DYNAMICAL SAMPLING FOR SHIFT-PRESERVING OPERATORS

A. AGUILERA, C. CABRELLI, D. CARBAJAL, AND V. PATERNOSTRO

Abstract. In this note, we solve the dynamical sampling problem for a class of shift-preserving operators $L : V \to V$ acting on a finitely generated shift-invariant space V. We find conditions on L and a finite set of functions of V so that the iterations of the operator L on the functions produce a frame generator set of V. This means that the integer translations of the generators form a frame of V.

1. Introduction

Dynamical Sampling addresses the problem of recovering a signal that evolves from its spatial-time samples. That is, let D be a bounded operator (the evolution operator) and f an unknown signal that we want to recover. Assume that we have insufficient samples of f. Will it be possible to compensate for this lack of information from f, if we sample the evolved signals at the same locations? i.e. sampling the signals $Df, D^2f, ...$?

Mathematically, this question can be reformulated as follows (see [6]): Let $\mathcal{H}$ be a separable Hilbert space, $D : \mathcal{H} \to \mathcal{H}$ a bounded operator and $\mathcal{F} = \{f_i : i \in I\} \subset \mathcal{H}$ a set of functions. Find conditions on D and $\mathcal{F}$ such that $\{D^j f_i : i \in I,\ j \in K\}$ is a basis or a frame of $\mathcal{H}$. Here, I and K are subsets of $\mathbb{N} \cup \{0\}$.

This problem has recently attracted a lot of attention and has been set in different scenarios. See [1],[7],[3],[2],[8],[6],[9] for different instances of the dynamical sampling problem. In [21],[22],[23],[26], the authors studied the problem of when a given frame can be represented as a discrete orbit of an operator. See also [10],[27] for posible applications.

One case that is mathematically very deep and rich is when $\mathcal{H}$ is infinite-dimensional, D is a bounded normal operator and $\mathcal{F}$ is finite. This case has been tackled using techniques from different areas such as spectral theory, Hardy spaces, and Carleson measures (see [6],[9],[17],[5],[18]). In particular, it has been proved that in this case the iterations of a finite set of functions under a bounded normal operator will never be a basis (see [6] and [17]), so only frames are possible and require many hypotheses on D and $\mathcal{F}$. On the other hand, the finite-dimensional case was solved completely for general linear transformations in [6], (see also [17]). However, in this case, no estimates of the frame bounds were given.

In this article, we study the *Dynamical Sampling* problem for *shift-preserving* operators acting on *shift-invariant* spaces of $L^2(\mathbb{R}^d)$, as we describe below.

2010 *Mathematics Subject Classification:* Primary 47A15, 94A20, 42C15

Keywords: Shift-invariant spaces, shift-preserving operators, range function, range operator, frames, Dynamical Sampling.

The research of the authors is partially supported by grants: UBACyT 20020170100430BA, PICT 2014-1480 (ANPCyT) and CONICET PIP 11220150100355. In particular VP is also supported by UBACyT 20020170200057BA and PICT-2016- 2616 (Joven).

A shift-invariant space is a closed subspace V of $L^2(\mathbb{R}^d)$ that is invariant under the action of translations along $\mathbb{Z}^d$ (or a lattice in a more general case). A set $\Phi \subset V$ is a set of generators of V if $V = S(\Phi) = \overline{\text{span}}\left\{T_k\varphi : \varphi \in \Phi,\, k \in \mathbb{Z}^d\right\}$, where $T_k f(x) = f(x-k)$. If $V = S(\Phi)$ for a finite set Φ then V is said to be *finitely generated*, and its *length* is the minimum cardinal of all sets of generators of V. If the integer translates of a set of generators form a frame of $S(\Phi)$ we say that Φ is a *frame generator set*. An operator $L : V \to V$ is said *shift preserving* if it commutes with integer translates (i.e. $LT_k = T_k L$ for every $k \in \mathbb{Z}^d$).

The dynamical sampling problem for shift-preserving operators that we study and solve in this article is the following.

Assume that V is a finitely generated shift-invariant space of length ℓ and that $\mathcal{F} = \{f_1, ..., f_m\} \subset V$. Let $L : V \to V$ be a bounded shift-preserving operator. *Find necessary and sufficient conditions on $\mathcal{F}$ and L in order that the collection $\{L^j f_i : i = 1, ..., m\,; j = 0, ..., \ell - 1\}$ is a frame generator set of V.*

As an application, we can think that we want to recover the shift-invariant space V and we only know some functions $f_1, ..., f_m$ in V (i.e. the set $\mathcal{F}$). If $\mathcal{F}$ is not a set of generators, then the integer translates of the functions in $\mathcal{F}$ are not enough to obtain the whole space V. So we resort to the evolution operator L to get a frame generator set.

To solve the dynamical sampling problem for shift-preserving operators, we make intensive use of the concept of *range function*, *range operator* and *fiberization techniques*, (see Section 2 for definitions and properties).

The results require a thorough understanding of the structure of shift-preserving operators. The invariant space V is isomorphic to a field of finite-dimensional subspaces of $\ell^2(\mathbb{Z}^d)$ (each one is a value of the range function) and the shift-preserving operator is isomorphic to a field of linear transformations acting on these subspaces (the range operator).

Since each value of a range operator is a linear transformation acting on a finite-dimensional space, the main idea behind this program is to translate the well known structure of these linear transformations to our shift-preserving operator. This requires certain uniformity across the different values of the range operator. In this regard, we will turn to the theory of s-diagonalization of shift-preserving operators that is fully developed in [4].

The key point in our analysis will be to reduce the problem of dynamical sampling for a shift-preserving operator acting on a finitely generated shift-invariant space, to a family of dynamical sampling problems where the operator that we iterate (the fiber of the range operator) acts on a finite-dimensional space (the fiber spaces). Then, we apply the known results on the finite-dimensional dynamical sampling to this case. Since our problem requires that the frame bounds are uniform across the fiber spaces, we obtain an estimate of the frame bounds for the finite-dimensional case, (see Section 3).

Using this estimate we obtain our main result that solves the dynamical problem. That is, we give necessary and sufficient conditions on L an $\mathcal{F}$ to extend the set $\mathcal{F}$ to a frame generator set by iterating L on the functions in $\mathcal{F}$.

The paper is organized as follows. In Section 2 we provide the basics of the theory of shift-invariant spaces and shift-preserving operators needed for the development

of the whole paper. In particular, we review the theory of s-diagonalization for shift-preserving operators. The main results are presented in Section 3. In Subsection 3.1 we give frame bound estimates for the finite-dimensional case of dynamical sampling. Then, in Subsection 3.2 we solve the dynamical sampling problem for shift-preserving operators.

2. Preliminaries

2.1. Shift-invariant spaces and frame generator sets.

In this subsection, we review some of the standard facts on shift-invariant subspaces of $L^2(\mathbb{R}^d)$ and recall a caracterization of frames on these spaces using a technique known as fiberization. These spaces have been used in approximation theory, sampling theory, and wavelets, and their structure is very well known. See [24],[12],[13],[25],[14] in the Euclidean case, and [19],[15],[11] in the context of topological groups. We now state the precise definitions and some properties of these spaces. For properties of frames see [20].

Definition 2.1. A closed subspace $V \subset L^2(\mathbb{R}^d)$ is *shift invariant* if for each $f \in V$ we have $T_k f \in V$ for any $k \in \mathbb{Z}^d$, where $T_k f(x) = f(x-k)$.

Given a countable set $\Phi \subset L^2(\mathbb{R}^d)$, the shift-invariant space generated by Φ is

$$V = S(\Phi) = \overline{\operatorname{span}} \left\{ T_k \varphi \, : \, \varphi \in \Phi, \, k \in \mathbb{Z}^d \right\}$$

and Φ is called a set of generators of V. When Φ is a finite set, we say that V is a *finitely generated* shift-invariant space.

We will denote by $E(\Phi)$ the family of translations of Φ, i.e.

$$E(\Phi) = \left\{ T_k \varphi \, : \, \varphi \in \Phi, \, k \in \mathbb{Z}^d \right\}.$$

When $E(\Phi)$ forms a frame of V we will say that Φ is a *frame generator set* of V.

The structure of a shift-invariant space can be studied in terms of its range function. This technique is known as fiberization.

Let $L^2([0,1)^d, \ell^2(\mathbb{Z}^d))$ be the Hilbert space of all vector-valued measurable functions $\psi : [0,1)^d \to \ell^2(\mathbb{Z}^d)$ with finite norm, where the norm is given by

$$\|\psi\| = \left(\int_{[0,1)^d} \|\psi(\omega)\|_{\ell^2}^2 \, d\omega \right)^{1/2}.$$

The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is given by

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \omega \rangle} \, dx,$$

and extends by density to an isometric isomorphism in $L^2(\mathbb{R}^d)$.

Proposition 2.2. [14, Proposition 1.2] *The map $\mathcal{T} : L^2(\mathbb{R}^d) \to L^2([0,1)^d, \ell^2(\mathbb{Z}^d))$ defined by*

$$\mathcal{T} f(\omega) = \{ \hat{f}(\omega + k) \}_{k \in \mathbb{Z}^d},$$

is an isometric isomorphism. We call $\mathcal{T} f(\omega)$ the fiber of f at ω. Moreover, it satisfies that

$$\mathcal{T} T_k f(\omega) = e_k(\omega) \, \mathcal{T} f(\omega),$$

where $e_k(\omega) = e^{-2\pi i \langle \omega, k \rangle}$, $k \in \mathbb{Z}^d$.

Definition 2.3. A *range function* is a mapping

$$J : [0,1)^d \to \{ \text{ closed subspaces of } \ell^2(\mathbb{Z}^d) \}$$
$$\omega \mapsto J(\omega).$$

It is said that a range function J is measurable if the scalar function $\omega \mapsto \langle P_{J(\omega)}u, v\rangle$ is measurable for every $u, v \in \ell^2(\mathbb{Z}^d)$, where $P_{J(\omega)}$ is the orthogonal projection from $\ell^2(\mathbb{Z}^d)$ onto $J(\omega)$.

Shift-invariant spaces are characterized in terms of range functions as the following theorem, due to Bownik, shows.

Theorem 2.4. [14, Proposition 1.5] *A closed subspace $V \subset L^2(\mathbb{R}^d)$ is shift invariant if and only if there exists a measurable range function J such that*

$$V = \{f \in L^2(\mathbb{R}^d) : \mathcal{T}f(\omega) \in J(\omega) \text{ for a.e. } \omega \in [0,1)^d\}.$$

Furthermore, if $V = S(\Phi)$ for some countable set $\Phi \subset L^2(\mathbb{R}^d)$, then

$$J(\omega) = \overline{span}\{\mathcal{T}f(\omega) \,:\, f \in \Phi\}$$

for a.e. $\omega \in [0,1)^d$.

We call the subspace $J(\omega)$ the fiber space of V at ω. Under the convention that two range functions are identified if they are equal a.e. $\omega \in [0,1)^d$, the correspondence between V and J is one-to-one.

In particular, when Φ is a finite set, the previous theorem allows us to translate some problems in infinite-dimensional shift-invariant spaces, into problems of finite dimension that can be treated with linear algebra.

The next lemma was proved by Helson (see [24]) and will be useful in the main results of this paper.

Lemma 2.5. *Let $V \subset L^2(\mathbb{R}^d)$ be a shift-invariant space. For each $f \in L^2(\mathbb{R}^d)$ we have that*

$$\mathcal{T}(P_V f)(\omega) = P_{J(\omega)}(\mathcal{T}f(\omega)).$$

It is possible to give a characterization of frames of a shift-invariant space V in terms of its fibers as shown in the next result which will be crucial for our problem.

Theorem 2.6. [14, Theorem 2.3] *Let $\Phi \subset L^2(\mathbb{R}^d)$ be a countable set. Then the following conditions are equivalent:*

(i) *The system $E(\Phi)$ is a frame of V with bounds $A, B > 0$;*
(ii) *The system $\{\mathcal{T}\varphi(\omega) : \varphi \in \Phi\} \subset \ell^2(\mathbb{Z}^d)$ is a frame of $J(\omega)$ with uniform bounds $A, B > 0$ for a.e. $\omega \in [0,1)^d$.*

Recall that the *length* of a finitely generated shift-invariant space $V \subset L^2(\mathbb{R}^d)$ is defined as the smallest natural number ℓ such that there exist $\varphi_1, ..., \varphi_\ell \in V$ with $V = S(\varphi_1, ..., \varphi_\ell)$. An equivalent definition of the length in terms of the range function J associated to V is $\ell = \text{ess sup}_{\omega\in[0,1)^d} \dim J(\omega)$.

The *spectrum* of V is defined by the set

$$\sigma(V) = \left\{\omega \in [0,1)^d \,:\, \dim J(\omega) > 0\right\}.$$

2.2. **Shift-preserving operators and s-diagonalization.** In this subsection, we give a brief exposition on the structure of shift-preserving operators and s-diagonalization. For a more comprehensive treatment of this topic we refer the reader to [4] and [14].

Definition 2.7. Let $V \subset L^2(\mathbb{R}^d)$ be a shift-invariant space and $L : V \to L^2(\mathbb{R}^d)$ be a bounded operator. We say that L is *shift preserving* if $LT_k = T_k L$ for all $k \in \mathbb{Z}^d$.

Shift-preserving operators are the natural operators acting on shift-invariant spaces. They were introduced by Bownik in [14], where he also studied their properties through the concept of *range operator*. The notion of range operator permits to decode the action of a shift-preserving operator throughout its fiber map. These last two concepts are a very natural tool for studying shift-invariant spaces. Shift-preserving operators are in one to one correspondence with range operators.

Definition 2.8. Let V be a shift-invariant space with range function J. A *range operator* on J is a mapping

$$R : [0,1)^d \to \left\{\text{bounded operators defined on closed subspaces of } \ell^2(\mathbb{Z}^d)\right\},$$

such that the domain of $R(\omega)$ is $J(\omega)$ for a.e. $\omega \in [0,1)^d$.

It is said that R is measurable if $\omega \mapsto \langle R(\omega)P_{J(\omega)}u, v\rangle$ is a measurable scalar function for every $u, v \in \ell^2(\mathbb{Z}^d)$.

In [14, Theorem 4.5], Bownik proved that given a shift-preserving operator $L : V \to L^2(\mathbb{R}^d)$, there exists a measurable range operator R on J such that

$$(\mathcal{T} \circ L)f(\omega) = R(\omega)\left(\mathcal{T}f(\omega)\right), \tag{1}$$

for a.e. $\omega \in [0,1)^d$ and $f \in V$. If range operators which are equal almost everywhere are identified, the range operator associated to L is unique.

Conversely, if R is a measurable range operator on J with

$$\operatorname*{ess\,sup}_{\omega\in[0,1)^d} \|R(\omega)\| < \infty$$

then, there exists a bounded shift-preserving operator $L : V \to L^2(\mathbb{R}^d)$ such that the intertwining property (1) holds, and its norm operator is given by

$$\|L\| = \operatorname*{ess\,sup}_{\omega\in[0,1)^d} \|R(\omega)\|. \tag{2}$$

We will consider the particular case where $L : V \to V$. Thus, this implies that L has a corresponding range operator R such that $R(\omega) : J(\omega) \to J(\omega)$ for a.e. $\omega \in [0,1)^d$. The following result is due to Bownik (see [14]).

Theorem 2.9. *Let V be a shift-invariant space and $L : V \to V$ a shift-preserving operator with associated range operator R. Then, the adjoint operator $L^* : V \to V$ is also shift preserving and its corresponding range operator R^* satisfies that $R^*(\omega) = (R(\omega))^*$ for a.e. $\omega \in [0,1)^d$. As a consequence, L is self-adjoint if and only if $R(\omega)$ is self-adjoint for a.e. $\omega \in [0,1)^d$, and L is a normal operator if and only if $R(\omega)$ is a normal operator for a.e. $\omega \in [0,1)^d$.*

The structure of a shift-preserving operator L acting on a shift-invariant space V can be studied in terms of its range operator, as suggested by (1). When V is a finitely generated shift-invariant space, R can be seen as a field of linear transformations acting on finite-dimensional spaces. We can exploit this fact to translate the structure of these linear transformations back to the shift-preserving operator L through the isometric isomorphism $\mathcal{T}$. In particular, this correspondence between L and R allows us to introduce a notion of diagonalization for shift-preserving operators, which is called s-diagonalization. This theory has been extensively developed in [4]. Here, we present the main definitions and results. For instance, we have a simpler representation of a bounded, normal, shift-preserving operator.

Definition 2.10. We say that a sequence $a = \{a(j)\}_{j\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ is of *bounded spectrum* if $\hat{a} \in L^\infty([0,1)^d)$, where $\hat{a}(\omega) = \sum_{j\in\mathbb{Z}^d} a(j)e_j(\omega)$.

Now, we give the definitions of s-eigenvalue and s-eigenspace of a shift-preserving operator.

Definition 2.11. Let V be a shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. Given $a \in \ell^2(\mathbb{Z}^d)$ a sequence of bounded spectrum, let $\Lambda_a : V \to V$ be the operator defined by $\Lambda_a = \sum_{j\in\mathbb{Z}^d} a(j)\, T_j$. We say that Λ_a is an *s-eigenvalue* of L if

$$V_a := \ker(L - \Lambda_a) \neq \{0\}.$$

We call V_a the *s-eigenspace* associated to Λ_a.

It is easy to see that the bounded spectrum assumption of a guarantees that Λ_a is well-defined and bounded (see [4, Proposition 4.1]). Furthermore, V_a is a shift-invariant subspace of V and for every $f \in V_a$, we have that $Lf = \Lambda_a f$. Since $\widehat{\Lambda_a f} = \hat{a}\hat{f}$, we see that for every $f \in V_a$,

$$R(\omega)(\mathcal{T}f(\omega)) = \mathcal{T}(Lf)(\omega) = \mathcal{T}(\Lambda_a f)(\omega) = \hat{a}(\omega)\mathcal{T}f(\omega),$$

for a.e. $\omega \in [0,1)^d$. This shows that the s-eigenvalues of L are closely related with the eigenvalues of its range operator, as we state in the next result whose proof is given in [4].

Proposition 2.12. *Let V be a shift-invariant space with range function J such that $\dim J(\omega) < \infty$ for a.e. $\omega \in [0,1)^d$, $L : V \to V$ a bounded shift-preserving operator and $a \in \ell^2(\mathbb{Z}^d)$ a sequence of bounded spectrum. Then, the following statements hold:*

(i) *If Λ_a is an s-eigenvalue of L, then $\lambda_a(\omega) := \hat{a}(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in \sigma(V_a)$.*
(ii) *The mapping $\omega \mapsto \ker(R(\omega) - \lambda_a(\omega)\mathcal{I})$, $\omega \in [0,1)^d$ is the measurable range function of V_a, which we will denote J_{V_a}.*

We remark that if $V_a \cap V_b = \{0\}$, then $\hat{a}(\omega) \neq \hat{b}(\omega)$ almost everywhere in $\sigma(V_a) \cap \sigma(V_b)$, that is, $\hat{a}(\omega)$ and $\hat{b}(\omega)$ correspond to different eigenvalues of $R(\omega)$ (see [4, Proposition 4.7]).

Definition 2.13. Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. We say that L is *s-diagonalizable* if there exist

$r \in \mathbb{N}$ and $a_1, \dots, a_r$ sequences of bounded spectrum such that $\Lambda_{a_1}, \dots, \Lambda_{a_r}$ are s-eigenvalues of L and

$$V = V_{a_1} \oplus \cdots \oplus V_{a_r},$$

where Λ_{a_j} and V_{a_j} for $j = 1, ..., r$ are given in Definition 2.11. In this case, we say that $(V, L, a_1, ...a_r)$ is an *s-diagonalization of L*.

When L is s-diagonalizable and $(V, L, a_1, ...a_r)$ is an s-diagonalization of L, then $R(\omega)$ is diagonalizable for a.e. $\omega \in \sigma(V)$ (see [4, Theorem 6.4]). In particular, the range function J associated to V has the following decomposition in direct sum

$$J(\omega) = J_{V_{a_1}}(\omega) \oplus \cdots \oplus J_{V_{a_r}}(\omega),$$

for a.e. $\omega \in [0,1)^d$, where $J_{V_{a_s}}(\omega) = \ker(R(\omega) - \hat{a}_s(\omega))$ is the range function associated to the shift-invariant subspace V_{a_s} for every $s = 1, ..., r$.

Remark 2.14. In [4, Theorem 6.13], it was proved that given an s-diagonalizable shift-preserving operator L acting on a finitely generated shift-invariant space V, there always exists an s-diagonalization of L, say $(V, L, a_1, ..., a_r)$, with the property that the spectra of the s-eigenspaces of L satisfy the condition $\sigma(V_{a_{s+1}}) \subseteq \sigma(V_{a_s})$ for every $s = 1, ..., r-1$.

We define for $s = 1, \dots, r-1$ the sets $B_s := \sigma(V_{a_s}) \setminus \sigma(V_{a_{s+1}})$ and $B_r := \sigma(V_{a_r})$. Then, $\sigma(V) = \bigcup_{s=1}^{r} B_s$ where the union is disjoint. Given $s \in \{1, \dots, r\}$, notice that for a.e. $\omega \in B_s$ we have that $J_{V_{a_j}}(\omega) \neq \{0\}$ for $1 \leq j \leq s$ and $J_{V_{a_j}}(\omega) = \{0\}$ for $s < j \leq r$, i.e. $R(\omega)$ has exactly s different eigenvalues in B_s.

Moreover, the construction of such s-diagonalization is based on [4, Theorem 6.8] where the s-eigenvalues obtained satisfy that

$$\hat{a}_s(\omega) = \begin{cases} \lambda_s(\omega), & \omega \in \sigma(V_a) \\ K + s & \text{otherwise} \end{cases}, \quad \omega \in [0,1)^d$$

where $\lambda_s(\omega) : \sigma(V_{a_s}) \to \mathbb{C}$ is a measurable function which is an eigenvalue of $R(\omega)$ almost everywhere in $\sigma(V_{a_s})$ and $K > 0$ is a constant such that $K \geq \|R(\omega)\|$ for a.e. $\omega \in [0,1)^d$.

This facts will be needed in Subsection 3.2.

The following generalized Spectral Theorem was obtained in [4] for bounded, shift-preserving operators which are normal.

Theorem 2.15. *Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. If L is normal, then it is s-diagonalizable and, if $(V, L, a_1, \dots, a_r)$ is an s-diagonalization of L, we have that*

$$L = \sum_{s=1}^{r} \Lambda_{a_s} P_{V_{a_s}},$$

where $P_{V_{a_s}}$ denotes the orthogonal projection of $L^2(\mathbb{R}^d)$ onto V_{a_s} for $s = 1, \dots, r$.

Finally, we add the next proposition which relates the s-diagonalization of a normal operator and its adjoint.

Proposition 2.16. *Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. If L is normal, the following statements hold:*

(i) *L and L^* are s-diagonalizable.*

(ii) *If Λ_a is an s-eigenvalue of L, then Λ_a^* is an s-eigenvalue of L^* and* $\ker(L^* - \Lambda_a^*) = V_a$. *Furthermore, $\Lambda_a^* = \Lambda_b$, where $b \in \ell^2(\mathbb{Z}^d)$ is defined by $b(j) := \overline{a(-j)}$ for $j \in \mathbb{Z}^d$.*

(iii) *If $(V, L, a_1, \dots, a_r)$ is an s-diagonalization of L, then $(V, L^*, b_1, \dots, b_r)$ is an s-diagonalization of L^*, where $\Lambda_{b_s} = \Lambda_{a_s}^*$ for $s = 1, \dots, r$.*

Proof. Since L is normal, so is L^* and by Theorem 2.15, L and L^* are s-diagonalizable, which proves (i).

Furthermore, assume that Λ_a is an s-eigenvalue of L. Let $f, g \in V$, we have that

$$\langle \Lambda_a f, g \rangle = \left\langle \hat{a}\hat{f}, \hat{g} \right\rangle = \left\langle \hat{f}, \bar{\hat{a}}\hat{g} \right\rangle. \tag{3}$$

On the other hand, it is easy to see that

$$\bar{\hat{a}}(\omega) = \sum_{j \in \mathbb{Z}^d} \overline{a(-j)} e_j(\omega).$$

Thus, if we define $b \in \ell^2(\mathbb{Z}^d)$ by $b(j) := \overline{a(-j)}$ for $j \in \mathbb{Z}^d$, we deduce from (3) that $\langle \Lambda_a f, g \rangle = \langle f, \Lambda_b g \rangle$, that is $\Lambda_a^* = \Lambda_b$.

On the other hand, $L - \Lambda_a$ is a normal operator and so $\{0\} \neq V_a = \ker(L - \Lambda_a) = \ker\left((L - \Lambda_a)^*\right) = \ker\left(L^* - \Lambda_b\right)$, from which we conclude that Λ_b is an s-eigenvalue of L^*, as we wanted to see in (ii).

For (iii), it only remains to observe that if $(V, L, a_1, \dots, a_r)$ is an s-diagonalization of L, then we have that $V = V_{a_1} \oplus \cdots \oplus V_{a_r}$, which is also a decomposition into s-eigenspaces of L^* since $\ker(L^* - \Lambda_{b_s}) = V_{a_s}$ for every $s = 1, \dots, r$ by item (ii). □

3. A Dynamical Sampling problem for shift-preserving operators

Given a set of functions $\mathcal{F} = \{f_i : i \in I\}$ in a Hilbert space $\mathcal{H}$ and a bounded operator $D : \mathcal{H} \to \mathcal{H}$, the dynamical sampling problem consists on finding conditions on D and $\mathcal{F}$ for $\{D^j f_i : i \in I, j \in K\}$ to be a frame of $\mathcal{H}$, where $I, K \subseteq \mathbb{N} \cup \{0\}$.

The finite-dimensional case of this problem has been solved in [6]. Whereas, for the infinite-dimensional case, a characterization theorem was given for normal operators iterating a finite set of functions in [6] and [17]. Moreover, the following condition was also proved in [17] for the finite-dimensional setting.

Theorem 3.1. *Let $\mathcal{H}$ be an n-dimensional Hilbert space and let $D : \mathcal{H} \to \mathcal{H}$ be a linear transformation. Let $I = \{1, \dots, m\}$, $K = \{0, \dots, n-1\}$ and let $\{f_i : i \in I\} \subseteq \mathcal{H}$. Then, $\{D^j f_i : i \in I, j \in K\}$ is a frame of $\mathcal{H}$ if and only if for each $\lambda \in spec(D^*)$,*

$$\left\{ P_{\ker(D^* - \lambda \mathcal{I})} f_i : i \in I \right\}$$

is a frame of $\ker(D^* - \lambda \mathcal{I})$. Here, $spec(D^*)$ denotes the set of eigenvalues of D^*.

In what follows, we formulate a dynamical sampling problem for shift-preserving operators. Let V be a finitely generated shift-invariant space whose length is ℓ. Suppose that $I = \{1, \dots, m\}$ is a finite index set and $K = \{0, \dots, \ell - 1\}$. Let $\{f_i : i \in I\}$ be a set of functions in V and let $L : V \to V$ be a bounded shift-preserving operator. We want to give conditions for the system

$$\left\{ L^j f_i : i \in I, j \in K \right\} \tag{4}$$

to be a frame generator set of V.

Our approach on solving this problem is through fiberization techniques. Let J be the range function of V and let R be the associated range operator of L. Recall that by Theorem 2.6, the system in (4) is a frame generator set of V if and only if its fibers form a frame of $J(\omega)$ with uniform bounds for a.e. $\omega \in [0,1)^d$. On the other hand, by the intertwinig property of $\mathcal{T}$ and R given in equation (1), we have that $\mathcal{T}(L^j f_i)(\omega) = R(\omega)^j(\mathcal{T} f_i(\omega))$.

This allows us to reduce our problem to a finite-dimensional dynamical sampling problem by studying conditions for the system

$$\left\{R(\omega)^j(\mathcal{T} f_i(\omega)) : i \in I,\, j \in K\right\}$$

to be a frame of $J(\omega)$ for a.e. $\omega \in [0,1)^d$. For this, one would like to apply Theorem 3.1 with $D = R(\omega)$ at almost every $\omega \in [0,1)^d$. To translate back results from the range operator to the shift-preserving operator L, we need uniformity in the frame bounds. However, in Theorem 3.1 frame bounds estimates were not provided. For this, we need to obtain frame bounds for the set of iterations in terms of the frame bounds of the frame of the projections of the elements of $\mathcal{F}$ and reciprocally.

We divide this section in two subsections. The first one is devoted to give frame bounds estimates for the finite-dimensional case of dynamical sampling. Then, in the second subsection, we apply these results to solve the problem of dynamical sampling for shift-preserving operators.

3.1. **Finite-dimensional dynamical sampling.** Along this subsection we will assume that $\mathcal{H}$ is an n-dimensional complex Hilbert space and $R : \mathcal{H} \to \mathcal{H}$ is a linear transformation. Let $I = \{1, \dots, m\}$ and $K = \{0, \dots, k\}$ where $k \geq n-1$. Moreover, for $\lambda \in \mathbb{C}$ we will denote by $E_\lambda = \ker(R^* - \lambda \mathcal{I})$ and by P_{E_λ} the orthogonal projection of $\mathcal{H}$ onto E_λ. As before, $\operatorname{spec}(R^*)$ will denote the set of eigenvalues of R^*.

To obtain frame bounds estimates for Theorem 3.1, we need to study each direction separately. In one direction, we have the following result.

Theorem 3.2. *If $\{R^j f_i : i \in I, j \in K\}$ is a frame of $\mathcal{H}$ with frame bounds $A, B > 0$, then for every $\lambda \in \operatorname{spec}(R^*)$, we have that $\{P_{E_\lambda} f_i : i \in I\}$ is a frame for E_λ, with bounds A/C_λ and B/C_λ, where $C_\lambda = \sum_{j \in K} |\lambda|^{2j}$.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of R^* and $f \in E_\lambda$. It is suffices to observe that

$$\begin{aligned}\sum_{j\in K}\sum_{i\in I} |\langle f, R^j f_i\rangle|^2 &= \sum_{j\in K}\sum_{i\in I} |\langle R^{*j} f, f_i\rangle|^2 \\ &= \sum_{j\in K}\sum_{i\in I} |\langle \lambda^j f, f_i\rangle|^2 \\ &= \sum_{j\in K} |\lambda|^{2j} \sum_{i\in I} |\langle f, P_{E_\lambda} f_i\rangle|^2.\end{aligned}$$

□

In the other direction, more work is required. We will ask for R to be a normal linear transformation in order to have a decomposition of $\mathcal{H}$ into an orthogonal sum of eigenspaces. The proof of the next theorem is inspired by [21, Theorem 5.1.], where frame bounds estimates where provided for the infinite-dimensional dynamical sampling problem, assuming that the operator is normal and only one function is

being iterated. For this, we make use of a quite technical lemma that we show below.

We will denote by $\mathcal{P}_k$ the Hilbert space of the polynomials with complex coefficients of degree less than or equal to k, provided with the inner product

$$\langle p, q\rangle = \sum_{j=0}^{k} c_j \overline{d_j},$$

where $p(z) = \sum_{j=0}^{k} c_j z^j$ and $q(z) = \sum_{j=0}^{k} d_j z^j$. The space $\mathcal{P}_k^{|I|}$ is the cartesian product of $|I|$ copies of $\mathcal{P}_k$ endowed with the norm

$$\|(p_i)_{i\in I}\|_{\mathcal{P}_k^{|I|}} = \left(\sum_{i\in I} \|p_i\|_{\mathcal{P}_k}^2\right)^{1/2} = \left(\sum_{i\in I}\sum_{j\in K} |c_{ij}|^2\right)^{1/2}.$$

Lemma 3.3. *Let $\lambda_1, \dots, \lambda_r$ be all different complex numbers. We define $T : \mathcal{P}_k^{|I|} \to \ell^2(I \times \{1, \dots, r\})$ as*

$$(p_i)_{i\in I} \mapsto (p_i(\lambda_s))_{i\in I, s\in\{1,\dots,r\}}, \tag{5}$$

and $M : \ell^2(I \times \{1, \dots, r\}) \to \mathcal{P}_k^{|I|}$ as the operator given by

$$(w_{is})_{i\in I, s\in\{1,\dots,r\}} \mapsto (p_i)_{i\in I}, \tag{6}$$

where for every $i \in I$, p_i denotes the Lagrange polynomial of degree $r-1$ which interpolates (λ_s, w_{is}), $s = 1, \dots, r$. Then, the following statements hold:

(i) *T is surjective.*
(ii) *$TM = \mathcal{I}$.*
(iii) *The operator norm of T and M have the following bounds:*

$$\|T\| \le \left(r \sum_{j=0}^{k} \beta^{2j}\right)^{1/2} \quad \text{and}$$

$$\|M\| \le \left(\frac{r}{\alpha} \sum_{u=0}^{r-1} \binom{r-1}{u}^2 \beta^{2u}\right)^{1/2}$$

with $\alpha = \min_{1\le s\le r} \prod_{\substack{u=1\\u\neq s}}^{r} |\lambda_s - \lambda_u|^2$ and $\beta = \max_{1\le s\le r} |\lambda_s|$.

Proof. Observe that given any vector $(w_1, \dots, w_r) \in \mathbb{C}^r$, one can construct the Lagrange polynomial of degree $r-1$ which interpolates the points $(\lambda_1, w_1), \dots, (\lambda_r, w_r)$. This proves that T is surjective. The relation $TM = \mathcal{I}$ follows from the definition of T and M.

To estimate the norm of T, let $(p_i)_{i\in I} \in \mathcal{P}_k^{|I|}$, using that for every $s = 1, \dots, r$, $|\lambda_s| \leq \beta$, we see that

$$\begin{aligned}
\|T(p_i)_{i\in I}\|_{\ell^2(I\times\{1,\dots,r\})} &= \left(\sum_{i\in I}\sum_{s=1}^{r} |p_i(\lambda_s)|^2\right)^{1/2} = \left(\sum_{i\in I}\sum_{s=1}^{r}\left|\sum_{j=0}^{k} c_{ij}\lambda_s^j\right|^2\right)^{1/2} \\
&\leq \left(\sum_{i\in I}\sum_{s=1}^{r}\left(\sum_{j=0}^{k}|c_{ij}||\lambda_s|^j\right)^2\right)^{1/2} \\
&\leq \left(\sum_{i\in I}\sum_{s=1}^{r}\left(\sum_{j=0}^{k}|c_{ij}|^2\right)\left(\sum_{j=0}^{k}|\lambda_s|^{2j}\right)\right)^{1/2} \\
&\leq \|(p_i)_{i\in I}\|_{\mathcal{P}_n^{|I|}}\left(r\sum_{j=0}^{k}\beta^{2j}\right)^{1/2}.
\end{aligned}$$

Now, let $w = (w_{is})_{i\in I, s\in\{1,\dots,r\}} \in \ell^2(I\times\{1,\dots,r\})$. Recall that for every $i \in I$, given the points (λ_s, w_{is}), $s = 1, \dots, r$, the Lagrange polynomial which interpolates them is given by $p_i(z) = \sum_{s=1}^{r} w_{is}B_s(z)$, where

$$B_s(z) = \prod_{\substack{u=1\\u\neq s}}^{r}\frac{z-\lambda_u}{\lambda_s-\lambda_u},$$

then,

$$\begin{aligned}
\|M(w)\|_{\mathcal{P}_k^{|I|}} = \|(p_i)_{i\in I}\|_{\mathcal{P}_k^{|I|}} &= \left(\sum_{i\in I}\left\|\sum_{s=1}^{r} w_{is}B_s\right\|_{\mathcal{P}_k}^2\right)^{1/2} \\
&\leq \left(\sum_{i\in I}\left(\sum_{s=1}^{r}|w_{is}|\|B_s\|_{\mathcal{P}_k}\right)^2\right)^{1/2} \\
&\leq \left(\sum_{i\in I}\left(\sum_{s=1}^{r}|w_{is}|^2\right)\left(\sum_{s=1}^{r}\|B_s\|_{\mathcal{P}_k}^2\right)\right)^{1/2} \\
&= \left(\sum_{s=1}^{r}\|B_s\|_{\mathcal{P}_k}^2\right)^{1/2}\|w\|_{\ell^2(I\times\{1,\dots,r\})}. \qquad (7)
\end{aligned}$$

Furthermore, since for every $s = 1, \dots, r$, $|\lambda_s| \leq \beta$, then

$$\begin{aligned}
\|B_s\|_{\mathcal{P}_k}^2 &= \left\| \prod_{\substack{u=1 \\ u \neq s}}^{r} \frac{z - \lambda_u}{\lambda_s - \lambda_u} \right\|_{\mathcal{P}_n}^2 = \frac{1}{\prod_{u \neq s} |\lambda_s - \lambda_u|^2} \left\| \prod_{u \neq s} (z - \lambda_u) \right\|_{\mathcal{P}_n}^2 \\
&\leq \frac{1}{\alpha} \left(1 + \left| \sum_{u \neq s} \lambda_u \right|^2 + \left| \sum_{\substack{1 \leq u < v \leq r \\ u, v \neq s}} \lambda_u \lambda_v \right|^2 + \cdots + \left| \prod_{u \neq s} \lambda_u \right|^2 \right) \\
&\leq \frac{1}{\alpha} \left(1 + (r-1)^2 \beta^2 + \binom{r-1}{2}^2 \beta^4 + \cdots + \beta^{2(r-1)} \right) \\
&= \frac{1}{\alpha} \sum_{u=0}^{r-1} \binom{r-1}{u}^2 \beta^{2u}.
\end{aligned}$$

Finally, it follows from the latter and (7) that

$$\|M\| \leq \left(\frac{r}{\alpha} \sum_{u=0}^{r-1} \binom{r-1}{u}^2 \beta^{2u} \right)^{1/2}.$$

□

Remark 3.4. Observe that using the equivalence between norms $\|\cdot\|_1$ and $\|\cdot\|_2$, we can obtain instead the following bound

$$\|B_s\|_{\mathcal{P}_k}^2 \leq \frac{1}{\alpha} \left(\sum_{u=0}^{r-1} \binom{r-1}{u} \beta^u \right)^2 = \frac{1}{\alpha} (1 + \beta)^{2r},$$

and therefore,

$$\|M\| \leq \left(\frac{r}{\alpha} \right)^{1/2} (1 + \beta)^r.$$

Theorem 3.5. *Assume that R is normal and let $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ be such that $R = \sum_{s=1}^{r} \lambda_s P_{E_{\lambda_s}}$. If for every $s = 1, \dots, r$, $\left\{ P_{E_{\lambda_s}} f_i : i \in I \right\}$ is a frame for E_{λ_s} with frame bounds $A_s, B_s > 0$, then $\{ R^j f_i : i \in I, j \in K \}$ is a frame for $\mathcal{H}$ with bounds*

$$A \left(\frac{r}{\alpha} \sum_{u=0}^{r-1} \binom{r-1}{u}^2 \|R\|^{2u} \right)^{-1} \quad \text{and} \quad B \left(r \sum_{j=0}^{k} \|R\|^{2j} \right),$$

where

$$\alpha = \min_{1 \leq s \leq r} \prod_{\substack{u=1 \\ u \neq s}}^{r} |\lambda_s - \lambda_u|^2 > 0, \tag{8}$$

and $A = \min_s A_s$, $B = \max_s B_s$.

Observe that the bounds estimates obtained only depend on the operator norm, the number of eigenvalues of R and, in the upper bound, the number of iterations $|K| = k$.

Proof. Consider $U : \ell^2(I \times K) \to \mathcal{H}$ the synthesis operator of $\{R^j f_i : i \in I, j \in K\}$ which is defined by

$$Uc = \sum_{i\in I} \sum_{j\in K} c_{ij} R^j f_i$$

for every $c \in \ell^2(I \times K)$. Observe that we have

$$\begin{aligned} Uc = \sum_{i\in I} \sum_{j\in K} c_{ij} R^j f_i &= \sum_{i\in I} \sum_{j\in K} c_{ij} \sum_{s=1}^{r} \lambda_s^j P_{E_{\lambda_s}} f_i \\ &= \sum_{i\in I} \sum_{s=1}^{r} \left(\sum_{j\in K} c_{ij} \lambda_s^j \right) P_{E_{\lambda_s}} f_i. \end{aligned} \tag{9}$$

The idea of this proof will be to estimate the frame bounds through the synthesis and the analysis operator. In order to do that, we will decompose U into three auxiliary operators C, P and T, which we define next.

We start by defining C. Since $\mathcal{H} = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}$, where the sums are orthogonal, and each E_{λ_s} has a frame with the same constants A and B, then the union of these frames, i.e.

$$\left\{ P_{E_{\lambda_s}} f_i : i \in I, s = 1, \dots, r \right\}, \tag{10}$$

is a frame for $\mathcal{H}$ with constants A and B. Thus, C will be the synthesis operator of (10), i.e., $C : \ell^2(I \times \{1, \dots, r\}) \to \mathcal{H}$ and

$$Ca = \sum_{i\in I} \sum_{s=1}^{r} a_{is} P_{E_{\lambda_s}} f_i$$

for $a \in \ell^2(I \times \{1, \dots, r\})$.

Second, define $P : \ell^2(I \times K) \to \mathcal{P}_k^{|I|}$ as $c = (c_{ij})_{i,j} \mapsto (p_i)_{i\in I}$ where $p_i(z) = \sum_{j=0}^{k} c_{ij} z^j$. Note that P is an isometry. Finally, we consider T defined as in (5), with $\lambda_1, \dots, \lambda_r$ the eigenvalues of R.

Using the operators already defined, the operator U have the expression

$$Uc = \sum_{i\in I} \sum_{s=1}^{r} (TPc)_{is} P_{E_{\lambda_s}} f_i = C\,TPc,$$

and therefore, if $f \in \mathcal{H}$,

$$\|U^* f\|^2 = \|P^* T^* C^* f\|^2 = \|T^* C^* f\|^2. \tag{11}$$

Given that T is, by Lemma 3.3, a surjective operator, then T^* is bounded from below by a positive constant. In particular,

$$\inf \left\{ \|T^* a\| : a \in \ell^2(I \times \{1, \dots, r\}),\ \|a\| = 1 \right\} = \gamma > 0.$$

Moreover, it is well known that given an operator M such that $TM = \mathcal{I}$, then $\gamma = 1/\|M\|$ (see, for instance, [16]). Let M be defined as in (6) with $\lambda_1, \dots, \lambda_r$ the eigenvalues of R. Thus, by Lemma 3.3 we get that $TM = \mathcal{I}$ and since $\|R\| = \beta = \max_{1\le s\le r} |\lambda_s|$, we have

$$\gamma \ge \left(\frac{r}{\alpha} \sum_{u=0}^{r-1} \binom{r-1}{u}^2 \|R\|^{2u} \right)^{-1/2}.$$

Therefore, from (11) we deduce that

$$\|U^*f\|^2 \geq \gamma\|C^*f\|^2 \geq \gamma A\|f\|^2 \geq A\left(\frac{r}{\alpha}\sum_{u=0}^{r-1}\binom{r-1}{u}^2\|R\|^{2u}\right)^{-1}\|f\|^2,$$

which gives us a lower frame bound for $\{R^j f_i : i \in I, j \in K\}$. On the other hand, by the estimate bound of $\|T\|$ obtained in Lemma 3.3 it follows that

$$\|U\|^2 = \|CTP\|^2 \leq B\|T\|^2 \leq B\left(r\sum_{j=0}^{k}\|R\|^{2j}\right),$$

from which we obtain an upper frame bound for $\{R^j f_i : i \in I, j \in K\}$. □

3.2. **Dynamical sampling for shift-preserving operators.** In this subsection we assume that V is a finitely generated shift-invariant space of length ℓ. We show the solution of the dynamical sampling problem for a bounded, normal, shift-preserving operator $L : V \to V$ acting on a set of functions $\mathcal{F} = \{f_i : i \in I\}$ of V. That is, we give conditions on L and $\mathcal{F}$ in order that the system

$$\left\{L^j f_i : i \in I\,; j \in K\right\}$$

is a frame generator set of V, where $I = \{1, \ldots, m\}$ and $K = \{0, \ldots, \ell - 1\}$.

The conditions found are reminiscent of the conditions for the finite-dimensional case. These are obtained through the theory of s-diagonalization of shift-preserving operators, which was developed in [4] and reviewed in the previous section.

We prove two theorems, one for the necessary conditions and one for the sufficient conditions. In the first theorem, the assumption of normality of the shift-preserving operator is not needed.

Theorem 3.6. *Let V be a finitely generated shift-invariant space of length ℓ, and let $L : V \to V$ be a bounded, shift-preserving operator with range operator R. Suppose $I = \{1, \ldots, m\}$ and $K = \{0, \ldots, \ell - 1\}$. Let $\{f_i : i \in I\}$ be a set of functions in V.*

If $\{L^j f_i : i \in I,\ j \in K\}$ is a frame generator set of V with bounds $A, B > 0$, then for every s-eigenvalue Λ_a of L^*, if $V_a = \ker(L^* - \Lambda_a)$, we have that $\{P_{V_a} f_i : i \in I\}$ is a frame generator set of V_a with frame bounds

$$A\left(\sum_{j=0}^{\ell-1}\|L\|^{2j}\right)^{-1} \quad \text{and} \quad B.$$

Proof. Suppose that $\{L^j f_i : i \in I, j \in K\}$ is a frame generator set of V with frame bounds $A, B > 0$. Then, by Theorem 2.6,

$$\left\{\mathcal{T}(L^j f_i)(\omega) : i \in I, j \in K\right\}$$

is a frame of $J(\omega)$ with uniform bounds A, B for a.e. $\omega \in [0,1)^d$. By (1) we have that

$$\left\{R(\omega)^j(\mathcal{T} f_i(\omega)) : i \in I,\ j \in K\right\}$$

is a frame of $J(\omega)$ with uniform bounds A, B for a.e. $\omega \in [0,1)^d$.

Let Λ_a be an s-eigenvalue of L^*. By Theorem 2.9 and item (i) of Proposition 2.12, we have that $\widehat{a}(\omega)$ is an eigenvalue of $R^*(\omega)$ for a.e. $\omega \in \sigma(V_a)$. Then, Theorem 3.2 implies that the set

$$\left\{P_{J_{V_a}(\omega)}(\mathcal{T} f_i(\omega)) : i \in I,\, j \in K\right\} \tag{12}$$

is a frame of $J_{V_a}(\omega)$ with bounds $A/C_a(\omega)$ and $B/C_a(\omega)$ for a.e. $\omega \in \sigma(V_a)$, where $C_a(\omega) = \sum_{j=0}^{\ell-1} |\widehat{a}(\omega)|^{2j}$.

Note that by (2) we have that

$$1 \leq C_a(\omega) = \sum_{j=0}^{\ell-1} |\widehat{a}(\omega)|^{2j} \leq \sum_{j=0}^{\ell-1} \|R(\omega)\|^{2j} \leq \sum_{j=0}^{\ell-1} \|L\|^{2j}.$$

Then, we are able to obtain uniform bounds for (12) for a.e. $\omega \in \sigma(V_a)$. Finally, using Lemma 2.5 and Theorem 2.6, we get that $\{P_{V_a} f_i : i \in I\}$ is a frame generator set of V_a with lower bound $A\left(\sum_{j=0}^{\ell-1} \|L\|^{2j}\right)^{-1}$ and B as an upper bound. □

Different from the finite-dimensional case of the dynamical sampling problem, for a sufficient condition we need to add an extra uniformity assumption.

Definition 3.7. Let V be a finitely generated shift-invariant space, and let $L : V \to V$ be a bounded shift-preserving operator with range operator R. We say that L has the *spectral gap property* if there exists a constant $c > 0$ such that $|\lambda - \lambda'| \geq c$ for every $\lambda \neq \lambda'$ in $\operatorname{spec}(R(\omega))$ and for a.e. $\omega \in \sigma(V)$.

Theorem 3.8. *Let V be a finitely generated shift-invariant space of length ℓ, and let $L : V \to V$ be a bounded, normal, shift-preserving operator with range operator R, satisfying the spectral gap property. Let $I = \{1, \ldots, m\}$, $K = \{0, \ldots, \ell - 1\}$ and let $\{f_i : i \in I\}$ be a set of functions in V.*

Then, $\{L^j f_i : i \in I,\, j \in K\}$ is a frame generator set of V if and only if for every s-eigenvalue Λ_a of L^*, if $V_a = \ker(L^* - \Lambda_a)$, we have that $\{P_{V_a} f_i : i \in I\}$ is a frame generator set of V_a with the same frame bounds.

Proof. The necessity has been proved in Theorem 3.6.

For the converse, recall that since L is a normal operator, then L^* is s-diagonalizable (see Proposition 2.16). As discussed in Remark 2.14, there exists an s-diagonalization of L^* $(V, L^*, a_1, \ldots, a_r)$ such that $\sigma(V_{a_{s+1}}) \subseteq \sigma(V_{a_s})$ for every $s = 1, \ldots, r-1$ and, given that L has the spectral gap property, there exists a constant $c > 0$ such that

$$\left|\hat{a}_s(\omega) - \hat{a}_u(\omega)\right| \geq c \tag{13}$$

for a.e. $\omega \in [0,1)^d$ and for every $u \neq s$, with $u, s = 1, \ldots, r$.

Now, suposse that $\{P_{V_{a_s}} f_i : i \in I\}$ is a frame generator set of V_{a_s} for every $s = 1, ..., r$ with bounds $A, B > 0$. Then, by Theorem 2.6,

$$\{\mathcal{T}(P_{V_{a_s}} f_i)(\omega) : i \in I\}$$

is a frame of $J_{V_{a_s}}(\omega)$ with uniform bounds A, B for a.e. $\omega \in [0,1)^d$ and for every $s = 1, ..., r$. By Lemma 2.5, we have that

$$\left\{P_{J_{V_{a_s}}(\omega)}(\mathcal{T} f_i(\omega)) : i \in I\right\}$$

is a frame of $J_{V_{a_s}}(\omega)$ with uniform bounds A, B for a.e. $\omega \in [0,1)^d$ and for every $s = 1, ..., r$.

Recall that the number of eigenvalues of $R(\omega)$ may vary through the different values of ω. In order to apply Theorem 3.5, we need to split $[0,1)^d$ into measurable sets where the number of eigenvalues is constant. For that, as in Remark 2.14, let us define for $h = 1, \dots, r-1$, the sets $B_h := \sigma(V_{a_h}) \setminus \sigma(V_{a_{h+1}})$ and $B_r := \sigma(V_{a_r})$. Then, we have that each B_h is the set of all $\omega \in \sigma(V)$ for which $R(\omega)$ has exactly h different eigenvalues. Furthermore,

$$[0,1)^d = \bigcup_{h=1}^{r} B_h \cup \left([0,1)^d \setminus \sigma(V)\right).$$

Fix $h \in \{1, \dots, r\}$. For a.e. $\omega \in B_h$, we have that $\left\{P_{J_{V_{a_s}}(\omega)}(\mathcal{T} f_i(\omega)) : i \in I\right\}$ is a frame of $J_{V_{a_s}}(\omega)$ for $s = 1, \dots, h$. Theorem 3.5 implies that

$$\left\{R(\omega)^j (\mathcal{T} f_i(\omega)) : i \in I,\, j \in K\right\} \tag{14}$$

is a frame of $J(\omega)$ for a.e. $\omega \in B_h$ with frame bounds

$$A\left(\frac{r}{\alpha(\omega)} \sum_{u=0}^{h-1} \binom{h-1}{u}^2 \|R(\omega)\|^{2u}\right)^{-1} \qquad \text{and} \qquad B\left(h \sum_{j=0}^{\ell-1} \|R(\omega)\|^{2j}\right),$$

$$\alpha(\omega) = \min_{1 \le s \le h} \prod_{\substack{u=1 \\ u \neq s}}^{h} |\widehat{a}_s(\omega) - \widehat{a}_u(\omega)|^2.$$

Observe that since (13) holds, we might as well asume that $c < 1$, and so $\alpha(\omega) \ge c^{2h} \ge c^{2r}$ for a.e. $\omega \in B_h$. On the other hand, by (2) we know that $\|R(\omega)\| \le \|L\|$ almost everywhere. In this way, we are able to obtain the following uniform bounds for a.e. $\omega \in B_h$,

$$A\left(\frac{h}{\alpha(\omega)} \sum_{u=0}^{h-1} \binom{h-1}{u}^2 \|R(\omega)\|^{2u}\right)^{-1} \ge A\left(\frac{r}{c^{2r}} \sum_{u=0}^{r-1} \binom{r-1}{u}^2 \|L\|^{2u}\right)^{-1}$$

and

$$B\left(h \sum_{j=0}^{\ell-1} \|R(\omega)\|^{2j}\right) \le B\left(r \sum_{j=0}^{\ell-1} \|L\|^{2j}\right).$$

Since these bounds are the same on every B_h for $h = 1, \dots, r$, we have that (14) is a frame of $J(\omega)$ for a.e. $\omega \in [0,1)^d$ with uniform bounds and therefore

$$\left\{L^j f_i : i \in I,\, j \in K\right\}$$

is a frame generator set of V with bounds $A\left(\frac{r}{c^{2r}} \sum_{u=0}^{r-1} \binom{r-1}{u}^2 \|L\|^{2u}\right)^{-1}$ and $B\left(r \sum_{j=0}^{\ell-1} \|L\|^{2j}\right)$. □

References

[1] Aceska, R., Aldroubi, A., Davis, J. and Petrosyan, A., Dynamical sampling in shift-invariant spaces. Contemp. Math. of the AMS 603, 139-148 (2013)

[2] Aceska, R., Petrosyan, A. and Tang, S., Multidimensional signal recovery in discrete evolution systems via spatiotemporal trade off. Sampl. Theory Signal Image Process, 14(2), 153-169 (2015)

[3] Aceska, R. and Tang, S., Dynamical sampling in hybrid shift invariant spaces. Contemp. Math. Amer. Math. Soc., volume 626, Providence, RI (2014)

[4] Aguilera, A., Cabrelli, C., Carbajal, D. and Paternostro. V., Diagonalization of shift-preserving operators. Preprint (2019).

[5] Aldroubi, A., Cabrelli, C., Çakmak, A.F., Molter, U. and Petrosyan, A., Iterative actions of normal operators. J. Funct. Anal., 272, 1121-1146 (2017)

[6] Aldroubi, A., Cabrelli, C., Molter, U. and Tang, S., Dynamical sampling. Applied and Computational Harmonic Analysis, 42(3), 378-401 (2017)

[7] Aldroubi, A., Davis, J. and Krishtal, I., Dynamical Sampling: Time Space Trade-off. Appl. Comput. Harmon. Anal., 34, 495-503 (2013)

[8] Aldroubi, A., Davis, J. and Krishtal, I., Exact reconstruction of signals in evolutionary systems via spatiotemporal trade-off. Journal of Fourier Analysis and Applications, 21(1), 11-31 (2015)

[9] Aldroubi, A. and Petrosyan, A., Dynamical sampling and systems from iterative actions of operators. In Frames and Other Bases in Abstract and Function Spaces, pp. 15-26. Birkhäuser, Cham. (2017)

[10] Aldroubi, A., Cabrelli, C., Molter, U. and Petrosyan, A., Local-to-global frames and applications to dynamical sampling problem. preprint arXiv: arXiv:1909.02987 ((2019)

[11] Barbieri, D., Hernández, E. and Paternostro, V., The Zak transform and the structure of spaces invariant by the action of an LCA group. J. Funct. Anal., 269(5), 1327-1358 (2015)

[12] de Boor, C., DeVore, R.A. and Ron, A., Approximation from shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Transactions of the American Mathematical Society 341(2), pp.787-806 (1994)

[13] de Boor, C., DeVore, R.A. and Ron, A., The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$. Journal of Functional Analysis, 119(1), 37-78 (1994)

[14] Bownik, M., The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$. Journal of Functional Analysis, 177(2), 282-309 (2000)

[15] Bownik, M. and Ross, K., The structure of translation-invariant spaces on locally compact abelian groups. J. Fourier Anal. Appl., 21(4), 849-884 (2015)

[16] Bouldin, R., The essential minimum modulus. Indiana University Mathematics Journal, vol. 30, no 4, p. 513-517 (1981)

[17] Cabrelli, C., Molter, U., Paternostro, V. and Philipp, F., Dynamical sampling on finite index sets. To appear in J. Anal. Math. (2019)

[18] Cabrelli, C., Molter, U. and Suarez, D. Multi-orbital frames through model spaces. Complex Analysis and Operator Theory, (2021) to appear.

[19] Cabrelli, C. and Paternostro, V., Shift-invariant spaces on LCA groups. J. Funct. Anal., 258(6), 2034-2059 (2010)

[20] Christensen, O., An introduction to frames and Riesz bases (Vol. 7) (2003), Boston: Birkhäuser.

[21] Christensen, O., Hasannasab, M. and Philipp, F., Frame properties of operator orbits. Mathematische Nachrichten, to appear.

[22] Christensen, O., Hasannasab, M. and Rashidi, E., Dynamical sampling and frame representations with bounded operators. Journal of Mathematical Analysis and Applications 463(2), 634-644 (2018)

[23] Christensen,O., Hasannasab, M., and Stoeva, D.T. Operator representations of sequences and dynamical sampling. Sampl. Theory Signal Image Process. 17, no. 1, 29-42 (2018)

[24] Helson, H., Lectures on invariant subspaces. Academic Press, London (1964)

[25] Ron, A. and Shen Z., Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Canad. J. Math., 47, no. 5, 1051-1094 (1995)

[26] Philipp, F.. Bessel orbits of normal operators. *Journal of Mathematical Analysis and Applications*, 448(2):767-785 (2017).

[27] Tang, S.. System identification in dynamical sampling. *Advances in Computational Mathematics*, 43(3):555-580 (2017).

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
E-mail address: aaguilera@dm.uba.ar

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
E-mail address: cabrelli@dm.uba.ar

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
E-mail address: dcarbajal@dm.uba.ar

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
E-mail address: vpater@dm.uba.ar