DIAGONALIZATION OF SHIFT-PRESERVING OPERATORS

A. AGUILERA, C. CABRELLI, D. CARBAJAL, AND V. PATERNOSTRO

ABSTRACT. In this note we study the structure of shift-preserving operators acting on a finitely generated shift-invariant space. We define a new notion of diagonalization for these operators, which we call s-diagonalization. We give necessary and sufficient conditions on a bounded shift-preserving operator in order to be s-diagonalizable. These conditions are in terms of its range operator. We also obtain a generalized Spectral Theorem for normal bounded shift-preserving operators.

1. INTRODUCTION

Shift-invariant spaces are subspaces of $L^2(\mathbb{R}^d)$ that are invariant under the action of translations by integer vectors. These spaces have been used in approximation theory, sampling theory, and wavelets, and their structure is very well known. See [6, 9, 10, 14, 15] in the euclidean case, and [4, 7, 8] in the context of topological groups.

Given an (at most countable) set of functions $\Phi \subset L^2(\mathbb{R}^d)$ the subspace

$$S(\Phi) := \overline{\operatorname{span}} \left\{ T_k \varphi \, : \, \varphi \in \Phi, \, k \in \mathbb{Z}^d \right\}$$

is shift invariant and Φ is called a *set of generators*. Moreover, every shift-invariant space is of this form. A *finitely generated* shift-invariant space is one that has a finite set of generators and the *length* is the minimum cardinal between all sets of generators.

Each shift-invariant space V has an associated *range function* (see Def. 2.3) which represents V as a measurable field of closed subspaces (the fiber spaces) of $\ell^2(\mathbb{Z}^d)$. The functions of V have its fibers in the fiber space (see Section 2) and the connection between V and its associated range function is through an isometric isomorphism that we denote by $\mathcal{T}$.

The representation of a shift-invariant space through its fiber spaces is a key to studying its structure. In fact, the integer translates of a set of functions in the space form a basis or a frame if and only if the fibers of such functions form a basis or a frame of the fiber spaces with uniform bounds (see [6]). On the other hand, when the shift-invariant space is finitely generated, all these fiber spaces are finite-dimensional. As a consequence, the representation by fiber spaces allows us to translate problems that involve shift-invariant spaces into linear algebra.

The natural operators acting on these spaces are those that commute with integer translates, i.e *shift-preserving* operators. In [6], Bownik considers shift-preserving operators acting on shift-invariant spaces and studies its properties through the concept of *range operator* (see Def. 3.2). A range operator is a representation of a shift-preserving operator as a measurable field of linear operators, obtained through the intertwining map $\mathcal{T}$ mentioned before. Each fiber of a range operator acts on the respective fiber space of the range function. In this way, the action of the range operator through the fiber spaces permits us to decode the behavior of the associated shift-preserving operator (see Section 3). For instance, Bownik proved in [6] that certain properties such as normality and unitaryness of a shift-preserving operator are inherited by its fibers.

When a shift-preserving operator acts on a finitely generated shift-invariant space, each fiber of the range operator is a linear transformation acting on a finite dimensional subspace of $\ell^2(\mathbb{Z}^d)$. So, considering appropriate bases of the fiber spaces, one can represent the range operator as a field of matrices. Given that, we want to translate the structure of these matrices (or the linear transformations that they induce) back to the shift-preserving operator.

There are two delicate issues when one tries to pursue this program. First, we need some consistency in the structure of the field of linear transformations. Here, the difficulty is to understand which is the precise uniformity condition to ask, to be able to obtain a similar property for the shift-preserving operator. The second issue is measurability. Since the field is measurable, the objects associated with our linear transformations, such as eigenvalues, eigenvectors and kernels, have to be measurable respect to its behavior along the field. This requires a significative effort.

In this paper, we focus on a special notion of diagonalization for a shift-preserving operator acting on a finitely generated shift-invariant space, which we call *s-diagonalization* (see Def. 6.1). This notion generalizes the usual concept of diagonalization. An *s*-diagonalizable shift-preserving operator is associated with a decomposition of the underlying shift-invariant space into a finite direct sum of closed subspaces, each of them being invariant under the shift-preserving operator and under integer translates. The shift-preserving operator has a simple form when it acts in each of the invariant subspaces and when it is *s*-diagonalizable it can be written as a sum of simpler operators.

Towards the concept of *s*-diagonalization, we define the notion of *s-eigenvalue*, a concept that generalizes the classical definition. Let $L : V \to V$ be a shift-preserving operator acting on a shift-invariant space V and $a = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ a sequence of *bounded spectrum* (i.e. $\hat{a} \in L^\infty([0,1)^d)$). Define $\Lambda_a := \sum_{k \in \mathbb{Z}^d} a_k T_k$ where T_k is the translation operator by k. We say that Λ_a is an *s*-eigenvalue of L if $\ker(L - \Lambda_a) \neq \{0\}$. An *s*-eigenvalue is a very simple shift-preserving operator that acts as a convolution in the following sense: Assume that $\{T_k \varphi_i : i = 1, \ldots, n, k \in \mathbb{Z}^d\}$ is a frame of V. If $f = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^d} b_i(k) T_k \varphi_i$ with $b_1, \ldots, b_n \in \ell^2(\mathbb{Z}^d)$, then $\Lambda_a f = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^d} (a * b_i)(k) T_k \varphi_i$.

One of our main results, states that a shift-preserving operator acting on a finitely generated shift-invariant space is *s*-diagonalizable if and only if its associated range operator is a field of diagonalizable matrices satisfying a uniformity angle condition.

As a consequence we obtain a type of spectral theorem for the case when the shift-preserving operator is bounded and normal:

Theorem 1.1. *Let $L : V \to V$ be a normal bounded shift-preserving operator on a finitely generated shift-invariant space V. Then there exist $m \in \mathbb{N}$, s-eigenvalues $\Lambda_1, \dots \Lambda_m$ and shift-invariant subspaces of V, $V_1, \dots, V_m$ with $V = V_1 \oplus \cdots \oplus V_m$ such that*

$$L = \sum_{j=1}^{m} \Lambda_j P_{V_j},$$

where P_S denotes the orthogonal projection onto a closed subspace S. Furthermore, the shift-invariant subspaces V_j are invariant under L which implies that $Lf = \Lambda_j f$ for $f \in V_j$.

On the way, we prove that an invertible bounded shift-preserving operator is s-diagonalizable if and only if its inverse is s-diagonalizable, and the decomposition of the inverse is obtained inverting the s-eigenvalues (see Proposition 6.3). We also show that a matrix with measurable entries has measurable eigenvalues. Based on that result, we explicitly construct s-eigenvalues for those shift-preserving operators acting on finitely shift-invariant spaces. Finally, we see that the minimum number of s-eigenvalues in a decomposition of a shift-preserving operator agrees with the essential supremum of the number of eigenvalues of the matrices of the associated field of operators.

The paper is organized as follows. In Section 2, we give the basic known results about shift-invariant spaces and range functions. Section 3 is devoted to shift-preserving operators. We recall their definition, some known properties and prove (under a mild condition) that they can be represented as matrices with measurable entries. Additionally, we characterize the invertibility of a shift-preserving operator in terms of the invertibility of its operator fibers. We introduce the concepts of s-eigenvalue and s-eigenspace for a shift-preserving operator in Section 4 and study their connection with the eigenvalues and eigenspaces of the associated range operator. Measurability issues related to s-eigenvalues and s-eigenspaces are developed in Section 5. Moreover, towards the existence of s-eigenvalues of a shift-preserving operator, we construct measurable functions in Theorem 5.9 by pasting eigenvalues of the associated range operator. Finally, in Section 6, we introduce the notion of s-diagonalization, show some of its properties and state and prove our main result, that relates s-diagonalization with the diagonalization of the associated field of operators, obtaining as a consequence Theorem 1.1.

2. Shift-invariant Spaces

In this section we collect some of the properties of the theory of shift-invariant spaces that we will need later.

Definition 2.1. We say that a closed subspace $V \subset L^2(\mathbb{R}^d)$ is *shift invariant* if for each $f \in V$ we have that $T_k f \in V$, $\forall k \in \mathbb{Z}^d$. Here, $T_k f(x) = f(x - k)$.

Given a countable set of functions $\Phi \subset L^2(\mathbb{R}^d)$, we will denote

$$S(\Phi) = \overline{\operatorname{span}} \left\{ T_k \varphi \, : \, \varphi \in \Phi, \, k \in \mathbb{Z}^d \right\}.$$

We say Φ is a set of generators of V if $V = S(\Phi)$. When Φ is a finite set, we say that V is a *finitely generated* shift-invariant space. Moreover, we will denote

$$E(\Phi) = \left\{ T_k\varphi \, : \, \varphi \in \Phi, \, k \in \mathbb{Z}^d \right\}.$$

An essential tool in the development of shift-invariant spaces theory is the technique known as fiberization that we present now. We follow the notation used in [6].

Proposition 2.2. *The map $\mathcal{T} : L^2(\mathbb{R}^d) \to L^2\left([0,1)^d, \ell^2(\mathbb{Z}^d)\right)$ defined by*

$$\mathcal{T}f(\omega) = \{\hat{f}(\omega + k)\}_{k \in \mathbb{Z}^d},$$

is an isometric isomorphism. We call $\mathcal{T}f(\omega)$ the fiber of f at ω. Moreover, it satisfies that

(1) $$\mathcal{T}T_k f(\omega) = e_k(\omega) \, \mathcal{T}f(\omega),$$

where $e_k(\omega) = e^{-2\pi i\langle \omega, k \rangle}$.

Here, the Hilbert space $L^2\left([0,1)^d, \ell^2(\mathbb{Z}^d)\right)$ consists of all vector-valued measurable functions $\psi : [0,1)^d \to \ell^2(\mathbb{Z}^d)$ with finite norm, where the norm is given by

$$\|\psi\| = \left(\int_{[0,1)^d} \|\psi(\omega)\|_{\ell^2}^2 \, d\omega \right)^{1/2}.$$

Definition 2.3. A *range function* is a mapping

$$J : [0,1)^d \to \{ \text{ closed subspaces of } \ell^2(\mathbb{Z}^d) \}$$
$$\omega \mapsto J(\omega).$$

We say J is measurable if the scalar function $\omega \mapsto \langle P_{J(\omega)}u, v \rangle$ is measurable for every $u, v \in \ell^2(\mathbb{Z}^d)$, where $P_{J(\omega)}$ is the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ onto $J(\omega)$. Given a range function J, the following space can be defined

$$M_J = \{\psi \in L^2\left([0,1)^d, \ell^2(\mathbb{Z}^d)\right) : \psi(\omega) \in J(\omega)\},$$

which is a closed *modulation-invariant* subspace of $L^2\left([0,1)^d, \ell^2(\mathbb{Z}^d)\right)$, i.e. for every $\psi \in M_J$ we have that $e_k\psi \in M_J$ for all $k \in \mathbb{Z}^d$.

In [6] and [14], it was proved that for every shift-invariant space $V \subset L^2(\mathbb{R}^d)$ there exists a measurable range function J_V which satisfies that

$$f \in V \text{ if and only if } \mathcal{T}f(\omega) \in J_V(\omega), \text{ for a.e. } \omega \in [0,1)^d,$$

that is $\mathcal{T}(V) = M_{J_V}$. On the other hand, every measurable range function J defines a shift-invariant space, namely $V := \mathcal{T}^{-1}(M_J)$. When identifying range functions a.e. $\omega \in [0,1)^d$, the correspondence between measurable range functions and shift-invariant spaces is one-to-one (see [6, Proposition 1.5]).

Furthermore, if $V = S(\Phi)$ for some countable set $\Phi \subset L^2(\mathbb{R}^d)$, then for a.e. $\omega \in [0,1)^d$,

$$J_V(\omega) = \overline{\text{span}}\{\mathcal{T}f(\omega) \, : \, f \in \Phi \}.$$

In particular, when Φ is a finite set, this allows us to translate problems in infinite dimensional shift-invariant spaces, into problems of finite dimension that can be treated with linear algebra.

We call the subspace $J_V(\omega)$ the fiber space of V at ω and we will simply denote J when it is evident that we are referring to the corresponding range function of V.

We have the following useful property.

Lemma 2.4. *(Helson, [14]) Let $V \subset L^2(\mathbb{R}^d)$ be a shift-invariant space with associated range function J. Then, for each $f \in L^2(\mathbb{R}^d)$ we have that*

$$\mathcal{T}(P_V f)(\omega) = P_{J(\omega)}(\mathcal{T}f(\omega)), \quad for\ a.e.\ \omega \in [0,1)^d.$$

Definition 2.5. The *length* of a finitely generated shift-invariant space $V \subset L^2(\mathbb{R}^d)$ is denoted by $\mathcal{L}(V)$ and defined as the smallest natural number ℓ such that there exist $\varphi_1, ..., \varphi_\ell \in V$ with $V = S(\varphi_1, ..., \varphi_\ell)$. It is possible to give an equivalent definition of $\mathcal{L}(V)$ in terms of the range function J associated to V, which is

$$\mathcal{L}(V) = \operatorname*{ess\ sup}_{\omega \in [0,1)^d} \dim J(\omega).$$

Since the range function of V can take the subspace $\{0\}$ as value, the *spectrum* of V is defined by

$$\sigma(V) = \left\{ \omega \in [0,1)^d \,:\, J(\omega) \neq \{0\} \right\}.$$

We now summarize some properties of shift-invariant spaces that we will need along the paper. We refer for details and proofs to [10].

Lemma 2.6. *Let V be a shift-invariant space of $L^2(\mathbb{R}^d)$. Then, there exists $\varphi \in V$ such that $\operatorname{supp} \|\mathcal{T}\varphi(\cdot)\|_{\ell^2} = \sigma(V)$.*

Lemma 2.7. *Let V, U be shift-invariant spaces of $L^2(\mathbb{R}^d)$ with associated range functions J_V, J_U respectively. Then we have:*

   (i) *The orthogonal complement of V, $V^\perp$, is shift invariant and $J_{V^\perp}(\omega) = (J_V(\omega))^\perp$ for a.e. $\omega \in [0,1)^d$.*

  (ii) *If $U \subseteq V$ and $J_V(\omega) = J_U(\omega)$ for a.e. $\omega \in [0,1)^d$, then $U = V$.*

 (iii) *The space $V \cap U$ is shift invariant and its range function satisfies $J_{V \cap U}(\omega) = J_V(\omega) \cap J_U(\omega)$, for a.e. $\omega \in [0,1)^d$.*

The last item of the above lemma was proved in [2]. The following Theorem is due by Bownik in [6].

Theorem 2.8. *Let V be a shift-invariant space of $L^2(\mathbb{R}^d)$. Then V can be decomposed as an orthogonal sum*

$$(2) \qquad\qquad V = \bigoplus_{i \in \mathbb{N}} S(\varphi_i),$$

where φ_i is a Parseval frame generator of $S(\varphi_i)$, and $\sigma(S(\varphi_{i+1})) \subset \sigma(S(\varphi_i))$ for all $i \in \mathbb{N}$. Moreover, $\dim J_{S(\varphi_i)}(\omega) = \|\mathcal{T}\varphi_i(\omega)\|$ for every $i \in \mathbb{N}$ and $\dim J_V(\omega) = \sum_{i \in \mathbb{N}} \|\mathcal{T}\varphi_i(\omega)\|$ for a.e. $\omega \in [0,1)^d$.

Observe that φ is a Parseval frame generator of $S(\varphi)$ if and only if $\|\mathcal{T}\varphi(\omega)\| = 1_{\sigma(S(\varphi))}(\omega)$ for a.e. $\omega \in [0,1)^d$ (see Theorem 2.3 in [6]).

The following lemma is a consequence of Theorem 2.8, which will be a key to working through measurability problems in Section 5. We require here, as well as in many other places along this article, that $\dim J(\omega) < \infty$ for a.e. $\omega \in [0,1)^d$. We remark that this condition does not mean that V must be finitely generated.

Lemma 2.9. *Let V be a shift-invariant space with range function J such that* $\dim(J(\omega)) < \infty$ *a.e.* $\omega \in [0,1)^d$. *Then, there exist functions* $\{\varphi_i\}_{i \in \mathbb{N}}$ *of* $L^2(\mathbb{R}^d)$ *and a family of disjoint measurable sets* $\{A_n\}_{n \in \mathbb{N}_0}$ *such that* $[0,1)^d = \bigcup_{n \in \mathbb{N}_0} A_n$ *and the following statements hold:*

- (i) $\{T_k \varphi_i : i \in \mathbb{N}, k \in \mathbb{Z}^d\}$ *is a Parseval frame of* V,
- (ii) $\mathcal{T}\varphi_i(\omega) = 0$ *for* $i > n$ *and a.e.* $\omega \in A_n$,
- (iii) $\{\mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega)\}$ *is an orthonormal basis of* $J(\omega)$ *for a.e.* $\omega \in A_n$,
- (iv) $\dim J(\omega) = n$ *for a.e.* $\omega \in A_n$.

Proof. Let $\{\varphi_i\}_{i \in \mathbb{N}} \subset L^2(\mathbb{R}^d)$ be the functions from the decomposition of V as in Theorem 2.8. As φ_i is a Parseval frame generator of $S(\varphi_i)$ for every $i \in \mathbb{N}$, it is easy to see that $\{T_k \varphi_i : i \in \mathbb{N}, k \in \mathbb{Z}^d\}$ is a Parseval frame of V. In addition, this implies that $\{\mathcal{T}\varphi_i(\omega)\}_{i \in \mathbb{N}}$ is a Parseval frame of $J(\omega)$ for a.e. $\omega \in [0,1)^d$ (see Theorem 2.3 in [6]).

Define $A_0 = [0,1)^d \setminus \sigma(V)$. Now, for $n > 0$ define the sets $A_n = \sigma(S(\varphi_n)) \setminus \sigma(S(\varphi_{n+1}))$. Since $\sigma(S(\varphi_{i+1})) \subset \sigma(S(\varphi_i))$ for all $i \in \mathbb{N}$, the sets $\{A_n\}_{n \in \mathbb{N}_0}$ are pairwise disjoint. Moreover, given that $\dim J(\omega) < \infty$ for a.e. $\omega \in [0,1)^d$, the sum $\sum_{i \in \mathbb{N}} \|\mathcal{T}\varphi_i(\omega)\|$ must be finite for a.e. $\omega \in [0,1)^d$ and so $\bigcap_{i \in \mathbb{N}} \sigma(S(\varphi_i)) = \emptyset$. From this, we can finally conclude that $[0,1)^d = \bigcup_{n \in \mathbb{N}_0} A_n$.

Fix $n > 0$. By definition of the sets A_n, we have that (ii) is satisfied. Thus, using that (i) holds, the sum in (2) is orthogonal and $\mathcal{T}$ is an isometry, we get that the system $\{\mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega)\}$ must be an orthonormal basis of $J(\omega)$ and so $\dim J(\omega) = n$ for a.e. $\omega \in A_n$. For $n = 0$, $J(\omega) = \{0\}$ for a.e. $\omega \in A_0$. $\square$

Remark 2.10. Let $n \in \mathbb{N}$. Given a measurable set $B \subseteq [0,1)^d$ such that $\dim J(\omega) = n$ for a.e. $\omega \in [0,1)^d$, then $B \subseteq A_n$. In other words, $A_n = \{\omega \in [0,1)^d : \dim J(\omega) = n\}$.

3. Shift-preserving operators

We describe here some properties of shift-preserving operators that we will use in the next section.

Definition 3.1. Let $V \subset L^2(\mathbb{R}^d)$ be a shift-invariant space and $L : V \to L^2(\mathbb{R}^d)$ be a bounded operator. We say that L is *shift preserving* if $LT_k = T_k L$ for all $k \in \mathbb{Z}^d$.

Definition 3.2. Let V be a shift-invariant space with range function J. A *range operator* on J is a mapping

$$R : [0,1)^d \to \left\{ \text{bounded operators defined on closed subspaces of } \ell^2(\mathbb{Z}^d) \right\},$$

such that the domain of $R(\omega)$ is $J(\omega)$ for a.e. $\omega \in [0,1)^d$.

We say that R is *measurable* if $\omega \mapsto \langle R(\omega) P_{J(\omega)} u, v \rangle$ is a measurable scalar function for every $u, v \in \ell^2(\mathbb{Z}^d)$.

In [6], Bownik proved that given a bounded, shift-preserving operator $L : V \to L^2(\mathbb{R}^d)$, there exists a measurable range operator R on J such that

$$(3) \qquad\qquad (\mathcal{T} \circ L)f(\omega) = R(\omega)\left(\mathcal{T}f(\omega)\right),$$

for a.e. $\omega \in [0,1)^d$ and $f \in V$.

Conversely, given a measurable range operator R on J with

$$\operatorname*{ess\,sup}_{\omega \in [0,1)^d} \|R(\omega)\| < \infty,$$

there exists a bounded shift preserving operator $L : V \to L^2(\mathbb{R}^d)$ such that (3) holds. The correspondence between L and R is one-to-one under the convention that the range operators are identified if they are equal a.e. $\omega \in [0,1)^d$. Furthermore, he also proved the following relation between the operator norm of L and its associated range operator,

$$(4) \qquad \|L\| = \operatorname*{ess\,sup}_{\omega \in [0,1)^d} \|R(\omega)\|.$$

Let us now focus on the particular case where $L : V \to V$. In this setting, $R(\omega) : J(\omega) \to J(\omega)$ for a.e. $\omega \in [0,1)^d$. The following result, that was proved in [6], shows that some properties of the operator L are intrinsically related to the properties of its fibers.

Theorem 3.3. *Let V be a shift-invariant space and $L : V \to V$ a shift-preserving operator with associated range operator R. Then, the adjoint operator $L^* : V \to V$ is also a shift-preserving operator and its corresponding range operator R^* satisfies that $R^*(\omega) = (R(\omega))^*$ for a.e. $\omega \in [0,1)^d$. As a consequence, L is self-adjoint if and only if $R(\omega)$ is self-adjoint for a.e. $\omega \in [0,1)^d$, and L is a normal operator if and only if $R(\omega)$ is a normal operator for a.e. $\omega \in [0,1)^d$.*

When a shift-preserving operator acts on a shift-invariant space satisfying that $\dim J(\omega) < \infty$ for a.e. $\omega \in [0,1)^d$, the fibers of its corresponding range operator can be identified with matrices. This is done explicitly in the next proposition.

Proposition 3.4. *Let V be a shift-invariant space with range function J such that $\dim(J(\omega)) < \infty$ a.e. $\omega \in [0,1)^d$ and $L : V \to V$ a shift-preserving operator with associated range operator R. Then, $R(\omega)$ has a matrix representation for a.e. $\omega \in [0,1)^d$ with measurable entries. More precisely, if $\dim(J(\omega)) = n$ a.e. $\omega \in B$ where $B \subseteq [0,1)^d$ is measurable, then, there exist n^2 measurable bounded $\mathbb{Z}^d$-periodic functions defined on B, $\{m_{i,j}\}_{i,j=1}^n$ such that the matrix representation of $R(\omega)$ is*

$$(5) \qquad \begin{bmatrix} m_{1,1}(\omega) & m_{1,2}(\omega) & \cdots & m_{1,n}(\omega) \\ m_{2,1}(\omega) & m_{2,2}(\omega) & \cdots & m_{2,n}(\omega) \\ \vdots & & \ddots & \\ m_{n,1}(\omega) & m_{n,2}(\omega) & \cdots & m_{n,n}(\omega) \end{bmatrix},$$

for a.e. $\omega \in B$.

Proof. Let $\{\varphi_i\}_{i \in \mathbb{N}} \subset L^2(\mathbb{R}^d)$ and the partition $\{A_n\}_{n \in \mathbb{N}_0}$ of $[0,1)^d$ be the ones given by Lemma 2.9. For a fixed $n \in \mathbb{N}$ we will see that for a.e. $\omega \in A_n$, $R(\omega)$ has a matrix representation as (5) relative to the basis $\{\mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega)\}$ of $J(\omega)$. The point will be to prove that the matrix can be chosen to have measurable entries.

Since $\{T_k\varphi_i : i \in \mathbb{N}, k \in \mathbb{Z}^d\}$ forms a Parseval frame for V, then for every $j \in \mathbb{N}$ we have that

$$L\varphi_j = \sum_{i \in \mathbb{N}} \sum_{k \in \mathbb{Z}^d} c_i^j(k) T_k \varphi_i,$$

where $\{c_i^j(k)\}_{i \in \mathbb{N}, k \in \mathbb{Z}^d} \in \ell^2(\mathbb{N} \times \mathbb{Z}^d)$. Thus, by (1), we obtain that $\mathcal{T}L\varphi_j(\omega) = \sum_{i \in \mathbb{N}} \left(\sum_{k \in \mathbb{Z}^d} c_i^j(k) e_k(\omega) \right) \mathcal{T}\varphi_i(\omega)$, for a.e $\omega \in [0,1)^d$. Since we are assuming that $\omega \in A_n$, we have that

$$\mathcal{T}L\varphi_j(\omega) = \sum_{i=1}^{n} \left(\sum_{k \in \mathbb{Z}^d} c_i^j(k) e_k(\omega) \right) \mathcal{T}\varphi_i(\omega)$$

and $m_{i,j} := \sum_{k \in \mathbb{Z}^d} c_i^j(k) e_k$ is a $\mathbb{Z}^d$-periodic measurable function.

Now, for $\omega \in A_n$, we denote by $[R](\omega)$ the matrix form of $R(\omega)$ relative to the basis $\{\mathcal{T}\varphi_1(\omega), \dots, \mathcal{T}\varphi_n(\omega)\}$. Then, we can describe $[R](\omega)$ in terms of $\{m_{i,j}(\omega)\}_{i,j}$ as follows: note that since $\omega \in A_n$,

$$([R](\omega))_{i,j} = (R(\omega)\mathcal{T}\varphi_j(\omega))_i = (\mathcal{T}L\varphi_j(\omega))_i = m_{i,j}(\omega).$$

Notice that since L is a bounded operator and (4) holds, then $|m_{i,j}(\omega)| \leq \|L\|$ for $i, j = 1, \dots, n$ and for a.e. $\omega \in A_n$.

Since every $B \subseteq [0,1)^d$ where $\dim J(\omega) = n$ for a.e. $\omega \in B$ is included in A_n, the statement of the proposition holds. $\qquad\square$

In the same spirit of Theorem 3.3, we will show that invertibility of a shift-preserving operator can be deduced from the invertibility of its fibers when $\dim(J(\omega)) < \infty$ for a.e. $\omega \in [0,1)^d$.

Theorem 3.5. *Let V be a shift-invariant space with range function J such that $\dim(J(\omega)) < \infty$ for a.e. $\omega \in [0,1)^d$ and $L : V \to V$ a shift-preserving operator with associated range operator R. Then, the following statements hold:*

(i) *If L is invertible, then L^{-1} is also shift preserving.*

(ii) *L is invertible if and only if $R(\omega)$ is invertible for a.e. $\omega \in [0,1)^d$ and there exist a constant $C > 0$ such that $R(\omega)$ is uniformly bounded from below by C (i.e. for a.e. $\omega \in [0,1)^d$ we have that $\|R(\omega)a\| \geq C\|a\|$ for all $a \in J(\omega)$). In that case, if we denote by R^{-1} the range operator associated to L^{-1}, we have that $(R(\omega))^{-1} = R^{-1}(\omega)$ for a.e. $\omega \in [0,1)^d$.*

Proof. Assume that L is invertible. Then, the inverse of L is a bounded operator. Now, for every $f \in V$ and $k \in \mathbb{Z}^d$ we see that

$$L^{-1}T_k f = L^{-1}T_k L L^{-1} f = L^{-1} L T_k L^{-1} f = T_k L^{-1} f,$$

then L^{-1} is a shift-preserving operator. This proves (i).

For (ii), first assume that L is invertible and denote by R^{-1} the range operator of L^{-1}. It is not difficult to prove that $\omega \mapsto R^{-1}(\omega)R(\omega)$ and $\omega \mapsto R(\omega)R^{-1}(\omega)$ are measurable and uniformly bounded range operators on J. Furthermore, their associated shift-preserving operators are $L^{-1}L$ and LL^{-1} respectively. Hence, we have that $R^{-1}(\omega)R(\omega) = R(\omega)R^{-1}(\omega) = \mathcal{I}$, where $\mathcal{I}$ denotes the identity range operator on $J(\omega)$ for a.e. $\omega \in [0,1)^d$. Thus, $R(\omega)$ is invertible and $(R(\omega))^{-1} = R^{-1}(\omega)$ for a.e. $\omega \in [0,1)^d$. In addition, $R(\omega)$ is uniformly bounded from below, since $(R(\omega))^{-1}$ is bounded uniformly from above by $\|L^{-1}\|$.

For the converse, assume that $R(\omega)$ is invertible for a.e. $\omega \in [0,1)^d$ and is uniformly bounded from below by $C > 0$. Then $\omega \mapsto (R(\omega))^{-1}$ is a measurable range operator on J (see Proposition 5.1) which is uniformly bounded by C. Hence, it

has a corresponding shift-preserving operator $\tilde{L}$ such that for every $f \in V$ and a.e. $\omega \in [0,1)^d$, $(R(\omega))^{-1}(\mathcal{T}f(\omega)) = \mathcal{T}\tilde{L}f(\omega)$. Given that

$$\mathcal{T}f(\omega) = (R(\omega))^{-1}R(\omega)(\mathcal{T}f(\omega)) = \mathcal{T}\tilde{L}Lf(\omega),$$

$$\mathcal{T}f(\omega) = R(\omega)(R(\omega))^{-1}(\mathcal{T}f(\omega)) = \mathcal{T}L\tilde{L}f(\omega),$$

for every $f \in V$ and a.e. $\omega \in [0,1)^d$, we deduce that $\tilde{L}L = L\tilde{L} = \mathcal{I}_V$, and so L is invertible and $\tilde{L} = L^{-1}$. $\qquad\square$

Remark 3.6. The condition of $R(\omega)$ being uniformly bounded from below is natural as we know that, in fact, L is bounded from below by a constant $C > 0$ (i.e. $\|Lf\| \geq C\|f\|$ for all $f \in V$), if and only if $R(\omega)$ is uniformly bounded from below by C (see [6], Theorem 4.6). For example, let R be a range operator on a range function J defined by $R(\omega)a = \omega a$ for $a \in J(\omega)$ and $\omega \in [0,1)^d$. This range operator is measurable and is uniformly bounded from above, thus it has a corresponding shift-preserving operator L. Notice that $R(\omega)$ is invertible for a.e. $\omega \in [0,1)^d$. However, since it is not uniformly bounded from below, then L is not bounded from below and is not invertible.

We will now show some examples of shift-preserving operators and their corresponding range operator. The first example was shown in [6].

Example 3.7. Let I be an index set. Suppose $\Phi = \{\varphi_i : i \in I\}$ is a set of functions such that $E(\Phi)$ is a Bessel family. Then, the frame operator of $E(\Phi)$ is self-adjoint and shift preserving with corresponding range operator $R(\omega)$ given by the frame operator of $\{\mathcal{T}\varphi_i(\omega) : i \in I\}$ for a.e. $\omega \in [0,1)^d$.

Example 3.8. Let V be a shift-invariant space of $L^2(\mathbb{R}^d)$ with range function J. Denote by $P_V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ the orthogonal projection of $L^2(\mathbb{R}^d)$ onto V. Then, P_V is shift preserving and, by Lemma 2.4, its range operator is given by $R(\omega) = P_{J(\omega)}$ for a.e. $\omega \in [0,1)^d$.

For the following example we need to introduce a new definition that will become crucial later on.

Definition 3.9. We say that a sequence $a = \{a_j\} \in \ell^2(\mathbb{Z}^d)$ is of *bounded spectrum* if $\hat{a} \in L^\infty([0,1)^d)$, where $\hat{a}(\omega) = \sum_{j \in \mathbb{Z}^d} a_j e_j(\omega)$.

Example 3.10. Let $\Phi = \{\varphi_1, ..., \varphi_n\} \subset L^2(\mathbb{R}^d)$ be a set of functions such that $V = S(\Phi)$ and $E(\Phi)$ is a Riesz basis of V. For every $f \in V$, there exist unique sequences $b_1, ...b_n$ in $\ell^2(\mathbb{Z}^d)$ such that $f = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^d} b_i(k)T_k\varphi_i$. Now, let $a_1, ..., a_n$ be sequences of bounded spectrum and let us define an operator $L : V \to V$ by

$$(6) \qquad Lf := \sum_{i=1}^n \sum_{k \in \mathbb{Z}^d} (b_i * a_i)(k)T_k\varphi_i.$$

Then, L is a bounded shift-preserving operator, and its range operator can be written as an $n \times n$ diagonal matrix with diagonal $\hat{a}_1(\omega), ..., \hat{a}_n(\omega)$ in the basis $\{\mathcal{T}\varphi_1(\omega), ..., \mathcal{T}\varphi_n(\omega)\}$ for a.e. $\omega \in [0,1)^d$.

Remark 3.11. Given a shift-invariant space V with range function J and a shift-preserving operator $L : V \to V$, define the operator $\tilde{L} := \mathcal{T}L\mathcal{T}^{-1} : M_J \to M_J$, where $\mathcal{T}$ is the map defined in the Proposition 2.2. Then, M_J can be written as

a *direct integral Hilbert space* and the operator $\widetilde{L}$ as a *direct integral operator*, (see [12] for definitions).

That is

$$M_J = \int_{[0,1)^d}^{\oplus} J(\omega)\,d\omega, \qquad \text{and} \qquad \widetilde{L} = \int_{[0,1)^d}^{\oplus} R(\omega)\,d\omega,$$

where R is the range operator related to L.

Thus, most of the results of this paper can be extended to the general case of direct integral operators acting on direct integral Hilbert spaces and hyper-invariant subspaces, not necessarily associated with shift-invariant spaces.

4. s-EIGENVALUES AND s-EIGENSPACES

In this section, we introduce the concepts of *s-eigenvalue* and *s-eigenspace* for a shift-preserving operator L and we study the relation between these and the eigenvalues and eigenspaces of its range operator. This requires to deal with delicate issues concerning the measurability of the eigenvalues of $R(\omega)$ and their associated eigenspaces that we will study in Section 5.

We begin with a proposition that will serve as a starting point to define s-eigenvalues of L.

Proposition 4.1. *Given $a = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, define the operator $\Lambda_a : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ as*

$$\Lambda_a = \sum_{k \in \mathbb{Z}^d} a_k T_k.$$

Then Λ_a is well-defined and bounded if and only if the sequence a is of bounded spectrum.

Proof. Observe that the operator $M_a : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by $M_a(\psi) = \hat{a}\psi$, is well defined and bounded if and only if $\hat{a} \in L^\infty([0,1)^d)$. The result is a consequence of the fact that M_a and Λ_a are unitarily equivalent via the Fourier transform. $\square$

Remark 4.2. If $a \in \ell^2(\mathbb{Z}^d)$ is of bounded spectrum, then Λ_a satisfies the intertwining property $\mathcal{T}(\Lambda_a f)(\omega) = \hat{a}(\omega)\mathcal{T}f(\omega)$ for every $f \in L^2(\mathbb{R}^d)$ and for a.e. $\omega \in [0,1)^d$. Observe that, if V is shift invariant, $\Lambda_a : V \to V$ is a shift-preserving operator and its corresponding range operator is $R_a(\omega) = \hat{a}(\omega)\mathcal{I}$ for a.e. $\omega \in [0,1)^d$, where $\mathcal{I}$ denotes the identity operator on $J(\omega)$.

Moreover, when $E(\Phi)$ is a frame of V where $\Phi = \{\varphi_1, \ldots, \varphi_n\} \subset L^2(\mathbb{R}^d)$ we have that every $f \in V$ can be written as $f = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^d} b_i(k) T_k \varphi_i$ with $b_i \in \ell^2(\mathbb{Z}^d)$ for all $i = 1, \ldots, n$. Then,

$$\widehat{\Lambda_a f}(\omega) = \hat{a}(\omega) \sum_{i=1}^n \hat{b}_i(\omega)\widehat{\varphi}_i(\omega) = \sum_{i=1}^n \widehat{a * b_i}(\omega)\widehat{\varphi}_i(\omega) \quad \text{for a.e. } \omega \in \mathbb{R}^d$$

and thus, $\Lambda_a f = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^d} (a * b_i)(k) T_k \varphi_i$.

Definition 4.3. Let V be a shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. Given $a \in \ell^2(\mathbb{Z}^d)$ a sequence of bounded spectrum, we say that Λ_a is an *s-eigenvalue* of L if

$$V_a := \ker(L - \Lambda_a) \neq \{0\}.$$

We call V_a the *s-eigenspace* associated to Λ_a.

Remark 4.4. Observe that if $\lambda \in \mathbb{C}$ is an eigenvalue of L (i.e. $\ker(L - \lambda\mathcal{I}) \neq \{0\}$), then it is an s-eigenvalue Λ_a of L, taking the sequence $a = \lambda e_0$, where $e_0(k) = \delta_{0,k}$, $k \in \mathbb{Z}^d$. Thus, s-eigenvalues generalize the eigenvalues of L.

Notice that V_a is always a shift-invariant subspace of V and it is L-invariant, that is, $LV_a \subseteq V_a$. Also, observe that since for every $f \in V_a$, $Lf = \Lambda_a f$, then

$$R(\omega)(\mathcal{T}f(\omega)) = \mathcal{T}(Lf)(\omega) = \mathcal{T}(\Lambda_a f)(\omega) = \hat{a}(\omega)\mathcal{T}f(\omega),$$

for a.e. $\omega \in [0,1)^d$. Hence, this new definition of s-eigenvalue is related with the eigenvalues of the range operator of L, as we state in the next proposition.

Proposition 4.5. *Let V be a shift-invariant space with range function J such that $\dim J(\omega) < \infty$ for a.e. $\omega \in [0,1)^d$, $L : V \to V$ a bounded shift-preserving operator with associated range operator R and $a \in \ell^2(\mathbb{Z}^d)$ a sequence of bounded spectrum. Then, the following statements hold:*

(i) *If Λ_a is an s-eigenvalue of L, then $\lambda_a(\omega) := \hat{a}(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in \sigma(V_a)$.*

(ii) *The mapping $\omega \mapsto \ker(R(\omega) - \lambda_a(\omega)\mathcal{I})$, $\omega \in [0,1)^d$ is the measurable range function of V_a, which we will denote J_{V_a}.*

Proof. (i). By Lemma 2.6, there exists $\varphi_a \in V_a$ such that $\operatorname{supp} \|\mathcal{T}\varphi_a(\cdot)\|_{\ell^2} = \sigma(V_a)$. By (4), $\mathcal{T}\varphi_a(\omega) \in \ker(R(\omega) - \lambda_a(\omega)\mathcal{I})$ for a.e. $\omega \in [0,1)^d$. Now, given that $\mathcal{T}\varphi_a(\omega) \neq 0$ almost everywhere in $\sigma(V_a)$, we have that for a.e. $\omega \in \sigma(V_a)$,

$$\ker(R(\omega) - \lambda_a(\omega)\mathcal{I}) \neq \{0\},$$

i.e. $\lambda_a(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in \sigma(V_a)$.

(ii). It is clear that the mapping $\omega \mapsto R(\omega) - \lambda_a(\omega)\mathcal{I}$ is a measurable range operator. Hence, by Proposition 5.2, the mapping $\omega \mapsto \ker(R(\omega) - \lambda_a(\omega)\mathcal{I})$ is a range function which is measurable.

It remains to see that it is, in fact, the range function associated to V_a. But this is a straightforward consequence of (2) and (4). $\qquad\square$

Note that we have proved that the range function associated to V_a takes as values the eigenspaces of $R(\omega)$ associated to the eigenvalues $\lambda_a(\omega)$ for a.e. $\omega \in \sigma(V_a)$.

On the other hand, measurable eigenvalues of $R(\omega)$ induce s-eigenvalues of L, as we state in the following lemma.

Lemma 4.6. *Let V be a shift-invariant space and $L : V \to V$ a bounded shift-preserving operator with range operator R. Let $B \subseteq [0,1)^d$ be a measurable set of positive measure. Suppose $\lambda : B \to \mathbb{C}$ is a measurable function such that $\lambda(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in B$, then there exists $a \in \ell^2(\mathbb{Z}^d)$ of bounded spectrum such that $\lambda(\omega) = \hat{a}(\omega)$ a.e. $\omega \in B$ and Λ_a is an s-eigenvalue of L.*

Proof. Observe that as R is the range operator of L, by (4) we get that $|\lambda(\omega)| \leq \|L\|$ for a.e. $\omega \in B$. Consider any bounded extension of λ to $[0,1)^d$, then, we have that $\lambda \in L^\infty([0,1)^d)$ and thus there exist $a \in \ell^2(\mathbb{Z}^d)$ of bounded spectrum such that $\hat{a}(\omega) = \lambda(\omega)$. On the other hand, since $\lambda(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in B$, then

$$\ker(R(\omega) - \lambda(\omega)\mathcal{I}) \neq \{0\}$$

for a.e. $\omega \in B$, from which we deduce that $V_a \neq \{0\}$, and therefore Λ_a is an s-eigenvalue of L. $\qquad\square$

A natural question that arises here is whether these kind of measurable functions exist. When $\dim J(\omega)$ is constant, we can consider $R(\omega)$ as a matrix of measurable entries (see Proposition 3.4). We will prove the existence of measurable eigenvalues for that case in Subsection 5.2. Then, for the more general case on which $\dim J(\omega) < \infty$ for a.e. $\omega \in [0,1)^d$ we will construct measurable functions and measurable sets on which such functions are eigenvalues of $R(\omega)$ (see Theorem 5.9).

In the next proposition we give conditions on two s-eigenvalues in order that its associated s-eigenspaces are in direct sum.

Proposition 4.7. *Let V be a shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. Let $a, b \in \ell^2(\mathbb{Z}^d)$ be two distinct sequences of bounded spectrum such that Λ_a and Λ_b are s-eigenvalues of L. The following statements are equivalent:*

   (i) *$V_a \cap V_b = \{0\}$,*
   (ii) *$\hat{a}(\omega) \neq \hat{b}(\omega)$ almost everywhere in $\sigma(V_a) \cap \sigma(V_b)$.*

Proof. We assume that $\sigma(V_a) \cap \sigma(V_b) \neq \emptyset$ since, otherwise, the equivalence holds.

(i) $\Rightarrow$ (ii). By (iii) in Lemma 2.7, we have that $J_{V_a}(\omega) \cap J_{V_b}(\omega) = J_{V_a \cap V_b}(\omega) = \{0\}$, i.e.

$$\ker\left(R(\omega) - \lambda_a(\omega)\mathcal{I}\right) \cap \ker\left(R(\omega) - \lambda_b(\omega)\mathcal{I}\right) = \{0\},$$

for a.e. $\omega \in [0,1)^d$.

Suppose there exists a measurable set $A \subset \sigma(V_a) \cap \sigma(V_b)$ such that $|A| > 0$ and $\hat{a}(\omega) = \hat{b}(\omega)$ when $\omega \in A$. Then, if $\omega \in A$, we have that

$$\ker\left(R(\omega) - \lambda_a(\omega)\mathcal{I}\right) = \ker\left(R(\omega) - \lambda_b(\omega)\mathcal{I}\right) = \{0\},$$

which is a contradiction.

(ii) $\Rightarrow$ (i). Let $f \in V_a \cap V_b$, then $Lf = \Lambda_a f = \Lambda_b f$ and thus,

$$(\hat{a} - \hat{b})(\omega)\mathcal{T}f(\omega) = 0,$$

for a.e. $\omega \in [0,1)^d$. By (ii), we get that $\mathcal{T}f(\omega) = 0$ for a.e. $\omega \in \sigma(V_a) \cap \sigma(V_b)$, given that $\sigma(V_a \cap V_b) \subseteq \sigma(V_a) \cap \sigma(V_b)$, we have that $f = 0$. $\square$

5. Measurability

This section is dedicated to discussing measurability questions that arose in the previous sections. It is divided into two subsections. In the first one, we prove the measurability of the inverse of a range operator and the measurability of the kernel of a range operator. The second subsection is devoted to studying the existence of measurable eigenvalues for a range operator.

5.1. **Measurability of $\omega \mapsto (R(\omega))^{-1}$ and $\omega \mapsto \ker(R(\omega))$.** We begin by proving the following proposition which was used in the proof of Theorem 3.5.

Proposition 5.1. *Let $R(\omega) : J(\omega) \to J(\omega)$ be a measurable range operator on a range function J satisfying $\dim J(\omega) < \infty$ a.e. $\omega \in [0,1)^d$. Assume that $R(\omega)$ is invertible for a.e. $\omega \in [0,1)^d$, then $\omega \mapsto (R(\omega))^{-1}$, $\omega \in [0,1)^d$ is a measurable range operator.*

Proof. Let $\{\varphi_i\}_{i \in \mathbb{N}} \subset L^2(\mathbb{R}^d)$ and the sets $\{A_n\}_{n \in \mathbb{N}_0}$ be the ones given by Lemma 2.9. In order to prove that $[0,1)^d \ni \omega \mapsto (R(\omega))^{-1}$ is a measurable range operator, it suffices to prove that so is $A_n \ni \omega \mapsto (R(\omega))^{-1}$ for every $n \in \mathbb{N}_0$.

It is clearly true for A_0. Now, for a fixed $n \geq 1$ we need to prove that $A_n \ni \omega \mapsto \langle (R(\omega))^{-1} P_{J(\omega)} u, v \rangle$ is measurable for every $u, v \in \ell^2(\mathbb{Z}^d)$. By Lemma 2.9, we have that $\{ \mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega) \}$ is an orthonormal basis of $J(\omega)$ for a.e. $\omega \in A_n$. Then, it is enough to see that $\omega \mapsto \langle (R(\omega))^{-1} \mathcal{T}\varphi_i(\omega), \mathcal{T}\varphi_j(\omega) \rangle$ is measurable for every $i, j = 1 \ldots, n$. In order to do so, let $\{ e_1, \ldots, e_n \}$ be the canonical basis of $\mathbb{C}^n$ and, for $\omega \in A_n$, define $\nu(\omega) : J(\omega) \to \mathbb{C}^n$ as

$$(7) \qquad \nu(\omega)\left(\mathcal{T}\varphi_i(\omega) \right) = e_i,$$

for $i = 1, \ldots, n$. This is the change-of-basis operator from $\{ \mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega) \}$ to $\{ e_1, \ldots, e_n \}$, which is unitary for a.e $\omega \in A_n$. We have that $\nu(\omega)(R(\omega))^{-1}\nu(\omega)^{-1} = ([R](\omega))^{-1}$, where $[R](\omega)$ is the matrix form of $R(\omega)$ relative to the basis $\{ \mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega) \}$.

Hence, given i and j, we see that

$$\left\langle R(\omega)^{-1} \mathcal{T}\varphi_i(\omega), \mathcal{T}\varphi_j(\omega) \right\rangle = \left\langle \nu(\omega)^{-1}([R](\omega))^{-1}\nu(\omega)\mathcal{T}\varphi_i(\omega), \mathcal{T}\varphi_j(\omega) \right\rangle$$
$$= \left\langle ([R](\omega))^{-1} e_i, e_j \right\rangle,$$

for a.e. $\omega \in A_n$. Now, for $\omega \in A_n$, recall that the coefficients of $([R](\omega))^{-1}$, namely $r_{i,j}(\omega)$, are given by the formula $r_{i,j}(\omega) = (-1)^{i+j} \det(([R](\omega))_{ij})/\det([R](\omega))$, where $[R](\omega)_{ij}$ denotes the minor matrix obtained from $[R](\omega)$ removing the i-th row and j-th column. Since these operations preserve measurability and $[R](\omega)$ has measurable functions as entries, we conclude that $([R](\omega))^{-1}$ is a matrix with measurable entries as well. Finally, $A_n \ni \omega \mapsto \langle ([R](\omega))^{-1} e_i, e_j \rangle$ is a measurable scalar function and so is $A_n \ni \omega \mapsto (R(\omega))^{-1}$, as we wanted to prove. $\qquad \square$

Now we prove the condition of measurability that we used in Proposition 4.5.

Proposition 5.2. *Let $R(\omega) : J(\omega) \to J(\omega)$ be a measurable range operator on a range function J satisfying $\dim J(\omega) < \infty$ a.e. $\omega \in [0,1)^d$. Then $\omega \mapsto \ker(R(\omega))$, $\omega \in [0,1)^d$ is a measurable range function.*

For this, we will need the subsequent two lemmas which were proved in [1]. For the sake of completeness we will include their proofs.

Lemma 5.3. *Let $M = M(\omega)$ be an $n \times n$ matrix of measurable entries defined on a measurable set $B \subset \mathbb{R}^d$. If for $k = 0, \ldots, n$*

$$B_k := \{ \omega \in B \, : \, \mathrm{rank}(M(\omega)) = k \},$$

then B_k is measurable and $B = \bigcup_{k=0}^n B_k$.

Proof. For $k = 0, \ldots, n$ consider $C_k(\omega) := \{ A(\omega) \in \mathbb{C}^{k \times k} : A(\omega) \text{ submatrix of } M(\omega) \}$, and let $f_k : B \to \mathbb{R}$ be defined as

$$f_k(\omega) = \sum_{A(\omega) \in C_k(\omega)} |\det(A(\omega))|.$$

Then, f_k is measurable for each k, and $f_k(\omega) > 0$ if and only if $\mathrm{rank}\,(M(\omega)) \geq k$. Since $B_k = \{ \omega \in B \, : \, f_k(\omega) > 0 \} \setminus \{ \omega \in B \, : \, f_{k+1}(\omega) > 0 \}$, it follows that B_k is measurable. $\qquad \square$

Lemma 5.4. *Let $B \subset \mathbb{R}^d$ be a measurable set and $M = M(\omega)$ be an $n \times n$ matrix of measurable entries defined on B. Suppose that* rank $(M(\omega)) = k$ *for a.e. $\omega \in B$. Then there exist measurable functions $v_1, \ldots, v_{n-k} : B \to \mathbb{C}^n$ such that the vectors $\{v_1(\omega), \ldots, v_{n-k}(\omega)\}$ form an orthonormal basis for* ker $(M(\omega))$, *for a.e. $\omega \in B$. In particular, if $P_{\ker(M(\omega))}$ denotes the orthogonal projection of $\mathbb{C}^n$ onto* ker $(M(\omega))$, *we have that*

$$B \ni \omega \mapsto \langle P_{\ker(M(\omega))} x, y \rangle$$

is measurable for every $x, y \in \mathbb{C}^n$.

Proof. Define

$$S_k = \{C \times D : C \subset \{1, \ldots, n\}, \, D \subset \{1, \ldots, n\}, \#C = \#D = k\}.$$

For $h \in S_k$, let M_h be the submatrix of M whose entries have subindexes in h. We have that $B = \bigcup_{h \in S_k} B_h$, where $B_h = \{\omega \in B : \det(M_h(\omega)) \neq 0\}$. The continuity of the determinant function implies that B_h are measurable for all $h \in S_k$. From the sets B_h we can construct measurable disjoint sets E_h, $h \in S_k$ such that $B = \bigcup_{h \in S_k} E_h$ and $\det(M_h(\omega)) \neq 0$ for a.e. $\omega \in E_h$. In this way, for each ω we selected a unique invertible $k \times k$ submatrix $M_h(\omega)$. Note that for a fixed $h \in S_k$ the indexes of the submatrices $M_h(\omega)$ are the same for all ω.

For each fixed $h \in S_k$ we want to choose measurable functions $v_1, \ldots, v_{n-k}$ defined on E_h, such that the vectors $\{v_1(\omega), \ldots, v_{n-k}(\omega)\}$ form a basis of ker$(M(\omega))$ for a.e. $\omega \in E_h$.

Without loss of generality we may assume that $h = \{1, \ldots, k\} \times \{1, \ldots, k\}$ (the general case can be reduced to this one by permutation of rows and columns). For $\omega \in E_h$ let us define the vector functions:

$$\begin{aligned}
v_1^h &= (x_{11}^h, \ldots, x_{1k}^h, 1, 0, \ldots, 0) \\
v_2^h &= (x_{21}^h, \ldots, x_{2k}^h, 0, 1, 0, \ldots, 0) \\
&\vdots \\
v_{n-k}^h &= (x_{(n-k)1}^h, \ldots, x_{(n-k)k}^h, 0, \ldots, 0, 1)
\end{aligned}$$

with $(x_{i1}^h(\omega), \ldots, x_{ik}^h(\omega)) = -M_h^{-1}(\omega) c_{i+k}(\omega)$ for $i = 1, \ldots, n - k$, where c_{i+k} is the column vector containing the first k entries of the $(i + k)$-column of $M(\omega)$. It is straightforward to see that the vectors $\{v_1^h(\omega), \ldots, v_{n-k}^h(\omega)\}$ are measurable functions and form a basis of ker$(M(\omega))$ for a.e. $\omega \in E_h$.

Finally, if we set $v_i = \sum_{h \in S_k} v_i^h \chi_{E_h}$, $i = 1, \ldots, m - k$, we obtain a measurable basis for ker$(M(\omega))$, for a.e. $\omega \in B$. Since the Gram-Schmidt orthogonalization process conserves measurability, we obtain the desired result. $\qquad\square$

Proof of Proposition 5.2. Let $\{\varphi_i\}_{i \in \mathbb{N}} \subset L^2(\mathbb{R}^d)$ and the sets $\{A_n\}_{n \in \mathbb{N}_0}$ be the ones given by Lemma 2.9. We will prove that $[0, 1)^d \ni \omega \mapsto \ker(R(\omega))$ is a measurable range function by showing that so is $A_n \ni \omega \mapsto \ker(R(\omega))$ for every $n \in \mathbb{N}_0$.

For A_0 there is nothing to prove. Fix $n \geq 1$. We need to prove that the scalar function $A_n \ni \omega \mapsto \langle P_{\ker(R(\omega))} u, v \rangle$ is measurable for every $u, v \in \ell^2(\mathbb{Z}^d)$. As in the proof of Proposition 5.1, it is enough to see that $\omega \mapsto \langle P_{\ker(R(\omega))} \mathcal{T}\varphi_i(\omega), \mathcal{T}\varphi_j(\omega) \rangle$ is measurable for every $i, j = 1 \ldots, n$.

Let $\nu(\omega) : J(\omega) \to \mathbb{C}^n$ be defined as in (7), then we have that

$$P_{\ker(R(\omega))} = \nu(\omega)^{-1} P_{\ker([R](\omega))} \nu(\omega),$$

where $[R](\omega)$ is the matrix form of $R(\omega)$ relative to the basis $\{\mathcal{T}\varphi_1(\omega), \dots, \mathcal{T}\varphi_n(\omega)\}$. Therefore, given i and j, we have that

$$\left\langle P_{\ker(R(\omega))} \mathcal{T}\varphi_i(\omega), \mathcal{T}\varphi_j(\omega) \right\rangle = \left\langle \nu(\omega)^{-1} P_{\ker([R](\omega))} \nu(\omega) \mathcal{T}\varphi_i(\omega), \mathcal{T}\varphi_j(\omega) \right\rangle$$
$$= \left\langle P_{\ker([R](\omega))} e_j, e_j \right\rangle,$$

for a.e. $\omega \in A_n$.

Consider now the partition of A_n into the sets $\{B_1, \cdots, B_n\}$ given by Lemma 5.3. Then, on each B_k, $[R](\omega)$ is a matrix of rank k and, by Lemma 5.4, $B_k \ni \omega \mapsto \langle P_{\ker([R](\omega))} e_i, e_j \rangle$ is a measurable scalar function for every $1 \le k \le n$. Then, so is $A_n \ni \omega \mapsto \langle P_{\ker([R](\omega))} e_i, e_j \rangle$ and this is what we wanted to prove. $\qquad\square$

5.2. **Measurable eigenvalues.** In this subsection, we first assume that $\dim J(\omega)$ is constant and finite a.e. $\omega \in [0, 1)^d$ in order to find measurable eigenvalues of $R(\omega)$. Then, for the general case where $\dim J(\omega) < \infty$ a.e. $\omega \in [0, 1)^d$ we obtain measurable functions on $[0, 1)^d$ whose values are eigenvalues of $R(\omega)$ when ω is restricted to appropriate measurable subsets of $[0, 1)^d$.

5.2.1. *Case dim $J(\omega) = n$.* As shown in Proposition 3.4, if $\dim J(\omega) = n$ almost everywhere in a measurable set $B \subseteq [0, 1)^d$, then, the range operator can be seen as an $n \times n$ matrix of measurable functions over B. Thus, what we want to study is how to find measurable eigenvalues of an $n \times n$ matrix $M = M(\omega)$ of measurable functions defined on a measurable set $B \subset \mathbb{R}^d$.

For every $\omega \in B$, let $\lambda_1(\omega), \dots, \lambda_n(\omega)$ be the eigenvalues of $M(\omega)$, repeated according to their multiplicity, and in some order, the order chosen for each ω being arbitrary. In this way, we have defined n functions of ω with domain B. However, it is not generally true that these selected functions are measurable.

In the case where $M(\omega)$ is a self-adjoint matrix for a.e. $\omega \in B$, the eigenvalues are real and it is known that ordering them by size $\lambda_1(\omega) \ge \cdots \ge \lambda_n(\omega)$ for each ω, one obtain measurable functions, see [15] and [1]. When the assumption of self-adjointness is removed, the problem of how to order the eigenvalues into measurable functions arise and the techniques used in [15] and [1] are not adjustable to the new setting.

This problem can be reduced to the following question: Let

$$p_\omega(z) = \sum_{i=0}^{n-1} a_i(\omega) z^i + z^n, \quad \omega \in B$$

be a family of polynomials, where $a_i : B \to \mathbb{C}$ are measurable functions for $i = 0, \dots, n-1$. Is it possible to find measurable roots? That is, is it possible to find measurable functions $\lambda_1, \dots, \lambda_n : B \to \mathbb{C}$, such that $\lambda_1(\omega), \dots, \lambda_n(\omega)$ are the roots of $p_\omega(z)$ counted with multiplicity for almost all $\omega \in B$?

To prove the existence of a way of ordering the roots into measurable functions we will use the following theorem whose proof can be found in [3].

Theorem 5.5. *Let X and Y be complete separable metric spaces and E, a closed σ-compact subset of $X \times Y$. Then $\pi_1(E)$ is a Borel set in X and there exists a Borel function $\psi : \pi_1(E) \to Y$ whose graph is contained in E.*

We obtain our first result on the measurability of roots of a polynomial with varying coefficients.

Theorem 5.6. *Let $n \in \mathbb{N}$. Then, there exist n Borel functions $\psi_j : \mathbb{C}^n \to \mathbb{C}$, $j = 1, \ldots, n$, such that for every $a_0, \ldots, a_{n-1} \in \mathbb{C}$, we have that $\psi_1(a_0, \ldots, a_{n-1}), \ldots, \psi_n(a_0, \ldots, a_{n-1})$ are the roots (counted with multiplicity) of*

$$p_{(a_0,\ldots,a_{n-1})}(z) = \sum_{i=0}^{n-1} a_i z^i + z^n.$$

Proof. We will prove this by induction on n.

When $n = 1$, $p_{a_0}(z) = a_0 + z$, and $\psi_1(a_0) = -a_0$ is clearly a Borel function.

Suppose now that the statement holds for $n - 1$. Let

$$E = \left\{ (a_0, \ldots, a_{n-1}, z) \in \mathbb{C}^n \times \mathbb{C} \; : \; \sum_{i=0}^{n-1} a_i z^i + z^n = 0 \right\}.$$

It is easy to see that E is closed in $\mathbb{C}^n \times \mathbb{C}$ and, since every polynomial of complex coefficients has a complex root, $\pi_1(E) = \mathbb{C}^n$. Then, if we apply Theorem 5.5 with $X = \mathbb{C}^n$ and $Y = \mathbb{C}$, we get a Borel function $\psi_n : \mathbb{C}^n \to \mathbb{C}$, such that

$$(a_0, \ldots, a_{n-1}, \psi_n(a_0, \ldots, a_{n-1})) \in E,$$

i.e. $\psi_n(a_0, \ldots, a_{n-1})$ is a root of $p_{(a_0,\ldots,a_{n-1})}$, for every $a_0, \ldots, a_{n-1} \in \mathbb{C}$.

Now, we can write

$$p_{(a_0,\ldots,a_{n-1})}(z) = (z - \psi_n(a_0, \ldots, a_{n-1})) q_{(a_0,\ldots,a_{n-1})}(z),$$

where $q_{(a_0,\ldots,a_{n-1})}$ is a monic polynomial of degree $n - 1$ with coefficients

$$b_0(a_0, \ldots, a_{n-1}), \ldots, b_{n-2}(a_0, \ldots, a_{n-1}),$$

where $b_i : \mathbb{C}^n \to \mathbb{C}$ are Borel functions for $i = 0, \ldots, n - 2$ (since the division algorithm has only Borel operations).

By induction hypothesis, there exist $n - 1$ Borel functions $\widetilde{\psi}_j : \mathbb{C}^{n-1} \to \mathbb{C}$, $j = 1 \ldots, n - 1$, such that for every $b_0, \ldots, b_{n-2} \in \mathbb{C}$, we know that $\widetilde{\psi}_1(b_0, \ldots, b_{n-2}), \ldots, \widetilde{\psi}_{n-1}(b_0, \ldots, b_{n-2})$ are the roots of $\sum_{i=0}^{n-2} b_i z^i + z^{n-1}$.

Then, let us define $\psi_j : \mathbb{C}^n \to \mathbb{C}$ for $j = 1, \ldots, n - 1$ as

$$\psi_j(a_0, \ldots, a_{n-1}) := \widetilde{\psi}_j(b_0(a_0, \ldots, a_{n-1}), \ldots, b_{n-2}(a_0, \ldots, a_{n-1})).$$

These are Borel functions since they are composition of Borel functions and, by construction, they are the roots (counted with multiplicity) of $q_{(a_0,\ldots,a_{n-1})}$, and hence of $p_{(a_0,\ldots,a_{n-1})}$. $\square$

As a consequence, we get the result we were looking for.

Corollary 5.7. *If $M = M(\omega)$ is an $n \times n$ matrix of measurable functions defined on a measurable set $B \subset \mathbb{R}^d$, then there exist n measurable functions $\lambda_j : B \to \mathbb{C}$, $j = 1, \ldots, n$, such that $\lambda_1(\omega), \ldots, \lambda_n(\omega)$ are the eigenvalues of $M(\omega)$ for a.e. $\omega \in B$, counted with multiplicity.*

Proof. Since the entries of $M(\omega)$ are measurable, the characteristic polynomial of $M(\omega)$ is $p_\omega(z) = \sum_{i=0}^{n-1} a_i(\omega)z^i + z^n$, where $a_i : B \to \mathbb{C}$ are measurable functions. Then, consider the measurable function $A : \omega \mapsto (a_1(\omega), \ldots, a_{n-1}(\omega), 1)$ defined on B. Composing the Borel functions obtained in Theorem 5.6 with A, we get for each $j = 1 \ldots, n$ a measurable function $\lambda_j = \psi_j \circ A$, such that $\lambda_j(\omega)$ is a root of $p_\omega(z)$, i.e. an eigenvalue of $M(\omega)$, for a.e. $\omega \in B$. $\qquad\square$

Finally we have:

Theorem 5.8. *Let $R(\omega) : J(\omega) \to J(\omega)$ be a measurable range operator on a range function J satisfying $\dim J(\omega) = n$ a.e. $\omega \in B$ where $B \subseteq [0,1)^d$ is measurable. Then, there exist n measurable functions $\lambda_j : B \to \mathbb{C}$, $j = 1, \ldots, n$, such that $\lambda_1(\omega), \ldots, \lambda_n(\omega)$ are the eigenvalues of $R(\omega)$ a.e. $\omega \in B$, counted with multiplicity.*

Proof. When $\dim J(\omega) = n$ almost everywhere in a measurable set $B \subseteq [0,1)^d$, the range operator $R(\omega)$ can be seen as an $n \times n$ matrix of measurable functions over B, which we denote as $[R](\omega)$ (see Proposition 3.4). Then, if $\nu(\omega) : J(\omega) \to \mathbb{C}^n$ is defined as in (7) we have that

$$R(\omega) = \nu(\omega)^{-1}[R](\omega)\nu(\omega), \quad \text{for a.e. } \omega \in B.$$

Thus, if $\lambda : B \mapsto \mathbb{C}$ is a measurable function, it holds that $\ker(R(\omega) - \lambda(\omega)\mathcal{I}) = \ker(\nu(\omega)^{-1}([R](\omega) - \lambda(\omega)\mathcal{I})\nu(\omega))$ for a.e. $\omega \in B$. As a consequence, measurable eigenvalues of $R(\omega)$ and $[R](\omega)$ agree, and Corollary 5.7 gives what we wanted. $\quad\square$

In particular, the theorem holds for $B = A_n$, where A_n is defined in Lemma 2.9.

5.2.2. *General case: dim $J(\omega) < \infty$.* For the more general case when $\dim J(\omega) < \infty$ a.e. $\omega \in [0,1)^d$ we can construct L^∞-functions such that when restricted to some measurable sets, they are eigenvalues of $R(\omega)$. This procedure is what we do next.

Assume that R is a measurable range operator on J that is a uniformly bounded from above by a constant $K > 0$. Let $\{A_n\}_{n \in \mathbb{N}_0}$ be the disjoint measurable sets given by Lemma 2.9. Then, we have that $\sigma(V) = \bigcup_{n \in \mathbb{N}} A_n$. For every $n \in \mathbb{N}$, by Theorem 5.8, there exist n measurable functions defined on A_n, namely

$$\lambda_1^n, \ldots, \lambda_n^n : A_n \to \mathbb{C},$$

such that at a.e. $\omega \in A_n$ these are the eigenvalues of $R(\omega)$, counted with multiplicity.

Now, fix $n \in \mathbb{N}$ and for every $1 \le i \le n$ define the sets $A_{n,i}$ on which we have exactly i different eigenvalues, that is,

$$(8) \qquad A_{n,i} := \{\omega \in A_n : \#\{\lambda_1^n(\omega), \ldots, \lambda_n^n(\omega)\} = i\}.$$

These are also disjoint measurable sets (possibly of measure zero) such that $A_n = \bigcup_{i=1}^n A_{n,i}$, for every $n \in \mathbb{N}$. On each of these sets $A_{n,i}$ we have i measurable functions

$$\lambda_1^{n,i}, \ldots, \lambda_i^{n,i} : A_{n,i} \to \mathbb{C},$$

such that $\lambda_j^{n,i}(\omega)$ is an eigenvalue of $R(\omega)$ and $\lambda_j^{n,i}(\omega) \ne \lambda_{j'}^{n,i}(\omega)$ when $j \ne j'$ for a.e. $\omega \in A_{n,i}$. The measurability of $\lambda_j^{n,i}$ deserves a moment of thought: partitioning $A_{n,i}$ into all possibilities on which i eigenvalues from $\{\lambda_1^n(\omega), \ldots, \lambda_n^n(\omega)\}$ are different and pasting them properly we get measurability.

Moreover, since R is uniformly bounded by K, we have that $|\lambda_j^{n,i}(\omega)| \leq K$ for a.e. $\omega \in A_{n,i}$, and for every $j \leq i \leq n$ and $n \in \mathbb{N}$. We now proceed to paste these functions in the following way:

For every $j \in \mathbb{N}$, let $\lambda_j : [0,1)^d \to \mathbb{C}$ be defined as

$$\lambda_j(\omega) := \begin{cases} \lambda_j^{n,i}(\omega), & \text{when } \omega \in A_{n,i}, \text{ for } n \geq i \geq j, \\ K + j & \text{otherwise.} \end{cases}$$

These are measurable functions satisfying that $\lambda_j(\omega) \neq \lambda_{j'}(\omega)$ when $j \neq j'$ for a.e. $\omega \in [0,1)^d$ and $\lambda_j \in L^\infty([0,1)^d)$ for $j \in \mathbb{N}$.

Finally, note that for $j \in \mathbb{N}$, $\ker(R(\omega) - \lambda_j(\omega)\mathcal{I}) = \ker(R(\omega) - \lambda_j^{n,i}(\omega)\mathcal{I})$ for a.e. $\omega \in A_{n,i}$, for every $n \geq i \geq j$. Therefore, $\lambda_j(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in A_{n,i}$ for every $n \geq i \geq j$. Otherwise, $\ker(R(\omega) - \lambda_j(\omega)\mathcal{I}) = \{0\}$ since $\lambda_j(\omega) = K + j$ is not an eigenvalue of $R(\omega)$.

The above argument proves the following result:

Theorem 5.9. *Let $R(\omega) : J(\omega) \to J(\omega)$ be a measurable range operator on a range function J satisfying $\dim J(\omega) < \infty$ for a.e. $\omega \in [0,1)^d$. If R is uniformly bounded from above, then there exist functions $\lambda_j \in L^\infty([0,1)^d)$, $j \in \mathbb{N}$, such that $\lambda_j(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in A_{n,i}$ and for every $n \geq i \geq j$, where $A_{n,i}$ are the sets defined in* (8).

Remark 5.10.

(i) Assume that J is the range function of a shift-invariant space V which is finitely generated. Then, for only finitely many values of n, the sets A_n have positive measure. In particular, $|A_n| = 0$ for all $n > \mathcal{L}(V)$. In that case, if we discard the functions λ_j such that, for a.e. $\omega \in [0,1)^d$, $\lambda_j(\omega)$ is not an eigenvalue of $R(\omega)$, the above procedure will generate at most $\mathcal{L}(V)$ functions.

More precisely, for $i \in \mathbb{N}$, we define

$$(9) \qquad B_i := \bigcup_{n=i}^{\infty} A_{n,i}, \quad \text{and} \quad \kappa_{max} := \max\{i \in \mathbb{N} : |B_i| > 0\}.$$

Observe that, in fact, B_i is the set of all the $\omega \in \sigma(V)$ for which $R(\omega)$ has exactly i different eigenvalues. Hence, it is easy to see that

$$(10) \qquad \kappa_{max} = \operatorname*{ess\,sup}_{\omega \in \sigma(V)} \#\{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } R(\omega)\}.$$

Then, we have that κ_{max} is finite, with $\kappa_{max} \leq \mathcal{L}(V)$ and the procedure will generate exactly κ_{max} functions.

(ii) For $j \in \mathbb{N}$, define the sets

$$C_j := \bigcup_{i=j}^{\infty} B_i,$$

where B_i are the ones defined in (9). Notice that $C_{j+1} \subseteq C_j \subseteq \sigma(V)$ for every $j \in \mathbb{N}$. Each C_j is the set of all the $\omega \in \sigma(V)$ for which $R(\omega)$ has at least j different eigenvalues. Furthermore, from the construction done in Theorem 5.9, we also observe that

$$C_j = \left\{\omega \in [0,1)^d : \ker(R(\omega) - \lambda_j(\omega)\mathcal{I}) \neq \{0\}\right\},$$

for every $j \in \mathbb{N}$ and $|C_j| = 0$ for every $j > \kappa_{max}$.

(iii) Later on, we will assume that J satisfies $\dim J(\omega) \leq \ell$ for a.e. $\omega \in [0,1)^d$, for some $0 \leq \ell < \infty$. In that case, the functions constructed in Theorem 5.9 will be used to define s-eigenvalues of the shift-preserving operator associated to R.

(iv) The functions $\{\lambda_j\}_{j \in \mathbb{N}}$ of Theorem 5.9 are, clearly, not unique.

6. s-DIAGONALIZATION

In this section, we introduce a new property for a bounded shift-preserving operator L acting on a finitely generated shift-invariant space V, which we call s-*diagonalization*. In order to establish conditions on L for being s-diagonalizable, we exploit the finite dimensional structure of its range function and its range operator: for almost each $\omega \in [0,1)^d$, we have that $R(\omega)$ is a linear transformation acting on a vector space of finite dimension. Thus, for instance, if L is a normal operator, $R(\omega)$ is normal and then diagonalizable for a.e. $\omega \in [0,1)^d$.

6.1. **Definition and properties of s-diagonalizations.**

Definition 6.1. Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. We say that L is s-*diagonalizable* if there exists a finite number of sequences of bounded spectrum $a_1, \ldots, a_m$ such that $\Lambda_{a_1}, \ldots, \Lambda_{a_m}$ are s-eigenvalues of L and

$$V = V_{a_1} \oplus \cdots \oplus V_{a_m}, \tag{11}$$

where Λ_{a_j} and V_{a_j} are the ones defined in Definition 4.3 for $j = 1, \ldots, m$. In that case, we say that $(V, L, a_1, \ldots, a_m)$ is an s-*diagonalization* of L.

Remark 6.2.

(i) Notice that if $(V, L, a_1, \ldots, a_m)$ is an s-*diagonalization* of L, then each $f \in V$ can be written in a unique way as $f = f_1 + \cdots + f_m$ with $f_j \in V_{a_j}$. On the other hand, we have $L f_j = \Lambda_{a_j} f_j$, for $j = 1, ..., m$. If we denote by $Q_{V_{a_j}} : V \to V$ the oblique projection onto V_{a_j} respect to this decomposition, i.e. $Q_{V_{a_j}} f = f_j$ we can write:

$$L = \sum_{j=1}^{m} \Lambda_{a_j} Q_{V_{a_j}}.$$

(ii) The decomposition in (11) is not unique. Furthermore, the number m of s-eigenspaces of L for which V admits a decomposition like in (11) could be arbitrarily large. Indeed, take for example the first s-eigenvalue Λ_{a_1}, then, by Proposition 4.5, $\lambda_{a_1}(\omega)$ is an eigenvalue for $R(\omega)$ for a.e. $\omega \in \sigma(V_{a_1})$. Now, split $\sigma(V_{a_1}) = A \cup B$ where $A \cap B = \emptyset$ and both sets are of positive measure. Then, define $\lambda_A(\omega) := \lambda_{a_1}(\omega) \chi_A(\omega)$ and $\lambda_B(\omega) := \lambda_{a_1}(\omega) \chi_B(\omega)$. These are measurable functions which are eigenvalues of $R(\omega)$ over a positive measure set. By Lemma 4.6, it is possible to construct sequences a_A and a_B of bounded spectrum such that $\lambda_A(\omega) = \hat{a}_A(\omega)$ for a.e. $\omega \in A$, $\lambda_B(\omega) = \hat{a}_B(\omega)$ for a.e. $\omega \in B$, $\Lambda_{a_A}, \Lambda_{a_B}$ are s-eigenvalues of L and $V_{a_1} = V_{a_A} \oplus V_{a_B}$. Finally, we obtain the decomposition $V = V_{a_A} \oplus V_{a_B} \oplus V_{a_2} \oplus \cdots \oplus V_{a_m}$.

The next proposition shows that when L is invertible and s-diagonalizable, L^{-1} must be s-diagonalizable as well.

Proposition 6.3. *Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator which is invertible. If L is s-diagonalizable then L^{-1} is s-diagonalizable. Furthermore, if $(V, L, a_1, \ldots, a_m)$ is an s-diagonalization of L, then $(V, L^{-1}, b_1, \ldots, b_m)$ is an s-diagonalization of L^{-1} with $\hat{b}_j(\omega) = (\hat{a}_j(\omega))^{-1}$ for a.e. $\omega \in [0, 1)^d$ and $V_{b_j} = \ker(L^{-1} - \Lambda_{b_j}) = V_{a_j}$ for $j = 1, \ldots, m$.*

Proof. Fix $j \in \{1, \ldots, m\}$. Since L is invertible and $L|_{V_{a_j}} = \Lambda_{a_j} : V_{a_j} \to V_{a_j}$, we see that Λ_{a_j} is invertible. Let R_{a_j} be the range operator corresponding to Λ_{a_j}, this is $R_{a_j}(\omega) = \hat{a}_j(\omega)\mathcal{I}$ for a.e. $\omega \in [0, 1)^d$. By Theorem 4.6 in [6], we get that $R_{a_j}(\omega)$ is uniformly bounded from below by a constant $A > 0$ for a.e. $\omega \in [0, 1)^d$, which implies that $|\hat{a}_j(\omega)| \geq A$ for a.e. $\omega \in [0, 1)^d$. Also, since a_j is of bounded spectrum, there exists a constant $B > 0$ such that $A \leq |\hat{a}_j(\omega)| \leq B$ for a.e. $\omega \in [0, 1)^d$.

Define $b_j \in \ell^2(\mathbb{Z}^d)$ by its Fourier transform as $\hat{b}_j(\omega) := (\hat{a}_j(\omega))^{-1}$ for a.e. $\omega \in [0, 1)^d$. Then, we have that $B^{-1} \leq |\hat{b}_j(\omega)| \leq A^{-1}$ for a.e. $\omega \in [0, 1)^d$ and so b_j is of bounded spectrum. Let us see that Λ_{b_j} is an s-eigenvalue of L^{-1}. Let $f \in V_{a_j}$, then $\mathcal{T}(Lf)(\omega) = \hat{a}_j(\omega)\mathcal{T}f(\omega)$ for a.e. $\omega \in [0, 1)^d$. Now, the function $g := Lf$ is in V_{a_j} and we have that $\mathcal{T}g(\omega) = \hat{a}_j(\omega)\mathcal{T}(L^{-1}g)(\omega)$ for a.e. $\omega \in [0, 1)^d$, which is the same that $\hat{b}_j(\omega)\mathcal{T}g(\omega) = \mathcal{T}(L^{-1}g)(\omega)$ for a.e. $\omega \in [0, 1)^d$. This proves that for every $g \in V_{a_j}$, $L^{-1}g = \Lambda_{b_j}g$ and thus $V_{a_j} \subseteq \ker(L^{-1} - \Lambda_{b_j})$. A similar argument shows that in fact $V_{a_j} = \ker(L^{-1} - \Lambda_{b_j})$. $\square$

One of our main goals is to explore the relation between the notion of s-diagonalization for a shift-preserving operator L and the diagonalization of the fibers of its range operator. In this direction, we have the following result.

Theorem 6.4. *Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator with range operator R. If L is s-diagonalizable, then $R(\omega)$ is diagonalizable for a.e. $\omega \in \sigma(V)$.*

Proof. Suppose that $(V, L, a_1, \ldots, a_m)$ is an s-diagonalization of L. Then, by Proposition 4.5, for every $j = 1, \ldots, m$, $\lambda_{a_j}(\omega) = \hat{a}_j(\omega)$ is an eigenvalue of $R(\omega)$ for a.e. $\omega \in \sigma(V_{a_j})$ and $J_{V_{a_j}}(\omega) = \ker(R(\omega) - \lambda_{a_j}(\omega)\mathcal{I})$ is its associated eigenspace. We will see that $J(\omega) = J_{V_{a_1}}(\omega) \oplus \cdots \oplus J_{V_{a_m}}(\omega)$.

First, notice that since $V_{a_j} \subseteq V$, we have that $J_{V_{a_j}}(\omega) \subseteq J(\omega)$ for every $j = 1, \ldots, m$, and hence $J_{V_{a_1}}(\omega) + \cdots + J_{V_{a_m}}(\omega) \subseteq J(\omega)$. For the other inclusion, let $f \in V$, then $f = f_1 + \cdots + f_m$, where $f_j \in V_{a_j}$ for $j = 1, \ldots, m$. Thus, $\mathcal{T}f(\omega) = \mathcal{T}f_1(\omega) + \cdots + \mathcal{T}f_m(\omega)$ for a.e. $\omega \in [0, 1)^d$, which implies that

$$J(\omega) = \overline{\text{span}}\{\mathcal{T}\varphi(\omega) : \varphi \in \Phi\} \subseteq \overline{J_{V_{a_1}}(\omega) + \cdots + J_{V_{a_m}}(\omega)},$$

where $\Phi \subset L^2(\mathbb{R}^d)$ is such that $V = S(\Phi)$. Since $J(\omega)$ has finite dimension for a.e. $\omega \in [0, 1)^d$, the sum is closed. By (iii) of Lemma 2.7, the sum is direct. $\square$

Remark 6.5.

(i) Note that in the above theorem we have proved that if J is the range function associated to V and if $(V, L, a_1, \ldots, a_m)$ is an s-diagonalization of L, then $J(\omega) = J_{V_{a_1}}(\omega) \oplus \cdots \oplus J_{V_{a_m}}(\omega)$, where $J_{V_{a_j}}(\omega)$ is the eigenspace associated to the eigenvalue $\lambda_{a_j}(\omega) = \hat{a}_j(\omega)$ of $R(\omega)$ for a.e. $\omega \in \sigma(V_{a_j})$ for $j = 1, \ldots, m$. From this, we can deduce that for a.e. $\omega \in \sigma(V)$, the number of eigenvalues of $R(\omega)$ is at most m.

(ii) Suppose $\mathcal{L}(V) = \ell$. In Remark 6.2 we showed that the number of s-eigenspaces for an s-diagonalization of L can be arbitrarily large. By Proposition 4.7 and by the fact that $\dim J(\omega) \leq \ell$ for a.e. $\omega \in [0,1)^d$ we obtain that, for a.e. $\omega \in \sigma(V)$, at most ℓ of the eigenspaces $J_{V_{a_j}}(\omega)$ will not be the zero subspace.

Let V be a finitely generated shift-invariant space, $L : V \to V$ a bounded shift-preserving operator and R its associated range operator. We will denote for $\omega \in \sigma(V)$,

$$(12) \qquad k(\omega) := \#\{\lambda \in \mathbb{C} \, : \, \lambda \text{ is an eigenvalue of } R(\omega)\}.$$

Observe that, as seen in (i) of Remark 6.5, given any s-diagonalization $(V, L, a_1, \ldots, a_m)$ of L, we have that $k(\omega) \leq m$ for a.e. $\omega \in \sigma(V)$. Moreover, we saw that if κ_{max} is as in (9), then $\kappa_{max} = \operatorname*{ess\,sup}_{\omega \in \sigma(V)} k(\omega)$ and therefore, $\kappa_{max} \leq m$.

The next lemma shows an interesting relation between the spectrum of the s-eigenspaces of an s-diagonalization and the function $k(\omega)$.

Lemma 6.6. *Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator which is s-diagonalizable. Let $(V, L, a_1, \ldots, a_m)$ be an s-diagonalization of L. Define for $\omega \in \sigma(V)$,*

$$(13) \qquad h(\omega) := \sum_{j=1}^{m} \chi_{\sigma(V_{a_j})}(\omega),$$

and let $k(\omega)$ be defined as in (12) for $\omega \in \sigma(V)$. Then $h(\omega) = k(\omega)$ for a.e. $\omega \in \sigma(V)$.

Proof. The function $h(\omega)$ is measurable and we have that $h(\omega) \in \{1, \ldots, m\}$ for a.e. $\omega \in \sigma(V)$. Let $s \in \{1, \ldots, m\}$ and define $\mathcal{P}_s := \{P \subseteq \{1, \ldots, m\} \, : \, \#P = s\}$. We have that $h^{-1}(s) = \bigcup_{P \in \mathcal{P}_s} B_P$ where $B_P := \{\omega \in \sigma(V) \, : \, \omega \in \sigma(V_{a_j}) \text{ if and only if } j \in P\}$.

Given a set $P = \{j_1, \ldots, j_s\} \in \mathcal{P}_s$ we see, by Proposition 4.5 and Theorem 6.4, that for a.e. $\omega \in B_P$, $\hat{a}_{j_1}(\omega), \ldots, \hat{a}_{j_s}(\omega)$ are eigenvalues of $R(\omega)$ and

$$J(\omega) = J_{V_{a_{j_1}}}(\omega) \oplus \cdots \oplus J_{V_{a_{j_s}}}(\omega) \oplus \bigoplus_{1 \leq j \leq m, \, j \notin P} \{0\}.$$

This shows that $k(\omega) = s$ for a.e. $\omega \in B_P$.

Since this holds for every $P \in \mathcal{P}_s$ and for every $s \in \{1, \ldots, m\}$, we can deduce that $h(\omega) = k(\omega)$ for a.e. $\omega \in \sigma(V)$. $\qquad \square$

Given an s-diagonalization $(V, L, a_1, \ldots, a_m)$ of L, the collection of the spectrums $\{\sigma(V_{a_j})\}_{j=1}^{m}$ is always a covering of $\sigma(V)$. We will prove in Theorem 6.13 that every s-diagonalizable shift-preserving operator L acting on V has always an s-diagonalization $(V, L, a_1, \ldots, a_m)$ whose spectrums satisfy $\sigma(V_{a_{j+1}}) \subseteq \sigma(V_{a_j})$ for every $j = 1, \ldots, m$. In the following proposition we show that when this inclusion holds, the number of s-eigenspaces in the asociated decomposition and its spectrums are uniquely determined. Later we will see (Theorem 6.15) that actually a decomposition that satisfies the above inclusions has a minimal posible number of s-eigenspaces.

Proposition 6.7. *Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator which is s-diagonalizable. Let $(V, L, a_1, \ldots, a_m)$ be an s-diagonalization of L. If $\sigma(V_{a_{j+1}}) \subseteq \sigma(V_{a_j})$ for every $j = 1, \ldots, m$, then the spectrums are uniquely determined by*

$$\sigma(V_{a_j}) = \{\omega \in \sigma(V) \,:\, k(\omega) \geq j\}$$

for $j = 1, \ldots, m$ and $m = \kappa_{max}$.

Proof. Since $\sigma(V_{a_{j+1}}) \subseteq \sigma(V_{a_j})$ for every $j = 1, \ldots, m$, it is easy to see that $\sigma(V_{a_j}) = \{\omega \in \sigma(V) \,:\, h(\omega) \geq j\}$ where h is the function defined in (13). Then, by Lemma 6.6, we immediately obtain that $\sigma(V_{a_j}) = \{\omega \in \sigma(V) \,:\, k(\omega) \geq j\}$ for every $j = 1, \ldots, m$.

We know that $\kappa_{max} \leq m$. Suppose that $\kappa_{max} < m$. Then, we have that $|\sigma(V_{a_j})| = 0$ for every $\kappa_{max} < j \leq m$. This implies that $V_{a_j} = \{0\}$ for every $\kappa_{max} < j \leq m$, which is not possible. Thus, $m = \kappa_{max}$. $\qquad\square$

6.2. **Range operators with diagonalizable fibers.** The converse of Theorem 6.4 requieres more work. If we assume that for almost each ω the range operator $R(\omega)$ is diagonalizable, then we can obtain a decomposition of $J(\omega)$ into a finite number of measurable eigenspaces as we show in the theorem below. However, in order that this decomposition corresponds to an s-diagonalization of the shift-preserving operator L in V, we need to add an extra uniformity hypothesis which we will explain in the next subsection.

Theorem 6.8. *Let V be a finitely generated shift-invariant space with associated range function J and $L : V \to V$ a shift-preserving operator with corresponding range operator R. If $R(\omega)$ is diagonalizable for a.e. $\omega \in \sigma(V)$, then there exist sequences of bounded spectrum $a_1, \ldots, a_{\kappa_{max}}$ such that the measurable range functions $J_{a_j}(\omega) := \ker(R(\omega) - \hat{a}_j(\omega)\mathcal{I})$ for $j = 1, \ldots, \kappa_{max}$ satisfy the following statements:*

(i) *$J(\omega) = J_{a_1}(\omega) \oplus \cdots \oplus J_{a_{\kappa_{max}}}(\omega)$ for a.e. $\omega \in [0,1)^d$.*

(ii) *The sets $C_j = \left\{\omega \in \sigma(V) \,:\, J_{a_j}(\omega) \neq \{0\}\right\}$ have positive measure for every $j = 1, \ldots, \kappa_{max}$ and $C_{j+1} \subseteq C_j$ for every $j = 1, \ldots, \kappa_{max} - 1$.*

In particular, $\Lambda_{a_1}, \ldots, \Lambda_{a_{\kappa_{max}}}$ are s-eigenvalues of L.

Proof. Assume that $\mathcal{L}(V) = \ell$. Then, by Theorem 5.9 and Remark 5.10 there exist measurable functions $\lambda_1, \ldots, \lambda_{\kappa_{max}} \in L^\infty([0,1)^d)$ based on the partition $\{A_{n,i}\}_{1 \leq i \leq n \leq \ell}$ of $\sigma(V)$, where $A_{n,i}$ is defined in (8).

Then, we have the decomposition

$$(14) \qquad J(\omega) = \bigoplus_{j=1}^{\kappa_{max}} \ker(R(\omega) - \lambda_j(\omega)\mathcal{I}),$$

for a.e. $\omega \in [0,1)^d$. Indeed, for a.e. $\omega \notin \sigma(V)$ we have that $J(\omega) = \{0\}$ and $\ker(R(\omega) - \lambda_j(\omega)\mathcal{I}) = \{0\}$ for $j = 1, \ldots, \kappa_{max}$. Whereas, for a.e. $\omega \in A_{n,i}$ for some n, i, we have that

$$\bigoplus_{j=1}^{\kappa_{max}} \ker(R(\omega) - \lambda_j(\omega)\mathcal{I}) = \bigoplus_{j=1}^{i} \ker(R(\omega) - \lambda_j^{n,i}(\omega)\mathcal{I}) \oplus \bigoplus_{j=i}^{\kappa_{max}} \{0\},$$

which is equal to $J(\omega)$ because $\{\lambda_j^{n,i}(\omega)\}_{j=1}^{i}$ are the eigenvalues of $R(\omega)$ on $A_{n,i}$ and $R(\omega)$ is diagonalizable for a.e. $\omega \in [0,1)^d$.

Finally, since $\lambda_j \in L^\infty([0,1)^d)$ for all $j = 1, \ldots, \kappa_{max}$, then, for each $j = 1, \ldots, \kappa_{max}$ there exists a sequence $a_j \in \ell^2(\mathbb{Z}^d)$ of bounded spectrum such that $\lambda_j(\omega) = \hat{a}_j(\omega)$ for a.e. $\omega \in [0,1)^d$ and a measurable range function $J_{a_j}(\omega) = \ker(R(\omega) - \hat{a}_j(\omega)\mathcal{I})$. Since we have that (14) is true for a.e. $\omega \in [0,1)^d$, these range functions decompose $J(\omega)$ as we state in (i). Moreover, by item (ii) in Remark 5.10, we see that the sets C_j have positive measure and $C_{j+1} \subseteq C_j$ for every $j = 1, \ldots, \kappa_{max} - 1$. □

In the conditions of Theorem 6.8, by Proposition 4.5, each J_{a_j} is the range function associated to the shift-invariant space V_{a_j}, i.e. $J_{a_j} = J_{V_{a_j}}$, and we get the decomposition $J(\omega) = J_{V_{a_1}}(\omega) \oplus \cdots \oplus J_{V_{a_{\kappa_{max}}}}(\omega)$ for a.e. $\omega \in [0,1)^d$. At this point, one would like to conclude that $V = V_{a_1} \oplus \cdots \oplus V_{a_{\kappa_{max}}}$ and hence deduce that L is s-diagonalizable. For this to be true, a few considerations regarding the sum of infinite dimensional spaces must be taken. Recall that the sum of infinite dimensional closed subspaces is not necessarily closed. In particular, the sum of shift-invariant spaces is not closed in general, although it is invariant under integer translations (see [2] for an example of two shift-invariant spaces whose sum is not closed). A condition which allows us to tackle this problem is described in the next subsection.

6.3. **Angle between subspaces.** We now turn to the definition of the angle between subspaces which provides a condition to determine when the sum of subspaces is closed (see [11] and the references therein).

Definition 6.9. Let $\mathcal{H}$ be a Hilbert space. The *angle* of the r-tuple of closed subspaces $(M_1, \ldots, M_r)$ in $\mathcal{H}$ is the angle in $[0, \pi/2]$ whose cosine is given by

$$c_b(M_1, M_2, \ldots, M_r) := \left\| P_{M_r} \ldots P_{M_2} P_{M_1} P_{\left(\bigcap\limits_{j=1}^{r} M_j\right)^\perp} \right\|.$$

Notice that when $r = 2$ this definition is equivalent to the definition of the Friedrichs angle between two spaces [13]. The following lemma gives a characterization of when $c_b(M_1, \ldots, M_r) < 1$ and was proved in [5].

Lemma 6.10. *Let $M_1, \ldots, M_r$ be closed subspaces of a Hilbert space $\mathcal{H}$. The subsequent statements are equivalent:*

   (i) $c_b(M_1, \ldots, M_r) < 1$,
   (ii) $M_1^\perp + \cdots + M_r^\perp$ *is closed.*

In the next two results the shift-invariant spaces considered are not necessarily finitely generated. The following lemma gives a relation between the angle of a tuple of shift-invariant spaces and the angle of a tuple of their respective range functions.

Lemma 6.11. *Let $V_1, \ldots, V_r$ be shift-invariant spaces of $L^2(\mathbb{R}^d)$ and $J_{V_1}, \ldots, J_{V_r}$ their respective range functions. Then*

$$(15) \qquad c_b(V_1, \ldots, V_r) = \operatorname*{ess\ sup}_{\omega \in [0,1)^d} c_b\left(J_{V_1}(\omega), \ldots, J_{V_r}(\omega)\right).$$

This lemma was proved in [2] for the case of $r = 2$. The proof is easy to extend to our context, as we show here.

Proof. Let us call $U := \left(\bigcap_{j=1}^{r} V_j \right)^{\perp}$. By virtue of (iii) in Lemma 2.7, U is a shift-invariant space and

$$J_U(\omega) = \left(\bigcap_{j=1}^{r} J_{V_j}(\omega) \right)^{\perp},$$

for a.e. $\omega \in [0,1)^d$.

Since orthogonal projections onto shift-invariant spaces are shift-preserving operators (see Example 3.8), we have that $P_{V_r} \ldots P_{V_2} P_{V_1} P_U$ is shift preserving and its range operator at ω is

$$P_{J_{V_r}(\omega)} \ldots P_{J_{V_2}(\omega)} P_{J_{V_1}(\omega)} P_{J_U(\omega)}.$$

Hence, by (4)

$$\operatorname*{ess\,sup}_{\omega \in [0,1)^d} \left\| P_{J_{V_r}(\omega)} \ldots P_{J_{V_2}(\omega)} P_{J_{V_1}(\omega)} P_{J_U(\omega)} \right\| = \left\| P_{V_r} \ldots P_{V_2} P_{V_1} P_U \right\|,$$

and consequently (15) holds. $\square$

As a consequence, we have:

Proposition 6.12. *Let $V_1, \ldots, V_r$ be shift-invariant spaces of $L^2(\mathbb{R}^d)$ and $J_{V_1}, \ldots, J_{V_r}$ their respective range functions. The following statements are equivalent:*

(i) *The space $U := V_1 + \cdots + V_r$ is closed. In particular, it is a shift-invariant space of $L^2(\mathbb{R}^d)$.*

(ii) $c_b(V_1^{\perp}, \ldots, V_r^{\perp}) < 1.$

(iii) $\operatorname*{ess\,sup}_{\omega \in [0,1)^d} c_b \left(J_{V_1^{\perp}}(\omega), \ldots, J_{V_r^{\perp}}(\omega) \right) < 1.$

(iv) $\operatorname*{ess\,sup}_{\omega \in [0,1)^d} c_b \left(J_{V_1}(\omega)^{\perp}, \ldots, J_{V_r}(\omega)^{\perp} \right) < 1.$

Moreover, if (i)-(iv) *hold, then we have that $J_U(\omega) = J_{V_1}(\omega) + \cdots + J_{V_r}(\omega), a.e. \omega.$*

Proof. From Lemma 6.10 we know that (i) and (ii) are equivalent. By Lemma 6.11 we see that (ii) is equivalent to (iii). Finally, the equivalence between (iii) and (iv) is immediate from item (i) in Lemma 2.7.

It remains to see that $J_U(\omega) = J_{V_1}(\omega) + \cdots + J_{V_r}(\omega)$. Analogously as in Theorem 6.4, we see that $J_{V_1}(\omega) + \cdots + J_{V_r}(\omega) \subseteq J_U(\omega)$ and $J_U(\omega) \subseteq \overline{J_{V_1}(\omega) + \cdots + J_{V_r}(\omega)}$. These spaces are not necessarily finite dimensional but, given that (iv) holds, the latter sum is closed for almost every ω. $\square$

6.4. **Sufficient conditions for s-diagonalization and minimality.** Using Proposition 6.12, we are able to give sufficient conditions for a shift-preserving operator to be s-diagonalizable.

As we already discussed, in order to deduce that L is s-diagonalizable from the hypothesis that $R(\omega)$ is diagonalizable for a.e. ω, an uniformity condition on the angle must be required.

For this, let R be a measurable range operator on J such that $R(\omega)$ is diagonalizable almost everywhere and let $Z \subseteq [0,1)^d$ be the exceptional zero measure set.

Then, for $\omega \in [0,1)^d \setminus Z$ and if $k(\omega)$ is as in (12), we know that

$$J(\omega) = E_{\mu_1(\omega)} \oplus \cdots \oplus E_{\mu_{k(\omega)}(\omega)},$$

where $\mu_1(\omega), \ldots, \mu_{k(\omega)}(\omega)$ are the different eigenvalues of $R(\omega)$ and $E_{\mu_1(\omega)} \ldots E_{\mu_{k(\omega)}(\omega)}$ are their associated eigenspaces. Note that at this point we do not care about measurability. Then, we can define

(16)
$$C_b(\omega) := c_b \left(E_{\mu_1(\omega)}^{\perp}, \ldots, E_{\mu_{k(\omega)}(\omega)}^{\perp} \right).$$

Suppose now that $a_1, \ldots, a_m \in \ell^2(\mathbb{Z}^d)$ are any sequences of bounded spectrum such that $\{J_{a_j}(\omega) := \ker(R(\omega) - \hat{a}_j(\omega)\mathcal{I})\}_{j=1}^m$ decomposes $J(\omega)$ as

$$J(\omega) = J_{a_1}(\omega) \oplus \cdots \oplus J_{a_m}(\omega)$$

for a.e. $\omega \in [0,1)^d$. The existence of such sequences is guaranteed by Theorem 6.8. Thus, for $\omega \in [0,1)^d \setminus Z$ we have that

$$\{E_{\mu_1(\omega)} \ldots E_{\mu_{k(\omega)}(\omega)}\} = \{J_{a_1}(\omega), \ldots, J_{a_m}(\omega)\} \setminus \{0\}.$$

Therefore, $C_b(\omega) = c_b \left(J_{a_1}(\omega)^{\perp}, \ldots, J_{a_m}(\omega)^{\perp} \right)$ almost everywhere and as a consequence C_b is measurable.

Theorem 6.13. *Let V be a finitely generated shift-invariant space with associated range function J and $L : V \to V$ a shift-preserving operator with corresponding range operator R. The following propositions are equivalent:*

(i) *L is s-diagonalizable.*
(ii) *$R(\omega)$ is diagonalizable for a.e. $\omega \in \sigma(V)$ and $\operatorname{ess\,sup}_{\omega \in [0,1)^d} C_b(\omega) < 1$.*

Furthermore, when (i) or (ii) holds, there exist $a_1, \ldots, a_{\kappa_{max}}$ sequences of bounded spectrum, such that $(V, L, a_1, \ldots, a_{\kappa_{max}})$ is an s-diagonalization of L and $\sigma(V_{a_{j+1}}) \subseteq \sigma(V_{a_j})$ for $j = 1, \ldots, \kappa_{max} - 1$.

Proof. Assume first that (ii) is satisfied and consider the sequences of bounded spectrum $a_1, \ldots, a_{\kappa_{max}}$ given by Theorem 6.8 and their respective measurable range functions $J_{a_j}(\omega) := \ker(R(\omega) - \hat{a}_j(\omega)\mathcal{I})$ for $j = 1, \ldots, \kappa_{max}$.

By Proposition 4.5, we have that $V_{a_j} = \ker(L - \Lambda_{a_j})$ is the shift-invariant space associated to the range function J_{a_j}, i.e. $J_{a_j} = J_{V_{a_j}}$. From (ii) in Theorem 6.8, we know that $\sigma(V_{a_j})$ has positive measure, Λ_{a_j} is an s-eigenvalue of L for every $j = 1, \ldots, \kappa_{max}$ and $\sigma(V_{a_{j+1}}) \subseteq \sigma(V_{a_j})$ for $j = 1, \ldots, \kappa_{max} - 1$.

Since $C_b(\omega) = c_b \left(J_{V_{a_1}}(\omega)^{\perp}, \ldots, J_{V_{a_{\kappa_{max}}}}(\omega)^{\perp} \right)$, by Proposition 6.12, we have that $V_{a_1} \oplus \cdots \oplus V_{a_{\kappa_{max}}}$ is a closed shift-invariant space which is contained in V and their range functions coincide. By (ii) in Lemma 2.7, this implies that $V = V_{a_1} \oplus \cdots \oplus V_{a_{\kappa_{max}}}$. Hence, L is s-diagonalizable and $(V, L, a_1, \ldots, a_{\kappa_{max}})$ is an s-diagonalization of L.

For the converse, Theorem 6.4 takes care of the diagonalization of $R(\omega)$ for a.e. $\omega \in \sigma(V)$. On the other hand, assume that $(V, L, a_1, \ldots, a_m)$ is an s-diagonalization of L. Since V is closed, by Proposition 6.12 we have that $c_b(V_{a_1}^{\perp}, \ldots, V_{a_m}^{\perp}) < 1$ and so, equivalently, $\operatorname{ess\,sup}_{\omega \in [0,1)^d} C_b(\omega) < 1$. $\qquad \square$

We already observed (see Remark 6.5) that κ_{max} never exceeds the number of components (or s-eigenvalues) in an s-diagonalization. Since, by Theorem 6.13 if L is s-diagonalizable then there always exists a diagonalization with κ_{max} components, then κ_{max} is the minimun possible number of components in an s-diagonalization. More precisely:

Definition 6.14. Given V a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator which is s-diagonalizable, we define $\beta(V, L)$ as the smallest natural number m for which there exist $a_1, \ldots, a_m$ sequences of bounded spectrum such that $(V, L, a_1, \ldots, a_m)$ is an s-diagonalization of L. We will say that an s-diagonalization is *minimal* if it has exactly $\beta(V, L)$ components.

Proposition 6.15. *Let V be a finitely generated shift-invariant space, $L : V \to V$ a bounded shift-preserving operator which is s-diagonalizable. Then*

$$\beta(V, L) = \kappa_{max}.$$

6.5. **Normal shift-preserving operators.** When L is normal, $R(\omega)$ is normal for a.e. $\omega \in [0, 1)^d$ and so its eigenspaces are orthonormal. This fact allows us to avoid the angle condition. Below is given a generalization of the finite dimensional Spectral Theorem, for bounded normal shift-preserving operators.

Theorem 6.16. *Let V be a finitely generated shift-invariant space and $L : V \to V$ a bounded shift-preserving operator. If L is normal then it is s-diagonalizable and, if $(V, L, a_1, \ldots, a_m)$ is an s-diagonalization of L, we have that*

$$(17) \qquad\qquad L = \sum_{j=1}^{m} \Lambda_{a_j} P_{V_{a_j}},$$

where $P_{V_{a_j}}$ denotes the orthogonal projection of $L^2(\mathbb{R}^d)$ onto V_{a_j} for $j = 1, \ldots, m$.

Proof. If L is normal, by Theorem 3.3 we know that $R(\omega)$ is a normal operator for a.e. $\omega \in [0, 1)^d$. Thus, $R(\omega)$ is diagonalizable a.e. $\omega \in [0, 1)^d$ and its eigenspaces are orthogonal (for a.e. $\omega \in [0, 1)^d$). Hence, if C_b is as in (16), we have that $C_b(\omega) = 0$ for a.e. $\omega \in [0, 1)^d$ and the angle condition always hold. Then, by Theorem 6.13, L is s-diagonalizable.

Moreover, let $(V, L, a_1, \ldots, a_m)$ be an s-diagonalization of L. Since the direct sum $J(\omega) = J_{V_{a_1}}(\omega) \oplus \cdots \oplus J_{V_{a_m}}(\omega)$ is orthogonal for a.e. $\omega \in [0, 1)^d$ and $\mathcal{T}$ is an isometry, this implies that the decomposition $V = V_{a_1} \oplus \cdots \oplus V_{a_m}$ must also be orthogonal and the decomposition of L in (17) holds forthwith. $\qquad\square$

Example 6.17. Let $V = S(\Phi)$ be a finitely generated shift-invariant space such that $E(\Phi)$ is a Bessel sequence in $L^2(\mathbb{R}^d)$. As we said in Example 3.7, the frame operator associated to $E(\Phi)$ is shift preserving. Since it is self-adjoint, by Theorem 6.16, we have that it is s-diagonalizable.

Example 6.18. In the setting of Example 3.10, assume that $E(\Phi)$ is an orthonormal basis and let L be the shift-preserving operator defined in (6). Then, since the matrix form of $R(\omega)$ written on the basis $\{\mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega)\}$ is the diagonal matrix $[R](\omega) = \operatorname{diag}(\hat{a}_1(\omega), \ldots, \hat{a}_n(\omega))$ for a.e. $\omega \in [0, 1)^d$, we have that $R(\omega)$ is normal a.e. $\omega \in [0, 1)^d$. Observe that this is not the case when we have a Riesz basis instead of an orthonormal basis. As a consequence, L is normal as well and

by Theorem 6.16, it is s-diagonalizable. Moreover, $\Lambda_{a_1} \ldots \Lambda_{a_n}$ are the s-eigenvalues of L and $V_{a_j} = S(\varphi_j)$ for every $j = 1, \ldots, n$.

When $E(\Phi)$ is a Riesz basis, we still have that, for a.e. $\omega \in [0,1)^d$, $\hat{a}_1(\omega), \ldots, \hat{a}_n(\omega)$ are the eigenvalues of $R(\omega)$ with associated eigenvectors $\mathcal{T}\varphi_1(\omega), \ldots, \mathcal{T}\varphi_n(\omega)$ and that

$$ J(\omega) = \mathrm{span}\{\mathcal{T}\varphi_1(\omega)\} \oplus \cdots \oplus \mathrm{span}\{\mathcal{T}\varphi_n(\omega)\}. $$

However, in order to conclude that L is s-diagonalizable we have to impose that $\mathrm{ess\,sup}_{\omega \in [0,1)^d} c_b\left(\mathrm{span}\{\mathcal{T}\varphi_1(\omega)\}^{\perp}, \ldots, \mathrm{span}\{\mathcal{T}\varphi_n(\omega)\}^{\perp}\right) < 1.$

References

[1] Aldroubi, A., Cabrelli, C., Hardin, D. and Molter, U., Optimal Shift Invariant Spaces and Their Parseval Generators, Appl. Comput. Harmon. Anal., 23(2), 273-283, (2007).

[2] Anastasio, M. and Cabrelli, C., Sampling in a union of frame generated subspaces. Sampling Theory in Signal and Image Processing, 8(3), 261-286 (2009).

[3] Azoff, E.A., Borel measurability in linear algebra. Proceedings of the American Mathematical Society, 42(2), pp.346-350 (1974).

[4] Barbieri, D., Hernández, E. and Paternostro, V., The Zak transform and the structure of spaces invariant by the action of an LCA group. J. Funct. Anal., 269(5), 1327-1358 (2015).

[5] Bauschke, H., Borwein, J. and Lewis, A., On the method of cyclic projections for convex sets in Hilbert space. In Research report. Simon Fraser University (1994).

[6] Bownik, M., The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$. Journal of Functional Analysis, 177(2), 282-309 (2000).

[7] Bownik, M. and Ross, K., The structure of translation-invariant spaces on locally compact abelian groups. J. Fourier Anal. Appl., 21(4), 849-884 (2015).

[8] Cabrelli, C. and Paternostro, V., Shift-invariant spaces on LCA groups. J. Funct. Anal., 258(6), 2034-2059 (2010).

[9] de Boor, C., DeVore, R.A. and Ron, A., Approximation from shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Transactions of the American Mathematical Society 341(2), pp.787-806 (1994).

[10] de Boor, C., DeVore, R.A. and Ron, A., The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$. Journal of Functional Analysis, 119(1), 37-78 (1994).

[11] Deutsch, F., The angle between subspaces of a Hilbert space. Approximation theory, wavelets and applications. Springer, Dordrecht, 107-130 (1995).

[12] Dixmier, J., Von Neumann Algebras. Nort-Holland Publishing Company (1981).

[13] Friedrichs, K., On certain inequalities and characteristic value problems for analytic functions and for functions of two variables. Trans. Amer. Math. Soc., 41, pp. 321-364 (1937).

[14] Helson, H., Lectures on invariant subspaces. Academic Press, London (1964).

[15] Ron, A. and Shen Z., Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Canad. J. Math., 47, no. 5, 1051-1094 (1995).

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
  E-mail address: aaguilera@dm.uba.ar

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
  E-mail address: cabrelli@dm.uba.ar

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
  E-mail address: dcarbajal@dm.uba.ar

Departamento de Matemática, Universidad de Buenos Aires, Instituto de Matemática "Luis Santaló" (IMAS-CONICET-UBA), Buenos Aires, Argentina
  E-mail address: vpater@dm.uba.ar