Advanced Nonlinear Studies **11** (2011), 201-220

Periodic Solutions of Systems with Singularities of Repulsive Type

Pablo Amster*

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina
and
Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)
e-mail: pamster@dm.uba.ar

Manuel Maurette*

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina
e-mail: maurette@dm.uba.ar

Received in revised form 06 February 2010
Communicated by Rafael Ortega

Abstract

Motivated by the classical Coulomb central motion model, we study the existence of T-periodic solutions for the nonlinear second order system of singular ordinary differential equations $u'' + g(u) = p(t)$. Using topological degree methods, we prove that when the nonlinearity $g : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}^N$ is continuous, repulsive at the origin and bounded at infinity, if an appropriate Nirenberg type condition holds then either the problem has a classical solution, or else there exists a family of solutions of perturbed problems that converge uniformly and weakly in H^1 to some limit function u. Furthermore, under appropriate conditions we prove that u is a classical solution.

2010 MSC. 34B16,34C25.
Key words. repulsive singularities; periodic solutions; topological degree.

*The authors acknowledge the support of Grant UBACyT X837 from the University of Buenos Aires.

1 Introduction

As a motivation for our work, let us firstly recall the T-periodic perturbed central motion problem

$$\begin{cases} u'' \mp \frac{u}{|u|^3} & = & p(t) & t \in \mathbb{R} \\ u(t+T) & = & u(t) & t \in \mathbb{R} \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \to \mathbb{R}^3$. We shall assume that the perturbation p has null average, that means that $\overline{p} := \frac{1}{T}\int_0^T p(t)dt = 0$, and that p is $T-$periodic, namely $p(t+T) = p(t)$. The $\mp$ sign leads to two essentially different physical problems; we shall focus on the '$-$' sign, which corresponds to the repulsive case. This is the case of the electrostatic Coulomb central motion problem with a charge being repelled by the source.

With this problem in mind, we study the more general problem for a function $u : \mathbb{R} \to \mathbb{R}^N$,

$$\begin{cases} u'' + g(u) & = & p(t) & t \in \mathbb{R} \\ u(t+T) & = & u(t) & t \in \mathbb{R} \end{cases} \tag{1.2}$$

where $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T-periodic, $\overline{p} = 0$, and $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ has a repulsive type singularity at $u = 0$. By this, we mean that $\langle g(u), u\rangle < 0$ when u is near the origin (see Definition 2.2).

There exists a vast bibliography on this kind of dynamical systems. Lazer and Solimini [11] have considered the scalar case $N = 1$, with $g(u) \to -\infty$ as $u \to 0$, and $\int_0^1 g(t)dt = -\infty$. Using a result proved by Lazer in [10], it is shown that a necessary and sufficient condition for the existence of a weak solution when $g < 0$ and $p \in L^1([0,T])$, is that $\overline{p} < 0$.

In [16], Solimini studied the case $g = \nabla G$, where the potential G has at zero a singularity of repulsive type: for example, the electrostatic potential between two charges of the same sign. More precisely, it is assumed that $G \in C^1(\mathbb{R}^N \backslash \{0\})$ satisfies $\lim_{|u|\to 0} G(u) = +\infty$ and ∇G is *strictly* repulsive at the origin, namely:

$$\limsup_{u\to 0} \left\langle g(u), \frac{u}{|u|} \right\rangle < 0.$$

Under the additional hypothesis

$$\exists\, \delta > 0 \text{ such that, if } \left| \frac{u}{|u|} - \frac{v}{|v|} \right| < \delta, \text{ then } \langle g(u), v\rangle < 0 \tag{1.3}$$

the existence is shown of a constant $\eta > 0$ such that if $\|p\|_\infty < \eta$ and $\overline{p} = 0$, then the problem has no classical solution. This includes the case of the repulsive central motion, where $G(u) = \frac{1}{|u|}$.

In the same work, the existence of a solution for $\overline{p} \neq 0$ under weaker assumptions is proved. Also, it is remarked that if $\|p\|_\infty$ is large enough, then condition $\overline{p} = 0$ does not imply that the problem is unsolvable. This is different from what happens in the case $N = 1$, in which u cannot turn around zero; thus, if the repulsive condition $g(u)u < 0$ is assumed for all $u \neq 0$, then the condition $\overline{p} \neq 0$ is necessary.

In a recent paper, Fonda and Toader [6] made an exhaustive analysis on radially symmetric Keplerian-like systems $u'' + h(t, |u|)u = 0$, where $h : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is T-periodic in t. Using a topological degree approach, the existence of classical T-periodic solutions is studied. This work provides also an excellent survey of the known results on the subject. It is focused in the attractive case, in which the main difficulty consists in avoiding collisions. It is also remarked that, for the repulsive case, the difficulty relies in the case $\overline{p} = 0$.

In [18], Zhang employed topological techniques in order to study the T-periodic problem for the system

$$u'' + (\nabla F(u))' + \nabla G(u) = p(t).$$

When $F \equiv 0$, the problem has variational structure and, as mentioned, the repulsive case was studied in [16]. The attractive case with $p \equiv 0$ and $N = 2$ was solved by Gordon [7], using critical point theory and imposing a strong force condition on G in order to get compactness properties for the involved functionals. Roughly speaking, this condition means that the potential G behaves as $\frac{1}{|u|^\gamma}$ near the origin, with $\gamma \geq 2$; thus, it is not satisfied by the Keplerian potential.

The same assumption is made in [5] for the repulsive case. In the recent works [17] and [4], the strong force condition is removed for the equation $u'' + a(t)u + g(t,u) = p(t)$, provided that the associated linear operator satisfies an anti-maximum principle.

In order to study the general problem (1.2), we shall proceed in two steps. Firstly, we introduce the approximated problem

$$\begin{cases} u'' + g_\varepsilon(u) & = & p(t) & t \in \mathbb{R} \\ u(t+T) & = & u(t) & t \in \mathbb{R}, \end{cases} \tag{1.4}$$

where g_ε is a continuous (nonsingular) perturbation of g, and obtain sufficient conditions for the existence of a family of solutions $\{u_\varepsilon\}$. Secondly, we study the convergence of particular sequences $\{u_{\varepsilon_n}\}$ as $\varepsilon_n \to 0$, and some properties of the limit function u. If $u \not\equiv 0$, then it shall be defined as a *generalized solution* of the problem (see Definition 2.1). In some cases, we shall consider specific choices of g_ε, for instance

$$g_\varepsilon(u) = \begin{cases} g(u) & |u| \geq \varepsilon \\ \rho_\varepsilon(|u|)g\left(\varepsilon\frac{u}{|u|}\right) & 0 < |u| < \varepsilon \\ 0 & u = 0, \end{cases} \tag{1.5}$$

where $\rho_\varepsilon : [0,\varepsilon] \to [0,+\infty)$ is continuous and satisfies $\rho_\varepsilon(0) = 0, \rho_\varepsilon(\varepsilon) = 1$ (more details shall be given in section 2).

For the first step, we extend a well-known result by Nirenberg [13], which in this context can be stated as follows:

Theorem 1.1 *Let $p \in C(\mathbb{R},\mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and let $g \in C(\mathbb{R}^N,\mathbb{R}^N)$ be bounded. Then problem (1.2) has a solution, provided that:*

(N_1) *The radial limits $g_v := \lim_{r\to+\infty} g(rv)$ exist uniformly for $v \in S^{N-1}$ and $g_v \neq 0$ $\forall v \in S^{N-1}$.*

(N_2) *There exists a constant $R_0 > 0$ such that $\deg(\Phi_r) \neq 0$ for $r \geq R_0$, where $\Phi_r : S^{N-1} \to S^{N-1}$ is given by $\Phi_r(v) := \frac{g(rv)}{|g(rv)|}$.*

Our result is based on two previous extensions of Theorem 1.1. On the one hand, a result by Ortega and Ward [14], originally in the context of partial differential equations, where (N_1) is replaced by the following condition, that allows g to vanish at infinity:

(H_1) The radial limits $\lim_{r\to+\infty} \Phi_r(v)$ exist uniformly for $v \in S^{N-1}$.

On the other hand, a result by Amster and De Nápoli [2], for a ϕ-laplacian operator, in which the asymptotic condition (N_1) is weakened to:

(F_1) There exists a family $\{(U_j, w_j)\}_{j=1,\dots,K}$, with U_j open subsets of S^{N-1} and $w_j \in S^{N-1}$ such that $\{U_j\}$ covers S^{N-1}, the upper limit

$$\limsup_{r\to+\infty} \langle g(ru), w_j \rangle := S_j(u)$$

is uniform for $u \in U_j$, and $S_j(u) < 0$.

Remark 1.1 (N_2) is equivalent to the original condition $\deg(\Phi) \neq 0$ in [13] and [14], where $\Phi : S^{N-1} \to S^{N-1}$ is given by $\Phi(v) := \frac{g_v}{|g_v|}$ in the first case, and by $\Phi(v) := \lim_{r\to+\infty} \Phi_r(v)$ in the second case. However, (N_2) makes sense also when the weaker assumption (F_1) is assumed, for which radial limits for g or $\frac{g}{|g|}$ do not necessarily exist.

It is worth mentioning that (N_2) can be also expressed in terms of the Brouwer degree of g, namely:

(N_2') There exists a constant $R_0 > 0$ such that $\deg(g, B_r(0), 0) \neq 0$ for $r \geq R_0$.

Indeed, the equivalence between (N_2) and (N_2') is clear from the following identity, valid for any continuous mapping $f : \overline{B_1(0)} \to \mathbb{R}^N$ such that f does not vanish on S^{N-1}:

$$deg(f, B_1(0), 0) = deg(\phi),$$

where $\phi : S^{N-1} \to S^{N-1}$ is given by $\phi(v) := \frac{f(v)}{|f(v)|}$.

However, in our context, the form (N_2) is preferable since our results shall be applied for a singular g, for which the Brouwer degree in (N_2') is not defined.

In the present work we state a further extension, which will be proved in section 3. For convenience, the boundedness condition on the (nonsingular) g shall be equivalently expressed as:

(B) $\limsup_{|u|\to\infty} |g(u)| < \infty$.

Moreover, it shall be seen that (B) may be replaced by

(B') $\limsup_{|u|\to\infty} \langle g(u), u \rangle < \infty$.

In particular, if $\liminf_{|u|\to\infty} |g(u)| > 0$, then condition (B') says that

$$\liminf_{|u|\to\infty} A(u) \geq \frac{\pi}{2},$$

where $A(u)$ denotes the angle between $g(u)$ and u. Our result for the nonsingular case reads:

Theorem 1.2 *Let $p \in C(\mathbb{R}, \mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfy (B) or (B'). Then problem (1.2) has a solution, provided that (N_2) and (P_1) hold, with:*

(P_1) *There exists a family* $\mathcal{F} = \{(U_j, w_j)\}_{j=1,\dots,K}$ *where* $\{U_j\}_{j=1,\dots,K}$ *is an open cover of* S^{N-1} *and* $w_j \in S^{N-1}$, *such that for some* $R_j > 0$ *and* $j = 1, \dots, K$:

$$\langle g(ru), w_j \rangle < 0 \qquad \forall r > R_j \qquad \forall u \in U_j.$$

Remark 1.2 It is easily seen that (P_1) generalizes (F_1), since the upper limits may vanish, or may not be uniform as $r \to +\infty$. For example, it is clear that (1.3) implies (P_1). More generally, it suffices to assume that (1.3) holds, but only when $|u|$ and $|v|$ are large.

On the other hand, following the ideas in [15] it is seen that (P_1) can be replaced by the following condition, of geometric nature:

(P_1') There exists an open cover $\{U_j\}_{j=1,\dots,K}$ of S^{N-1} such that for some $R_j > 0$ and $j = 1, \dots, K$:

$$0 \notin co\left(g\left(C_j\right)\right),$$

where $co(A)$ denotes the convex hull of $A \subset \mathbb{R}^N$, and $C_j := \bigcup_{r>R_j} rU_j$.

Indeed, from the geometric version of the Hahn-Banach theorem, for any compact subset $C \subset C_j$ we deduce the existence of a vector w_j such that $\langle g(u), w_j \rangle < 0$ for every $u \in C$ and, as we shall see, this suffices for obtaining a priori bounds for the equation.

With Theorem 1.2 in mind, we proceed to the second step. Our main existence results can be stated as follows:

Theorem 1.3 *Let* $p \in C(\mathbb{R}, \mathbb{R}^N)$ *be* T*-periodic such that* $\overline{p} = 0$, *and let* $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ *be repulsive at the origin. Further, assume that g satisfies* (B) *or* (B'), *and that conditions* (P_1) *and* (N_2) *hold. Then either (1.2) has a classical solution, or else for any choice of* g_ε *as in (1.5) there exists a sequence* u_n *of solutions of problem (1.4) and* $\varepsilon_n \to 0$ *that converges uniformly and weakly in* H^1.

Theorem 1.4 *Let* $p \in C(\mathbb{R}, \mathbb{R}^N)$ *be* T*-periodic such that* $\overline{p} = 0$, *and assume that* $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ *is repulsive at the origin and satisfies* (B) *or* (B'). *Further, assume that condition* (P_1) *holds, and that*

$$\|p\|_\infty + \sup_{|u|=\tilde{r}} \left\langle g(u), \frac{u}{|u|} \right\rangle < 0$$

for some $\tilde{r} > 0$. *If also*

(P_2) *There exists a constant* $R_0 > 0$ *such that* $\deg(\Phi_r) \neq (-1)^N$ *for* $r \geq R_0$,

then either (1.2) has a classical solution, or a generalized solution u such that $\|u\|_\infty \geq \tilde{r}$. *Moreover, if g is strictly repulsive at the origin (see Definition 2.2), then the boundary of the set of zeros of u is finite, and if* $g = \nabla G$ *with* $\lim_{u\to 0} G(u) = +\infty$, *then (1.2) has a classical solution.*

The work is organized as follows. In the next section we study the singular problem (1.2), and prove our main existence results and some auxiliary lemmas, making use of Theorem 1.2. As mentioned, this result concerns the nonsingular case, and it is independent of the results in the next section. Although it might have some interest on its own, it does not constitute the main topic of this work: for this reason, we postpone its proof for section 3. Moreover, we state a corollary under more explicit assumptions, which generalize the well-known Landesman-Lazer conditions for the scalar case (see Theorem 3.1). Also, we show an example of a system satisfying the conditions of Theorem 1.2 but neither those of Nirenberg [13], nor those of [14] and [2].

2 Singular repulsive problems

Throughout the rest of the paper we shall always assume that $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T-periodic, and $\overline{p} = 0$.

In order to present our results, let us start making some simple comments on the central motion repulsive problem stated in the introduction:

$$\begin{cases} u'' - \frac{u}{|u|^3} = p(t) & t \in \mathbb{R} \\ u(t+T) = u(t) & t \in \mathbb{R}. \end{cases}$$

Here, the first difficulty arises on the fact that g is singular at 0; a reasonable way to overcome it consists in considering, for $\varepsilon > 0$, the function $g_\varepsilon(u) = -\frac{u}{\varepsilon+|u|^3}$ and then studying the convergence of the solutions u_ε of the perturbed systems (1.4).

The second difficulty relies on the fact that g_ε vanishes at infinity; however, in this case the existence of at least one solution u_ε of (1.4) for each $\varepsilon > 0$ follows as an immediate consequence of the results in [14]. Indeed, as

$$\langle g_\varepsilon(u), u\rangle = \left\langle -\frac{u}{\varepsilon + |u|^3}, u \right\rangle = -\frac{|u|^2}{\varepsilon + |u|^3} < 0$$

for $u \neq 0$, it follows that conditions (B') and (N_2) are trivially satisfied. Moreover, for every $w \in S^{N-1}$ define $U_w = \{u \in S^{N-1} : \langle u, w\rangle > 0\}$. Then $\{U_w\}$ covers S^{N-1}, and clearly $\langle g(ru), w\rangle < 0$ for $u \in U_w$ and $r > 0$. From the compactness of S^{N-1}, condition (P_1) is satisfied. Thus, we may pass to the next step. The following computations provide some information concerning the behavior of the family $\{u_\varepsilon\}$ as $\varepsilon \to 0$:

Multiplying (1.4) by $u_\varepsilon - \overline{u_\varepsilon}$, the facts that $\langle g_\varepsilon(u), u\rangle \leq 0$ and $\overline{p} = 0$ imply that

$$\|u_\varepsilon'\|_{L^2} \leq C, \qquad \|u_\varepsilon - \overline{u_\varepsilon}\|_\infty \leq C$$

where the constant C does not depend on ε. On the other hand, it is easy to prove that $\{\overline{u}_\varepsilon\}$ is also bounded. Indeed, integrating the equation we obtain

$$\int_0^T \frac{u_\varepsilon}{\varepsilon + |u_\varepsilon|^3} dt = 0,$$

and we deduce that

$$-\int_0^T \frac{\overline{u_\varepsilon}}{\varepsilon + |u_\varepsilon|^3} dt = \int_0^T \frac{u_\varepsilon - \overline{u_\varepsilon}}{\varepsilon + |u_\varepsilon|^3} dt.$$

Now, taking norm in $\mathbb{R}^N$:

$$|\overline{u_\varepsilon}| \int_0^T \frac{1}{\varepsilon + |u_\varepsilon|^3} dt \leq \|u_\varepsilon - \overline{u_\varepsilon}\|_\infty \int_0^T \frac{1}{\varepsilon + |u_\varepsilon|^3} dt.$$

Thus, $|\overline{u_\varepsilon}| \leq C$. Hence, for every sequence $\varepsilon_n \to 0$ we may choose a solution $u_n := u_{\varepsilon_n}$ and from the previous bounds there exists a subsequence (still denoted $\{u_n\}$) and a function u such that $u_n \to u$ uniformly and weakly in H^1. Moreover, it is easily seen that if $u \neq 0$ over an open interval I, then $u'' - \frac{u}{|u|^3} = p$ in I, in the classical sense.

So, our last problem concerns the study of the set of zeros of the limit function u. As we shall prove for a more general case, the boundary of the zero set $\{t \in [0,T] : u(t) = 0\}$ is finite. However, in the central motion problem it can be seen, further, that if $u \not\equiv 0$ then the zero set is empty, i. e. u is a classical solution.

A detailed proof of the preceding remarks will be done below, for the general case (1.2).

In order to define the perturbed problem (1.4) in an appropriate way, let us firstly observe that the 'natural' extension of the previous situation would consist in considering

$$g_\varepsilon(u) = \begin{cases} \frac{|u|}{\varepsilon|g(u)|+|u|} g(u) & u \neq 0 \\ 0 & u = 0. \end{cases} \tag{2.1}$$

Nevertheless, there are other possible choices of g_ε such as the ones defined by (1.5). In particular, for the central motion problem, taking $\rho_\varepsilon(s) = \frac{s}{\varepsilon}$ it simply reduces to $g_\varepsilon(u) = -\frac{u}{(\max\{|u|,\varepsilon\})^3}$. For the moment, we shall prove some general properties that hold for any approximation that is 'admissible', in the sense that $g_\varepsilon \to g$ uniformly over compact subsets of $\mathbb{R}^N \backslash \{0\}$ as $\varepsilon \to 0$.

According to the previous comments, we shall also define the concept of *generalized solution*:

Definition 2.1 A function $u \in H^1_{per}(\mathbb{R}, \mathbb{R}^N)$ is said to be a *generalized solution* of (1.2) if $u \not\equiv 0$, and for some admissible choice of g_ε there exists a sequence $\varepsilon_n \to 0$ and u_{ε_n} a solution of (1.4) for $\varepsilon = \varepsilon_n$ such that $u_{\varepsilon_n} \to u$ uniformly and weakly in H^1.

Remark 2.1 For convenience, in the previous situation we shall denote $u_n := u_{\varepsilon_n}$ $g_n := g_{\varepsilon_n}$.

Remark 2.2 When $g = \nabla G$, a different concept of solution (called *collision solution*) is introduced in [3] (see also [1]). As we shall prove (see lemma 2.3 below), under the assumption that $G(u) \to +\infty$ as $u \to 0$, both generalized and collision solutions are in fact classical. Conversely, taking g_ε as in (1.5), it is obvious that classical solutions are also generalized solutions.

Before establishing the main results of this section, we shall prove some lemmas concerning the properties of those functions defined as the limit of a sequence of perturbed problems.

Lemma 2.1 *Let u_n and u be defined as before, and assume that $u \neq 0$ over an open interval I. Then u satisfies*

$$u'' + g(u) = p(t) \qquad \forall\ t \in I$$

in the classical sense.

Proof. Let $\phi \in C_0^\infty(I)$, then

$$\int_I \langle u_n'' + g_n(u_n), \phi \rangle dt = \int_I \langle p, \phi \rangle dt.$$

Integrating by parts

$$-\int_I \langle u_n', \phi' \rangle dt + \int_I \langle g_n(u_n), \phi \rangle dt = \int_I \langle p, \phi \rangle dt,$$

and from the weak convergence in H^1, we deduce that

$$\int_I \langle u_n', \phi' \rangle dt \to \int_I \langle u', \phi' \rangle dt.$$

Thus, it suffices to check that

$$\int_I \langle g_n(u_n), \phi \rangle dt \to \int_I \langle g(u), \phi \rangle dt.$$

As $u_n \to u$ uniformly on I, we may assume that there exist $M, c > 0$ with $M \geq |u_n| \geq c > 0$ on the support of ϕ. Moreover, as $g_n \to g$ uniformly on $\{c \leq |u| \leq M\} \subset \mathbb{R}^N \backslash \{0\}$, it follows that

$$\Big| \int_I \langle g_n(u_n) - g(u), \phi \rangle dt \Big| \leq$$

$$\int_I |\langle g_n(u_n) - g(u_n), \phi \rangle| dt + \int_I |\langle g(u_n) - g(u), \phi \rangle| dt \to 0.$$

This proves that u is a weak solution, and the result follows from standard regularity arguments. ■

From now on, we shall always consider nonlinearities with singularities of repulsive type at the origin, namely:

Definition 2.2 $g : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}^N$ is said to be *repulsive at the origin* if, for some $\kappa > 0$

$$\langle g(u), u \rangle < 0 \qquad \text{for } 0 < |u| < \kappa. \tag{2.2}$$

If, furthermore

$$\limsup_{u \to 0} \left\langle g(u), \frac{u}{|u|} \right\rangle := -c < 0, \tag{2.3}$$

then g shall be called *strictly repulsive* at the origin.

As mentioned, condition (2.3) is the same as in [16] for the case $g = \nabla G$. It is observed that it does not imply the strong force condition: in particular, for any value of $\gamma > -1$ the singularity $g(u) = \frac{-u}{|u|^{\gamma+2}}$ is strictly repulsive, with $c = +\infty$. In such a situation, it can be proved that the boundary of the set of zeros of the limit function u is discrete; more generally:

Lemma 2.2 *Let u_n and u be defined as before, and assume that g is strictly repulsive at the origin. Then the boundary of the set $\{t \in [0, T] : u(t) = 0\}$ is finite, provided that $\|p\|_\infty < c$, with $c \in (0, +\infty]$ as in (2.3).*

Proof. Suppose $u(t_0) = 0$, and fix $\mu > 0$ such that $\|p\|_\infty + \left\langle g(u), \frac{u}{|u|} \right\rangle < 0$ for $0 < |u| < \mu$.

Next, fix $\delta > 0$ such that $|u(t)| < \mu$ for $t \in (t_0 - \delta, t_0 + \delta)$, and suppose for example that u does not vanish in (a, b) for some non-trivial interval $[a, b] \subset [t_0, t_0 + \delta)$. By Lemma 2.1 u is a classical solution of the equation $u'' = p - g(u)$ in (a, b). Moreover, if $\phi(t) = |u(t)|^2$ then on (a, b) we have:

$$\phi'' = 2\langle u'', u \rangle + 2|u'|^2 \geq 2\langle p - g(u), u \rangle =$$

$$2[\langle p, u\rangle - \langle g(u), u\rangle] \geq -2|u| \left[||p||_\infty + \left\langle g(u), \frac{u}{|u|} \right\rangle \right] > 0.$$

Thus, ϕ cannot vanish both on a and b, and it follows that either u does not vanish on $(t_0, t_0 + \delta)$ or $u \equiv 0$ on $[t_0, t_1]$ for some $t_1 > t_0$. The same conclusion holds for $(t_0 - \delta, t_0]$, and the result follows from the compactness of $[0, T]$. ■

The following result improves Lemma 2.2 for the variational case studied in [16]. However, we do not make use of the variational structure of the problem: more generally, it may be assumed that $g = \nabla G$ only near the origin.

Lemma 2.3 *Assume there exists a neighborhood U of the origin and a function $G \in C^1(U \backslash \{0\}, \mathbb{R})$ such that $g = \nabla G$ on $U\backslash\{0\}$. Further, assume that $\lim_{|u|\to 0} G(u) = +\infty$. Then every generalized solution of (1.2) is classical.*

Proof. Let u be a generalized solution, and suppose that u vanishes at some point. Fix $\tilde{t}$ such that $u(\tilde{t}) \neq 0$, and define $t_1 = \inf\{t > \tilde{t} : u(t) = 0\}$. Next, fix a value $t_0 \in (\tilde{t}, t_1)$ such that $u(t) \in U\backslash\{0\}$ and $G(u(t)) > 0$ for $t \in [t_0, t_1)$. As u is a classical solution of the equation on $[t_0, t_1)$, multiplying by u' we deduce, for $t \in [t_0, t_1)$ that

$$\frac{|u'(t)|^2}{2} + G(u(t)) = \frac{|u'(t_0)|^2}{2} + G(u(t_0)) + \int_{t_0}^{t} \langle p(s), u'(s)\rangle \, ds. \tag{2.4}$$

As $G(u(t)) > 0$, for any $\tilde{t}_1 \in (t_0, t_1)$ and $t \in [t_0, \tilde{t}_1]$ we obtain:

$$\frac{|u'(t)|^2}{2} \leq A + B||u'|_{[t_0, \tilde{t}_1]}||_\infty$$

where the constants $A := \frac{|u'(t_0)|^2}{2} + G(u(t_0))$ and $B := (t_1 - t_0)||p||_\infty$ do not depend on the choice of $\tilde{t}_1$. This implies that $u'(t)$ is bounded on $[t_0, t_1)$, and taking limit as $t \to t_1^-$ in (2.4) a contradiction yields. ■

Remark 2.3 It is worth noticing that in this context the repulsive condition (2.2) implies that $G(u)$ increases when u moves on rays that point towards the origin. However, this specific condition was not necessary in the preceding result, which only uses the fact that $G(0) = +\infty$, since it is not required for the proof of Lemma 2.1.

Taking into account the previous comments on the central motion problem, we are able to establish an existence result for the particular radial case $g(u) = h(|u|)u$:

Theorem 2.1 *Let $g(u) = h(|u|)u$, with $h : (0, +\infty) \to (-\infty, 0)$ continuous, and let*

$$g_\varepsilon(u) = \frac{h(|u|)u}{1 - \varepsilon h(|u|)}.$$

Then there exists a sequence $\{u_n\}$ of solutions of (1.4) with $\varepsilon_n \to 0$ that converges uniformly and weakly in H^1 to some limit function u. Furthermore, if $\limsup_{r\to 0^+} rh(r) + ||p||_\infty < 0$, then the set $\partial\{t \in [0, T] : u(t) = 0\}$ is finite, and if $\int_0^1 sh(s)ds = -\infty$, then either $u \equiv 0$ or u is a classical solution.

Proof. As in the particular case of the central motion problem, existence of solutions of (1.4) follows from Theorem 1.2 with condition (B'). Moreover, a bound for $\|u_\varepsilon'\|_{L^2}$ is also obtained as before and, again, the fact that $\int_0^T g_\varepsilon(u_\varepsilon)dt = 0$ implies that

$$-\int_0^T \frac{h(|u_\varepsilon|)\overline{u_\varepsilon}}{1-\varepsilon h(|u_\varepsilon|)}dt = \int_0^T \frac{h(|u_\varepsilon|)(u_\varepsilon - \overline{u_\varepsilon})}{1-\varepsilon h(|u_\varepsilon|)}dt.$$

Thus, a bound for $\overline{u}_\varepsilon$ is also obtained and the conclusion follows from Arzelá-Ascoli theorem and the Banach-Alaoglu Theorem.

Moreover, if $\left\langle g(u), \frac{u}{|u|}\right\rangle = h(r)r < -\|p\|_\infty$ for $|u| = r$ small, then Lemma 2.2 applies. Finally, as $g = \nabla G$, with $G(u) = f(|u|)$ for $f(\sigma) := \int_1^\sigma sh(s)ds$, Lemma 2.3 applies. ■

Example 2.1 The following elementary example shows that the assumption $\lim_{|u|\to 0} G(u) = +\infty$ in Lemma 2.3 is sharp. Let us consider the equation

$$u'' = \frac{u}{|u|^{\gamma+2}} + p, \tag{2.5}$$

which corresponds to the potential

$$G(u) = \begin{cases} \frac{1}{\gamma |u|^\gamma} & \text{if } \gamma \neq 0 \\ -\log|u| & \text{if } \gamma = 0. \end{cases}$$

If $\gamma > -1$, the equation is singular, although for $\gamma \in (-1, 0)$ the potential is continuous up to 0.

For simplicity, let us consider the case $N = 1$, and $p = \chi_{[\frac{T}{2},T]} - \chi_{[0,\frac{T}{2})}$ (note that although p is only piecewise continuous, Lemma 2.3 still applies). As $\overline{p} = 0$, then there are no classical solutions. Moreover, if we set g_ε as

$$g_\varepsilon(u) = -\frac{|u|^{\gamma-2}u}{(\varepsilon + |u|^\gamma)^2},$$

then from the energy conservation law

$$\frac{u_\varepsilon'^2}{2} = E_\varepsilon - u_\varepsilon - \frac{1}{\gamma(\varepsilon + |u_\varepsilon|^\gamma)}, \qquad 0 < t < \frac{T}{2}.$$

A standard computation proves that if T is sufficiently large, then there exist $M_\varepsilon > 0$ and v_ε a positive solution of the equation over $(0, \frac{T}{2})$ such that $v_\varepsilon(0) = v_\varepsilon(\frac{T}{2}) = 0$, with energy $E_\varepsilon = M_\varepsilon + \frac{1}{\gamma(\varepsilon + M_\varepsilon^\gamma)}$ and $\|v_\varepsilon\|_\infty = v_\varepsilon(\frac{T}{4}) = M_\varepsilon$.

We obtain a periodic solution of the perturbed problem by reflection, namely:

$$u_\varepsilon(t) = \begin{cases} v_\varepsilon(t) & \text{if} \quad 0 \leq t \leq \frac{T}{2} \\ -v_\varepsilon(t - \frac{T}{2}) & \text{if} \quad \frac{T}{2} < t \leq T. \end{cases}$$

In particular, for $\varepsilon = 0$ we obtain a solution u of the problem with a collision at $t = \frac{T}{2}$. Furthermore, it is easily checked that $u_\varepsilon \to u$; thus, u is a generalized but non-classical solution.

Remark 2.4 Lemma 2.3 can be regarded as an alternative, in the following way: for g satisfying the assumption, if a sequence u_{ε_n} of solutions of (1.4) for $\varepsilon = \varepsilon_n \to 0$ converges uniformly and weakly in H^1 to some function u, then either $u \equiv 0$, or u is a classical solution of the problem.

It is worth noticing that both situations may occur: for instance, we may consider again equation (2.5), now with $\gamma \geq 0$. If $p \equiv 0$, then there are no generalized solutions (since they should be classical): in some sense, this is expectable since if g_ε is given as in (2.1) or (1.5), then $u_\varepsilon \equiv 0$ is the unique solution of the perturbed problem. On the other hand, for $N = 2$ we may consider the case in which $p(t) = -\lambda(\cos(\omega t), \sin(\omega t))$ with $\omega = \frac{2\pi}{T}$, and the circular solution given by $u(t) = r(\cos(\omega t), \sin(\omega t))$, where $\lambda = r\omega^2 + \frac{1}{r^{\gamma+1}}$. After a simple computation, we conclude that the problem has classical solutions for $\lambda \geq (\gamma+2)\left(\frac{\omega^2}{(\gamma+1)}\right)^{\frac{\gamma+1}{\gamma+2}}$.

Following the ideas in [16], for the preceding case (2.5) with $\gamma \geq 0$ a non-existence result holds when $\|p\|_\infty$ is small. It is interesting to observe that this result can be extended for the L^1-norm: if $\|p\|_{L^1} \leq \eta$ for some η sufficiently small, then the problem admits no classical solutions.

For simplicity, we shall consider only the case $\gamma = 1$ and prove that $\eta \geq \left(\frac{16}{T}\right)^{1/3}$. On the other hand, as we always have circular solutions for any $\lambda \geq 3\left(\frac{2\pi^2}{T^2}\right)^{2/3}$ (and any $N \geq 2$), we also know that $\eta \leq 3\left(\frac{4\pi^4}{T}\right)^{1/3}$.

In order to obtain the previously mentioned explicit lower bound for η, let us assume that u is a classical solution, and fix t_0 such that $|u(t_0)| = \|u\|_\infty$. Multiplying the equation by u and integrating, it follows that

$$\|u'\|_{L^2}^2 = -\int_0^T \left(\frac{1}{|u|} + \langle p, u\rangle\right) dt \leq -\frac{T}{\|u\|_\infty} + \|p\|_{L^1}\|u\|_\infty,$$

and in particular, as u is non-constant,

$$\|p\|_{L^1} > \frac{T}{\|u\|_\infty^2}.$$

Also, for the j-th coordinate of u we have:

$$u_j(t) - u_j(t_0) = \int_{t_0}^t u_j'(s)ds \leq \int_0^T (u_j')^+(s)ds = \frac{1}{2}\|u_j'\|_{L^1} \leq \frac{T^{1/2}}{2}\|u_j'\|_{L^2},$$

and an analogous inequality follows using $(u_j')^-$. Then

$$\|u - u(t_0)\|_\infty^2 \leq \frac{T}{4}\|u'\|_{L^2}^2 \leq \frac{T}{4}\left(\|p\|_{L^1}\|u\|_\infty - \frac{T}{\|u\|_\infty}\right)$$

and in particular

$$|u(t)| \geq |u(t_0)| - \left[\frac{T}{4}\left(\|p\|_{L^1}\|u\|_\infty - \frac{T}{\|u\|_\infty}\right)\right]^{1/2}.$$

Thus,

$$\langle u(t), u(t_0)\rangle = \frac{1}{2}\left(|u(t)|^2 + |u(t_0)|^2 - |u(t) - u(t_0)|^2\right) \geq$$

$$\geq \|u\|_\infty\left(\|u\|_\infty - \left[\frac{T}{4}\left(\|p\|_{L^1}\|u\|_\infty - \frac{T}{\|u\|_\infty}\right)\right]^{1/2}\right).$$

If $\|p\|_{L^1}^3 \leq \frac{16}{T}$, we deduce that $\|p\|_{L^1}^2 \leq \frac{16}{T^2}\frac{T}{\|p\|_{L^1}} < \left(\frac{4}{T}\|u\|_\infty\right)^2$. Hence $\frac{T}{4}\|p\|_{L^1}\|u\|_\infty < \|u\|_\infty^2$, and we conclude that $\langle u(t), u(t_0)\rangle > 0$ for every t.

Finally, integrating the equation we obtain

$$0 = \left\langle u(t_0), \int_0^T u''(t)\, dt\right\rangle = \int_0^T \frac{1}{|u(t)|^3}\langle u(t_0), u(t)\rangle\, dt > 0,$$

a contradiction.

Remark 2.5 It might be worth observing that the geometric idea behind the last proof is that for any $w \in \mathbb{R}^N\backslash\{0\}$ the range of a classical solution of the problem cannot be contained in the half-space $H_w := \{u : \langle u, w\rangle > 0\}$.

Together with the preceding results, the previous computations imply that, for the central motion case, if $\|p\|_{L^1} \leq \left(\frac{16}{T}\right)^{1/3}$ then there exist sequences of solutions of the perturbed problems (1.4) with $g_\varepsilon(u) = -\frac{u}{\varepsilon+|u|^3}$ that converge uniformly to 0. However, it is worth to observe that this fact is immediate if we do not impose restrictions on the choice of g_ε. Indeed, we may recall that for any $\lambda > 0$, the unique T-periodic solution of the linear problem $u'' - \lambda^2 u = p$ is given by

$$u(t) = \int_0^T G(t, s)p(s)ds,$$

where G is the Green function defined by

$$G(t, s) = \frac{-\cosh\left(\lambda\left(\frac{T}{2} - |t - s|\right)\right)}{2\lambda\sinh\left(\lambda\frac{T}{2}\right)}.$$

A simple computation shows, moreover, that $\|G(t, \cdot)\|_{L^1} = \frac{1}{\lambda^2}$. Thus, if $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is any continuous function satisfying $\varepsilon\mu(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, then we may define, using Tietze's theorem, a function $g_\varepsilon \in C(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$g_\varepsilon(u) = \begin{cases} g(u) & \text{if } |u| \geq 2\varepsilon \\ -\mu(\varepsilon)u & \text{if } |u| \leq \varepsilon. \end{cases}$$

Then, for every $\varepsilon > 0$ with $\varepsilon\mu(\varepsilon) > \|p\|_\infty$, the unique solution of the linear problem $u'' - \mu(\varepsilon)u = p$ satisfies:

$$|u(t)| \leq \frac{\|p\|_\infty}{\mu(\varepsilon)} < \varepsilon,$$

and hence it solves (1.4).

The rest of the section is devoted to the particular case in which g_ε is defined by (1.5) for some ρ_ε. The reason of this specific choice is that, unlike the case of Theorem 2.1, the existence of a priori bounds for u_ε cannot be established for a general nonlinearity g. Note also that, if $g(u) = h(|u|)u$, then the 'linear' cutoff function defined by $\rho_\varepsilon(s) = \frac{s}{\varepsilon}$ in (1.5) would lead to the previous situation, with $\mu = -h$, and the conclusions in our existence results would become trivial. However, we do not need to impose any restriction on the function $\rho(\varepsilon) := \rho_\varepsilon$.

Theorem 2.2 *Let $g : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}^N$ be continuous, and assume that (2.2) holds. Further, assume that g satisfies (B) or (B'). Then either problem (1.2) has a classical solution, or else for every sequence $\{u_n\}$ of solutions of (1.4) with $\varepsilon_n \to 0$ and $g_n := g_{\varepsilon_n}$ as in (1.5), there exists a subsequence that converges uniformly and weakly in H^1.*

Proof. If the problem has a classical solution, then there is nothing to prove. Next, assume that (1.4) admits no classical solutions, and let u_n be a T-periodic solution of

$$u_n'' + g_n(u_n) = p.$$

Multiplying by $u_n - \overline{u}_n$ and integrating:

$$\int_0^T \langle u_n'', u_n - \overline{u}_n \rangle dt + \int_0^T \langle g_n(u_n), u_n - \overline{u}_n \rangle dt = \int_0^T \langle p(t), u_n - \overline{u}_n \rangle dt,$$

and hence

$$-\int_0^T |u_n'|^2 dt + \int_0^T \langle g_n(u_n), u_n - \overline{u_n} \rangle dt = \int_0^T \langle p(t), u_n - \overline{u}_n \rangle dt.$$

Then we have:

$$\|u_n'\|_{L^2}^2 \leq \|p\|_{L^2} \|u_n - \overline{u}_n\|_{L^2} + \int_0^T \langle g_n(u_n), u_n - \overline{u_n} \rangle dt. \tag{2.6}$$

If (B) holds, then we may split the last term in two terms as:

$$\int_{\{|u_n| > \kappa\}} \langle g_n(u_n), u_n - \overline{u_n} \rangle dt + \int_{\{|u_n| \leq \kappa\}} \langle g_n(u_n), u_n - \overline{u_n} \rangle dt,$$

with κ as in (2.2).

For the first term, we use the definition of g_n: $g_n(u) = g(u)$ if $|u| > \varepsilon_n$. We may assume that $\varepsilon_n < \kappa$, and hence:

$$\left| \int_{\{|u_n| > \kappa\}} \langle g_n(u_n), u_n - \overline{u_n} \rangle dt \right| \leq \int_{\{|u_n| > \kappa\}} |g(u_n)| |u_n - \overline{u_n}| dt \leq C \|u_n - \overline{u_n}\|_{L^2}.$$

On the other hand, the remaining term can be written as:

$$\int_{\{|u_n| \leq \kappa\}} \langle g_n(u_n), u_n \rangle dt - \left\langle \int_{\{|u_n| \leq \kappa\}} g_n(u_n) dt, \overline{u_n} \right\rangle.$$

From the repulsive condition (2.2), the first term is non-positive; moreover, as $\overline{g_n(u_n)} = 0$ we deduce: $\int_{\{|u_n| \leq \kappa\}} g_n(u_n) dt = -\int_{\{|u_n| > \kappa\}} g_n(u_n) dt$. Hence,

$$\int_{\{|u_n| \leq \kappa\}} \langle g_n(u_n), u_n - \overline{u_n} \rangle dt \leq |\overline{u_n}| \int_{\{|u_n| > \kappa\}} |g_n(u_n)| dt.$$

Again, the integral in the right-hand side term is bounded, because g_n may be replaced by g. Gathering all together:

$$\|u_n'\|_{L^2}^2 \leq C_1 \|u_n - \overline{u_n}\|_{L^2} + C_2 |\overline{u_n}|.$$

Finally, using Wirtinger's inequality we get:

$$\|u_n'\|_{L^2} \leq C|\overline{u_n}|^{\frac{1}{2}}, \qquad \|u_n - \overline{u_n}\|_\infty \leq C|\overline{u_n}|^{\frac{1}{2}}.$$

Now, we can state that $\{\overline{u}_n\}$ is bounded. If this was not the case, we would have, for some value of n, that $|\overline{u_n}|^{\frac{1}{2}} > C + 1 \geq \varepsilon_n$. Then

$$|u_n(t)| \geq |\overline{u_n}| - \|u_n - \overline{u_n}\|_\infty \geq |\overline{u_n}| - C|\overline{u_n}|^{\frac{1}{2}} > C + 1.$$

Thus, u_n is a solution of the original problem, a contradiction.

If, instead, we assume that (B') holds, then from the fact that $\overline{g_n(u_n)} = 0$ we deduce that the last term of (2.6) is bounded, and a bound for $\|u_n'\|_{L^2}$ and $\|u_n - \overline{u}_n\|_\infty$ yields. As before, this implies that $\{\overline{u}_n\}$ is also bounded. Hence, there is a subsequence (still denoted $\{u_n\}$) and a function $u \in H^1$ such that $u_n \to u$ uniformly and weakly in H^1. ■

In the previous proof, note that the bounds for $\|u_n\|_{H^1}$ do not depend on the choice of ρ_ε. This is the reason why Theorem 1.3, with ρ arbitrarily chosen, follows as an immediate consequence of the preceding results:

Proof of Theorem 1.3:

Given $0 < \varepsilon_n \to 0$ then either $g_n \in C(\mathbb{R}^N, \mathbb{R}^N)$ is bounded or satisfies (B') for each n. Theorem 1.2 guarantees the existence of a sequence $\{u_n\}$ of classical solutions of problem (1.4). Finally, Theorem 2.2 is applied. ■

The last part of this section is devoted to Theorem 1.4, which assumes a different asymptotic condition on g. In order to understand its meaning, let us firstly observe that if

$$\|p\|_\infty + \sup_{|u|=\varepsilon} \left\langle g_\varepsilon(u), \frac{u}{|u|} \right\rangle \leq 0 \tag{2.7}$$

then a Hartman type condition (see [8]) holds, and the existence of a solution u_ε of (1.4) with $\|u_\varepsilon\|_\infty \leq \varepsilon$ is deduced. In particular, if g satisfies (2.3) with $c > \|p\|_\infty$, then condition (2.7) holds strictly when ε is small and, again, there exists a sequence of solutions of (1.4) that converges to 0. However, in this case we may take advantage of the fact that $\deg(\Phi_\varepsilon) = (-1)^N$, and replace condition (N_2) by (P_2), namely that $\deg(\Phi_R) \neq (-1)^N$ for R sufficiently large. Indeed, if we consider now the Brouwer degree of g_ε, from the excision property it follows that

$$\deg(g_\varepsilon, B_R(0)\backslash B_\varepsilon(0), 0) = \deg(\Phi_R) - \deg(\Phi_\varepsilon) \neq 0.$$

Thus, Mawhin's continuation theorem [12] implies the existence of a second solution u_ε of (1.4) such that $\|u_\varepsilon\|_\infty > \varepsilon$, provided that the homotopy does not vanish when $\|u\|_\infty = \varepsilon$ or $\|u\|_\infty = R$. More generally, if we assume only that (2.7) holds strictly for some fixed $\tilde{r}$, then we are able to prove Theorem 1.4.

Proof of Theorem 1.4:

From Theorem 2.2, it suffices to show that for each $\varepsilon \leq \tilde{r}$ problem (1.4) has a solution u_ε such that $\|u_\varepsilon\|_\infty > \tilde{r}$. To this end, we may follow the general outline of the proof of Theorem 1.2 (which will be presented in the next section), but now taking the domain $\Omega = \{u \in C([0,T], \mathbb{R}^N) : \tilde{r} < \|u\|_\infty < R\}$.

The proof of the fact that $u'' \neq \lambda(p - g_\varepsilon(u))$ for any T-periodic function u with $\|u\|_\infty = R \gg 0$ and $\lambda \in (0, 1]$ follows as in the proof of Theorem 1.2. On the other hand, if u is T-periodic and satisfies

$$u'' = \lambda(p - g_\varepsilon(u))$$

with $\|u\|_\infty = \tilde{r}$, then consider $\phi(t) := |u(t)|^2$ and t_0 a maximum of ϕ. Hence $|u(t_0)| = \tilde{r}$, and

$$0 \geq \phi''(t_0) \geq -2\lambda r \left[\|p\|_\infty + \left\langle g(u(t_0)), \frac{u(t_0)}{|u(t_0)|} \right\rangle \right] > 0,$$

a contradiction. Finally, from the remarks previous to this proof we deduce that the Brouwer degree $\deg(g, \Omega \cap \mathbb{R}^N, 0) \neq 0$, and the conclusion follows. ■

Example 2.2 If there exist $v \in S^{N-1}$ and $r_0 > 0$ such that $g(u) \in H_v$ for $|u| \geq r_0$, where H_v is the half-space defined as before, then condition (P_1) is satisfied taking $w = -v$ and $\mathcal{F} = \{(S^{N-1}, w)\}$. Moreover, it is also clear that $\deg(\Phi_R) = 0$ for $R \geq r_0$: hence, if g satisfies (B) or (B') and (2.3), the existence of a generalized solution follows for any p continuous and T-periodic such that $\overline{p} = 0$ and $\|p\|_\infty < c$.

More generally, if g satisfies (B) or (B'), (P_1) and (2.3) with $\|p\|_\infty < c$, then it suffices to assume that $g(u) \neq \lambda v$ for $|u| \geq r_0$ and $\lambda \geq 0$.

Remark 2.6 Under the assumptions of Theorem 2.2, if (P_1) and (P_2) are satisfied, and g is sequentially strongly repulsive at the origin, namely

$$\sup_{|u|=r_n} \left\langle g(u), \frac{u}{|u|} \right\rangle \to -\infty$$

for some $r_n \to 0$, then existence of a generalized solution holds for any p continuous and T-periodic such that $\overline{p} = 0$.

Remark 2.7 It is interesting to observe that condition (1.3) implies that $\deg(\Phi_r) = (-1)^N$ for all values of r; thus, Theorem 1.4 does not apply to this case. This is consistent with the non-existence result obtained in [16]. On the other hand, condition (P_1) is still satisfied if (1.3) is reversed, namely:

$$\exists\, \delta, r_0 > 0 : \text{ if } |u|, |v| \geq r_0 \text{ and } \left| \frac{u}{|u|} - \frac{v}{|v|} \right| < \delta, \text{ then } \langle g(u), v \rangle > 0. \tag{2.8}$$

In some sense, (2.8) says that g is repulsive at ∞, and that it cannot rotate too fast. We have already used the fact that repulsiveness at the origin implies that the Brouwer degree of g_ε over small balls is $(-1)^N$; on the other hand, repulsiveness at ∞ implies that its degree over large balls is 1. Hence, if the assumptions of Theorem 2.2 are satisfied and g is (sequentially) strongly repulsive at the origin and (2.8) holds, then there exist generalized solutions for any p continuous and T-periodic such that $\overline{p} = 0$, provided that N is odd.

In particular, for the radial case we have:

Corollary 2.1 *let N be odd, p as before, and let g be given by*

$$g(u) = \varphi(|u|)\psi\left(\frac{u}{|u|}\right)$$

with $\psi : S^{N-1} \to S^{N-1}$ *continuous,* $\varphi : \mathbb{R}^+ \to \mathbb{R}$ *continuous and bounded from below, and*

$$\langle \psi(v), v \rangle < 0 \qquad \forall v \in S^{N-1},$$

$$\lim_{r\to 0^+} \varphi(r) = +\infty, \qquad \varphi(r) < 0 \quad \text{if } r > r_0$$

for some $r_0 > 0$.

Then, for any p, either (1.2) has a classical solution, or a generalized solution u. Moreover, the boundary of the set of zeros of u is finite. For the case $\psi(v) = -v$, *if furthermore* $\int_0^1 \varphi(s)ds = +\infty$, *then (1.2) has a classical solution.*

Proof. Condition (B) is clear. Moreover, as ψ is continuous, for each $u \in S^{N-1}$ there exists an open neighborhood $U \subset S^{N-1}$ of u such that :

$$\langle \psi(w), u \rangle < 0 \qquad \forall w \in U.$$

Then taking $w_u = -u$, for $r > r_0$ and $w \in U$ we obtain:

$$\langle g(rw), w_u \rangle = |\varphi(r)| \langle \psi(w), u \rangle < 0.$$

From the compactness of S^{N-1}, condition (P_1) is satisfied.

Finally, define the homotopy $H : \mathbb{R}^N \backslash \{0\} \times [0, 1] \to \mathbb{R}^N$ given by $H(u, \lambda) = \lambda g(u) + (1 - \lambda)u$. Then, for $|u| = R > r_0$,

$$\langle H(u, \lambda), u \rangle = \lambda \langle g(u), u \rangle + (1 - \lambda)R^2 > 0.$$

By the homotopy invariance of the degree, we conclude that

$$\deg(\Phi_R) = \deg(Id) = 1 \neq (-1)^N.$$

Hence, condition (P_2) is then also satisfied, and the conclusion follows from Theorem 1.4. ■

3 A general theorem for the non-singular case

Proof of Theorem 1.2:

It suffices to verify that the hypotheses of Mawhin's Continuation Theorem [12] are satisfied over the domain $\Omega = \{u \in C([0, T], \mathbb{R}^N) : \|u\|_\infty < R\}$. As (N_2) holds, we know that $\deg(g, B_R(0), 0) \neq 0$ for large values of R. Thus, we only need to prove that for $\lambda \in (0, 1]$, the problem

$$u'' = \lambda(p(t) - g(u)) \tag{3.1}$$

does not have a T-periodic solution on $\partial B_R(0) \subset C([0, T], \mathbb{R}^N)$, for some R large enough.

Assume firstly that (B) holds, and let us suppose that problem (3.1) has an unbounded sequence of solutions; namely, there exist $\lambda_n \in (0, 1]$ and T-periodic functions u_n such that $\|u_n\|_\infty \to \infty$ and

$$u_n''(t) = \lambda_n(p(t) - g(u_n(t))).$$

Taking average on both sides, it follows that

$$\int_0^T g(u_n(t))dt = 0. \tag{3.2}$$

On the other hand, from the boundedness of g we obtain:

$$\|u_n'\|_\infty \le T\|u_n''\|_\infty \le T(\|p\|_\infty + \|g\|_\infty) = M.$$

Hence, $u_n - \overline{u}_n$ is bounded; in particular, as $\|u_n\|_\infty \to \infty$, we conclude that $|\overline{u_n}| \to \infty$ and $r_n(t) := |u_n(t)| \ge |\overline{u_n}| - \|u_n - \overline{u}_n\|_\infty \to \infty$ uniformly.

Next, define

$$z_n(t) = \frac{u_n(t)}{|u_n(t)|} \in S^{N-1}.$$

Passing to a subsequence, we may assume that $\frac{\overline{u}_n}{|\overline{u}_n|}$ converges to some $u \in S^{N-1}$, and hence $z_n \to u \in S^{N-1}$ uniformly. From (P_1), $u \in U_j$ for some $j = 1, \dots, K$.

Also, fix n_0 such that if $r_n(t) > r_0$ then $z_n(t) \in U_j$ for all $n \ge n_0$ and all $t \in [0, T]$. For $n \ge n_0$, we deduce that

$$\langle g(r_n(t)z_n(t)), w_j\rangle < 0$$

for all $t \in [0, T]$. Hence

$$0 = \left\langle \int_0^T g(u_n(t))dt, w_j \right\rangle = \int_0^T \langle g(u_n(t)), w_j\rangle dt$$

$$= \int_0^T \langle g(r_n(t)z_n(t)), w_j\rangle dt < 0 \qquad \text{for } n \ge n_0,$$

a contradiction.

Finally, if condition (B') holds instead of (B), then multiplying the equality $u_n'' = \lambda_n(p - g(u_n))$ by $u_n - \overline{u}_n$ and using the fact that $\overline{g(u_n)} = 0$ we deduce:

$$\|u_n'\|_{L^2}^2 \le \|p\|_{L^2}\|u_n - \overline{u}_n\|_{L^2} + \lambda \int_0^T \langle g(u_n), u_n\rangle dt \le \frac{T}{2\pi}\|p\|_{L^2}\|u_n'\|_{L^2} + kT.$$

Hence, $\|u_n'\|_{L^2}$ is bounded which, in turn, $\|u_n - \overline{u}_n\|_\infty$ is bounded, and the rest of the proof follows as before. ■

Remark 3.1 Under an appropriate Nagumo type condition, a more general result may be obtained for $g = g(t, u, u')$.

Perhaps it is hard to see the improvement in the previous technical hypothesis (P_1). The crucial point is that we can guarantee existence of solutions in the absence of radial limits for g or even for $\frac{g}{|g|}$. To visualize this fact, let us consider the following Landesman-Lazer type condition [9], motivated by an analogous result in [2]:

(P_1') Let $\{e_i\}_{i=1}^N, \{w_j\}_{j=1}^N \subset S^{N-1}$ be two bases of $\mathbb{R}^N$, and assume there exists $s_0 > 0$ such that

$$\langle g(x - se_i), w_i\rangle > 0 > \langle g(x + se_i), w_i\rangle \qquad \forall\, s \geq s_0$$

for all $x \in \text{span}\{e_j : j \neq i\}$ and $1 \leq i \leq N$.

Remark 3.2 It is easy to see that condition (P_1') implies (P_1). Indeed, let $u \in S^{N-1}$, $u = x + \alpha e_i$, with $x \in \text{span}\{e_j : j \neq i\}$, $\alpha \neq 0$. Now, fix $\delta < |\alpha|$ and consider $\tilde{u} = \tilde{x} + \tilde{\alpha} e_i \in U := B_\delta(u) \cap S^{N-1}$. If $\alpha > 0$, then as $s\tilde{x} \in \text{span}\{e_j : j \neq i\}$ we obtain:

$$\langle g(s\tilde{u}), w_i\rangle = \langle g(s\tilde{x} + s\tilde{\alpha} e_i), w_i\rangle < 0 \qquad \text{for } s\tilde{\alpha} \geq s_0.$$

In the same way, for $\alpha < 0$:

$$\langle g(s\tilde{u}), -w_i\rangle = -\langle g(s\tilde{x} - s|\tilde{\alpha}|e_i), w_i\rangle < 0 \qquad \text{for } s|\tilde{\alpha}| \geq s_0.$$

As $|\tilde{\alpha}| > \alpha - \delta$, both inequalities hold for $\tilde{u} \in U$ when $s \geq \frac{s_0}{\alpha-\delta}$. The result follows now from the compactness of S^{N-1}.

Theorem 3.1 *Let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfy (B) or (B'), and let $p \in C(\mathbb{R}^N, \mathbb{R}^N)$ be T-periodic with $\overline{p} = 0$. If condition (P_1') is satisfied, then problem (1.2) has at least one solution.*

Proof. From the previous remark, we only need to prove (N_2). Without loss of generality we may assume that $\{w_i\} = \{e_i\}_{i=1}^N$ is the canonical basis. From (P_1'), there exists s_0 such that if $s \geq s_0$, then

$$g_i(x - se_i) > 0 > g_i(x + se_i) \qquad \forall x \in \text{span}\{e_j : j \neq i\},\ i = 1, \ldots, N$$

Let $R \geq s_0$, and consider the cube $C_R := [-R, R]^N$ and the homotopy $h(\lambda, u) := \lambda g(u) - (1 - \lambda)u$. Suppose there exists $u \in \partial C_R$ such that $h(\lambda, u) = 0$ for some $\lambda \in [0, 1]$: for example $u = x + Re_i$ with $x \in \text{span}\{e_j : j \neq i\}$. Then, looking at the i−th coordinate:

$$\lambda g_i(x + Re_i) = (1 - \lambda)R.$$

From (P_1'), the left hand-side term is negative, unless $\lambda = 0$, a contradiction. An analogous argument can be used in the case $u = x - Re_i$. We then conclude that for any $R \geq s_0$:

$$\deg(g, C_R, 0) = \deg(-Id, C_R, 0) \neq 0.$$

This is obviously equivalent to (N_2), and so all the assumptions of Theorem 1.2 are fulfilled.

Example 3.1 Let $N = 2$ and g given by

$$g(x, y) = \left(\frac{1 + x + r(y)}{1 + x^2}, \frac{1 + y}{1 + y^2}\left(1 + \frac{\sin x}{1 + |y|}\right)\right)$$

where $r : \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Taking $e_1 = (1, 0) = -w_1; e_2 = (0, 1) = -w_2$:

$$\langle g(s, y), w_1\rangle = -\frac{1 + s + r(y)}{1 + s^2} < 0 \quad \forall s > ||r||_\infty - 1\ (\forall y)$$

$$\langle g(-s, y), w_1 \rangle = \frac{s - 1 - r(y)}{1 + s^2} > 0 \quad \forall s > \|r\|_\infty + 1\,(\forall y)$$

and

$$\langle g(x, s), w_2 \rangle = -\frac{1 + s}{1 + s^2}\left(1 + \frac{\sin x}{1 + s}\right) < 0 \quad \forall s > 0\,(\forall x)$$

$$\langle g(x, -s), w_2 \rangle = \frac{s - 1}{1 + s^2}\left(1 + \frac{\sin x}{1 + s}\right) > 0 \quad \forall s > 1\,(\forall x).$$

Thus, g verifies (P_1'), although it does not verify the assumptions of Ortega and Ward [14]. Indeed, the radial limits for $\frac{g}{|g|}$ do not necessarily exist. For example, let us consider the direction $(1, 0) \in S^1$: then, $(sx, sy) = (s, 0)$ and

$$g(s, 0) = \left(\frac{1 + s + r(0)}{1 + s^2}, 1 + \sin s\right);$$

$$|g(s, 0)| = \sqrt{\left(\frac{1 + s + r(0)}{1 + s^2}\right)^2 + (1 + \sin s)^2}.$$

Let $s = \frac{4k-1}{2}\pi$, $k \in \mathbb{N}$, $\gamma_{\frac{4k-1}{2}} = \frac{g(\frac{4k-1}{2}\pi, 0)}{|g(\frac{4k-1}{2}\pi, 0)|}$. Here, $\sin\left(\frac{4k-1}{2}\pi\right) = -1$, then

$$\gamma_{\frac{4k-1}{2}} = (1, 0) \qquad \text{for } k \text{ large enough.}$$

Now, let $s = k\pi$, $k \in \mathbb{N}$, $\gamma_k = \frac{g(k\pi, 0)}{|g(k\pi, 0)|}$. As $\sin(k\pi) = 0$,

$$\gamma_k \to (0, 1) \qquad \text{as } k \to \infty.$$

This shows that the limit of $\frac{g(s,0)}{|g(s,0)|}$ as $s \to +\infty$ does not exist. Note also that this example does not satisfy the assumptions in [2], because g vanishes as $|x|$ and $|y|$ tend to ∞.

4 Acknowledgments

We would like to thank Prof. Rafael Ortega for the enlightening conversations we had both at the Department of Mathematics at the University of Buenos Aires, and the Department of Applied Mathematics at the University of Granada. Also, we thank Prof. Pedro Torres for his fruitful comments on the manuscript.

References

[1] V. Ambrosetti, A. Coti Zelati. Periodic Solutions of Singular Lagrangian Systems, Birkhäuser, Boston, 1993.

[2] P. Amster and P. De Nápoli. *Landesman-Lazer type conditions for a system of p-Laplacian like operators*, J. Math. Anal. Appl. **326**(2):1236–1243, 2007.

[3] A. Bahri and P. Rabinowitz. *A minimax method for a class of hamiltonian systems with singular potentials*, J. Funct. Anal. **82**(2):412–428, 1989.

[4] J. Chu, P. Torres, and M. Zhang. *Periodic solutions of second order non-autonomous singular dynamical systems*, J. Differential Equations **239**(12):196–212, 2007.

[5] V. Coti Zelati. *Dynamical systems with effective-like potentials*, Nonlinear Anal. **12**(2):209–222, 1988.

[6] A. Fonda and R. Toader. *Periodic orbits of radially symmetric Keplerian-like systems: a topological degree approach,* J. Differential Equations **244**(12):3235–3264, 2008.

[7] W. Gordon. *Conservative dynamical systems involving strong forces*, Trans. Amer. Math. Soc. **204**:113–135, 1975.

[8] P. Hartman. *On boundary value problems for systems of ordinary nonlinear second order differential equations*, Trans. Amer. Math. Soc. **96**:493–509, 1960.

[9] E. M. Landesman and A. C. Lazer. *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19**:609–623, 1969/1970.

[10] A. C. Lazer. *On Schauder's fixed point theorem and forced second-order nonlinear oscillations*, J. Math. Anal. Appl. **21**:421–425, 1968.

[11] A. C. Lazer and S. Solimini. *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc. **99**(1):109–114, 1987.

[12] J. Mawhin. Topological Degree Methods in Nonlinear B oundary Value Problems, volume **40** of CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, R.I., 1979. Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977.

[13] L. Nirenberg. Generalized degree and nonlinear problems, In Contributions to Nonlinear Functional Analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971), pages 1–9. Academic Press, New York, 1971.

[14] R. Ortega and J. R. Ward Jr. *A semilinear elliptic system with vanishing nonlinearities*, Discrete Contin. Dyn. Syst., (suppl.):688–693, 2003. Dynamical Systems and Differential Equations (Wilmington, NC, 2002).

[15] D. Ruiz and J. R. Ward Jr. *Some notes on periodic systems with linear part at resonance,* Discrete Contin. Dyn. Syst. **11**(2& 3):337–350, 2004.

[16] S. Solimini. *On forced dynamical systems with a singularity of repulsive type*, Nonlinear Anal. **14**(6):489–500, 1990.

[17] P. Torres. *Non-collision periodic solutions of forced dynamical systems with weak singularities*, Discrete Contin. Dyn. Syst. **11**(2 & 3):693–698, 2004.

[18] M. Zhang. *Periodic solutions of damped differential systems with repulsive singular forces,* Proc. Amer. Math. Soc. **127**(2):401–407, 1999.