A general limit lifting theorem for 2-dimensional monad theory (but don't let the long title scare you!)

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Limit lifting along the forgetful functor

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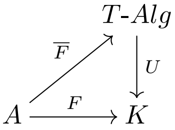

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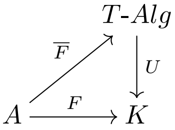

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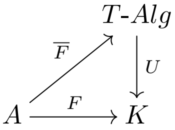

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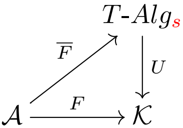

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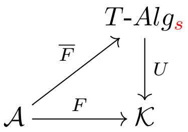

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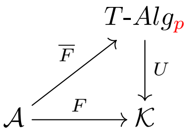

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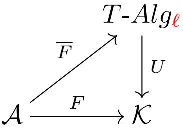

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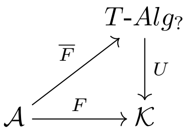

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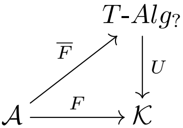

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We will present a theorem which unifies and generalizes these results.

Ω-morphisms of T-algebras

A lax morphism $A \xrightarrow{f} B$ between T-algebras has a structural 2-cell

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \Downarrow \overline{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

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Considering $\Omega_\ell = \text{2-cells}(\mathcal{K})$, $\Omega_p = \{\text{invertible 2-cells}\}$, $\Omega_s = \{\text{identities}\}$, we recover the three cases above.

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- σ-ω-natural transformation: $\mathcal{A} \underset{G}{\overset{F}{\underset{\theta\Downarrow}{\rightrightarrows}}} \mathcal{B}$, θ is a lax natural transformation

$$\begin{array}{ccc} FA & \xrightarrow{\theta_A} & GA \\ {\scriptstyle Ff}\downarrow & \Downarrow\theta_f & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\theta_B]{} & GB \end{array}$$

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- σ-ω-cone (for F, with vertex $E \in \mathcal{B}$): is a σ-ω-natural $\mathcal{A} \underset{F}{\overset{\triangle E}{\underset{\theta\Downarrow}{\rightrightarrows}}} \mathcal{B}$,
i.e.
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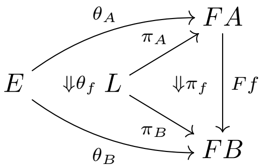

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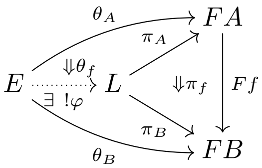

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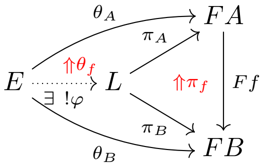

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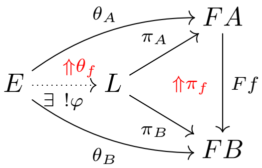

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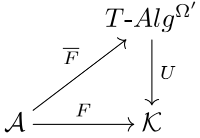
• The notions of lax, pseudo and strict limits are recovered with particular choices of Ω (and Σ).

Our limit lifting theorem (finding the hypotheses)

We consider $\Sigma \subseteq \text{Arrows}(\mathcal{A})$, $\Omega, \Omega' \subseteq \text{2-cells}(\mathcal{K})$. The σ-ω-limits are always taken with respect to Σ and Ω.



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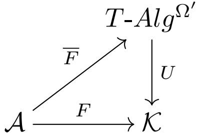

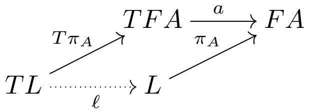

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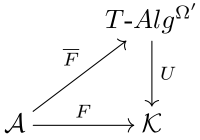

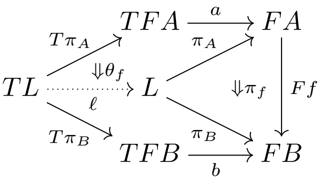

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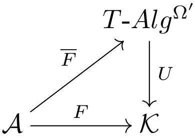

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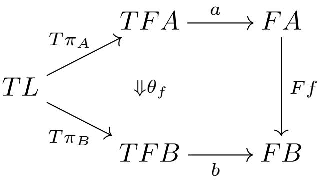

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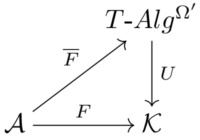

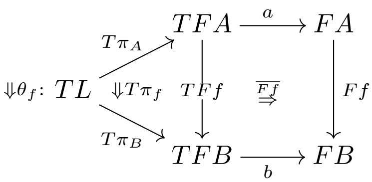

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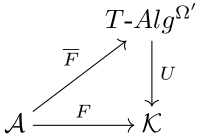

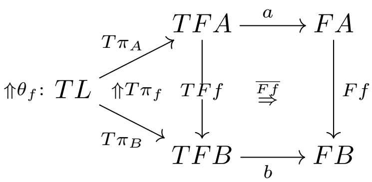

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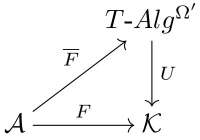

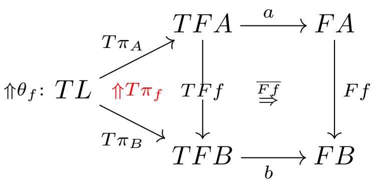

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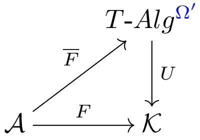

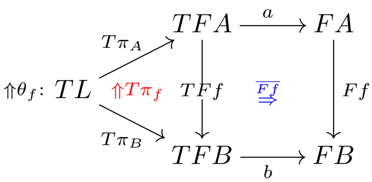

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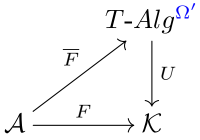

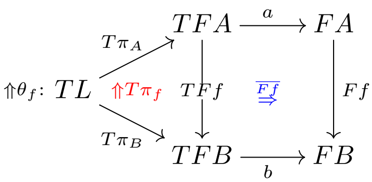

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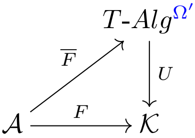
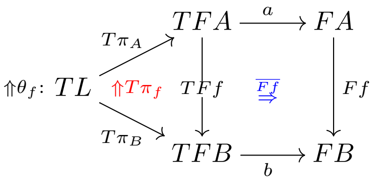


Can we give $L = \sigma$-ω-oplim$F$ a structure of algebra such that the projections are strict morphisms?

We need the 2-cells θ_f yielding a σ-ω-opcone:



$\theta_f \in \Omega$ if $f \in \Sigma$: $T(\Omega) \subseteq \Omega$, $\Omega' \subseteq \Omega \Rightarrow TL \xrightarrow{\ell} L$.

The limit L is Ω'-*compatible* $\Rightarrow (TL, \ell)$ is the desired lifted limit.

Our limit lifting theorem (properly stated)

Theorem: Let $\Sigma \subseteq \text{Arrows}(\mathcal{A})$, $\Omega, \Omega' \subseteq \text{2-cells}(\mathcal{K})$. Assume $T(\Omega) \subseteq \Omega$ and $\Omega' \subseteq \Omega$. Then $T\text{-}Alg^{\Omega'} \xrightarrow{U} \mathcal{K}$ creates Ω'-compatible σ-ω-oplimits.

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We deduce the result for weighted σ-ω-limits, by showing that they can be expressed as conical σ-ω-limits.

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We deduce the result for weighted σ-ω-limits, by showing that they can be expressed as conical σ-ω-limits.

The case $\Omega, \Omega' \in \{\Omega_\ell, \Omega_p, \Omega_s\}$

$T(\Omega) \subseteq \Omega$ ✓, Ω'-compatible ✓

1. (with $\Omega = \Omega' = \Omega_s$) $T\text{-}Alg_s \xrightarrow{U} \mathcal{K}$ creates all (strict) limits.
2. (with $\Omega = \Omega' = \Omega_p$) $T\text{-}Alg_p \xrightarrow{U} \mathcal{K}$ creates σ-limits (thus in particular lax and pseudolimits).
3. (with $\Omega = \Omega' = \Omega_\ell$) $T\text{-}Alg_\ell \xrightarrow{U} \mathcal{K}$ creates oplax limits.

Thank you for your attention!

References

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[Lack,05] Lack S., *Limits for lax morphisms*, ACS 13.

A general limit lifting theorem for 2-dimensional monad theory is available as arXiv:1702.03303.