential space.

(d) $\mathcal{S}(\mathbb{R}^d) = \cap_{t \geq 0} M_{v_t}^\infty$ and $\mathcal{S}'(\mathbb{R}^d) = \cup_{t \leq 0} M_{v_t}^\infty$.

Modulation spaces are coorbit spaces of the Heisenberg group (cf. Section 1.5.2). The abstract wavelet transform $\overline{\mathcal{V}}_\phi$ associated with the representation $\overline{\pi}$ from Section 1.5.2 is related to the short-time Fourier transform by,

$$\overline{\mathcal{V}}_\phi f(x, w, \lambda) = \overline{\lambda}\mathcal{V}_\phi f(x, w).$$

Since the mapping $f \mapsto \overline{f}$ given by $\overline{f}(x, w, \lambda) = \overline{\lambda} f(x, w)$ preserves weighted L^p norms, it follows that the abstract construction from Section 1.10 coincides with the more concrete presentation given above.

1.11.1 Gabor frames

A Gabor system generated by a function $g \in L^2(\mathbb{R}^d)$ is a set of functions of the form,

$$\mathcal{G}(g, \lambda) := \{\, \pi(\lambda)g \,\big|\, \lambda \in \Lambda \,\},$$

where $\pi(x, w) = M_w T_x$ and the set $\Lambda \subseteq \mathbb{R}^{2d}$ is called the set of *phases*.

As an example, if we let $g = \chi_{[0,1]}$ and $\Lambda = \mathbb{Z} \times \mathbb{Z}$, the corresponding Gabor system is an orthonormal basis of $L^2(\mathbb{R})$. Several classic results show that this is only possible when the generating atom g has poor time-frequency localization. Hence, redundancy is essential in good time-frequency representations.

Theorem 1.11.1 ([72, 11])**.** *If $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then both $g \notin W(C_0, L^1)$ and $\hat{g} \notin W(C_0, L^1)$.*

Theorem 1.11.2 (Balian-Low, Daubechies, [25, 11])**.** *If $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a Riesz basis for $L^2(\mathbb{R}^d)$ then $xf \notin L^2(\mathbb{R})$ or $f' \notin L^2(\mathbb{R})$.*[4]

Gabor systems provide atomic decompositions for modulation spaces. This was first noted in [40]. Using the realization of modulation spaces as coorbit spaces we can deduce this from Theorem 1.10.1.

Theorem 1.11.3. *Let v be a polynomially moderated weight on $\mathbb{R}^{2d}$ and let $h \in M^1_v$ be a non-zero function. Then, for every sufficiently dense, relatively separated set Λ, the Gabor system $\mathcal{G}(g, \lambda)$ is a Banach frame for $M^{p,q}_m$, for all $1 \leq p, q \leq \infty$ and all v-moderated weights m.*

Here, the terms Banach frame can be understood in both of the senses discussed in Section 1.9, so the elements of the Gabor system $\mathcal{G}(g, \lambda)$ play simultaneously the roles of atoms and analyzing functionals.

Constructing Gabor frames in $L^2(\mathbb{R}^d)$ is significantly easier than constructing atomic decompositions for the whole family of modulation spaces. As an example, we quote the following construction known as "the painless method" [26] (see also [62, Prop. 6.4.1]).

[4] Here, xf denotes the function taking the value $xf(x)$ at a point $x \in \mathbb{R}$. The assertion $f' \notin L^2(\mathbb{R})$ means that f does not have a weak-derivative in L^2.

Theorem 1.11.4. *Let $g \in L^\infty(\mathbb{R}^d)$ be supported on the cube $[0, L]^d$. Let $0 < \alpha \leq L$ and $0 < \beta \leq 1/L$. Then, the Gabor system $\mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ is a frame for $L^2(\mathbb{R}^d)$ with bounds $\beta^{-d}a$, $\beta^{-d}b$ if and only if,*

$$0 < a \leq \sum_{k\in\mathbb{Z}^d} |g(x - \alpha k)|^2 \leq b, \text{ for almost every } x \in \mathbb{R}^d.$$

In particular, $\mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ is a tight frame if and only if,

$$\sum_{k\in\mathbb{Z}^d} |g(x - \alpha k)|^2 \equiv \beta^d, \text{ for almost every } x \in \mathbb{R}^d.$$

In the “painless” case it is even possible to get an explicit formula for the dual frame of the Gabor system. This is almost the only case where this is possible. Most existence results for Gabor frames rely on establishing the frame inequality but do not provide explicit constructions of the dual system.

The other important case where the frame condition for a Gabor system has been completely characterized is the case of the time-frequency translates of the one-dimensional Gaussian function over a lattice [86, 99, 100].

Theorem 1.11.5. *Let $\phi(x) := e^{-\pi x^2}$, $(x \in \mathbb{R})$. Then, the Gabor system $\mathcal{G}(\phi, \alpha\mathbb{Z}\times\beta\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 1$.*

The proof of this theorem resorts to sampling estimates for entire functions and does not provide an explicit formula for the canonical dual system.

Theorems 1.11.4 and 1.11.5 provide sharp conditions for a Gabor system to be a frame of $L^2(\mathbb{R}^d)$. In contrast, the result in Theorem 1.11.3 is only qualitative, it does not give such sharp sufficient conditions. As a trade-off, Theorem 1.11.3 yields a much stronger conclusion: it gives a the simultaneous atomic decomposition of the whole class of modulations spaces. In order to bridge this gap, it is desirable to known when a Gabor frame for $L^2(\mathbb{R}^d)$ extends to a Banach frame for all modulation spaces. The key technical obstacle is the lack of information on the dual system, whose existence is granted by the frame inequality. This difficulty has been addressed in the last years by resorting to spectral invariance results (or non-commutative variants of Wiener's lemma) like the ones in Section 1.7 [46, 66, 63, 6]. We cite the following result as an illustration.

Theorem 1.11.6 ([66])**.** *Let v be a subexponential weight, $g \in M^1_v(\mathbb{R}^d)$, and $\Lambda \subseteq \mathbb{R}^{2d}$ a lattice. Suppose that the Gabor system $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. Then, its canonical dual frame has the form $\mathcal{G}(h, \Lambda)$, for some function $h \in M^1_v(\mathbb{R}^d)$.*

As a consequence, each of the pairs $(\mathcal{G}(g, \Lambda), \mathcal{G}(h, \Lambda))$, $(\mathcal{G}(h, \Lambda), \mathcal{G}(g, \Lambda))$ provides, simultaneously, an atomic decomposition for every modulation space $M^{p,q}_m$, with $1 \leq p, q \leq +\infty$ and m a v-moderated weight. [5]

[5] In order to define modulation spaces as subsets of the class of tempered distributions, we must further assume that the weight v is polynomially moderated. If this is not the case, modulation spaces can still be constructed by the abstract method of Section 1.10.

More generally, the set of phases Λ need not be assumed to be a lattice [6], but in this case the dual frame is not a Gabor system. Instead it is a set of *time-frequency molecules* $\{ h_\lambda \mid \lambda \in \Lambda \}$, i.e., it satisfies,

$$\left|\mathcal{V}_\phi h_\lambda((x,w)\right| \leq H((x,w)-\lambda)),$$

for some function $H \in W(L^\infty, L^1_v)$. This is enough to deduce the Banach frame condition in Theorem 1.11.6 above.

1.12 Besov and Triebel-Lizorkin spaces

1.12.1 Besov spaces

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\operatorname{supp}\hat{\varphi} \subseteq \{ w \in \mathbb{R}^d \mid 1/2 \leq |w| \leq 1 \}$ and that $|\hat{\varphi}(w)| \gtrsim 1$, for $3/5 \leq |w| \leq 5/3$. If we let $\varphi_j(x) := 2^{-jd}\varphi(2^{-j}x)$, for $j \in \mathbb{Z}$, it follows that $\sum_j \left|\hat{\varphi}_j\right| \approx 1$.

For $1 \leq p, q \leq +\infty$, $\alpha \in \mathbb{R}$ and a distribution $f \in \mathcal{S}'$, we define the (semi)norm,

$$\|f\|_{\dot{B}^\alpha_{p,q}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{-\alpha q j}\|f * \varphi_j\|^q_{L^p}\right)^{1/q}, & \text{if } q < +\infty, \\ \sup_{j\in\mathbb{Z}} 2^{-\alpha j}\|f * \varphi_j\|_{L^p}, & \text{if } q = +\infty. \end{cases} \tag{1.35}$$

Note that in the definition above, $f * \varphi_j$ is a smooth function because it is a distribution with compactly-supported Fourier transform. Observe also, that $\|f\|_{\dot{B}^\alpha_{p,q}} = 0$ if and only if $\hat{f}$ is supported at $\{0\}$. This happens if and only if f is a polynomial. Thus, in order to have a Banach space, elements in $\dot{B}^\alpha_{p,q}$ have to be considered modulo polynomials. Let $\mathcal{P}$ denote the class of all polynomials on $\mathbb{R}^d$. The space *homogeneous Besov space* $\dot{B}^\alpha_{p,q}$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that $\|f\|_{\dot{B}^\alpha_{p,q}} < +\infty$.

The inhomogeneous Besov spaces are defined by distinguishing the positive and negatives scales. Let $\phi(x) := \sum_{j<0} \varphi_j(x)$. For $1 \leq p, q \leq +\infty$, $\alpha \in \mathbb{R}$ and a distribution $f \in \mathcal{S}'$, we define,

$$\|f\|_{B^\alpha_{p,q}} := \|f * \phi\|_{L^p} + \begin{cases} \left(\sum_{j\geq 0} 2^{-\alpha q j}\|f * \varphi_j\|^q_{L^p}\right)^{1/q}, & \text{if } q < +\infty, \\ \sup_{j\geq 0} 2^{-\alpha j}\|f * \varphi_j\|_{L^p}, & \text{if } q = +\infty. \end{cases} \tag{1.36}$$

The *inhomogeneous Besov space* $B^\alpha_{p,q}$ is defined as the set of all distributions $f \in \mathcal{S}'$ such that $\|f\|_{B^\alpha_{p,q}} < +\infty$. In this case there is no need to cancel out the class of polynomials.

For $\alpha > 0$, inhomogeneous Besov spaces consists of "ordinary" functions and can be characterized in terms of moduli of continuity (see [107, Section 2.6]). (For simplicity we only mention the inhomogeneous case; a similar statement holds for the homogeneous case). For a function $f : \mathbb{R}^d \to \mathbb{C}$ and $h \in \mathbb{R}^d$, the differences $\Delta^n_h f$ are defined recursively by,

$$\Delta^1_h f(x) := f(x) - f(x-h),$$
$$\Delta^n_h f(x) := \Delta^1_h(\Delta^{n-1}_h f)(x).$$

For $1 \le p \le \infty$, the *n-th order L^p modulus of continuity* of f is defined as,

$$w_n(f,t)_p := \sup_{|h|\le t} \|\Delta_h^n f\|_{L^p}.$$

For $1 \le p, q \le +\infty$ and $\alpha > 0$ the Besov space $B^\alpha_{p,q}$ consists of all the locally-integrable functions such that the quantity,

$$\|f\|_{L^p} + \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{-\alpha q j} w_{\lceil\alpha\rceil+1}(f,2^j)_p^q\right)^{1/q}, & \text{if } q < +\infty, \\ \sup_{j\in\mathbb{Z}} 2^{-\alpha j} w_{\lceil\alpha\rceil+1}(f,2^j)_p, & \text{if } q = +\infty, \end{cases} \tag{1.37}$$

is finite. Moreover, the expression in Equation (1.37) is an equivalent norm on $B^\alpha_{p,q}$.

1.12.2 Triebel-Lizorkin spaces

Triebel-Lizorkin spaces are defined similarly to Besov spaces. We keep using the notation from Section 1.12.1. For $1 \le q \le +\infty$, $1 \le p < +\infty$, $\alpha \in \mathbb{R}$ and a distribution $f \in \mathcal{S}'$, we define the norm,

$$\|f\|_{\dot{F}^\alpha_{p,q}} := \begin{cases} \left\|\left(\sum_{j\in\mathbb{Z}} 2^{-\alpha q j} \left|f * \varphi_j\right|^q\right)^{1/q}\right\|_{L^p}, & \text{if } q < +\infty, \\ \left\|\sup_{j\in\mathbb{Z}} 2^{-\alpha j} \left|f * \varphi_j\right|\right\|_{L^p}, & \text{if } q = +\infty. \end{cases} \tag{1.38}$$

The definition for $p = +\infty$ is more complicated: the L^∞ norm should be replaced by a Carleson measure condition. We refrain from discussing the details. As noted before, pointwise evaluation of $f * \varphi_j$ makes sense since this is a smooth function. As in the case of Besov spaces, the *homogeneous Triebel-Lizorkin* space is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that $\|f\|_{\dot{F}^\alpha_{p,q}} < +\infty$.

The inhomogeneous version is defined using the norm,

$$\|f\|_{F^\alpha_{p,q}} := \|f * \phi\|_{L^p} + \begin{cases} \left\|\left(\sum_{j\ge 0} 2^{-\alpha q j} \left|f * \varphi_j\right|^q\right)^{1/q}\right\|_{L^p}, & \text{if } q < +\infty, \\ \left\|\sup_{j\ge 0} 2^{-\alpha j} \left|f * \varphi_j\right|\right\|_{L^p}, & \text{if } q = +\infty. \end{cases} \tag{1.39}$$

Many classical function spaces lay in the range of Triebel-Lizorkin spaces (see [57] and [60, Chapter 6]). For example, for $1 < p < \infty$, $F^0_{p,2} = \dot{F}^0_{p,2} = L^p(\mathbb{R}^d)$. More generally, for $1 < p < \infty$ and $\alpha \in \mathbb{R}$,

$$F^\alpha_{p,2} = L^p_\alpha(\mathbb{R}^d), \qquad \dot{F}^\alpha_{p,2} = \dot{L}^p_\alpha(\mathbb{R}^d),$$

where $L^p_\alpha(\mathbb{R}^d)$ and $\dot{L}^p_\alpha(\mathbb{R}^d)$ are the homogeneous and inhomogeneous Sobolev spaces given by the norms,

$$\|f\|_{\dot{L}^p_\alpha(\mathbb{R}^d)} = \left\|\mathcal{F}^{-1}(|\cdot|^s \mathcal{F}(f))\right\|_{L^p},$$
$$\|f\|_{\dot{L}^p_\alpha(\mathbb{R}^d)} = \left\|\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\mathcal{F}(f))\right\|_{L^p}.$$

More precisely, the homogeneous and inhomogeneous Sobolev spaces are defined as the class of all tempered distributions having finite norm. In the homogeneous case, the space should be considered modulo polynomials. When α is a nonnegative integer, the inhomogeneous Sobolev space consist of all the functions in L^p having weak derivatives of order up to α in L^p.

1.12.3 Identification as coorbit spaces

Besov and Triebel-Lizorkin spaces are coorbit spaces of the affine group acting by translations and dilations. Let $\mathcal{G} = \mathbb{R}^d \times (0, +\infty)$ be the affine group (cf. Section 1.5.1). For $(x, s) \in \mathcal{G}$ let $\pi(x, s)$ be the operator on $L^2(\mathbb{R}^d)$ given by,

$$\pi(x, s) f(y) := T_x D_s f(y) = s^{-d/2} f\left(\frac{y - x}{s}\right).$$

The associated Wavelet transform is,

$$W_h f(x, s) = s^{-d/2} \int_{\mathbb{R}^d} f(t) \overline{h\left(\frac{t - x}{s}\right)} dt,$$

for $f, h \in L^2(\mathbb{R}^d)$, whereas the inverse wavelet transform is given by,

$$W_h^* F(x) = \int_0^{+\infty} \int_{\mathbb{R}^d} F(y, s) \overline{h\left(\frac{x - y}{s}\right)} dx \frac{ds}{s^{\frac{3}{2}d+1}},$$

for $F \in L^2(\mathcal{G})$.[6]

In [68, Section 4.2] it is shown that a function $h \in L^2(\mathbb{R}^d)$ is an admissible window in the sense of Section 1.10 if it satisfies the classical "smooth molecule" conditions [55, 56, 57],

$$\left|D^\beta h(x)\right| \lesssim (1 + |x|)^{-M}, \qquad \text{for all } |\beta| \leq M,$$
$$\int_{\mathbb{R}^d} x^\beta h(x) dx = 0, \qquad \text{for all } |\beta| \leq N,$$

for sufficiently large $N, M > 0$. In particular, any Schwartz h with all moments vanishing is adequate.

Triebel's work [106] (see also [61, 108]) yields a characterization of Besov and Triebel-Lizorkin spaces in terms of the wavelet transform, giving:

$$\dot{B}^\alpha_{pq}(\mathbb{R}^d) = \mathrm{Co}\left(L^{p,q}_{\alpha+d/2-d/q}(\mathcal{G})\right), \qquad \text{for all } 1 \leq p, q \leq +\infty,\ \alpha \in \mathbb{R}, \tag{1.40}$$
$$\dot{F}^\alpha_{pq}(\mathbb{R}^d) = \mathrm{Co}\left(L^{\alpha+d/2}_{p,q}(\mathcal{G})\right), \qquad \text{for all } 1 \leq q \leq +\infty,\ 1 \leq p < +\infty,\ \alpha \in \mathbb{R}, \tag{1.41}$$

[6]The integral converges in the weak-sense. The possibility of evaluating it pointwise requires further hypothesis.

where the space $L_\alpha^{p,q}$ and $L_{p,q}^\alpha$ are defined in Equations (1.13) and (1.14). There is however a serious caveat in the case of Tribel-Lizorkin spaces. The spaces $L_{p,q}^\alpha$ do not statisfy the general assumptions of Section 1.10 since they are not right-invariant. The spaces $\dot{F}_{pq}^\alpha$ can however be regarded as coorbit spaces of the so-called Tent spaces (cf. [20]) which are left and right translation invariant. Recently, it has been shown that Tribel-Lizorkin spaces can also be regarded as coorbits of the so-called Peetre spaces (see [108]).

A characterization of the corresponding inhomogeneous spaces as coorbit spaces of π is also possible, although it requires a more geneneral setting (see [109]), where the domain of the wavelet transform is not the group $\mathcal{G}$ anymore.

1.12.4 Wavelets

Given a function $h \in L^2(\mathbb{R}^d)$ and a relatively separated set of the affine group $\Lambda \subseteq \mathcal{G}$, the set

$$\{\, T_x D_s h \,\big|\, (x,s) \in \Lambda \,\} = \{\, s^{-d/2} h\left(\frac{\cdot - x}{s}\right) \,\big|\, (x,s) \in \Lambda \,\}$$

is called the *wavelet system* generated by the window h and the set of phases Λ. Theorem 1.10.1 implies that given a function h with adequate decay and smoothness and sufficiently many vanishing moments, the exist $\beta > 1$ and $\alpha > 0$ such that the wavelet system

$$\{\beta^{-jd/2} h\left(\beta^{-jd} \cdot -\alpha k\right) \,\big|\, j \in \mathbb{Z}, k \in \mathbb{Z}^d \,\},$$

provides a simultaneous atomic decomposition for the spaces $\dot{B}_{pq}^\alpha, (1 \le p, q \le +\infty, \alpha \in \mathbb{R})$ and $\dot{F}_{pq}^\alpha, (1 \le q \le +\infty, 1 \le p < +\infty, \alpha \in \mathbb{R})$. The corresponding sequence spaces are given by Equations (1.16) and (1.17). In fact, Theorem 1.10.1 allows for much more irregular sets of phases and gives a more robust statement (where, for example, any sufficiently small choice for β and α is granted to be adequate).

In contrast to the case of time-frequency decompositions, non-redundant well-behaved time-scale decompositions do exist. In [80], Lemarié and Meyer constructed a wavelet system,

$$\{\, 2^{-j/2} h\left(2^{-j} \cdot -k\right) \,\big|\, j,k \in \mathbb{Z}\,\},$$

that is an orthonormal basis for $L^2(\mathbb{R})$. The method was refined and generalized by Mallat [87, 88], introducing the method known as multi-resolution analysis. In [23], Daubechies obtained a compactly supported orthonormal wavelet basis with integer translates and dyadic scales. All these systems extend to atomic decompositions of the Besov and Triebel-Lizorkin spaces by a density argument.[7]

[7] The historical references are taken from [57].

1.13 Coorbit spaces of localized frames

Let $\mathbb{H}$ be a separable Hilbert space and let $(\{f_k\}_{k\in I}, \{g_k\}_{k\in I})$ a frame pair for it (cf. Section 1.8.3). This means that $\{f_k\}_k$,$\{g_k\}_k$ are Bessel sequences and that every $f \in \boldsymbol{H}$ admits the expansion

$$f = \sum_k \langle f, g_k \rangle f_k. \tag{1.42}$$

There are two abstract ways to extend this expansion to other Banach spaces. The *orbit method* consist of "pushing forward" a certain sequence space through the synthesis map $c \mapsto \sum_k c_k f_k$, while the *coorbit method* consists of "pulling back" a sequence space by the analysis map $f \mapsto (\langle f, g_k \rangle)_k$. The second method is similar to the one used in Section 1.10 to define coorbit spaces. In that context, the orbit method is also possible and the equivalence of both methods was shown in [44, Corollary 4.5].

In abstract setting of Hilbert space frames the same constructions were considered in [63, 53] (see also [5]). The frame pair $(\{f_k\}_{k\in I}, \{g_k\}_{k\in I})$ is called *localized* with respect to a subalgebra $\boldsymbol{A}$ of $B(\ell^2(I))$ if the Gramian matrices of $\{f_k\}_k$ and $\{g_k\}_k$ belong to $\boldsymbol{A}$ (cf. Section 1.8.5). In this case, the *cross-gramian matrix* $\mathcal{K}$ given by,

$$\mathcal{K}_{k,j} := \langle f_k, g_k \rangle, \qquad (k, j \in I),$$

also belongs to $\boldsymbol{A}$. Localized frame pairs with respect to adequate matrix algebras (like for example the ones considered in Section 1.7) are appropriate for the construction of orbit and coorbit spaces.

We illustrate this construction for some concrete decay conditions. Let $\boldsymbol{A}_w$ be the weighted Schur from Section 1.7.2. Here, the index set $I = \Lambda \subseteq \mathbb{R}^d$ is a relatively separated set and the weight w is subexponential (cf. Equation (1.9)) and satisfies $w(x) \gtrsim (1 + |x|)^\delta$, for some $\delta > 0$. Note that $\boldsymbol{A}_w$ is an algebra of matrices with entries in Λ that acts boundedly on $\ell^1_w(\Lambda)$ and $\ell^\infty_{1/w}(\Lambda)$.

Suppose that $(\{f_k\}_{k\in\Lambda}, \{g_k\}_{k\in\Lambda})$ is a frame pair for a Hilbert space $\mathbb{H}$ that is $\boldsymbol{A}_w$-localized. Let v be a w-moderated weight on $\mathbb{R}^d$ (cf. Equation (1.8)). Let $\mathbb{H}^{00}$ be the linear space (algebraically) generated by the atoms $\{f_k\}_k$ within $\mathbb{H}$. For $f \in \mathbb{H}^{00}$ and $1 \leq p < +\infty$ we define,

$$\|f\|_{\mathbb{H}^p_v} := \|(\langle f, g_k \rangle)_k\|_{\ell^p_v},$$

and let the space $\mathbb{H}^p_v$ be the completion of $\mathbb{H}^{00}$ with respect to that norm. Since $\ell^p_w \hookrightarrow \ell^\infty_{1/w}$, the space $\mathbb{H}^p_v$ can be described as,

$$\mathbb{H}^p_v = \{\, f \in \mathbb{H}^\infty_{1/w} \,\big|\, (\langle f, g_k \rangle)_k \in \ell^p_v \,\}.$$

The pair $(\{f_k\}_{k\in I}, \{g_k\}_{k\in I})$ yields a Banach frame for $\mathbb{H}^p_v$ with corresponding sequence space ℓ^p_v (cf. Section 1.9). In particular, each $f \in \mathbb{H}^p_v$ admits the expansion,

$$f = \sum_k \langle f, g_k \rangle f_k, \tag{1.43}$$

with unconditional convergence in the $\mathbb{H}^p_v$-norm. Moreover, the norm of $\mathbb{H}^p_v$ is equivalent to the quantity,

$$\inf \{ \|c\|_{\ell^p_v} \,\big|\, f = \sum_k c_k f_k \}.$$

This shows that the space $\mathbb{H}^p_v$, originally constructed as a coorbit space of the map $f \mapsto (\langle f, g_k \rangle)_k$, is also an orbit space of the map $c \mapsto \sum_k c_k f_k$.

When $p = \infty$ the construction above yields a space that will denoted by $\mathbb{H}^\infty_v$. All the statements above apply to this space, but the sequence space ℓ^p_v should be replaced by c^0_v, the closure of the set of finitely supported sequences within ℓ^∞_v. It is also possible to consider a space associated with the full sequence space ℓ^∞_v; in this case the expansion in Equation (1.43) converges only in the $(\mathbb{H}^\infty_{1/w}, \mathbb{H}^1_w)$ topology.

Two localized frame pairs $(\{f_k\}_{k\in I}, \{g_k\}_{k\in I})$, $(\left\{f'_k\right\}_{k\in I}, \left\{g'_k\right\}_{k\in I})$ yield the same spaces with equivalent norms if they are localized with respect to each other in the sense that the cross-gramian matrices $\mathcal{L}$, $\mathcal{M}$ given by,

$$\mathcal{L}_{k,j} := \langle f_k, g'_k \rangle, \mathcal{M}_{k,j} := \langle f'_k, g_k \rangle, \qquad (k, j \in I),$$

belong to $\boldsymbol{A}$.

A frame $\{f_k\}_k$ for a Hilbert space $\mathbb{H}$ is called *intrinsically localized* with respect to $\boldsymbol{A}$ if its Gramian matrix belongs to $\boldsymbol{A}$ (see [53]). If the algebra $\boldsymbol{A}$ is one of the algebras from Section 1.7, it follows that the Gramian matrix of the canonical dual frame $\{g_k\}_k$ also belongs to $\boldsymbol{A}$ and consequently $(\{f_k\}_{k\in I}, \{g_k\}_{k\in I})$ is a localized frame pair. Indeed, the Gramian matrix of $\{g_k\}_k$ is the pseudo-inverse of the Gramian matrix of $\{f_k\}_k$ so Remark 1.7.1 applies.

For Gabor frames generated by windows that are well-localized in space and frequency the corresponding coorbit spaces are modulation spaces. Theorem 1.11.6, for example, can be seen as an instance of the abstract theory presented in this section. This setting does not apply however to the case of time-scale decompositions because the assumptions on the geometry of the index sets in Section 1.7 are not well suited to that scenario.

1.14 Shift-invariant spaces

Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. A closed subspace $\boldsymbol{S} \subseteq L^2(\mathbb{R}^d)$ is called Λ-*shift-invariant* if whenever $f \in \boldsymbol{S}$ and $\lambda \in \Lambda$, then $T_\lambda f = f(\cdot - \lambda) \in \boldsymbol{S}$. The structure of shift-invariant spaces has been extensively studied in [96, 28, 14] through the so-called *fiberization theory*.

Let $\Lambda^\perp$ be the *orthogonal lattice* of Λ given by,

$$\Lambda^\perp := \{ \lambda^\perp \in \mathbb{R}^d \,\big|\, \langle \lambda, \lambda^\perp \rangle \in \mathbb{Z}, \text{ for all } \lambda \in \Lambda \}. \tag{1.44}$$

For a function $f \in L^2(\mathbb{R}^d)$, the *fiber* of f at a point $w \in \mathbb{R}^d$ is the sequence $f^w \in \ell^2(\Lambda^\perp)$ given by,

$$f^w := \left(\hat{f}(w + \lambda^\perp)\right)_{\lambda^\perp \in \Lambda^\perp}. \tag{1.45}$$

Of course the family of sequences $\{ f^w \,|\, w \in \mathbb{R}^d \}$ is only well-defined up to null measure sets. For any countable family $\mathcal{X} \subseteq L^2(\mathbb{R}^d)$ we further define,

$$\mathcal{X}_w := \{ f^w \,|\, f \in \mathcal{X} \}.$$

The fibers of shift-invariant spaces are defined as follows. Given a shift-invariant space $\boldsymbol{S}$, it is always possible to obtain a countable family $\mathcal{X} \subseteq \boldsymbol{S}$ that *generates* $\boldsymbol{S}$ in the sense that $\boldsymbol{S}$ is the closed linear span of the set,

$$E(\mathcal{X}, \Lambda) := \{ T_\lambda f \,|\, f \in \mathcal{X}, \lambda \in \Lambda \}.$$

For example we may take $\mathcal{X}$ to be a countable dense subset of $\boldsymbol{S}$. For $w \in \mathbb{R}^d$, we let $\boldsymbol{S}_w$ be the subspace of $\ell^2(\Lambda^\perp)$ generated by the set $\mathcal{X}_w$. The family $\{ S_w \,|\, w \in \mathbb{R}^d \}$ is called the set of *fibers* of $\boldsymbol{S}$. A different choice for $\mathcal{X}$ produces the same set of fibers up to null measure. As the following theorem shows, they completely characterize the space $\boldsymbol{S}$.

Theorem 1.14.1. *Let $\boldsymbol{S} \subseteq L^2(\mathbb{R}^d)$ be a shift-invariant space and let $\{ S_w \,|\, w \in \mathbb{R}^d \}$ be its set of fibers. Then the following holds.*

(a) A function $f \in L^2(\mathbb{R}^d)$ belongs to $\boldsymbol{S}$ if and only if for almost every $w \in \mathbb{R}^d$, $f^w \in S_w$.

(b) $E(\mathcal{X}, \Lambda)$ is complete in $\boldsymbol{S}$ (i.e., it generates a dense subspace) if and only if for almost every $w \in \mathbb{R}^d$ the set $\mathcal{X}_w$ is complete in $\boldsymbol{S}_w$.

(c) $E(\mathcal{X}, \Lambda)$ is a Riesz basis (resp. frame) of $\boldsymbol{S}$ with bounds A, B if and only if for almost every $w \in \mathbb{R}^d$, the set $\mathcal{X}_w$ is Riesz basis of $\boldsymbol{S}_w$ (resp. frame) with bounds A, B.

The conditions in Theorem 1.14.1 can be further reformulated by computing the Gramian matrix of the fibers. For $f, g \in L^2(\mathbb{R}^d)$, the bracket product is defined by,

$$[f, g]_\Lambda(w) := \sum_{\lambda^\perp \in \Lambda^\perp} \hat{f}(w + \lambda^\perp)\overline{\hat{g}(w + \lambda^\perp)}, \qquad (w \in \mathbb{R}^d).$$

The Gramian matrix of the system $\mathcal{X}_w$ is,

$$\hat{G}_w = \left([f_x, f_y]_\Lambda(w)\right)_{x,y \in \mathcal{X}}.$$

Theorem 1.8.6 then yields the following.

Theorem 1.14.2. *Let $\mathcal{X} \subseteq L^2(\mathbb{R}^d)$ and consider the family,*

$$E(\mathcal{X}, \Lambda) = \{\, T_\lambda f \,\big|\, f \in \mathcal{X}, \lambda \in \Lambda \,\},$$

and the family of matrices,

$$\hat{G}_w = \Big([f_x, f_y]_\Lambda(w) \Big)_{x,y\in\mathcal{X}}, \qquad (w \in \mathbb{R}^d).$$

Let $0 < A \leq B < +\infty$. Then the following holds.

- *$E(\mathcal{X}, \Lambda)$ is a Riesz sequence with bounds A, B if and only if for almost every $w \in \mathbb{R}^d$, $\inf(\mathrm{spec}(\hat{G}_w)) = A$ and $\sup(\mathrm{spec}(\hat{G}_w)) = B$.*
- *$E(\mathcal{X}, \Lambda)$ is a frame sequence with bounds A, B if and only if for almost every $w \in \mathbb{R}^d$, $\inf(\mathrm{spec}(\hat{G}_w) \setminus \{0\}) = A$ and $\sup(\mathrm{spec}(\hat{G}_w)) = B$.*

Theorems 1.14.2 and 1.14.1 answer the fundamental questions concerning shift-invariant spaces. Using these tools a complete characterization of the structure of shift invariant spaces in terms of fibers is possible (see [14]).

Finally, we illustrate the theory of Section 1.13 in the case of shift-invariant spaces. For a complete proof in a more general context see Section 3.3. Suppose that $\mathcal{X}$ is a finite set $\mathcal{X} = \{f_1, \dots, f_n\}$ and that $E(\mathcal{X}, \Lambda)$ is a frame sequence. Let us denote by $\boldsymbol{S}$ the closed linear span of $E(\mathcal{X}, \Lambda)$. Assume further that each f_i belongs to $W(L^\infty, L^1)$ (cf. Section 1.6). Then the frame $E(\mathcal{X}, \Lambda)$ is easily seen to be self-localized with respect to the unweighted Baskakov-Sjöstrand algebra $\mathcal{C}$ from Section 1.7.3. The coorbit space corresponding to the sequence space ℓ^p is just the L^p-closure $E(\mathcal{X}, \Lambda)$ within L^p (as before, for $p = \infty$ the right sequence space is c^o, not ℓ^∞). The canonical dual frame of $E(\mathcal{X}, \Lambda)$ has the form $E(\mathcal{Y}, \Lambda)$ where $\mathcal{Y} = \{g_1, \dots, g_n\}$. The self-localization of the dual frame and Equation 1.33 imply that each g_i also belongs to $W(L^\infty, L^1)$. Consequently we have the following.

Theorem 1.14.3. *Let $\{f_1, \dots, f_n\} \subseteq W(L^\infty, L^1)$. Assume that $E = \{\, T_\lambda f_1, \dots, T_\lambda f_n \,\big|\, \lambda \in \Lambda \,\}$ forms a frame of $\boldsymbol{S}$, its closed linear space within $L^2(\mathbb{R}^d)$. Then the canonical dual frame is given by $E' = \{\, T_\lambda g_1, \dots, T_\lambda g_n \,\big|\, \lambda \in \Lambda \,\}$, for some family $\{g_1, \dots, g_n\} \subseteq W(L^\infty, L^1)$.*

As a consequence, if for $1 \leq p < +\infty$ we let $\boldsymbol{S}^p$ be the closure of E within L^p, then both (E, E') and (E', E) are Banach frames for $\boldsymbol{S}^p$ with associated sequence space $\ell^p(\Lambda)$. For $p = \infty$ the same is true, replacing ℓ^∞ with c^0. In particular, for $1 \leq p \leq +\infty$, every $f \in \boldsymbol{S}^p$ admits the expansions,

$$f = \sum_{k=1}^{n} \sum_{\lambda\in\Lambda} \langle f, g_k(\cdot - \lambda)\rangle\, f_k(\cdot - \lambda) = \sum_{k=1}^{n} \sum_{\lambda\in\Lambda} \langle f, f_k(\cdot - \lambda)\rangle\, g_k(\cdot - \lambda),$$

with unconditional convergence in L^p-norm.

Chapter 2

Amalgam spaces

In this chapter we introduce two variants of the amalgam space norms with the aim of obtaining stronger statements in several applications. The first one concerns general locally-compact groups and is irrelevant in the case of IN groups. We introduce a *weak amalgam space* and prove that certain convolution relations for amalgam spaces can be improved by introducing this space. The second one, in contrast, concerns only the Euclidean space. It is designed to model the concept of smooth spatial molecule.

2.1 Weak and strong amalgam norms

Let $\mathcal{G}$ be a locally-compact group. We now introduce some variations of the amalgam spaces $W(L^\infty, L^1_w)$, $W_R(L^\infty, L^1_w)$. We do so in order to handle certain technicalities involving right convolution actions on the spaces $W(L^\infty, \boldsymbol{E})$. For an IN group, the spaces $W(L^\infty, \boldsymbol{E})$ are right L^1_w modules, but for a general group $\mathcal{G}$, they are only right $W(L^\infty, L^1_w)$ modules. We will now introduce a space between L^1_w and $W(L^\infty, L^1_w)$ that acts on the spaces $W(L^\infty, \boldsymbol{E})$ from the right and collapses to L^1_w in the case that $\mathcal{G}$ is an IN group. Similarly, we will introduce a certain subspace of $W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$ that reduces to $W(L^\infty, L^1_w)$ when $\mathcal{G}$ is an IN group.

For an admissible weight w, let the left and right *weak amalgam spaces* be defined by

$$W^{\text{weak}}(L^\infty, L^1_w) := \{\, f \in L^1_{\text{loc}} \,\big|\, \chi_V * |f| \in W(L^\infty, L^1_w) \,\},$$
$$W^{\text{weak}}_R(L^\infty, L^1_w) := \{\, f \in L^1_{\text{loc}} \,\big|\, |f| * \chi_V \in W_R(L^\infty, L^1_w) \,\},$$

and endow them with the norms,

$$\|f\|_{W^{\text{weak}}(L^\infty, L^1_w)} := \|\chi_V * |f|\|_{W(L^\infty, L^1_w)} = \|(\chi_V * |f|)^\#\|_{L^1_w},$$
$$\|f\|_{W^{\text{weak}}_R(L^\infty, L^1_w)} := \||f| * \chi_V\|_{W_R(L^\infty, L^1_w)} = \|(|f| * \chi_V)_\#\|_{L^1_w}.$$

These spaces are related by $\|f\|_{W^{\text{weak}}(L^\infty, L^1_w)} = \|f^\vee\|_{W^{\text{weak}}_R(L^\infty, L^1_w)}$.

Consider also the *strong amalgam space* defined as,

$$W^{\mathrm{st}}(L^\infty, L^1_w) := W_R(L^\infty, W(L^\infty, L^1_w)).$$

Hence, the norm of a function $f \in W^{\mathrm{st}}(L^\infty, L^1_w)$ is given by,

$$\|f\|_{W^{\mathrm{st}}(L^\infty, L^1_w)} = \|(f_\#)^\#\|_{L^1_w}.$$

We now observe how these new spaces are related to the classical ones.

Proposition 2.1.1. *Let w be an admissible weight. Then the following holds.*

(a)

$$W(L^\infty, L^1_w) \hookrightarrow W^{\mathrm{weak}}(L^\infty, L^1_w) \hookrightarrow L^1_w,$$

and

$$W_R(L^\infty, L^1_w) \hookrightarrow W_R^{\mathrm{weak}}(L^\infty, L^1_w) \hookrightarrow L^1_w.$$

(b) If $\mathcal{G}$ is an IN group then,

$$W_R^{\mathrm{weak}}(L^\infty, L^1_w) = W^{\mathrm{weak}}(L^\infty, L^1_w) = L^1_w.$$

(c) $W^{\mathrm{st}}(L^\infty, L^1_w) \hookrightarrow W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$.

(d) If $\mathcal{G}$ is an IN group then,

$$W(L^\infty, L^1_w) = W_R(L^\infty, L^1_w) = W^{\mathrm{st}}(L^\infty, L^1_w).$$

Proof. For (a) and (b) we only prove the statements concerning the "right" spaces; the corresponding statements for "left" spaces follow by using the involution $^\vee$.

Let $f \in W_R(L^\infty, L^1_w)$. Since $(|f| * \chi_V)_\# \leq (f_\# * \chi_V)$, we have that,

$$\begin{aligned}\|f\|_{W_R^{\mathrm{weak}}(L^\infty, L^1_w)} &= \|(|f| * \chi_V)_\#\|_{L^1_w} \leq \|f_\# * \chi_V\|_{L^1_w} \\ &\leq \|f_\#\|_{L^1_w} \|\chi_V\|_{L^1_w} \lesssim \|f\|_{W_R(L^\infty, L^1_w)}.\end{aligned}$$

This proves the first embedding of (a). For the second one, let $f \in W_R^{\mathrm{weak}}(L^\infty, L^1_w)$ and estimate,

$$\begin{aligned}\int_{\mathcal{G}} |f(x)|\, w(x) dx &\lesssim \int_{\mathcal{G}} |f(x)|\, w(x) \int_{\mathcal{G}} \chi_V(x^{-1}y) dy dx \\ &\leq \int_{\mathcal{G}} \int_{\mathcal{G}} |f(x)|\, w(y^{-1}x) \chi_V(x^{-1}y) dx w(y) dy.\end{aligned}$$

Since w is locally bounded, in the last integral $w(y^{-1}x) \lesssim 1$ and we conclude that $\|f\|_{L^1_w} \lesssim \||f| * \chi_V\|_{L^1_w}$. Now the conclusion follows from the fact that $|f| * \chi_V \leq (|f| * \chi_V)_\#$.

Part (b) follows from the convolution relation,

$$W(L^\infty, L^1_w) * L^1_w \hookrightarrow W(L^\infty, L^1_w),$$

which holds when $\mathcal{G}$ is an IN group. This follows easily from the fact that, for an IN group, $f^\# = f_\#$. It also follows from Theorem 1.6.2. Indeed, Theorem 1.6.2 imples that $W(L^\infty, L^1_w) * W(L^1, L^1_w) \hookrightarrow W(L^\infty, L^1_w)$. It is straightforward to see that $W(L^1, L^1_w) = L^1_w$.

Part (c) follows from the observation that $f_\# \leq (f_\#)^\#$ and $f^\# \leq (f^\#)_\#$. Finally if $\mathcal{G}$ is an IN group, for $x \in \mathcal{G}$, $VxV = VVx$, and therefore,

$$(f_\#)^\#(x) = \sup_{v \in V} f_\#(xv) = \sup_{v \in V} \sup_{w \in V} |f(wxv)| = \sup_{y \in VV} |f(yx)|\,.$$

Hence the conclusion follows from the fact that a different choice for the neighborhood V induces an equivalent norm in $W(L^\infty, L^1_w)$. □

For the weak norm, we now derive the following convolution relation (cf. Lemma 1.6.1).

Proposition 2.1.2. *Let $\boldsymbol{E}$ be a solid, translation invariant BF space and let w be an admissible weight for it. Then,*

$$W(L^\infty, \boldsymbol{E}) * W^{\text{weak}}(L^\infty, L^1_w) \hookrightarrow W(C_0, \boldsymbol{E}),$$

together with the corresponding norm estimate.

Proof. Let $f \in W(L^\infty, \boldsymbol{E})$ and $g \in W^{\text{weak}}(L^\infty, L^1_w)$. For almost every $y \in \mathcal{G}$ and $t \in V$, $|f(y)| \leq f^\#(yt)$. Hence for $x \in \mathcal{G}$,

$$\begin{aligned} |f| * |g|\,(x) &\leq \int_{\mathcal{G}} \int_{\mathcal{G}} f^\#(yt)\chi_V(t^{-1})dt \left|g(y^{-1}x)\right| dy \\ &= \int_{\mathcal{G}} f^\#(t) \int_{\mathcal{G}} \chi_V(t^{-1}y) \left|g(y^{-1}x)\right| dydt \\ &= \int_{\mathcal{G}} f^\#(t)(\chi_V * |g|)(t^{-1}x)dt = f^\# * (\chi_V * |g|)(x). \end{aligned}$$

Therefore Lemma 1.6.1 implies that,

$$\begin{aligned} \|f * g\|_{W(L^\infty,\boldsymbol{E})} &\leq \|f^\# * (\chi_V * |g|)\|_{W(L^\infty,\boldsymbol{E})} \\ &\leq \|f^\#\|_{\boldsymbol{E}} \|\chi_V * |g|\|_{W(L^\infty,L^1_w)} = \|f\|_{W(L^\infty,\boldsymbol{E})}\|g\|_{W^{\text{weak}}(L^\infty,L^1_w)}. \end{aligned}$$

It only remains to note that $f * g$ is a continuous function. This follows from the embedding $W^{\text{weak}}(L^\infty, L^1_w) \hookrightarrow L^1_w$ in Proposition 2.1.1 and Lemma 1.6.3. □

Using Proposition 2.1.2, we can derive a variant of Lemma 1.6.2 (b) that only requires g to be in $W^{\text{weak}}_R(L^\infty, L^1_w)$.

Lemma 2.1.1. *Let $\boldsymbol{E}$ be a solid, translation invariant BF space and let w be an admissible weight for it. Let $\Lambda \subseteq \mathcal{G}$ be a relatively separated set. Then, for $f \in W(L^\infty, E)$ and $g \in W_R^{\mathrm{weak}}(L^\infty, L^1_w)$, the sequence $(\langle f, L_\lambda g\rangle)_{\lambda\in\Lambda}$ belongs to $\boldsymbol{E}^d(\Lambda)$ and satisfies*

$$\|(\langle f, L_\lambda g\rangle)_\lambda\|_{E^d} \lesssim \|f\|_{W(L^\infty,E)}\|g\|_{W_R^{\mathrm{weak}}(L^\infty,L^1_w)},$$

where the implicit constant depends on the set Λ.

Proof. As in the proof on Lemma 1.6.2 (b), $\|(\langle f, L_\lambda g\rangle)_\lambda\|_{E^d} \lesssim \|f * g^\vee\|_{W(C_0,\boldsymbol{E})}$. Now we can invoke Proposition 2.1.2 and the fact that the involution $^\vee$ maps $W_R^{\mathrm{weak}}(L^\infty, L^1_w)$ into $W^{\mathrm{weak}}(L^\infty, L^1_w)$ to obtain the desired conclusion. □

2.2 Schur-type amalgam norms

When moving from shift-invariant spaces (cf. Section 1.14) to the setting of spaces generated by general atoms, the standard tools for amalgam spaces are not directly applicable and require an extension. In the study of shift-invariant spaces (or more generally, spaces generated by translates) the relevant operators can be expressed as products and convolutions with possibly distributional kernels. Wiener amalgam spaces, have proved to be a powerful tool to quantify this formalism. The abstract convolution multiplier theorems allow to deal with smoothness and approximation problems in the context of atoms generated by irregular shifts. In the context of general spline-type spaces, the relevant operations are not convolutions but, nonetheless, they are convolution-like. For example, in the intended applications to time-frequency analysis we will consider the image of a modulation space (cf. Section 1.11) through the short-time Fourier transform (with a fixed window). In this case, the relevant operations are not convolutions on the Euclidean space but twisted convolutions (which nevertheless are closely related to convolutions on the Heisenberg group).

Convolution dominated operators and enveloping conditions for irregular atoms are widely used concepts. Here, we will consider an enveloping condition for atoms, not in a pointwise sense, but in the sense of a local - possibly non solid - quantity. We will extend the amalgam norm of a function f to families of functions F in such a way that the condition $\|F\|_{W(B,E)} < \infty$ grants to F the same properties shared by a set of translates of f, when $\|f\|_{W(B,E)} < \infty$. When the local norm measures size, the condition $\|F\|_{W(B,E)} < \infty$ will amount to certain spatial localization for the family F; when the local norm measures smoothness, it will amount to certain equismoothness property for the family F. Using this extension of the amalgam norm and a simple interpolation argument, we obtain replacements for some of the convolution inequalities in amalgam spaces. These will be used, for example, in Section 3.3 to extend to the general setting the principle that in a finitely-generated shift-invariant space the smoothness of the generating windows is inherited by the whole space.

We will consider a relatively separated set of points $\Lambda \subseteq \mathbb{R}^d$, which will be called *nodes* and a symmetric, submultiplicative, continuous weight $w : \mathbb{R}^d \to (0, +\infty)$. We will also consider a family of measurable functions $f_k : \mathbb{R}^d \to \mathbb{C}$ indexed by the set of nodes Λ.

Let $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ be a uniformly localizable, isometrically translation invariant Banach space on $\mathbb{R}^d$ (cf. Section 1.6.1). For a family $F \equiv \{f_k\}_{k\in\Lambda} \subseteq \boldsymbol{B}_{loc}$ we define its $W(\boldsymbol{B}, L^1_w)$ norm by

$$\|F\|_{W(\boldsymbol{B},L^1_w)} := \max\left\{\sup_k \|g_k\|_1, \operatorname{supess}_x \sum_k |g_k(x)|\right\},$$
$$\text{where } g_k(x) := \|f_k \eta(\cdot - x)\|_{\boldsymbol{B}} w(x-k), \quad (x \in \mathbb{R}^d, k \in \Lambda).$$

Here, $\eta \in \mathcal{D}(\mathbb{R}^d)$ is any nonzero window function (see Proposition 2.2.1 below).

Observe that if $\|F\|_{W(\boldsymbol{B},L^1_w)} < +\infty$, then each f_k belongs to $W(\boldsymbol{B}, L^1_w)$. The estimate $\|F\|_{W(\boldsymbol{B},L^1_w)} < +\infty$ implies, in addition, certain uniformity for the set $\{f_k\}_k$, similar to that shared by the translates of an individual atom. Some results to come will give evidence of that. The following proposition shows that, at least, the hypothesis $\|F\|_{W(\boldsymbol{B},L^1_w)} < +\infty$ indeed extends to more general families F, the condition $\|f\|_{W(\boldsymbol{B},L^1_w)} < \infty$ normally imposed on families produced by translation of a single generator f. Before showing that, we must prove the independence of the window function in the definition above.

Proposition 2.2.1. *Let a family $F \equiv \{f_k\}_{k\in\Lambda} \subseteq \boldsymbol{B}_{loc}$ be given.*

(a) Let $\|F\|_{W_i(\boldsymbol{B},L^1_w)}$ be the norm defined using a nonzero window function $\eta_i \in \mathcal{D}(\mathbb{R}^d)$, (i=1,2). Then $\|F\|_{W_1(\boldsymbol{B},L^1_w)} \approx \|F\|_{W_2(\boldsymbol{B},L^1_w)}$.

(b) For any bounded set $Q \subset \mathbb{R}^d$ with non-empty interior, the norm $\|F\|_{W(\boldsymbol{B},L^1_w)}$ is also equivalent to the norm $\|F\|_{\widetilde{W}(\boldsymbol{B},L^1_w)}$ defined by

$$\|F\|_{\widetilde{W}(\boldsymbol{B},L^1_w)} := \max\left\{\sup_k \|g_k\|_1, \operatorname{supess}_x \sum_k |g_k(x)|\right\},$$
$$\textit{where } g_k(x) := \|f_k\|_{\boldsymbol{B}(Q+x)} w(x-k), \quad x \in \mathbb{R}^d, k \in \Lambda,$$
$$\textit{and } \|f\|_{\boldsymbol{B}(Q)} := \inf\{\|g\|_{\boldsymbol{B}} : g \equiv f \textit{ on } Q\}.$$

(c) If the family F is given by $f_k = f(\cdot - k), k \in \Lambda$ and Λ is relatively separated, then $\|F\|_{W(\boldsymbol{B},L^1_w)} \approx \|f\|_{W(\boldsymbol{B},L^1_w)}$.

Remark 2.2.1. *The implicit constant on (c) depends on the relative separation of Λ.*

Proof. For (a), since η_2 is compactly supported and not identically 0, it is possible to choose $\alpha > 0$ such that $\sum_{j\in\mathbb{Z}^d} |\eta_2|^2 (\cdot - \alpha j) \approx 1$. This series is locally finite, so the function $m := \eta_1 \left(\sum_{j\in\mathbb{Z}^d} |\eta_2|^2 (\cdot - \alpha j)\right)^{-1}$ is smooth. Choose $\theta \in \mathcal{D}(\mathbb{R}^d)$ such that $\theta \equiv 1$ on the support of η_1. Now,

$$\eta_1 = \theta\eta_1 = \sum_{j\in\mathbb{Z}^d} \theta m |\eta_2|^2 (\cdot - \alpha j).$$

Since both θ and η_2 are compactly supported, only finitely many terms are not zero and we may write

$$\eta_1 = \sum_{j=1}^{n} m_j \eta_2(\cdot - x_j),$$

where $x_j \in \alpha\mathbb{Z}^d$ and $m_j := \theta m \overline{\eta_2(\cdot - x_j)} \in \mathcal{D}(\mathbb{R}^d)$.

Now, for $x \in \mathbb{R}^d$, and $k \in \Lambda$,

$$\begin{aligned}\|f_k\eta_1(\cdot - x)\|_{\boldsymbol{B}} w(x-k) &\lesssim \sum_{j=1}^{n} \|f_k\eta_2(\cdot - x - x_j)\|_{\boldsymbol{B}} w(x-k) \\ &\lesssim \sum_{j=1}^{n} \|f_k\eta_2(\cdot - (x + x_j))\|_{\boldsymbol{B}} w((x+x_j)-k) w(x_j).\end{aligned}$$

Consequently,

$$\|F\|_{W_1(\boldsymbol{B},L^1_w)} \lesssim \|F\|_{W_2(\boldsymbol{B},L^1_w)}.$$

The other inequality follows by symmetry.

To prove (b), consider first a window $\eta \in \mathcal{D}(\mathbb{R}^d)$ such that $\eta \equiv 1$ on Q. Then for any $k \in \Lambda$ and $x \in \mathbb{R}^d$, $\|f_k\|_{\boldsymbol{B}(Q+x)} \leq \|f_k\eta(\cdot - x)\|_{\boldsymbol{B}}$ and it follows that $\|F\|_{\widetilde{W}(\boldsymbol{B},L^1_w)} \lesssim \|F\|_{W(\boldsymbol{B},L^1_w)}$.

For the other inequality, since Q has non-empty interior, there exists a non-zero window function $\eta \in \mathcal{D}(\mathbb{R}^d)$ supported on Q. For any $k \in \Lambda$, $x \in \mathbb{R}^d$ and any $h \in \boldsymbol{B}$ such that $h \equiv f_k$ on $Q + x$, we have

$$\|f_k\eta(\cdot - x)\|_{\boldsymbol{B}} = \|h\eta(\cdot - x)\|_{\boldsymbol{B}} \lesssim \|h\|_{\boldsymbol{B}}.$$

Therefore, $\|f_k\eta(\cdot - x)\|_{\boldsymbol{B}} \lesssim \|f_k\|_{\boldsymbol{B}(Q+x)}$, and the desired inequality follows.

Let us now prove (c). For $x \in \mathbb{R}^d$ and $k \in \Lambda$, since $\boldsymbol{B}$ is isometrically translation invariant,

$$g_k(x) = \|f(\cdot - k)\eta(\cdot - x)\|_{\boldsymbol{B}} w(x-k) = \|f\eta(\cdot - (x-k))\|_{\boldsymbol{B}} w(x-k).$$

Integrating over x we get that for any $k \in \Lambda$,

$$\|f\|_{W(\boldsymbol{B},L^1_w)} = \|g_k\|_1. \tag{2.1}$$

This shows that $\|f\|_{W(\boldsymbol{B},L^1_w)} \leq \|F\|_{W(\boldsymbol{B},L^1_w)}$.

Since by (2.1) we know that $\sup_k \|g_k\|_1 \leq \|f\|_{W(\boldsymbol{B},L^1_w)}$, it suffices to show that $\operatorname{supess}_x \sum_k g_k(x) \lesssim \|f\|_{W(\boldsymbol{B},L^1_w)}$.

To this end, let us call Q the unitary cube centered at 0 and let $\theta \in \mathcal{D}(\mathbb{R}^d)$ be such that

$\theta \equiv 1$ on $\mathrm{supp}(\eta) + Q$. For $x \in \mathbb{R}^d$, and $k \in \Lambda$,

$$\begin{aligned} g_k(x) = \|f\eta(\cdot - (x-k))\|_{\boldsymbol{B}} w(x-k) &= \int_Q \|f\eta(\cdot - (x-k))\|_{\boldsymbol{B}} w(x-k) dy \\ &= \int_Q \|f\eta(\cdot - (x-k))\theta(\cdot - (x+y-k))\|_{\boldsymbol{B}} w(x-k) dy \\ &\lesssim \int_Q \|f\theta(\cdot - (x+y-k))\|_{\boldsymbol{B}} w(x-k) dy \\ &= \int_{Q+x-k} \|f\theta(\cdot - y)\|_{\boldsymbol{B}} w(x-k) dy. \end{aligned}$$

Since w is bounded on Q, for $y \in Q + x - k$, $w(x-k) \leq w(y) \sup_Q w$. Therefore, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} \sum_k g_k(x) &\lesssim \sum_k \int_{Q+x-k} \|f\theta(\cdot - y)\|_{\boldsymbol{B}} w(y) dy \\ &= \int_{\mathbb{R}^d} \|f\theta(\cdot - y)\|_{\boldsymbol{B}} w(y) \sum_k \chi_{Q+x-k}(y) dy. \end{aligned}$$

Finally, observe that $\sum_k \chi_{Q+x-k}(y)$ is bounded by the relative separation of the set of nodes Λ. This completes the proof. □

Example 2.2.1. *As an easy example of amalgam norm of families, consider a relatively separated set of nodes $\Lambda \subseteq \mathbb{R}^d$, and a family of measurable functions $f_k : \mathbb{R}^d \to \mathbb{C}$, $k \in \Lambda$ satisfying the concentration condition,*

$$|f_k(x)| \leq C w_{-(s+\alpha)}(x-k), \quad x \in \mathbb{R}^d, k \in \Lambda, \tag{2.2}$$

for some $s > d$ and $\alpha \geq 0$.

Let $Q := [0,1]^d$ be the unit cube. From equation (2.2) we get that for any $x \in \mathbb{R}^d$,

$$\|f_k\|_{L^\infty(Q+x)} \leq C\|w_{-(s+\alpha)}\|_{L^\infty(Q+(x-k))} \lesssim C w_{-(s+\alpha)}(x-k),$$

where the implicit constant depends on $s+\alpha$. Therefore,

$$\|f_k\|_{L^\infty(Q+x)} w_\alpha(x-k) \lesssim C w_{-s}(x-k).$$

Hence by Proposition 2.2.1 and Lemma 1.3.2, $\|F\|_{W(L^\infty, L^1_{w_\alpha})} \lesssim C\rho(\Lambda)$.

The concentration condition in Equation (2.2) is however much more precise than the last statement.

2.3 Estimates for Schur-type norms

We now introduce a number of multiplier estimates that will replace in certain applications the convolution relations for amalgam spaces. These are easily established for some endpoint spaces and then generalized by interpolation. Throughout this section we will assume the following.

- A relatively separated set of nodes $\Lambda \subseteq \mathbb{R}^d$ is given.
- $\boldsymbol{B}$ is a uniformly localizable, isometrically translation invariant, Banach space (cf. Section 1.6.1).
- $w : \mathbb{R}^d \to (0, +\infty)$ is a symmetric, submultiplicative, continuous weight.
- $v : \mathbb{R}^d \to (0, +\infty)$ is a symmetric weight moderated by w.

We first show that the synthesis of well-localized atoms is bounded with respect to amalgam space norms.

Proposition 2.3.1. *Let a family $F \equiv \{f_k\}_{k\in\Lambda} \subseteq \boldsymbol{B}_{loc}$ such that $\|F\|_{W(\boldsymbol{B},L^1_w)} < +\infty$ be given. Let $c \in \ell^p_v$, for some $1 \le p < \infty$, or $c \in c^0$, for $p = +\infty$. Then, the series*

$$c \cdot F := \sum_k c_k f_k,$$

converges in $W(\boldsymbol{B}, L^p_v)$ and satisfies the following estimate,

$$\|c \cdot F\|_{W(\boldsymbol{B},L^p_v)} \lesssim \|c\|_{\ell^p_v} \|F\|_{W(\boldsymbol{B},L^1_w)}.$$

Remark 2.3.1. *The implicit constant is the constant in Equation (1.8).*

Remark 2.3.2. *If $c \in \ell^\infty_v$, then the same conclusion holds but the series is only weak* convergent.*

Remark 2.3.3. *In contrast to Lemma 1.6.2, the space $\boldsymbol{B}$ is not assumed to be solid.*

Proof. We will assume that the sequence c is finitely supported. The general case follows from this one by approximation and the completeness of $W(\boldsymbol{B}, L^1_v)$.

Let us set $f := c \cdot F = \sum_k c_k f_k$. For a window function $\eta \in \mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have $f\eta(\cdot - x) = \sum_k c_k f_k \eta(\cdot - x)$. Therefore

$$\|f\eta(\cdot - x)\|_{\boldsymbol{B}} v(x) \le C \sum_k |c_k|\, v(k) \|f_k \eta(\cdot - x)\|_{\boldsymbol{B}} w(x - k),$$

where the constant C is the constant in (1.8).

Now Schur's lemma (see below) yields the desired inequality. □

In the proof we used part (a) of the following interpolation lemma which we quote for completeness. For a proof see [60, Theorem 1.3.4].

Lemma 2.3.1. *Let $F \equiv \{f_k\}_k$ be a family of measurable functions on $\mathbb{R}^d$ and let $1 \leq p \leq \infty$.*

(a) Let $\{c_k\}_k \subseteq \mathbb{C}$ be a sequence. Then,

$$\|c \cdot F\|_{L^p} \leq \|c\|_{\ell^p} \left(\sup_k \|f_k\|_{L^1}\right)^{1/p} \left(\operatorname*{supess}_{x \in \mathbb{R}^d} \sum_k |f_k(x)|\right)^{1/p'},$$

where,

$$c \cdot F := \sum_k c_k f_k.$$

(b) Let $g : \mathbb{R}^d \to [0, \infty]$ be a measurable function. Then,

$$\|g \cdot F\|_{\ell^p} \leq \|g\|_{L^p} \left(\sup_k \|f_k\|_{L^1}\right)^{1/p'} \left(\operatorname*{supess}_{x \in \mathbb{R}^d} \sum_k |f_k(x)|\right)^{1/p},$$

where,

$$(g \cdot F)_k := \int_{\mathbb{R}^d} g(x) f_k(x) dx.$$

For both statements, if $p = 1$, we interpret $1/p' = 0$.

We now give estimates for transformations operating on families of well-localized atoms. For a matrix of complex numbers $C \equiv (c_{k,j})_{k,j \in \Lambda}$, we consider the following weighted Schur-type norm,

$$\|C\|_{\mathcal{S}_w} := \max \left\{ \sup_k \sum_j \left|c_{k,j}\right| w(k-j), \sup_j \sum_k \left|c_{k,j}\right| w(k-j) \right\}.$$

Furthermore, we denote by $\mathcal{S}_w$ the set of all such matrices having finite norm.

Let us show that these matrices act boundedly on well-concentrated families of atoms.

Proposition 2.3.2. *Let a family $F \equiv \{f_k\}_{k \in \Lambda} \subseteq \boldsymbol{B}_{loc}$ such that $\|F\|_{W(\boldsymbol{B}, L^1_w)} < +\infty$ and a matrix $C \in \mathcal{S}_w$ be given.*

Let $C \cdot F \equiv \{g_k\}_k$ be the family defined by,

$$g_k := \sum_j c_{k,j} f_j.$$

Then, each of the series defining g_k converges in $W(\boldsymbol{B}, L^1_w)$ and we have the following estimate,

$$\|C \cdot F\|_{W(\boldsymbol{B}, L^1_w)} \leq \|C\|_{\mathcal{S}_w} \|F\|_{W(\boldsymbol{B}, L^1_w)}.$$

Proof. Again, by an approximation argument we may assume that C is finitely supported.

First observe that for fixed $k \in \Lambda$, the sequence $\left\{c_{k,j}\right\}_j$ belongs to ℓ^1_m, where m is the weight given by $m(j) := w(k-j)$. Since $w(j) \leq m(j)w(k)$, it follows from Proposition 2.3.1 that the series defining g_k converges in $W(\boldsymbol{B}, L^1_w)$.

Fix a window function $\eta \in \mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. For each $k \in \Lambda$, $g_k\eta(\cdot - x) = \sum_j c_{k,j} f_j \eta(\cdot - x)$. Consequently, if we set $h_k(x) := \|g_k\eta(\cdot - x)\|_{\boldsymbol{B}} w(x-k)$, we get,

$$h_k(x) \leq \sum_j \left|c_{k,j}\right| w(k-j) \|f_j\eta(\cdot - x)\|_{\boldsymbol{B}} w(x-j). \tag{2.3}$$

Integrating this equation yields,

$$\|h_k\|_1 \leq \sum_j \left|c_{k,j}\right| w(k-j) \|F\|_{W(\boldsymbol{B}, L^1_w)}.$$

Hence, $\sup_k \|h_k\|_1 \leq \|C\|_{\mathcal{S}_w} \|F\|_{W(\boldsymbol{B}, L^1_w)}$. From Equation (2.3) we also get,

$$\begin{aligned}\sum_k h_k(x) &\leq \sum_j \sum_k \left|c_{k,j}\right| w(k-j) \|f_j\eta(\cdot - x)\|_{\boldsymbol{B}} w(x-j) \\ &\leq \|C\|_{\mathcal{S}_w} \sum_j \|f_j\eta(\cdot - x)\|_{\boldsymbol{B}} w(x-j).\end{aligned}$$

Therefore, $\operatorname{supess}_x \sum_k h_k(x) \lesssim \|C\|_{\mathcal{S}_w} \|F\|_{W(\boldsymbol{B}, L^1_w)}$. This completes the proof. □

We now give a dual estimate.

Proposition 2.3.3. *Let two families $F \equiv \{f_k\}_{k\in\Lambda}$, $G \equiv \{g_k\}_{k\in\Lambda} \subseteq \boldsymbol{B}_{loc}$ such that $\|F\|_{W(\boldsymbol{B}, L^1_w)}, \|G\|_{W(\boldsymbol{B}, L^1_w)} < +\infty$ be given. Suppose that $\boldsymbol{B}$ is continuously embedded in L^∞_{loc}. Then the cross-correlation matrix C, defined by*

$$C_{k,j} := \left\langle f_k, g_j \right\rangle,$$

satisfies $\|C\|_{\mathcal{S}_w} \lesssim \|F\|_{W(\boldsymbol{B}, L^1_w)} \|G\|_{W(\boldsymbol{B}, L^1_w)}$.

Remark 2.3.4. *The implicit constant depends on the embedding $\boldsymbol{B} \hookrightarrow L^\infty_{loc}$.*

Proof. Fix $\eta \in \mathcal{D}(\mathbb{R}^d)$ supported on an open ball B around 0 and such that $\eta \equiv 1$ on a smaller concentric ball B'. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a locally integrable function. Given $x \in \mathbb{R}^d$, for almost every $y \in B' + x$,

$$|f(y)| = |f(y)\eta(y-x)| \leq \|f\eta(\cdot - x)\|_{L^\infty(B)} \lesssim \|f\eta(\cdot - x)\|_{\boldsymbol{B}}.$$

Hence

$$|B_r(x)|^{-1} \int_{B_r(x)} |f(y)|\, dy \lesssim \|f\eta(\cdot - x)\|_{\boldsymbol{B}},$$

for all sufficiently small $r > 0$. This shows that

$$|f(x)| \lesssim \|f\eta(\cdot - x)\|_{\boldsymbol{B}},$$

at every $x \in \mathbb{R}^d$ that is a Lebesgue point of f. Consequently,

$$\left|c_{kj}\right| w(k-j) \lesssim \int_{\mathbb{R}^d} \|f_k\eta(\cdot - x)\|_{\boldsymbol{B}} w(x-k) \|g_j\eta(\cdot - x)\|_{\boldsymbol{B}} w(x-j) dx.$$

Taking $\sup_k \sum_j$ and $\sup_j \sum_k$, it follows that $\|C\|_{\mathcal{S}_w} \lesssim \|F\|_{W(\boldsymbol{B},L^1_w)} \|G\|_{W(\boldsymbol{B},L^1_w)}$. □

Finally we show that well-localized atoms induce bounded analysis operators.

Proposition 2.3.4. *Let a family $F \equiv \{f_k\}_{k\in\Lambda} \subseteq \boldsymbol{B}_{loc}$ such that $\|F\|_{W(B,L^1_w)} < +\infty$ be given. Suppose that $\boldsymbol{B}$ is continuously embedded in L^∞_{loc}. For $f \in L^p_v$ $(1 \le p \le \infty)$ define the analysis sequence,*

$$c_k := \langle f, f_k \rangle, \quad (k \in \Lambda).$$

Then c is well-defined, belongs to ℓ^p_v and satisfies

$$\|c\|_{\ell^p_v} \lesssim \|f\|_{L^p_v} \|F\|_{W(B,L^1_w)}.$$

Remark 2.3.5. *The implicit constant depends on the embedding $\boldsymbol{B} \hookrightarrow L^\infty_{loc}$ and the constant in Equation* (1.8).

Proof. Fix $\eta \in \mathcal{D}(\mathbb{R}^d)$ supported on an open ball B around 0 and such that $\eta \equiv 1$ on a smaller concentric ball B'. As in the proof of Proposition 2.3.3, the sequence c satisfies

$$|c_k| \lesssim \int_{\mathbb{R}^d} |f(x)| \, \|f_k\eta(x-k)\|_{\boldsymbol{B}} dx,$$

so

$$|c_k| \, v(k) \lesssim \int_{\mathbb{R}^d} |f(x)| \, v(x) \|f_k\eta(x-k)\|_{\boldsymbol{B}} w(x-k) dx.$$

The conclusion now follows from part (b) of Lemma 2.3.1. □

Chapter 3

Atomic spaces: the model for phase-space

Considered in full generality, coorbit spaces are functional spaces defined by imposing size conditions to a certain transform. More precisely, considering a functional space $\boldsymbol{X}$ as a coorbit space consists of giving a transform $T : \boldsymbol{X} \rightarrow \boldsymbol{E}$ that embeds $\boldsymbol{X}$ into another functional space $\boldsymbol{E}$ that is *solid*. This means that the membership in $\boldsymbol{E}$ is determined by size conditions. The space $\boldsymbol{E}$ consists of functions defined on a measure space with some underlying geometrical structure.

The coorbit theory presented in Section 1.10 studies the case when T arises as the representation coefficients of a unitary action of a locally compact group. The examples of this theory include a wide range of classical function spaces. In the case of the affine group acting on $L^2(\mathbb{R}^d)$ by translations and dilations, T is the continuous wavelet transform and the corresponding class of coorbit spaces includes the Lebesgue spaces L^p $(1 < p < \infty)$, Sobolev spaces and, more generally, the whole class of Besov and Triebel-Lizorkin spaces (see Section 1.12). In the case of the Heisenberg group acting on $L^2(\mathbb{R}^d)$ by time-frequency shifts, the transform T is - up to a phase-factor - the short-time Fourier transform and the corresponding coorbit spaces are the modulation spaces from Section 1.11. Another example is the coorbit theory from Section 1.13. In this case, the map T consists of the coefficient mapping of an abstract frame.

When a functional space $\boldsymbol{X}$ is identified as a coorbit space, the properties of an element $f \in \boldsymbol{X}$ are reformulated in terms of decay or integrability conditions of the function $T(f) \in \boldsymbol{E}$, that is sometimes referred to as the *phase-space representation* of f. The elements of $\boldsymbol{X}$ can be resynthesized from their phase-space representation by means of an operator $U : \boldsymbol{E} \rightarrow \boldsymbol{X}$ that is a left-inverse for T (i.e. $f = UT(f)$).

We now introduce a general setting that will model phase-space in several situations. The model consists of a solid BF space $\boldsymbol{E}$ over a locally-compact group $\mathcal{G}$ (called the environment) and a certain distinguished subspace $\boldsymbol{S}$ that is the range of an idempotent integral operator $P : \boldsymbol{E} \rightarrow \boldsymbol{S}$. The kernel of the operator P is assumed to be concentrated around its diagonal. Hence its range $\boldsymbol{S}$ enjoys certain regularity properties (see Proposition 3.1.1 below). This setting models phase space in the following way. If a functional space $\boldsymbol{X}$ is presented as

$\boldsymbol{X} = T^{-1}(\boldsymbol{E})$, we let $\boldsymbol{S} := T(\boldsymbol{X})$ and $P := TU$.

After introducing the general setting we introduce a number of more particular scenarios. We will study the case where the atomic space has a distinguished atomic decomposition. We do so because some applications will require a "fine-tuning" of the general results based on this extra structure. Secondly, we consider the case when the group $\mathcal{G}$ is the Euclidean space, where smoothness matters are relevant.

3.1 The general model

Let $\mathcal{G}$ be a locally-compact group. We list a number of ingredients in the form of two assumptions: (A1) and (A2).

(A1)
- $\boldsymbol{E}$ is a solid, translation invariant BF space, called *the environment*.
- w is an admissible weight for $\boldsymbol{E}$.
- $\boldsymbol{S}$ is a closed complemented subspace of $\boldsymbol{E}$, called *the atomic subspace*.

The second assumption is that the retraction $\boldsymbol{E} \to \boldsymbol{S}$ is given by an operator that is dominated by right convolution with a kernel in $W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$.

(A2) We have an operator P and a function H satisfying the following.
- $P : W(L^1, L^\infty_{1/w}) \to L^\infty_{1/w}$ is a (bounded) linear operator,
- $P(\boldsymbol{E}) = \boldsymbol{S}$,
- $P(f) = f$, for all $f \in \boldsymbol{S}$,
- $H \in W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$,
- For $f \in W(L^1, L^\infty_{1/w})$,

$$|P(f)(x)| \le \int_{\mathcal{G}} |f(y)|\, H(y^{-1}x)dy, \qquad (x \in \mathcal{G}). \tag{3.1}$$

We now observe some consequences of these assumptions.

Proposition 3.1.1. *Under Assumptions (A1) and (A2) the following holds.*

(a) P maps $\boldsymbol{E}$ boundedly into $W(L^\infty, \boldsymbol{E})$.

(b) $\boldsymbol{S} \hookrightarrow W(L^\infty, \boldsymbol{E})$.

(c) If $f \in W(L^1, L^\infty_{1/w})$, then $\|P(f)\|_{L^\infty_{1/w}} \lesssim \|f\|_{W(L^1, L^\infty_{1/w})} \|H\|_{W_R(L^\infty, L^1_w)}$.

(d) If $f \in W(L^1, L^\infty)$, then $\|P(f)\|_{L^\infty} \lesssim \|f\|_{W(L^1, L^\infty)} \|H\|_{W_R(L^\infty, L^1_w)}$.

Remark 3.1.1. *Since $w \gtrsim 1$, $L^\infty \hookrightarrow L^\infty_{1/w}$.*

Proof. Part (a), (c) and (d) follow from Equation (3.1) and Lemma 1.6.1. For (b), observe that by part (a), P maps $\boldsymbol{E}$ into $W(L^\infty, \boldsymbol{E})$ and coincides with the identity operator on $\boldsymbol{S}$. □

3.1.1 Example

We now show preciseley how the coorbit theory of Section 1.10 fits into this model. Let π be a (strongly continuous) unitary representation of a locally compact group $\mathcal{G}$ on a Hilbert space $\mathbb{H}$ and let h be an admissible vector and let $\boldsymbol{E}$ be a solid BF space.

With the notation of Section 1.10 let $\boldsymbol{S} = V_h(\mathrm{Co}\boldsymbol{E})$. It is proved in [44, Proposition 4.3] that $\boldsymbol{S}$ is a closed subspace of $\boldsymbol{E}$ and that, moreover, $P(F) := F * V_h h$ defines a projector onto $\boldsymbol{S}$.

By the admissibility of h (cf. Section 1.10), $V_h h \in W_R(L^\infty, L^1_w)$. Since $V_h h(x^{-1}) = \overline{V_h h(x)}$, it follows that $V_h h$ also belongs to $W(L^\infty, L^1_w)$. Hence, if we let $H := V_h h$, Assumptions (A1) and (A2) are verified. When $\boldsymbol{E}$ is $L^2(\mathcal{G})$, the operator P is in fact the orthogonal projector onto $\boldsymbol{S}$.

3.2 The case of atomic decompositions

We now consider a setting where the atomic space from Section 3.1 has a distinguished atomic decomposition. We prove a number of technical results that will allow us to finely adjust the results obtained in the general setting in order to get sharper statements for certain applications.

It is known that under certain conditions any instance of the model introduced in Section 3.1 has an associated atomic decomposition (see [89]). However, the point here is not the fact that $\boldsymbol{S}$ has an atomic decomposition, but they way in which this extra structure relates to the general model.

Let us assume that Assumption (A1) from Section 3.1 holds. We now state Assumption (A2') introducing new ingredients to the model.

(A2')
- $\Lambda \subseteq \mathcal{G}$ is a relatively separated set. Its points will be called *nodes*.
- $\{ \varphi_\lambda \,|\, \lambda \in \Lambda \}$ and $\{ \psi_\lambda \,|\, \lambda \in \Lambda \}$ are sets of $W^{\mathrm{st}}(L^\infty, L^1_w)$ molecules, enveloped by a function h. That is,
  - $|\varphi_\lambda(x)|, |\psi_\lambda(x)| \le h(\lambda^{-1}x), \qquad (x \in \mathcal{G}, \lambda \in \Lambda)$,
  - $h \in W^{\mathrm{st}}(L^\infty, L^1_w)$.

  The sets $\{\varphi_\lambda\}_\lambda$ and $\{\psi_\lambda\}_\lambda$ will be called *atoms* and *dual atoms* respectively.
- $\boldsymbol{S} \subseteq \boldsymbol{E}$ has the following atomic decomposition.
  (a) For every $c \in \boldsymbol{E}^d(\Lambda)$, the series $\sum_\lambda c_\lambda \varphi_\lambda$ belong to $\boldsymbol{S}$. [1]
  (b) For all $f \in \boldsymbol{S}$, the following expansion holds,

$$f = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \varphi_\lambda. \tag{3.2}$$

[1] The convergence of the series is clarified in Lemma 1.6.2.

Associated with the atoms we consider the *analysis* and *synthesis* operators given by,

$$C : \boldsymbol{E} \to \boldsymbol{E}_d, \quad C(f) := (\langle f, \psi_\lambda \rangle)_\lambda,$$
$$S : \boldsymbol{E}_d \to \boldsymbol{E}, \quad S(c) := \sum_\lambda c_\lambda \varphi_\lambda.$$

Under Assumptions (A1) and (A2'), these operators are well-defined and bounded by Lemma 1.6.2 and the fact that $h \in W^{\text{st}}(L^\infty, L^1_w) \subseteq W_R(L^\infty, L^1_w)$.

We also consider the operator $P : \boldsymbol{E} \to \boldsymbol{S}$ defined by $P := S \circ C$. Hence,

$$P(f) = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \varphi_\lambda. \tag{3.3}$$

According to (A2'), P is a projector from $\boldsymbol{E}$ onto $\boldsymbol{S}$.

We will now see that the setting introduced by (A1) and (A2') can be regarded as an instance of the one set by (A1) and (A2). We first introduce the function H required by (A2). Let $H : \mathcal{G} \to [0, +\infty)$ be defined by

$$H(x) := \sup_{y \in \mathcal{G}} \sum_{\lambda \in \Lambda} h(\lambda^{-1} y) h(\lambda^{-1} y x). \tag{3.4}$$

The following lemma shows that P and H satisfy the conditions in (A2).

Lemma 3.2.1. *Under Assumptions (A1) and (A2') the following statements hold.*

- *The function H (cf. Equation (3.4)) belongs both to $W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w)$.*
- *For every $f \in W(L^1, L^\infty_{1/w})$, the function $P(f) = \sum_\lambda \langle f, \psi_\lambda \rangle \varphi_\lambda$ is well-defined (with absolute convergence at every point) and satisfies the following pointwise estimate,*

$$|P(f)(x)| \leq \int_{\mathcal{G}} |f(y)|\, H(y^{-1} x) dy, \qquad (x \in \mathcal{G}).$$

Moreover, $\|P(f)\|_{L^\infty_{1/w}} \lesssim \|f\|_{W(L^1, L^\infty_{1/w})} \|H\|_{W_R(L^\infty, L^1_w)}$.

Proof. Let us prove (a). Let $x, y \in \mathcal{G}$ be given. We estimate,

$$\sum_\lambda h(\lambda^{-1} y) h(\lambda^{-1} y x) \lesssim \sum_\lambda \int_{\mathcal{G}} h_\#(t^{-1} \lambda^{-1} y) h_\#(t^{-1} \lambda^{-1} y x) \chi_V(t) dt.$$

Making the change of variables $t \mapsto \lambda^{-1} y t$ we get,

$$\sum_\lambda h(\lambda^{-1} y) h(\lambda^{-1} y x) \lesssim \int_{\mathcal{G}} h_\#(t^{-1}) h_\#(t^{-1} x) \sum_\lambda \chi_V(\lambda^{-1} y t) dt.$$

Since Λ is relatively separated and $V = V^{-1}$,

$$\sum_{\lambda} \chi_V(\lambda^{-1} yt) = \sum_{\lambda} \chi_{(ytV)}(\lambda) \lesssim 1.$$

Hence, taking supremum on y we get that

$$H(x) \lesssim \int_{\mathcal{G}} h_{\#}(t^{-1}) h_{\#}(t^{-1}x) dt.$$

Using this inequality we can estimate the local maximum functions of H. For $v \in V$, we have

$$H(xv) \lesssim \int_{\mathcal{G}} h_{\#}(t^{-1}) h_{\#}(t^{-1}xv) dt \leq \int_{\mathcal{G}} h_{\#}(t^{-1}) (h_{\#})^{\#}(t^{-1}x) dt.$$

Hence,

$$H^{\#}(x) \lesssim \int_{\mathcal{G}} h_{\#}(t^{-1}) (h_{\#})^{\#}(t^{-1}x) dt. \tag{3.5}$$

Likewise, for $v \in V$,

$$\begin{aligned} H(vx) &\lesssim \int_{\mathcal{G}} h_{\#}(t^{-1}) h_{\#}(t^{-1}vx) dt = \int_{\mathcal{G}} h_{\#}(t^{-1}v^{-1}) h_{\#}(t^{-1}x) dt \\ &\leq \int_{\mathcal{G}} (h_{\#})^{\#}(t^{-1}) (h_{\#})(t^{-1}x) dt. \end{aligned}$$

So,

$$H_{\#}(x) \lesssim \int_{\mathcal{G}} (h_{\#})^{\#}(t^{-1}) (h_{\#})(t^{-1}x) dt. \tag{3.6}$$

Using Equation (3.5) and the submultiplicativity of w we get,

$$H^{\#}(x) w(x) \lesssim \int_{\mathcal{G}} h_{\#}(t^{-1}) w(t) (h_{\#})^{\#}(t^{-1}x) w(t^{-1}x) dt.$$

Hence,

$$\begin{aligned} \|H^{\#}\|_{L^1_w} &\lesssim \|(h_{\#})^{\#}\|_{L^1_w} \int_{\mathcal{G}} h_{\#}(t^{-1}) w(t) dt \\ &= \|(h_{\#})^{\#}\|_{L^1_w} \int_{\mathcal{G}} h_{\#}(t) w(t^{-1}) \Delta(t^{-1}) dt \\ &= \|(h_{\#})^{\#}\|_{L^1_w} \|h_{\#}\|_{L^1_w}. \end{aligned}$$

Therefore, $\|H\|_{W(L^\infty, \boldsymbol{E})} \lesssim \|h\|_{W^{\mathrm{st}}(L^\infty, L^1_w)} \|h\|_{W(L^\infty, L^1_w)}$, and the desired conclusion follows from the embedding $W^{\mathrm{st}}(L^\infty, L^1_w) \hookrightarrow W(L^\infty, L^1_w)$ in Proposition 2.1.1. The bound for $\|H\|_{W_R(L^\infty, \boldsymbol{E})}$ follows similarly, this time using Equation (3.6).

For part (b), we use the enveloping condition in (A2') we get the desired pointwise estimate for P. The rest of the claim the follows from part (a) and Lemma 1.6.1. $\square$

3.2.1 Weak continuity of the atomic decomposition

Suppose that Assumptions (A1) and (A2') hold. Lemmas 1.6.1 and 1.6.2 give the embeddings $\boldsymbol{E} \hookrightarrow W(L^1, L^\infty_{1/w})$ and $\boldsymbol{E}^d \hookrightarrow \ell^\infty_{1/w}$. We denote by $(\boldsymbol{E}^d, \ell^1_w)$ the space $\boldsymbol{E}^d$ considered with the restriction of the weak* star topology of $\ell^\infty_{1/w}$. Likewise, since by Lemma 1.6.1, $W(L^1, L^\infty_{1/w})$ embeds into the dual space of $W(L^\infty, L^1_w)$, we let $(\boldsymbol{E}, W(L^\infty, L^1_w))$ stand for space $\boldsymbol{E}$ considered with the topology induced by the linear functionals obtained by integration against $W(L^\infty, L^1_w)$ functions. Observe that, since this family of functionals separates points, the corresponding topology is Hausdorff.

We will now establish the continuity of the maps that implement the atomic decomposition of $\boldsymbol{S}$ with respect to these coarser topologies. This will allow us to use density arguments for $\boldsymbol{S}$. This is irrelevant when the atomic decomposition in Equation (3.2) converges in the norm of $\boldsymbol{E}$, but is important to make the abstract results fully applicable.

Proposition 3.2.1. *Under Assumptions (A1) and (A2') the following statements hold.*

(a) The map $C : (\boldsymbol{E}, W(L^\infty, L^1_w)) \to (\boldsymbol{E}^d, \ell^1_w)$ *is continuous.*

(b) For $c \in \boldsymbol{E}^d$*, the series defining* $S(c)$ *converge unconditionally in the* $(\boldsymbol{E}, W(L^\infty, L^1_w))$ *topology. Moreover, the map* $S : (\boldsymbol{E}^d, \ell^1_w) \to (\boldsymbol{E}, W(L^\infty, L^1_w))$ *is continuous.*

Proof. For $\lambda \in \Lambda$, $\|\psi_\lambda\|_{W(L^\infty, L^1_w)} \leq \|L_\lambda h\|_{W(L^\infty, L^1_w)} \leq w(\lambda)\|h\|_{W(L^\infty, L^1_w)}$. Hence, for any $b \in \ell^1_w(\Lambda)$, the series $\sum_\lambda b_\lambda \psi_\lambda$ are absolutely convergent in $W(L^\infty, L^1_w)$. Moreover, by Lemma 1.6.1 (c), for $f \in \boldsymbol{E} \subseteq W(L^1, L^\infty_{1/w})$ we can interchange summation and integration to obtain: $\langle C(f), b\rangle = \langle f, \sum_\lambda b_\lambda \psi_\lambda\rangle$. Part (a) now follows from this formula.

For (b), let $c \in \boldsymbol{E}^d$ and let us show that the series $S(c) = \sum_\lambda c_\lambda \varphi_\lambda$ converge unconditionally in the $(\boldsymbol{E}, W(L^\infty, L^1_w))$ topology. For $f \in W(L^\infty, L^1_w)$, we need to show that,

$$\left\langle \sum_{\lambda \in \Lambda'} c_\lambda \varphi_\lambda, f \right\rangle \to \left\langle \sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda, f \right\rangle,$$

as $\Lambda' \to \Lambda$ in the directed order of finite subsets of Λ. Since $c \in \boldsymbol{E}^d$, Lemma 1.6.2 implies that $\sum_{\lambda \in \Lambda} |c_\lambda| L_\lambda h \in \boldsymbol{E} \subseteq W(L^1, L^\infty_{1/w})$. Hence by Lemma 1.6.1,

$$\sum_{\lambda \in \Lambda} |c_\lambda| \langle |f|, L_\lambda h\rangle = \left\langle |f|, \sum_{\lambda \in \Lambda} |c_\lambda| L_\lambda h \right\rangle < +\infty,$$

where the interchange of summation and integration is justified by the Fubini-Tonelli Theorem. Consequently,

$$\left| \left\langle \sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda, f \right\rangle - \left\langle \sum_{\lambda \in \Lambda'} c_\lambda \varphi_\lambda, f \right\rangle \right| \leq \sum_{\lambda \in \Lambda \setminus \Lambda'} |c_\lambda| \langle |f|, L_\lambda h\rangle \to 0,$$

as $\Lambda' \to \Lambda$.

To establish the continuity of S, let $c \in \boldsymbol{E}^d$ and $f \in W(L^\infty, L^1_w)$ and observe that, since the series defining $S(c)$ converge in the $(\boldsymbol{E}, W(L^\infty, L^1_w))$ topology, we can interchange summation and integration against f to obtain,

$$\langle S(c), f\rangle = \langle c, (\langle f, \varphi_\lambda\rangle)_\lambda\rangle .$$

We will now see that the sequence $(\langle f, \varphi_\lambda\rangle)_\lambda$ belongs to ℓ^1_w and this will yield the desired continuity of S. Using the bound $|\varphi_\lambda| \le L_\lambda h$ and Lemma 1.6.3 with $\boldsymbol{E} = L^1_w$ we see that $(\langle f, \varphi_\lambda\rangle)_\lambda \in W(L^\infty, L^1_w)^d$. Since $W(L^\infty, L^1_w) \hookrightarrow L^1_w$, we have that $W(L^\infty, L^1_w)^d \hookrightarrow \ell^1_w$ and the conclusion follows. □

3.3 Spline-type spaces in the Euclidean space

We now consider the setting of the Euclidean space. We make concrete choices for the function and sequence spaces and emphasise the matters that pertain the Euclidean space. In this context it will be better to use Schur-type conditions instead of domination by convolution.

As function spaces we use weighted L^p spaces and as sequence spaces the corresponding ℓ^p spaces. To avoid having to distinguish the case $p = +\infty$ in every statement we let z^p_v stand for the closure of the set of finitely-supported sequences within ℓ^p_v. For $p < +\infty$ this is just ℓ^p_v, and for $p = +\infty$ it is c^0_v.

We consider a relatively separated set of points $\Lambda \subseteq \mathbb{R}^d$ which will be called *nodes* and a family of functions $F \equiv \{f_k\}_{k\in\Lambda} \subseteq L^\infty_{loc}$ that will be called *atoms*.

Let $\boldsymbol{V}^{00}$ be the set of finite linear combination of elements of F. For a weight function v, and $1 \le p \le \infty$, we denote by $\boldsymbol{V}^p_v$ the L^p_v-closure of $\boldsymbol{V}^{00}$. If the weight v is the trivial weight 1 we drop it in the notation.

Following the spirit of coorbit theory, we do not want to consider each of the spaces $\boldsymbol{V}^p_v$ individually, but to treat all the range of spaces $\boldsymbol{V}^p_v$ as a whole. We think of each $\boldsymbol{V}^p_v$ as variant of a single spline-type space $\boldsymbol{V} = \boldsymbol{V}(F, \Lambda)$.

It will be assumed that the family F is a Banach frame for each $\boldsymbol{V}^p_v$. The general theory of localized frames (see Section 1.13) ensures that this is indeed the case provided that F is a Hilbert space frame for $\boldsymbol{V}^2$ and that F satisfies a localization property. In our context this property amounts to spatial localization.

We now formulate precisely the assumptions that we will make on the set of atoms F and show that under those assumptions, F is a Banach frame for the whole range of spaces $\boldsymbol{V}^p_v$.

- We assume that we have chosen a uniformly localizable and isometrically translation invariant Banach space $\boldsymbol{B}$, that is continuously embedded into L^∞_{loc} (see Section 1.6.1). An example to keep in mind is the one of fractional Sobolev spaces L^q_s. These spaces are embedded in L^∞_{loc} if either $q = +\infty$ or if $s > d/q$ (see [1]).

- We also assume that F satisfies the uniform concentration and smoothness condition $\|F\|_{W(\boldsymbol{B},L^1_w)} < +\infty$, for some subexponential weight $w : \mathbb{R}^d \to (0,\infty)$ that verifies $w(x) \gtrsim (1+\|x\|)^\delta$, for some $\delta > 0$ (cf. Section 2.2)

- Finally, we assume that F forms a frame sequence in $\boldsymbol{L}^2(\mathbb{R}^d)$.

If all the above assumptions are met we say that $\boldsymbol{V} = \boldsymbol{V}(F,\Lambda)$ is a *spline type space*.

Remark 3.3.1. *Remember that, under the above assumptions, the weight w satisfies:* $w(0) = 1$, $w(x) = w(-x)$ *and is submultiplicative. The polynomial weights* w_α *with* $\alpha > 0$ *and the subexponential weights* $w(x) := e^{\alpha|x|^\beta}$ *with* $\alpha > 0$ *and* $0 < \beta < 1$ *satisfy the assumptions above (cf. Section 1.3).*

The first items of the next proposition are just an application of the theory in Section 1.13.

Theorem 3.3.1. *Let* $\boldsymbol{V} = \boldsymbol{V}(F,\Lambda)$ *be a spline type space, then the following holds.*

(a) $G \equiv \{g_k\}_k$*, the canonical dual family of F satisfies* $\|G\|_{W(\boldsymbol{B},L^1_w)} < +\infty$.

(b) For any $1 \le p \le \infty$ *and any symmetric, w-moderated weight v, the pair* (F,G) *is a Banach frame for* $\boldsymbol{V}^p_v$ *with associated sequence space* z^p_v.

(c) For any $1 \le p \le \infty$ *and any symmetric, w-moderated weight v, we have the inclusion* $\boldsymbol{V}^p_v \subseteq W(\boldsymbol{B}, L^p_v)$*. Moreover, on* $\boldsymbol{V}^p_v$*, the* L^p_v *and* $W(\boldsymbol{B}, L^p_v)$ *norms are equivalent.*

Remark 3.3.2. *If the norm of* $\boldsymbol{B}$ *measures smoothness (eg.* $\boldsymbol{B}$ *is a Sobolev space), item (c) implies that all the elements of* $\boldsymbol{V}^p_v$ *are as smooth as the set of atoms. Moreover, size estimates for a function* $f \in \boldsymbol{V}^p_v$ *can be turned into smoothness estimates.*

Remark 3.3.3. *In the situation of the theorem, the frame expansion arising from the pair* (F,G) *can be extended to the weak* closure of* $\boldsymbol{V}^{00}$ *within* L^∞_v *using coeffcients in* ℓ^∞*, but the series converge only in the weak* topology.*

Proof. Consider the self-correlation (Gramian) matrix C, given by $c_{kj} := \left\langle f_k, f_j \right\rangle$. By proposition 2.3.3 we have that $\|C\|_{\mathcal{S}_w} < +\infty$, where $\mathcal{S}_w$ is the weighted Schur class. Since F is a frame sequence in $\boldsymbol{L}^2(\mathbb{R}^d)$, the matrix C has a pseudo-inverse $C^\dagger \in B(\ell^2)$. By the result in Section 1.7.2 and Remark 1.7.1 it follows that $\|C^\dagger\|_{\mathcal{S}_w} < +\infty$. The formula for the dual frame in Equation 1.33 yields $G = C^\dagger \cdot F$. Hence, it follows from Proposition 2.3.2 that $\|G\|_{W(\boldsymbol{B},L^1_w)} < +\infty$. This proves (a).

For part (b) let v be a symmetric, w-moderated weight and let $1 \le p \le \infty$. The reconstruction operator $R : z^p_v \to \boldsymbol{V}^p_v$, $c \mapsto c \cdot F$, is well defined and bounded by Proposition 2.3.1. Moreover $\|R\| \lesssim \|F\|_{W(\boldsymbol{B},L^1_w)}$. Proposition 2.3.4 implies that the coefficients mapping $C : L^p_v \to \ell^p_v$, given by $f \mapsto \{\langle f, g_k\rangle\}_k$ is well defined and satisfies $\|C\| \lesssim \|G\|_{W(\boldsymbol{B},L^1_w)}$. Moreover, if f is a finite linear combination of functions of F, we have that $RC(f) = f$. It follows that (F,G) determines a Banach frame pair.

Now (c) follows easily from Proposition 2.3.1. Since $\boldsymbol{B} \hookrightarrow L^{\infty,loc} \hookrightarrow L^{p,loc}$, we have the inclusion $W(\boldsymbol{B}, L^p_v) \hookrightarrow W(L^p, L^p_v) = L^p_v$. Therefore, for $f \in \boldsymbol{V}^p_v$, $f = RC(f)$ and,

$$\begin{aligned}\|f\|_{L^p_v} &\lesssim \|f\|_{W(\boldsymbol{B},L^p_v)} \lesssim \|F\|_{W(\boldsymbol{B},L^1_w)} \|C(f)\|_{\ell^p_v} \\ &\lesssim \|F\|_{W(\boldsymbol{B},L^1_w)} \|G\|_{W(\boldsymbol{B},L^1_w)} \|f\|_{L^p_v}.\end{aligned}$$

□

We now observe that, in order to bound an operator on a spline-type space, we just need to control its behavior on the atoms.

Proposition 3.3.1. *Let $\boldsymbol{V} = \boldsymbol{V}(F, \Lambda)$ be a spline-type space. Let v be a symmetric, w-moderated weight, $1 \le p \le \infty$ and let $T : \boldsymbol{V}^p_v \to L^p_v$ be a linear operator. Then,*

$$\|T\|_{\boldsymbol{V}^p_v \to L^p_v} \lesssim \|T(F)\|_{W(\boldsymbol{B},L^1_w)}.$$

Proof. If $f = c \cdot F$ for a finitely supported sequence $c \in \ell^p(\Lambda)$, then $T(f) = c \cdot T(F)$. Theorem 3.3.1 implies that,

$$\begin{aligned}\|T(f)\|_{L^p_v} &\lesssim \|T(f)\|_{W(L^\infty,L^p_v)} \lesssim \|T(f)\|_{W(\boldsymbol{B},L^p_v)} \\ &\lesssim \|c\|_{\ell^p_v} \|T(F)\|_{W(\boldsymbol{B},L^1_w)} \lesssim \|f\|_{L^p_v} \|T(F)\|_{W(\boldsymbol{B},L^1_w)}.\end{aligned}$$

The conclusion extends to general f by an approximation argument. □

Finally we observe that, as a consequence of Theorem 3.3.1, there is a universal projector $P : L^p_v \to \boldsymbol{V}^p_v$, for all $1 \le p \le \infty$ and w-moderated weights v. More precisely, we have the following statement.

Theorem 3.3.2. *Let $\boldsymbol{V} = \boldsymbol{V}(F, \Lambda)$ be a spline-type space and let $P : L^2 \to \boldsymbol{V}^2$ be the orthogonal projector onto $\boldsymbol{V}^2$. Then, for all $1 \le p < \infty$ and w-moderated weights v, the restriction of P to $\mathcal{S}(\mathbb{R}^d)$ extends by density to a bounded projector $P : L^p_v \to \boldsymbol{V}^p_v$. For $p = \infty$ the same statement is true replacing L^∞_v with C^0_v.*

Moreover, the norm of P is uniformly bounded for $1 \le p \le +\infty$ and any class of w-moderated weights for which the constant in Equation (1.8) is bounded.

Proof. We only need to check that the restriction of P to $\mathcal{S}(\mathbb{R}^d)$ is bounded in the norm of L^p_v. The projector P is given by,

$$P(f) = \sum_{k \in \Lambda} \langle f, g_k \rangle f_k,$$

where G is the family of dual atoms given by Theorem 3.3.1 (a). Using Propositions 2.3.1 and 2.3.4 we get,

$$\|P(f)\|_{L^p_v} \lesssim \|P(f)\|_{W(\boldsymbol{B},L^p_v)} \lesssim \|F\|_{W(\boldsymbol{B},L^1_w)} \|G\|_{W(\boldsymbol{B},L^1_w)} \|f\|_{L^p_v}.$$

□

Chapter 4

Localization of dual atoms

One of the fundamental parts of the theory of localized frames (cf. Section 1.13) is the fact that self-localized frames have localized dual frames. This is particularly important because it is rarely the case that dual system are explicitely exhibited. The construction in Chapter 5 will require however more refined information. In addition to the existence of localized dual frames we will need to know what qualities of the original atoms influence the concentration of their respective dual atoms. This problem is closely related to the one of quantifying spectral invariance for matrix algebras, as discussed in Section 1.7.4.

More concretely, the general problem to address is the one of determining what qualities of a matrix influence the off-diagonal decay of its inverse, or more generally, its pseudo-inverse. Among the vast literature on preservation of off-diagonal decay under inversion [27, 29, 30, 7, 59, 76, 8, 102, 9, 10, 66, 103, 4, 67, 104, 3] only a small portion uses completely constructive methods. For the kind of application we will need in Chapter 5 the most suitable result is the one of Qiyu Sun in [104]. There, the author establishes the inverse-closedness of certain algebras of matrices concentrated around their diagonal, under very general geometric conditions on the corresponding index set. His methods are mainly constructive and, as a by-product, he obtains a quantitative conclusion. Below we quote an application of his result to polynomial off-diagonal decay conditions that yields a quantitative version of Jaffard's result (cf. Section 1.7.1). We then give a slight adaptation of this result to cover pseudo-inversion.

Before noticing that the quantification of Jaffard's Theorem was contained in Qiyu Sun's result, I studied that same problem [93]. In Section 4.2 we give certain estimates on the preservation of polynomial off-diagonal decay from a matrix to its inverse. The techniques and precise form of the estimates might be of independet interest, although the result in [104] yields a much better qualitative conclusion.

4.1 Controlled inversion

The following is a particular case of Theorem 4.1 in [104]. It also follows from a careful reading of the proof in [103].

Theorem 4.1.1. *Let $\Gamma \subseteq \mathbb{R}^d$ be a relatively separated set and let $M \in B(\ell^2(\Gamma))$ be an invertible operator. Assume the following.*

- *M satisfies,*

$$\left|M_{k,j}\right| \leq C(1 + |k - j|)^{-s} \qquad (k, j \in \Gamma),$$

for some constants $C > 0$ and $s > d$.

- *$\|M^{-1}\|_{\ell^2 \to \ell^2} \leq A$, for some constant $A > 0$.*
- *$\rho(\Gamma) \leq R$, for some $0 \leq R < \infty$.*

Then M^{-1} satisfies,

$$\left|M^{-1}_{k,j}\right| \leq C'(1 + |k - j|)^{-s} \qquad (k, j \in \Gamma),$$

for some constant C' that only depends on C, s, d, A and R.

We now generalize this to cover pseudo-inversion. The case of the pseudo-inverse is treated in [104], but no explicit reference to the qualities involved in the off-diagonal decay of the pseudo-inverse is made. However, the proof given in [104, Theorem 5.1] (see also [53]) can be slightly adapted to obtain a quantitative conclusion. We only sketch the modifications.

Theorem 4.1.2. *Let $\Gamma \subseteq \mathbb{R}^d$ be a relatively separated set and let $M \in B(\ell^2(\Gamma))$ be a positive operator. Assume the following.*

- *M satisfies,*

$$\left|M_{k,j}\right| \leq C(1 + |k - j|)^{-s} \qquad (k, j \in \Gamma),$$

for some constants $C > 0$ and $s > d$.

- *The spectrum of M, satisfies,*

$$(\sigma(M) \setminus \{0\}) \cap B_A(0) = \emptyset,$$

for some $A > 0$. (Here $B_A(0) \subseteq \mathbb{C}$ is the ball of radius A centered at 0).

- *$\rho(\Gamma) \leq R$, for some $0 \leq R < \infty$.*

Then $M^\dagger$, the Moorse-Penrose pseudo-inverse of M, satisfies,

$$\left|M^\dagger_{k,j}\right| \le C'(1+|k-j|)^{-s} \qquad (k, j \in \Gamma),$$

for some constant C' that only depends on C, s, d, A and R.

Proof. Under the assumptions of the theorem,

$$M^\dagger = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,(zI-M)^{-1}\,dz, \tag{4.1}$$

where the curve γ is the rectangle with vertices $A/2 \pm i$, $\|M\| + A/2 \pm i$ oriented anti-clockwise; here $\|M\|$ denotes the norm of M in $B(\ell^2(\Gamma))$. Consequently, for $k, j \in \Gamma$,

$$M^\dagger_{kj} = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,(zI-M)^{-1}_{kj}\,dz. \tag{4.2}$$

Observe that $\|M\|$ can be bounded in terms of d, s, C and R (by interpolating its $\ell^1 \to \ell^1$ and $\ell^\infty \to \ell^\infty$ norm) and that the length of γ is $2\|M\| + 2$. For z in the curve γ, $|z| \lesssim \|M\| + 1$ and $|1/z| \le 2/A$. Hence, it suffices to bound the off-diagonal decay of the resolvent $(zI - M)^{-1}$ in terms of the allowed parameters.

Let z lie in the curve γ. The distance from z to $\sigma(M)$ is at least $m := \min\{1, A/2\}$, so $(\sigma(zI - M) \setminus \{0\}) \cap B_m(0) = \emptyset$. Moreover, for $k, j \in \Gamma$,

$$\begin{aligned}\left|(zI-M)_{kj}\right| &\le |z|\,\delta_{kj} + C(1+|k-j|)^{-s}\\ &\lesssim (C + \|M\| + 1)(1+|k-j|)^{-s}.\end{aligned}$$

By Theorem 4.1.1, the off-diagonal decay of $(zI - M)^{-1}$ is bounded by a constant depending only on allowed parameters. □

4.2 Explicit polynomial off-diagonal decay bounds

We now prove some exlicit estimates for the off-diagonal decay of inverse matrices, from where a quantitative conclusion like the one in Theorem 4.3.1 can be derived.

The intuiton of the techinique is the following one. If $M \in B(\ell^2(\mathbb{Z}^d))$ is an invertible convolution operator then,

$$M_{k,j} = a_{k-j}, \qquad (k, j \in \mathbb{Z}^d),$$

for some sequence a. The inverse matrix M^{-1} is similarly given by,

$$M^{-1}_{k,j} = b_{k-j}, \qquad (k, j \in \mathbb{Z}^d),$$

where the sequence b satisfies $a * b = \delta$. The off-diagonal decay of M and M^{-1} is therefore equivalent to the decay of their kernels a and b. Since the decay of a sequence x can be characterized by the smoothness of its Fourier transform $\hat{x}$, the problem can be reformulated as the preservation of the smoothness of the function $\hat{a}$ under pointwise inversion.

We can measure the smoothness of $\hat{a}$ by considering weak-derivatives and use repeatedly a chain-rule argument for Sobolev spaces to obtain similar smoothness conditions for $\hat{b}$.

In the general case, where M need not be a convolution operator, we try to imitate this reasoning, but we avoid using the Fourier transform. Given a matrix M and $1 \leq h \leq d$, consider the matrix $D_h(M)$ defined as,

$$D_h(M)_{k,j} := (k_h - j_h)M_{k,j}.$$

Observe that, up to some multiplicative constant, the map D_h acts on a convolution operator by taking a partial derivative of its symbol (that is, the Fourier transform of its kernel). The domain of D_h consists of those matrices M such that $D_h(M)$ defines a bounded operator on ℓ^2. We call $D_h(M)$ the *partial derivative* of M (with respect to x_h).

D_h is a derivation in the sense that it satisfies Leibniz's rule: $D_h(AB) = D_h(A)B + AD_h(B)$, provided that $D_h(A)$ and $D_h(B)$ are both defined. We can then try to imitate the computations related to derivatives of functions in this setting. A reasoning of this kind seems to be implicit in Jaffard's proof [76]. For more on the use of derivations in the field of operator algebras (see [15], [78] and [79]).

The use of the derivation D_h to measure off-diagonal decay in matrix algebras was also recently introduced by Gröchenig and Klotz in [65], with the aim of relating off-diagonal decay to rates of approximation by banded matrices. We refer the reader to that article for a discussion about the connection between this technique and Jaffard's and Baskakov's approaches [76, 8].

We now formally state and prove the result.

Theorem 4.2.1. *Let $\Gamma \subseteq \mathbb{R}^d$ be a relatively separated set and let $M \in B(\ell^2(\Gamma))$ be an invertible operator. Assume the following.*

- *M satisfies,*

$$\left|M_{k,j}\right| \leq C(1 + |k - j|)^{-s} \prod_{h=1}^{d} (1 + |k_h - j_h|)^{-t_h} \qquad (k, j \in \Gamma), \tag{4.3}$$

for some constants $C > 0$ and $s > d$, and integers $t_i \geq 0$.

- $\|M^{-1}\|_{\ell^2 \to \ell^2} \leq A$, *for some constant $A > 0$.*

- $\rho(\Gamma) \leq R$, *for some $0 \leq R < \infty$.*

Then M^{-1} satisfies,

$$\left|M^{-1}_{k,j}\right| \leq C' \prod_{h=1}^{d} (1 + |k_h - j_h|)^{-t_h} \qquad (k, j \in \Gamma),$$

where

$$C' = E^{t^2} A^{t+1} C^t R^t (1 + (s - t)^{-1})^t, \qquad t = t_1 + \ldots + t_d,$$

and $E > 0$ is a constant that only depends on the dimension d.

Remark 4.2.1. *The constant E can be explicitly determined from the proof.*

Remark 4.2.2. *Observe that although the theorem does not assert the full preservation of the rate of decay from M to M^{-1}, it shows that if M has a priviledged off-diagonal decay in a certain direction, then this is also the case for M^{-1}.*

Before proving the theorem, we introduce the following notation. For a multi-index $\beta \in \mathbb{N}_0^d$, let the operator D^β act on a matrix $T \in \mathbb{C}^{\Gamma\times\Gamma}$ by,

$$D^\beta(T)_{k,j} := (k - j)^\beta T_{k,j}, \qquad (k, j \in \Gamma),$$

where, for an index $k \in \mathbb{R}^d$,

$$k^\beta := \prod_{h=1}^{d} k_h^{\beta_h}.$$

Also, denote by $|\beta| := \beta_1 + \ldots + \beta_d$, the length of a multi-index. If α and β are multi-indexes, $\alpha \leq \beta$ means that $\alpha_k \leq \beta_k$, for all k.

Proof of Theorem 4.2.1. Let $K := 1 + (s - t)^{-1}$ and $\alpha := (t_1, \ldots, t_d)$. Throughout the proof we denote by $\|M\|$ the norm of a matrix $T \in \mathbb{C}^{\Gamma\times\Gamma}$ as an operator $T : \ell^2 \to \ell^2$. We make and prove a number of claims that will lead to the desired conclusion.

Claim 4.2.1. *For $\beta \leq \alpha$, $\|D^\beta(M)\| \lesssim CRK$.*

Proof of Claim 4.2.1. For $\beta \leq \alpha$, using the estimate in Equation (4.3) we see that,

$$\left|D^\beta(M_{k,j})\right| \leq C(1 + |k - j|)^{-s} \qquad (k, j \in \Gamma).$$

The conclusion follows from Schur's lemma (interpolation) and Lemma 1.3.2. □

Claim 4.2.2. *For all $\beta \leq \alpha$, $\|D^\beta(M^{-1})\|$ is a bounded operator on $\ell^2(\Gamma)$.*

Proof of Claim 4.2.2. By Jaffard's Theorem (cf. Section 1.7.1), M^{-1} satisfies the off-diagonal decay,

$$\left|M^{-1}_{k,j}\right| \leq C''(1+|k-j|)^{-s}\prod_{h=1}^{d}(1+|k_h-j_h|)^{-t_h} \qquad (k,j\in\Gamma),$$

for some (unknown) constant C''. Proceeding as in Claim 4.2.1 we see that the derivatives of M^{-1} are bounded (and obtain a bound depending on the unknown constant C''). □

Claim 4.2.3. *For all* $0\neq\beta\leq\alpha$*,*

$$\|D^\beta(M^{-1})\| \lesssim ACRK2^{|\beta|}\max_{\beta'<\beta}\|D^{\beta'}(M^{-1})\|.$$

Proof of Claim 4.2.3. Using Leibniz's rule for D^β we get,

$$0 = D^\beta(M^{-1}M) = \sum_{\beta'\leq\beta}\binom{\beta}{\beta'}D^{\beta'}(M^{-1})D^{\beta-\beta'}(M). \tag{4.4}$$

Since by Claim 4.2.2 all the operators involed in the last formula are bounded, we can associate factors to obtain,

$$D^\beta(M^{-1}) = -\sum_{\beta'<\beta}\binom{\beta}{\beta'}D^{\beta'}(M^{-1})D^{\beta-\beta'}(M)M^{-1}. \tag{4.5}$$

Using Claim 4.2.1 we the get,

$$\begin{aligned}\|D^\beta(M^{-1})\| &\leq A\sum_{\beta'<\beta}\binom{\beta}{\beta'}\|D^{\beta'}(M^{-1})\|\|D^{\beta-\beta'}(M)\|\\ &\lesssim ACRK\sum_{\beta'<\beta}\binom{\beta}{\beta'}\|D^{\beta'}(M^{-1})\|\end{aligned}$$

The claim now follows using that $\sum_{\beta'\leq\beta}\binom{\beta}{\beta'} = 2^{|\beta|}$. □

Claim 4.2.4.

$$\max_{\beta\leq\alpha}\|D^\beta(M^{-1})\| \leq E^{t^2}A^{t+1}(CRK)^t,$$

for some constant E that only depend on the dimension d.

Proof of Claim 4.2.4. Consider now the numbers

$$v_k := \max_{\beta\leq\alpha,|\beta|\leq k}\|D^\beta(M^{-1})\| \qquad (k\geq 0).$$

Since by Claim 4.2.1, $\|M\| \lesssim CRK$, we have that $1 \leq \|M^{-1}\|\|M\| \lesssim ACRK$. Using this and Claim 4.2.3 we see that the numbers v_t satisfy,

$$v_k \lesssim ACRK2^k v_{k-1}, \qquad (k \geq 1)$$
$$v_0 \leq A.$$

Iterating these relations $t = |\alpha|$ times we get,

$$v_t \leq (E')^t 2^{t(t+1)/2} A^{t+1} (CRK)^t \leq E^{t^2} A^{t+1} (CRK)^t,$$

where E and E' are constants that only depend on the dimension d. □

Having proved the claims we now finish the proof of Theorem 4.2.1. Observe that Claim 4.2.4 implies that for each $k, j \in \Gamma$,

$$\max_{\beta\leq\alpha} \prod_{h=1}^{d} |k_h - j_h|^{\beta_h} \left|M^{-1}_{k,j}\right| \leq \max_{\beta\leq\alpha} \|D^\beta(M^{-1})\| \leq E^{t^2} A^{t+1} (CRK)^t.$$

Hence, it suffices to observe that,

$$\prod_{h=1}^{d} (1 + |k_h - j_h|)^{t_h} \lesssim E^t \max_{\beta\leq\alpha} \prod_{h=1}^{d} |k_h - j_h|^{\beta_h},$$

for some constant E, that only dependes on the dimension. Given an index (k, j), if $|k_h - j_h| < 1$ we let $\beta_h := 0$, so that,

$$(1 + |k_h - j_h|)^{t_h} \leq 2^{t_h} = 2^{t_h} |k_h - j_h|^{\beta_h}.$$

If $|k_h - j_h| \geq 1$, we let $\beta_h = t_h = \alpha_h$, so that

$$(1 + |k_h - j_h|)^{t_h} \leq 2^{t_h} |k_h - j_h|^{\beta_h}.$$

Then $\beta \leq \alpha$ and $\prod_{h=1}^{d}(1 + |k_h - j_h|)^{t_h} \leq 2^t \prod_{h=1}^{d} |k_h - j_h|^{\beta_h}$. □

4.2.1 Some remarks on the proof

The most delicate part of the proof is the justification of the formal computations in Claim 4.2.3, that allowed us to solve $\|D_h^\beta(M^{-1})\|$ recursively from the binomial formula. In order to associate factors, we needed to know that M^{-1} belongs to the domain of D_h^β.

To see why this is important, let us consider the case when M is a convolution operator, having some sequence a as kernel. The matrix M^{-1} is also a convolution operator and has a kernel b that satisfies,

$$a * b = \delta. \tag{4.6}$$

The decay of a and b can be reformulated in terms of smoothness estimates for their Fourier transforms $\hat{a}$ and $\hat{b}$. As we pointed out before, in this case, the argument in the proof of Theorem 4.2.1 amounts to transferring smoothness estimates from $\hat{a}$ to its pointwise inverse $\hat{b}$ by an iterated application of the Leibniz product rule (cf. Equation (4.4)).

The obstacle to derive Equation (4.5) formally from Equation (4.4) is that the latter equation does not determine, by itself, the derivatives of M^{-1}. For example, when a is a finitely supported sequence, Equation (4.6) is a recurrence equation in b, that has many solutions even if $\hat{a}$ has no zeros. The sequence b that we are looking for (that is, the kernel of M^{-1}) can be singled out as the only solution of Equation (4.6) that belongs to ℓ^2.

In the case that M is a convolution operator, the justification we need follows from some careful regularization argument for Sobolev spaces. In our case, this justification was done in Claim 4.2.2, by resorting to Jaffard's result [76], where derivations are implicitly used. Another possible approach would be to use the general theory of unbounded derivations, in particular the results in [15] and [78]. However, this would require adapting those results to non-densely defined derivations.

As observed before, in Theorem 4.2.1, the decay condition on the original matrix M is not shown to be fully shared by the inverse matrix M^{-1} (although the result in [76] shows that the full decay condition is actually preserved). This is due to the kind of objects used to bound the decay of the entries of M and M^{-1}. According to the previous remark, in the case of a convolution operator with symbol τ, the estimates given amount to smoothness estimates for τ. In Claims 4.2.1 and 4.2.3 we bounded the size of the entries of a matrix by means of its $\ell^2 \to \ell^2$ operator norm and controlled that norm by interpolating its $\ell^1 \to \ell^1$ and $\ell^\infty \to \ell^\infty$ norms (by Schur's lemma). This would correspond in the case of a convolution operator to bounding the L^∞ norm of its symbol τ, from above by its $\mathcal{F}(\ell^1)$ norm and from below by its L^2 norm. This accounts, in that case, for the loss of some precision in the estimates.

4.3 Applications to spline-type spaces

Theorem 4.3.1. *Let $V \equiv V^2(F, \Lambda)$ be a spline-type space, where the atoms F satisfy,*

$$|f_k(x)| \leq C\,(1 + |x - k|)^{-s} \qquad (x \in \mathbb{R}^d, k \in \Lambda),$$

for some constant $C > 0$ and $s > 0$. Assume the following.

- *For each $i \in I$, we have a family of measurable functions $\left\{\varphi^i_k\right\}_{k\in\Lambda_i}$ that satisfy the following uniform concentration condition around their nodes Λ_i:*

$$\left|\varphi^i_k(x)\right| \leq C'\,(1 + |x - k|)^{-s} \qquad (x \in \mathbb{R}^d, k \in \Lambda_i), \tag{4.7}$$

for some constant $C' > 0$ (independent of i).

- *The set of nodes* Λ_i *are* uniformly relatively separated. *That is,*

$$\sup_{i\in I} \rho(\Lambda_i) < \infty, \qquad \text{(cf. Equation (1.3))}.$$

- *Each family* $\left\{\varphi_k^i\right\}_k$ *satisfies the (exterior) frame inequality*[1],

$$A\|f\|_2^2 \le \sum_k \left|\left\langle f, \varphi_k^i \right\rangle\right|^2 \le B\|f\|_2^2, \tag{4.8}$$

for $f \in \boldsymbol{V}^2$ *and constants* $0 < A \le B < \infty$ *that are independent of i.*

Then, the respective families of canonical dual frame sequences $\left\{\psi_k^i\right\}_k \subseteq \boldsymbol{V}^2$ *satisfy,*

$$\left|\psi_k^i(x)\right| \le D\,(1+|x-k|)^{-s} \qquad (x\in\mathbb{R}^d, k\in\Lambda_i), \tag{4.9}$$

for some constant D, independent of i.

Proof. Let $G \equiv \{g_k\}_k$ be the canonical dual frame of F. By the theory of localized frames (cf. Sections 1.13 and 1.7.1), there exists a constant $C'' > 0$ such that

$$|g_k(x)| \le C''\,(1+|x-k|)^{-s} \qquad (x\in\mathbb{R}^d, k\in\Lambda).$$

For each $i \in I$ and $k \in \Lambda_i$, let $\overline{\varphi}_k^i$ be the orthogonal projection of φ_k^i on $\boldsymbol{V}^2$. Each of the functions has the expansion,

$$\overline{\varphi}_k^i = \sum_{j\in\Lambda} \left\langle \varphi_k^i, g_j \right\rangle f_j.$$

Consequently using Lemmas 1.3.1 and 1.3.2,

$$\begin{aligned}\left|\overline{\varphi}_k^i(x)\right| &\lesssim CC'C'' \sum_{j\in\Lambda} w_{-s}(k-j)w_{-s}(x-j)\\ &\lesssim CC'C''\rho(\Lambda)w_{-s}(x-k).\end{aligned}$$

Since the exterior frame condition in the hypothesis is also satisfied by the functions $\left\{\overline{\varphi}_k^i\right\}_k$, we can replace each φ_k^i by $\overline{\varphi}_k^i$ and assume without loss of generality that $\varphi_k^i \in \boldsymbol{V}^2$.

For each $i \in I$, consider the Gram matrix M^i given by,

$$M_{kj}^i := \left\langle \varphi_k^i, \varphi_j^i \right\rangle \qquad (k, j\in\Lambda_i).$$

By Lemma 1.3.2, it follows that

$$\left|M_{k,j}^i\right| \le K(1+|k-j|)^{-s} \qquad (k, j\in\Lambda_i),$$

[1] Note that the functions φ_k^i need not to belong to $\boldsymbol{V}^2$.

for some constant K that depends on s and C'. Moreover, since each $\left\{\varphi_k^i\right\}_k$ is a frame with bounds A and B, the spectrum of M^i satisfies,

$$\sigma(M^i) \subseteq \{0\} \cup [A, B].$$

By Lemma 4.1.2, the pseudo-inverse of M^i satisfies

$$\left|(M^i)^\dagger_{k,j}\right| \leq K'(1 + |k - j|)^{-s} \qquad (k, j \in \Lambda_i),$$

for some constant K' independent of i.

Each of the dual elements ψ_k^i is given by,

$$\psi_k^i = \sum_{j \in \Lambda_i} (M^i)^\dagger_{k,j} \varphi_j^i, \qquad \text{(cf. Equation (1.33))}.$$

Therefore,

$$\left|\psi_k^i(x)\right| \leq CK' \sum_{j \in \Lambda_i} w_{-s}(k - j) w_{-s}(j - x).$$

Using Lemma 1.3.2 (c) with $\Gamma := \Lambda_i - \{x\}$ and $k' := k - x$, it follows that

$$\left|\psi_k^i(x)\right| \leq K'' \rho(\Gamma) w_{-s}(x - k) = K'' \rho(\Lambda_i) w_{-s}(x - k).$$

For some constant that K'' independent of i. Since the sets of nodes are uniformly relatively separated, the conclusion follows. □

Similarly, the estimates in Section 4.2 yield the following result. For simplicity, we only illustrate the case of isotropic decay.

Theorem 4.3.2. *Let $\boldsymbol{V} \equiv \boldsymbol{V}^2(F, \mathbb{Z}^d)$ be a spline-type space, where the atoms F form a Riesz sequence within $L^2(\mathbb{R}^d)$ with lower bound A and satisfy,*

$$|f_k(x)| \leq C\,(1 + |x - k|)^{-s}\,, \qquad (x \in \mathbb{R}^d),$$

for some constants $C > 0$ and $s > 2d + 1$. Let t be an integer such that $d < t < s - d$. Then, the dual system $G \equiv \{g_k\}_k \subseteq \boldsymbol{V}^2$ satisfies

$$|g_k(x)| \leq C'\,(1 + |x - k|)^{-t}\,, \qquad (x \in \mathbb{R}^d).$$

where,

$$C' = \frac{E^{t^2} C^{2t+1}}{A^{t+1}} \left(1 + \frac{1}{s - t - d}\right)^t,$$

for some constant $E > 0$ that only depends on the dimension d.

Proof. The proof is similar to that of Theorem 4.3.1, this time using Theorem 4.2.1 instead of Theorem 4.1.1. □

Chapter 5

Frame surgery

In this chapter prove a locality principle for spline-type spaces in the form of a *surgery scheme* for well-localized frames. Then, using spline-type spaces as models for the range of certain wavelet transforms we obtain consequences for various kinds of atomic decompositions. Our main result asserts that, given a family of frames for a spline-type space, it is possible to construct a new frame for the same space by piecing together arbitrary portions of the original frames, provided that the overlaps between these portions are large enough. Although the result we prove is qualitative, special emphasis is made on how the qualities of the ingredients affect the surgery procedure and what kind of uniformity is to be expected. This is one reason why we work on the Euclidean space and not on a general locally-compact group (although much of the elements involved in the construction have a counterpart in the abstract setting). The other - more important - reason is that we make use of localization theory (cf. Section 1.13) and the results for matrix algebras on which that theory relies (cf. Section 1.7) are not available for arbitrary groups.

For the applications we consider mainly two transforms. The first is the Short Time Fourier Transform (STFT) with a fixed (good) window. This transform maps modulation spaces into spline-type spaces - considered in the general sense - and then yields an application of the surgery scheme to Gabor frames. These results imply a general existence condition for the recently introduced concept of *quilted Gabor frame* (see [31, 32]). Since the STFT does not exactly map time-frequency shifts into translations - there is an extra phase factor or twist on the STFT side - we see that shift-invariant spaces are not a sufficient model for the range of the transform: we must us general spline-type spaces. As a by-product of this general treatment, the result we obtain holds not only for pure time-frequency shifts but also for Gabor molecules concentrated around a general set of nodes.

The second transform we consider is the Kohn-Nirenberg map, which - as shown in [42] - establishes a correspondence between the class of Gabor multipliers (related to different Gabor frames) and the class of (shift-invariant) spline-type spaces (see also [47, Chapter 5]). Gabor multipliers are operators that arise from applying a mask to the coefficients associated with a Gabor frame expansion; hence each of these operators has the form

$$T = \sum_{\lambda \in \Lambda} c_\lambda P_\lambda,$$

where $c_\lambda \in \mathbb{C}$ and P_λ is a rank-one operator (essentially a projector onto the subspace generated by a time-frequency atom). Each operator in a given class of Gabor multipliers can be identified by its associated *lower symbol* which consists of the Hilbert-Schmidt inner products $\{ \langle T, P_\lambda \rangle \,\big|\, \lambda \in \Lambda \}$. Combining the surgery scheme with the KN map and known tools for shift-invariant spaces we get a sufficient condition to identify a class of Gabor multipliers by a *mixed lower symbol* constructed by using different types of rank-one operators P_λ for λ in different regions of the time-frequency plane.

Finally, we give an application to irregular sampling. Given a family of sampling sets for which a sampling inequality is known, we can construct new sets for which the sampling inequality still holds. Moreover, given explicit reconstruction formulas for the original sets, we get an approximate reconstruction formula for the new sets.

The construction in this Chapter motivated a great part of the previous study of spline-type spaces and amalgam norms. For example, to identify modulation spaces with certain spline-type spaces we needed a sufficiently general treatment of spline-type spaces, allowing for general spatial molecules. Moreover, in the proposed applications to Gabor frames, instead of the usual convolution inequalities for Wiener amalgams we would need *twisted* convolution inequalities. These are covered by the "multiplier" estimates for Schur-type amalgam families in Section 2.2. Secondly, the surgery scheme requires specific information on the dual atoms of the frames being glued. In practice, this would greatly compromise the applicability of the result. This problem motivated the study carried out in Chapter 4.

5.1 Frame surgery for spline-type spaces

In this section we consider the following locality problem. We are given a spline type space $\boldsymbol{V}$ and several exterior frames $\left\{\varphi^i_k\right\}_{k \in \Lambda_i}$, $i \in I$, for it. For each of these frames, we arbitrarily select a region of the Euclidean space E_i where we want to use it. The family $\{E_i\}_i$ must form a covering of $\mathbb{R}^d$. We argue that, if for each $i \in I$ we pick from the frame $\left\{\varphi^i_k\right\}_{k \in \Lambda_i}$ those elements that are concentrated near E_i, then the resulting family $\left\{\varphi^i_k\right\}_{k \in \Delta_i, i \in I}$ forms an exterior frame for $\boldsymbol{V}$. Moreover, given (possibly non-canonical) dual frames for each of the original exterior frames, we provide an approximate reconstruction operator for the new exterior frame.

Since we are not dealing with frames for the whole space $L^2(\mathbb{R}^d)$, we cannot take a function f, break it into pieces f_i supported on E_i, expand each f_i using the exterior frame $\left\{\varphi^i_k\right\}_k$, and then add all those expansions. This approach does not work in our context because for $f \in \boldsymbol{V}$, the localized pieces f_i do not belong to $\boldsymbol{V}$ and consequently cannot be expanded using the frame $\left\{\varphi^i_k\right\}_k$.

Instead, we argue that for each member of the covering E_i, the norm of a function $f \in \boldsymbol{V}$ restricted to E_i should depend mainly on the atoms concentrated around E_i. Then we glue these local estimates together by means of an almost-orthogonality principle, which is implicit in the computations below.

To be able to quantify the approximation scheme we will work with frames that are polynomially localized in space.

5.1.1 The approximation scheme

We now give the precise assumptions for this section.

- We assume that $\boldsymbol{V} = \boldsymbol{V}(F, \Lambda)$ is a spline-type space where the atoms $F \equiv \{f_k\}_{k\in\Lambda}$ and a given system of dual atoms $G \equiv \{g_k\}_{k\in\Lambda}$ satisfy,
$$|f_k(x)|, |g_k(x)| \leq C\,(1 + |x - k|)^{-(s+\alpha)} \qquad (x \in \mathbb{R}^d, k \in \Lambda), \tag{5.1}$$
for some constants $C > 0$, $s > d$ and $\alpha \geq 0$. It is well-known [63] that if this condition holds for the atoms F, then it is automatically satisfied by some system of dual atoms G (see Sections 1.7.1 and 1.13).

- We are given a family of frame pairs for $\boldsymbol{V}^2$. [1]
$$\left(\left\{\psi_k^i\right\}_{k\in\Lambda_i}, \left\{\varphi_k^i\right\}_{k\in\Lambda_i}\right) \qquad (i \in I),$$
that satisfy the following uniform concentration condition around their nodes Λ_i,
$$\left|\varphi_k^i(x)\right|, \left|\psi_k^i(x)\right| \leq C\,(1 + |x - k|)^{-(s+\alpha)} \qquad (x \in \mathbb{R}^d, k \in \Lambda_i, i \in I), \tag{5.2}$$
for some constant $C > 0$, that, for simplicity, is assumed to be equal to the constant in (5.1).

  Observe that we are requiring all the frames *and the dual frames* to be uniformly localized. Given a concrete family of uniformly localized (exterior) frames, it can be difficult to decide if they possess a corresponding family of dual frames sharing a common spatial localization. This was the motivation for Chapter 4.

- We have a (measurable) covering of $\mathbb{R}^d$, $\mathcal{E} \equiv \{E_i\}_{i\in I}$ that is *uniformly locally finite*. This means that for some (or any) cube Q,
$$\#_{\mathcal{E},Q} := \max_{x\in\mathbb{R}^d} \#\{i \in I/(Q + x) \cap E_i \neq \emptyset\} < \infty. \tag{5.3}$$
Observe that this assumption in particular implies that number of overlaps of $\mathcal{E}$ is finite. That is,
$$\#_{\mathcal{E}} := \max_{x\in\mathbb{R}^d} \#\{i \in I/x \in E_i\} < \infty. \tag{5.4}$$

[1] Remember that the analyzing atoms $\left\{\varphi_k^i\right\}_k$ need not belong to $\boldsymbol{V}^2$.

- We suppose that the set of nodes Λ_i are *uniformly relatively separated*. That is,

$$\sup_{i\in I}\rho(\Lambda_i) = \sup_{i\in I}\max_{x\in\mathbb{R}^d}\#(\Lambda_i \cap ([-1/2, 1/2]^d + x)) < +\infty. \tag{5.5}$$

  Observe that this assumption, together with the uniform localization of the dual frames, implies that the original frames have a uniform common lower bound.

We now prove the central approximation result. In Section 5.1.2 we apply this result to the construction of new frames.

Theorem 5.1.1. *Let $\{\eta_i\}_{i\in I}$ be a (measurable) partition of unity subordinated to $\mathcal{E}$. That is, each η_i is nonnegative, $\sum_i \eta_i \equiv 1$ and $\operatorname{supp}(\eta_i) \subseteq E_i$.*

For each $r > 0$ consider the sets

$$\Lambda_i^r := \{k \in \Lambda_i / d(k, E_i) \le r\}$$

and choose any set Δ_i^r such that $\Lambda_i^r \subseteq \Delta_i^r \subseteq \Lambda_i$. Consider also, the following approximate reconstruction operator*,*

$$A^r(f) := \sum_{i\in I}\sum_{k\in\Delta_i^r}\left\langle f, \varphi_k^i\right\rangle \psi_k^i \eta_i.$$

Then, for any $1 \le p \le \infty$, any w_α-moderated weight v and any $f \in \boldsymbol{V}_v^p$,

$$\|A^r(f) - f\|_{L_v^p} \le K\#_{\mathcal{E}}\|f\|_{L_v^p} r^{-(s-d)},$$

where $K > 0$ is a constant that only depends on d, C, s, α, the set of nodes Λ, the common relative separation of all the sets of nodes Λ_i and the constant in Equation (1.8) using the weight w_α as moderator.

Remark 5.1.1. *The fact that the covering is uniformly locally finite is not used in the proof of the theorem; only the weaker condition in Equation (5.4) is needed. However, the stronger assumption of Equation (5.3) is required for the applications.*

Proof. Observe first that we can always add more nodes to the set Λ and extend the set of atoms F and dual atoms G by associating 0 to the new nodes. All the assumptions on the atoms are preserved by this extension, but the relative separation of the set of nodes changes. By adding to Λ any fixed relatively separated and relatively dense set Γ, we can assume that Λ is L-dense, for some $L > 0$ (cf. Section 1.2.1). The relative separation of the resulting set can be bounded by $\rho(\Lambda) + \rho(\Gamma)$.

For all $i \in I$, every $f \in \boldsymbol{V}^{00}$ admits the expansion,

$$f = \sum_{j\in\Lambda_i}\left\langle f, \varphi_j^i\right\rangle \psi_j^i.$$

Averaging all these expansions yields,

$$f = \sum_{i\in I}\sum_{j\in\Lambda_i}\left\langle f,\varphi_j^i\right\rangle \psi_j^i\eta_i.$$

Since f also admits the expansion $f = \sum_k \langle f, g_k\rangle f_k$, it follows that

$$f = \sum_{k\in\Lambda}\langle f, g_k\rangle \sum_{i\in I}\sum_{j\in\Lambda_i}\left\langle f_k,\varphi_j^i\right\rangle \psi_j^i\eta_i.$$

Similarly,

$$\begin{aligned} A^r(f) &= \sum_{i\in I}\sum_{j\in\Delta_i^r}\left\langle f,\varphi_j^i\right\rangle \psi_j^i\eta_i \\ &= \sum_{k\in\Lambda}\langle f, g_k\rangle \sum_{i\in I}\sum_{j\in\Delta_i^r}\left\langle f_k,\varphi_j^i\right\rangle \psi_j^i\eta_i. \end{aligned}$$

Therefore, for $f \in \boldsymbol{V}^{00}$,

$$f - A^r(f) = \sum_{k\in\Lambda}\langle f, g_k\rangle \sum_{i\in I}\sum_{j\notin\Delta_i^r}\left\langle f_k,\varphi_j^i\right\rangle \psi_j^i\eta_i.$$

Consequently, if we set $c_k := \langle f, g_k\rangle$, by Lemma 1.3.1,

$$\begin{aligned} |f - A^r(f)| &\lesssim \sum_{k\in\Lambda}|c_k|\sum_{i\in I}\sum_{j\notin\Delta_i}\left\langle w_{-(s+\alpha)}(\cdot - k), w_{-(s+\alpha)}(\cdot - j)\right\rangle w_{-(s+\alpha)}(\cdot - j)\chi_{E_i} \\ &\lesssim \sum_{k\in\Lambda}|c_k|\sum_{i\in I}\sum_{j\notin\Lambda_i^r} w_{-(s+\alpha)}(k - j)w_{-(s+\alpha)}(\cdot - j)\chi_{E_i}. \end{aligned}$$

If we define,

$$E_k^r := \sum_{i\in I}\sum_{j\notin\Lambda_i^r} w_{-(s+\alpha)}(k - j)w_{-(s+\alpha)}(\cdot - j)\chi_{E_i},$$

Lemma 2.3.1 implies that,

$$\|f - A^r(f)\|_{L^p_v} \lesssim \|c\|_{\ell^p_v}\left(\sup_k\|E_k^r w_\alpha(\cdot - k)\|_1\right)^{1/p}\left(\operatorname*{supess}_{x\in\mathbb{R}^d}\sum_k\left|E_k^r(x)w_\alpha(x - k)\right|\right)^{1/p'}.$$

Since (F, G) is a frame pair for $\boldsymbol{V}^p_v$, $\|c\|_{\ell^p_v} \leq K\|f\|_{L^p_v}$, for some constant K that only depends on d, C, s, α, the constant in Equation (1.8) (taking $w = w_\alpha$) and Λ. Consequently,

$$\|f - A^r(f)\|_{L^p_v} \leq K\|f\|_{L^p_v}\left(\sup_k\|E_k^r w_\alpha(\cdot - k)\|_1\right)^{1/p}\left(\operatorname*{supess}_{x\in\mathbb{R}^d}\sum_k\left|E_k^r(x)w_\alpha(x - k)\right|\right)^{1/p'}. \tag{5.6}$$

Now observe that, since $w_\alpha(x-k) \le w_\alpha(x-j)w_\alpha(k-j)$,

$$\left|E_k^r(x)w_\alpha(x-k)\right| \le \sum_{i\in I}\sum_{j\notin\Lambda_i^r} w_{-s}(k-j)w_{-s}(x-j)\chi_{E_i}(x).$$

For every $i \in I$, since Λ is now assumed to be L-dense, there exists a map $\mu_i : \Lambda_i \to \Lambda$ such that $|k-\mu_i(k)| \le L$, for all $k \in \Lambda_i$. This map will be used to reduce the proof to the case where all the index sets are equal. This same argument was used in [5], where irregularly distributed phase-space points are assigned a near point in a regular reference system by means of a 'round-up' map.

Since the sets Λ_i are assumed to be uniformly relatively separated, there exists a number $N \in \mathbb{N}$, that depends only on L and the relative separation of all the sets of nodes, such that

$$\#\mu_i^{-1}(\{j\}) \le N, \text{ for every } j \in \Lambda.$$

Suppose initially that $r > 2L$, define $R := r - L$ and estimate,

$$\left|E_k^r(x)w_\alpha(x-k)\right| \le \sum_{i\in I}\sum_{j\in\Lambda}\sum_{\substack{l\in\mu_i^{-1}(j),\\ l\notin\Lambda_i^r}} w_{-s}(k-l)w_{-s}(x-l)\chi_{E_i}(x).$$

If $l \in \mu_i^{-1}(j)$, then $|j-l| \le L$, so $w_{-s}(k-l) \lesssim w_{-s}(k-j)$ and $w_{-s}(x-l) \lesssim w_{-s}(x-j)$. (Here the implicit constants depend on L and s). If in addition $l \notin \Lambda_i^r$, then $j \notin \Omega_i^R$, where

$$\Omega_i^R := \{\, k \in \Lambda \,\big|\, d(k, E_i) \le R \,\}.$$

Consequently,

$$\left|E_k^r(x)w_\alpha(x-k)\right| \lesssim N\sum_{i\in I}\sum_{j\notin\Omega_i^R} w_{-s}(k-j)w_{-s}(x-j)\chi_{E_i}(x). \tag{5.7}$$

Using this estimate, we bound the weighted Schur norm of the kernel E^r.

For every $k \in \Lambda$,

$$\begin{aligned}
\|E_k^r w_\alpha(\cdot-k)\|_1 &\lesssim \sum_{i\in I}\sum_{j\notin\Omega_i^R} w_{-s}(j-k)\int_{E_i} w_{-s}(x-j)dx \\
&= \sum_{j\in\Lambda}\sum_{\substack{i\in I\\ d(j,E_i)>R}} w_{-s}(j-k)\int_{E_i} w_{-s}(x-j)dx \\
&\le \#_{\mathcal{E}}\sum_{j\in\Lambda} w_{-s}(j-k)\int_{\bigcup E_i} w_{-s}(x-j)dx,
\end{aligned}$$

where the union in the last integral ranges over all $i \in I$ such that $d(j, E_i) > R$. Since the complement of the cube $Q_R(j)$ contains that union, we get,

$$\begin{aligned}\|E_k^r w_\alpha(\cdot - k)\|_1 &\le \#_\mathcal{E} \sum_{j\in\Lambda} w_{-s}(j-k) \int_{\mathbb{R}^d \setminus Q_R(j)} w_{-s}(x-j)dx \\ &= \#_\mathcal{E} \sum_{j\in\Lambda} w_{-s}(j-k) \int_{\mathbb{R}^d \setminus Q_R(0)} w_{-s}(x)dx.\end{aligned}$$

The set $\Lambda - k$ has the same relative separation that Λ, so Lemma 1.3.2 implies that

$$\begin{aligned}\|E_k^r w_\alpha(\cdot - k)\|_1 &\lesssim \#_\mathcal{E} \int_{\mathbb{R}^d \setminus Q_R(0)} w_{-s}(x)dx \\ &\lesssim \#_\mathcal{E} R^{-(s-d)}.\end{aligned}$$

Since $r > 2L$, it follows that $R > r/2$ and

$$\sup_k \|E_k^r w_\alpha(\cdot - k)\|_1 \lesssim \#_\mathcal{E} r^{-(s-d)}. \tag{5.8}$$

Using again the estimate in Equation (5.7), we now bound $\operatorname{supess}_x \sum_k \left|E_k^r(x) w_\alpha(x-k)\right|$.

Fix $x \in \mathbb{R}^d$ and let

$$I_x := \{\, i \in I \,\big|\, x \in E_i \,\}.$$

From Equation (5.4) we know that $\# I_x \le \#_\mathcal{E}$. We now estimate,

$$\sum_{k\in\Lambda} \left|E_k^r(x) w_\alpha(x-k)\right| \lesssim \sum_{i\in I_x} \sum_{j\notin\Omega_i^R} \sum_{k\in\Lambda} w_{-s}(k-j) w_{-s}(x-j).$$

Since Λ and $\Lambda - \{j\}$ have the same relative separation, Lemma 1.3.2 implies that,

$$\sum_{k\in\Lambda} w_{-s}(k-j) \lesssim 1,$$

so,

$$\sum_{k\in\Lambda} \left|E_k^r(x) w_\alpha(x-k)\right| \lesssim \sum_{i\in I_x} \sum_{j\notin\Omega_i^R} w_{-s}(x-j).$$

For $i \in I_x$ and $j \notin \Omega_i^R$, we have that $|x-j| \ge d(j, E_i) > R$. It follows that,

$$\sum_{k\in\Lambda} \left|E_k^r(x) w_\alpha(x-k)\right| \le \sum_{i\in I_x} \sum_{j:|x-j|>R} w_{-s}(x-j).$$

Since the sets Γ and $x - \Gamma$ have the same relative separation, Lemma 1.3.2 implies that,

$$\sum_{k \in \Lambda} \left| E^r_k(x) w_\alpha(x-k) \right| \lesssim \#_{\mathcal{E}} R^{-(s-d)}.$$

Using again the fact that $r > 2L$, it follows that,

$$\operatorname*{supess}_{x \in \mathbb{R}^d} \sum_{k \in \Lambda} \left| E^r_k(x) w_\alpha(x-k) \right| \lesssim \#_{\mathcal{E}} r^{-(s-d)}. \tag{5.9}$$

Combining the estimates in Equations (5.8), (5.9) and (5.6), it follows that

$$\|A^r(f) - f\|_{L^p_v} \lesssim \|f\|_{L^p_v} \#_{\mathcal{E}} r^{-(s-d)},$$

for $r > 2L$.

It remains to show that a similar estimate holds for $0 < r \leq 2L$. In this case, $r^{-(s-d)} \gtrsim 1$. So, it suffices to observe that $\|A^r\|_{V^p_v \to L^p_v} \lesssim \#_{\mathcal{E}}$, uniformly on r. Reexamining the estimates given for the error kernel E^r, the desired conclusion follows. □

Remark 5.1.2. *The technique in the proof of the theorem of using the frame expansion twice is somehow analogous to the use of reproducing formulas in the classical decomposition results for function spaces (see for example [57] and [44]).*

The formula defining the operator A^r makes sense in L^p_v, but the bound given in the theorem is only valid in the smaller subspace $\boldsymbol{V}^p_v$, where the "reproducing formula" (the frame expansion) is valid. By means of it, the task of bounding the operator is reduced in the proof to the one of controlling its behavior on atoms, much in the spirit of the classical atomic decompositions (see [57] and also [68]).

5.1.2 Constructing new frames

We now interpret the approximation result of Section 5.1.1 as a method to produce new frames. Observe that, however, for some applications, the estimate provided by Theorem 5.1.1 is all that is needed. If concrete atoms and dual atoms are known, then the estimate in the theorem provides an approximate reconstruction operator for the new system of atoms.

Consider again the ingredients of Section 5.1.1 and let $\{\eta_i\}_{i \in I}$ be a (measurable) partition of unity subordinated to $\mathcal{E}$ (e.g. $\eta_i = (\sum_j \chi_{E_j})^{-1} \chi_{E_i}$). Let v be a w_α-moderated weight and let $P : L^p_v \to \boldsymbol{V}^p_v$ be the universal projector onto $\boldsymbol{V}^p_v$ (cf. Theorem 3.3.2).

Fix a value of $r > 0$ and consider the operator $B^r : \boldsymbol{V}^p_v \to \boldsymbol{V}^p_v$ given by $B^r := P \circ A^r$, where

$$A^r(f) := \sum_{i \in I} \sum_{k \in \Lambda^r_i} \left\langle f, \varphi^i_k \right\rangle \psi^i_k \eta_i,$$

and, as before, $\Lambda^r_i := \{k \in \Lambda_i / d(k, E_i) \leq r\}$.

For each $i \in I$, let $(\boldsymbol{V}^p_\nu)_i$ be the L^p_ν-closed linear space generated by the atoms $\left\{\psi^i_k \eta_i\right\}_{k \in \Lambda^r_i}$. These spaces, of course, depend on r.

Consider the direct sum $\oplus_i (\boldsymbol{V}^p_\nu)_i$ as a subspace of $\ell^p_{L^p_\nu}$, the space of L^p_ν-valued ℓ^p families; more precisely, $\oplus_i (\boldsymbol{V}^p_\nu)_i$ is the closure of the algebraic direct sum within $\ell^p_{L^p_\nu}$. Let $\iota : \oplus_i (\boldsymbol{V}^p_\nu)_i \to L^p_\nu$ be the operator given by

$$\iota((f^i)_i) := \sum_i f^i.$$

Since $\mathcal{E}$ is locally finite, ι is well-defined and bounded uniformly on p and ν. Indeed, for $1 \le p < \infty$,

$$\begin{aligned} \|\sum_i f^i\|^p_{L^p_\nu} &= \int_{\mathbb{R}^d} \left|\sum_i f_i(x)\right|^p \nu(x)^p dx \\ &\le \int_{\mathbb{R}^d} (\sum_{i \in I_x} |f_i(x)|)^p \nu(x)^p dx, \end{aligned}$$

where $I_x := \{\, i \in I \,\big|\, x \in E_i \,\}$. Since $\# I_x \le \#_{\mathcal{E}}$,

$$\begin{aligned} \|\iota((f^i)_i)\|^p_{L^p_\nu} &\le \#^p_{\mathcal{E}} \int_{\mathbb{R}^d} \sum_i |f_i(x)|^p \, \nu(x)^p dx \\ &= \#^p_{\mathcal{E}} \sum_i \|f_i\|^p_{L^p_\nu} \\ &= \#^p_{\mathcal{E}} \|(f^i)_i)\|^p_{\ell^p_{L^p_\nu}}. \end{aligned}$$

So, $\|\sum_i f^i\|_{L^p_\nu} \le \#_{\mathcal{E}} \|(f^i)_i)\|_{\ell^p_{L^p_\nu}}$. For $p = \infty$, a similar computation establishes the same estimate. Composing ι with the projector P, we obtain a *synthesis operator* $Sy : \oplus_i (\boldsymbol{V}^p_\nu)_i \to \boldsymbol{V}^p_\nu$.

For each $i \in I$, let $Q_i : \boldsymbol{V}^p_\nu \to (\boldsymbol{V}^p_\nu)_i$ be given by

$$Q_i(f) := \sum_{k \in \Lambda^r_i} \left\langle f, \varphi^i_k \right\rangle \psi^i_k \eta_i.$$

The concentration conditions on Equation (5.2) imply that all these operators are uniformly bounded. Moreover, they determine a map $Q : \boldsymbol{V}^p_\nu \to \oplus_i (\boldsymbol{V}^p_\nu)_i$, given by $Q(f) := (Q_i(f))_i$. We will prove below that Q is well defined and bounded. Assuming this for the moment, we have a commutative diagram,

$$\begin{array}{ccc} \boldsymbol{V}^p_\nu & \xrightarrow{Q} & \bigoplus_i (\boldsymbol{V}^p_\nu)_i \\ & {\scriptstyle B^r} \searrow & \downarrow {\scriptstyle Sy} \\ & & \boldsymbol{V}^p_\nu \end{array} \tag{5.10}$$

It follows from Theorem 5.1.1 that for a sufficiently large value of $r > 0$, B^r is invertible and consequently Q is left-invertible and Sy is right-invertible. This provides two ways of viewing $\boldsymbol{V}^p_v$ as a retract of $\oplus_i (\boldsymbol{V}^p_v)_i$. One is Q (with retraction $(B^r)^{-1} S y$) and the other is $Q(B^r)^{-1}$ (with retraction Sy). In the spirit of [91] and [16], this can be called an *exterior Banach fusion frame* or an *exterior stable splitting* (see also [43] and [39]).

Now observe that each of the maps Q_i can be factored through z^p_v [2],

$$\begin{array}{ccc} \boldsymbol{V}^p_v & \xrightarrow{Q_i} & (\boldsymbol{V}^p_v)_i \\ & {\searrow}^{C_i} & \uparrow R_i \\ & & z^p_v(\Lambda^r_i) \end{array}$$

where $C_i(f) := \left(\left\langle f, \varphi^i_k \right\rangle\right)_k$ and $R_i(c) := \sum_k c_k \psi^i_k \eta_i$.

This induces a commutative diagram,

$$\begin{array}{ccc} \boldsymbol{V}^p_v & \xrightarrow{Q} & \bigoplus_i (\boldsymbol{V}^p_v)_i \\ & {\searrow}^{C} & \uparrow R \\ & & \bigoplus_i z^p_v(\Lambda^r_i) \end{array}$$

where in $\bigoplus_i z^p_v(\Lambda^r_i)$ we use the p-norm; that is $\|(c^i)_i\| := \|(\|c^i\|_{\ell^p_v})_i\|_{\ell^p}$. This is just a weighted ℓ^p norm; this way of presenting it is due to the structure of the index sets. The boundedness of the operators C and R is proved in Theorem 5.1.2 below. Assuming this fact for the moment, observe that if B^r is invertible, then Q is left-invertible and so is C. We formalize this in the following theorem.

Theorem 5.1.2. *Suppose that the assumptions of Section 5.1.1 are satisfied. Let v be a w_α-moderated weight. Then for all sufficiently large values of $r > 0$,*

$$\left\{\varphi^i_k : i \in I, k \in \Lambda^r_i\right\}$$

is a Banach frame for $\boldsymbol{V}^p_v$.

More precisely, if we define the index set $\Gamma := \bigcup_{i\in I} \Lambda^r_i \times \{i\}$ and the weight $V(k,i) := v(k)$, then the analysis map

$$\begin{aligned} \boldsymbol{V}^p_v &\to z^p_V(\Gamma) \\ f &\mapsto \left(\left\langle f, \varphi^i_k \right\rangle\right)_{(k,i)} \end{aligned}$$

is bounded and left-invertible, for all sufficiently large values of $r > 0$.

Moreover, the value of r may be chosen uniformly for all $1 \leq p \leq \infty$ and every class of w_α-moderated weights for which the respective constant (cf. Equation (1.8)) is uniformly bounded.

[2] Recall that the space z^p_v is ℓ^p_v when $p < +\infty$ and c^0_v for $p = +\infty$.

Remark 5.1.3. *Observe that although we are constructing a new frame* $\left\{\varphi_k^i\right\}_{i\in I,k\in\Lambda_i^r}$ *out of the pieces* $\left\{\varphi_k^i\right\}_{k\in\Lambda_i^r}$*, we do not claim that each of these pieces forms a frame sequence. This construction should be compared to the methods in [2], [52] and [16] where a global frame is built* from *local (possibly exterior) frames for certain subspaces.*

Remark 5.1.4. *As a related result, we mention Lemma 4.7 in [97] where it is shown that if* $\{\, 2^{k/2}\psi(2^k\cdot - j) \,|\, k, j \in \mathbb{Z} \,\}$ *is a wavelet frame for* $L^2(\mathbb{R})$ *and the wavelet* ψ *satisfies a mild smoothness condition, then for all sufficiently large values of* $r > 0$*, the system of fine scales* $\{\, 2^k\psi(2^k\cdot + j) \,|\, k \in \mathbb{Z}, j \geq 0 \,\}$ *forms an exterior frame for the subspace* $H_r := \{\, f \in L^2(\mathbb{R}) \,|\, \hat{f} \equiv 0$ *on* $[-r, r] \,\}$.

Proof. Using Theorem 5.1.1, choose a value of $r > 0$ such that the operator B^r is invertible. By the discussion above, it only remains to bound the operators C and R. Consider the index set Γ as a set with multiplicity, where we map each of the sets $\Lambda_i^r \times \{i\}$ into $\mathbb{R}^d$ by discarding the second coordinate.

The fact that the sets Λ_i are uniformly relatively separated (cf. Equation (5.5)) and that the covering $\mathcal{E}$ is uniformly locally finite (cf. Equation (5.3)) implies that Γ is relatively separated. Indeed, let Q be the unit cube and $Q' := Q + [-r, r]^d$. For any $x \in \mathbb{R}^d$,

$$\begin{aligned}\{\, (k, i) \in \Gamma \,|\, k \in Q + \{x\} \,\} &= \bigcup_{i\in I} \{\, k \in \Lambda_i \cap (Q + \{x\}) \,|\, d(k, E_i) \leq r \,\} \times \{i\} \\ &\subseteq \bigcup_{i\in I_x} (\Lambda_i \cap (Q + \{x\})) \times \{i\},\end{aligned}$$

where $I_x := \{\, i \in I \,|\, E_i \cap (Q' + \{x\}) \neq \emptyset \,\}$. Hence $\rho(\Gamma) \leq \#_{\mathcal{E},Q'} \sup_i \rho(\Lambda_i) < \infty$ (cf. Equation (5.3)).

The family of atoms $\left\{\varphi_k^i\right\}_{(k,i)\in\Gamma}$ satisfies a polynomial concentration condition. By Example 2.2.1, the family has finite $W(L^\infty, L^1_{w_\alpha})$ norm. The boundedness of the operator C now follows from Propositions 2.3.4.

For the boundedness of R, observe that the families $\left\{\psi_k^i\eta_i\right\}_{k\in\Lambda_i^r}$ satisfy a uniform polynomial concentration condition and their nodes are uniformly relatively separated. Hence, by Example 2.2.1,

$$M := \sup_i \|\left\{\psi_k^i\eta_i\right\}_k\|_{W(L^\infty, L^1_{w_\alpha})} < \infty.$$

Consequently, by Proposition 2.3.1, for $1 \leq p < \infty$,

$$\begin{aligned}\|R(c)\|^p_{\ell^p_{L^p_v}} &\leq \sum_i \|\sum_{k\in\Lambda_i^r} c_k^i \psi_k^i \eta_i\|^p_{L^p_v} \\ &\lesssim M^p \sum_i \|c^i\|^p_{\ell^p_v} \\ &= M^p \|c\|^p_{\ell^p_V}.\end{aligned}$$

So, $\|R(c)\|_{\ell^p_{L^p_v}} \lesssim \|c\|_{\ell^p_V}$. A similar computation shows that the same estimate is valid for $p = \infty$. □

5.1.3 Application to spline-type spaces

We now combine the results of Section 5.1.2 and Chapter 4 in a concrete statement.

Theorem 5.1.3. *Let $\boldsymbol{V} = \boldsymbol{V}(F, \Lambda)$ be a spline-type space. Assume the following.*

- *The atoms F satisfy the polynomial concentration condition around their nodes,*

$$|f_k(x)| \leq C\,(1 + |x-k|)^{-(s+\alpha)} \qquad (x \in \mathbb{R}^d, k \in \Lambda), \tag{5.11}$$

  for some constants $C > 0$, $s > d$ and $\alpha \geq 0$.

- *We are given a family of exterior frames for $\boldsymbol{V}^2$, $\left\{\varphi^i_k\right\}_{k\in\Lambda_i}$, $i \in I$, that satisfy the following uniform polynomial concentration condition around their nodes Λ_i,*

$$\left|\varphi^i_k(x)\right| \leq C'\,(1 + |x-k|)^{-(s+\alpha)} \qquad (x \in \mathbb{R}^d, k \in \Lambda_i, i \in I), \tag{5.12}$$

  for some constant $C' > 0$.

- *The exterior frames $\left(\left\{\varphi^i_k\right\}_k\right)_{i\in I}$ share a uniform lower (and upper) bound. That is,*

$$A\|f\|_2^2 \leq \sum_k \left|\left\langle f, \varphi^i_k \right\rangle\right|^2 \leq B\|f\|_2^2 \qquad (f \in \boldsymbol{V}^2), \tag{5.13}$$

  holds for some constants $0 < A \leq B < \infty$. [3]

- *The sets of nodes Λ_i are* uniformly relatively separated *(cf. Equation (5.5)).*

- *We have a measurable covering of $\mathbb{R}^d$, $\mathcal{E} \equiv \{E_i\}_{i\in I}$ that is* uniformly locally finite *(cf. Equation (5.3)).*

Then, for all sufficiently large values of $r > 0$,

$$\left\{\varphi^i_k : i \in I, d(k, E_i) \leq r\right\}$$

is an exterior Banach frame for $\boldsymbol{V}^p_v$.

More precisely, if we define the index set $\Gamma^r := \bigcup_{i\in I} \Lambda^r_i \times \{i\}$ and the weight $V(k,i) := v(k)$, then the analysis map

$$\begin{aligned} \boldsymbol{V}^p_v &\to z^p_V(\Gamma^r) \\ f &\mapsto \left(\left\langle f, \varphi^i_k \right\rangle\right)_{(k,i)} \end{aligned}$$

[3]Observe that Equation (5.12) already implies the existence of a uniform upper bound B.

is bounded and left-invertible.

Moreover, the value of r may be chosen uniformly for all $1 \leq p \leq \infty$ *and every class of* w_α*-moderated weights for which the respective constants (cf. Equation (1.8)) are uniformly bounded.*

Proof. Combine Theorems 5.1.2 and 4.3.1. □

5.2 Applications

5.2.1 Shift invariant spaces

As a corollary of Theorem 5.1.3 we describe a method to piece together bases of lattice translates. First we recall some notation and facts for shift-invariant spaces (see Section 1.14). Given a lattice $\Lambda \subseteq \mathbb{R}^d$ and $f, g \in L^2(\mathbb{R}^d)$, the bracket product is defined by,

$$[f, g]_\Lambda(x) := \sum_{\lambda^\perp \in \Lambda^\perp} \hat{f}(x + \lambda^\perp)\overline{\hat{g}(x + \lambda^\perp)} \qquad (x \in \mathbb{R}^d).$$

The Λ translates of a finite set of functions $\{f_1, \dots, f_N\}$ form a Riesz sequence in $L^2(\mathbb{R}^d)$ if an only if the matrix of functions $\hat{G} \equiv \left(\hat{G}_{n,m}\right)_{1 \leq k,j \leq N}$ given by

$$\hat{G}(x)_{n,m} := [f_n, f_m]_\Lambda(x) \qquad (x \in \mathbb{R}^d),$$

is uniformly invertible in the sense that all its eigenvalues are bounded away from 0 and ∞, uniformly on x (up to sets of null measure).

Combining the theory of shift-invariant spaces with Theorem 5.1.3 we get the following.

Theorem 5.2.1. *Let* $\Lambda \subseteq \mathbb{R}^d$ *be a lattice and let* $\boldsymbol{V}^2 = \boldsymbol{V}^2(F, \Lambda \times \{1, \dots, N\})$ *be a finitely-generated shift invariant space where the atoms are given by,*

$$F \equiv \{f_n(\cdot - \lambda) : 1 \leq n \leq N, \lambda \in \Lambda\}.$$

Assume the following.

- *The atoms form a Riesz basis of* $\boldsymbol{V}^2$ *and satisfy the following decay condition,*

$$|f_n(x)| \leq C(1 + |x|)^{-(s+\alpha)} \qquad (1 \leq n \leq N), \tag{5.14}$$

  for some constants $C > 0$, $\alpha \geq 0$ *and* $s > d$.

- *We have a measurable covering of* $\mathbb{R}^d$, $\mathcal{E} \equiv \{E_i\}_{i \in I}$ *that is uniformly locally finite (cf. Equation (5.3)).*

- *We are given a family of measurable functions*

$$\{ g_n^i : \mathbb{R}^d \to \mathbb{C} \,\big|\, i \in i, 1 \leq n \leq N \}$$

satisfying the decay condition,

$$\left|g_n^i(x)\right| \leq C'(1+|x|)^{-(s+\alpha)} \qquad (1 \leq n \leq N), \tag{5.15}$$

for some constant $C' > 0$ (independent of i and n).

- *The matrices of functions $\left(\hat{G}^i_{n,m}\right)_{1\leq n,m\leq N}$ given by*

$$\hat{G}^i(x)_{n,m} := [f_n, g_m^i]_\Lambda(x) \qquad (x \in \mathbb{R}^d/\Lambda^\perp),$$

are uniformly bounded and invertible in the sense that each $\hat{G}^i(x)$ is invertible and

$$\sup_{x,i} \|\hat{G}^i(x)\|, \sup_{x,i} \|\hat{G}^i(x)^{-1}\| < \infty.$$

Then, for all sufficiently large values of $r > 0$, the set

$$\{ g_n^i(\cdot - \lambda) \,\big|\, i \in I, 1 \leq n \leq N, \lambda \in \Lambda, d(\lambda, E_i) \leq r \},$$

is a Banach frame for $\boldsymbol{V}_v^p$, for all $1 \leq p \leq \infty$ and all strictly w_α-moderated weights v. More precisely, if we define the index set

$$\Gamma^r := \{ (i,n,\lambda) \in I \times \{1,\dots,N\} \times \Lambda \,\big|\, d(\lambda, E_i) \leq r \}$$

and the weight $V(i,n,\lambda) := v(\lambda)$ on it, then the analysis map

$$\begin{aligned} \boldsymbol{V}_v^p &\to z_V^p(\Gamma^r) \\ f &\mapsto \left(\left\langle f, g^i_{n,\lambda}\right\rangle\right)_{(i,n,\lambda)} \end{aligned}$$

is bounded and left-invertible.

Remark 5.2.1. *The theorem is stated for bases just for simplicity. Using the tools from [96], [28] and [14] it can be reformulated for frames.*

Proof. Let A and B be the Riesz basis bounds of F. Also let $A' := \sup_{x,i} \|\hat{G}^i(x)^{-1}\|$ and $B' := \sup_{x,i} \|\hat{G}^i(x)\|$. Using the fiberization theory from Section 1.14, for each $x \in \mathbb{R}^d/\Lambda^\perp$, the system $\left\{(\hat{f}_1(x+k))_k, \dots, (\hat{f}_N(x+l))_k\right\}$ is a Riesz basis with bounds A and B for some subspace $\boldsymbol{V}_x^2 \subseteq \ell^2(\Lambda^\perp)$. Since its cross-gramian matrix with the system $\left\{(\hat{g_1^i}(x+k))_k, \dots, (\hat{g_N^i}(x+k))_k\right\}$ is invertible, it follows that this latter system is an exterior Riesz basis for the subspace $\boldsymbol{V}_x^2$ with bounds $B^{-1}A'^{-2}$ and $(B')^2 A^{-1}$. Invoking again the fiberization theory, it follows that the Λ translates of $\left\{g_1^i, \dots, g_N^i\right\}$ are an exterior basis for $\boldsymbol{V}^2$ with bounds $\approx B^{-1}A'^{-2}$ and $(B')^2A^{-1}$ (the implicit constant depends on the volume of the lattice Λ). We can now apply Theorem 5.1.3. ◻

Remark 5.2.2. *In [14] no results for exterior bases or frames are explicitly given. However, it is proved there (and also in [28]) that the orthogonal projector onto a shift-invariant space operates fiberwise, so the desired extension follows. For further results on exterior frames for shift-invariant spaces see [18] and [19].*

5.2.2 Sampling

Applying Theorem 5.1.3 to the reproducing kernels of a (smooth) spline-type space we get the following.

Theorem 5.2.2. *Let $\boldsymbol{V} = \boldsymbol{V}(F, \Lambda)$ be a spline-type space generated by a family of continuous atoms $F \subseteq C^0(\mathbb{R}^d)$ that satisfy,*

$$|f_k(x)| \leq C\,(1 + |x-k|)^{-(s+\alpha)}\,, \quad (x \in \mathbb{R}^d, k \in \Lambda),$$

for some $s > d$, $C > 0$ and $\alpha \geq 0$.

Assume the following.

- *$\mathcal{E} \equiv \{E_i\}_{i \in I}$ is a uniformly locally finite measurable covering of $\mathbb{R}^d$ (cf. Equation (5.3)).*
- *For each $i \in I$, we have a set $X_i \subseteq \mathbb{R}^d$ and this collection of sets is uniformly relatively separated (i.e $\sup_i \rho(X_i) < \infty$).*
- *For each of the sets X_i, the following sampling inequality*

  $$A\|f\|_2^2 \leq \sum_{x \in X_i} |f(x)|^2 \leq B\|f\|_2^2, \tag{5.16}$$

  holds for all $f \in \boldsymbol{V}^2$ and some constants $0 < A \leq B < \infty$ independent of i.

For each $r > 0$, let

$$X^r := \{(i, x) : i \in I, x \in X_i, d(x, E_i) \leq r\}\,.$$

Then, for all sufficiently large $r > 0$, there exists constants $0 < A^r \leq B^r < \infty$ such that the sampling inequality,

$$A^r\|f\|_{L^p_v} \leq \left(\sum_{(i,x) \in X^r} |f(x)|^p\, v(x)^p \right)^{1/p} \leq B^r\|f\|_{L^p_v}, \tag{5.17}$$

holds for all $1 \leq p \leq \infty$ (with the usual adjustment for $p = \infty$), all strictly w_α-moderated weights v, and all $f \in \boldsymbol{V}^p_v$.

Remark 5.2.3. *For any class of w_α-moderated weights for which the respective constants (cf. Equation (1.8)) are uniformly bounded, the conclusion of the theorem still holds.*

Proof. First observe that since $F \subseteq C^0(\mathbb{R}^d)$, Theorem 3.3.1 applies with $\boldsymbol{B} = C^0$ and consequently $\boldsymbol{V}_v^p \subseteq C^0$. The norm equivalence of Theorem 3.3.1 also implies that $\boldsymbol{V}^2$ is a reproducing-kernel Hilbert space. We already know that F has a dual frame $G \equiv \{g_k\}_k$ satisfying a polynomial decay condition,

$$|g_k(x)| \leq C' \left(1 + |x-k|\right)^{-(s+\alpha)},$$

for some constant $C' > 0$. The functional $f \mapsto f(x_0)$ is represented by the function $K_{x_0} \in \boldsymbol{V}^2$ given by

$$K_{x_0} = \sum_{k \in \Lambda} \overline{g_k}(x_0) f_k. \tag{5.18}$$

We will apply Theorem 5.1.3 to the family of frames,

$$\{K_x\}_{x \in X_i} \qquad (i \in I).$$

To this end, observe that Equation (5.16) implies that this family satisfies the condition on Equation (5.13) of Theorem 5.1.3. We only need to check the condition on Equation (5.12) for the family of reproducing kernels.

For $x \in X_i$, using Equation (5.18), we estimate,

$$|K_x| \leq CC' \sum_{k \in \Lambda} w_{-(s+\alpha)}(x-k) w_{-(s+\alpha)}(\cdot - k).$$

Using Lemma 1.3.2 (c) with $\Gamma := \Lambda - \{x\}$, it follows that

$$|K_x| \leq CC' \rho(\Gamma) w_{-(s+\alpha)}(\cdot - x) = K'' \rho(\Lambda) w_{-(s+\alpha)}(\cdot - x).$$

Now we can apply Theorem 5.1.3 to obtain the desired conclusion. □

5.2.3 Gabor molecules

Let us recall some notation and facts about time-frequency analysis and Gabor frames (cf. Section 1.11). Let $\phi : \mathbb{R}^d \to \mathbb{R}$, $\phi(x) := \pi^{-d/4} e^{-\frac{|x|^2}{2}}$ be the Gaussian normalized in L^2. The Short-Time Fourier Transform with respect to ϕ of a test function $f \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$\mathcal{V}_\phi f(x,w) := \langle f, M_w T_x \phi \rangle. \tag{5.19}$$

Here, T_x is the *translation operator* given by

$$T_x(f)(y) := f(y-x),$$

and M_w is the *modulation operator* given by

$$M_w(f)(y) := e^{2\pi i w y} f(y).$$

For $1 \le p \le \infty$ and a weight v, the *modulation space* M_v^p is defined as

$$M_v^p := \{\, f \in \mathcal{S}'(\mathbb{R}^d) \,\big|\, \mathcal{V}_\phi f \in L_v^p(\mathbb{R}^{2d}) \,\},$$

and given the norm $\|f\|_{M_v^p} := \|\mathcal{V}_\phi f\|_{L_v^p}$. M_v^0 is similarly defined, this time using C_v^0 instead of L_v^∞.

For an adequate lattice, $\Lambda \subseteq \mathbb{R}^d \times \mathbb{R}^d$, the Gabor system $\{M_w T_x \phi : (x,w) \in \Lambda\}$ is a frame for $L^2(\mathbb{R}^d)$. Consider the family of functions $F \equiv \{f_k\}_{k\in\Lambda} \subseteq L^2(\mathbb{R}^d \times \mathbb{R}^d)$ defined by $f_k := \mathcal{V}_\phi(M_w T_x \phi)$, where $k = (x,w)$. Since $\mathcal{V}_\varphi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R}^d)$ is an isometry, it follows that F forms a frame sequence in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Since $\mathcal{V}_\phi \phi \in \mathcal{S}(\mathbb{R}^d)$ (cf. Prop. 1.11.1), for any $s > 0$ there exists a constant $C_s > 0$ such that

$$\left|\mathcal{V}_\phi \phi(z)\right| \le C_s \,(1 + |z|)^{-s}\,.$$

Since $|f_k| = \left|\mathcal{V}_\phi \phi(\cdot - k)\right|$ it follows that,

$$|f_k(z)| \le C_s \,(1 + |z - k|)^{-s} \qquad (z \in \mathbb{R}^{2d}, k \in \Lambda). \tag{5.20}$$

Consequently, by Example 2.2.1, we know that $\boldsymbol{V} = \boldsymbol{V}(F, \Lambda)$ is a spline-type space.

Observe that for polynomially moderated weights v and $1 \le p < \infty$, $\mathcal{V}_\phi$ maps, by definition, the modulation space M_v^p isometrically onto $\boldsymbol{V}_v^p$. For $p = \infty$, the same statement is true replacing M_v^∞ for M_v^0.

In view of this, Theorem 5.1.2 can be reformulated for Gabor molecules.

Theorem 5.2.3.

- *Let $\mathcal{E} \equiv \{E_i\}_{i\in I}$ be a uniformly locally finite measurable covering of $\mathbb{R}^d \times \mathbb{R}^d$ (cf. Equation (5.3)).*

- *For each $i \in I$, let $G^i \equiv \left\{g_k^i\right\}_{k\in\Lambda_i}$ be a frame for $L^2(\mathbb{R}^d)$ with lower bound A_i and suppose that $A := \inf_i A_i > 0$.*

- *Suppose that the sets of time-frequency nodes $\Lambda_i \subseteq \mathbb{R}^d \times \mathbb{R}^d$ are uniformly relatively separated (i.e $\sup_{i\in I} \rho(\Lambda_i) < \infty$).*

- *Assume that the molecules G^i satisfy the following uniform time-frequency concentration condition,*

$$\left|\mathcal{V}_\phi g_k^i(z)\right| \le C \,(1 + |z - k|)^{-(s+\alpha)} \qquad (z \in \mathbb{R}^d \times \mathbb{R}^d, k \in \Lambda_i), \tag{5.21}$$

for some constants $C > 0$, $s > 2d$ and $\alpha \ge 0$ (independent of i).

Then, for all sufficiently large $r > 0$, the system

$$\left\{g_k^i : i \in I, k \in \Lambda_i, d(k, E_i) \le r\right\}$$

is a Banach frame simultaneously for all the modulation spaces M_v^p, for all strictly w_α-moderated weights v and $1 \le p < \infty$. The same is true for $p = \infty$, replacing M_v^∞ for M_v^0.

More precisely, if we set,

$$\Gamma^r := \{(i,k) : i \in I, k \in \Lambda_i, d(k, E_i) \le r\},$$

and define a weight V on Γ^r by

$$V(k,i) := v(k),$$

then, the coefficients map given by

$$\begin{aligned} M_v^p &\to z_V^p(\Gamma^r) \\ f &\mapsto \left(\left\langle f, g_k^i \right\rangle\right)_{(i,k)} \end{aligned}$$

is bounded and left-invertible, for all sufficiently large values of r.

Remark 5.2.4. *For any class of w_α-moderated weights for which the respective constants (cf. Equation (1.8)) are uniformly bounded, it is also possible to choose a value of $r > 0$ for which the conclusion of the theorem holds.*

Proof. Consider the spline-type space $\boldsymbol{V}^2 = \mathcal{V}_\phi(L^2(\mathbb{R}^d))$ from the discussion above. Define the functions,

$$\varphi_k^i := \mathcal{V}_\phi(g_k^i) \qquad (i \in I, k \in \Lambda_i).$$

Since $\mathcal{V}_\varphi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R}^d)$ is an isometry, each of the families $\left\{\varphi_k^i\right\}_k$ is a frame for $\boldsymbol{V}^2$ with lower bound A. Moreover, Equation (5.21) implies that these families share a uniform polynomial concentration condition. This condition is also shared by the atoms $\{f_k\}_k$ because Equation (5.20) holds for any value of $s > 0$. The theorem now follows from Theorem 5.1.3 and the fact that : $\mathcal{V}_\phi : M_v^p \to \boldsymbol{V}_v^p$ is a surjective isometry (with the discussed modification for $p = \infty$). □

Remark 5.2.5. *Observe that since we have identified the range of the STFT (with a fixed window) with a spline-type space, it follows from Theorem 3.3.1 that, on the range of the STFT, the L_v^p and $W(L^\infty, L_v^p)$ norms are equivalent (the class of weights v for which this is true depends on the time-frequency localization of the window function; in the case of the Gaussian window, any polynomial weight w_α with $\alpha \ge 0$ will work). Results of this kind can be found in Chapter 12 of [62], see for example Proposition 12.1.11 there.*

Remark 5.2.6. *Finally observe that the argument given can be used to combine not only time-frequency concentrated frames for $L^2(\mathbb{R}^d)$ but also frames for* proper subspaces $S \subseteq L^2(\mathbb{R}^d)$. *Simply let $\boldsymbol{V}^2 = \mathcal{V}_\phi(S)$ and apply the same argument as above.*

For completeness, we give a version of Theorem 5.2.3 for pure time-frequency atoms. This gives general sufficient conditions for the existence of the so called *quilted Gabor frames*, recently introduced in [32].

Corollary 5.2.1.

- *Let $\mathcal{E} \equiv \{E_i\}_{i\in I}$ be a uniformly locally finite measurable covering of $\mathbb{R}^d \times \mathbb{R}^d$ (cf. Equation (5.3)).*

- *For each $i \in I$, let $G^i \equiv \left\{M_j T_k g^i : (k, j) \in \Lambda_i\right\}$ be a Gabor frame for $L^2(\mathbb{R}^d)$ with lower bound A_i and suppose that $A := \inf_i A_i > 0$.*

- *Suppose that the sets of time-frequency nodes $\Lambda_i \subseteq \mathbb{R}^d \times \mathbb{R}^d$ are uniformly relatively separated.*

- *Assume that the windows $\left\{g^i\right\}_i$ satisfy the following uniform time-frequency concentration condition,*

$$C := \sup_i \|g^i\|_{M^\infty_{w_{s+\alpha}}} < +\infty,$$

  for some constants $s > 2d$ and $\alpha \geq 0$ (independent of i).

Then, for all sufficiently large $r > 0$, the system

$$\left\{M_j T_k g^i : i \in I, (k, j) \in \Lambda_i, d((k, j), E_i) \leq r\right\}$$

is a Banach frame simultaneously for all the modulation spaces M^p_v, for all strictly w_α-moderated weights v and $1 \leq p < \infty$. The same is true for $p = \infty$, replacing M^∞_v for M^0_v.

Proof. Observe that,

$$\left|\mathcal{V}_\phi(M_j T_k g^i)\right| = \left|\mathcal{V}_\phi(g^i)(\cdot - \lambda)\right| \leq C w_{-(s+\alpha)}(z - \lambda), \tag{5.22}$$

where $\lambda := (k, j) \in \Lambda_i$. Therefore, we can apply Theorem 5.2.3. □

5.2.4 Gabor multipliers

Now we give an application of the frame surgery scheme to Gabor multipliers. We follow largely the approach in [42]. For a general background on Gabor multipliers see [47, Chapter 5].

Given a lattice in the time-frequency plane $\Lambda \subseteq \mathbb{R}^{2d}$ and two families of functions $F \equiv \{f_1, \ldots, f_N\}, G \equiv \{g_1, \ldots, g_N\} \subseteq L^2(\mathbb{R}^d)$ we consider the class of operators,

$$\mathbf{G}_{F,G} := \{ \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} m_n(\lambda) \langle -, \pi(\lambda) g_n \rangle \pi(\lambda) f_n \,\Big|\, m_n \in \ell^2(\Lambda) \},$$

where $\pi(\lambda)$ is the time-frequency shift $\pi(\lambda) := M_w T_x$, if $\lambda = (x, w)$. The convergence of the series defining the class $\mathbf{G}$ requires additional assumptions (see below). The operators in this class are called the *Gabor multipliers* associated with the time-frequency atoms (F, G) and the lattice Λ.

For $f, g \in L^2(\mathbb{R}^d)$ we use the notation $P_{f,g} := \langle -, g \rangle f$ for the corresponding rank-one operator. Furthermore, for a point $(x, w) \in \mathbb{R}^d \times \mathbb{R}^d$ we let the time-frequency shifts act on an operator T by

$$\rho(x, w)(T) := M_w T_x T T_{-x} M_{-w} = \pi(x, w) T \pi(x, w)^*.$$

Every linear operator T mapping continuously $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ admits a distributional kernel $K(T) \in \mathcal{S}'(\mathbb{R}^{2d})$. The *Kohn-Nirenberg symbol* of T is defined in terms of K by

$$\sigma(T)(x, w) := \int_{\mathbb{R}^d} K(T)(x, x - s) e^{-2\pi i s w} ds.$$

From this definition it follows that the Kohn-Nirenberg map defines an isometry between the class of Hilbert-Schmidt operators and $L^2(\mathbb{R}^{2d})$. The important property for us is that the Kohn-Nirenberg map interwines the action ρ with the regular action of $\mathbb{R}^d \times \mathbb{R}^d$ (by translations). That is,

$$\sigma(\rho(z)T) = \sigma(T)(\cdot - z) \qquad (z \in \mathbb{R}^d \times \mathbb{R}^d).$$

We see then that the Kohn Nirenberg map $KN : T \mapsto \sigma(T)$ relates the class $\mathbf{G}$ to a shift-invariant space $\boldsymbol{V}^2(F, G) := KN(\mathbf{G}_{F,G})$ given by,

$$\boldsymbol{V}^2 = \{ \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} m_n(\lambda) \sigma(P_{f_n, g_n})(\cdot - \lambda) \,\Big|\, (m_n) \in \ell^2(\Lambda) \}.$$

The Kohn-Nirenberg symbol of the projector $P_{f,g}$ is explicitly given by,

$$\sigma(P_{f,g})(x, w) = f(x) \overline{\hat{g}}(w) e^{-2\pi i x w}, \tag{5.23}$$

so its 2d Fourier transform is

$$\widehat{\sigma(P_{f,g})}(x,w) = \mathcal{V}_g f(-w,x).$$

Consequently, the inner product between the building blocks of $\boldsymbol{V}^2$ is given by,

$$\left\langle \sigma(P_{f_n,g_n}), \sigma(P_{f_m,g_m}) \right\rangle = \left\langle \mathcal{V}_{g_n} f_n, \mathcal{V}_{g_m} f_m \right\rangle,$$

whereas, with the notation $z^* = (-w,x)$ for $z = (x,w)$, their bracket product (see Section 1.14 and 5.2.1) is given by

$$[\sigma(P_{f_n,g_n}), \sigma(P_{f_m,g_m})]_\Lambda(z) = \sum_{\lambda^\perp \in \Lambda^\perp} \mathcal{V}_{g_n} f_n(z^* - \lambda^\perp) \overline{\mathcal{V}_{g_m} f_m(z^* - \lambda^\perp)}. \tag{5.24}$$

Hence, the theory of shift-invariant spaces (cf. Section 1.14) implies the following.

Proposition 5.2.1. *The set* $\{ \langle -, \pi(\lambda) g_n \rangle \pi(\lambda) f_n \,|\, \lambda \in \Lambda \}$ *is a Riesz sequence in the space of Hilbert-Schmidt operators if and only if the matrix of functions* $\hat{G} = \hat{G}(F,G) \equiv (\hat{G}_{n,m})_{1 \le n,m \le N}$*, given by,*

$$\hat{G}_{n,m}(z) = \sum_{\lambda^\perp \in \Lambda^\perp} \mathcal{V}_{g_n} f_n(z - \lambda^\perp) \overline{\mathcal{V}_{g_m} f_m(z - \lambda^\perp)}, \tag{5.25}$$

is uniformly bounded and invertible (that is, its eigenvalues are bounded away from 0 and ∞*, uniformly on* z*).*

Remark 5.2.7. *Observe that, for time-frequency concentrated windows, since by Remark 5.2.5 the STFT of an* L^2 *function belongs to the amalgam space* $W(C_0, L^2)$*, it follows that the entries of the matrix in Equation (5.25) are continuous periodic functions. Therefore, that matrix will be uniformly invertible if it is invertible at every point.*

Proof. The only observation to complete the proof is that, since the condition in Equation (5.25) is required for every $z \in \mathbb{R}^{2d}$, we can drop the change of coordinates $z \mapsto z^*$ in the bracket product. □

Consequently, in the situation of Proposition 5.2.1, any operator $T \in \mathbf{G}(F,G)$ can be stably recovered from its *lower symbol*

$$\left(\left\langle T, P_{\pi(\lambda) f_n, \pi(\lambda) g_n} \right\rangle_{HS} : \lambda \in \Lambda \right),$$

where $\langle \cdot, \cdot \rangle_{HS}$ denotes the Hilbert-Schmidt inner product. We can now reformulate Theorem 5.2.1 in this context.

Theorem 5.2.4. *Let a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ and a uniformly locally finite measurable covering of $\mathbb{R}^{2d}$, $\mathcal{E} \equiv \{E_i\}_{i\in I}$, be given.*

Let $f_1, \dots, f_N, g_1, \dots g_N \in L^2(\mathbb{R}^d)$ be such that the matrix $\hat{G}(F, G)$ on Equation (5.25) is uniformly invertible and suppose that these atoms satisfy,

$$|f_n(x)| \leq C(1 + |x|)^{-s}, \tag{5.26}$$
$$|\hat{g}_n(w)| \leq C(1 + |w|)^{-s}, \tag{5.27}$$

for some constants $C > 0$ and $s > 2d$.

Let families $\left\{f_1^i, \dots, f_N^i\right\}, \left\{g_1^i, \dots g_N^i\right\} \subseteq L^2(\mathbb{R}^d)$, $i \in I$ be given. Assume the following.

- *The given families satisfy,*

$$\left|f_n^i(x)\right| \leq C'(1 + |x|)^{-s}, \tag{5.28}$$
$$\left|\hat{g}_n^i(w)\right| \leq C'(1 + |w|)^{-s}, \tag{5.29}$$

  for some constant $C' > 0$ (independent of i and n).

- *The matrices of functions $\left(\hat{G}^i_{n,m}\right)_{1\leq n,m\leq N}$ given by*

$$\hat{G}^i(z)_{n,m} := \sum_{\lambda^\perp \in \Lambda^\perp} \mathcal{V}_{g_n} f_n(z - \lambda^\perp) \overline{\mathcal{V}_{g^i_m} f^i_m(z - \lambda^\perp)} \qquad (z \in \mathbb{R}^d \times \mathbb{R}^d),$$

  are uniformly bounded and invertible in the sense that each $\hat{G}^i(z)$ is invertible and

$$\sup_{z,i} \|\hat{G}^i(z)\|, \sup_{z,i} \|\hat{G}^i(z)^{-1}\| < \infty.$$

Then, for all sufficiently large values of $r > 0$, any Gabor multiplier $T \in \mathbf{G}(F, G)$ can be stably recovered in Hilbert-Schmidt norm from its mixed lower symbol

$$\left(\left\langle T, P_{\pi(\lambda) f^i_n, \pi(\lambda) g^i_n} \right\rangle_{HS} : i \in I, 1 \leq n \leq N, \lambda \in \Lambda, d(\lambda, E_i) \leq r\right). \tag{5.30}$$

Remark 5.2.8. *As we have seen, the theorem also establishes an uniform equivalence between the ℓ^p_v norm of the coefficients in Equation (5.30) and the L^p_v norm of the Kohn-Nirenberg symbol of T, for $1 \leq p \leq \infty$ and a certain class of weights.*

Proof. By the discussion above, in order to apply Theorem 5.2.1 we need to observe that the Kohn-Nirenberg symbols of all the atoms are adequately localized. This follows from Equation (5.23) and the fact that $w_{-s}(x) w_{-s}(w) \leq w_{-s}(x, w)$, for $x, w \in \mathbb{R}^d$. ◻

Chapter 6

Phase-space coverings

In this chapter we study a second locality problem in phase-space. We now go back to the most abstract setting of locally-compact groups rather than the Euclidean space. This is required by certain applications. Recall that a coorbit space, in the most general sense of the term, is a functional space $\boldsymbol{X}$ that is defined by imposing size conditions to a certain transform. More precisely, considering a functional space $\boldsymbol{X}$ as a coorbit space consists of giving a transform $T : \boldsymbol{X} \to \boldsymbol{E}$ that embeds $\boldsymbol{X}$ as a complemented subspace of a solid BF space $\boldsymbol{E}$.

When a functional space $\boldsymbol{X}$ is identified as a coorbit space, the properties of an element $f \in \boldsymbol{X}$ are reformulated in terms of decay or integrability conditions of the function $T(f) \in \boldsymbol{E}$ - called the *phase-space representation* of f. The elements of $\boldsymbol{X}$ can be resynthesized from their phase-space representations by means of an operator $U : \boldsymbol{E} \to \boldsymbol{X}$ that is a left-inverse for T (i.e. $f = UT(f)$).

In an attempt to finely adjust the properties of a function f that are shown by $T(f)$ one can consider operators of the form $M_m(f) = U(mT(f))$ that apply a mask m to the phase-space representation $T(f)$. We will call these operators *phase-space multipliers*. Of course, the rigorous interpretation of $M_m(f)$ is problematic since, in general, $TM_m(f) \neq mT(f)$. When T is the abstract wavelet transform (representation-coefficients function) associated with an unitary representation of a group, these operators are know as *localization operators* or *wavelet multipliers* [71, 113, 83]. In the case of time-frequency analysis these operators are known as time-frequency localization operators or multipliers of the short-time Fourier transform [24, 21, 22, 13].

The main result of the chapter is a characterization of the norm of a coorbit space in terms of families of phase-space multipliers associated with a partition of unity in $\mathcal{G}$. Specifically, suppose that $\boldsymbol{X}$ is a Banach space that is regarded as a coorbit space by means of a transform $T : \boldsymbol{X} \to \boldsymbol{E}$, having a left-inverse $U : \boldsymbol{E} \to \boldsymbol{X}$. Let $\left\{\theta_\gamma\right\}_\gamma$ be a partition of unity on $\mathcal{G}$ and consider the corresponding phase-space multipliers given by $M_\gamma(f) = U(\theta_\gamma T(f))$. The partition of unity is only assumed to satisfy certain spatial localization conditions but it is otherwise arbitrary. We prove that $\|f\|_{\boldsymbol{X}}$ is equivalent to the norm of the sequence $\left\{\|M_\gamma(f)\|_{\boldsymbol{B}}\right\}_\gamma$ in a discrete version of the space $\boldsymbol{E}$, where the space $\boldsymbol{B}$ can be chosen among a large class of function spaces. Moreover, we prove that the map $f \mapsto \left\{M_\gamma(f)\right\}_\gamma$ embeds $\boldsymbol{X}$ as a complemented subspace of a space of $\boldsymbol{B}$-valued sequences, obtained as a discretization of $\boldsymbol{E}$. This quantifies the relation between an element $f \in \boldsymbol{X}$ and the phase-space localized pieces $\left\{M_\gamma(f)\right\}_\gamma$.

For the case of time-frequency analysis, Dörfler and Gröchenig have recently obtained this kind of characterization of modulation spaces [34], using techniques from rotation algebras (non-commutative tori) developed in [66] and [64] and spectral theory for Hilbert spaces.[1] Here, we use a different approach to obtain consequences for settings where the techniques in [34] are not applicable, such as time-scale decompositions and Besov spaces. As a by-product we derive a stronger version of the main result in [34] where the admissible partitions of unity are restricted to be lattice shifts of a non-negative function and the space $\boldsymbol{B}$ is L^2. These restrictions seem to be essential for the applicability of the techniques in [34].

However, to fully recover and generalize the results in [34], we will need to refine the results of this chapter using tools that are only available on certain groups. We carry out that task in Chapter 7. There, instead of the tools from rotation algebras used in [34], we will resort to related results for matrix algebras.

6.1 Approximation of phase-space projections

In this section we prove the main technical estimate of the chapter. Given the setting from Section 3.1 and a partition of unity $\sum_\gamma \eta_\gamma \equiv 1$, we will show that the phase-space projection $P(f)$ from Section 3.1 can be resynthesized from the phase-space localized pieces $\left\{P(f\eta_\gamma)\right\}_\gamma$. Note that $P(f)$ can be trivially recovered from $\left\{P(f\eta_\gamma)\right\}_\gamma$ by simply summing all these functions. We will prove that this reconstruction can also be achieved by placing the localized pieces on top of the (morally) corresponding regions of the phase-space. This controlled synthesis will then allow us to quantify the relation between $P(f)$ and $\left\{P(f\eta_\gamma)\right\}_\gamma$ and yield the main result on the characterization of the norm of $\boldsymbol{S}$.

6.1.1 Setting

Let us recall the model from Section 3.1. Let $\mathcal{G}$ be a locally-compact group. We assume the following.

(A1)
- $\boldsymbol{E}$ is a solid, translation invariant BF space, called *the environment*.
- w is an admissible weight for $\boldsymbol{E}$.
- $\boldsymbol{S}$ is a closed complemented subspace of $\boldsymbol{E}$, called *the atomic subspace*.

(A2) We have an operator P and a function H satisfying the following.

[1] For more about the relation between time-frequency analysis and non-commutative tori see [85].

- $P : W(L^1, L^\infty_{1/w}) \to L^\infty_{1/w}$ is a (bounded) linear operator,
- $P(\boldsymbol{E}) = \boldsymbol{S}$,
- $P(f) = f$, for all $f \in \boldsymbol{S}$,
- $H \in W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$,
- For $f \in W(L^1, L^\infty_{1/w})$,

$$|P(f)(x)| \leq \int_{\mathcal{G}} |f(y)|\, H(y^{-1}x)dy, \qquad (x \in \mathcal{G}). \tag{6.1}$$

We now state Assumption (B1) introducing the partition of unity covering phase-space and the norm used to measure it.

(B1)
- $\Gamma \subseteq \mathcal{G}$ is a relatively separated set.
- $\{\, \eta_\gamma \,|\, \gamma \in \Gamma \,\}$ is a set of $W_R^{\text{weak}}(L^\infty, L^1_w)$-molecules enveloped by a function g. More precisely,
  - $\left|\eta_\gamma(x)\right| \leq g(\gamma^{-1}x), \qquad (x \in \mathcal{G}, \gamma \in \Gamma)$,
  - $g \in W_R^{\text{weak}}(L^\infty, L^1_w)$.
- $\left\{\eta_\gamma\right\}_\gamma$ is a bounded partition of unity. That is,

$$\sum_\gamma \eta_\gamma \equiv 1, \quad \text{and} \quad \sum_\gamma \left|\eta_\gamma\right| \in L^\infty(\mathcal{G}).$$

- $\boldsymbol{B}$ is a solid, isometrically left-translation invariant Banach space such that,

$$W(L^\infty, L^1_w) \hookrightarrow \boldsymbol{B}.$$

Remark 6.1.1. *By Lemma 1.6.1, $\boldsymbol{B} \hookrightarrow W(L^1, L^\infty)$. In addition, by the definition of translation invariant space $L^1 * \boldsymbol{B} \hookrightarrow \boldsymbol{B}$.*

6.1.2 Vector-valued analysis and synthesis

Let us now describe the operators mapping a function f into the phase-space localized pieces, by means of the partition of unity $\left\{\eta_\gamma\right\}_\gamma$. Let the operator $C^{\boldsymbol{B}}$ be formally defined by,

$$C^{\boldsymbol{B}}(f) := \left(P(f\eta_\gamma)\right)_{\gamma\in\Gamma}.$$

For each U, a relatively compact neighborhood of the identity in $\mathcal{G}$, we also formally define the operator $S_U^{\boldsymbol{B}}$, acting on a sequence of functions by,

$$S_U^{\boldsymbol{B}}(\left(f_\gamma\right)_{\gamma\in\Gamma}) := \sum_\gamma P(f_\gamma)\chi_{\gamma U}.$$

The operator $S_U^{\boldsymbol{B}}$ will be used as an approximate left-inverse of the vector valued analysis operator $C^{\boldsymbol{B}}$. Let us now establish the well-definition and mapping properties of these operators.

Proposition 6.1.1. *Under Assumptions (A1), (A2) and (B1) the following statements holds.*

(a) *The analysis operator $C^{\boldsymbol{B}}$ maps $W(L^\infty, \boldsymbol{E})$ boundedly into $\boldsymbol{E}^d_{\boldsymbol{B}}(\Gamma)$. In particular (cf. Proposition 3.1.1) it maps $\boldsymbol{S}$ boundedly into $\boldsymbol{E}^d_{\boldsymbol{B}}(\Gamma)$.*

(b) *For every relatively compact neighborhood of the identity U, and every sequence $F \in \boldsymbol{E}^d_B$, the series defining $S_U^{\boldsymbol{B}}(F)$ converges absolutely in the norm of $\boldsymbol{B}$ at every point. Moreover, the synthesis operator $S_U^{\boldsymbol{B}}$ maps $\boldsymbol{E}^d_{\boldsymbol{B}}(\Gamma)$ boundedly into $\boldsymbol{E}$ (with a bound that depends on U).*

Proof. To prove (a) let $f \in W(L^\infty, \boldsymbol{E})$. Since η_γ is bounded, $f\eta_\gamma \in W(L^\infty, \boldsymbol{E}) \subseteq \boldsymbol{E}$. By the pointwise bound for P (cf. Equation (6.1)),

$$\left|P(f\eta_\gamma)(x)\right| \leq \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y) H(y^{-1}x) dy.$$

Since $\boldsymbol{B}$ is solid and $L^1 * \boldsymbol{B} \hookrightarrow \boldsymbol{B}$, we have,

$$\|P(f\eta_\gamma)\|_{\boldsymbol{B}} \leq \|H\|_{\boldsymbol{B}} \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y) dy \lesssim \|H\|_{W(L^\infty, L^1_w)} \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y) dy.$$

Now the solidity of $\boldsymbol{E}$ and Lemma 2.1.1 yield,

$$\|C^{\boldsymbol{B}}(f)\|_{\boldsymbol{E}^d_B} \lesssim \|f\|_{W(L^\infty, \boldsymbol{E})} \|g\|_{W_R^{\text{weak}}(L^\infty, L^1_w)}.$$

To prove (b) consider a family $F \equiv (f_\gamma)_\gamma \in \boldsymbol{E}^d_{\boldsymbol{B}}$. For each $\gamma \in \Gamma$, $f_\gamma \in \boldsymbol{B} \subseteq W(L^1, L^\infty)$, so by Proposition 3.1.1, $P(f_\gamma)$ is well-defined and satisfies,

$$\left|P(f_\gamma)(x)\right| \lesssim \|f_\gamma\|_{W(L^1, L^\infty)} \|H\|_{W_R(L^\infty, L^1_w)} \lesssim \|f_\gamma\|_{\boldsymbol{B}} \|H\|_{W_R(L^\infty, L^1_w)}.$$

Hence, for every $x \in \mathcal{G}$, $\left|S^{\boldsymbol{B}}(F)(x)\right| \lesssim \sum_\gamma \|f_\gamma\|_{\boldsymbol{B}} \chi_U(\gamma^{-1}x)$. Since U is relatively compact, $\chi_U \in W_R(L^\infty, L^1_w)$ and consequently Lemma 1.6.2 together with the solidity of $\boldsymbol{E}$ imply that $\|R_U(F)\|_{\boldsymbol{E}} \lesssim \|\chi_U\|_{W_R(L^\infty, L^1_w)} \|F\|_{\boldsymbol{E}^d_B}$. □

6.1.3 Approximation of the projector

Now we can state the main result on the approximation of P. For every U, relatively compact neighborhood of the identity in $\mathcal{G}$, consider the approximate projector $P_U : W(L^\infty, \boldsymbol{E}) \to \boldsymbol{E}$ given by,

$$P_U(f) := \sum_{\gamma \in \Gamma} P(f\eta_\gamma) \chi_{\gamma U}. \tag{6.2}$$

Since $P_U = S_U^{\boldsymbol{B}} \circ C^{\boldsymbol{B}}$, P_U is well-defined. We will prove that P_U approximates P in the following way.

Theorem 6.1.1. *Given $\varepsilon > 0$, there exists U_0, a relatively compact neighborhood of e such that for all $U \supseteq U_0$,*

$$\|P(f) - P_U(f)\|_{\boldsymbol{E}} \leq \varepsilon \|f\|_{W(L^\infty, \boldsymbol{E})}, \quad (f \in W(L^\infty, \boldsymbol{E})).$$

In order to prove Theorem 6.1.1 we introduce the following auxiliary function. For each U, let $G_U : \mathcal{G} \to [0, +\infty)$ be defined by,

$$G_U(x) := \sup_{y \in \mathcal{G}} \sum_{\gamma \in \Gamma} (g * \chi_V)(\gamma^{-1} y) \chi_{\gamma(\mathcal{G} \setminus U)}(yx). \tag{6.3}$$

Observe that G_U is defined as a supremum of a family of sums. The estimates for P that we will derive in terms of G_U are sharper than the usual convolution estimates involving Wiener amalgam norms of g. This extra precision is crucial for the proof of Theorem 6.1.1. Before proving that theorem we establish some necessary estimates for the auxiliary function.

Lemma 6.1.1. *The function G_U satisfies $\|G_U\|_{L^\infty(\mathcal{G})} \lesssim 1$ (with a bound independent of U). Moreover, for every compact set $K \subseteq \mathcal{G}$,*

$$\|G_{UK}\|_{L^\infty(K)} \lesssim \int_{V(\mathcal{G} \setminus U)} (g * \chi_V)_\#(x) w(x) dx.$$

Proof. Let a compact set K and an element $x \in K$ be given. For $y \in \mathcal{G}$, if $yx \in \gamma(\mathcal{G} \setminus (UK))$, then $\gamma^{-1} yx \notin UK$, so $\gamma^{-1} y \notin U$.

Therefore,

$$\begin{aligned}
\sum_\gamma (g * \chi_V)(\gamma^{-1} y) \chi_{\gamma(\mathcal{G} \setminus (UK))}(yx) &\leq \sum_{\gamma : \gamma^{-1} y \notin U} (g * \chi_V)(\gamma^{-1} y) \\
&\lesssim \sum_{\gamma : \gamma^{-1} y \notin U} \int_{\mathcal{G}} (g * \chi_V)_\#(t^{-1} \gamma^{-1} y) \chi_V(t) dt \\
&= \int_{\mathcal{G}} (g * \chi_V)_\#(t^{-1}) \sum_{\gamma : \gamma^{-1} y \notin U} \chi_V(\gamma^{-1} yt) dt.
\end{aligned}$$

Since Γ is relatively separated, $\sum_\gamma \chi_V(\gamma^{-1} yt) = \sum_\gamma \chi_V(t^{-1} y^{-1} \gamma) \lesssim 1$. In addition, if $\gamma^{-1} yt \in V$ and $\gamma^{-1} y \notin U$ then $t = (\gamma^{-1} y)^{-1} \gamma^{-1} yt \in (\mathcal{G} \setminus U)^{-1} V$.

Hence,

$$G_{UK}(x) \lesssim \int_{(\mathcal{G} \setminus U)^{-1} V} (g * \chi_V)_\#(t^{-1}) dt = \int_{V(\mathcal{G} \setminus U)} (g * \chi_V)_\#(t) \Delta(t^{-1}) dt.$$

Since $\Delta(t^{-1}) \lesssim \Delta(t^{-1}) w(t^{-1}) = w(t)$ the desired bound follows. Reexamining the computations above we see that, $\|G_U\|_{L^\infty(\mathcal{G})} \lesssim \int_{\mathcal{G}} (g * \chi_V)_\#(t) w(t) dt$. Since $g \in W_R^{\text{weak}}(L^\infty, L^1_w)$, the last integral is finite and we obtain the desired uniform bound. ◻

Now we can prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Let $f \in W(L^\infty, \boldsymbol{E})$ and let U be a relatively compact neighborhood of e. Since $\sum_\gamma \eta_\gamma \equiv 1$,

$$P(f) - P_U(f) = \sum_\gamma P(f\eta_\gamma) - \sum_\gamma P(f\eta_\gamma)\chi_{\gamma U} = \sum_\gamma P(f\eta_\gamma)\chi_{\gamma(\mathcal{G}\setminus U)}.$$

Consequently, by the pointwise bound for P (cf. Equation (6.1)), for $x \in \mathcal{G}$,

$$|P(f)(x) - P_U(f)(x)| \leq \sum_\gamma \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y)H(y^{-1}x)\chi_{\gamma(\mathcal{G}\setminus U)}(x)dy.$$

Since $|f(y)| \lesssim \int f^\#(z)\chi_V(y^{-1}z)dz$, we have,

$$|P(f)(x) - P_U(f)(x)| \lesssim \int_{\mathcal{G}} f^\#(z) \sum_\gamma \int_{\mathcal{G}} \chi_V(y^{-1}z)g(\gamma^{-1}y)H(y^{-1}x)\chi_{\gamma(\mathcal{G}\setminus U)}(x)dydz.$$

Note that if $y^{-1}z \in V$, then $y^{-1}x = y^{-1}zz^{-1}x \in V(z^{-1}x)$, and therefore $H(y^{-1}x) \leq H_\#(z^{-1}x)$. Hence,

$$\begin{aligned} |P(f)(x) - P_U(f)(x)| &\lesssim \int_{\mathcal{G}} f^\#(z)H_\#(z^{-1}x) \sum_\gamma \int_{\mathcal{G}} g(\gamma^{-1}y)\chi_V(y^{-1}z)\chi_{\gamma(\mathcal{G}\setminus U)}(x)dydz. \\ &= \int_{\mathcal{G}} f^\#(z)H_\#(z^{-1}x) \sum_\gamma (g * \chi_V)(\gamma^{-1}z)\chi_{\gamma(\mathcal{G}\setminus U)}(x)dz. \\ &\leq \int_{\mathcal{G}} f^\#(z)H_\#(z^{-1}x)G_U(z^{-1}x)dz. \end{aligned}$$

Consequently,

$$\|P(f) - P_U(f)\|_{\boldsymbol{E}} \lesssim \|f^\#\|_{\boldsymbol{E}}\|H_\# G_U\|_{L^1_w} = \|f\|_{W(L^\infty,\boldsymbol{E})}\|H_\# G_U\|_{L^1_w}.$$

Therefore, it suffices to show that $\|H_\# G_U\|_{L^1_w} \longrightarrow 0$, as U grows to $\mathcal{G}$. For every compact set $K \subseteq \mathcal{G}$, Lemma 6.1.1 implies that

$$\begin{aligned} &\int_{\mathcal{G}} H_\#(z)G_U(z)w(z)dz \\ &\qquad \leq \|G_U\|_{L^\infty(K)}\|H_\#\|_{L^1_w} + \|G_U\|_{L^\infty(\mathcal{G})} \int_{\mathcal{G}\setminus K} H_\#(z)w(z)dz \\ &\qquad \lesssim \|G_U\|_{L^\infty(K)} + \int_{\mathcal{G}\setminus K} H_\#(z)w(z)dz. \end{aligned}$$

Given $\varepsilon > 0$, we choose a compact set K such that the second term in the last inequality is less that ε. Since $g \in W_R^{\text{weak}}(L^\infty, L^1_w)$, we can also choose a compact set $Q \subseteq \mathcal{G}$ such that $\int_{\mathcal{G}\setminus Q}(g * \chi_V)_\#(x)w(x)dx < \varepsilon$. Set $U_0 := VQK$. If $U \supseteq U_0$ is a relatively compact neighborhood of e, then, using Lemma 6.1.1,

$$\|G_U(z)\|_{L^\infty(K)} \leq \|G_{VQK}\|_{L^\infty(K)} \lesssim \int_{V(\mathcal{G}\setminus(VQ))} (g * \chi_V)_\#(x)w(x)dx.$$

Since $V = V^{-1}$, we have that $V(\mathcal{G} \setminus (VQ)) \subseteq (\mathcal{G} \setminus Q)$ and consequently $\|G_U(z)\|_{L^\infty(K)} \lesssim \varepsilon$. Hence, we have shown that for $U \supseteq U_0$, $\|H_\# G_U\|_{L^1_w} \lesssim \varepsilon$. This completes the proof. □

6.2 Approximation of phase-space multipliers

We will now interpret Theorem 6.1.1 as a result about approximation of *phase-space multipliers*. Let us suppose that Assumptions (A1), (A2) and (B1) hold.

For $m \in L^\infty(\mathcal{G})$, the *multiplier* $M^m : \boldsymbol{S} \to \boldsymbol{S}$ with *symbol* m is defined by,

$$M^m(f) := P(mf), \qquad (f \in \boldsymbol{S}). \tag{6.4}$$

The operator M^m is clearly bounded by Proposition 3.1.1 and the solidity of $\boldsymbol{E}$. When the space $\boldsymbol{S}$ is taken to be the range of the abstract wavelet transform associated with an unitary representation of $\mathcal{G}$, these operators are called localization operators or wavelet multipliers (see for example [71, 113, 83]). (To be precise, the operators M^m are unitary equivalent to localization operators, see Section 6.4.1 for further details). When $\boldsymbol{S}$ is the range of the Short-time Fourier transform the corresponding operators are known as STFT multipliers or Time-Frequency localization operators ([24, 21, 22]).

Using the approximation of the projector from the previous section, we construct an approximation of the multiplier M^m. For a relatively compact neighborhood of the identity U, let $M^m_U : \boldsymbol{S} \to \boldsymbol{S}$ be defined by,

$$M^m_U(f) := PP_U(mf), \qquad (f \in \boldsymbol{S}).$$

Now Theorem 6.1.1 implies the following.

Theorem 6.2.1. *For each $m \in L^\infty(\mathcal{G})$, $M^m_U \longrightarrow M^m$ in operator norm, as U ranges over the class of relatively compact neighborhoods of the identity, ordered by inclusion. Moreover, convergence is uniform on any bounded class of symbols.*

Proof. By Proposition 3.1.1, for $f \in \boldsymbol{S}$,

$$\|M^m_U(f) - M^m(f)\|_{\boldsymbol{E}} = \|PP_U(mf) - PP(mf)\|_{\boldsymbol{E}} \lesssim \|P_U(mf) - P(mf)\|_{\boldsymbol{E}}.$$

By Theorem 6.1.1, $\|P_U(mf) - P(mf)\|_{\boldsymbol{E}} \lesssim \delta(U)\|mf\|_{W(L^\infty,\boldsymbol{E})}$, for some function δ such that $\delta(U) \longrightarrow 0$, as U grows to $\mathcal{G}$. Finally, since $m \in L^\infty(\mathcal{G})$ and $f \in \boldsymbol{S}$, the embedding $\boldsymbol{S} \hookrightarrow W(L^\infty, \boldsymbol{E})$ in Proposition 3.1.1 implies that $\|mf\|_{W(L^\infty,\boldsymbol{E})} \lesssim \|f\|_{W(L^\infty,\boldsymbol{E})} \lesssim \|f\|_{\boldsymbol{E}}$, and the conclusion follows. Observe that if m belongs to a certain bounded subset of L^∞, then the last estimate holds uniformly on that set. □

6.3 Characterization of the atomic space

We can finally prove the main abstract result on the characterization of the atomic space with phase-space multipliers.

Theorem 6.3.1. *Under Assumptions (A1), (A2) and (B1), the map*

$$\begin{aligned} C^{\boldsymbol{B}} : \boldsymbol{S} &\to \boldsymbol{E}^d_{\boldsymbol{B}} \\ f &\mapsto (P(f\eta_\gamma))_\gamma \end{aligned}$$

is left-invertible. Consequently, the following norm equivalence holds for $f \in \boldsymbol{S}$,

$$\|f\|_{\boldsymbol{E}} \approx \|(\|P(f\eta_\gamma)\|_{\boldsymbol{B}})_\gamma\|_{\boldsymbol{E}^d}.$$

Remark 6.3.1. *The fact that there is such a liberty to choose the BF space $\boldsymbol{B}$ is analogous to the fact that for coorbit spaces only the "global behavior" of the norm imposed on the wavelet transform matters. See [45, Theorem 8.3].*

Proof. With the notation of Section 6.2, using Theorem 6.2.1 with symbol $m \equiv 1$, we choose a relatively compact neighborhood of the identity U such that M^1_U is invertible. Since the operator P_U (cf. Equation (6.2)) can be factored as $P_U = S^{\boldsymbol{B}}_U C^{\boldsymbol{B}}$, we have that, $M^1_U = PS^{\boldsymbol{B}}_U C^{\boldsymbol{B}}$. Since M^1_U is invertible, $C^{\boldsymbol{B}}$ is left-invertible, as claimed. This implies that $\|f\|_{\boldsymbol{E}} \lesssim \|C^{\boldsymbol{B}}(f)\|_{\boldsymbol{E}^d_{\boldsymbol{B}}}$, for $f \in \boldsymbol{S}$. The converse inequality is just the boundedness of $C^{\boldsymbol{B}}$ and was proved in Proposition 6.1.1. □

6.4 Applications

6.4.1 Coorbit spaces

We now give some applications to the theory of coorbit spaces of group representations (cf. Section 1.10). Let π be a (strongly continuous) unitary representation of a locally compact group $\mathcal{G}$ on a Hilbert space $\mathbb{H}$. Let $\boldsymbol{E}$ be a solid BF space, w an admissible weight for it, and let $h \in \mathbb{H}$ be an admissible vector (in the sense of Section 1.10).

Let $\boldsymbol{S} = V_h(\mathrm{Co}\boldsymbol{E})$. It is proved in [44, Proposition 4.3] that $\boldsymbol{S}$ is a closed subspace of $\boldsymbol{E}$ and moreover $P(F) := F * V_h h$ defines a projector onto $\boldsymbol{S}$. When $\boldsymbol{E}$ is $L^2(\mathcal{G})$, the operator P is in fact the orthogonal projector onto $\boldsymbol{S}$.

In order to apply Theorem 6.3.1 to this setting, let a partition of unity $\{\eta_\gamma\}_\gamma$ and a BF space $\boldsymbol{B}$ satisfying (B1) be given. Let the operators $M_\gamma : \mathrm{Co}\boldsymbol{E} \to \mathrm{Co}\boldsymbol{E}$ be defined as,

$$M_\gamma(f) := V^*_h(\eta_\gamma V_h(f)).$$

Observe that, since $V_h : \mathbb{H} \to L^2$ is an isometry, V_h^* is the projection onto the range of V_h followed by the inverse of V_h on its range. Hence,

$$M_\gamma(f) := V_h^{-1}(M_{\eta_\gamma} V_h(f)),$$

where $M_{\eta_\gamma} : \boldsymbol{S} \to \boldsymbol{S}$ is the multiplier form Section 6.2. Now Theorem 6.3.1 yields the following.

Theorem 6.4.1 (Characterization of coorbit spaces)**.** *Let a partition of unity* $\{\eta_\gamma\}_\gamma$ *and a BF space* $\boldsymbol{B}$ *satisfying (B1) be given. Then, for* $f \in \mathrm{Co}\boldsymbol{E}$*, the following norm equivalence holds,*

$$\|f\|_{\mathrm{Co}\boldsymbol{E}} \approx \|\left\{\|M_\gamma(f)\|_{\mathrm{Co}\boldsymbol{B}}\right\}_\gamma\|_{\boldsymbol{E}^d}.$$

In particular, $f \in (\mathbb{H}_w^1)^\urcorner$ *belongs to* $\mathrm{Co}\boldsymbol{E}$ *if and only if* $\left\{\|M_\gamma(f)\|_{\mathrm{Co}\boldsymbol{B}}\right\}_\gamma \in \boldsymbol{E}^d(\Gamma)$.

Remark 6.4.1. *One possible choice for* $\boldsymbol{B}$ *is* $L^2(\mathcal{G})$ *yielding* $\mathrm{Co}\boldsymbol{B} = \mathbb{H}$ *(cf. [44, Corollary 4.4]).*

Proof. The norm equivalence follows directly from Theorem 6.3.1 and the fact that $V_h : \mathrm{Co}\boldsymbol{E} \to \boldsymbol{S}$ is an isometry. The "in particular" part follows from a standard approximation argument. □

6.4.2 Time-Scale analysis

We now consider the affine group $\mathcal{G} = \mathbb{R}^d \times (0, +\infty)$, where multiplication is given by $(x, s) \cdot (x', s') = (x + sx', ss')$ (cf. Section 1.5.1). We recall some facts. The measure with density $dx\frac{ds}{s^{d+1}}$ with respect to the Lebesgue measure is a left Haar measure. The modular function is given by $\Delta(x, s) = s^{-d}$. The affine group acts on $L^2(\mathbb{R}^d)$ by translations and dilations,

$$\pi(x, s)f(y) = s^{-d/2} f\left(\frac{y - x}{s}\right).$$

The Wavelet transform associated with π is,

$$W_h f(x, s) = s^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{h\left(\frac{t - x}{s}\right)}dt,$$

for $f, h \in L^2(\mathbb{R}^d)$, whereas the inverse wavelet transform is given by,

$$W_h^* F(x) = \int_0^{+\infty} \int_{\mathbb{R}^d} F(y, s)\overline{h\left(\frac{x - y}{s}\right)}dx\frac{ds}{s^{\frac{3}{2}d+1}},$$

for $F \in L^2(\mathcal{G})$.[2]

[2]The integral converges in the weak-sense. The possibility of evaluating it pointwise requires further hypothesis.

The wavelet multiplier with symbol $m \in L^\infty(\mathcal{G})$ is given by,

$$\mathrm{WM}_m f(x) = W_h^*(mW_hF), \tag{6.5}$$

for $f \in L^2(\mathbb{R}^d)$.

We illustrate Theorem 6.4.1 for homogeneous Besov spaces (cf. Section 1.12.1). Recall that for $1 \le p, q \le +\infty$ and $\sigma \in \mathbb{R}$, the homogeneous Besov space $\dot{B}^\sigma_{pq}(\mathbb{R}^d)$ is the set of all tempered distributions (modulo polynomials) $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^d)$ such that

$$\|f\|^q_{\dot{B}^\sigma_{pq}} = \sum_{j\in\mathbb{Z}} 2^{j\sigma q} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}(f))\|_{L^p}\Big\|^q$$

is finite (with the usual modification for $q = \infty$), where $\mathcal{F}$ is the Fourier transform and $\left\{\varphi_j\right\}_j$ is an adequate Schwartz class partition of unity subordinated to dyadic crowns.

As mentioned in Section 1.12.1, $\dot{B}^\sigma_{pq}(\mathbb{R}^d) = \mathrm{Co}(L^{p,q}_{\sigma+d/2-d/q}(\mathcal{G}))$, where,

$$\|F\|_{L^{p,q}_\sigma} = \left(\int_0^{+\infty} \left(\int_{\mathbb{R}^d} |F(x,s)|^p dx\right)^{q/p} s^{-\sigma q} \frac{ds}{s^{d+1}}\right)^{1/q}.$$

In addition, the admissibility of the window h is implied by the classical "smooth molecule" conditions involving decay of derivatives and vanishing moments (see [55, 56, 57]). For example, any Schwartz function h with all moments vanishing is adequate.[3]

In order to illustrate Theorem 6.4.1, we can consider a covering of $\mathbb{R}^d \times (0, +\infty)$ of the form,

$$U_{k,j} := 2^j((-1,1)^d + k) \times (2^{j-1}, 2^{j+1}), \qquad (k \in \mathbb{Z}^d, j \in \mathbb{Z}),$$

and let $\left\{\eta_{k,j}\right\}_{k,j}$ be a (measurable) partition of unity subordinated to it. The discrete norms corresponding to the spaces $L^{p,q}_\sigma$ are given in Equation (1.16). We now obtain the following result.

Theorem 6.4.2. *The quantity,*

$$\left(\sum_{j\in\mathbb{Z}} 2^{-j\sigma' q} \left(\sum_{k\in\mathbb{Z}^d} \|\mathrm{WM}_{\eta_{k,j}} f\|^p_{L^2}\right)^{q/p}\right)^{1/q},$$

where $\sigma' := \sigma + d/2 - d/p$, is an equivalent norm on $\dot{B}^\sigma_{pq}$ (with the usual modifications when p or q are ∞).

Remark 6.4.2. *Observe that Theorem 6.4.1 also allows for non-compactly supported partitions of unity, as long as its members are enveloped by a well-concentrated function. Also observe that in the norm equivalence above we can measure the norms of* $\mathrm{WM}_{\eta_{k,j}} f$ *in other Besov spaces besides L^2.*

[3] To satisfy the general assumptions of Section 3.1 we can use the weight $w(x,s) := \max\left\{s^{-\sigma}, \Delta(x,s)^{-1}s^{\sigma}\right\} = \max\left\{s^{-\sigma}, s^{d+\sigma}\right\}$.

Chapter 7

More general phase-space coverings

Under additional assumptions we can extend Theorem 6.3.1 to the case where the condition on the partition of unity, $\sum_\gamma \eta_\gamma \equiv 1$ is relaxed to $0 < A \leq \sum_\gamma \eta_\gamma \leq B < \infty$.

To be able to use the result from the previous chapter we keep the assumption that $\sum_\gamma \eta_\gamma \equiv 1$ and introduce a new (generalized) partition of unity $\left\{\theta_\gamma : \gamma \in \Gamma\right\}$ related to the one in Assumption (B1) of Section 6.1.1 by $\theta_\gamma = m\eta_\gamma$, where $0 < A \leq m \leq B < \infty$. This is the general form of a family of functions $\left\{\theta_\gamma : \gamma \in \Gamma\right\}$ enveloped by g and whose sum is nonnegative and bounded away from zero and infinity.

Consider the setting of Section 3.2. The problem of extending Theorem 6.3.1 to this new partition of unity can be reduced to the one of establishing the invertibility of the multiplier M^m (cf. Equation (6.4)). To this end, we will extend the atomic decomposition on Equation (3.2) to an adequate Hilbert space $\mathbb{H}$, then prove the invertibility of M^m on $\mathbb{H}$ and finally use the spectral invariance of a certain subalgebra of the algebra of bounded operators on ℓ^2 to deduce the invertibility of M^m on $\boldsymbol{S}$. This is where certain restrictions on the geometry of $\mathcal{G}$ need to be imposed. For the case of time-frequency decompositions and modulation spaces, this line of reasoning is hinted on the final remark of [22] and developed for a very general class of symbols and weighted modulation spaces in [69].

7.1 Setting

We now assume that we are in the situation of Section 3.2. Specifically we assume the following.

(A1)
- $\boldsymbol{E}$ is a solid, translation invariant BF space, called *the environment*.
- w is an admissible weight for $\boldsymbol{E}$.
- $\boldsymbol{S}$ is a closed complemented subspace of $\boldsymbol{E}$, called *the atomic subspace*.

(A2')
- $\Lambda \subseteq \mathcal{G}$ is a relatively separated set. Its points will be called *nodes*.

- $\{\varphi_\lambda \,|\, \lambda \in \Lambda\}$ and $\{\psi_\lambda \,|\, \lambda \in \Lambda\}$ are sets of $W^{\mathrm{st}}(L^\infty, L^1_w)$ molecules, enveloped by a function h. That is,
  - $|\varphi_\lambda(x)|, |\psi_\lambda(x)| \leq h(\lambda^{-1}x), \qquad (x \in \mathcal{G}, \lambda \in \Lambda),$
  - $h \in W^{\mathrm{st}}(L^\infty, L^1_w)$.

Let us also assume that we have a partition of unity like the one considered in Section 6.1.1.

(B1)
- $\Gamma \subseteq \mathcal{G}$ is a relatively separated set.
- $\{\eta_\gamma \,|\, \gamma \in \Gamma\}$ is a set of $W^{\mathrm{weak}}_R(L^\infty, L^1_w)$-molecules enveloped by a function g. More precisely,
  - $\left|\eta_\gamma(x)\right| \leq g(\gamma^{-1}x), \qquad (x \in \mathcal{G}, \gamma \in \Gamma),$
  - $g \in W^{\mathrm{weak}}_R(L^\infty, L^1_w)$.
- $\left\{\eta_\gamma\right\}_\gamma$ is a bounded partition of the unity. That is,

$$\sum_\gamma \eta_\gamma \equiv 1, \quad \text{and} \quad \sum_\gamma \left|\eta_\gamma\right| \in L^\infty(\mathcal{G}).$$

- $\boldsymbol{B}$ is a solid, isometrically left-translation invariant Banach space such that,

$$W(L^\infty, L^1_w) \hookrightarrow \boldsymbol{B}.$$

We will now introduce Assumptions (C1) and (C2) and present the extension of Theorem 6.3.1.

7.2 Assumption (C1)

We will use a key result from from [49]. To this end we introduce the following conditions for a discrete group Ω and a weight u on it.

Definition 7.2.1. *We say that the pair (Ω, u) satisfies the FGL-conditions if the following holds.*

- *Ω is a discrete, amenable, rigidly symmetric group.*
- *$u : \Omega \to [1, \infty)$ is a submultiplicative, symmetric weight that satisfies,*

$$\lim_{n\to+\infty} \sup_{x\in U^n} u(x)^{1/n} = 1, \textit{ and,}$$
$$\inf_{x\in U^n\setminus U^{n-1}} u(x) \approx \sup_{x\in U^n\setminus U^{n-1}} u(x), \quad (n \in \mathbb{N}),$$

for some generating subset U of Ω, containing the identity element.

For an explanation of the FGL-conditions and their relation to other concepts for groups (such as polynomial growth) see [49, 48] and the references therein. In Proposition 7.2.2 we give more concrete sufficient conditions for the applicability of the FGL conditions to our setting.

We introduce the following assumption on the geometry of $\mathcal{G}$ and the set of nodes Λ that provides the atomic decomposition of $\boldsymbol{S}$. This will be satisfied in the applications to time-frequency analysis but not in the case of time-scale decompositions.

(C1) We assume the following.

- $\mathcal{G}$ is an IN group.[1]
- The set Λ is a closed, discrete subgroup of $\mathcal{G}$ that, considered as a topological group in itself, satisfies the FGL-conditions with respect to the restriction of the weight w.

Remark 7.2.1. *The fact that $\mathcal{G}$ is an IN group implies that it is unimodular (i.e. $\Delta \equiv 1$) (see [92]). As a consequence, the weight w is symmetric (i.e. $w(x) = w(x^{-1})$).*

The submultiplicativity of w now implies that $(1/w)(xy) \leq w(x)(1/w)(y)$. This equation in turn implies that the weight w is admissible for all the spaces L^p_w and $L^p_{1/w}$, $(1 \leq p \leq +\infty)$.

Remark 7.2.2. *In (C1), $\mathcal{G}$ is assumed to be an IN group. Then, according to Theorem 2.1.1, the space $W^{\mathrm{st}}(L^\infty, L^1_w)$ in (A2') is just $W(L^\infty, L^1_w) = W_R(L^\infty, L^1_w)$ while the space $W_R^{\mathrm{weak}}(L^\infty, L^1_w)$ in (B1) is L^1_w.*

Since under Assumption (C1) Λ is a subgroup, it is possible to consider convolution operators on $\boldsymbol{E}^d$. The space $\boldsymbol{E}^d$ is always left-invariant, but for a general group $\mathcal{G}$ it may not be right-invariant (even if $\boldsymbol{E}$ is). Using the fact that in (C1) $\mathcal{G}$ is assumed to be an IN group, we have the following proposition.

Proposition 7.2.1. *Under Assumption (C1), $\boldsymbol{E}^d * \ell^1_w(\Lambda) \subseteq \boldsymbol{E}^d$, with the corresponding norm estimate.*

Proof. For $a \in \boldsymbol{E}^d$, $b \in \ell^1_w(\Lambda)$ and $x \in \mathcal{G}$, we estimate,

$$
\begin{aligned}
\sum_{\lambda} |(a * b)_\lambda| \chi_{\lambda V}(x) &\leq \sum_{\lambda} \sum_{\lambda'} |a_{\lambda'}| \, |b_{\lambda'^{-1}\lambda}| \chi_{\lambda V}(x) \\
&= \sum_{\lambda} \sum_{\lambda'} |a_{\lambda'}| \, |b_\lambda| \chi_{\lambda'\lambda V}(x) \\
&= \sum_{\lambda} |b_\lambda| \sum_{\lambda'} |a_{\lambda'}| \chi_{\lambda' V}(x\lambda^{-1}),
\end{aligned}
$$

[1] Remember that, by convention, we also assume that the distinguished neighborhood V is invariant under inner automorphisms.

where we used that $\lambda'\lambda V = \lambda' V\lambda$. Since $a \in \boldsymbol{E}^d$, the function $C(a) := \sum_{\lambda'} |a_{\lambda'}| \chi_{\lambda' V}$ belongs to $\boldsymbol{E}$. Moreover, $\|R_{\lambda^{-1}} C\|_{\boldsymbol{E}} \leq w(\lambda)\|C\|_{\boldsymbol{E}} = w(\lambda)\|a\|_{\boldsymbol{E}^d}$. Hence, then sum

$$\sum_{\lambda} |b_\lambda|\, R_{\lambda^{-1}} C,$$

converges absolutely in $\boldsymbol{E}$ and by the solidity of $\boldsymbol{E}$,

$$\|a * b\|_{\boldsymbol{E}^d} \leq \sum_{\lambda} |b_\lambda|\, w(\lambda) \|a\|_{\boldsymbol{E}^d}.$$

Hence the conclusion follows. □

Before introducing the second assumption we give some sufficient conditions for Assumption (C1) to hold. Recall that a group is called almost connected if the quotient by the connected component of the identity element is compact.

Proposition 7.2.2. *Suppose that Assumptions (A1), (A2') and (B1) hold and that, in addition, $\mathcal{G}$ is an almost connected IN group. Suppose that Λ is a discrete, closed, finitely-generated subgroup of $\mathcal{G}$ and that the weight w satisfies the Gelfand-Raikov-Shilov condition,*

$$\lim_{n\to+\infty} w(\lambda^n)^{1/n} = 1, \textit{ for all } \lambda \in \Lambda, \tag{7.1}$$

and the condition,

$$\inf_{x\in U^n \setminus U^{n-1}} w(x) \approx \sup_{x\in U^n \setminus U^{n-1}} w(x), \textit{ for all } n \in \mathbb{N},$$

for some generating subset U of Λ, containing the identity.

Then, the conditions in (C1) are satisfied.

Proof. The group $\mathcal{G}$ is an almost connected IN group and therefore has polynomial growth (see [92]). Since Λ is discrete and closed in $\mathcal{G}$ it also has polynomial growth (with respect to the counting measure). Indeed, let $F \subseteq \Lambda$ be a finite set containing the identity. Since Λ is discrete and closed in $\mathcal{G}$ there exists W', a relatively compact neighborhood of the identity in $\mathcal{G}$ such that $W' \cap (\Lambda \setminus \{e\}) = \emptyset$. Let W be a relatively compact neighborhood of the identity in $\mathcal{G}$ such that $W = W^{-1}$ and $WW \subseteq U'$. We then have that $\lambda W \cap \lambda' W = \emptyset$ for any two distinct elements $\lambda, \lambda' \in \Lambda$.

Since $e \in W$, we have that for any $n \geq 0$, $F^n W \subseteq (FW)^n$. So, using the polynomial growth condition for the neighborhood FW we have that,

$$|F^n W| \leq |(FW)^n| \lesssim n^k,$$

for some positive integer k. Since $F \subseteq \Lambda$, $F^n W = \cup_{\lambda\in F^n} \lambda W$ and the union is disjoint. Hence $|F^n W| = \#(F^n)\,|W|$ and we deduce that Λ also has polynomial growth.

Hence, Λ is a finitely-generated discrete group of polynomial growth. Therefore Λ is amenable (see [92]). In addition, by Gromov's structure theorem [70], Λ has a nilpotent subgroup of finite index. Corollary 3 from [82] implies that Λ is rigidly symmetric (see also [49]). Finally, since Λ is a finitely-generated discrete group of polynomial growth, Theorem 1.3 from [48] implies that the GRS condition in Equation (7.1) implies the condition required in (C1). □

7.3 Assumption (C2)

In order to introduce the second assumption, suppose that Assumptions (A1), (A2') and (C1) hold and let $\mathbb{H}$ be the closed linear subspace of $L^2(\mathcal{G})$ generated by the atoms $\{\varphi_\lambda : \lambda \in \Lambda\}$.

Since $\mathcal{G}$ is now assumed to be unimodular, left and right translations are isometries on $L^2(\mathcal{G})$. Hence, the weight w is also admissible for $L^2(\mathcal{G})$ (cf. Equation 1.11) and consequently the operators C and S from Section 3.2 map $L^2(\mathcal{G})$ into $\ell^2(\Lambda)$ and $\ell^2(\Lambda)$ into $L^2(\mathcal{G})$, respectively. For clarity, when considered with this domain and codomain we will denote these operators by $C_{\mathbb{H}}$ and $S_{\mathbb{H}}$. We also consider the operator $P_{\mathbb{H}} := S_{\mathbb{H}} C_{\mathbb{H}}$, which coincides with P on $L^2 \cap \boldsymbol{E}$.

Recall that a *frame* for a Hilbert space $\boldsymbol{L}$ is a collection of vectors $\{e_k\}_k$ such that $\|v\|_{\boldsymbol{L}} \approx \|(\langle v, e_k\rangle)_k\|_{\ell^2}$, for $v \in \boldsymbol{L}$ (see Section 1.8). We now observe that the atoms of $\boldsymbol{S}$ form a frame for $\mathbb{H}$.

Claim 7.3.1. *The set* $\{\varphi_\lambda : \lambda \in \Lambda\}$ *is a frame for* $\mathbb{H}$.

Proof. Since $f = P(f) = P_{\boldsymbol{H}}(f) = S_{\boldsymbol{H}} C_{\boldsymbol{H}}(f)$ for finite linear combinations of the atoms $\{\varphi_\lambda\}_\lambda$, and $C_{\boldsymbol{H}}$ and $S_{\boldsymbol{H}}$ are bounded, it follows that $f = S_{\boldsymbol{H}} C_{\boldsymbol{H}}(f)$, for all $f \in \boldsymbol{H}$. This implies that $f = Q C_{\boldsymbol{H}}^* S_{\boldsymbol{H}}^*(f)$, for all $f \in \boldsymbol{H}$, where Q is the orthogonal projection onto $\boldsymbol{H}$. Hence, $\|f\|_{L^2(\mathcal{G})} \approx \|S_{\boldsymbol{H}}^*(f)\|_{\ell^2(\Lambda)} = \|(\langle f, \varphi_\lambda\rangle)_\lambda\|_{\ell^2(\Lambda)}$, for all $f \in \boldsymbol{H}$. □

Since $\{\varphi_\lambda : \lambda \in \Lambda\}$ is a frame for $\mathbb{H}$, it has an associated canonical dual frame, that provides an expansion with coefficients having minimal ℓ^2-norm (cf. Section 1.8). This dual frame need not coincide with our distinguished set of dual atoms $\{\psi_\lambda : \lambda \in \Lambda\}$. We will now assume that they do coincide. This assumption will be justified in a large number of examples.

(C2) We assume that the set $\{\psi_\lambda : \lambda \in \Lambda\}$ is the canonical dual frame of $\{\varphi_\lambda : \lambda \in \Lambda\}$, considered as a frame for $\mathbb{H}$.

Under Assumption (C2), the operator $P_{\boldsymbol{H}}$ is the orthogonal projector $L^2(\mathcal{G}) \to \boldsymbol{H}$. Also $C_{\boldsymbol{H}}$ and $S_{\boldsymbol{H}}$ are related by $C_{\boldsymbol{H}}^\dagger = S_{\boldsymbol{H}}$ and $S_{\boldsymbol{H}}^\dagger = C_{\boldsymbol{H}}$. (Here $L^\dagger$ denotes the Moore-Penrose pseudo-inverse of an operator L).

7.4 Convolution-dominated operators

For the remainder of this chapter we assume that conditions (A1), (A2'), (B1), (C1) and (C2) hold. Using the fact that Λ is a subgroup, it is possible to dominate operators on $\boldsymbol{E}^d$ by convolutions. We consider the class of operators dominated by left convolution,

$$CD(\Lambda, w) := \{ T \in \mathbb{C}^{\Lambda\times\Lambda} \mid |T_{\lambda,\lambda'}| \leq a_{\lambda\lambda'^{-1}}, \text{ for some } a \in \ell^1_w(\Lambda) \},$$

and we endow it with the norm,

$$\|T\|_{CD(\Lambda,w)} := \inf \{ \|a\|_{l^1_w} \mid |T_{\lambda,\lambda'}| \leq a_{\lambda\lambda'^{-1}}, \text{ for all } \lambda, \lambda' \in \Lambda \}.$$

$CD(\Lambda, w)$ is a Banach *-algebra (see [49]). We also consider the Banach *-algebra of operators dominated by right convolution,

$$CD_R(\Lambda, w) := \{ T \in \mathbb{C}^{\Lambda\times\Lambda} \mid |T_{\lambda,\lambda'}| \leq a_{\lambda'^{-1}\lambda}, \text{ for some } a \in \ell^1_w(\Lambda) \},$$

and we endow it with a norm in a similar manner. We will use a slightly adapted version of the main result from [49].

Proposition 7.4.1. *The inclusion $CD_R(\Lambda, w) \hookrightarrow B(\ell^2(\Lambda))$ is spectral (i.e. it preserves the spectrum of each element).*[2] *Moreover, if $L \in CD_R(\Lambda, w)$ is a self-adjoint operator with closed range, then its pseudo-inverse $L^\dagger$ also belongs to $CD_R(\Lambda, w)$.*

Proof. Let Λ^{op} denote the set Λ considered with the opposite group operation, given by $\lambda ._{op} \lambda' = \lambda'\lambda$. Since $x \mapsto x^{-1}$ is an algebraic and topological isomorphism between Λ and Λ^{op} and the weight w is symmetric, if follows that Λ^{op} also satisfies the FGL-conditions with respect to the restriction of the weight w. Hence, [49, Corollary 6] implies $CD(\Lambda^{op}, w)$ is a spectral subalgebra of $B(\ell^2(\Lambda^{op}))$. Finally note that $CD_R(\Lambda, w) = CD(\Lambda^{op}, w)$ and that $B(\ell^2(\Lambda^{op})) = B(\ell^2(\Lambda))$.

The second part of the theorem is a consequence of the first one (cf. Remark 1.7.1). Since the inclusion $CD_R(\Lambda, w) \hookrightarrow B(\ell^2(\Lambda))$ is closed under inversion, it is also closed under holomorphic functional calculus. For a self-adjoint operator with closed range $L \in CD_R(\Lambda, w)$, its pseudo-inverse is given by $L^\dagger = f(L)$, where $f(z) = z^{-1}$, for $z \neq 0$ and $f(0) = 0$. f is holomorphic on the spectrum of L because, since the range of L is closed, 0 is an isolated point of its spectrum. □

Remark 7.4.1. *The result in [49] seems to be the most appropriate one for this context but in some cases it is also possible to apply the results in [104, 101] to the same end. If the group Λ is $\mathbb{Z}^d$, then the desired result also follows from [8], [67] and [102] with the advantage of slightly improving the assumptions on the weight.*

We now observe that $CD_R(\Lambda, w)$ acts on $\boldsymbol{E}^d$.

[2] Here, $B(\ell^2(\Lambda))$ denotes the algebra of bounded operators on $\ell^2(\Lambda)$.

Proposition 7.4.2. *Let $T \in CD_R(\Lambda, w)$. Then the following holds.*

(a) T maps $\boldsymbol{E}^d$ into $\boldsymbol{E}^d$ and $\|T\|_{\boldsymbol{E}^d\to\boldsymbol{E}^d} \leq \|T\|_{CD_R(\Lambda,w)}$.

(b) $T : (\boldsymbol{E}^d, \ell^1_w) \to (\boldsymbol{E}^d, \ell^1_w)$ is continuous.

Proof. Part (a) follows from Proposition 7.2.1 and the solidity of $\boldsymbol{E}^d$. For part (b), observe that the spaces L^1_w and $L^\infty_{1/w}$ satisfy the same assumptions that $\boldsymbol{E}$ (cf. Remark 7.2.1) and consequently, by part (a), every operator in $CD_R(\Lambda, w)$ maps ℓ^1_w into ℓ^1_w and $\ell^\infty_{1/w}$ into $\ell^\infty_{1/w}$. Since the class $CD_R(\Lambda, w)$ is closed under taking adjoints it follows that $T : \ell^\infty_{1/w} \to \ell^\infty_{1/w}$ is weak* continuous, so part (b) follows. □

7.5 Invertibility of multipliers

We will now prove the invertibility of M^m on $\boldsymbol{S}$. We assume that $m \in L^\infty(\mathcal{G})$ is real-valued and satisfies,

$$0 < A \leq m \leq B < \infty,$$

for some constants A, B, and we will establish a number of claims that will lead to the desired conclusion.

Claim 7.5.1. *The operator $M^m : \mathbb{H} \to \mathbb{H}$ is invertible.*

Proof. Observe that, since $P_{\mathbb{H}} : L^2(\mathcal{G}) \to \mathbb{H}$ is the orthogonal projector, and m is real-valued, the operator $M^m : \mathbb{H} \to \mathbb{H}$ is self-adjoint. Moreover, for $f \in \mathbb{H}$,

$$\begin{aligned} \|M^m(f)\|_{\mathbb{H}}\|f\|_{\mathbb{H}} &\geq \langle P(mf), f\rangle = \langle mf, f\rangle \\ &= \int_{\mathcal{G}} m(x)\,|f(x)|^2\,dx \geq A\|f\|^2_{\mathbb{H}}. \end{aligned}$$

Hence, $M^m : \mathbb{H} \to \mathbb{H}$ is self-adjoint and bounded below and therefore invertible. □

Remark 7.5.1. *Claim 7.5.1 may not be true without the assumption that m is nonnegative. Indeed, if $\mathcal{G} = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $\varphi_\lambda = \psi_\lambda = \chi_{[\lambda,\lambda+1]}$ and $m = \chi_{(-\infty,1/2)} - \chi_{[1/2,+\infty)}$, then $M^m(\varphi_0) = 0$.*

Let $L \in \mathbb{C}^{\Lambda\times\Lambda}$ be the matrix representing the operator $S^*_{\boldsymbol{H}} M^m S_{\boldsymbol{H}} : \ell^2(\Lambda) \to \ell^2(\Lambda)$. Hence, L is given by,

$$L_{\lambda,\lambda'} := \langle m\varphi_{\lambda'}, \varphi_\lambda\rangle\,.$$

Claim 7.5.2. *The matrix L belongs to $CD_R(\Lambda, w)$ and has a Moore-Penrose pseudo-inverse $L^\dagger$ that also belongs to $CD_R(\Lambda, w)$. In addition, $(M^m)^{-1} : \boldsymbol{H} \to \boldsymbol{H}$ can be decomposed as $(M^m)^{-1} = S_{\boldsymbol{H}} L^\dagger S^*_{\boldsymbol{H}}$.*

Proof. To see that $L \in CD_R(\Lambda, w)$ let us estimate,

$$\left|L_{\lambda,\lambda'}\right| \lesssim \int_{\mathcal{G}} h(\lambda^{-1}x)h(\lambda'^{-1}x)dx = a_{\lambda'^{-1}\lambda},$$

where $a_\lambda := h * h^\vee(\lambda)$. Using Lemmas 1.6.1 and 1.6.2 we see that $a \in \ell^1_w$.

The operator $S_{\boldsymbol{H}}$ has range $\boldsymbol{H}$ because $\{\varphi_\lambda\}_\lambda$ is a frame for $\boldsymbol{H}$ (cf. Claim 7.3.1). Since $M^m : \boldsymbol{H} \to \boldsymbol{H}$ is invertible by Claim 7.5.1, the range of $L = S^*_{\boldsymbol{H}} M^m S_{\boldsymbol{H}}$ equals $S^*_{\boldsymbol{H}}(\boldsymbol{H})$. This subspace is closed because $S^*_{\boldsymbol{H}}$ is bounded below on $\boldsymbol{H}$ (that is the frame condition). Hence, L has closed range and consequently has a pseudo-inverse $L^\dagger$. Since M^m is self-adjoint, so is L. In addition, $L^\dagger$ is given by,

$$L^\dagger = C_{\boldsymbol{H}}(M^m)^{-1}C^*_{\boldsymbol{H}}.$$

Hence, $(M^m)^{-1} = S_{\boldsymbol{H}} L^\dagger S^*_{\boldsymbol{H}}$, (where the operator $S^*_{\boldsymbol{H}}$ is restricted to $\boldsymbol{H}$). Finally, by Proposition 7.4.1, $L^\dagger \in CD_R(\Lambda, w)$. □

Now we can prove the invertibility of M^m on $\boldsymbol{S}$.

Proposition 7.5.1. *Let $m \in L^\infty(\mathcal{G})$ be real-valued and satisfy $0 < A \le m \le B < \infty$, for some constants A, B. Then, the multiplier $M^m : \boldsymbol{S} \to \boldsymbol{S}$ is invertible.*

Proof. Let $N^m : \boldsymbol{S} \to \boldsymbol{S}$ be the operator defined by $N^m := S L^\dagger S'$. It follows from Proposition 7.4.2 and Claim 7.5.2 that N^m is bounded. Moreover, by Claim 7.5.2, for $f \in \boldsymbol{S} \cap \mathbb{H}$,

$$M^m N^m(f) = N^m M^m(f) = f. \tag{7.2}$$

By Propositions 3.2.1 and 7.4.2, the operators M^m and N^m are continuous in the $(\boldsymbol{E}, W(L^\infty, L^1_w))$ topology. Since by Proposition 3.2.1, any $f \in \boldsymbol{S}$ can be approximated by a net of elements of $\boldsymbol{S} \cap \mathbb{H}$ in the $(\boldsymbol{E}, W(L^\infty, L^1_w))$ topology (by considering the partial sums of the expansion in Equation (3.2)) it follows that Equation (7.2) holds for arbitrary $f \in \boldsymbol{S}$. Hence $M^m : \boldsymbol{S} \to \boldsymbol{S}$ is invertible. □

7.6 Characterization of the atomic space

Finally we can derive the extension of Theorem 6.3.1 to more general partitions of unity.

Theorem 7.6.1. *Suppose that Assumptions (A1), (A2'), (B1), (C1) and (C2) are satisfied. Let $\left\{\theta_\gamma : \gamma \in \Gamma\right\}$ be given by $\theta_\gamma = m\eta_\gamma$, where $0 < A \le m \le B < \infty$.*

Then the operator,

$$\begin{aligned} \widetilde{C^{\boldsymbol{B}}} : \boldsymbol{S} &\to \boldsymbol{E}^d_{\boldsymbol{B}}(\Gamma) \\ f &\mapsto (P(f\theta_\gamma))_\gamma \end{aligned}$$

is left-invertible. Consequently, the following norm equivalence holds for $f \in \boldsymbol{S}$,

$$\|f\|_{\boldsymbol{E}} \approx \|(\|P(f\theta_\gamma)\|_{\boldsymbol{B}})_\gamma\|_{\boldsymbol{E}^d}.$$

Remark 7.6.1. *Any family $\left\{\theta_\gamma\right\}_\gamma$ that is enveloped by g and whose sum is a real-valued function that is bounded away from 0 and ∞, has the prescribed form for some adequate choice of the partition of unity $\left\{\eta_\gamma\right\}_\gamma$ and the function m.*

Proof. First observe that $\widetilde{C^{\boldsymbol{B}}}(f) = C^{\boldsymbol{B}}(mf)$, so $\widetilde{C^{\boldsymbol{B}}}$ is bounded on $\boldsymbol{S}$ by Propositions 3.1.1 and 6.1.1. By Proposition 7.5.1, M^m is invertible, so by Theorem 6.2.1 we can choose a relatively compact neighborhood of the identity U such that M^m_U is also invertible. Since the operator P_U (cf. Equation (6.2)) can be factored as $P_U = S^{\boldsymbol{B}}_U C^{\boldsymbol{B}}$, we have that, $M^m_U(f) = PS^{\boldsymbol{B}}_U C^{\boldsymbol{B}}(mf) = PS^{\boldsymbol{B}}_U \widetilde{C^{\boldsymbol{B}}}(f)$. Since M^m_U is invertible, $\widetilde{C^{\boldsymbol{B}}}$ is is left-invertible, as claimed. This also implies the desired norm equivalence. □

7.7 Applications

7.7.1 Time-Frequency analysis

Let us recall some notation and facts from time-frequency analysis (cf. Section 1.11). For $f, h \in L^2(\mathbb{R}^d)$, the *Short-Time Fourier Transform* (STFT) (or *windowed Fourier Transform*) is defined by,

$$\mathcal{V}_h f(x,\varsigma) = \int_{\mathbb{R}^d} f(y) e^{-2\pi i \varsigma y} \overline{h(y-x)} dy.$$

The translation and modulation operators are given by $T_x f(y) := f(y-x)$ and $M_\varsigma f(y) := e^{2\pi i \varsigma y} f(y)$, so that,

$$\mathcal{V}_h f(x,\varsigma) := \langle f, M_\varsigma T_x h \rangle . \tag{7.3}$$

If h is suitably normalized, $\mathcal{V}_h : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is an isometry. The adjoint (inverse) STFT is given by,

$$\mathcal{V}_h^* F(x) = \int_{\mathbb{R}^{2d}} F(y,\varsigma) M_\varsigma T_y h(x) dy d\varsigma,$$

so the localization operator with symbol $m \in L^\infty(\mathbb{R}^{2d})$ is given by,

$$\mathrm{H}_m f(x) = \mathcal{V}_h^*(m \mathcal{V}_h f)(x) = \int_{\mathbb{R}^{2d}} m(y,\varsigma) \mathcal{V}_h f(y,\varsigma) M_\varsigma T_y h(x) dy d\varsigma.$$

If h belongs to the Schwartz class, the definition in Equation (7.3) extends to tempered distributions. Modulation spaces are then defined by imposing integrability conditions of the STFT. Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be a submultiplicative, even weight that satisfies the GRS condition: $\lim_{n\to\infty} w(nx)^{1/n} = 1$, for all $x \in \mathbb{R}^{2d}$. Let $v : \mathbb{R}^{2d} \to (0, +\infty)$ be a w-moderated weight; that is: $v(x + y) \lesssim w(x)v(y)$, for all $x, y \in \mathbb{R}^{2d}$. Assume further that w is moderated by a polynomial weight[3] . For $1 \leq p, q \leq +\infty$, the modulation space $M^{p,q}_v$ is defined as,

$$M^{p,q}_v := \{\, f \in \mathcal{S}'(\mathbb{R}^d) \,\big|\, \mathcal{V}_h f \in L^{p,q}_v(\mathbb{R}^{2d}) \,\}$$

where,

$$\|F\|_{L^{p,q}_v} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,\varsigma)|^p v(x,\varsigma)^p dx \right)^{q/p} d\varsigma \right)^{1/q},$$

with the usual modifications when p or q are $+\infty$. $M^{p,q}_v$ is of course given the norm $\|f\|_{M^{p,q}_v} = \|\mathcal{V}_h f\|_{L^{p,q}_v}$.

After some normalizations and identifications, modulation spaces can be regarded as coorbit spaces of the Schrödinger representation of the Heisenberg group. We chose however to consider them in the context of Section 3.2. For $h \in M^{1,1}_w$, $1 \leq p, q \leq \infty$, and w, v as above, we let $\mathcal{G}$ be $\mathbb{R}^d \times \mathbb{R}^d$, $\boldsymbol{E} := L^{p,q}_v(\mathcal{G})$ and $\boldsymbol{S} := \mathcal{V}_h(M^{p,q}_v)$.

For an adequate lattice $\Lambda \subseteq \mathbb{R}^{2d}$ the system $\{\, M_\varsigma T_x h \,\big|\, (x,\varsigma) \in \Lambda \,\}$ gives rise to an atomic decomposition of $M^{p,q}_v$. Moreover, on $M^{2,2}_v$ the dual atoms consist of the Hilbert-space dual frame of $\{\, M_\varsigma T_x h \,\big|\, (x,\varsigma) \in \Lambda \,\}$ and are of the form $\{\, M_\varsigma T_x \tilde{h} \,\big|\, (x,\varsigma) \in \Lambda \,\}$ for some function $\tilde{h} \in M^{1,1}_v$ (see Section 1.11). Hence, if we define $\varphi_{(x,\varsigma)} := \mathcal{V}_h(M_\varsigma T_x h)$ and $\psi_{(x,\varsigma)} := \mathcal{V}_h(M_\varsigma T_x \tilde{h})$, the atoms $\{\, \varphi_\lambda \,\big|\, \lambda \in \Lambda \,\}$ and dual atoms $\{\, \psi_\lambda \,\big|\, \lambda \in \Lambda \,\}$ provide an atomic decomposition for $\boldsymbol{S}$.

Since $\mathcal{G}$ is abelian, left and right amalgam spaces coincide. The envelopes for the atoms and dual atoms are the functions $\mathcal{V}_h h$ and $\mathcal{V}_h \tilde{h}$.[4] These functions indeed envelope the atoms because of the straightforward relation: $\left|\mathcal{V}_h M_\varsigma T_x f\right| = |\mathcal{V}_h f(\cdot - (x,\varsigma))|$. The fact that h and $\tilde{h}$ belong to $M^{1,1}_w$ means that $V_h h$ and $V_h \tilde{h}$ belong to L^1_w, but it is well-know that in this case they also belong to $W(L^\infty, L^1_w)$ (see Remark 5.2.5). This fact can also be derived from the norm equivalence in Proposition 3.1.1.

Let us now consider a family of functions $\{\, \theta_\gamma \,\big|\, \gamma \in \Gamma \,\}$ that satisfy

$$0 < A \leq \sum_\gamma \theta_\gamma \leq B < \infty.$$

Let us also assume that Γ is a relatively separated subset of $\mathbb{R}^{2d}$ and that there exists a function $g \in L^1_w(\mathbb{R}^{2d})$ such that $\left|\theta_\gamma(x)\right| \leq g(x - \gamma)$, for all $x \in \mathbb{R}^{2d}$ and $\gamma \in \Gamma$. We will let the space $\boldsymbol{B}$ that measures the localized pieces be an unweighted Lebesgue space $L^{r,s}$. We are then in the situation of Theorem 7.6.1 (remember that, since $\mathcal{G}$ is abelian, $L^1_w = W^{\text{weak}}_R(L^\infty, L^1_w)$ - cf. Proposition 2.1.1).

To illustrate Theorem 7.6.1 more clearly we further assume that $\Gamma = \Gamma_1 \times \Gamma_2$ for two relatively separated sets $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^d$. Then we get the following.

[3] This assumption is only made in order to define modulation spaces as subsets of distributions. For a general weight, the same results hold, but the space $M^{p,q}$ has to be constructed as an abstract coorbit space.

[4] For simplicity Assumption (A2') requires the same envelope for both the atoms and the dual atoms, but clearly if they have different envelopes then their sum serves as a common envelope.

Theorem 7.7.1. *For all* $1 \leq s, t \leq \infty$, *the quantity,*

$$\left(\sum_{\gamma_2 \in \Gamma_2} \left(\sum_{\gamma_1 \in \Gamma_1} \|\mathrm{H}_{\theta_{(\gamma_1,\gamma_2)}} f\|^p_{M^{s,t}} \nu(\gamma_1, \gamma_2)^p \right)^{q/p} \right)^{1/q},$$

is an equivalent norm on $M^{p,q}_\nu$ *(with the usual modifications when* p *or* q *are* ∞*).*

This generalizes the main result in [34] in two directions. The results in [34] apply only to partitions of unity produced by lattice translations of a single function, whereas Theorem 7.7.1 allows for irregular partitions. Secondly, in [34] the space measuring the localized pieces is restricted to be L^2. In contrast, in Theorem 7.7.1 it is possible to measure the localized pieces using the whole range of unweighted modulations spaces.

The proof in [34] resorts to techniques from rotation algebras and spectral theory to construct an atomic decomposition that is simultaneously adapted to all the localization operators $\{\mathrm{H}_\theta\}_\theta$. Part of our motivation came from the observation that such an atomic decomposition could be obtained in a more constructive manner by using the technique of frame surgery from Chapter 5.

7.7.2 Localized frames

Theorem 7.6.1 also applies to coorbit spaces of localized frames (cf. Section 1.13). Let $\boldsymbol{H}$ be a Hilbert space and let $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}^d}$ be a frame for it. Assume that $\mathcal{F}$ satisfies the following localization property,

$$\left| \left\langle f_k, f_j \right\rangle \right| \leq a_{k-j}, \qquad (k, j \in \mathbb{Z}^d),$$

where $a \in \ell^1_w(\Lambda)$ and w is a subexponential weight that satisfies $w(x) \gtrsim (1 + |x|)^\delta$, for some $\delta > 0$.

Let $\mathcal{G} = \{g_k\}_{k\in\mathbb{Z}^d}$ be the canonical dual frame of $\mathcal{F}$. Every element $f \in \mathbb{H}$ then admits the expansion $f = \sum_k \langle f, f_k\rangle\, g_k$. The coorbit spaces $\mathbb{H}^p_\nu$ are defined by imposing ℓ^p_ν sumability conditions to that expansion (see Section 1.13 for the details).

Frame multipliers are defined by applying a mask to the canonical frame expansion. For $m \in \ell^\infty(\mathbb{Z}^d)$, let

$$M_m(f) := \sum_k m_k \langle f, f_k\rangle\, g_k,$$

where $m \in l^\infty(\mathbb{Z}^d)$. Theorem 7.6.1 can be applied using $\mathcal{G} = \Lambda = \mathbb{Z}^d$ and yields a characterization of the spaces $\mathbb{H}^p_\nu$ in terms of frame multipliers.

When $\mathbb{H} = L^2(\mathbb{R}^d)$ and $\mathcal{F}$ is a Gabor frame, then the corresponding coorbit spaces are modulation spaces (see Section 1.11) and the corresponding multipliers are the Gabor multipliers from Section 5.2.4. The fact that the index set is $\mathbb{Z}^d$ is no limitation. The case of a general relatively separated set as index set can be reduced to this one by a well-known trick (see [5]).

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