Continuous population models with diffusion

Single species

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Main aim

We discuss continuous population models starting from ODEs and passing by PDEs and arriving in nonlocal models. In this first class we will talk about the single species models. In the second one we will go from single species to two talking about systems.

Referências:

1. Murray, J. D.; Math. Biology, Springer-Verlag, N. York. 1989.
2. Cosner, C.; Reaction-Diffusion Eq. and Ecological Modeling. Tutorials in Math. Biosciences IV, Springer (2008) 77-116.
3. Henry, D.; Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. 840, Springer, 1981.

Continuous growth population

Let $N(t)$ be the population at time t. We can set its rate as

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}$$

introducing a kind of *conservation equation* for the population.

These models are of relevance to **laboratory studies**, and also to identify phenomena which can influence the **population dynamics**.

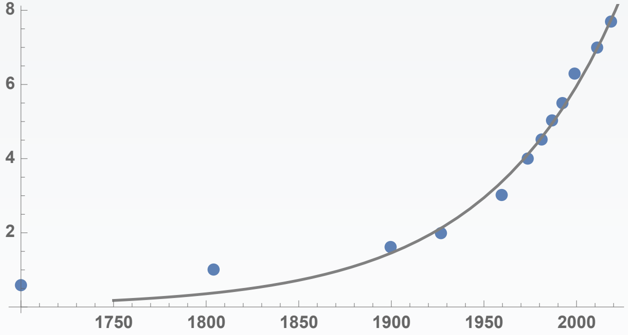
Without migration and setting birth and death proporcional to N, we get the **unrealistic approach** due to *Malthus* in 1789.

$$\frac{dN}{dt} = a\,N - b\,N \quad \Rightarrow \quad N(t) = N_0\, e^{(a-b)t}$$

where $a, b > 0$ and $N(0) = N_0$ is the population at $t = 0$.

- If $a > b$, the populations **grows exponentially**.
- If $a < b$, it is **annihilated**.

But, according to United Nations estimates, from 17th to 21st centuries, it is perhaps less unrealistic.



- It can be noticed exponential growth from 1900.[2]
- In 2017, the estimated annual growth rate was 1.1%.

[2] Here, we got a growth rate of 1.4%.

A self-limiting process called **logistic growth** was proposed by Verhulst (1838, 1845) in population models:

$$\frac{dN}{dt} = r\,N\left(1 - \frac{N}{K}\right); \qquad r,\ K > 0.$$

a) Here the per capita birth rate depends on N:

$$r\left(1 - \frac{N}{K}\right).$$

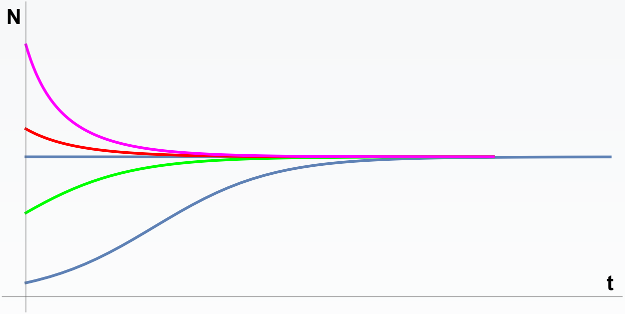
b) K is the **carrying capacity** of the environment which is set by the available sustaining resources.

Notice that K **determines** the size of the stable steady while r is a measure of the **rate** at which it is reached.

1. If $N(0) = N_0 > 0$ the **solution** is

$$N(t) = \frac{N_0\, K\, e^{r\,t}}{K + N_0\,(e^{r\,t} - 1)} \to K \quad \text{as } t \to \infty.$$

See the graph of N for some values of N_0.

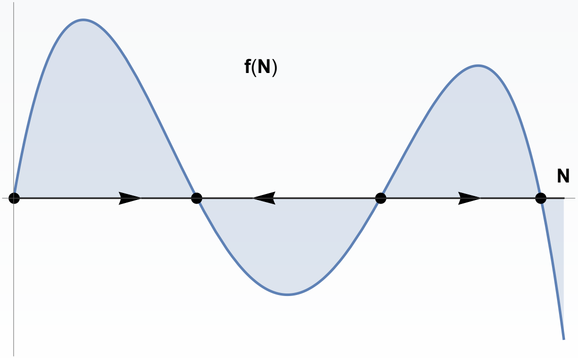


2. Here, the **phase space** of the system.

In general, we can consider a **single population** governed by

$$\frac{dN}{dt} = f(N).$$

- $f(N)$ is a typical **nonlinear** function;
- the **equilibrium** solutions N^* are solutions of $f(N) = 0$;
- N^* is **stable** if $f'(N^*) < 0$, and **unstable** as $f'(N^*) > 0$.

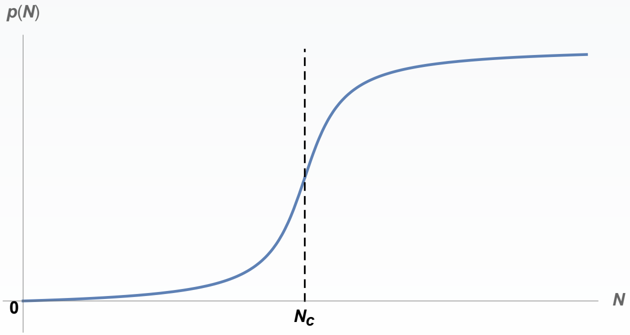


Insect outbreak model

In 1978 Ludwig et al. proposed the following equation to model budworm population

$$\frac{dN}{dt} = r_B\, N\,(1 - N/K_B) - p(N).$$

- r_B is the **linear birth rate**, and K_B the carrying capacity set by **density of foliage** available.
- $p(N)$ represents **predation**.



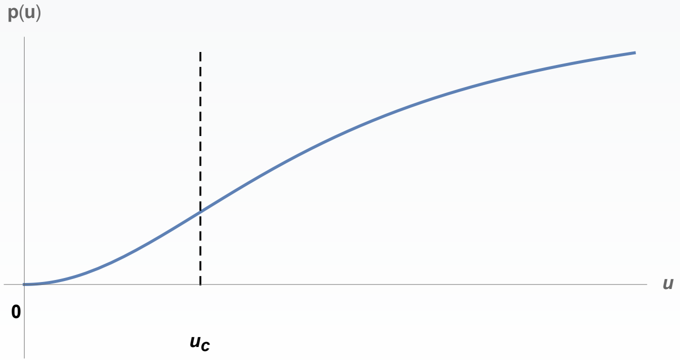
The original Ludwig et al. model:

$$\frac{dN}{dt} = r_B \, N \, (1 - N/K_B) - \frac{B \, N^2}{A^2 + N^2}$$

which can be rewritten as

$$\frac{du}{d\tau} = r \, u \left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}$$

taking $u = \frac{N}{A}$, $r = \frac{A \, r_B}{B}$, $q = \frac{K_B}{A}$ and $\tau = \frac{B \, t}{A}$.



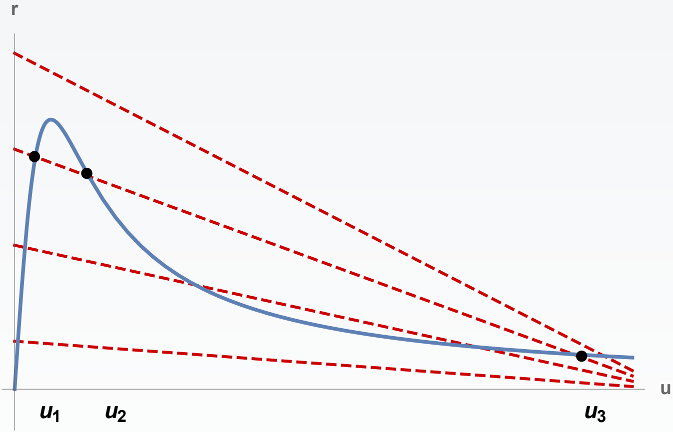
The steady state solutions are given by

$$f(u) = r\,u\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2} = 0.$$

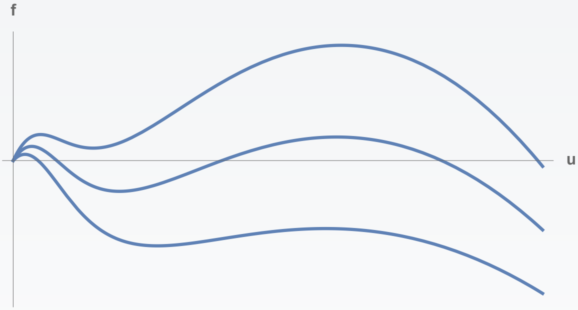
- $u_0 = 0$ is an equilibrium for any r and q.
- The other ones can be computed by

$$r\left(1 - \frac{u}{q}\right) = \frac{u}{1+u^2}$$

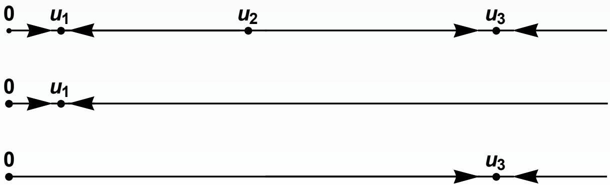
  and they will depend on parameters r and q.



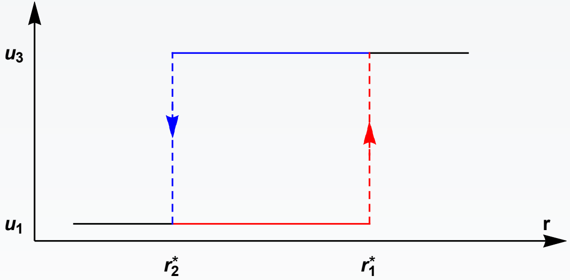
$$Blue(u) = \frac{u}{1+u^2} \qquad Red(u;r,q) = r\left(1 - \frac{u}{q}\right) \quad \text{assuming} \quad u \geq 0.$$



★ The space phase is gotten using the graph of f.



This model exhibits a **hysteresis effect**.



a) If r increases from zero, we have a jump from equilibrium u_1 to the **outbreak** equilibrium u_3 at $r_1^* >> 0$.

b) From large r to zero, we notice a jump from u_3 to u_1 at $r_2^* < r_1^*$.

c) This discontinuous behavior is called **catastrophe**.

Diffusion and random walks

'Diffusion is a description of movement that arises as a result of an object or organism making many short movements in random directions.'[a]

[a] C. Cosner.

If $p(x,t)$ is the **probability** of being at **location** x at time t, we have

$$p(x,t) = \frac{1}{2}p(x+\Delta x, t-\Delta t) + \frac{1}{2}p(x-\Delta x, t-\Delta t)$$

which leads us to

$$\frac{p(x,t) - p(x,t-\Delta t)}{\Delta t} = \frac{D}{(\Delta x)^2}\left[p(x+\Delta x, t-\Delta t) - 2p(x,t-\Delta t) + p(x-\Delta x, t-\Delta t)\right]$$

assuming

$$D = \frac{(\Delta x)^2}{2\Delta t}.$$

Taking limit as Δx and $\Delta t \to 0$, we get the **diffusion equation**

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}$$

which describes the probable location of a **single organism**.

- The **diffusion coefficient** D is $1/2$ of the square of the distance moved per unit time.
- Using **Fourier transform**, one can solve the equation for $p(x,t)$ starting at time $t = 0$ and at position $x = y$ as a **Dirac functional**

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-y)^2/4Dt}.$$

- Then, if we start with a **collection of organisms** at $t = 0$ with density $u_0(x)$, then the **density** at t is obtained by **averaging**

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4Dt} u_0(y)\, dy.$$

- Thus, if the organisms are just **moving** and **not dying or reproducing**, we get the **diffusion equation**

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad t > 0 \text{ and } x \in \mathbb{R},$$

for the density $u(x,t)$.

Diffusion and transport from Fick's law

i) Suppose a substance **flowing** and **diffusing** along a tube.

ii) A flux J sets the rate per unit area at which the substance is transported across a cross-section of the tube at point x.

iii) Call A the cross-sectional area of the tube.

iv) Assume J is **constant** on the cross-section.

★ If $u(x,t)$ is its **density**, then the rate of change of the amount of substance in $(x, x+\Delta x)$ at the time interval Δt is given by:

$$\frac{(A\Delta x)u(x,t+\Delta t) - (A\Delta x)u(x,t)}{\Delta t} = A[J(x) - J(x+\Delta x)]$$

which leads us to[3]

$$\frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x} \quad \text{as} \quad \Delta x,\, \Delta t \to 0.$$

[3]This expression can also be derived from the continuity equation which states that a change in density in any part of the system is due to inflow and outflow of material into and out of that part of the system.

a) If the transport is by **diffusion** Fick's law gives us

$$J = -D\frac{\partial u}{\partial x}.$$

b) If the transport arises from **advection** with velocity v

$$J = vu.$$

It yields a diffusion equation with advection:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \frac{\partial (vu)}{\partial x}.$$

Notice that the advective term **can be seen** as a description of **directed movement** along environment gradients.

Boundary conditions

For models involving diffusion in a **bounded interval** $\Omega \subset \mathbb{R}$, it is necessary to specify what happens at the boundary.

Typical boundary conditions with **null advection** involve setting:

- the density at the boundary, **Dirichlet** boundary conditions

$$u(x,t) = g(x), \quad x \in \partial\Omega;$$

- if the flux is specified by Fick's law, **Neumann** b.c.

$$J = -D\frac{\partial u}{\partial x} = g(x), \quad \text{on } \partial\Omega.$$

- As the flux is proportional to the density, we get **Robin** b.c.

$$-D\frac{\partial u}{\partial x} = \gamma u, \quad x \in \partial\Omega.$$

Reaction-diffusion models

Let $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$ and $u(x,t)$ setting the **density** of a substance or population at a position $x \in \Omega$ and time t:

$$\frac{\partial u}{\partial t} = D\Delta u + f(u) \quad \text{in } \Omega \times (0,\infty)$$
$$\alpha \frac{\partial u}{\partial n} + (1-\alpha)u = 0 \quad \text{on } \partial\Omega \times (0,\infty)$$
$$u(x,0) = u_0(x) \quad \text{on } \Omega$$

$\alpha \in [0,1]$; n is the **normal** vector of $\partial\Omega$; and Δ is the **Laplacian**

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}.$$

- Under smoothness conditions, u **exists** and is **unique** in $\Omega \times [0,T]$. Also, if $T < \infty$ and $f : \mathbb{R} \mapsto \mathbb{R}$, we have

$$\lim_{t \to T} \max_{\bar{\Omega}} |u(x,t)| = \infty.$$

Minimal patch size. [a]

[a] C. Cosner (2008).

We consider

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru \quad x \in (0, l)$$

with $u(0) = u(l) = 0$.

a) u is the **density** of a population in a patch of **length** l.
b) **Dirichlet** BC sets the region outside the patch is lethal.
c) D is the diffusion rate and r intrinsic **growth rate** of population.

⋆ Notice that we can solve this problem in terms of **eigenvalues** and **eigenfunctions**.

We have

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{(r - Dn^2\pi^2/l^2)t} \sin(n\pi x/l) \quad (x,t) \in (0,l) \times [0,\infty)$$

where c_n depends on the initial density $u(x,0)$,

$$\lambda_n = -n^2\pi^2/l^2 \quad \text{and} \quad \phi_n(x) = \sin(n\pi x/l) \qquad n = 1, 2, \ldots$$

are respectively the eigenvalues and eigenfunctions of

$$\begin{aligned} \phi_{xx} + \lambda\phi &= 0 \qquad x \in (0,l) \\ \phi(0) = \phi(l) &= 0. \end{aligned}$$

- ⋆ See that $u(x,t) \to 0$ as $t \to \infty$ if $r - Dn^2\pi^2/l^2 < 0$.
- ⋆ To predict **population growth**, we need $r - D\pi^2/l^2 > 0$, that is:

$$l > \pi\sqrt{D/r} \quad \text{or} \quad r > -\lambda_1 D$$

which gives us the **minimum patch size**.

A logistic example.[a]

[a] Page 95; D. Henry 1982.

Consider

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ru\left(1 - \frac{\rho}{r}u\right), \quad x \in (0,\pi)$$

with $u(0) = u(\pi) = 0$ for r and $\rho > 0$.

Maximum principle arguments shows

$$C = \{\phi \in H_0^1(0,\pi) \,:\, \phi \geq 0\}$$

is a positively invariant set. Indeed $A : D(A) \subset L^2(0,\pi) \mapsto L^2(0,\pi)$ with

$$Au(x) = -u_{xx} \quad \text{and} \quad D(A) = H^2 \cap H_0^1(0,\pi)$$

satisfies, whenever $u \in C$,

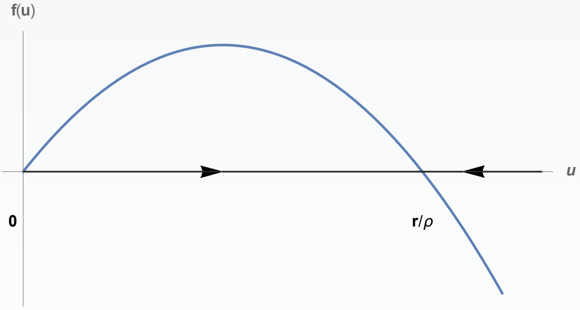
$$Au(x) \geq 0 \Rightarrow u(x) \geq 0, \quad \text{and then,} \quad e^{-At}u \geq 0 \quad \text{for all } t \geq 0.$$ [4]

[4] e^{-At} denotes the analytic semigroup of A.

Also, since $f(u) = ru\left(1 - \frac{\rho}{r}u\right) \geq 0$ for $u \in [0, r/\rho]$,

$$u(t, u_0) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(u(s))\, ds \geq 0$$

whenever $u_0 \in C$.



- See that $u(x) = r/\rho$ **does not belong** to $H_0^1(0, \pi)$! And then **it is not** an equilibrium solution.

Using the **Lyapunov function**

$$V(\phi) = \int_0^\pi \left\{ {\phi_x}^2 - r\phi^2 + \frac{2\rho}{3}\phi^3 \right\} dx \quad \text{for } \phi \in C$$

we get a **gradient** dynamical system. If $\phi(t)$ is a solution

$$\begin{aligned}
\frac{d}{dt}V(\phi(t)) &= 2\int_0^\pi \left\{ \phi_x\phi_{xt} - r\phi\phi_t + \rho\phi^2\phi_t \right\} dx \\
&= 2\int_0^\pi \left\{ -\phi_{xx}\phi_t - r\phi\phi_t + \rho\phi^2\phi_t \right\} dx + \phi_t\phi_x \Big|_{x=0}^{x=\pi} \\
&= 2\int_0^\pi \phi_t \left\{ -\phi_{xx} - r\phi + \rho\phi^2 \right\} dx \\
&= -2\int_0^\pi {\phi_t}^2 dx \leq 0.
\end{aligned}$$

Further, by Poincaré and Hölder Inequality, one can get

$$V(\phi) \geq (1-r)\|\phi\|^2_{L^2(0,\pi)} + \frac{2\rho}{3\sqrt{\pi}}\|\phi\|^3_{L^2(0,\pi)}.$$

Then, by **La Salle invariance principle**, we get

$$\lim_{t\to+\infty} u(t,u_0) \in \mathcal{M}$$

where

$$\mathcal{M} = \{u \in C \ : \ u \text{ is a critical point of } V\}$$

which is the non-negative solutions of

$$\begin{cases} \phi_{xx} + r\phi - \rho\phi^2 = 0 \\ \phi(0) = \phi(\pi) = 0 \end{cases} \qquad (*).$$

More precisely:

- If $0 < r \leq 1$, $V(\phi) \geq \frac{2\rho}{3\sqrt{\pi}} ||\phi||^3_{L^2(0,\pi)}$, and then $\mathcal{M} = \{0\}$.
- As $r > 1$, $\mathcal{M}$ is 0 and the unique positive solution ϕ^+ of $(*)$.

⋆ Previous example ensures $r > \lambda_1 D = 1$ implies **zero unstable**.[a]

[a] λ_1 here is the first eigenvalue of $-\phi_{xx}$ with Dirichlet BC.

Indeed, if $\phi \in \mathcal{M}$, it satisfies equation

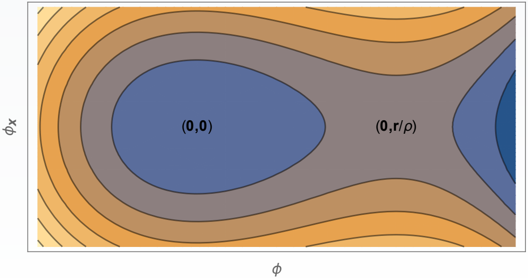
$$\phi_x^2 + r\phi^2 - 2\rho/3\phi^3 = \text{Const.}$$

which leads us to **level curves** of

$$F(u,v) = v^2 + ru^2 - 2\rho/3u^3$$

and ODE

$$\begin{cases} u_x = v \\ v_x = -ru + \rho u^2 \end{cases} .$$



Since $\phi(0) = \phi(\pi) = 0$, we are looking for a **closed curve** with

$$\phi_x = \sqrt{rc^2 - 2\rho/3c^3 - r\phi^2 + 2\rho/3\phi^3}$$

where $c = \max_{x\in(0,\pi)} \phi(x)$ is **attached** at $x = \pi/2$.

In a **neighborhood** of $(0, 0)$

$$\begin{aligned} x &= \int_0^x ds \\ &= \int_0^c \frac{d\phi}{\sqrt{rc^2 - 2\rho/3c^3 - r\phi^2 + 2\rho/3\phi^3}} \\ &\simeq \int_0^c \frac{d\phi}{\sqrt{rc^2 - r\phi^2}} = \frac{1}{\sqrt{r}}\frac{\pi}{2} \end{aligned}$$

with **change of variable** $\sin\theta = \frac{\phi}{c}$ which **works** since $0 < \frac{\phi}{c} \leq 1$.

★ Here **we get** $0 < x < \pi/2$ for $r > 1$.

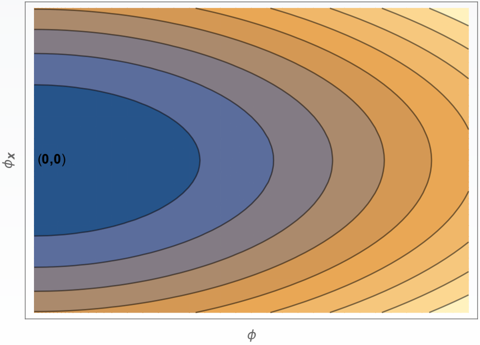
As, for $0 < \phi < r/\rho$, **we have**

$$\frac{d\phi_x}{dx} = -r\phi + \rho\phi^2 \quad \Rightarrow \quad \frac{dx}{d\phi_x} = \frac{1}{\phi(\rho\phi - r)}$$

there exist unique $(0, v)$ with $v > 0$ in the **phase space** such that

$$\begin{cases} \phi_{xx} + r\phi - \rho\phi^2 = 0 \\ \phi(0) = \phi(\pi) = 0 \end{cases}$$

since x **assume values** less than $\pi/2$ and near to $+\infty$ for $r > 1$.



Now let us **see** that **zero** is an **unstable** equilibrium. Set

$$s(t) = \int_0^{\pi} \sin x\, u(x,t)\, dx$$

where u is the **solution** of

$$u_t = u_{xx} + ru - \rho u^2, \quad x \in (0,\pi)$$
$$u(0) = u(\pi) = 0$$

with **initial condition** $u(x,0) = \phi(x)$. Then

$$\begin{aligned} s'(t) &= \int_0^{\pi} \Big(u_{xx}(x,t) + ru(x,t) + \rho u^2(x,t)\Big) \sin x\, dx \\ &= s(t)(r-1) - \rho \int_0^{\pi} u^2(x,t) \sin x\, dx \\ &\simeq s(t)(r-1) \qquad \text{if } \phi \simeq 0. \end{aligned}$$

Thus,

$$\lim_{t \to +\infty} s(t) = \lim_{t \to +\infty} s(0) e^{(r-1)t} = +\infty$$

since $r > 1$ and $\phi > 0$ is **small enough** which implies that

$u(x,t) \equiv 0$ is **unstable** for any **small** initial condition $\phi > 0$.

We conclude from La Salle principle

- $lim_{t \to +\infty} ||u(t)||_{H_0^1(0,\pi)} = 0$ as $0 < r \leq 1$.
- $lim_{t \to +\infty} ||u(t)||_{H_0^1(0,\pi)} = \phi^+$ as $1 < r$

★ **Recent results** in bounded sets of $\mathbb{R}^N$ can be see at Arrieta, Pardo, Rodriguez-Bernal, JDE (2015).

Continuous population models with diffusion

Nonlocal models

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[1]Partially supported by FAPESP 2017/02630-2 and CNPq 303253/2017-7.

Main aim

In this second class we discuss eigenvalue problems to nonlocal equations going to nonlocal equations and systems.

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2. Hutson, V.; Martinez, S.; Mischaikow, K.; Vickers, G. T.; The evolution of dispersal, J. Math. Biology (2003).
3. Fife, P.; Some nonclassical trends in parabolic and parabolic-like evolutions (2003).

A nonlocal model for dispersal

'a class of model more general than difusion'[a]

[a] Hutson, Martinez et al. (2003)

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - \int_{\mathbb{R}^N} J(x-y)u(x,t)dy, \qquad x \in \Omega \subset \mathbb{R}^N$$

- ⋆ Here u is the **density** at $x \in \Omega$ and $J(x-y)$ is the **probability distribution** of jumping from location y to location x.
- ⋆ **Dirichlet** condition is set by $u(x,t) \equiv 0$ whenever $x \in \mathbb{R}^N \setminus \Omega$.

A nonlocal model for dispersal

'a class of model more general than difusion'[a]

[a] Hutson, Martinez et al. (2003).

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - \int_{\mathbb{R}^N} J(x-y)u(x,t)dy, \qquad x \in \Omega \subset \mathbb{R}^N$$

We assume

$$(\mathbf{H_J}) \qquad \begin{array}{c} J \in C(\mathbb{R}^N, \mathbb{R}) \text{ is non-negative with } J(0) > 0, \\ J(-x) = J(x) \text{ for every } x \in \mathbb{R}^N \text{ and } \int_{\mathbb{R}^N} J(x)\,dx = 1. \end{array}$$

1. $\int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate which individuals arrive at position x.
2. $-\int_{\mathbb{R}^N} J(x-y)u(x,t)dy$ is the rate which they leave location x.

From local to nonlocal models

'natural nonlocal counterpart of Δu'[a]

[a] P. Fife (2003).

$-\Delta u$ on Dirichlet BC can also be characterized by

$$\mathcal{E}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 dx, \qquad u \in H_0^1(\Omega),$$

since

$$\frac{d}{dt}\mathcal{E}[u(t)] = \int_\Omega \nabla u(t) \cdot \nabla u_t(t)\, dx = -\int_\Omega \Delta u\, \phi\, dx$$

for instance to $u(t) = u + t\phi \in H_0^1(\Omega)$. Hence, its Gateaux derivative is

$$\mathcal{E}'[u] = -\Delta u.$$

⋆ In this context, $\mathcal{E}$ **measures** how much u **deviates** from zero.

From local to nonlocal models

'natural nonlocal counterpart of Δu'[a]

[a] P. Fife (2003).

A natural energy being an alternative measure is

$$\mathcal{E}_l = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{4} J(x-y)(u(x)-u(y))^2 dxdy$$

for $u(x) \equiv 0$ whenever $x \in \mathbb{R}^N \setminus \Omega$ since

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_l[u(t)] &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))(u_t(x)-u_t(y))\, dydx \\
&= \frac{1}{2}\Big[\int_{\mathbb{R}^N} u_t(x)\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))dydx \\
&\qquad - \int_{\mathbb{R}^N} u_t(y)\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))dydx\Big] \\
&= \int_{\mathbb{R}^N} u_t(x)\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))dydx.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{E}_l'[u] &= \int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))dydx \\
&= -\int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))dydx \\
&= -\left[\int_{\mathbb{R}^N} J(x-y)u(y)dy - \int_{\mathbb{R}^N} J(x-y)u(x)dy\right]
\end{aligned}$$

for $x \in \Omega$ taking as before

$$u(t) = u + t\phi \quad \text{in } L^2(\mathbb{R}^N)$$

with $u(t) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

⋆ Thus, $\mathcal{E}_l$ also **measures** how much u **deviates** from zero.

Indeed, from **Andreu-Vaillo et al. (2010)**, if Ω is smooth and **bounded**,

$$J_\delta(x) = C_1 \frac{1}{\delta^{N+2}} J(x/\delta) \quad \text{with} \quad C_1 = \left(\frac{1}{2} \int_{\mathbb{R}^N} J(x) x_N^2 \, dx \right)^{-1}$$

and u^δ is **solution** of

$$u_t = \int_{\mathbb{R}^N} J_\delta(x-y)(u(y,t) - u(x,t)) dy dx \quad x \in \Omega$$

with $u(x,0) = u_0(x)$ in Ω and $u(x,t) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, then

$$\sup_{t\in[0,T]} \|u^\delta(t) - v(t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } \delta \to 0$$

where v is the **solution** of

$$v_t = \Delta v \quad \text{in } \Omega$$

with $v = 0$ on $\partial\Omega$ and $v(x,0) = u_0(x)$ in Ω.

⋆ **Clue**: Performing the change $z = (x-y)/\delta$ and using **Taylor expansion**

$$
\begin{aligned}
&\int_{\mathbb{R}^N} J_\delta(x-y)(u(y)-u(x))dy \\
&= C_1 \frac{1}{\delta^{2+N}} \int_{\mathbb{R}^N} J\left(\frac{x-y}{\delta}\right)(u(y)-u(x))\, dy \\
&= \frac{C_1}{\delta^2} \int_{\mathbb{R}^N} J(z)\,(u(x-\delta z)-u(x))\, dz \\
&= \frac{C_1}{\delta^2}\left[\delta \sum_{i=1}^{N} \partial_i u(x) \int_{\mathbb{R}^N} J(z) z_i\, dz + \frac{\delta^2}{2} \sum_{i,j=N} \partial^2_{ij} u(x) \int_{\mathbb{R}^N} J(z) z_i z_j\, dz\right] + O(\delta) \\
&= \Delta u(x) + O(\delta)
\end{aligned}
$$

since J is **even**, and then,

$$
\int_{\mathbb{R}^N} J(z)\, z_i\, dz = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} J(z)\, z_i^2\, dz = \int_{\mathbb{R}^N} J(z)\, z_N^2\, dz < \infty.
$$

Principal eigenvalue to Dirichlet problem.[a]

[a] J. García-Melián and J. D. Rossi (2009)

We consider

$$\int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))dy = -\lambda u(x) \quad x \in \Omega \qquad (*)$$

with $u(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

First let us notice that λ and u satisfies $(*)$ **if and only if** satisfies

$$L_0 u = (1-\lambda)u$$

with

$$L_0 u(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy, \quad u(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

As L_0 is **self-adjoint**, positive and **compact**[2] in $L^2(\Omega)$

[2] By Arzelà-Ascoli.

Then

- ⋆ L_0 possesses a unique simple **eigenvalue** λ associated to a **positive eigenfunction** $u \in C(\bar{\Omega})$ such that $|\lambda| = \|L_0\|$. Thus

$$\lambda_1(\Omega) = 1 - \|L_0\| < 1.$$

Moreover

$$\begin{aligned}
-\lambda_1(\Omega) \int_{\mathbb{R}^N} u^2(x)\, dx &= \int_{\mathbb{R}^N} u(x)\, ((L_0 u)(x) - u(x))\, dx \\
&= \int_{\mathbb{R}^N} u(x) \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x)) dy dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))^2 dy dx < 0
\end{aligned}$$

which implies

$$\lambda_1(\Omega) > 0.$$

Further

$$||L_0||^2 = \sup_{\phi\neq 0} \frac{||L_0\phi||^2_{L^2}}{||\phi||^2_{L^2}} = \sup_{\phi\neq 0} \frac{\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} J(x-y)\phi(y)dy\right)^2 dx}{||\phi||^2_{L^2}}$$

$$||L_0|| = \sup_{\phi\neq 0} \frac{|<L_0\phi,\phi>|}{||\phi||^2_{L^2}} = \sup_{\phi\neq 0} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\phi(x)\phi(y)dydx}{||\phi||^2_{L^2}}$$

which gives us

$$\begin{aligned}\lambda_1(\Omega) &= 1-\left(\sup_{\phi\neq 0} \frac{\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} J(x-y)\phi(y)dy\right)^2 dx}{||\phi||^2_{L^2}}\right)^{1/2} \\ &= 1-\sup_{\phi\neq 0} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\phi(x)\phi(y)dydx}{||\phi||^2_{L^2}}.\end{aligned}$$

Also

$$\frac{1}{2}\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(\phi(x)-\phi(y))^2 dxdy}{||\phi||_{L^2}^2} = 1 - \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\phi(x)\phi(y)dydx}{||\phi||_{L^2}^2}$$

and then,

$$\lambda_1(\Omega) = \inf_{\phi\neq 0}\frac{1}{2}\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(\phi(x)-\phi(y))^2 dxdy}{||\phi||_{L^2}^2}.$$

$\lambda_1(\Omega)$ is decreasing.

- ★ If $\Omega_1 \subsetneq \Omega_2$, we have $L^2(\Omega_1) \subset L^2(\Omega_2)$. **Hence** $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$. Indeed, $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$ since the eigenfunctions are **strictly positive**.
- ★ Indeed, $\lambda_1(\Omega) \to 1$ as $|\Omega| \to 0$ and $\lambda_1(\Omega) \to 0$ as $\Omega \to \mathbb{R}^N$.

Existence and uniqueness

Let us consider

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))dy + f(x,t) \quad x \in \Omega$$

with $u(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, $u(x,0) = u_0(x)$ and $f \in C(\mathbb{R}; L^2(\Omega))$.

Then, there exists $u : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ such that for all $[a,b] \subset \mathbb{R}$

$$u \in C^1([a,b], L^2(\mathbb{R}^N)) \quad \text{with} \quad u(x) \equiv 0 \text{ for } x \in \mathbb{R}^N \setminus \Omega,$$

satisfying the equation in an **integral sense**

$$u(x,t) = e^{-t}u_0(x) + \int_0^t e^{-(t-s)} \int_{\mathbb{R}^N} J(x-y)\,u(y,s)\,dyds + \int_0^t e^{-(t-s)} f(x,s)\,ds \qquad (x,t) \in \mathbb{R} \times \Omega$$

since

$$\int_{\mathbb{R}^N} J(x-y)\,dy = 1 \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover, **there exist** positive α and C, such that

$$\|u(\cdot,t)\|_{L^2(\Omega^\epsilon)} \leq e^{-\alpha t}\left[\|u_0\|_{L^2(\Omega)} + C\int_0^t \|f(\cdot,s)\|^2_{L^2(\Omega)}\,ds\right], \quad t>0.$$

It is not difficult to see that the solutions can be written in the **integral form** being obtained as the **fixed point** of

$$F(u)(x,t) = e^{-t}u_0(x) + \int_0^t e^{-(t-s)}\int_{\mathbb{R}^N} J(x-y)\,u(y,s)\,dyds + \int_0^t e^{-(t-s)} f(x,s)\,ds, \quad x\in\Omega.$$

The proof that F possesses a fixed point **globally defined** is standard.

Also, **estimate** is obtained from the **energy**

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^N} (u(x,t))^2\, dx = \frac{1}{2}\, \|u(\cdot,t)\|^2_{L^2(\Omega)}\,.$$

For any $\delta > 0$, we get by Young Inequality

$$\begin{aligned}
H'(t) &= \int_{\mathbb{R}^N} u(x,t)u_t(x,t)\, dx \\
&= \int_{\mathbb{R}^N} u(x,t)\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dydx + \int_{\mathbb{R}^N} u(x,t)f(x,t)dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))^2 dydx + \int_{\mathbb{R}^N} u(x,t)f(x,t)\, dx \\
&\quad \le 2(\delta^2 - \lambda_1(\Omega))H(t) + \delta^{-2}\|f(\cdot,t)\|^2_{L^2(\Omega)}
\end{aligned}$$

where λ_1 is the **principal eigenvalue**. Thus,

$$H(t) \le e^{2(\delta^2-\lambda_1(\Omega))t}\left[H(0) + \delta^{-2}\int_0^t \|f(s,\cdot)\|^2_{L^2(\Omega)} ds\right]$$

completing the **proof** since $\lambda_1(\Omega) > 0$.

An **analogous proof** can be performed assuming

$$f : \mathbb{R} \mapsto \mathbb{R} \text{ \textbf{globally Lipschitz}}.$$

Semilinear equations

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))dy + f(u) \quad x \in \Omega$$

with $u(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and **initial condition** $u(x,0) = u_0(x)$.

1. **Nonlinearities** as $f(u) = u^p$ are also studied. Solutions are global when $0 \leq p \leq 1$ **blowing up** for $p > 1$ and initial conditions in $L^\infty(\Omega)$.[3]
2. Indeed, existence, uniqueness and **monotonicity properties** setting f **locally Lipschitz** satisfying **sign condition** as

$$f(u)u \leq Cu^2 + D|u|$$

for $C \in \mathbb{R}$ and $D > 0$ are also considered.[4]

---

[3] Pérez-Llanos and Rossi (2009).

[4] Sastre-Gomez, PhD thesis (2016).

Nonlocal minimal patch size.

Now, let us consider

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy + ru(x,t) \quad x \in \Omega$$

with $u(x,t) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

a) u is the **density** of a population in a patch Ω.

b) Dirichlet BC sets the region outside the patch is **lethal**.

c) r is the **intrinsic growth** rate of population.

As before, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}\right) = \int_{\mathbb{R}^N} u(x,t)\left[\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy + ru(x,t)\right]dx$$
$$= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))^2 dydx + r\int_{\mathbb{R}^N} u^2(x,t)\,dx$$
$$\leq 2(r-\lambda_1(\Omega))\,\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}$$

which **implies**

$$\|u(t)\|_{L^2(\Omega)} \leq e^{(r-\lambda_1(\Omega))t}\|u(0)\|_{L^2(\Omega)}.$$

Moreover, if $\phi(x)$ is the **eigenfunction** associated to $\lambda_1(\Omega)$ and

$$v(x,t) = e^{(r-\lambda_1(\Omega))t}\phi(x)$$

we have

$$\begin{aligned} v_t(x,t) &= (r-\lambda_1(\Omega))\phi(x)e^{(r-\lambda_1(\Omega))t} \\ &= \left[r\phi(x) + \int_{\mathbb{R}^N} J(x-y)(\phi(y)-\phi(x))dy \right] e^{(r-\lambda_1(\Omega))t} \\ &= rv(x,t) + \int_{\mathbb{R}^N} J(x-y)(v(y,t)-v(x,t))dy \end{aligned}$$

with **initial condition** $v(x,0) = \phi(x)$ and $v(x,t) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

Therefore, from

1. $||u(t)||_{L^2(\Omega)} \leq e^{(r-\lambda_1(\Omega))t}||u(0)||_{L^2(\Omega)}$
2. $v(x,t) = e^{(r-\lambda_1(\Omega))t}\phi(x)$ being a solution

we get

a) If the intrinsic rate growth $r < \lambda_1(\Omega)$ the null function is **asymptotically stable** and population is **extinct**.

b) As $r > \lambda_1(\Omega)$, zero is **unstable** with

$$\lim_{t\to+\infty} v(x,t) \to +\infty, \qquad \text{for all } x \in \Omega$$

since $\phi(x) > 0$ in Ω.

c) Intrinsic rate growth **bigger than** 1 implies null function **unstable** whatever Ω is. In this sense nonlocal equations **further** population growth.

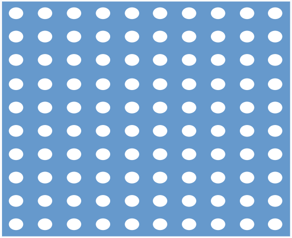
We discuss perforated domains on Dirichlet BC

$$u_t^\epsilon(x,t) = \int_{\mathbb{R}^N} J(x-y)(u^\epsilon(y,t) - u^\epsilon(x,t))dy + ru^\epsilon(x,t) \qquad x \in \Omega^\epsilon,\ t \in \mathbb{R}$$

with $u^\epsilon(x,t) \equiv 0$ in $x \in \mathbb{R}^N \setminus \Omega^\epsilon$ and $u^\epsilon(0,x) = u_0(x)$.

Let $\Omega^\epsilon \subset \Omega \subset \mathbb{R}^N$ be a **family of bounded sets** for $\epsilon > 0$.

- ⋆ $\chi_\epsilon \in L^\infty$ is the **characteristic function** of Ω^ϵ with $\chi_\epsilon \rightharpoonup \mathcal{X}$ weakly* in $L^\infty(\Omega)$.
- ⋆ **Holes** are given by $A^\epsilon = \Omega \setminus \Omega^\epsilon$.



Ω^ϵ

Recall $\chi_\epsilon \rightharpoonup \mathcal{X}$ weakly* in $L^\infty(\Omega)$ as $\epsilon \to 0$ as

$$\int_\Omega \chi_\epsilon(x)\,\varphi(x)\,dx \to \int_\Omega \mathcal{X}(x)\,\varphi(x)\,dx \qquad \forall \varphi \in L^1(\Omega).$$

Periodically Perforated Domain. Let $Q \subset \mathbb{R}^N$ be the *representative cell*

$$Q = (0, l_1) \times (0, l_2) \times ... \times (0, l_N).$$

★ We perforate Ω removing a set A^ϵ of periodically distributed holes.

1. Take any open set $A \subset Q$ such that $T = Q \setminus A$ and $|T| \neq 0$.
2. Denote by $\tau_\epsilon(A)$ the set of all **translated images** of $\epsilon\bar{A}$ of the form $\epsilon(kl + A)$ where $k \in \mathbb{Z}^N$ and $kl = (k_1 l_1, ..., k_N l_N)$.
3. Now define the holes inside Ω by

$$A^\epsilon = \Omega \cap \tau_\epsilon(A).$$

We introduce our **perforated domain** as

$$\Omega^\epsilon = \Omega \setminus A^\epsilon.$$

★ Considering Ω^ϵ we have removed from Ω a large number of holes of size $|\epsilon\bar{A}|$ which are ϵ-periodically distributed.

★ Let us get the characteristic function χ_ϵ to Ω^ϵ.

If χ_A be the characteristic function of A **periodically extended** in $\mathbb{R}^N$, and χ_{A^ϵ} is **the characteristic function** of A^ϵ, for each $x \in A^\epsilon$

$$\chi_{A^\epsilon}(x) = \chi_A\left(\frac{x - \epsilon k l}{\epsilon}\right) = \chi_A(x/\epsilon), \quad \text{for some } k \in \mathbb{Z}^N.$$

Therefore, if χ_Ω and χ_ϵ are the characteristic functions of Ω and Ω^ϵ

$$\chi_\epsilon(x) = \chi_\Omega(x) - \chi_{A^\epsilon}(x).$$

It follows from the **Average Theorem**

$$\chi_{A^\epsilon} \rightharpoonup \frac{1}{|Q|}\int_Q \chi_A(s)\, ds = \frac{|A|}{|Q|} \quad \text{weakly}^* \text{ in} L^\infty(\Omega). \tag{1}$$

Hence,

$$\chi_\epsilon \rightharpoonup \frac{|Q \setminus A|}{|Q|} \quad \text{weakly}^* \text{ in } L^\infty(\Omega).$$

Thus, we can set

$$\mathcal{X}(x) = \frac{|Q \setminus A|}{|Q|}\chi_\Omega(x) \quad \text{in } \mathbb{R}^N.$$

Let us pass to the limit

$$u_t^\epsilon(x,t) = \int_{\mathbb{R}^N} J(x-y)(u^\epsilon(y,t) - u^\epsilon(x,t))dy + ru^\epsilon(x,t) \qquad x \in \Omega^\epsilon, \ t \in \mathbb{R}$$

with $u^\epsilon(x,t) \equiv 0$ in $x \in \mathbb{R}^N \setminus \Omega^\epsilon$, $u^\epsilon(0,x) = u_0(x)$ and

$$\chi_\epsilon \rightharpoonup \mathcal{X} \quad \text{weakly}^* \text{ in } L^\infty(\Omega).$$

⋆ First, we notice there exists $K > 0$, **independent** of $\epsilon > 0$, such that

$$\sup_{t\in[a,b]} \|u^\epsilon(t,\cdot)\|_{L^2(\Omega)} = \sup_{t\in[a,b]} \|u^\epsilon(t,\cdot)\|_{L^2(\Omega^\epsilon)} \leq K$$

for **any** bounded $[a,b] \subset \mathbb{R}$.

⋆ Now, since $L^1\left([a,b]; L^2(\Omega)\right)$ is **separable**, we can extract a **subsequence**, still denoted by u^ϵ, such that

$$u^\epsilon \rightharpoonup u^* \text{ weakly}^* \text{ in } L^\infty([a,b]; L^2(\Omega)),$$

for some $u^* \in L^\infty\left([a,b]; L^2(\Omega)\right)$.

⋆ Set $[a, b] = [0, T]$. We pass to the limit in the variational formulation

$$\begin{aligned}\int_\Omega \varphi(x)\, u^\epsilon(x,t)\, dx &= \int_\Omega \varphi(x)\, e^{-t}\, \chi_\epsilon(x)\, u_0(x)\, dx \\ &+ \int_\Omega \varphi(x) \int_0^t e^{-(t-s)} r u^\epsilon(x,s)\, dsdx \\ &+ \int_\Omega \varphi(x)\, \chi_\epsilon(x) \int_0^t e^{-(t-s)} \int_{\mathbb{R}^N} J(x-y)\, u^\epsilon(y,s)\, dydsdx \\ &= I_1^\epsilon + I_2^\epsilon + I_3^\epsilon\end{aligned}$$

as $\epsilon \to 0$ for any $\varphi \in L^2(\Omega)$.

⋆ One can show that:

$$I_1^\epsilon = \int_\Omega \varphi(x)\, e^{-t}\, \chi_\epsilon(x)\, u_0(x)\, dx \to \int_\Omega \varphi(x)\, e^{-t}\, \mathcal{X}(x)\, u_0(x)\, dx$$

$$I_2^\epsilon = \int_\Omega \varphi(x) \int_0^t e^{-(t-s)} r u^\epsilon(x,s)\, dsdx \to \int_\Omega \varphi(x) \int_0^t e^{-(t-s)} r u^*(x,s)\, dsdx$$

and

$$\begin{aligned} I_3^\epsilon &= \int_\Omega \varphi(x)\, \chi_\epsilon(x) \int_0^t e^{-(t-s)} \int_{\mathbb{R}^N} J(x-y)\, u^\epsilon(y,s)\, dydsdx \\ &\to \int_\Omega \varphi(x)\, \mathcal{X}(x) \int_0^t e^{-(t-s)} \int_{\mathbb{R}^N} J(x-y)\, u^*(y,s)\, dydxds \end{aligned}$$

as $\epsilon \to 0$.

Thus, we obtain

$$\int_\Omega \varphi(x)\, u^*(x,t)\, dx = \int_\Omega \varphi(x) \left[e^{-t}\, \mathcal{X}(x) u_0(x) + \int_0^t e^{-(t-s)}\, ru(x,s)\, ds \right] dx$$
$$+ \int_\Omega \varphi(x)\, \mathcal{X}(x) \int_0^t e^{-(t-s)} \int_{\mathbb{R}^N} J(x-y)\, u^*(y,s)\, dydsdx,$$

which implies

$$u^*(x,t) = e^{-t}\, \mathcal{X}(x)\, u_0(x) + \int_0^t e^{-(t-s)}\, ru(x,s)\, ds$$
$$+ \int_0^t e^{-(t-s)}\, \mathcal{X}(x) \int_{\mathbb{R}^N} J(x-y)\, u^*(y,s)\, dyds$$

for all $t \in [0,T]$ and a.e. x in Ω.

Notice $u^* \in C^1([0,T]; L^2(\Omega))$ satisfies

$$u_t^*(x,t) = \mathcal{X}(x) \int_{\mathbb{R}^N} J(x-y)\, u^*(t,y)\, dy + ru^*(x,t) - u^*(x,t),$$
$$u^*(0,x) = \mathcal{X}(x)\, u_0(x),$$

with $u^*(x,t) \equiv 0$ in $x \in \mathbb{R}^N \setminus \Omega$ whose equation can be rewritten as

$$\begin{aligned} u_t^*(x,t) &= \mathcal{X}(x) \int_{\mathbb{R}^N} J(x-y)(u^*(t,y) - u^*(x,t))dy \\ &\quad + u^*(x,t)(r - 1 + \mathcal{X}(x)) \qquad x \in \Omega,\ t \in \mathbb{R}. \end{aligned}$$

- ⋆ At the limit **perfurations** change the operator and **reaction term**.
- ⋆ And about **minimal patch** size in **periodically perforated domains**?

If we take **periodically perforated domains**, we know that

$$\mathcal{X}(x) = \frac{|Q \setminus A|}{|Q|} \chi_\Omega(x) \quad \text{in } \mathbb{R}^N$$

is a **positive constant** in Ω with

$$u_t^*(x,t) = \mathcal{X} \int_{\mathbb{R}^N} J(x-y)(u^*(t,y) - u^*(x,t))dy + u(x,t)(r - 1 + \mathcal{X}).$$

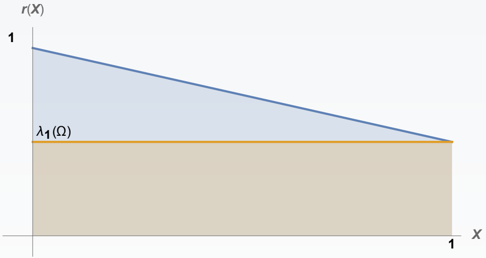
⋆ In order to get population growth we need
intrinsic growth bigger than **first eigenvalue**.

$$r - 1 + \mathcal{X} > \mathcal{X}\lambda_1(\Omega)$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the **original** nonlocal operator.

Then

$$r > \mathcal{X}(\lambda_1(\Omega) - 1) + 1.$$



⋆ Let us recall

$$0 < \lambda_1(\Omega) < 1 \quad \text{and } 0 \leq \mathcal{X} \leq 1.$$

Thus, perforations **do not further** population growth.

A logistic nonlocal equation.[a]

[a] Hutson et al. (2003).

Consider

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy + u(x,t)\,(a(x)-u(x,t)) \qquad (lp)$$

for $x \in [0,l]$ with $u(x,t) \equiv 0$ for $x \in \mathbb{R} \setminus [0,l]$.

We assume

- $J \in C^1(\mathbb{R},\mathbb{R})$ non-negative, with $J(0) > 0$, $J(-x) = J(x)$ for every $x \in \mathbb{R}$ and $\int_{\mathbb{R}^N} J(x)\,dx = 1$.
- $a \in C^1[0,l]$ strictly positive.

★ Thus, the intrinsic growth rate **depends on** $x \in \Omega = [0,l]$.

Strong Maximal Principle.

Let $c : \Omega \times [0, +\infty) \mapsto \mathbb{R}$ be continuous and $T > 0$. Assume u satisfies

$$\frac{\partial u}{\partial t} \geq \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy + c(x,t)\, u(x,t)$$

with $u(x,0) \geq 0$. Then

- **Either** $u > 0$ in $Q_T \cup S_T$
- **or** $\exists\, t^* \leq T$ such that $u = 0$ in $t \in [0, t^*]$ and $u > 0$ if $t > t^*$.

Here

$$Q_T = [0, l] \times \{T\} \quad \text{and} \quad S_T = [0, l] \times (0, T).$$

⋆ We say u_- is a **subsolution** if

$$\frac{\partial u}{\partial t} \leq \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy + u(x,t)\,(a(x)-u(x,t))$$

Supersolution is set similarly by reversing the inequality.

Comparison

Let u_- and u^- sub/supersolutions with $u_-(x,0) \leq u^-(x,0)$. Then

- **Either**

$$u_- < u^- \quad \text{in } Q_T \cup S_T$$

- **or** $\exists\, t^* \leq T$ such that

$$u_- = u^- \quad \text{in } Q_{t^*} \cup S_{t^*}.$$

Moreover:

1. Let u_- be a **stationary subsolution** and u a **solution** with $u(x,0) = u_-$. Then
   - **Either** u_- is a **solution**,
   - **or** u is **strictly increasing** in t for each $x \in [0,l]$.
2. If $u_0(x) \geq 0$ in $[0,l]$, then **there exists** u nonnegative solution to the **logistic equation** (lp), defined for all $t > 0$ with $u(x,0) = u_0(x)$.

Stability

Suppose **there exists** a nontrivial subsolution u_-.
Then (lp) has **exactly one**, strictly positive stationary solution, **globally stable** in the sense of **pointwise convergence**.

⋆ How can we get a subsolution?

If the **linearization about zero** possesses a **positive** principal eigenvalue

$$\int_{\mathbb{R}^N} J(x-y)(\phi(y)-\phi(x))dy + \phi(x)a(x) = \lambda_1 \phi(x), \quad x \in \Omega$$

we can set a **subsolution**. Indeed, let $\phi^\epsilon = \epsilon\phi$, then

$$\begin{aligned}
&\int_{\mathbb{R}^N} J(x-y)(\phi^\epsilon(y)-\phi^\epsilon(x))dy + \phi^\epsilon(x)a(x) - \phi^\epsilon(x)^2 \\
&= \int_{\mathbb{R}^N} J(x-y)(\phi^\epsilon(y)-\phi^\epsilon(x))dy + \phi^\epsilon(x)a(x) - \phi^\epsilon(x)^2 \pm \lambda_1\phi^\epsilon(x) \\
&= \phi^\epsilon(x)(\lambda_1 - \phi^\epsilon(x)) \\
&= \epsilon\phi(x)(\lambda_1 - \epsilon\phi(x)) \geq 0 = \frac{\partial}{\partial t}\phi(x)
\end{aligned}$$

as ϵ is **small** and $\lambda_1 > 0$.

Time-scale analysis non-local diffusion systems

We consider a **vector-borne** disease modeled by the Ross-Macdonald model incorporating spatial movements. The hosts can move **non-locally** and vectors can move **locally**.

a) Assuming **birth** and **mortality** rates are equal, we have that populations remain **constant** over time.

b) We also set host and vector dynamics in **different scales**.

We lead to the following **system**

$$
\begin{aligned}
di/dt &= \alpha_h(1-i)j - \beta_h i \\
\varepsilon\, dj/dt &= \alpha_v(1-j)i - \beta_v j
\end{aligned}
$$

for positive constants α_h, α_v, β_h and β_v.

- i and j are **human** and **vector** density.

The spatial movement for the vector will be modeled by the Laplacian with Neumann boundary condition and for the hosts our non-local operator K_J

$$K_J\, i(x) = \int_\Omega J(x-y)(i(y)-i(x))dy, \quad x \in \Omega.$$

Putting the local disease dynamics with the spatial dynamics, we get

$$\begin{cases} \dfrac{\partial i}{\partial t} = \alpha_h(1-i)j - \beta_h i + d_1 K_J\, i, \\ \\ \dfrac{\partial j}{\partial t} = \dfrac{\alpha_v}{\varepsilon}(1-j)i - \dfrac{\beta_v}{\varepsilon} j + d_2 \Delta j, \end{cases} \qquad x \in \Omega,\ t > 0.$$

with $\frac{\partial j}{\partial \vec{n}} = 0$ on $x \in \partial\Omega$.

Asymptotic Expansion

Here we use power series expansion to analyze in a formal way the asymptotic behavior of the system with respect to $\varepsilon > 0$.

We assume

$$i = i_0 + \varepsilon i_1 + \dots \quad \text{and} \quad j = j_0 + \varepsilon j_1 + \dots$$

and then,

$$\frac{di}{dt} = \frac{di_0}{dt} + \varepsilon \frac{di_1}{dt} + \dots \quad \text{and} \quad \frac{dj}{dt} = \frac{dj_0}{dt} + \varepsilon \frac{dj_1}{dt} + \dots$$

which gives us that

$$\begin{aligned}
\frac{di}{dt} &= [\alpha_h(1 - i_0)j_0 - \beta_h i_0 + d_1 K_J i_0] \\
&\qquad + \varepsilon [\alpha_h(j_1 - i_0 j_1 - i_1 j_0) - \beta_h i_1 + d_1 K_J i_1] + O(\varepsilon^2) \\
\frac{dj}{dt} &= [\alpha_v(1 - j_0)i_0 - \beta_v j_0] \\
&\qquad + \varepsilon [\alpha_v(i_1 - j_0 i_1 - j_1 i_0) - \beta_v i_1 + d_2 \Delta j_0] + O(\varepsilon^2)
\end{aligned}$$

Hence, if we plug these expressions in the system, we get at $\varepsilon = 0$ that

$$\begin{cases} \dfrac{\partial i_0}{\partial t} & = \alpha_h(1 - i_0)j_0 - \beta_h i_0 + d_1 K_J\, i_0, \\ \\ 0 & = \alpha_v(1 - j_0)i_0 - \beta_v j_0. \end{cases}$$

Consequently,

$$j_0 = m(i_0) = \frac{\alpha_v i_0}{\alpha_v i_0 + \beta_v}$$

and then, we deduce the reduced equation

$$\frac{\partial i_0}{\partial t} = \alpha_h(1 - i_0)m(i_0) - \beta_h i_0 + d_1 K_J\, i_0$$

with initial condition $i_0(t, x) = i_0(0, x)$.

$$i \approx i_0 \qquad \text{in } L^2(\Omega)$$

under **smooth** initial conditions and **finite time**.

The limit problem

Strong Maximum Principle works to the limit equation for **non-negative** continuous functions. Hence, we get a dynamical system behaving as

$$\frac{dz}{dt} = \alpha_h(1 - z)m(z) - \beta_h z.$$

- Thus, if

$$(\mathbf{H_C}) \qquad \alpha_h\alpha_v > \beta_h\beta_v$$

  the constant i_0^* is the unique **stationary** solution **globally stable** for solutions with non-trivial and non-negative **initial** conditions.

- On the other hand, if $\alpha_h\alpha_v \leq \beta_h\beta_v$, then the null function is the unique stationary and non-negative solution **globally stable**.

Indeed, we consider a more general system as:

$$\begin{cases} \dot{x} = f(x,y) + K_J x \\ \varepsilon \dot{y} = g(x,y) + \varepsilon \Delta y \end{cases} \quad \text{in } \Omega, \quad \varepsilon > 0,$$

with **homogeneous Neumann boundary** condition

$$\frac{\partial y}{\partial N} = 0 \quad \text{on } \partial\Omega.$$

- The nonlinearities f and $g : \mathbb{R}^2 \mapsto \mathbb{R}$ are smooth functions and will include the class of those ones discussed in the previous sections.
- As $\varepsilon \to 0$, we show **slow component** $x(t)$ converges to $X(t)$ which is governed by the **effective equation**

  $$\dot{X} = f(X, m(X)) + K_J X, \quad \text{in } \Omega,$$

  where $y = m(x)$ is the **graph** representation of a set given by

  $$g(x, m(x)) = 0.$$

- **Other work** with nonlocal systems: Bai, X.; Li, F.; Calculus of Variations (2018).

¡MUCHAS GRACIAS!