The Grothendieck Construction for Double Categories

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Double Categories, the concise and the expanded definition

- A **double category** is an internal category in **Cat**,

$$C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0 \,.$$

- It has
  - objects (objects of C_0),
  - vertical arrows (arrows of C_0), denoted $A \xrightarrow{u} A'$,
  - horizontal arrows (objects of C_1), denoted $A \xrightarrow{f} B$,
  - double cells (arrows of C_1), denoted

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

Compositions and identities

- Since C_0 is a category, we have vertical compositions $u' \bullet u : A \overset{u}{\bullet\!\!\!\longrightarrow} A' \overset{u'}{\bullet\!\!\!\longrightarrow} A''$ and identities $A \overset{id_A}{\bullet\!\!\!\longrightarrow} A$.
- Since C_1 is a category too, we have vertical compositions (pastings)

$$\alpha' \bullet \alpha : \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"] \arrow[d, "u'"'] \arrow[dr, phantom, "\alpha'"] & B' \arrow[d, "v'"] \\ A'' \arrow[r, "f''"'] & B'' \end{tikzcd} \quad \text{and identities} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "id_A"'] \arrow[dr, phantom, "id_f"] & B \arrow[d, "id_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Compositions and identities

Since $C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \xleftarrow{1} C_0$ are functors:

- horizontal arrows form a category too, we have thus $g \circ f : A \xrightarrow{f} B \xrightarrow{g} C$ and identities $A \xrightarrow{1_A} A$.
- double cells can also be pasted horizontally

$$\beta \circ \alpha : \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "g"] \arrow[d, "v"] \arrow[dr, phantom, "\beta"] & C \arrow[d, "w"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd} \quad \text{with identities} \quad \begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "u"'] \arrow[dr, phantom, "1_u"] & A \arrow[d, "u"] \\ A' \arrow[r, "1_{A'}"'] & A' \end{tikzcd}$$

Middle four interchange

Since $\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \xrightarrow{\circ} \mathsf{C}_1$ is a functor, it commutes with composition. That means that given two arrows of $\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$

$$\begin{tikzcd}
(g, f) \arrow[d, "{(\beta,\alpha)}"'] \\
(g', f') \arrow[d, "{(\beta',\alpha')}"'] \\
(g'', f'')
\end{tikzcd}$$

i.e. a configuration

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "g"] \arrow[d, "v"] \arrow[dr, phantom, "\beta"] & C \arrow[d, "w"] \\
A' \arrow[r, "f'"] \arrow[d, "u'"'] \arrow[dr, phantom, "\alpha'"] & B' \arrow[r, "g'"] \arrow[d, "v'"] \arrow[dr, phantom, "\beta'"] & C' \arrow[d, "w'"] \\
A'' \arrow[r, "f''"'] & B'' \arrow[r, "g''"'] & C''
\end{tikzcd}$$

we have $(\beta' \circ \alpha') \bullet (\beta \circ \alpha) = (\beta' \bullet \beta) \circ (\alpha' \bullet \alpha)$

Examples

1. A 2-category can be defined as a double category in which every horizontal arrow is an identity. It has 2-cells $\alpha : v \Longrightarrow u$.
2. For any 2-category $\mathcal{C}$, $\mathbb{Q}(\mathcal{C})$ is the double category of quintets in $\mathcal{C}$,

with double cells
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$
for each $\alpha \colon vf \Rightarrow gu$ in $\mathcal{C}$.

3. $\mathbb{V}(\mathcal{C})$ is the double category with double cells

$$\begin{tikzcd} \cdot \arrow[r, "id"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "id"'] & \cdot \end{tikzcd}$$
for each $\alpha \colon v \Rightarrow u$ in $\mathcal{C}$.

4. More generally, if Σ is a 1-subcategory of $\mathcal{C}$, in $\mathbb{Q}^{\Sigma}(\mathcal{C}) \subseteq \mathbb{Q}(\mathcal{C})$ we require the horizontal arrows to be in Σ. ($\mathbb{Q}^{\{id\}}(\mathcal{C}) = \mathbb{V}(\mathcal{C})$)
5. The double category $\mathbb{H}(\mathcal{C})$ is defined analogously.

The category **DblCat** - Definition

The category **DblCat** of double categories has:

- objects: double categories $\mathbb{C}, \mathbb{D}, \ldots$;
- arrows: double functors $F, G, \ldots$;
- transformations: these come in two *flavors*:
  - a horizontal transformation $\gamma \colon F \Rightarrow G$ is given by

$$\begin{tikzcd} FA \arrow[r, "\gamma_A"] \arrow[d, "Fv"'] \arrow[dr, phantom, "\gamma_v"] & GA \arrow[d, "Gv"] \\ FB \arrow[r, "\gamma_B"'] & GB \end{tikzcd} \quad \text{for each } A \text{ in } \mathrm{dom}(F)$$

    functorial in the vertical direction and natural in the horizontal direction.
  - vertical transformations $\nu \colon F \Rightarrow G$ are defined dually;
- modifications given by a family of double cells.

The category **DblCat** - Properties

- **DblCat** is not a double category;
- a double category has two types of arrows, and **DblCat** has only one;
- a double category has one type of 2-cell, and **DblCat** has two;
- **DblCat** is enriched in double categories: $\mathbf{DblCat}(\mathbb{C}, \mathbb{D})$ is a double category.

Review of the Grothendieck construction $\rightsquigarrow$ Questions

Whiteboard

- Can we do this for $F\colon \mathbb{D} \to \mathbf{DblCat}$?
- What sort of colimit do we get?
- What's the relation to other pre-existing constructions?

Double Index Functors

- We would like to have a double functor $F\colon \mathbb{D} \to \mathbf{DblCat}$.
- So we need to build a double category out of **DblCat**.
- We first make the 2-category $\mathbf{DblCat}_v$ of double categories, double functors and vertical transformations.
- And then apply the quintet construction to get the double category $\mathbb{Q}\mathbf{DblCat}_v$.
- So a **double index functor** is a double functor $F\colon \mathbb{D} \to \mathbb{Q}\mathbf{DblCat}_v$.
- We will also call this a **vertical double functor** $\mathbb{D} \to \mathbf{DblCat}$.
- It looks as if at this point we have lost most of the horizontal data.

Double Transformations

- We regain use of some of the horizontal structure in the definition of **double transformation** between double index functors

$$F, G \colon \mathbb{D} \to \mathbb{Q}\mathbf{DblCat}_v.$$

- Analogous to the set-up for bicategories, we will define these transformations in terms of a double category of *cylinders*, $\mathrm{Cyl}_v(\mathbf{DblCat})$.

Whiteboard

The Double Category of (Vertical) Cylinders

The double category $\mathrm{Cyl}_v(\mathbf{DblCat})$ of vertical cylinders has

- Objects f are arrows of **DblCat**, $\downarrow f$.
- Vertical arrows $f \xrightarrow[\bullet]{(u,\mu,v)} \overline{f}$ are given by vertical transformations,

$$
\begin{tikzcd}
{} \arrow[dr, "u"] \arrow[dd, "f"'] & \\
& {} \arrow[dd, "\overline{f}"] \\
{} \arrow[dr, "v"'] \arrow[ur, phantom, "\Uparrow\!\!\bullet\,\mu"] & \\
& {}
\end{tikzcd}
$$

- Horizontal arrows are given by horizontal transformations,

$$
\begin{tikzcd}
& {} \arrow[dd, "f'"] \\
{} \arrow[ur, "h"] \arrow[dd, "f"'] \arrow[dr, phantom, "\Uparrow\kappa"] & \\
& {} \\
{} \arrow[ur, "k"'] &
\end{tikzcd}
$$

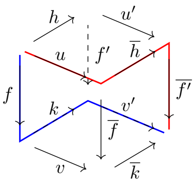
Double Cylinders

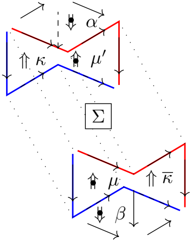
A double cell, $\begin{tikzcd} f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha,\Sigma,\beta)}"] & f' \arrow[d, "{(u',\mu',v')}"] \\ \overline{f} \arrow[r, "{(\overline{h},\overline{\kappa},\overline{k})}"'] & \overline{f'} \end{tikzcd}$ is given by two vertical

transformations $\begin{tikzcd} & \cdot \arrow[dr, "u'"] & \\ \cdot \arrow[ur, "h"] \arrow[dr, "u"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & \cdot \\ & \cdot \arrow[ur, "\overline{h}"'] & \end{tikzcd}$, $\begin{tikzcd} & \cdot \arrow[dr, "v'"] & \\ \cdot \arrow[ur, "k"] \arrow[dr, "v"'] \arrow[rr, phantom, "\Downarrow \beta"] & & \cdot \\ & \cdot \arrow[ur, "\overline{k}"'] & \end{tikzcd}$

and a modification Σ,

$$\begin{tikzcd} v'kf \arrow[r, Rightarrow, "v'\kappa"] \arrow[d, Rightarrow, "\beta f"] \arrow[dd, phantom, bend right=0, ""] & v'f'h \arrow[d, Rightarrow, "\mu' h"] \\ \overline{k}vf \arrow[r, phantom, "\Sigma"] \arrow[d, Rightarrow, "\overline{k}\mu"] & \overline{f'}u'h \arrow[d, Rightarrow, "\overline{f'}\alpha"] \\ \overline{k}\,\overline{f}u \arrow[r, Rightarrow, "\overline{\kappa}u"] & \overline{f'}\,\overline{h}u \end{tikzcd}$$





Cylinders lead to double lax transformations

- There are vertical double functors $\pi_0, \pi_1 \colon \mathrm{Cyl}_v(\mathbf{DblCat}) \to \mathbf{DblCat}$, projecting to the top and the bottom of each cylinder respectively;
- A **double lax transformation** $\theta \colon F \Rightarrow G$ between vertical double functors $F, G \colon \mathbb{D} \to \mathbf{DblCat}$ is given by a double functor

$$\theta \colon \mathbb{D} \to \mathrm{Cyl}_v(\mathbf{DblCat}), \quad \text{with } \ \pi_0\theta = F, \ \pi_1\theta = G.$$

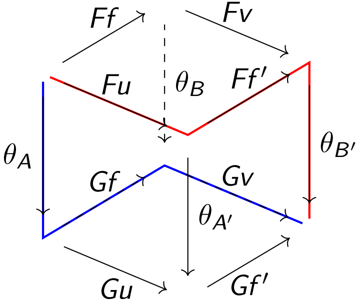
- For each A: $$\begin{tikzcd} FA \arrow[d, "\theta_A"'] \\ GA \end{tikzcd}$$. For each u: $$\begin{tikzcd} FA \arrow[r, "Fu"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\Uparrow\theta_u"] & FA' \arrow[d, "\theta_{A'}"] \\ GA \arrow[r, "Gu"'] & GA' \end{tikzcd}$$

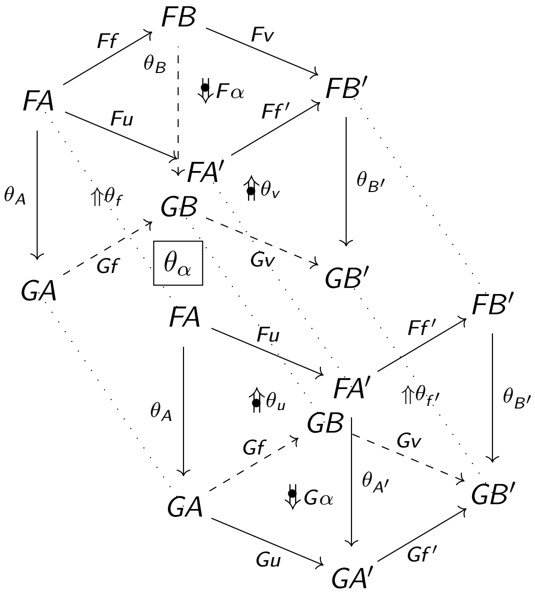
  For each f: $$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\Uparrow\theta_f"] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$, and...

A Double Lax Transformation $\theta\colon F \Rightarrow G$

For each double cell $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$:

$$\begin{tikzcd} GvGf\theta_A \arrow[r, Rightarrow, "Gv\theta_f"] \arrow[d, Rightarrow, "G\alpha\theta_A"'] & Gv\theta_B Ff \arrow[d, Rightarrow, "\theta_v Ff"] \\ Gf'Gu\theta_A \arrow[d, Rightarrow, "Gf'\theta_u"'] \arrow[r, phantom, "\theta_\alpha"] & \theta_{B'}FvFf \arrow[d, Rightarrow, "\theta_{B'}F\alpha"] \\ Gf'\theta_{A'}Fu \arrow[r, Rightarrow, "\theta_{f'}Fu"'] & \theta_{B'}Ff'Fu \end{tikzcd}$$




FB
Ff
Fv
θ_B
⇓Fα
FA
FB′
Fu
Ff′
FA′
θ_A
⇑θ_f
⇑θ_v
θ_B′
GB
Gf
θ_α
Gv
GA
GB′
FA
FB′
Fu
Ff′
FA′
⇑θ_u
⇑θ_f′
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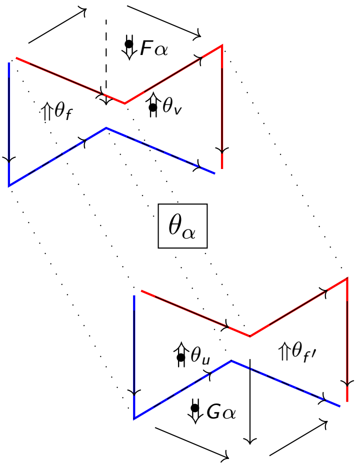

⇓Fα
⇑θ_f
⇑θ_v
θ_α
⇑θ_u
⇑θ_f′
⇓Gα

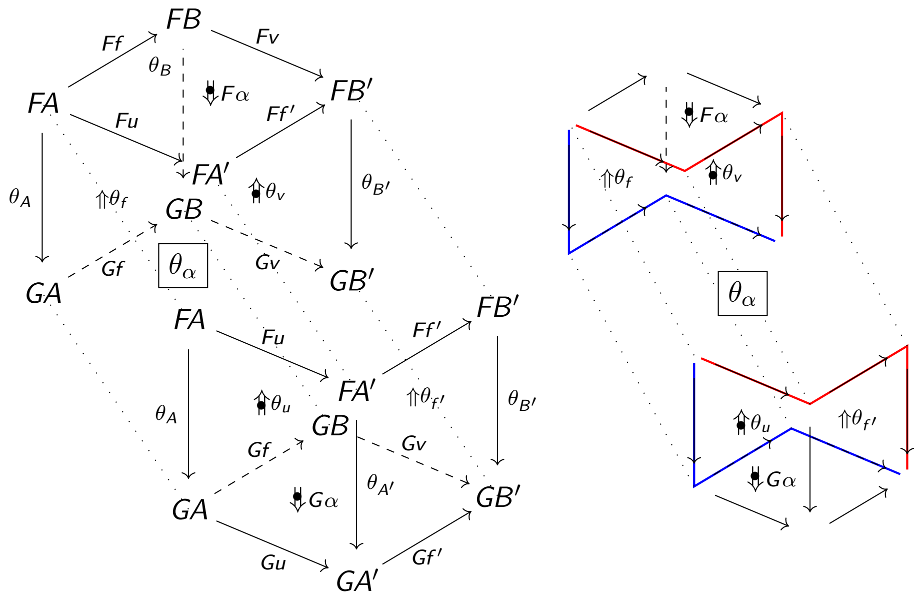
Double Lax Cones

Let $F : \mathbb{D} \to \mathbf{DblCat}$ be a vertical double functor and $\mathbb{E} \in \mathbf{DblCat}$. A **double lax (co)cone** for F, with vertex $\mathbb{E}$, is a double lax transformation

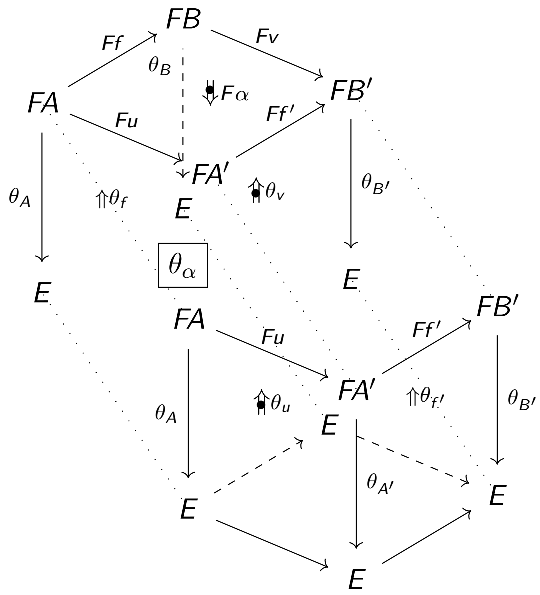
$$F \overset{\theta}{\Rightarrow} \triangle\mathbb{E}$$

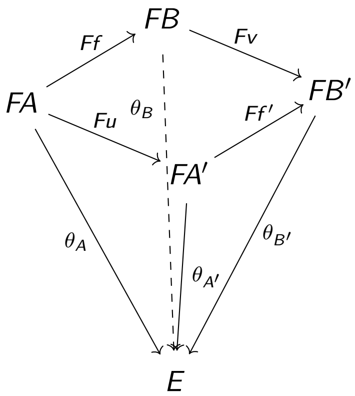
A **double lax colimit** of F is a *universal* double lax cone $F \overset{\lambda}{\Rightarrow} \triangle\mathbb{L}$.

Recall the general double lax natural (6 double cells, one triple cell):

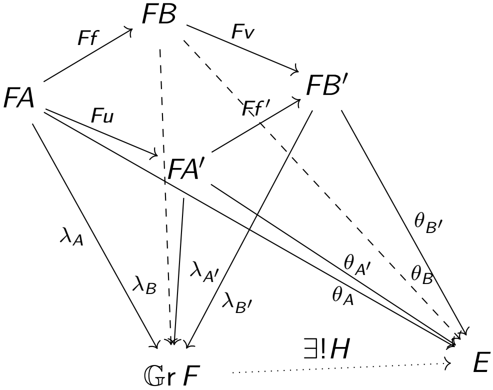


A double lax cone (5 double cells, one triple cell):





The double lax colimit of F:



$$H\lambda_A = \theta_A, \quad H\lambda_u = \theta_u, \quad H\lambda_f = \theta_f, \quad H\lambda_\alpha = \theta_\alpha$$

(for all A, u, f, α!)

Examples "It matters how we index"

- If $\mathcal{A}$ is a 2-category, then a 2-functor $\mathcal{A} \xrightarrow{F} \mathbf{DblCat}_v$ can be seen as a vertical double functor $\mathbb{V}\mathcal{A} \xrightarrow{F} \mathbf{DblCat}$ and as $\mathbb{H}\mathcal{A} \xrightarrow{F} \mathbf{DblCat}$.
- For $\mathbb{H}\mathcal{A} \xrightarrow{F} \mathbf{DblCat}$, we see that a lax cone is given by:
  - double functors $FA \xrightarrow{\theta_A} E$,
  - horizontal transformations $\theta_A \overset{\theta_f}{\Longrightarrow} \theta_B Ff$,
  - for each 2-cell $f \overset{\alpha}{\Longrightarrow} f'$ of $\mathcal{A}$, a modification θ_α,

$$\begin{tikzcd}
\theta_A \arrow[r, Rightarrow, "\theta_f"] \arrow[d, Rightarrow, "id"'] \arrow[dr, phantom, "\theta_\alpha"] & \theta_B Ff \arrow[d, Rightarrow, "\theta_B F\alpha"] \\
\theta_A \arrow[r, Rightarrow, "\theta_{f'}"'] & \theta_B Ff'
\end{tikzcd}$$

- For $\mathbb{V}\mathcal{A} \xrightarrow{F} \mathbf{DblCat}$, the 2-cells θ_f are required to be vertical, thus the triple cells θ_α are triple cells of $\mathbf{DblCat}_v$. Now everything is vertical! This is a *lax tricolimit* in the 3-category $\mathbf{DblCat}_v$.

The Double Grothendieck Construction: Objects and Arrows

Let $\mathbb{D}$ be a double category, and let $\mathbb{D} \xrightarrow{F} \mathbb{Q}\mathbf{DblCat}_v$ be a double functor. The **double category of elements**, $\mathbb{G}\mathrm{r}\, F = \mathbb{E}\mathrm{l}\, F = \int_{\mathbb{D}} F$, is defined by:

- Objects: (C, X) with C in $\mathbb{D}$ and X in FC,
- Vertical arrows:
$$(C, X) \xrightarrow[\bullet]{(u,\rho)} (C', X'),$$
where $C \xrightarrow[\bullet]{u} C'$ in $\mathbb{D}$ and $FuX \xrightarrow[\bullet]{\rho} X'$ in FC'.
- Horizontal arrows:
$$(C, X) \xrightarrow{(f,\varphi)} (D, Y),$$
where $C \xrightarrow{f} D$ in $\mathbb{D}$, and $FfX \xrightarrow{\varphi} Y$ in FD.

The Double Grothendieck Construction: Double Cells

- Double cells: $$\begin{tikzcd} (C,X) \arrow[r, "{(f,\varphi)}"] \arrow[d, "{(u,\rho)}"' , "\bullet" marking] \arrow[dr, phantom, "{(\alpha,\Phi)}"] & (D,Y) \arrow[d, "{(v,\lambda)}", "\bullet" marking] \\ (C',X') \arrow[r, "{(f',\varphi')}"'] & (D',Y') \end{tikzcd}$$, where $\alpha : \left(u \; {}^{f}_{f'} \; v\right)$ is a double cell in $\mathbb{D}$ and Φ is a double cell in FD':

$$\begin{tikzcd} FvFfX \arrow[r, "Fv\varphi"] \arrow[d, "(F\alpha)_X"', "\bullet" marking] & FvY \arrow[dd, "\lambda", "\bullet" marking] \\ Ff'FuX \arrow[d, "Ff'\rho"', "\bullet" marking] \arrow[r, phantom, "\Phi"] & \\ Ff'X' \arrow[r, "\varphi'"'] & Y' \end{tikzcd}$$

Examples

Let A be a 2-category and $F\colon A \to \mathbf{2\text{-}Cat}$ a 2-functor. There are several ways to construct a double index functor: "first compose, then apply $\mathbb{Q}$, and then (optional) restrict"

1. $A \xrightarrow{F} \mathbf{2\text{-}Cat} \xrightarrow{\mathbb{V},\ \mathbb{Q},\ \mathbb{Q}^{op}} \mathbf{DblCat}_v$
2. $\mathbb{Q}(A) \to \mathbb{Q}(\mathbf{DblCat}_v)$
3. Restrict to $\mathbb{H}(A)$ or $\mathbb{V}(A)$

- $\int_{\mathbb{V}A} \mathbb{Q}(\mathbb{V} \circ F) = \int_{\mathbb{V}A} \mathbb{V}(\mathbb{V} \circ F) = \mathbb{V} \int_A F$ (Bakovic, Buckley)
- $\int_{\mathbb{Q}A} \mathbb{Q}(\mathbb{Q} \circ F) = \mathbb{Q} \int_A F$
- $\int_{\mathbb{Q}A} \mathbb{Q}(\mathbb{V} \circ F) = \mathbb{Q}^{\{cart\}} \int_A F$ (only the cartesian arrows horizontally)
- $\int_{\mathbb{H}A} \mathbb{Q}(\mathbb{V} \circ F) = \mathbb{E}l(F)$ (Pare)
- $\int_{\mathbb{Q}A} \mathbb{Q}(\mathbb{Q}^{op} \circ F) = F \wr F^{op}$ (Myers)

Factorization

- Any horizontal arrow (f, φ) can be factored as $(A, x) \xrightarrow{(f,id)} (B, Ffx) \xrightarrow{(id,\varphi)} (B, y)$.
- Any vertical arrow (u, ρ) can be factored as $(A, x) \xrightarrow{(u,id)} (A', Fux) \xrightarrow{(id,\rho)} (A', x')$.
- And any double cell (α, Φ) can be factored as

$$
\begin{tikzcd}
(A,x) \arrow[r, "(f,id)"] \arrow[dd, "(u,id)"'] \arrow[ddr, phantom, "\color{red}(\alpha,id)"] & (B,Ffx) \arrow[r, "(id,\varphi)"] \arrow[d, "(v,id)"] \arrow[dr, phantom, "\color{blue}(id,id)"] & (B,y) \arrow[d, "(v,id)"] \\
 & (B',FvFfx) \arrow[r, "(id,Fv\varphi)"'] \arrow[d, "(id,(F\alpha)_x)"] \arrow[ddr, phantom, "(id,\Phi)"] & (B',Fvy) \arrow[dd, "(id,\lambda)"] \\
(A',Fux) \arrow[r, "(f',id)"] \arrow[d, "(id,\rho)"'] \arrow[dr, phantom, "\color{green}(id,id)"] & (B',Ff'Fux) \arrow[d, "(id,Ff'\rho)"] & \\
(A',x') \arrow[r, "(f',id)"'] & (B',Ff'x') \arrow[r, "(id,\varphi')"'] & (B',y')
\end{tikzcd}
$$

$\mathbb{G}\mathrm{r}\, F$ as a double lax cone

For $F\colon \mathbb{D} \to \mathbf{DblCat}$ a vertical double functor, construct $F \overset{\lambda}{\Longrightarrow} \triangle \mathbb{G}\mathrm{r}\, F$ as follows:

- Double functors $FA \xrightarrow{\lambda_A} \mathbb{G}\mathrm{r}\, F$: $\lambda_A(-) = (A, -)$.
- Vertical transformations $\lambda_A \overset{\lambda_u}{\Longrightarrow} \lambda_{A'} Fu$: for each $x \in FA$, resp. $x \xrightarrow{\varphi} y$:

$$\begin{tikzcd} (A,x) \arrow[d, "(\lambda_u)_x = (u, id)"] \\ (A', F\varphi x) \end{tikzcd} , \qquad \begin{tikzcd} (A,x) \arrow[r, "(id,\varphi)"] \arrow[d, "(u,id)"'] \arrow[dr, phantom, "(\lambda_u)_\varphi = (id,id)"] & (A,y) \arrow[d, "(u,id)"] \\ (A', Fux) \arrow[r, "(id, Fu\varphi)"'] & (A', Fuy) \end{tikzcd}$$

$\mathbb{G}\mathrm{r}\, F$ as a double lax cone

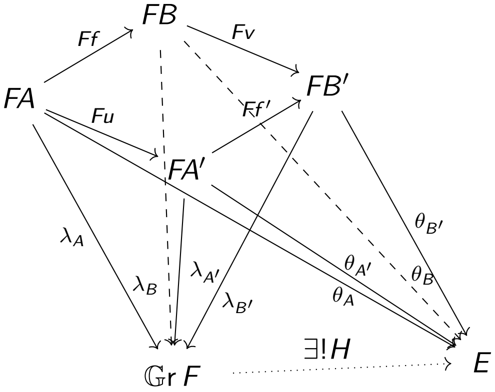
- Hor. transformations $\lambda_A \overset{\lambda_f}{\Longrightarrow} \lambda_B Ff$: for each $x \in FA$, resp. $x \overset{\rho}{\to} x'$:

$$(A,x) \xrightarrow{(\lambda_f)_x = (f,id)} (B, Ffx), \qquad \begin{tikzcd} (A,x) \arrow[r, "{(f,id)}"] \arrow[d, "{(id,\rho)}"'] \arrow[dr, phantom, "{(\lambda_f)_\rho = (id,id)}"] & (B, Ffx) \arrow[d, "{(id, Ff\rho)}"] \\ (A,x') \arrow[r, "{(f,id)}"'] & (B, Ffx') \end{tikzcd}$$

- The modifications λ_α, given for each $x \in FA$:

$$\begin{tikzcd} \lambda_A \arrow[r, Rightarrow, "\lambda_f"'] \arrow[dd, Rightarrow, "\lambda_u"'] \arrow[ddr, phantom, "\lambda_\alpha"] & \lambda_B Ff \arrow[d, Rightarrow, "\lambda_v Ff"] \\ & \lambda_{B'} FvFf \arrow[d, Rightarrow, "\lambda_{B'} F\alpha"] \\ \lambda_{A'} Fu \arrow[r, Rightarrow, "\lambda_{f'} Fu"] & \lambda_{B'} Ff'Fu \end{tikzcd} \qquad \begin{tikzcd} (A,x) \arrow[r, "{(f,id)}"'] \arrow[dd, "{(u,id)}"'] \arrow[ddr, phantom, "{(\lambda_\alpha)_x = (\alpha, id)}"] & (B, Ffx) \arrow[d, "{(v,id)}"] \\ & (B', FvFfx) \arrow[d, "{(id,(F\alpha)_x)}"] \\ (A', Fux) \arrow[r, "{(f',id)}"] & (B', Ff'Fux) \end{tikzcd}$$

Theorem: $\mathbb{G}\mathrm{r}\, F$ is the double lax colimit of F (in DblCat)



$$H\lambda_A = \theta_A, \quad H\lambda_u = \theta_u, \quad H\lambda_f = \theta_f, \quad H\lambda_\alpha = \theta_\alpha$$

Factorization gives H on double cells

Recall that any double cell (α, Φ) can be factored as

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[dd] \arrow[ddr, phantom, "(\alpha{,}id)"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(id{,}id)"] & \cdot \arrow[d] \\
 & \cdot \arrow[r] \arrow[d] \arrow[ddr, phantom, "(id{,}\Phi)"] & \cdot \arrow[dd] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(id{,}id)"] & \cdot \arrow[d] & \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad \overset{H}{\mapsto} \quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[dd] \arrow[ddr, phantom, "(\theta_{\alpha})_x"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(\theta_v)_{\varphi}"] & \cdot \arrow[d] \\
 & \cdot \arrow[r] \arrow[d] \arrow[ddr, phantom, "\theta_{B'}(\Phi)"] & \cdot \arrow[dd] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(\theta_f)_{\rho}"] & \cdot \arrow[d] & \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

this is why this works

A corollary: Recall that $\int_{\mathbb{V}\mathsf{A}} \mathbb{V}(\mathbb{V} \circ F) = \mathbb{V} \int_{\mathsf{A}} F$. We obtain that $\int_{\mathsf{A}} F$ is the *lax tricolimit* of F in 2-Cat. Looking at the other examples gives other universal properties of those constructions.

Thank you!