A Tannakian context for Galois

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Preliminaries

Galois context

Tannakian context

Conclussions

Relations in $s\ell$

- $s\ell$ = Category of Sup-lattices.
- $\mathcal{E}ns \xrightarrow{\ell} s\ell$, $X \mapsto \ell X = \mathcal{P}(X)$, $f \mapsto f$ free Sup-lattice functor.
- Rel = image of ℓ ($\ell X \xrightarrow{R} \ell Y$ corresponds to $X \times Y \xrightarrow{R} \{0, 1\}$).

Hopf algebras in $s\ell$

- $s\ell$ is a tensor category with $\otimes$ and $I = 2$.
- $Alg_{s\ell}$:= commutative algebras in $s\ell =$ $\{(S, S \otimes S \to S, 2 \to S)\}$.
- $Hopf$:= group objects in $Alg_{s\ell}^{op} =$ $\{(A, A \to A \otimes A, A \to 2, A \to A)\}$.

Localic Groups

- $Loc := \{(S, \wedge, 1)\} \therefore \text{Loc} \subset Alg_{s\ell}$.
- $Gr\text{-}Loc :=$ group objects in $Loc^{op} \subset Alg_{s\ell}^{op}$.
- Therefore: $Gr\text{-}Loc \subset Hopf$.

Their representations

- G localic group $\rightsquigarrow \beta^G :=$ sets with an action of G.
- G Hopf algebra $\rightsquigarrow \mathrm{Cmd}_0(G) :=$ G-comodules in $s\ell$ of the form ℓX.
- Theorem 1:

$$G \text{ localic group } \Rightarrow \ Rel(\beta^G) = Cmd_0(G).$$

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Hypotheses

- $\mathcal{E}$ locally conected topos with a point F.
- $F : \mathcal{E} \to \mathcal{E}ns$ can be though of as:
- $F : \mathcal{C} \to \mathcal{E}ns$, $\mathcal{C} =$ small site of connected objects.

Localic Galois Theory

$$\begin{array}{c} \mathcal{C} \\ \downarrow F \\ \mathcal{E}ns \end{array} \quad \rightsquigarrow \quad G = Aut(F) \text{ localic group.}$$

Lifting

$$\begin{array}{ccc} \beta^G & \xleftarrow{\widetilde{F}} & \mathcal{E} \\ & \searrow & \downarrow F \\ & & \mathcal{E}ns \end{array}$$

<u>Theorem $\mathcal{G}$</u> : $\mathcal{E}$ atomic if and only if $\widetilde{F}$ equivalence.

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$\mathcal{V}$-Tannaka theory

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow T & \rightsquigarrow & H = End^{\vee}(T) \text{ Hopf algebra.} \\ \mathcal{V}_0 & & \end{array}$$

Lifting

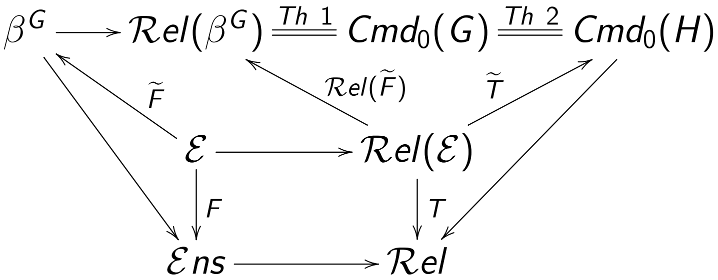
$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\widetilde{T}} & Cmd_0(H) \\ \downarrow T & \swarrow & \\ \mathcal{V}_0 & & \end{array}$$

known: $\mathcal{V}_0 = Vec_{<\infty}$ + hypotheses $\Rightarrow$ $\widetilde{T}$ equivalence.

It is an open problem if $\widetilde{T}$ is an equivalence in general.

Tannakian context associated to Galois

$$\mathcal{V}_0 = Rel \subset s\ell; \;\; \mathcal{X} = Rel(\mathcal{E}); \;\; T = Rel(F)$$



$$G = Aut(F); \;\; H = End^{\vee}(T)$$

$$\underline{\text{Theorem 2}} : G = H$$

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In the Tannakian context associated to the Galois context (that is a locally connected topos $\mathcal{E}$ with a point F), we have

$$\begin{array}{ccc} Rel(\mathcal{E}) & \xrightarrow{\widetilde{T}} & Cmd_0(H) \\ \downarrow T & \swarrow & \\ Rel & & \end{array}$$

Therefore $\widetilde{T}$ is an equivalence $\overset{Teo\ \mathcal{G}}{\Longleftrightarrow}$ $\mathcal{E}$ is atomic.

Thank you!