CLUSTERING FOR METRIC GRAPHS USING THE $p-$LAPLACIAN

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Abstract. We deal with the clustering problem in a metric graph. We look for two clusters, to this end we study the first non-zero eigenvalue of the $p-$Laplacian on a quantum graph with Newmann or Kirchoff boundary conditions on the nodes. Then, an associated eigenfunction u_p provides two sets inside the graph, $\{u_p > 0\}$ and $\{u_p < 0\}$ that define the clusters. Moreover, we describe in detail the limit cases $p \to \infty$ and $p \to 1$.

1. Introduction

One of the mayor problems for networks is that of clustering. Clustering in a network means that we want to identify dense regions of it maximizing or minimizing some criterium. Here we deal with metric graphs, Γ, that are graphs in which we have a length for the edges and try to identify two clusters. Our approach to find two clusters in Γ is based on the following idea: given u a sign-changing function defined on the graph just take $A = \{u > 0\}$ and $B = \{u < 0\}$ as clusters (note that the set $\{u = 0\}$ may be nontrivial and therefore it may happen that $A \cup B \neq \Gamma$). In this work we take u as being an eigenfunction for some differential operator, we take a $p-$Laplacian, $-(|u'|^{p-2}u')'$, defined on the graph and study properties of this approach. We find two extreme cases: for $p = \infty$ (this is understood as the limit as $p \to \infty$), A and B are sets that have diameter as large as possible (each one of them has diameter equal to ${}^{\mathrm{diam}(\Gamma)}/_2$); while for $p = 1$ (understood as the limit as $p \to 1$) we find that A and B are sets with large total length and small number of "cuts" in the graph (small perimeter).

A quantum graph is a graph in which we associate a differential law with each edge, that models the interaction between the two nodes defining each edge. The use of quantum graphs (in contrast to more elementary graph models, such as simple unweighted or weighted graphs) opens up the possibility of modeling the interactions between agents identified by the graph's vertices in a more detailed manner than with standard graphs. Quantum graphs are used to model thin tubular structures, so-called graph-like spaces, they are their natural limits, when the radius of a graph-like space tends to zero. On both, the graph-like spaces and the metric graph, we can naturally define Laplace-like differential operators. See [2, 16, 26].

Among properties that are relevant in the study of quantum graphs is the study of the spectrum of the associated differential operator. In particular, the so-called spectral gap (this concerns bounds for the first non-zero eigenvalue for the Laplacian

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with Neumann boundary conditions) has physical relevance and was extensively studied in recent years. We quote, for example, [16, 17, 19, 20].

In this paper we are interested in the eigenvalue problem that naturally arises when we consider the $p-$Laplacian, $(|u'|^{p-2}u')'$, as the differential law on each side of the graph together with Newmann or Kirchoff boundary conditions, see [15], at the nodes. To be concrete, we deal with the following problem: in a finite metric graph Γ we consider the minimization problem

$$\lambda_{2,p}(\Gamma) = \inf \left\{ \frac{\displaystyle\int_\Gamma |u'(x)|^p dx}{\displaystyle\int_\Gamma |u(x)|^p dx} : u \in W^{1,p}(\Gamma), \int_\Gamma |u|^{p-2}u(x)\,dx = 0,\, u \neq 0 \right\}. \tag{1.1}$$

There is a minimizer, see Theorem 1.1 below, that is a nontrivial sign-changing weak solution to

$$\begin{cases} -(|u'|^{p-2}u')'(x) = \lambda_{2,p}(\Gamma)|u|^{p-2}u(x) & \text{on the edges of } \Gamma, \\ \displaystyle\sum_{\mathrm{e}\in \mathrm{E_v}(\Gamma)} \left|\frac{\partial u}{\partial x_{\mathrm{e}}}\right|^{p-2} \frac{\partial u}{\partial x_{\mathrm{e}}}(\mathrm{v}) = 0 & \text{on the nodes.} \end{cases} \tag{1.2}$$

Our main results for this eigenvalue problem can be summarized as follows:

- For $1 < p < \infty$, we show that the infimum in (1.1) is attained at a sign-changing function. We provide examples that show that the set $\{u_p = 0\}$ may have nontrivial measure.
- We study the limit cases $p \to \infty$ and $p \to 1$. For $p = \infty$ we find a geometric characterization of the first non-zero eigenvalue and for $p = 1$ we prove that there exist the analogous of Cheeger sets in quantum graphs.

Now, let us present precise statements of our results. First, the following result follows by a standard compactness argument.

Theorem 1.1. *Let Γ be a connected compact metric graph. Then, the infimum in (1.1), $\lambda_{2,p}(\Gamma)$, is attained and is the first non-zero Neumann eigenvalue for the $p-$Laplacian in Γ, that is, $\lambda_{2,p}(\Gamma)$ is the smallest positive number such that there exists $u_p \in W^{1,p}(\Gamma)$ such that*

$$\int_\Gamma |u_p'(x)|^{p-2}u_p'(x)v'(x)dx = \lambda \int_\Gamma |u_p(x)|^{p-2}u_p(x)v(x)dx \tag{1.3}$$

for all $v \in W^{1,p}(\Gamma)$.

Concerning the limit as $p \to \infty$ we have the following result:

Theorem 1.2. *Let Γ be a connected compact metric graph, and u_p be a minimizer for (1.1) normalized by $\|u_p\|_{L^p(\Gamma)} = 1$. Let*

$$\Lambda_{2,\infty}(\Gamma) = \inf \left\{ \|v'\|_{L^\infty(\Gamma)} \colon \max_\Gamma v = \max_\Gamma -v = 1 \right\}. \tag{1.4}$$

Then,

$$\lim_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} = \Lambda_{2,\infty}(\Gamma)$$

and there exists a subsequence $p_j \to \infty$ such that

$$u_{p_j} \to u_\infty$$

uniformly in Γ *and weakly in* $W^{1,q}(\Gamma)$ *for every* $q < \infty$. *Moreover, any possible limit* u_∞ *is a minimizer for* (1.4).

This value $\Lambda_{2,\infty}(\Gamma)$ *can be characterized as*

$$\Lambda_{2,\infty}(\Gamma) = \frac{2}{\operatorname{diam}(\Gamma)}.$$

While for the limit as $p \to 1$ we have:

Theorem 1.3. *Let* Γ *be a connected compact metric graph, and* u_p *be a minimizer for* (1.1) *normalized by* $\|u_p\|_{L^1(\Gamma)} = 1$. *Then, there exists a subsequence* $p_j \to 1^+$ *and* $u_1 \in BV(\Gamma)$ *such that*

$$u_{p_j} \to u_1$$

in $L^1(\Gamma)$.

Moreover, any possible limit u_1 *is a minimizer for*

$$\Lambda_{2,1}(\Gamma) = \inf\left\{ \frac{\|v'\|_{BV(\Gamma)}}{\|v\|_{L^1(\Gamma)}} : v \in BV(\Gamma), \int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0, v \neq 0 \right\}. \tag{1.5}$$

This value $\Lambda_{2,1}(\Gamma)$ *is the limit of* $\lambda_{2,p}(\Gamma)$, *it holds that*

$$\lim_{p\to 1} \lambda_{2,p}(\Gamma) = \Lambda_{2,1}(\Gamma).$$

We also have an analogous to Cheeger sets for metric graphs.

Theorem 1.4. *Let* Γ *be a connected compact metric graph and* A *be a subset of* Γ *such that* $|A| = {}^{\ell(\Gamma)}/_2$ *and* $\operatorname{Per}(A) < \infty$. *Then*

$$\frac{2\operatorname{Per}(A)}{\ell(\Gamma)} = \inf\left\{ \frac{\operatorname{Per}(E)}{\min\{|E|, |\Gamma \setminus E|\}} : E \subsetneq \Gamma, E \neq \emptyset \right\} \tag{1.6}$$

if only if

$$u = \chi_A - \chi_{\Gamma\setminus A}$$

is a minimizer for $\Lambda_{2,1}(\Gamma)$.

As we have mentioned at the beginning of this introduction, for a metric graph one important problem is clustering. We want to identify two disjoint subsets of the graph Γ, A and B that are similar in size (here we have to define in which sense we measure the size of a subset of a metric graph) and such that the resulting partition of Γ minimizes or maximizes some criterium (also to be specified). We remark again that, in general, we are not prescribing that $\Gamma = A \cup B$, we can have $\Gamma \setminus (A \cup B) \neq \emptyset$.

For the case $p = +\infty$ we let $A_\infty = \{u_\infty > 0\}$ and $B_\infty = \{u_\infty < 0\}$ and we have that A_∞ and B_∞ are two subsets of Γ with the same diameter that maximizes this common diameter, that is,

$$\operatorname{diam}(A_\infty) = \operatorname{diam}(B_\infty) = \frac{\operatorname{diam}(\Gamma)}{2}.$$

For $p = 1$ we let $A_1 = \{u_1 > 0\}$ and $B_1 = \{u_1 < 0\}$ and we obtain two subsets with total length $|A_1|$ and $|B_1|$ less or equal to $|\Gamma|/2$ with maximizes the sum $|A_1| + |B_1|$ and such that the perimeter of them inside Γ is minimized.

In general, for intermediate p, $1 < p < \infty$, if we let $A = \{u_p > 0\}$ and $B = \{u_p < 0\}$ we obtain something that interpolates between the two previous situations.

Let us end this introductions with a brief description of the ideas and techniques used in the proofs and of the previous bibliography.

Existence of eigenfunctions can be easily obtained from a compactness argument as for the usual $p-$Laplacian in a bounded domain of $\mathbb{R}^N$, see [12]. However, here we show examples that show that the set $\{u_p = 0\}$ may have nontrivial measure (it may contain some edges).

Eigenvalues on quantum graphs are by now a classical subject with an increasing number of recent references, we quote [5, 11, 17, 20]. The literature on eigenfunctions of the $p-$Laplacian in a one dimensional interval, also called $p-$trigonometric functions, is now quite extensive: we refer in particular to [22, 23, 24] and references therein.

Concerning the limit as $p \to \infty$ for the eigenvalue problem for the $p-$Laplacian in the usual PDE case we refer to [3, 4, 13, 14, 27]. To study this limit the main point is to use adequate test functions to obtain bounds that are uniform in p in order to gain compactness on a sequence of eigenfunctions.

Finally, for $p = 1$ we refer to [7, 10, 25] that deal with Cheeger sets in the Euclidean space. In this limit problem the natural space that appear is that of bounded variation functions (that are not necessarily continuous, see [1]).

The paper is organized as follows: in Section 2 we collect some preliminaries; in Section 3 we deal with the first eigenvalue on a quantum graph and prove its upper and lower bounds; in Section 4 we study the limit as $p \to \infty$ of the first eigenvalue while in the final section, Section 5 we look for the limit as $p \to 1$.

2. Preliminaries.

We start with a brief review of the basic results that will be needed in subsequent sections. The known results are generally stated without proofs, but we provide references where the proofs can be found. Also, we introduce our notational conventions.

2.1. **Neumann Eigenvalues for the $p-$Laplacian in one dimension.** First, we introduce a review about the one-dimensional Neumann eigenvalue problem for the $p-$Laplacian. For more details, see [21]. Let $p \in (1, +\infty)$ and $L > 0$. We consider the following eigenvalue problem for the $p-$Laplacian in an interval,

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \lambda |u|^{p-2}u(x) & \text{in } (0, L), \\ u'(0) = u'(L) = 0. \end{cases}$$

The eigenvalues λ are of the form

$$\lambda_{n+1,p} = \left(\frac{n\pi_p}{L}\right)^p \frac{p}{p'} \qquad \forall n \in \mathbb{N}_0,$$

where $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$, and $1/p + 1/p' = 1$. The eigenfunctions corresponding to the zero eigenvalue are the non-zero constants; those corresponding to $\lambda_{n,p}$ with $n > 0$ are

$$u_{n+1}(x) = \frac{\alpha L}{n\pi_p} \sin_p\left(\frac{n\pi_p}{L}\left(x - \frac{L}{2n}\right)\right), \quad \alpha \in \mathbb{R} \setminus \{0\},$$

where $\sin_p$ is the $p-$sine function.

Note that $\{\lambda_{n,p}\}$ coincides with the usual Neumann eigenvalues of the Laplacian when $p = 2$.

Finally, we want to remark that the first non-zero Neumann eigenvalue is

$$\lambda_{2,p} = \left(\frac{\pi_p}{L}\right)^p \frac{p}{p'}, \tag{2.7}$$

and the eigenfunctions u_2 corresponding to $\lambda_{2,p}$ have the following property

$$\int_0^L |u_2(x)|^{p-2}u_2(x)\,dx = 0.$$

2.2. **Quantum Graphs.** We now remind here some basic knowledge about quantum graphs, see for instance [2] and references therein.

A graph Γ consists of a finite or countable infinite set of vertices $\mathrm{V}(\Gamma) = \{\mathrm{v}_i\}$ and a set of edges $\mathrm{E}(\Gamma) = \{\mathrm{e}_j\}$ connecting the vertices. A graph Γ is said a finite graph if the number of edges and the number of vertices are finite.

Two vertices u and v are called adjacent (denoted u $\sim$ v) if there is an edge connecting them. An edge and a vertex on that edge are called incident. We will denote $\mathrm{v} \in \mathrm{e}$ when e and v are incident. We define $\mathrm{E}_{\mathrm{v}}(\Gamma)$ as the set of all edges incident to v. The degree $d_{\mathrm{v}}(\Gamma)$ of a vertex $\mathrm{V}(\Gamma)$ is the number of edges that incident to it, where a loop (an edge that connects a vertex to itself) is counted twice.

A graph Γ is said connected if a path exists between every pair of vertices, that is a graph which is connected in the sense of a topological space.

A graph Γ is called a directed graph if each of its edges is assigned a direction. In the remainder of the section, Γ is a directed graph.

Each edge e can be identified with an ordered pair $(\mathrm{v_e}, \mathrm{u_e})$ of vertices.The vertices $\mathrm{v_e}$ and $\mathrm{u_e}$ are the initial and terminal vertex of e. The edge ê is called the reversal of the edge e if $\mathrm{v}_{\hat{\mathrm{e}}} = \mathrm{u_e}$ and $\mathrm{u}_{\hat{\mathrm{e}}} = \mathrm{v_e}$.

Definition 2.1 (See Definition 1.2.3 in [2])**.** A graph Γ is called a metric graph, if

(1) each edge e is assigned a positive length $\ell_{\mathrm{e}} \in (0, +\infty]$;
(2) the lengths of the edges that are reversals of each other are assumed to be equal, that is $\ell_{\mathrm{e}} = \ell_{\hat{\mathrm{e}}}$;
(3) a coordinate $x_{\mathrm{e}} \in I_{\mathrm{e}} = [0, \ell_{\mathrm{e}}]$ increasing in the direction of the edge is assigned on each edge;
(4) the relation $x_{\hat{\mathrm{e}}} = \ell_{\mathrm{e}} - x_{\mathrm{e}}$ holds between the coordinates on mutually reserved edges.

A finite metric graph whose edges all have finite lengths will be called compact.

If a sequence of edges $\{\mathrm{e}_j\}_{j=1}^n$ forms a path, its length is defined as $\sum_{j=1}^n \ell_{\mathrm{e}_j}$. For two vertices v and u, the distance $d(\mathrm{v}, \mathrm{u})$ is defined as the minimal length of the path connected them. A compact metric graph Γ becomes a metric measure space by defining the distance $d(x, y)$ of two points x and y of the graph (that are not necessarily vertices) to be the short path on Γ connected these points, that is

$$d(x,y) := \inf\left\{\int_0^1 |\gamma'(t)|\,dt \colon \gamma\colon [0,1] \to \Gamma \text{ Lipschitz},\ \gamma(0) = x,\ \gamma(1) = y\right\}.$$

The total length of a metric graph (denoted $\ell(\Gamma)$) is the sum of the length of all edges and its diameter (denoted by $\mathrm{diam}(\Gamma)$) is the maximum length between two points in Γ.

A function u on a metric graph Γ is a collection of functions u_{e} defined on $(0, \ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$, not just at the vertices as in discrete models.

Let $1 \le p \le +\infty$. We say that u belongs to $L^p(\Gamma)$ if u_{e} belongs to $L^p(0, \ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$ and

$$\|u\|^p_{L^p(\Gamma)} := \sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \|u_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} < +\infty.$$

The Sobolev space $W^{1,p}(\Gamma)$ is defined as the space of continuous functions u on Γ such that $u_{\mathrm{e}} \in W^{1,p}(I_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$ and

$$\|u\|^p_{W^{1,p}(\Gamma)} := \sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \|u_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} + \|u'_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} < +\infty.$$

Observe that the continuity condition in the definition of $W^{1,p}(\Gamma)$ means that for each $\mathrm{v} \in \mathrm{V}(\Gamma)$, the function on all edges $\mathrm{e} \in \mathrm{E}_{\mathrm{v}}(\Gamma)$ assume the same value at v.

The space $W^{1,p}(\Gamma)$ is a Banach space for $1 \le p \le \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$.

Theorem 2.2. *Let Γ be a compact graph and $1 < p < +\infty$. The injection $W^{1,p}(\Gamma) \subset L^q(\Gamma)$ is compact for all $1 \le q \le +\infty$.*

A quantum graph is a metric graph Γ equipped with a differential operator $\mathcal{H}$, accompanied by a vertex conditions. In this work, we will consider

$$\mathcal{H}(u)(x) := -\Delta_p u(x) = -(|u'(x)|^{p-2}u'(x))'.$$

Our vertex conditions are the following

$$\sum_{\mathrm{e}\in\mathrm{E}_{\mathrm{v}}(\Gamma)} \left|\frac{\partial u}{\partial x_{\mathrm{e}}}\right|^{p-2} \frac{\partial u}{\partial x_{\mathrm{e}}}(\mathrm{v}) = 0, \qquad \forall \mathrm{v} \in \mathrm{V}(\Gamma), \tag{2.8}$$

where the derivatives are assumed to be taken in the direction away from the vertex.

Throughout this work, $\int_\Gamma u(x)dx$ denotes $\sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \int_0^{\ell_{\mathrm{e}}} u_{\mathrm{e}}(x)dx$.

3. The first non-zero eigenvalue in Γ.

Let Γ be a compact connected quantum graph and $p \in (1,\infty)$. We say that the value $\lambda \in \mathbb{R}$ is a Neumann eigenvalue of the $p-$Laplacian in Γ if there exists non trivial function $u \in W^{1,p}(\Gamma)$ such that

$$\int_\Gamma |u'(x)|^{p-2}u'(x)v'(x)dx = \lambda \int_\Gamma |u(x)|^{p-2}u(x)v(x)dx \tag{3.9}$$

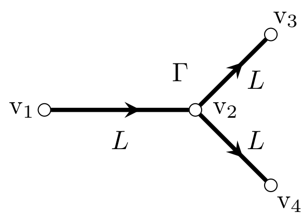
for all $v \in W^{1,p}(\Gamma)$. In which case, u is called an eigenfunction associated to λ.

Of course, the first eigenvalue is $\lambda = 0$ with eigenfunction $u \equiv 1$. Moreover, if $\lambda > 0$ is an eigenvalue and u is an associated eigenfunction, then, taking $v \equiv 1$ as a test function in (3.9), we have $\int_\Gamma |u(x)|^{p-2}u(x)\,dx = 0$.

Thus, the existence of the first non-zero eigenvalue $\lambda_{2,p}(\Gamma)$ is related to the problem of minimizing the quotient $\int_\Gamma |v'(x)|^p dx / \int_\Gamma |v(x)|^p dx$ among all functions $v \in W^{1,p}(\Gamma)$ such that $v \neq 0$ and $\int_\Gamma |v(x)|^{p-2}v(x)\,dx = 0$. This is exactly the content of Theorem 1.1 that we prove next.

Proof of Theorem 1.1. Take a minimizing sequence u_n for $\lambda_{2,p}(\Gamma)$ and normalize it according to $\|u_n\|_{L^p(\Gamma)} = 1$. This sequence verifies that $\int_\Gamma |u_n(x)|^{p-2}u_n(x)\,dx = 0$ and its $W^{1,p}$-norm is bounded. Hence, by a standard compactness argument, using the compactness result Theorem 2.2, it follows that there exists a subsequence u_{n_j} that converges strongly in $L^p(\Gamma)$ and weakly in $W^{1,p}(\Gamma)$. The limit of this subsequence verifies $\|u\|_{L^p(\Gamma)} = 1$, $\int_\Gamma |u(x)|^{p-2}u(x)\,dx = 0$ and $\|u\|^p_{W^{1,p}(\Gamma)} = \lambda_{2,p}(\Gamma)$. Therefore, $\lambda_{2,p}(\Gamma)$ is attained and it is the first non-zero Neumann eigenvalue of the $p-$Laplacian in Γ. The fact that a minimizer verifies (1.3) is standard and therefore we omit its proof. □

Remark 3.1. In general, the second eigenvalue $\lambda_{2,p}(\Gamma)$ is not simple. For instance, let Γ be a simple graph with 4 vertices and 3 edges, that is $\mathrm{V}(\Gamma) = \{\mathrm{v}_1, \mathrm{v}_2, v_3, v_4\}$ and $\mathrm{E}(\Gamma) = \{[\mathrm{v}_1, \mathrm{v}_2], [\mathrm{v}_2, \mathrm{v}_3], [\mathrm{v}_2, \mathrm{v}_4]\}$,



$$\lambda_{2,p}(\Gamma) = \left(\frac{\pi_p}{2L}\right)^p \frac{p}{p'}.$$

Observe that

$$u(x) = \begin{cases} \dfrac{2L}{\pi_p} \sin_p\left(\dfrac{\pi_p}{2L}(x-L)\right), & \text{if } x \in I_{[\mathrm{v}_1,\mathrm{v}_2]} = [0, L], \\ \dfrac{2L}{\pi_p} \sin_p\left(\dfrac{\pi_p}{2L}x\right), & \text{if } x \in I_{[\mathrm{v}_2,\mathrm{v}_3]} = [0, L], \\ 0 & \text{otherwise}, \end{cases}$$

$$v(x) = \begin{cases} \dfrac{2L}{\pi_p} \sin_p\left(\dfrac{\pi_p}{2L}(x-L)\right), & \text{if } x \in I_{[\mathrm{v}_1,\mathrm{v}_2]} = [0, L], \\ \dfrac{2L}{\pi_p} \sin_p\left(\dfrac{\pi_p}{2L}x\right), & \text{if } x \in I_{[\mathrm{v}_2,\mathrm{v}_4]} = [0, L], \\ 0 & \text{otherwise}, \end{cases}$$

are two linearly independent eigenfunctions associated to $\lambda_{2,p}(\Gamma)$.

Also remark that in this example, the above described eigenfunctions associated with $\lambda_{2,p}(\Gamma)$ vanishes on an entire edge. Therefore here we have that the set $\{u = 0\}$ is nontrivial.

These features correspond to a highly symmetric case. If we change the graph just by taking the same configuration but with three different lengths L_1, L_2, L_3 for the three different edges we have a an eigenvalue whose associated eigenfunction vanishes only at one point (hence its zero set has zero length). In fact, that an eigenfunction associated with the first nontrivial eigenvalue vanishes at the vertex v_2 is impossible since for different lengths we have different values of the first eigenvalue of the $p-$Laplacian with mixed boundary conditions ($u = 0$ at one endpoint and $u' = 0$ at the other endpoint). By the same reason, an eigenfunction must vanish only inside the longest edge and there is only one possibility for this point x_p (it must be the only one such that the first eigenvalue with mixed boundary conditions in the interval between the vertex v_i and the point x_p in the longest edge equals $\lambda_{2,p}(\Gamma)$).

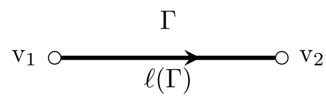
Our next result shows an upper bound and a lower bound for $\lambda_{2,p}(\Gamma)$ which depend on p, the length of a metric graph and the number of elements in $\mathrm{E}(\Gamma)$. The prove is similar to the one of [8, Theorems 3.5 and 3.8]. See also [18, Theorem 1].

Theorem 3.2. *Let Γ be a connected compact metric graph, and $p \in (1, +\infty)$. Then*

$$\left(\frac{\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'} \le \lambda_{2,p}(\Gamma) \le \left(\frac{\mathrm{card}(E(\Gamma))\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'},$$

where $\mathrm{card}(E(\Gamma))$ *is the number of elements in* $\mathrm{E}(\Gamma)$.

Note that the bounds given in the previous theorem are optimal. For instance, let Γ be a graph with only one edge, that is, $\mathrm{V}(\Gamma) = \{\mathrm{v}_1, \mathrm{v}_2\}$ and $\mathrm{E}(\Gamma) = \{[\mathrm{v}_1, \mathrm{v}_2]\}$,



Then, by (2.7), we have that

$$\lambda_{2,p}(\Gamma) = \left(\frac{\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'},$$

and then the upper and lower bound in Theorem 3.2 are attained and coincide.

4. The limit as $p \to \infty$.

In this section we deal with the limit as $p \to \infty$ of the eigenvalue problem (1.1). We split the proof of Theorem 1.2 in several steps.

Lemma 4.1. *There holds*

$$\limsup_{p \to +\infty} \lambda_{2,p}(\Gamma)^{1/p} \leq \Lambda_{2,\infty}(\Gamma). \tag{4.10}$$

Proof. Let $w \in W^{1,\infty}(\Gamma)$ be admissible for $\Lambda_{2,\infty}(\Gamma)$ i.e. $\max_\Gamma w = -\min_\Gamma w = 1$. Now, multiply the positive part of w, w^+, by $a_p \in \mathbb{R}$ and the negative part of w, w^-, by $b_p \in \mathbb{R}$ to obtain

$$\int_\Gamma |z(x)|^{p-2} z(x)\, dx = 0,$$

with

$$z(x) = a_p w^+(x) - b_p w^-(x).$$

Note that z is continuous in Γ and we can always assume that

$$\max_\Gamma |z| = 1,$$

hence $a_p = 1$ or $b_p = 1$. Also note that we have

$$a_p \left(\int_\Gamma (w^+(x))^{p-1}\, dx\right)^{1/(p-1)} = b_p \left(\int_\Gamma (w^-(x))^{p-1}\, dx\right)^{1/(p-1)}$$

and hence

$$\lim_{p\to\infty} a_p = \lim_{p\to\infty} b_p = 1$$

since

$$\lim_{p\to\infty} \left(\int_\Gamma (w^+(x))^{p-1}\, dx\right)^{1/(p-1)} = \lim_{p\to\infty} \left(\int_\Gamma (w^-(x))^{p-1}\, dx\right)^{1/(p-1)} = 1.$$

Then, z is an admissible function for the minimization problem defining $\lambda_{2,p}(\Gamma)$, hence we get

$$\lambda_{2,p}^{1/p}(\Gamma) \leq \frac{\|z'\|_{L^p(\Gamma)}}{\|z\|_{L^p(\Gamma)}}.$$

Now, we just observe that

$$\lim_{p\to\infty} \|z\|_{L^p(\Gamma)} = \|w\|_{L^\infty(\Gamma)} = 1,$$

and

$$\lim_{p\to\infty} \|z'\|_{L^p(\Gamma)} = \|w'\|_{L^\infty(\Gamma)} = \max\left\{\|(w^+)'\|_{L^\infty(\Gamma)}; \|(w^-)'\|_{L^\infty(\Gamma)}\right\}.$$

Hence, it follows that,

$$\limsup_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} \le \|w'\|_{L^\infty(\Gamma)},$$

and we conclude

$$\limsup_{p\to+\infty} \lambda_{2,p}(\Gamma)^{1/p} \le \Lambda_{2,\infty}(\Gamma).$$

□

As a second step, we prove that, up to a subsequence, u_p converges uniformly to a minimizer of $\Lambda_{2,\infty}(\Gamma)$.

Lemma 4.2. *Let u_p be an eigenfunction associated to $\lambda_{2,p}(\Gamma)$ normalized with $\|u_p\|_{L^p(\Gamma)} = 1$. Then, up to a subsequence, u_p converge uniformly in Γ and weakly in $W^{1,r}(\Gamma)$ for any $1 < r < \infty$ to some $u_\infty \in W^{1,\infty}(\Gamma)$ which is a minimizer of $\Lambda_{2,\infty}(\Gamma)$.*

Moreover, we have

$$\lim_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} = \Lambda_{2,\infty}(\Gamma).$$

Proof. We first notice that $\{u_p\}_{p\ge r}$ is bounded in $W^{1,r}(\Gamma)$ for any r. Indeed, by Holder's inequality,

$$\int_\Gamma |u_p'(x)|^r\,dx \le \|u_p'\|^p_{L^p(\Gamma)} |\Gamma|^{1-r/p}$$

so that by (4.10),

$$\|u_p'\|_{L^r(\Gamma)} \le \lambda_{2,p}(\Gamma)^{1/p} |\Gamma|^{1/r-1/p} \le C. \tag{4.11}$$

By Morrey's inequality $\{u_p\}_{p>r}$ is bounded in some Holder space $C^{0,\alpha}(\Gamma)$, and then, up to a subsequence, that $u_p \to u_\infty$ in $C(\Gamma)$. We can also assume that this convergence holds weakly in $W^{1,r}(\Gamma)$ for any r.

Let us prove that $\|u_\infty\|_{L^\infty(\Gamma)} = 1$. We have

$$\int_\Gamma |u_p(x)|^r\,dx \le \|u_p\|^p_{L^p(\Gamma)} |\Gamma|^{1-r/p}$$

so that by the normalization $\|u_p\|_{L^p(\Gamma)} = 1$, we get

$$\|u_p\|_{L^r(\Gamma)} \le |\Gamma|^{1/r-1/p}. \tag{4.12}$$

Letting $p, r \to \infty$ in (4.12), we see that $\|u_\infty\|_{L^\infty(\Gamma)} \le 1$. Now, suppose that $\|u_\infty\|_{L^\infty(\Gamma)} \le 1-2\varepsilon < 1$ for some $\varepsilon > 0$. Since $\|u_p\|_{L^\infty(\Gamma)} \to \|u_\infty\|_{L^\infty(\Gamma)}$ as $p \to \infty$, we have $\|u_p\|_{L^\infty(\Gamma)} \le 1-\varepsilon$ for p large. Then

$$1 = \int_\Gamma |u_p(x)|^p\,dx \le (1-\varepsilon)^p |\Gamma| \to 0,$$

as $p \to +\infty$, which is a contradiction with the normalization $\|u_p\|_{L^p(\Gamma)} = 1$.

We now verify that $\max_\Gamma u_\infty + \min_\Gamma u_\infty = 0$. From $\int_\Gamma |u_p(x)|^{p-2} u_p(x)\,dx = 0$ we obtain that

$$\int_{\{u_p\ge 0\}} |u_p(x)|^{p-1}\,dx = \int_{\{u_p\le 0\}} |u_p(x)|^{p-1}\,dx\,.$$

We already know that $\|u_\infty\|_{L^\infty(\Gamma)} = 1$. Let us show that $\max_\Gamma u_\infty = 1$ and $\min_\Gamma u_\infty = -1$. We argue by contradiction. Assume, e.g., that $\max_\Gamma u_\infty = 1$ but $\min_\Gamma u_\infty \geq -1 + 2\varepsilon$ for some $\varepsilon > 0$. Since $u_p \to u_\infty$ in $C(\Gamma)$, we also have $\min_\Gamma u_p \geq -1 + \varepsilon$ for p large. Then

$$\int_{\{u_p \geq 0\}} |u_p(x)|^{p-1}\, dx = \int_{\{u_p \leq 0\}} |u_p(x)|^{p-1}\, dx \leq (1-\varepsilon)^{p-1}|\Gamma| \to 0$$

as $p \to \infty$. Since $\{u_p\}$ is bounded in $C(\Gamma)$ (because it converges), we obtain

$$1 = \int_\Gamma |u_p(x)|^p\, dx \leq C \int_\Gamma |u_p(x)|^{p-1}\, dx \to 0$$

which is a contradiction.

As $\|u_\infty\|_{L^\infty(\Gamma)} = 1$ and $\max_\Gamma u_\infty + \min_\Gamma u_\infty = 0$, we have that u_∞ is an admissible test-function for $\Lambda_{2,\infty}(\Gamma)$. It follows that $\Lambda_{2,\infty}(\Gamma) \leq \|u'_\infty\|_{L^\infty(\Gamma)}$. Since $u_p \to u_\infty$ weakly in $W^{1,r}(\Gamma)$ for any $\infty > r > 1$, we also have from (4.11) that

$$\|u'_\infty\|_{L^r(\Gamma)} \leq \liminf_{p\to+\infty} \|u'_p\|_{L^r(\Gamma)} \leq |\Gamma|^{1/r} \liminf_{p\to+\infty} \lambda_{2,p}(\Gamma)^{1/p}.$$

Letting $r \to \infty$, we obtain, using (4.10), that

$$\Lambda_{2,\infty}(\Gamma) \leq \|u'_\infty\|_{L^\infty(\Gamma)} \leq \liminf_{p\to+\infty} \lambda_{2,p}(\Gamma)^{1/p} \leq \limsup_{p\to+\infty} \lambda_{2,p}(\Gamma)^{1/p} \leq \Lambda_{2,\infty}(\Gamma)$$

from where we deduce the claim. □

Now, our goal is to show that $\Lambda_{2,\infty}(\Gamma) = 2/\mathrm{diam}(\Gamma)$. As a first step, we prove an inequality.

Lemma 4.3. *There holds* $\Lambda_{2,\infty}(\Gamma) \geq 2/\mathrm{diam}(\Gamma)$.

Proof. Given some admissible test-function u for the minimum defining $\Lambda_{2,\infty}(\Gamma)$, let $x \in \Gamma$ be a point where u attains its maximum and $y \in \Gamma$ a point where u attains a minimum so that $u(x) = 1$ and $u(y) = -1$. Consider also some curve $\gamma : [0,T] \to \Gamma$ joining y and x. Then

$$2 = u(x) - u(y) = u(\gamma(T)) - u(\gamma(0)) = \int_0^T u'(\gamma(s))\gamma'(s)\, ds = \|u'\|_{L^\infty(\Gamma)}\mathrm{Long}(\gamma).$$

Taking the infimum over all such curves γ and all admissible u, we obtain

$$2 \leq \Lambda_{2,\infty}(\Gamma) d(x,y),$$

from where we deduce the claim. □

We now prove the reverse inequality.

Lemma 4.4. *There holds* $\Lambda_{2,\infty}(\Gamma) \leq 2/\mathrm{diam}(\Gamma)$.

Proof. Take two points $x_0, y_0 \in \Gamma$ such that $\mathrm{diam}(\Gamma) = d(x_0, y_0)$. Consider the function

$$u(z) = \frac{2}{\mathrm{diam}(\Gamma)}\left(d(z,x_0) - \frac{\mathrm{diam}(\Gamma)}{2}\right), \qquad z \in \Gamma.$$

This function is admissible for the minimization problem for $\Lambda_{2,\infty}$ and has

$$\|u'\|_{L^\infty(\Gamma)} = \frac{2}{\mathrm{diam}(\Gamma)}.$$

This gives the desired upper bound.

Another possible choice of a test-function is

$$u(z) = C_y(z)_+ - C_x(z)_+$$

where

$$C_y(z) = 1 - \frac{2}{\mathrm{diam}(\Gamma)} d(z,y) \qquad \text{and} \qquad C_x(z) = 1 - \frac{2}{\mathrm{diam}(\Gamma)} d(z,x)$$

are the cones centered at x and y of height 1 and slope $\frac{2}{\mathrm{diam}(\Gamma)}$. □

Remark 4.5. In the example described in Remark 3.1 with three edges of the same length L, we have that this limit selects (extracting a subsequence u_{p_j} with $p_j \to \infty$) two edges as A_∞, B_∞ and the third edge is just $\{u = 0\}$. Here the diameter of Γ is $2L$ and we obtain two sets of maximum diameter as A_∞, B_∞.

When we consider the same configuration of the graph, but with three different lengths L_1, L_2, L_3 (assume that $L_1 > L_2 > L_3$) for the three different edges we get that the diameter of Γ is $L_1 + L_2$ and our limit as $p \to \infty$ gives A_∞ as the segment of the longest edge of length $(L_1 + L_2)/2$ starting at v_1 and B_∞ as the rest of the graph.

5. The limit as $p \to 1^+$.

In this section we study the other limit case, $p = 1$. We will use functions of bounded variation on the graph (that we will denote by $BV(\Gamma)$) and the perimeter of a subset of the graph (denoted by $\mathrm{Per}(D)$). We refer to [1] for precise definitions and properties of functions and sets in this context.

We start by showing two technical lemmas that are required in the proof of Theorem 1.3.

Lemma 5.1. *Let Γ be a connected compact metric graph and $v \in BV(\Gamma)$ such that*

$$\int_\Gamma \mathrm{sgn}(v)(x)\, dx = 0. \tag{5.13}$$

If there exists a constant $c \neq 0$ such

$$\int_\Gamma \mathrm{sgn}(v - c)(x)\, dx = 0, \tag{5.14}$$

then $\|v - c\|_{L^1(\Gamma)} = \|v\|_{L^1(\Gamma)}$ and

$$|\{x\colon v(x) \geq c\}| = |\{x\colon v(x) \leq 0\}| \text{ and } |\{x\colon 0 < v(x) < c\}| = 0 \quad \text{if } c > 0;$$
$$|\{x\colon v(x) \leq c\}| = |\{x\colon v(x) \geq 0\}| \text{ and } |\{x\colon c < v(x) < 0\}| = 0 \quad \text{if } c < 0.$$

Proof. We will consider the case $c > 0$. The other case is analogous. We begin by introducing the following notation $E_0^+ = \{x\colon v(x) > 0\}$, $E_0^- = \{x\colon v(x) < 0\}$, $E_0 = \{x\colon v(x) = 0\}$, $E_c^+ = \{x\colon v(x) > c\}$, $E_c^- = \{x\colon v(x) < c\}$, $E_c = \{x\colon v(x) = c\}$, and $E_{0,c} = \{x\colon 0 < v(x) < c\}$. By (5.13) and (5.14), there exist $w_1 \in \mathrm{sgn}(v)$ and $w_2 \in \mathrm{sgn}(v - c)$ such that

$$0 = \int_\Gamma w_1(x)\, dx = |E_0^+| + \int_{E_0} w_1(x)\, dx - |E_0^-|, \tag{5.15}$$

and

$$
\begin{aligned}
0 &= \int_\Gamma w_2(x)\,dx = |E_c^+| - |E_c^-| + \int_{E_c} w_2(x)\,dx \\
&= |E_c^+| - |E_{0,c}| - |E_0| - |E_0^-| + \int_{E_c} w_2(x)\,dx \\
&\le |E_c^+| - |E_{0,c}| + \int_{E_0} w_1(x)\,dx - |E_0^-| + \int_{E_c} w_2(x)\,dx \qquad (\|w_1\|_{L^\infty(\Gamma)} = 1) \\
&= |E_c^+| - |E_{0,c}| - |E_0^+| + \int_{E_c} w_2(x)\,dx \qquad \text{(by (5.15))} \\
&= -2|E_{0,c}| + \int_{E_c} (w_2 - 1)(x)\,dx \\
&\le -2|E_{0,c}| \qquad \text{(note that } \|w_2\|_{L^\infty(\Gamma)} = 1).
\end{aligned}
$$

Observe if we assume that $|E_{0,c}| > 0$ we arrive to a contradiction in the last inequality. Then $|E_{0,c}| = 0$. Therefore

$$
\begin{aligned}
0 &= \int_\Gamma w_1(x)\,dx = |E_0^+| + \int_{E_0} w_1(x)\,dx - |E_0^-| \\
&= |\{x\colon v(x) \ge c\}| + \int_{E_0} w_1(x)\,dx - |E_0^-|,
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \int_\Gamma w_2(x)\,dx = |E_c^+| + |E_c^-| + \int_{E_c} w_2(x)\,dx \\
&= |E_c^+| - |\{x\colon v(x) \le 0\}| + \int_{E_c} w_2(x)\,dx.
\end{aligned}
$$

Subtracting these equations we get

$$
\begin{aligned}
0 &= |E_c| + \int_{E_0} w_1(x)\,dx + |E_0| - \int_{E_c} w_2(x)\,dx \\
&= \int_{E_0} (w_1 + 1)(x)\,dx + \int_{E_c} (1 - w_2)(x)\,dx.
\end{aligned}
$$

Therefore, $w_1 = -1$ in E_0 and $w_2 = 1$ in E_c due to $\|w_i\|_{L^\infty(\Gamma)} \le 1$ for $i = 1, 2$. Thus,

$$
\begin{aligned}
0 &= \int_\Gamma w_1(x)\,dx = |\{x\colon v(x) \ge c\}| + \int_{E_0} w_1(x)\,dx - |E_0^-| \\
&= |\{x\colon v(x) \ge c\}| - |\{x\colon v(x) \le 0\}|,
\end{aligned}
$$

that is $|\{x\colon v(x) \ge c\}| = |\{x\colon v(x) \le 0\}|$.

Finally,

$$
\begin{aligned}
\int_\Gamma |v - c|(x)\,dx &= \int_{\{x\colon v(x) \ge c\}} (v - c)(x)\,dx + \int_{\{x\colon v(x) \le 0\}} (c - v)(x)\,dx \\
&= \int_\Gamma |v(x)|\,dx + c|\{x\colon v(x) \ge c\}| - c|\{x\colon v(x) \le c\}| \\
&= \int_\Gamma |v(x)|\,dx,
\end{aligned}
$$

the proof is complete. □

Lemma 5.2. *Let Γ be a connected compact metric graph and $v \in L^1(\Gamma)$ and $\{v_n\}_{n\in\mathbb{N}}$ such that*

$$\int_\Gamma \operatorname{sgn}(v_n)(x)\,dx = 0 \quad \forall n \in \mathbb{N}, \text{ and } v_n \to v \text{ strongly in } L^1(\Gamma). \tag{5.16}$$

Then,

$$\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0.$$

Proof. For any $n \in \mathbb{N}$, by (5.16), there exists $w_n \in \operatorname{sgn}(v_n)$ such that

$$\int_\Gamma w_n(x)\,dx = 0. \tag{5.17}$$

Moreover $\|w_n\|_{L^\infty(\Gamma)} \le 1$ for all $n \in \mathbb{N}$. Therefore, there exist a function w and a subsequence still denoted $\{w_n\}_{n\in\mathbb{N}}$ such that

$$w_n \rightharpoonup w \quad \text{weakly in } L^q(\Omega) \quad \text{for any } 1 < q < \infty.$$

Thus, using (5.17),

$$\int_\Gamma w(x)\,dx = \lim_{n\to\infty}\int_\Gamma w_n(x)\,dx = 0,$$

and for any $\varphi \in C^\infty(\Gamma)$ we have

$$\left|\int_\Gamma w(x)\varphi(x)\,dx\right| = \left|\lim_{n\to\infty}\int_\Gamma w_n\varphi(x)\,dx\right| \le \int_\Gamma |\varphi(x)|\,dx\,.$$

Then $w \in L^\infty(\Omega)$, and $\|w\|_{L^\infty(\Omega)} \le 1$. In addition, by (5.16),

$$w_n \to \operatorname{sgn}(v) \text{ a.e. in } \{x \in \Gamma\colon v(x) \neq 0\}$$

as $n \to \infty$. Thus, $w \in \operatorname{sgn}(u_1)$ and $\int_\Gamma w(x)\,dx = 0$, that is $\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0$. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. We split the proof in 3 steps.

Setp 1. First we show that $\{u_p\}_{1<p\le 2}$ is bounded in $W^{1,1}(\Gamma)$.

Let $\varphi \in C^\infty(\Gamma)$ such that φ_e is odd respect to the center of I_e for any $e \in \mathrm{E}(\Gamma)$. Then

$$\int_\Gamma |\varphi(x)|^{p-2}\varphi(x)\,dx = 0 \qquad \forall p \in (1,+\infty).$$

Then, by Hölder's inequality and using that and u_p be a minimizer for $\lambda_{2,p}(\Gamma)$ and $\|u_p\|_{L^p(\Omega)=1}$, we get

$$\|u_p'\|^p_{L^1(\Gamma)} \le \|u_p'\|^p_{L^p(\Gamma)}|\Gamma|^{p-1} \le \frac{\|\varphi'\|^p_{L^p(\Gamma)}}{\|\varphi\|^p_{L^p(\Gamma)}}|\Gamma|^{p-1}.$$

Therefore, $\{u_p\}_{1<p\le 2}$ is bounded in $W^{1,1}(\Gamma)$.

Setp 2. Next, we show that

$$\liminf_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} \ge \Lambda_{2,1}(\Gamma).$$

Let $\{u_{p_n}\}_{n\in\mathbb{N}}$ be a subsequence of $\{u_p\}_{p\in(1,2)}$ such that $p_n \to 1^+$ as $n \to \infty$ and

$$\lim_{n\to\infty}\|u_{p_n}'\|_{L^{p_n}(\Gamma)} = \liminf_{p\to 1^+}\lambda_{2,p}(\Gamma)^{1/p}. \tag{5.18}$$

By step 1, $\{u_{p_n}\}_{n\in\mathbb{N}}$ is bounded in $W^{1,1}(\Gamma)$. Then, by Theorem 8.8 in [6] and Theorem 1 in [9, Section 5.2.1], there exist a constant $C>0$, u_1 and a subsequence that we still call $\{u_{p_n}\}_{n\in\mathbb{N}}$ such that

$$\|u_{p_n}\|_{L^\infty(\Omega)} \le C \quad \forall n \in \mathbb{N}, \tag{5.19}$$

$$u_{p_n} \to u_1 \quad \text{strongly in } L^q(\Gamma) \quad \text{for any } q\in[1,\infty), \tag{5.20}$$

$$u_{p_n} \to u_1 \quad \text{a.e. in } \Gamma, \tag{5.21}$$

and

$$\begin{aligned} \|u_1'\|(\Gamma) &\le \liminf_{n\to\infty} \|u_{p_n}'\|_{L^1(\Gamma)} \le \liminf_{n\to\infty} \|u_{p_n}'\|_{L^p(\Gamma)} |\Gamma|^{\frac{p_n-1}{p_n}} \\ &= \liminf_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p}. \end{aligned} \tag{5.22}$$

Moreover, by (5.19), (5.20) and Holder's inequality, we have that

$$\begin{aligned} \int_\Gamma |u_1(x)|\,dx = \lim_{n\to+\infty} \int_\Gamma |u_{p_n}(x)|\,dx &\le \lim_{n\to+\infty} \|u_{p_n}\|_{L^{p_n}(\Omega)} |\Gamma|^{\frac{p_n-1}{p_n}} = 1 \\ &\le \lim_{n\to+\infty} C^{p_n-1}\|u_{p_n}\|_{L^1(\Omega)}|\Gamma|^{\frac{p_n-1}{p_n}} = \int_\Gamma |u_1(x)|\,dx. \end{aligned}$$

Then $\|u_1\|_{L^1(\Gamma)} = 1$.

On the other hand, by (5.19), we have that $\{|u_{p_n}|^{p_n-2}u_{p_n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Gamma)$. Therefore, there exist a function w and a subsequence still denoted $\{u_{p_n}\}_{n\in\mathbb{N}}$ such that

$$|u_{p_n}|^{p_n-2}u_{p_n} \rightharpoonup w \quad \text{weakly in } L^q(\Omega) \quad \text{for any } 1<q<\infty.$$

Thus

$$\int_\Gamma w(x)\,dx = \lim_{n\to\infty} \int_\Gamma |u_{p_n}(x)|^{p_n-2}u_{p_n}(x)\,dx = 0$$

and for any $\varphi \in C^\infty(\Gamma)$ we have

$$\begin{aligned} \left|\int_\Gamma w(x)\varphi(x)\,dx\right| &= \left|\lim_{n\to\infty} \int_\Gamma |u_{p_n}(x)|^{p_n-2}u_{p_n}(x)\varphi(x)\,dx\right| \\ &\le \lim_{n\to\infty} C^{p_n-1} \int_\Gamma |\varphi(x)|\,dx \qquad \text{(by (5.19))} \\ &= \int_\Gamma |\varphi(x)|\,dx\,. \end{aligned}$$

Then $w\in L^\infty(\Omega)$, and $\|w\|_{L^\infty(\Omega)} \le 1$. In addition, by (5.21),

$$|u_{p_n}|^{p_n-2}u_{p_n} \to \operatorname{sgn}(u_1)$$

a.e. in $\{x\colon u_1(x)\neq 0\}$ as $n\to\infty$. Thus, $w\in\operatorname{sgn}(u_1)$ and $\int_\Gamma w(x)\,dx = 0$, that is $\int_\Gamma \operatorname{sgn}(u_1)(x)\,dx = 0$.

Finally, since $u\in BV(\Gamma)$ and $\int_\Gamma \operatorname{sgn}(u_1)(x)\,dx = 0$, we get

$$\Lambda_{2,p}(\Gamma) \le \|u_1'\|(\Gamma) \le \liminf_{p\to1^+} \lambda_{2,p}(\Gamma)^{1/p},$$

since $\|u_1\|_{L^1(\Gamma)} = 1$ and (5.22).

Setp 3. Finally, we show that

$$\limsup_{p\to1^+} \lambda_{2,p}(\Gamma)^{1/p} \le \Lambda_{2,1}(\Gamma).$$

Let $\{p_n\}_{n\in\mathbb{N}} \subset (1,2)$ such that $p_n \to 1^+$ and

$$\limsup_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} = \lim_{n\to\infty} \lambda_{2,p_n}(\Gamma)^{1/p_n}. \tag{5.23}$$

Given $v \in BV(\Gamma)\backslash\{0\}$ such that $\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0$, by Theorem 2 in [9, Subsection 5.2.2], there exists $\{\varphi_j\}_{j\in\mathbb{N}} \subset C^\infty(\Gamma)$ such that

$$\varphi_j \to v \quad \text{strongly in } L^1(\Gamma), \tag{5.24}$$

$$\varphi_j \to v \quad \text{a.e in } \Gamma, \tag{5.25}$$

$$\|\varphi_j'\|_{L^1(\Gamma)} \to \|v'\|(\Gamma). \tag{5.26}$$

Moreover, there exists a constant $C > 0$ such that

$$\|\varphi_j\|_{L^\infty(\Gamma)} \le C \quad \forall j \in \mathbb{N}. \tag{5.27}$$

Fix $j \in \mathbb{N}$, for any $n \in \mathbb{N}$ there exists $c_{j,n} \in [\min_{x\in\Gamma} \varphi_j(x), \max_{x\in\Gamma} \varphi_j(x)]$ such that

$$\int_\Gamma |\varphi_j(x) - c_{j,n}|^{p_n-2}(\varphi_j(x) - c_{j,n})\,dx = 0. \tag{5.28}$$

By (5.27), there exist $c_j \in [\min_{x\in\Gamma} \varphi_j(x), \max_{x\in\Gamma} \varphi_j(x)]$ and a subsequence that we still call $\{c_{j,n}\}_{n\in\mathbb{N}}$ such that $c_{j,n} \to c_j$ as $n \to \infty$. Moreover, proceeding as in the step 2, one can check that there exists $w_j \in \operatorname{sgn}(\varphi_j - c_j)$ such that $\int_\Gamma w_j(x)\,dx = 0$, that is $\int_\Gamma \operatorname{sgn}(\varphi_j(x) - c_j)\,dx = 0$.

Then,

$$\begin{aligned}
\limsup_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} &= \lim_{n\to\infty} \lambda_{2,p_n}(\Gamma)^{1/p_n} \le \liminf_{n\to\infty} \frac{\|(\varphi_j - c_{j,n})'\|_{L^{p_n}(\Gamma)}}{\|(\varphi_n - c_{j,n})\|_{L^{p_n}(\Gamma)}} \\
&\le \liminf_{n\to\infty} \frac{\|(\varphi_j - c_{j,n})'\|_{L^\infty(\Gamma)}^{\frac{p_n-1}{p_n}} \|(\varphi_{n_i} - c_{n_i,p_i})'\|_{L^1(\Gamma)}^{\frac{1}{p_n}}}{|\Gamma|^{\frac{1-p_n}{p_n}} \|(\varphi_j - c_j)\|_{L^1(\Gamma)}} \\
&= \frac{\|(\varphi_j - c_j)'\|_{L^1(\Gamma)}}{\|\varphi_j - c_j\|_{L^1(\Gamma)}}.
\end{aligned} \tag{5.29}$$

On the other hand, by (5.27) and since $c_j \in [\min_{x\in\Gamma} \varphi_j(x), \max_{x\in\Gamma} \varphi_j(x)]$ for all $j \in \mathbb{N}$, there exists $c \in \mathbb{R}$ and a subsequence still call $\{c_j\}_{j\in\mathbb{N}}$ such that $c_j \to c$ as $j \to \infty$. Then, by (5.24), we have that $\varphi_j - c_j \to v - c$ strongly in $L^1(\Gamma)$. Therefore, by Lemma 5.2, $\int_\Gamma \operatorname{sgn}(v(x) - c)dx = 0$. Hence, by (5.29), (5.26) and Lemma 5.2, we obtain

$$\begin{aligned}
\limsup_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} &\le \lim_{j\to\infty} \frac{\|(\varphi_j - c_j)'\|_{L^1(\Gamma)}}{\|\varphi_j - c_j\|_{L^1(\Gamma)}} = \lim_{j\to\infty} \frac{\|\varphi_j'\|_{L^1(\Gamma)}}{\|\varphi_j - c_j\|_{L^1(\Gamma)}} \\
&= \frac{\|v'\|(\Gamma)}{\|v - c\|_{L^1(\Gamma)}} = \frac{\|v'\|(\Gamma)}{\|v\|_{L^1(\Gamma)}}.
\end{aligned}$$

Since v is arbitrary, we have that

$$\limsup_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} \le \Lambda_{2,1}(\Gamma).$$

Therefore, from this inequality and step 2, we conclude that

$$\lim_{p\to 1^+} \lambda_{2,p}(\Gamma) = \Lambda_{2,1}(\Gamma)$$

and that u_1 is a minimizer for (1.5). □

The next result gives a curious property that we include here just for completeness but is not needed in the proof of our main results.

Lemma 5.3. *Let* Γ *be a connected compact metric graph and* $\varphi \in C^\infty(\Gamma)$ *such that*

$$\int_\Gamma \operatorname{sgn}(\varphi)(x)\, dx = 0, \tag{5.30}$$

and $\{c_p\}_{p>1}$ *be a subset of* $(\min_{x\in\Gamma}\varphi(x), \max_{x\in\Gamma}\varphi(x))$ *such that*

$$\int_\Gamma |\varphi(x) - c_p|^{p-2}(\varphi(x) - c_p)dx = 0. \tag{5.31}$$

Then $c_p \to 0$ *as* $p \to 1^+$.

Proof. We show that all convergent subsequence of $\{c_p\}_{p>1}$ converge to 0. Let $\{c_{p_i}\}_{i\in\mathbb{N}}$ be a subsequence of $\{c_p\}_{p>1}$ such that

$$p_i \to 1^+ \text{ and } c_{p_i} \to c \in \left[\min_{x\in\Gamma}\varphi(x), \max_{x\in\Gamma}\varphi(x)\right]$$

as $i \to \infty$. We will see that $c = 0$.

It is clear that there exists a constant $C > 0$ such that

$$\|\varphi - c_{p_i}\|_{L^\infty(\Omega)} \le C \quad \forall i \in \mathbb{N}. \tag{5.32}$$

Then $\{|\varphi - c_{p_i}|^{p_i-2}(\varphi - c_{p_i})\}_{i\in\mathbb{N}}$ is bounded in $L^q(\Gamma)$ for all $q \in [1,\infty]$. Therefore, there exist $v \in L^q(\Omega)$ and a subsequence that will still call $\{|\varphi-c_{p_i}|^{p_i-2}(\varphi-c_{p_i})\}_{i\in\mathbb{N}}$ such that

$$|\varphi - c_{p_i}|^{p_i-2}(\varphi - c_{p_i}) \rightharpoonup v \quad \text{weakly in } L^q(\Omega)$$

for any $1 < q < \infty$. Thus

$$\int_\Gamma v(x)\, dx = \lim_{i\to\infty}\int_\Gamma |\varphi(x) - c_{p_i}|^{p_i-2}(\varphi(x) - c_{p_i})\, dx = 0 \quad \text{(by (5.31))}$$

and for any $\phi \in C^\infty(\Gamma)$ we have

$$\begin{aligned}\left|\int_\Gamma v(x)\phi(x)\, dx\right| &= \left|\lim_{i\to\infty}\int_\Gamma |\varphi(x) - c_{p_i}|^{p_i-2}(\varphi(x) - c_{p_i})\phi(x)\, dx\right| \\ &\le \lim_{i\to\infty} C^{p_i-1}\int_\Gamma |\phi(x)|\, dx \qquad \text{(by (5.32))} \\ &= \int_\Gamma |\phi(x)|\, dx\,.\end{aligned}$$

Then $v \in L^\infty(\Omega)$, $\|v\|_{L^\infty(\Omega)} \le 1$ and

$$\int_\Gamma v(x)\, dx = 0. \tag{5.33}$$

In addition,

$$|\varphi - c_{p_i}|^{p_i-2}(\varphi - c_{p_i}) \to \operatorname{sgn}(\varphi - c)$$

a.e. in $\{x\colon \varphi(x) - c \neq 0\}$ as $i \to \infty$. Therefore, $v \in \operatorname{sgn}(\varphi - c)$.

On the other hand, by (5.30), there exists $w \in \operatorname{sgn}(\varphi)$ such that

$$0 = \int_\Gamma w(x)\, dx = |E_0^+| + \int_{E_0} w(x)\, dx - |E_0^-|, \tag{5.34}$$

where $E_0^+ = \{x\colon \varphi(x) > 0\}$, $E_0^- = \{x\colon \varphi(x) < 0\}$, and $E_0 = \{x\colon \varphi(x) = 0\}$.

We now suppose by contraction that $c \neq 0$. We will only consider the case $c > 0$. The case $c < 0$ is analogous.

Taking $E_c^+ = \{x \colon \varphi(x) > c\}$, $E_c^- = \{x \colon \varphi(x) < c\}$, $E_c = \{x \colon \varphi(x) = c\}$, and $E_{0,c} = \{x \colon 0 < \varphi(x) < c\}$, we have that

$$\begin{aligned}
0 &= \int_\Gamma v(x)\,dx \qquad \text{(by (5.33))}\\
&= |E_c^+| - |E_c^-| + \int_{E_c} v(x)\,dx \qquad (v \in \operatorname{sgn}(\varphi - c))\\
&= |E_c^+| - |E_{0,c}| - |E_0| - |E_0^-| + \int_{E_c} v(x)\,dx\\
&\leq |E_c^+| - |E_{0,c}| + \int_{E_0} w(x)\,dx - |E_0^-| + \int_{E_c} v(x)\,dx \quad (\|w\|_{L^\infty(\Gamma)} \leq 1)\\
&= |E_c^+| - |E_{0,c}| - |E_0^+| + \int_{E_c} v(x)\,dx \qquad \text{(by (5.34))}\\
&= -2|E_{0,c}| + \int_{E_c} (v-1)(x)\,dx \leq -2|E_{0,c}| \qquad (\|v\|_{L^\infty(\Gamma)} \leq 1).
\end{aligned}$$

If $|E_{0,c}| > 0$, we arrive to a contradiction. If $|E_{0,c}| = 0$, we have two possibilities: either $\varphi \geq c$ or $\varphi \leq 0$. In the case $\varphi \geq c$ we get a contradiction with (5.34). Finally, if $\varphi \leq 0$ we arrive to a contradiction with (5.33). Consequently, $c = 0$. □

Proof of Theorem 1.4. We begin by observing that

$$\Lambda_{2,1}(\Gamma) \leq \inf\left\{\frac{\operatorname{Per}(E)}{\min\{|E|, |\Gamma \setminus E|\}} : D \subsetneq \Gamma, E \neq \emptyset\right\}.$$

Therefore, if $u = \chi_A - \chi_{\Gamma\setminus A}$ is a minimizer for $\Lambda_{2,1}(\Gamma)$ then

$$\Lambda_{2,1}(\Gamma) = \frac{\|u'\|(\Gamma)}{\|u\|_{L^1(\Gamma)}} = \frac{2\operatorname{Per}(A)}{\ell(\Gamma)} \geq \inf\left\{\frac{\operatorname{Per}(E)}{\min\{|E|, |\Gamma \setminus E|\}} : D \subsetneq \Gamma, E \neq \emptyset\right\},$$

that is

$$\Lambda_{2,1}(\Gamma) = \frac{2\operatorname{Per}(A)}{\ell(\Gamma)} = \inf\left\{\frac{\operatorname{Per}(E)}{\min\{|E|, |\Gamma \setminus E|\}} : E \subsetneq \Gamma, E \neq \emptyset\right\}.$$

On the other hand, suppose that (1.6) is valid. For any $v \in BV(\Gamma)$ such that $\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0, v \neq 0$, by coarea formula (see [9, Theorem 1 in Section 5.5]), we have that

$$\|v'\|(\Gamma) = \int_{-\infty}^{\infty} \operatorname{Per}(E_t^+)dt \tag{5.35}$$

where $E_t^+ = \{x \colon v(x) > t\}$.

Since $\int_\Gamma \mathrm{sgn}(v)(x)\,dx = 0$, we also have that $|E_t^+| \le |\{x\colon v(x) \le t\}|$ for all $t \ge 0$, and $|E_t^+| \ge |\{x\colon v(x) \le t\}|$ for all $t < 0$. Then, by (5.35) and (1.6), we get

$$\begin{aligned}\|v'\|(\Gamma) &= \int_0^\infty \mathrm{Per}(E_t^+)dt + \int_{-\infty}^0 \mathrm{Per}(E_t^+)dt\\ &\ge \frac{2\,\mathrm{Per}(A)}{\ell(\Gamma)}\left(\int_0^\infty |E_t^+|dt + \int_{-\infty}^0 |\{x\in\Gamma\colon v(x)\le t\}|dt\right)\\ &\ge \frac{2\,\mathrm{Per}(A)}{\ell(\Gamma)}\|v\|_{L^1(\Gamma)}.\end{aligned}$$

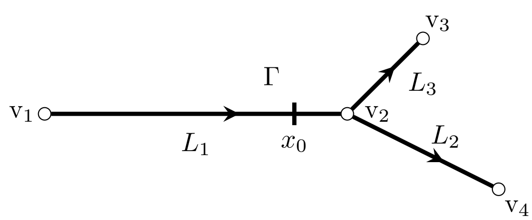
Then,

$$\Lambda_{2,1}(\Gamma) \ge \frac{2\,\mathrm{Per}(A)}{\ell(\Gamma)} = \frac{\|u'\|(\Gamma)}{\|u\|_{L^1(\Gamma)}}. \tag{5.36}$$

Finally, we observe that $|x\colon u(x) > 0| = |A| = |\Gamma \setminus A| = |\{x\colon u(x) \le 0\}|$. Then $\int_\Gamma \mathrm{sgn}(u)(x)\,dx = 0$, and by (5.36), u is a minimizer for $\Lambda_{2,1}(\Gamma)$. □

Remark 5.4. In the example described in Remark 3.1 with three edges of the same length L, we have that this limit selects (as for the case $p = \infty$) two edges as A_∞, B_∞ and the third edge is just $\{u = 0\}$. Here we have only one "cut" in our graph Γ (the perimeter of A and B inside Γ is one).

Now, let us consider the same configuration of the graph, but with three different lengths L_1, L_2, L_3 for the three different edges and let us assume that $L_1 > L_2 > L_3$ with $L_1 > L_2 + L_3$. In this case we get that this limit finds a point $x_0 \in \Gamma$ that divides Γ in two sets A and B with the same total length. The position of x_0 is the point in L_1 whose distance to v_1 is $(L_1 + L_2 + L_3)/2$.



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