SOME CONSIDERATIONS REGARDING THE UNIVERSAL W^*-ALGEBRA OF A TOPOLOGICAL GROUP

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Abstract. For a topological group G it is possible to obtain its universal von Neumann algebra $W^*(G)$ as a set of "fields" over the category of representations of G. This idea was first developped by John Ernest in [1]. Here we propose several improvements to the original construction. We point out that G doesn't have to be locally compact or second countable and analyze the functorial aspects: W^* turns out to be a functor from the category of topological groups to that of von Neumann algebras. It is left adjoint of U, the functor which assigns to each algebra its unitary group with the σ-weak topology (this adjunction was previously proved by Martin Wanvik for a different description of the functor W^*). We include a section on the analogous construction for C^*-algebras instead of groups, mainly as an auxiliary tool. For a locally compact group, G is a topological subspace of $W^*(G)$. We study the closure of G for it gives a compactification. This leads us to a paradox (4.4).

Conventions. "Representation" will mean:

- For a group G: a weakly continuous unitary representation on a Hilbert space, i.e. a weakly continuous group homomorphism $G \to U(H)$.
- For a von Neumann algebra M: a unit preserving $*$-algebra morphism $M \to B(H)$ continuous for the σ-weak topologies.
- For a C^*-algebra A: a nondegenerate $*$-algebra morphism $A \to B(H)$.

Consistently, the σ-weak (or weak-*) topology will be the standard topology for a von Neumann algebra. Thus, morphisms in the category of von Neumann algebras will be those continuous for this topology (and unit preserving). "Von Neumann algebra" and "W^*-algebra" are taken as synonyms, meaning a C^*-algebra with predual.

1. Universal W^*-algebra for general topological groups

The universal von Neumann algebra $W^*(G)$ of a locally compact second countable Hausdorff group is a very big algebra containing $L^1(G)$, $C^*(G)$, $M(G)$ [(1)] as "immerse" subalgebras and G as a subgroup of its unitary group. G and $W^*(G)$ have the same representations, and $W^*(G)$ is equal to $C^*(G)^{**}$, the double dual of the full group C^*-algebra. All this has been done by John Ernest in [1]. The same procedure can be applied to every topological group though some properties might be lost. For example, the canonical application $G \to W^*(G)$ won't be injective. The importance of

[1] $M(G)$ is the algebra of all complex valued finite regular measures on G.

removing such hypothesis for us is that it allows to take the W^*-algebra of $U(M)^{(2)}$, the unitary group of a von Neumann algebra M, so we can prove that W^* is a left adjoint functor for U.

Let us start with the definition of $W^*(G)$.

1.1. **Definition.** *For a topological group G, let $cyc(G)$ be the category of cyclic representations of G (just one for each equivalence class) with the usual morphisms: bounded linear interwining operators. We call "field" over $cyc(G)$ a function T assigning to each $\pi \in cyc(G)$, $G \xrightarrow{\pi} U(H_\pi)$, an element $T(\pi) \in B(H_\pi)$ in a bounded and coherent with morphisms way. Explicitely: $\underset{\pi}{sup}||T(\pi)|| < \infty$, and if $H_{\pi_1} \xrightarrow{S} H_{\pi_2}$ is a morphism of representations ($S\pi_1(a) = \pi_2(a)S$) then $ST(\pi_1) = T(\pi_2)S$. In other words, fields are just bounded endomorphisms of the forgetful functor $cyc(G) \to \mathcal{H}$, where $\mathcal{H}$ is the category of Hilbert spaces.*

It is easy to see that $W^*(G)$ is a C^*-algebra with product and involution defined pointwise and the norm is $||T|| = \underset{\pi}{sup}||T(\pi)||$.

1.2. **Proposition.** *$W^*(G)$ is a von Neumann algebra.*

Proof. Take the Hilbert

$$H = \bigoplus_{\pi \in cyc(G)} H_\pi \quad , \quad W^*(G) \xrightarrow{\Pi} B(H) \quad , \quad \Pi(T) = \bigoplus_{\pi \in cyc(G)} T(\pi)$$

Π is clearly a faithful representation. Let us see that the image is strongly closed. Assume $\Pi(T_\mu) \to S$ for the *sot*. If $\alpha \in H_\pi$, $\Pi(T_\mu)\alpha = T_\mu(\pi)\alpha \to S\alpha$, then $S\alpha \in H_\pi$. This means $S = \bigoplus S_\pi$. Putting $T(\pi) = S_\pi$, it follows easily that T is a field and $\Pi(T) = S$. □

1.3. **Proposition.** *If T is a field over $cyc(G)$, $T(\pi)$ belongs to the von Neumann algebra generated by $\pi(G)$ for every $\pi \in cyc(G)$.*

Proof. An operator $S \in \pi(G)'$ is an endomorphism of π, so it commutes with $T(\pi)$ because of compatibility with morphisms. Therefore, $T(\pi) \in \pi(G)''$. □

1.4. **Proposition.** *A field over $cyc(G)$ can be uniquely defined over any representation of G in a compatible way with morphisms. In other words, replacing $cyc(G)$ by a category of representations $rep(G) \supset cyc(G)$ containing all of the interwiners between its objects, the set of fields remains the same.*

[2] The set of unitaries $U(M)$ with the σ-weak topology is a Hausdorff topological group. The product $U(M) \times U(M) \to U(M)$ is continuous: consider a faithful representation of M. The strong topology coincides with the weak and σ-weak in $U(M)$. Composition is jointly continuous for the strong topology over bounded sets, while the "inverse" application is the involution $*$, continuous for the weak topology.

Proof. Clearly, a field over $rep(G)$ can be restricted to a field over $cyc(G)$. Now let T be a field over $cyc(G)$, and $(\pi, H) \in rep(G)$. π can be expressed as a direct sum of cyclic representations, so we define $T(\pi)$ as the direct product of the operators associated to these subrepresentations. This definition is correct because of the following. Assume we have two decompositions into cyclic subrepresentations: $H = \bigoplus A_i = \bigoplus B_j$. Consider P_i and Q_j the orthogonal projections to the subspaces A_i and B_j. We have the following morphisms of cyclic representations, $B_j \xrightarrow{P_i|_{B_j}} A_i$. Compatibility of T says $T(A_i)P_i|_{B_j} = P_i|_{B_j}T(B_j)$ (we abuse harmlessly identifying the subspace with the subrepresentation).

$$\sum_i T(A_i)P_i = (\sum_i T(A_i)P_i)(\sum_j Q_j) = \sum_{i,j} T(A_i)P_iQ_j =$$
$$= \sum_{i,j} P_iT(B_j)Q_j = \sum_j T(B_j)Q_j$$

The sums converge strongly. It is valid to interchange the order of summation because composition of operators is jointly continuous for the strong topology when restricted to bounded sets. This proves that T is well defined.

The extended field is clearly bounded. To see compatibility, take a morphism between π_1 and π_2, $H_1 \xrightarrow{S} H_2$, and any vector $\alpha \in H_1$. Now take decompositions of these representations as sum of cyclic subrepresentations, containing the cyclic representations generated by α and $S(\alpha)$ respectively. Because of the original compatibility in $cyc(G)$, we have $ST(\pi_1)(\alpha) = T(\pi_2)S(\alpha)$. □

1.5. **Observation.** Consider Π, the faithful representation from proposition 1.2, and $\Pi_G = \bigoplus_{\pi \in cyc(G)} \pi$ the representation of G acting on the same Hilbert as Π, $H = \bigoplus_{\pi \in cyc(G)} H_\pi$. For $T \in W^*(G)$ we have $T(\Pi_G) = \Pi(T)$. This is can be easily checked for $\alpha \in H_\pi$ because of compatibility with the inclusion morphism $H_\pi \hookrightarrow H$.

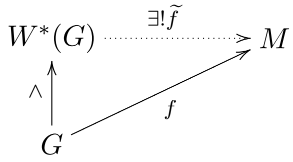
1.6. **Proposition.** *There exists a canonical continuous function* $G \xrightarrow{\wedge} W^*(G)$. *The elements* $\hat{g}$ *are unitaries and generate* $W^*(G)$ *as a von Neumann algebra.*

Proof. $\hat{g}(\pi) := \pi(g)$ defines a unitary field. Next we show continuity. All the weak topologies in a von Neumann algebra coincide on the unitary group. $g_\mu \to g$ implies $\Pi(\hat{g_\mu})\alpha = \pi(g_\mu)\alpha \to \pi(g)\alpha$ for $\alpha \in H_\pi$ (because π is continuous), and this is easily generalized for all $\alpha \in H$. Now we'll see $\Pi(W^*(G)) = \Pi(\hat{G})''$. Let $T \in W^*(G)$. If $S \in \Pi(\hat{G})'$, it is an endomorphism of Π_G. Compatibility says:

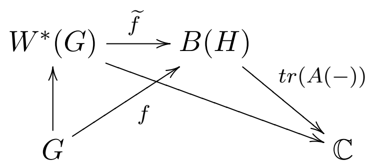
$$ST(\Pi_G) = T(\Pi_G)S$$
$$S\Pi(T) = \Pi(T)S$$

proving $\Pi(T) \in \Pi(\hat{G})''$ and therefore $\Pi(W^*(G)) \subset \Pi(\hat{G})''$. The other inclusion holds because the bicommutant is the smallest von Neumann algebra containing $\Pi(\hat{G})$. $\square$

1.7. **Proposition.** *Let M be a von Neumann algebra and $G \xrightarrow{f} U(M)$ a continuous morphism of groups. There exists a unique morphism of W^*-algebras $W^*(G) \xrightarrow{\widetilde{f}} M$ such that the triangle commutes.*



Proof. Uniqueness is clear, since G generates $W^*(G)$ as a W^*-algebra. Let us prove existence. We first consider the case $M = B(H)$ (H is any Hilbert space). f is a representation. According to proposition 1.4, we can define $\widetilde{f}(T) = T(f)$. Thus defined, $\widetilde{f}$ clearly preserves the operations of sum, product and involution. We must prove that it is continuous for the σ-weak topologies. In order to do so, take an element of the predual of $B(H)$. We write it as $tr(A(-))$, where A is trace class.

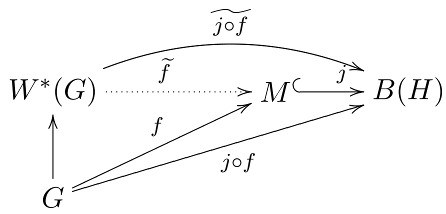


If we show that $tr(A\widetilde{f}(-))$ is in the predual of $W^*(G)$ (i.e: it is a normal functional) we are done. But this follows at once if we faithfully represent $W^*(G)$ on the Hilbert

$$(\bigoplus_{\pi \in cyc(G)} H_\pi) \oplus H$$

where an element $T \in W^*(G)$ acts on each H_π and H according to $T(\pi)$ and $T(f)$ respectively. Just like in 1.2, this representation is faithful and the image is strongly closed.

If we now have a general W^*-algebra M, we can take a faithful representation $M \overset{j}{\hookrightarrow} B(H)$ and extend $j \circ f$ to $\widetilde{j \circ f}$. The argument in 1.3 applied to $j \circ f$ proves that $\widetilde{j \circ f}(T)$ belongs to the von Neumann algebra generated by $f(G)$, so $\widetilde{j \circ f}(T) \in M$.

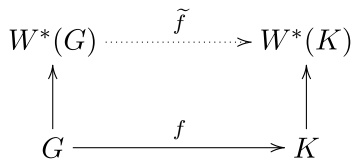


∎

1.8. **Observation.** From previous proposition it follows that the category of representations of G coincides with the category of representations of $W^*(G)$.

1.9. **Corollary.** *W^* is a functor from the category of groups to that of von Neumann algebras and it is a left adjoint for the functor U which assigns the unitary group to each algebra.*

Proof. Functoriality is a direct consequence of the previous proposition applied to:



Composition is preserved thanks to uniqueness.

The adjunction $W^* \dashv U$ also follows immediately. A morphism $G \to U(M)$ induces $W^*(G) \to M$, and a morphism $W^*(G) \to M$ can be restricted to $G \to U(M)$ composing with the canonical map $G \to W^*(G)$. Again uniqueness allows us to prove that these correspondences $[W^*(G), M] \rightleftarrows [G, U(M)]$ are mutually inverse, and natural in both variables. ∎

1.10. **Observation.** If π is a representation of G and $\alpha, \beta \in H_\pi$, the linear function

$$W^*(G) \to \mathbb{C}$$
$$T \longmapsto \langle T(\pi)\alpha, \beta\rangle$$

is continuous. This is because it is continuous for the *wot* through a faithful representation of $W^*(G)$ containing π, which is achieved just like in the proof of 1.7.

Comparison with [1]**.** Our definition of field presents some differences with the one from [1]. In the first place, Ernest works with representations on a fixed big enough Hilbert space. We prefer to work with a category because it allows a neater formulation. As proposition 1.4 shows, it is enough to consider cyclic representations only. For the notion of compatibility required for fields, Ernest assumes compatibility with direct sums and (implicitly) unitary equivalences. We next show that it is actually enough to require just compatibility with those morphisms of representations which are partial isometries. Then it is left to the reader the verification that Ernest's condition is stronger (thus equivalent).

1.11. **Proposition.** *Let T be an object that assigns to each $\pi \in cyc(G)$ a bounded operator on H_π in a compatible way with morphisms of representations that are partial isometries. Then T is compatible with every morphism.*

Proof. Given an arbitrary interwiner S, let $S = UP$ be its polar decompositionon. $P = (S^*S)^{1/2}$ is a morphism of representations and the partial isometry U is a morphism as well (U maps $(S^*S)^{1/2}y$ to Sy and the orthogonal complement to 0). Therefore, T is compatible with U by hypothesis and it only remains to prove that T is compatible with any positive morphism P. Taking $r > 0$ small enough, rP has its spectrum inside $[0, 2\pi)$. e^{irP} is a unitary equivalence, so it is compatible (commutes) with T. But rP is the logarithm of e^{irP}, so rP also commutes with T. □

2. Enveloping W^*-algebra of a C^*-algebra

Starting with a C^*-algebra A it is possible to imitate the same procedure and obtain good results. [5]

2.1. **Definition.** *Let $cyc(A)$ be the category of cyclic representations of A. As in the case for groups, a field T over $cyc(A)$ is a function which assigns to each $\pi \in cyc(A)$ a bounded operator $T(\pi) \in B(H_\pi)$ in a bounded and compatible with morphisms way.*

Fields over $cyc(A)$ form a von Neumann algebra that we call A^F. The proof of this fact is completely analogous to the one for $W^*(G)$, i.e., showing that the representation Π made with the sum of all cyclic representations of A gives a faithful representation whose image is strongly closed. A is a subalgebra of A^F through the canonical map $A \overset{\wedge}{\hookrightarrow} A^F$, $\widehat{a}(\pi) = \pi(a)$. The analogous to 1.4 is valid with same proof.

2.2. **Proposition.** *A^F is the enveloping von Neumann algebra of the C^*-algebra A.*

Proof. Let $\Pi_U = \bigoplus_{\varphi \in \mathcal{S}(A)} \pi_\varphi$ be the universal representation of A. $\mathcal{S}(A)$ is the set of states and $A \overset{\pi_\varphi}{\curvearrowright} H_\varphi$ are the GNS representations. For A^F, consider the natural faithful representation $A^F \overset{\widetilde{\Pi}}{\curvearrowright} \bigoplus_{\varphi \in \mathcal{S}(A)} H_\varphi$. Let $T \in A^F$. Compatibility with the inclusion interwiner $H_{\varphi_0} \hookrightarrow \bigoplus H_\varphi$ implies $\widetilde{\Pi}(T)\alpha = T(\Pi_U)\alpha$ for $\alpha \in H_{\varphi_0}$. Then $\widetilde{\Pi}(T) = T(\Pi_U)$. Now we can prove $\widetilde{\Pi}(A^F) = \Pi_U(A)''$. Let $T \in A^F$. If $S \in \Pi_U(A)'$, it is an endomorphism of Π_U. Compatibility says:

$$ST(\Pi_U) = T(\Pi_U)S$$

$$S\widetilde{\Pi}(T) = \widetilde{\Pi}(T)S$$

which means $\widetilde{\Pi}(T) \in \Pi_U(A)''$ and therefore $\widetilde{\Pi}(A^F) \subset \Pi_U(A)''$. The other inclusion holds because the bicommutant is the smallest von Neumann algebra containing $\Pi_U(A)$. □

The enveloping von Neumann algebra of A is equal to the bidual A^{**} ([4], theorem 1.17.2) with Arens multiplication (see [3] for the definition of this product).

Also mimicking the proof for the case of groups we obtain $(-)^F \dashv O$, where $\mathcal{C}^* \overset{O}{\leftarrow} \mathcal{W}^*$ is the forgetful functor.

3. Locally compact Hausdorff groups

If G is a locally compact Hausdorff group, it is known that its category of representations is isomorphic to the category of representations of its universal C^*-algebra $C^*(G)$. From this fact it can be easily deduced that the respective categories of cyclic representations are isomorphic as well. Therefore, if $A = C^*(G)$, A^F coincides with $W^*(G)$.

As cyclic representations of G separate points, the canonical map $G \to W^*(G)$ is injective. Besides this inclusion is topological, as we now proceed to show.(3)

3.1. **Lemma.** *For a locally compact Hausdorff group G, the topology of G is the initial topology with respect to the family of positive type functions.*

Proof. Let τ_p be the topology generated by the positive type functions. Every element in τ_p is an open set of G. So it is enough to prove that for every $x \in G$, U open set of G containing x, there exists an open set $W \in \tau_p$ such that $x \in W \subset U$. First we assume $x = 1$. Let V be an open set of G with compact closure such that $V^2 \subset U$ y $V^{-1} = V$. The function $\chi_V * \chi_V$ is continuous, positive type, it annihilates outside U and takes the value $|V| > 0$ on 1. With this function it is easy to find a W as required. If we now take any $x \in G$, we can translate it to 1. A translation of a function of positive type is a linear combination of positive type functions, as the following calculation shows:

$$\begin{aligned} \langle \pi(g^{-1}x)\xi, \xi\rangle &= \langle \pi(x)\xi, \pi(g)\xi\rangle = \langle \pi(x)\alpha, \beta\rangle \\ &= {}^1\!/_4\Big(\langle \pi(x)(\alpha+\beta), \alpha+\beta\rangle - \langle \pi(x)(\alpha-\beta), \alpha-\beta\rangle + \\ &\quad + i\langle \pi(x)(\alpha+i\beta), \alpha+i\beta\rangle - i\langle \pi(x)(\alpha-i\beta), \alpha-i\beta\rangle\Big) \end{aligned}$$

where $\alpha = \xi$ y $\beta = \pi(g)\xi$. □

3.2. **Proposition.** *Let G be a locally compact Hausdorff group. G is a topological subspace of $W^*(G)$ through the canonical inclusion $G \overset{\wedge}{\hookrightarrow} W^*(G)$.*

(3)This is done in [1] but we include it here for the sake of completeness.

Proof. The σ-weak topology of $W^*(G)$ is, by definition, initial with respect to $C^*(G)^*$. Since $C^*(G)^*$ is linearly generated by the positive functionals ([4] proposition 1.17.1), these suffice to generate the topology. The topological inclusion $\widehat{G} \hookrightarrow W^*(G)$ is of course initial, so if we compose it with those positive functionals we have an initial family that is equal to the class of all positive type functions. Let's check this. For a $\varphi \in C^*(G)^*$, $0 \neq \varphi \geq 0$, there is a representation $\widetilde{\pi}$ of $W^*(G)$ such that $\varphi = \langle \widetilde{\pi}(-)\xi, \xi \rangle$. The restriction to G is the positive type function f associated to the representation $\pi = \widetilde{\pi} \circ \wedge$. Conversely, for a positive type function $f \neq 0$ over G, there is an associated representation whose extention to $W^*(G)$ gives a positive $\varphi \in C^*(G)^*$ extending f.

Now the result follows from previous lemma. □

4. Closure of G inside $W^*(G)$

In $W^*(G)$ balls are compact because of Banach-Alaoglu's theorem. They are closed too because $W^*(G)$ is Hausdorff. Since G is contained in the unit ball, its closure is a compactification of G that we want to understand. Compactifications can be classified according to the algebra of bounded continuous functions $G \to \mathbb{C}$ extendible to $\overline{G}$. The inclusion $G \hookrightarrow W^*(G)$ extends every representation of G, so at least $\overline{G}$ extends every function of positive type.

The category $cyc(G)$ isn't closed by tensor products. For this reason, we will now consider the fields $T \in W^*(G)$ over the category $rep(G)$ of representations whose dimensions are bounded by an infinite cardinal big enough to contain all the cyclic representations. Thus, $rep(G)$ is closed by tensor products.

4.1. **Definition.** *Let*

$$G_{\otimes} = \{T \in W^*(G) \setminus \{0\} / T(\pi_1 \otimes \pi_2) = T(\pi_1) \otimes T(\pi_2) \, \forall \pi_1, \pi_2 \in rep(G)\}$$

Clearly, $G_{\otimes}$ contains G.

4.2. **Proposition.** *Elements in $G_{\otimes}$ are unitary.*

Proof. Let $T \in G_{\otimes}$. If 1 is the trivial representation, we might think $T(1) \in \mathbb{C}$. Since $T(1) = T(1 \otimes 1) = T(1) \otimes T(1) = T(1)^2$, $T(1)$ equals 0 or 1. If $T(1) = 0$ then $T(\pi) = T(\pi \otimes 1) = T(\pi) \otimes T(1) = 0$, so $T = 0$, absurd. We then have $T(1) = 1$.

Consider the interwiner $\pi \otimes \overline{\pi} \xrightarrow{\epsilon} 1$ [4] given by $x \otimes \overline{y} \mapsto < x, y >$. Since $T(\pi \otimes \overline{\pi}) = T(\pi) \otimes T(\overline{\pi})$, compatibility with morphisms gives:

$$< T(\pi)(x), \overline{T(\overline{\pi})}(y) > = < x, y >$$

From here we can easily conclude $\overline{T(\overline{\pi})}^* T(\pi) = Id_H$. But this last equality holds for any element of $G_{\otimes}$ and any representation. Therefore we can

[4] $\overline{\pi}$ is the conjugate representation of π. It is defined by $\overline{\pi}(g)(\overline{x}) = \overline{\pi(g)(x)}$ for $\overline{x} \in \overline{H}$, the conjugate Hilbert space of H.

choose T^* and $\overline{\pi}$, leading to: $\overline{T(\pi)}T(\overline{\pi})^* = Id_{\overline{H}}$. Conjugating we have: $T(\pi)\overline{T(\overline{\pi})}^* = Id_H$. So $T(\pi)$ is invertible. This allows to define the field $T^{-1}(\pi) = T(\pi)^{-1} = \overline{T(\overline{\pi})}^*$ which is compatible with morphisms because T is, and it is bounded because $||T(\pi)^{-1}|| = ||\overline{T(\overline{\pi})}^*|| = ||T(\overline{\pi})||$, from where also follows $||T|| = ||T^{-1}||$. Under this conditions, an argument of Ernest in [1] shows that T is unitary:

$$1 = ||TT^{-1}|| \leq ||T||.||T^{-1}|| = ||T||^2$$

implies $||T|| \geq 1$. On the other hand, if $||T|| > 1$ then

$$||T(\pi^{\otimes n})|| = ||T(\pi)^{\otimes n}|| \geq ||T(\pi)||^n$$

wouldn't be bounded for some π such that $||T(\pi)|| > 1$. Thus, $||T|| = ||T^{-1}|| = 1$. For operators on a Hilbert space this implies T unitary. Just as a commentary, observe that $T(\overline{\pi}) = \overline{T(\pi)}$. □

Notice that $G_\otimes$ is contained in the unit ball.

4.3. **Proposition.** *$G_\otimes$ is closed and therefore compact. Besides it is closed for the product, so it is a subgroup of $U(W^*(G))$.*

Proof. Let $T_\mu \in G_\otimes$ be a convergent net, $T_\mu \to T$. Let π_1 and π_2 be two representations of G.

$$\langle T(\pi_1 \otimes \pi_2)x \otimes y, w \otimes z\rangle = \lim_\mu \langle T_\mu(\pi_1 \otimes \pi_2)x \otimes y, w \otimes z\rangle =$$
$$= \lim_\mu \langle T_\mu(\pi_1) \otimes T_\mu(\pi_2)x \otimes y, w \otimes z\rangle = \lim_\mu \langle T_\mu(\pi_1)x, w\rangle \langle T_\mu(\pi_2)y, z\rangle =$$
$$= \langle T(\pi_1)x, w\rangle \langle T(\pi_2)y, z\rangle = \langle T(\pi_1) \otimes T(\pi_2)x \otimes y, w \otimes z\rangle$$

This shows $T(\pi_1 \otimes \pi_2) = T(\pi_1) \otimes T(\pi_2)$. Besides $T \neq 0$, because $1 = \langle T_\mu(1)1, 1\rangle \to \langle T(1)1, 1\rangle$, so $T(1) = 1$. Then, $T \in G_\otimes$.

The other claim is clear. □

Tannaka's theorem [6] [2] affirms that it is possible to recover a compact group from its category of representations. More specifically, the original group is equal to the group of unitary tensor preserving fields. Tatsuuma's theorem ([7], proposition 2) generalizes this result for locally compact groups.

4.4. **Paradox.** Applying Tatsuuma's duality theorem we get $G = G_\otimes$, so G is compact. We can reach to a contradiction even without Tatsuuma's duality theorem. We have the following chain of subgroups of $U(W^*(G))$:

$$G \subset \overline{G} \subset G_\otimes \subset U(W^*(G))$$

Since every representation of G extends to a representation of $W^*(G)$, and this can be subsequently restricted to $\overline{G}$, we deduce that every representation of G extends uniquely to a representation of $\overline{G}$. Even more, the categories of representations of G and $\overline{G}$ are isomorphic. But representations of compact groups are different from representations of noncompact groups. For example, for certain noncompact groups there are infinite dimensional irreducible representations.

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