THE FIRST EIGENVALUE OF THE $p-$LAPLACIAN ON QUANTUM GRAPHS

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ABSTRACT. We study the first eigenvalue of the $p-$Laplacian (with $1 < p < \infty$) on a quantum graph with Dirichlet or Kirchoff boundary conditions on the nodes. We find lower and upper bounds for this eigenvalue when we prescribe the total sum of the lengths of the edges and the number of Dirichlet nodes of the graph. Also we find a formula for the shape derivative of the first eigenvalue (assuming that it is simple) when we perturb the graph by changing the length of an edge. Finally, we study in detail the limit cases $p \to \infty$ and $p \to 1$.

1. Introduction

A quantum graph is a graph in which we associate a differential law with each edge. This differential law models the interaction between the two nodes defining each edge. The use of quantum graphs (as opposed to more elementary graph models, such as simple unweighted or weighted graphs) opens up the possibility of modeling the interactions between agents identified by the graph's vertices in a far more detailed manner than with standard graphs. Quantum graphs are now widely used in physics, chemistry and engineering (nanotechnology) problems, but can also be used, in principle, in the analysis of complex phenomena taking place on large complex networks, including social and biological networks. Such graphs are characterized by highly skewed degree distributions, small diameter and high clustering coefficients, and they have topological and spectral properties that are quite different from those of the highly regular graphs, or lattices arising in physics and chemistry applications. Quantum graphs are also used to model thin tubular structures, so-called graph-like spaces, they are their natural limits, when the radius of a graph-like space tends to zero. On both, the graph-like spaces and the metric graph, we can naturally define Laplace-like differential operators. See [3, 4, 19, 29].

Among properties that are relevant in the study of quantum graphs is the study of the spectrum of the associated differential operator. In particular, the so-called spectral gap (this concerns bounds for the first nontrivial eigenvalue for the Laplacian with Neumann boundary conditions) has physical relevance and was extensively studied in recent years. See, for example, [19, 20, 22, 23] and references therein.

In this paper we are interested in the eigenvalue problem that naturally arises when we consider the $p-$Laplacian, $(|u'|^{p-2}u')'$, as the differential law on each side of the graph together with Dirichlet boundary conditions on a subset of nodes of

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the graph and pure transmission (known as Kirchoff boundary conditions, [18]) in the rest of the nodes. To be concrete, given $1 < p < \infty$, we deal with the following problem: in a finite metric graph Γ we consider a set of nodes V_D and look for the minimization problem

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) = \inf \left\{ \frac{\displaystyle\int_\Gamma |u'(x)|^p \, dx}{\displaystyle\int_\Gamma |u(x)|^p \, dx} : u \in \mathcal{X}(\Gamma, \mathrm{V}_D), u \neq 0 \right\}, \tag{1.1}$$

where $\mathcal{X}(\Gamma, \mathrm{V}_D) \coloneqq \{v \in W^{1,p}(\Gamma) \colon v \text{ is continuous in } \Gamma,\ v = 0 \text{ on } \mathrm{V}_D\}$.

There is a minimizer, see Section 3, that is a nontrivial weak solution to

$$\begin{cases} -(|u'|^{p-2}u')'(x) = \lambda_{1,p}(\Gamma, \mathrm{V}_D)|u|^{p-2}u(x) & \text{on the edges of } \Gamma, \\ u(\mathrm{v}) = 0 & \forall \mathrm{v} \in \mathrm{V}_D, \\ \displaystyle\sum_{\mathrm{e} \in \mathrm{E}_\mathrm{v}(\Gamma)} \left| \frac{\partial u}{\partial x_\mathrm{e}}(\mathrm{v}) \right|^{p-2} \frac{\partial u}{\partial x_\mathrm{e}}(\mathrm{v}) = 0 & \forall \mathrm{v} \in \mathrm{V}(\Gamma) \setminus \mathrm{V}_D. \end{cases} \tag{1.2}$$

Our main results for this eigenvalue problem can be summarized as follows (we refer to the corresponding sections for precise statements):

- We show that there is a first eigenvalue with an associated nonnegative eigenfunction, that is, the infimum in (1.1) is attained at a nonnegative function. We provide examples that show that $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$ can be a multiple eigenvalue or a simple eigenvalue depending on the graph.

- We find a sharp lower bound for the first eigenvalue that depends only on the total sum of the lengths of the edges of the graph, $\ell(\Gamma)$, namely
$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) \geq C(p) \left(\frac{1}{\ell(\Gamma)} \right)^p,$$
here the constant $C(p)$ is explicit and depends only on p.

- We find a sharp upper bound for the first eigenvalue depending on the total sum of the lengths of the edges, $\ell(\Gamma)$, and the number of edges of the graph, $\mathrm{card}(E(\Gamma))$,
$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) \leq C(p) \left(\frac{\mathrm{card}(E(\Gamma))}{\ell(\Gamma)} \right)^p,$$
again the constant $C(p)$ is explicit and depends only on p.

- Under the assumption that the first eigenvalue is simple, we find a formula for its shape derivative when we perturb the graph by changing the length of an edge. In the case of a multiple eigenvalue, we provide examples that show that the first eigenvalue is not differentiable with respect to the lengths of the edges of the graph (but it is Lipschitz).

- We study the limit cases $p \to \infty$ and $p \to 1$. For $p = \infty$ we find a geometric characterization of the first eigenvalue and for $p = 1$ we prove that there exist the analogous of Cheeger sets in quantum graphs.

Note that without a bound on the total length of the graph the first eigenvalue is unbounded from above and from below the optimal bound is zero and without a bound on the number of Dirichlet nodes it is not bounded above even if we prescribe the total length. Therefore our results are also sharp in this sense. Also remark that our results are new even for the linear case $p = 2$.

Let us end this introduction with a brief discussion on ideas and techniques used in the proofs as well as a description of the previous bibliography.

Existence of eigenfunctions can be easily obtained from a compactness argument as for the usual $p-$Laplacian in a bounded domain of $\mathbb{R}^N$, see [13]. However, in contrast to what happens in the usual case of a bounded domain, see [2], the first eigenvalue is not simple, we show examples of this phenomena.

Eigenvalues on quantum graphs are by now a classical subject with an increasing number of recent references, we quote [7, 12, 20, 23]. The literature on eigenfunctions of the $p-$Laplacian, also called $p-$trigonometric functions, is now quite extensive: we refer in particular to [25, 26, 27] and references therein.

The upper and lower bounds comes from test functions arguments together with some analysis of the possible configurations of the graphs.

For the shape derivative when we modify the length of one edge we borrow ideas from [14].

Concerning the limit as $p \to \infty$ for the eigenvalue problem of the $p-$Lapla-cian in the usual PDE case we refer to [5, 6, 16, 17]. To obtain this limit the main point is to use adequate test functions to obtain bounds that are uniform in p in order to gain compactness on a sequence of eigenfunctions.

Finally, for $p = 1$ we refer to [8, 11, 28]. In this limit problem the natural space that appear is that of bounded variation functions, see [1]. Remark that when considering bounded variation functions we loose continuity.

The paper is organized as follows: in Section 2 we collect some preliminaries; in Section 3 we deal with the first eigenvalue on a quantum graph and prove its upper and lower bounds; in Section 4 we perform a shape derivative approach of the first eigenvalue showing that it is differentiable when we change the length of one edge and providing an explicit formula for this derivative; in Section 5 we study the limit as $p \to \infty$ of the first eigenvalue while in the final section, Section 6 we look for the limit as $p \to 1$.

2. Preliminaries.

2.1. **Quantum Graphs.** We collect here some basic knowledge about quantum graphs, see for instance [4] and references therein.

A graph Γ consists of a finite or countable infinite set of vertices $\mathrm{V}(\Gamma) = \{\mathrm{v}_i\}$ and a set of edges $\mathrm{E}(\Gamma) = \{\mathrm{e}_j\}$ connecting the vertices. A graph Γ is said a finite graph if the number of edges and the number of vertices are finite.

Two vertices u and v are called adjacent (denoted $\mathrm{u} \sim \mathrm{v}$) if there is an edge connecting them. An edge and a vertex on that edge are called incident. We will denote $\mathrm{v} \in \mathrm{e}$ when e and v are incident. We define $\mathrm{E}_{\mathrm{v}}(\Gamma)$ as the set of all edges incident to v. The degree $d_{\mathrm{v}}(\Gamma)$ of a vertex $\mathrm{V}(\Gamma)$ is the number of edges that incident to it, where a loop (an edge that connects a vertex to itself) is counted twice.

We will say that v is a terminal vertex if there exists an unique vertex $\mathrm{u} \in \mathrm{V}(\Gamma)$ such that $\mathrm{u} \sim \mathrm{v}$. Let us denote by $\mathrm{T}(\Gamma)$ the set of all terminal vertices.

A walk is a sequence of edges in which the end of each edge (except the last) is the beginning of the next. A trail is a walk in which no edge is repeated. A path is a trail in which no vertex is repeated. A graph Γ is said connected if a path exists between every pair of vertices, that is a graph which is connected in the sense of a topological space.

A graph Γ is called a directed graph if each of its edges is assigned a direction. In the remainder of the section, Γ is a directed graph.

Each edge e can be identified with an ordered pair $(\mathrm{v_e}, \mathrm{u_e})$ of vertices.The vertices $\mathrm{v_e}$ and $\mathrm{u_e}$ are the initial and terminal vertex of e. The edge ê is called the reversal of the edge e if $\mathrm{v_{\hat{e}}} = \mathrm{u_e}$ and $\mathrm{u_{\hat{e}}} = \mathrm{v_e}$. We define

$$\widehat{\mathrm{E}}(\Gamma) \coloneqq \{\hat{e} \colon e \in \mathrm{E}(\Gamma)\}.$$

The edge e is called outgoing (incoming) at a vertex v if v is the initial (terminal) vertex of e. The number of outgoing (incoming) edges at a vertex v is called outgoing (incoming) degree and denoted $d^o_{\mathrm{v}}(\Gamma)$ $(d^i_{\mathrm{v}}(\Gamma))$. Observe that $d_{\mathrm{v}}(\Gamma) = d^o_{\mathrm{v}}(\Gamma) + d^i_{\mathrm{v}}(\Gamma)$.

Definition 2.1 (See Definition 1.2.3 in [4]). *A graph Γ is said to be a metric graph, if*

(1) *each edge* e *is assigned a positive length* $\ell_{\mathrm{e}} \in (0, +\infty]$;
(2) *the lengths of the edges that are reversals of each other are assumed to be equal, that is* $\ell_{\mathrm{e}} = \ell_{\hat{\mathrm{e}}}$;
(3) *a coordinate* $x_{\mathrm{e}} \in I_{\mathrm{e}} = [0, \ell_{\mathrm{e}}]$ *increasing in the direction of the edge is assigned on each edge;*
(4) *the relation* $x_{\hat{\mathrm{e}}} = \ell_{\mathrm{e}} - x_{\mathrm{e}}$ *holds between the coordinates on mutually reserved edges.*

A finite metric graph whose edges all have finite lengths will be called compact. If a sequence of edges $\{\mathrm{e}_j\}_{j=1}^n$ forms a path, its length is defined as $\sum_{j=1}^n \ell_{\mathrm{e}_j}$. For two vertices v and u, the distance $d(\mathrm{v}, \mathrm{u})$ is defined as the minimal length of the path connected them. A compact metric graph Γ becomes a metric measure space by defining the distance $d(x, y)$ of two points x and y of the graph (that are not necessarily vertices) to be the short path on Γ connected these points, that is

$$d(x, y) \coloneqq \inf \left\{ \int_0^1 |\gamma'(t)|\, dt \colon \gamma \colon [0,1] \to \Gamma \text{ Lipschitz},\ \gamma(0) = x,\ \gamma(1) = y \right\}.$$

The length of a metric graph (denoted $\ell(\Gamma)$) is the sum of the length of all edges.

A function u on a metric graph Γ is a collection of functions u_{e} defined on $(0, \ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$, not just at the vertices as in discrete models.

Let $1 \le p \le \infty$. We say that u belongs to $L^p(\Gamma)$ if u_{e} belongs to $L^p(0, \ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$ and

$$\|u\|^p_{L^p(\Gamma)} \coloneqq \sum_{\mathrm{e} \in \mathrm{E}(\Gamma)} \|u_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} < \infty.$$

The Sobolev space $W^{1,p}(\Gamma)$ is defined as the space of continuous functions u on Γ such that $u_{\mathrm{e}} \in W^{1,p}(I_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$ and

$$\|u\|^p_{W^{1,p}(\Gamma)} \coloneqq \sum_{\mathrm{e} \in \mathrm{E}(\Gamma)} \|u_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} + \|u'_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} < \infty.$$

Observe that the continuity condition in the definition of $W^{1,p}(\Gamma)$ means that for each $\mathrm{v} \in \mathrm{V}(\Gamma)$, the function on all edges $\mathrm{e} \in \mathrm{E_v}(\Gamma)$ assume the same value at v.

The space $W^{1,p}(\Gamma)$ is a Banach space for $1 \le p \le \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$.

Theorem 2.2. *Let Γ be a compact graph and $1 < p < \infty$. The injection $W^{1,p}(\Gamma) \subset L^q(\Gamma)$ is compact for all $1 \le q \le \infty$.*

A quantum graph is a metric graph Γ equipped with a differential operator $\mathcal{H}$, accompanied by a vertex conditions. In this work, we will consider

$$\mathcal{H}(u)(x) := -\Delta_p u(x) = -(|u'(x)|^{p-2}u'(x))'.$$

Given V_D a non empty subset of $\mathrm{V}(\Gamma)$, our vertex conditions are the following

$$\begin{cases} u(x) \text{ is continuous in } \Gamma, \\ u(\mathrm{v}) = 0 \quad \forall \mathrm{v} \in \mathrm{V}_D, \\ \displaystyle\sum_{\mathrm{e}\in \mathrm{E}_{\mathrm{v}}(\Gamma)} \left|\frac{\partial u}{\partial x_{\mathrm{e}}}(\mathrm{v})\right|^{p-2} \frac{\partial u}{\partial x_{\mathrm{e}}}(\mathrm{v}) = 0 \quad \forall \mathrm{v} \in \mathrm{V}(\Gamma) \setminus \mathrm{V}_D, \end{cases} \tag{2.3}$$

where the derivatives are assumed to be taken in the direction away from the vertex.

Throughout this work, $\int_\Gamma u(x)\,dx$ denotes $\sum_{\mathrm{e}\in \mathrm{E}(\Gamma)} \int_0^{\ell_{\mathrm{e}}} u_{\mathrm{e}}(x)\,dx$.

2.2. **Eigenvalues of the $p-$Laplacian in $\mathbb{R}$.** Here we present a brief review concerning eigenvalues of the 1-dimensional $p-$Laplacian. For a more elaborate treatment we refer the reader to [24].

Let $p \in (1, +\infty)$. Given $L > 0$, all eigenvalues λ of the Dirichlet problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda |u|^{p-2}u & \text{in } (0, L), \\ u(0) = u(L) = 0, \end{cases}$$

are of the form

$$\lambda_{n,p} = \left(\frac{n\pi_p}{L}\right)^p \frac{p}{p'} \quad \forall n \in \mathbb{N}$$

with corresponding eigenfunctions

$$u_n(x) = \frac{\alpha L}{n\pi_p} \sin_p\left(\frac{n\pi_p}{L}x\right), \quad \alpha \in \mathbb{R} \setminus \{0\}$$

where $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$, $1/p + 1/p' = 1$, and $\sin_p$ is the $p-$sine function.

Then the first Dirichlet eigenvalue is

$$\lambda_{1,p} = \left(\frac{\pi_p}{L}\right)^p \frac{p}{p'}, \tag{2.4}$$

and has a positive eigenfunction (any other eigenvalue has eigenfunctions that change sign).

Remark 2.3. *Observe that $\{\lambda_{n,p}\}$ coincides with the Dirichlet eigenvalues of the Laplacian when $p = 2$.*

3. The first eigenvalue on a quantum graph.

Let Γ be a compact connected quantum graph and V_D be a non-empty subset of $\mathrm{V}(\Gamma)$. We say that the value $\lambda \in \mathbb{R}$ is an eigenvalue of the $p-$Laplacian if there exists non trivial function $u \in \mathcal{X}(\Gamma, \mathrm{V}_D) := \{v \in W^{1,p}(\Gamma) \colon v = 0 \text{ on } \mathrm{V}_D\}$ such that

$$\int_\Gamma |u'(x)|^{p-2}u'(x)w'(x)\,dx = \lambda \int_\Gamma |u(x)|^{p-2}u(x)w(x)\,dx$$

for all $w \in \mathcal{X}$. In which case, u is called an eigenfunction associated to λ.

Recall from the introduction that the first eigenvalue of the $p-$Laplacian is given by

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) = \inf \left\{ \frac{\displaystyle\int_\Gamma |u'(x)|^p \, dx}{\displaystyle\int_\Gamma |u(x)|^p \, dx} : u \in \mathcal{X}(\Gamma, \mathrm{V}_D), u \neq 0 \right\}. \tag{3.5}$$

By a standard compactness argument, it follows that there exists an eigenfunction associated to $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$. Note that when $V_D \neq \emptyset$ the norm in $W^{1,p}(\Gamma)$ is equivalent to $(\int_\Gamma |u'|^p)^{1/p} = (\sum_{\mathrm{e}\in \mathrm{E}(\Gamma)} \|u'_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})})^{1/p}$.

Theorem 3.1. *Let Γ be a compact connected quantum graph, V_D be a non-empty subset of $\mathrm{V}(\Gamma)$ and $p \in (1, +\infty)$. Then there exists a non-negative $u_0 \in \mathcal{X}(\Gamma, \mathrm{V}_D)$ such that*

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) = \frac{\displaystyle\int_\Gamma |u_0'(x)|^p \, dx}{\displaystyle\int_\Gamma |u_0(x)|^p \, dx}.$$

Moreover, u_0 is an eigenfunction associated to $\lambda_{1,p}(\Gamma, V_D)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}(\Gamma, V_D)$ be a minimizing sequence for $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$, that is,

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) = \lim_{n\to\infty} \int_\Gamma |u_n'(x)|^p \, dx, \qquad \int_\Gamma |u_n(x)|^p \, dx = 1 \quad \forall n \in \mathbb{N}.$$

Note that we can assume that $u_n \geq 0$. Then, there exists $C > 0$ such that $\|u_n\|_{W^{1,p}(\Gamma)} \leq C$ for all $n \in \mathbb{N}$. Therefore, using that $\mathcal{X}(\Gamma, \mathrm{V}_D)$ is a reflexive space and Theorem 2.2, there exist $u_0 \in \mathcal{X}(\Gamma, V_D)$ and a subsequence that will still call $\{u_n\}_{n\in\mathbb{N}}$ such that

$$u_n \rightharpoonup u_0, \text{ weakly in } \mathcal{X}(\Gamma, \mathrm{V}_D), \tag{3.6}$$

$$u_n \to u_0, \text{ strongly in } L^p(\Gamma). \tag{3.7}$$

As $\|u_n\|_{L^p(\Gamma)} = 1$ for all $n \in \mathbb{N}$, by (3.7), we have that $\|u_0\|_{L^p(\Gamma)} = 1$. Then $u_0 \neq 0$.

On the other hand, by (3.6),

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) = \lim_{n\to\infty} \int_\Gamma |u_n'(x)|^p \, dx \geq \int_\Gamma |u_0'(x)|^p \, dx.$$

Then, by (3.5), we get

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) = \int_\Gamma |u_0'(x)|^p \, dx.$$

Finally, it is clear that u_0 is an eigenfunction of the $p-$Laplacian associated to $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$. □ □

Remark 3.2. *Note that, if $\mathrm{V}_D \subset \mathrm{V}'_D \subset \mathrm{V}(\Gamma)$ then $\lambda_1(\Gamma, \mathrm{V}_D) \leq \lambda_1(\Gamma, \mathrm{V}'_D)$, due to $\mathcal{X}(\Gamma, V'_D) \subset \mathcal{X}(\Gamma, V_D)$.*

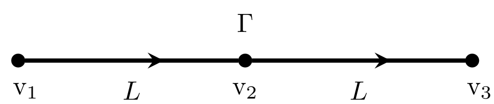
Our next result shows that the first eigenvalue is simple if the Dirichlet vertices are terminal vertices.

Theorem 3.3. *Let Γ be a compact connected quantum graph such that $\mathrm{T}(\Gamma) \neq \emptyset$, and $p \in (1, +\infty)$. If $\mathrm{V}_D \subseteq \mathrm{T}(\Gamma)$ is non-empty then the eigenfunctions associated to $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$ do not change sign and, in addition, $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$ is simple. Here* $\mathrm{card}(\mathrm{V}(\Gamma))$ *is the cardinal number of* $\mathrm{V}(\Gamma)$.

Proof. Let u be an eigenfunction associated to $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$. We have that $|u|$ is also a minimizer of (3.5). Then, without loss of generality, we can assume that $u \geq 0$ in Γ.

Let $\mathrm{v} \in \mathrm{V}_D$ and $\mathrm{u} \in \mathrm{V}(D)$ such that $\mathrm{v} \sim \mathrm{u}$ and $u \neq 0$ in I_{e_0} where $e_0 \in \mathrm{E}(\Gamma)$ and $\mathrm{v}, \mathrm{u} \in e_0$. Then, by the maximum principle (see [30]), we have that $u > 0$ in $(0, \ell_e)$. Moreover if $u(\mathrm{u}) = 0$, by Hopf's lemma, $u'(\mathrm{u}) > 0$, and this contradicts the Kirchhoff conditions at u. Hence $u(\mathrm{u}) > 0$. Then $u > 0$ in $(0, \ell_e)$ for all $e \in E_{\mathrm{u}}(\Gamma)$. We continue in this fashion obtaining $u > 0$ in Γ. Once we have that every eigenfunction does not change sign we get simplicity for $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$ arguing as in [25]. □ □

Remark 3.4. *In general, the first eigenvalue is not simple. For example, let Γ be a simple graph with 3 vertices and 2 edges, that is $\mathrm{V}(\Gamma) = \{\mathrm{v}_1, \mathrm{v}_2, v_3\}$ and $\mathrm{E}(\Gamma) = \{[\mathrm{v}_1, \mathrm{v}_2], [\mathrm{v}_2, \mathrm{v}_3]\}$. Let $\mathrm{V}_D = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$.*



Then $\lambda_{1,p}(\Gamma, V_D) = \left(\frac{\pi_p}{L}\right)^p \frac{p}{p'}$ *and*

$$u(x) = \begin{cases} \dfrac{L}{\pi_p} \sin_p\left(\dfrac{\pi_p}{L} t\right), & \textit{if } x \in I_{[\mathrm{v}_1, \mathrm{v}_2]} = [0, L], \\ 0 & \textit{otherwise}, \end{cases}$$

$$v(x) = \begin{cases} \dfrac{L}{\pi_p} \sin_p\left(\dfrac{\pi_p}{L} t\right), & \textit{if } x \in I_{[\mathrm{v}_2, \mathrm{v}_3]} = [0, L], \\ 0 & \textit{otherwise}, \end{cases}$$

are two linearly independent eigenfunctions associated to $\lambda_{1,p}(\Gamma, V_D)$. The reason for this lack of simplicity is that the vertex v_2 can be understood as a node that disconnects Γ.

Now, we give a lower bound for the first eigenvalue of the $p-$Laplacian which does not depend on $\mathrm{V}(\Gamma)$, $\mathrm{E}(\Gamma)$ and V_D. For the proof of the next theorem we follow the ideas of [21].

Theorem 3.5. *Let Γ be a connected compact metric graph, V_D be a non-empty subset of* $\mathrm{V}(\Gamma)$ *and $p \in (1, +\infty)$. Then*

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) \geq \left(\frac{\pi_p}{2\ell(\Gamma)}\right)^p \frac{p}{p'}.$$

Proof. Let $\widetilde{\Gamma}$ be a metric graph obtained from Γ by doubling each edge. Then $\mathrm{E}(\widetilde{\Gamma}) = \mathrm{E}(\Gamma) \cup \widehat{\mathrm{E}}(\Gamma)$, $\mathrm{V}(\widetilde{\Gamma}) = \mathrm{V}(\Gamma)$, and $d_{\mathrm{v}}(\widetilde{\Gamma})$ is even for all $\mathrm{v} \in \mathrm{V}(\widetilde{\Gamma})$.

On the other hand, given $u \in \mathcal{X}(\Gamma, \mathrm{V}_D)$ we can define $\widetilde{u} \in \mathcal{X}(\widetilde{\Gamma}, \mathrm{V}_D)$ such that

$$\begin{aligned} \widetilde{u}_e(x_{\mathrm{e}}) &= u_e(x_{\mathrm{e}}) && \forall x_{\mathrm{e}} \in I_{\mathrm{e}} \quad \text{if } \mathrm{e} \in \mathrm{E}(\Gamma) \\ \widetilde{u}_e(x_{\mathrm{e}}) &= u_e(\ell_{\mathrm{e}} - x_{\mathrm{e}}) && \forall x_{\mathrm{e}} \in I_{\mathrm{e}} \quad \text{otherwise} . \end{aligned}$$

Moreover

$$\int_{\widetilde{\Gamma}} |\widetilde{u}'(x)|^p \, dx = 2\int_{\Gamma} |u'(x)|^p \, dx, \quad \text{and} \quad \int_{\widetilde{\Gamma}} |\widetilde{u}(x)|^p \, dx = 2\int_{\Gamma} |u(x)|^p \, dx.$$

Then

$$\lambda_{1,p}(\widetilde{\Gamma}, \mathrm{V}_D) \leq \lambda_{1,p}(\Gamma, \mathrm{V}_D). \tag{3.8}$$

On the other hand, there exists a closed path on $\widetilde{\Gamma}$ coming along every edge in $\widetilde{\Gamma}$ precisely one time, due to $d_{\mathrm{v}}(\widetilde{\Gamma})$ is even for all $\mathrm{v} \in \mathrm{V}(\widetilde{\Gamma})$, see [9, 15]. We identify this path with a loop $\mathfrak{L}$ on a vertex $\mathrm{v}_0 \in \mathrm{V}_D$ of length less than or equal to $2\ell(\Gamma)$. Observe that $\mathfrak{L}$ is a metric graph,

$$\ell(\mathfrak{L}) \leq 2\ell(\Gamma) \quad \text{and} \quad \lambda_{1,p}(\mathfrak{L}, , \{\mathrm{v}_0\}) \leq \lambda_{1,p}(\widetilde{\Gamma}, \mathrm{V}_D). \tag{3.9}$$

Moreover,

$$\begin{aligned}
\lambda_{1,p}(\mathfrak{L}, \{\mathrm{v}_0\}) &= \inf\left\{ \frac{\displaystyle\int_{\mathfrak{L}} |u'|^p \, dx}{\displaystyle\int_{\mathfrak{L}} |u|^p \, dx} : u \in \mathcal{X}(\mathfrak{L}, \{\mathrm{v}_0\}), u \neq 0 \right\} \\
&= \inf\left\{ \frac{\displaystyle\int_0^{\ell(\mathfrak{L})} |u'|^p \, dx}{\displaystyle\int_0^{\ell(\mathfrak{L})} |u|^p \, dx} : u \in W_0^{1,p}(0, \ell(\mathfrak{L})), u \neq 0 \right\} \\
&= \left(\frac{\pi_p}{\ell(\mathfrak{L})} \right)^p \frac{p}{p'} \quad \text{(by (2.4))}.
\end{aligned}$$

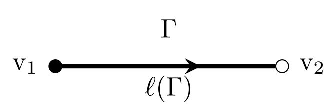
Therefore, by (3.8) and (3.9),

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D) \geq \lambda_{1,p}(\widetilde{\Gamma}, \mathrm{V}_D) \geq \lambda_{1,p}(\mathfrak{L}, \{v_0\}) = \left(\frac{\pi_p}{\ell(\mathfrak{L})} \right)^p \frac{p}{p'} \geq \left(\frac{\pi_p}{2\ell(\Gamma)} \right)^p \frac{p}{p'},$$

which is the desired conclusion. □ □

The lower bound given in the above theorem is optimal as the following example shows.

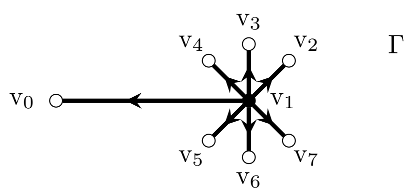
Example 3.6. *Let* Γ *be a simple graph with 2 vertices and an edge, that is* $\mathrm{V}(\Gamma) = \{\mathrm{v}_1, \mathrm{v}_2\}$ *and* $\mathrm{E}(\Gamma) = \{[\mathrm{v}_1, \mathrm{v}_2]\}$. *Let* $\mathrm{V}_D = \{\mathrm{v}_1\}$.



Then

$$\begin{aligned}\lambda_{1,p}(\Gamma,\mathrm{V}_D) &= \inf\left\{\frac{\displaystyle\int_\Gamma |u'|^p\,dx}{\displaystyle\int_\Gamma |u|^p\,dx} : u\in\mathcal{X}(\Gamma,\mathrm{V}_D), u\neq 0\right\}\\ &= \inf\left\{\frac{\displaystyle\int_0^{\ell(\Gamma)} |u'|^p\,dx}{\displaystyle\int_0^{\ell(\Gamma)} |u|^p\,dx} : u\in W^{1,p}(0,\ell(\Gamma)), u(0)=0, u\neq 0\right\}\\ &= \inf\left\{\frac{\displaystyle\int_0^{2\ell(\Gamma)} |u'|^p\,dx}{\displaystyle\int_0^{2\ell(\Gamma)} |u|^p\,dx} : u\in W_0^{1,p}(0,2\ell(\Gamma)), u\neq 0\right\}\\ &= \left(\frac{\pi_p}{2\ell(\Gamma)}\right)^p\frac{p}{p'}.\end{aligned}$$

Example 3.7. *Let* Γ *be a star graph with* $n+1$ *vertices and* n *edges, that is* $\mathrm{V}(\Gamma)=\{\mathrm{v}_0,\mathrm{v}_1,\dots,\mathrm{v}_n\}$ *and* $\mathrm{E}(\Gamma)=\{[\mathrm{v}_1,\mathrm{v}_0],[\mathrm{v}_1,\mathrm{v}_2],\dots,[\mathrm{v}_1,\mathrm{v}_n]\}$. *Let* $\mathrm{V}_D=\{\mathrm{v}_1\}$, $\varepsilon>0$ *and* $\ell([\mathrm{v}_1,\mathrm{v}_0])=L-(m-1)\varepsilon$ *and* $\ell([\mathrm{v}_1,\mathrm{v}_i])=\varepsilon$ *for all* $i\in\{2,\dots,n\}$. *Then* $\ell(\Gamma)=L$.



Then

$$\lambda^{\varepsilon}_{1,p}(\Gamma,V_D)=\left(\frac{\pi_p}{2(L-(n-1)\varepsilon)}\right)^p\frac{p}{p'}\to\left(\frac{\pi_p}{2L}\right)^p\frac{p}{p'}=\left(\frac{\pi_p}{2L}\right)^p\frac{p}{p'}$$

as $\varepsilon\to 0^+$. *Hence, given* $L>0$ *we have that*

$$\inf\{\lambda_{1,p}(\Gamma,\mathrm{V}_D)\colon \Gamma \text{ is a star graph}, \ell(\Gamma)=L, \emptyset\neq\mathrm{V}_D\subset\mathrm{V}(\Gamma)\}$$

is equal to $\left(\frac{\pi_p}{2L}\right)^p\frac{p}{p'}$.

Finally, we give an upper bound for the first eigenvalue of the $p-$Laplacian.

Theorem 3.8. *Let* Γ *be a connected compact metric graph,* V_D *be a non-empty subset of* $\mathrm{V}(\Gamma)$ *and* $p\in(1,+\infty)$. *Then*

$$\lambda_{1,p}(\Gamma,\mathrm{V}_D)\le\left(\frac{\mathrm{card}(E(\Gamma))\pi_p}{\ell(\Gamma)}\right)^p\frac{p}{p'},$$

where $\mathrm{card}(E(\Gamma))$ *is the number of elements in* $\mathrm{E}(\Gamma)$.

Proof. Let $e_0\in\mathrm{E}(\Gamma)$ such that $\ell_{e_0}=\max\{\ell_e\colon e\in\mathrm{E}(\Gamma)\}$. Then

$$\ell_{e_0}\ge\frac{\mathrm{card}(E(\Gamma))}{\ell(\Gamma)}. \tag{3.10}$$

On the other hand, taking

$$u(x) = \begin{cases} \dfrac{\ell_{e_0}}{\pi_p} \sin_p\left(\dfrac{\pi_p}{\ell_{e_0}} x\right) & \text{if } x \in I_{e_0} \\ 0 & \text{otherwise,} \end{cases}$$

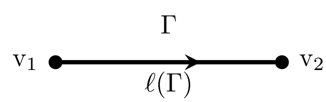
and using (3.10), we have that

$$\lambda_{1,p}(\Gamma, V_D) \leq \frac{\displaystyle\int_{e_0} |u'(x)|\, dx}{\displaystyle\int_{e_0} |u'(x)|\, dx} = \left(\frac{\pi_p}{\ell_{e_0}}\right)^p \frac{p}{p'} \leq \left(\frac{\text{card}(E(\Gamma))\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'}.$$

This completes the proof. □ □

The upper bound is also optimal.

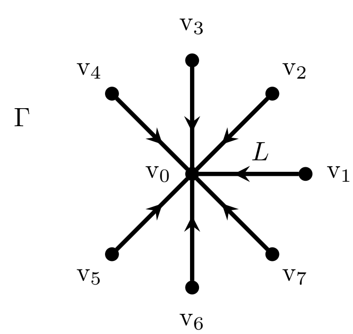
Example 3.9. *Let* Γ *as in Eample 3.6 and* $\text{V}_D = \{\text{v}_1, \text{v}_2\}$.



Then

$$\text{card}(E(\Gamma)) = 1 \quad \textit{and} \quad \lambda_{1,p}(\Gamma, V_D) = \left(\frac{\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'}.$$

Example 3.10. *Let* Γ *be a star graph with* $n+1$ *vertices and* n *edges, that is* $\text{V}(\Gamma) = \{\text{v}_0, \text{v}_1, \dots, \text{v}_n\}$ *and* $\text{E}(\Gamma) = \{[\text{v}_1, \text{v}_0], [\text{v}_2, \text{v}_0], \dots, [\text{v}_n, \text{v}_0]\}$. *Let* $\text{V}_D = \text{V}(\Gamma)$ *and* $\ell([\text{v}_i, \text{v}_0]) = \ell$ *for all* $i \in \{1, \dots, n\}$. *Then* $\ell(\Gamma) = n\ell = \text{card}(\text{E}(\Gamma))\ell$.



Then

$$\lambda_{1,p}(\Gamma, V_D) = \left(\frac{\pi_p}{\ell}\right)^p \frac{p}{p'} = \left(\frac{\text{card}(\text{E}(\Gamma))\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'}.$$

Hence, given $L > 0$ *and* $n \in \mathbb{N}$ *we have that*

$$\max\{\lambda_{1,p}(\Gamma, \text{V}_D) \colon \Gamma \textit{ is a star graph}, \ell(\Gamma) = L, \text{card}(\text{E}(\Gamma)) = n, \emptyset \neq \text{V}_D\}$$

is equal to $\left(\dfrac{n\pi_p}{L}\right)^p \dfrac{p}{p'}$.

4. The shape derivative of $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$.

The aim of this section is to study the perturbation properties of $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$ with respect to the edges.

More precisely, let $e_0 \in \mathrm{E}(\Gamma)$ such that $e_0 = [\mathrm{u}, \mathrm{v}]$, we consider the following family of graphs $\{\Gamma_\delta\}_{\delta \in \mathbb{R}}$ where for any δ

$$\mathrm{V}(\Gamma_\delta) = \mathrm{V}(\Gamma), \quad \mathrm{E}(\Gamma_\delta) = \mathrm{E}(\Gamma_\delta)$$

and the length assigned to $e \in \mathrm{E}(\Gamma_\delta)$ is

$$\ell_e^\delta = \begin{cases} \ell_{e_0} + \delta, & \text{if } e = e_0, \\ \ell_e, & \text{otherwise.} \end{cases}$$

The problem of perturbation of eigenvalues consists in analyzing the dependence of $\lambda(\delta) \coloneqq \lambda_{1,p}(\Gamma_\delta, \mathrm{V}_D)$ with respect to δ. Note that $\lambda(0) = \lambda_{1,p}(\Gamma, \mathrm{V}_D)$.

Lemma 4.1. *Let* Γ *be a connected compact metric graph,* V_D *be a non-empty subset of* $\mathrm{V}(\Gamma)$ *and* $p \in (1, +\infty)$*. Then function* $\lambda(\delta)$ *is continuous at* $\delta = 0$*.*

Proof. Let u be an eigenfunction associated to $\lambda(0)$ with $\|u\|_{L^p(\Gamma)} = 1$. Then

$$w_\delta(x) = \begin{cases} u\left(\dfrac{\ell_{e_0}}{\ell_{e_0} + \delta} x\right) & \text{if } x \in I_{e_0}, \\ u(x) & \text{otherwise,} \end{cases}$$

belongs to $\mathcal{X}(\Gamma_\delta, \mathrm{V}_D)$ for all δ. Therefore for any δ

$$\begin{aligned} \lambda(\delta) &\leq \frac{\displaystyle\int_{\Gamma_\delta} |w_\delta'(x)|^p \, dx}{\displaystyle\int_{\Gamma_\delta} |w_\delta(x)|^p \, dx} \\ &= \frac{\displaystyle\sum_{e \in \mathrm{E}(\Gamma) \setminus \{e_0\}} \int_0^{\ell_e} |u'(x)|^p \, dx + \int_0^{\ell_{e_0} + \delta} \left| u'\left(\frac{\ell_{e_0}}{\ell_{e_0} + \delta} x\right) \right|^p \left(\frac{\ell_{e_0}}{\ell_{e_0} + \delta}\right)^p dx}{\displaystyle\sum_{e \in \mathrm{E}(\Gamma) \setminus \{e_0\}} \int_0^{\ell_e} |u(x)|^p \, dx + \int_0^{\ell_{e_0} + \delta} \left| u\left(\frac{\ell_{e_0}}{\ell_{e_0} + \delta} x\right) \right|^p dx} \\ &= \frac{\displaystyle\sum_{e \in \mathrm{E}(\Gamma) \setminus \{e_0\}} \int_0^{\ell_e} |u'(x)|^p \, dx + \int_0^{\ell_{e_0}} |u'(x)|^p \, dx \left(\frac{\ell_{e_0}}{\ell_{e_0} + \delta}\right)^{p-1}}{\displaystyle\sum_{e \in \mathrm{E}(\Gamma) \setminus \{e_0\}} \int_0^{\ell_e} |u(x)|^p \, dx + \int_0^{\ell_{e_0}} |u(x)|^p \, dx \frac{\ell_{e_0} + \delta}{\ell_{e_0}}}. \end{aligned}$$

Since u is an eigenfunction associated to $\lambda(0)$ and $\|u\|_{L^p(\Gamma)} = 1$, we have that

$$\lambda(\delta) \leq \frac{\lambda(0) + \left[\left(\dfrac{\ell_{e_0}}{\ell_{e_0} + \delta}\right)^{p-1} - 1\right] \displaystyle\int_0^{\ell_{e_0}} |u'(x)|^p \, dx}{1 + \dfrac{\delta}{\ell_{e_0}} \displaystyle\int_0^{\ell_{e_0}} |u(x)|^p \, dx} \qquad \forall \delta. \tag{4.11}$$

Therefore

$$\limsup_{\delta \to 0} \lambda(\delta) \leq \lambda(0). \tag{4.12}$$

Then to show that $\lambda(\delta)$ is continuous at $\lambda = 0$, it remains to prove that

$$\liminf_{\delta\to 0} \lambda(\delta) \geq \lambda(0). \tag{4.13}$$

Let u_δ be an eigenfunction associated to $\lambda(\delta)$ normalized by $\|u_\delta\|_{L^p(\Gamma_\delta)} = 1$. Then, for any δ

$$v_\delta(x) = \begin{cases} u_\delta\left(\dfrac{\ell_{e_0}+\delta}{\ell_{e_0}}x\right) & \text{if } x \in I_{e_0}, \\ u_\delta(x) & \text{otherwise,} \end{cases}$$

belongs to $\mathcal{X}(\Gamma, \mathrm{V}_D)$. Moreover

$$\begin{aligned} \|v_\delta\|^p_{L^p(\Gamma)} &= \int_\Gamma |v_\delta(x)|^p\,dx \\ &= \sum_{e\in \mathrm{E}(\Gamma)\setminus\{e_0\}} \int_0^{\ell_e} |u_\delta(x)|^p\,dx + \int_0^{\ell_{e_0}} \left|u_\delta\left(\frac{\ell_{e_0}+\delta}{\ell_{e_0}}x\right)\right|^p dx \\ &= \sum_{e\in \mathrm{E}(\Gamma)\setminus\{e_0\}} \int_0^{\ell_e} |u_\delta(x)|^p\,dx + \left(\frac{\ell_{e_0}}{\ell_{e_0}+\delta}\right)\int_0^{\ell_{e_0}+\delta} |u_\delta(x)|^p\,dx \\ &= 1 - \frac{\delta}{\ell_{e_0}+\delta}\int_0^{\ell_{e_0}+\delta} |u_\delta(x)|^p\,dx \end{aligned} \tag{4.14}$$

for all δ, and

$$\begin{aligned} \|v'_\delta\|^p_{L^p(\Gamma)} &= \int_\Gamma |v'_\delta(x)|^p\,dx \\ &= \sum_{e\in \mathrm{E}(\Gamma)\setminus\{e_0\}} \int_0^{\ell_e} |u'_\delta(x)|^p\,dx + \int_0^{\ell_{e_0}} \left|u'_\delta\left(\frac{\ell_{e_0}+\delta}{\ell_{e_0}}x\right)\right|^p \left(\frac{\ell_{e_0}+\delta}{\ell_{e_0}}\right)^p dx \\ &= \sum_{e\in \mathrm{E}(\Gamma)\setminus\{e_0\}} \int_0^{\ell_e} |u'_\delta(x)|^p\,dx + \left(1+\frac{\delta}{\ell_{e_0}}\right)^{p-1}\int_0^{\ell_{e_0}+\delta} |u'_\delta(x)|^p\,dx. \end{aligned}$$

Hence

$$\|v'_\delta\|^p_{L^p(\Gamma)} = \lambda(\delta) + \left[\left(1+\frac{\delta}{\ell_{e_0}}\right)^{p-1} - 1\right]\int_0^{\ell_{e_0}+\delta_j} |u'_\delta(x)|^p\,dx. \tag{4.15}$$

Then

$$\lambda(0) \leq \frac{\lambda(\delta) + \left[\left(1+\dfrac{\delta}{\ell_{e_0}}\right)^{p-1} - 1\right]\displaystyle\int_0^{\ell_{e_0}+\delta_j} |u'_\delta(x)|^p\,dx}{1 - \dfrac{\delta}{\ell_{e_0}+\delta}\displaystyle\int_0^{\ell_{e_0}+\delta} |u_\delta(x)|^p\,dx} \tag{4.16}$$

for all δ.

Let $\{\delta_j\}_{j\in\mathbb{N}}$ such that $\delta_j \to 0$ as $j\to\infty$ and

$$\lim_{j\to+\infty} \lambda(\delta_j) = \liminf_{\delta\to 0} \lambda(\delta). \tag{4.17}$$

Then, by (4.11), (4.17), (4.14), and (4.15), $\{v_{\delta_j}\}_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(\Gamma)$. Hence there exist a subsequence (still denote $\{v_{\delta_j}\}_{j\in\mathbb{N}}$) and $u_0 \in \mathcal{X}(\Gamma, \mathrm{V}_D)$ such that

$$\begin{aligned} v_{\delta_j} &\rightharpoonup u_0 && \text{weakly in } W^{1,p}(\Gamma), \\ v_{\delta_j} &\to u_0 && \text{strongly in } L^p(\Gamma). \end{aligned}$$

Then, by (4.14), we have $\|u_0\|_{L^p(\Gamma)} = 1$. In addition, by (4.15) and (4.17), we get

$$\begin{aligned}\lambda(0) &\leq \int_\Gamma |u_0'(x)|^p \, dx \\ &\leq \liminf_{j\to+\infty} \int_\Gamma |v_{\delta_j}'(x)|^p \, dx \\ &\leq \liminf_{j\to+\infty} \lambda(\delta_j) + \left[\left(1+\frac{\delta_j}{\ell_{e_0}}\right)^{p-1} - 1\right] \int_0^{\ell_{e_0}+\delta_j} \left|u_{\delta_j}'(x)\right|^p \, dx \\ &= \liminf_{\delta\to 0} \lambda(\delta).\end{aligned}$$

Therefore (4.13) holds.

Thus, by (4.12) and (4.13), the function $\lambda(\delta)$ is continuous at $\delta = 0$. □ □

Corollary 4.2. *Let Γ be a connected compact metric graph, V_D be a non-empty subset of $\mathrm{V}(\Gamma)$, $p \in (1, +\infty)$ and u_δ be an eigenfunction associated to $\lambda(\delta)$ normalized by $\|u_\delta\|_{L^p(\Gamma_\delta)} = 1$. Then there exists a subsequence $\delta_j \to 0$ and an eigenfunction u_0 associated to $\lambda(0)$ such that*

$$v_{\delta_j} \to u_0 \text{ strongly in } \mathcal{X}(\Gamma, \mathrm{V}_D)$$

as $j \to +\infty$ where

$$v_{\delta_j}(x) = \begin{cases} u_{\delta_j}\left(\dfrac{\ell_{e_0}+\delta}{\ell_{e_0}} x\right) & \text{if } x \in I_{e_0} \\ u_{\delta_j}(x) & \text{otherwise.}\end{cases}$$

Moreover $\|u_0\|_{L^p(\Gamma)} = 1$ and

$$\lim_{j\to\infty} \int_0^{\ell_{e_0}+\delta_j} \left|u_{\delta_j}(x)\right|^p \, dx = \int_0^{\ell_{e_0}} |u_0(x)|^p \, dx,$$
$$\lim_{j\to\infty} \int_0^{\ell_{e_0}+\delta_j} \left|u_j'(x)\right|^p \, dx = \int_0^{\ell_{e_0}} |u_0'(x)|^p \, dx.$$

Proof. Let $\{\delta_j\}_{j\in\mathbb{N}}$ such that $\delta_j \to 0$ as $j \to \infty$. By (4.14) and (4.15), we have that

$$\|v_{\delta_j}\|^p_{L^p(\Gamma)} = 1 - \frac{\delta_j}{\ell_{e_0}+\delta_j} \int_0^{\ell_{e_0}+\delta_j} |u_j(x)|^p \, dx$$
$$\|v_{\delta_j}'\|^p_{L^p(\Gamma)} = \lambda(\delta_j) + \left[\left(1+\frac{\delta_j}{\ell_{e_0}}\right)^{p-1} - 1\right] \int_0^{\ell_{e_0}+\delta_j} \left|u_{\delta_j}'(x)\right|^p \, dx.$$

By Lemma 4.1, we have that $\lambda(\delta_j) \to \lambda(0)$. Then $\{v_{\delta_j}\}_{j\in\mathbb{N}}$ is bounded in $\mathcal{X}(\Gamma, V_D)$,

$$\|v_{\delta_j}\|^p_{L^p(\Gamma)} \to 1, \tag{4.18}$$
$$\|v_{\delta_j}\|^p_{L^p(\Gamma)} \to \lambda(0), \tag{4.19}$$

as $j \to \infty$. Therefore there exists a subsequence (still denoted $\{v_{\delta_j}\}_{j\in\mathbb{N}}$) and $u_0 \in \mathcal{X}(\Gamma, \mathrm{V}_D)$ such that

$$\begin{aligned} v_{\delta_j} &\rightharpoonup u_0 && \text{weakly in } W^{1,p}(\Gamma), \\ v_{\delta_j} &\to u_0 && \text{strongly in } L^p(\Gamma).\end{aligned}$$

Then, by (4.11), we have $\|u_0\|_{L^p(\Gamma)} = 1$. In addition, by (4.12), we get

$$\lambda(0) = \int_\Gamma |u_0'(x)|^p \, dx \le \liminf_{j\to+\infty} \int_\Gamma |v_j'(x)|^p \, dx = \lambda(0).$$

Therefore u_0 is an eigenfunction associated to $\lambda(0)$ and

$$\|v_{\delta_j}\|_{W^{1,p}(\Gamma)} \to \|u_0\|_{W^{1,p}(\Gamma)}$$

as $j \to \infty$. Since $v_{\delta_j} \rightharpoonup u_0$ weakly in $W^{1,p}(\Gamma)$, we have that $v_{\delta_j} \to u_0$ strongly in $W^{1,p}(\Gamma)$. Then $v_{\delta_j} \to u_0$ strongly in $W^{1,p}(I_{e_0})$ and hence

$$\int_0^{\ell_{e_0}} |u_0|^p \, dx = \lim_{j\to\infty} \int_0^{\ell_{e_0}} |v_{\delta_j}|^p \, dx = \lim_{j\to\infty} \left(\frac{\ell_{e_0}}{\ell_{e_0} + \delta_j} \right) \int_0^{\ell_{e_0}+\delta_j} \left|u_{\delta_j}(x)\right|^p \, dx,$$

$$\int_0^{\ell_{e_0}} |u_0'|^p \, dx = \lim_{j\to\infty} \int_0^{\ell_{e_0}} |v_{\delta_j}'|^p dx = \lim_{j\to\infty} \left(1 + \frac{\delta_j}{\ell_{e_0}}\right)^{p-1} \int_0^{\ell_{e_0}+\delta_j} \left|u_{\delta_j}'(x)\right|^p \, dx,$$

that is

$$\int_0^{\ell_{e_0}} |u_0(x)|^p \, dx = \lim_{j\to\infty} \int_0^{\ell_{e_0}+\delta_j} \left|u_{\delta_j}(x)\right|^p dx,$$

$$\int_0^{\ell_{e_0}} |u_0'(x)|^p \, dx = \lim_{j\to\infty} \int_0^{\ell_{e_0}+\delta_j} \left|u_j'(x)\right|^p dx,$$

which completes the proof. □ □

Before proving that the function λ is differentiable at $\delta = 0$ when the first eigenvalue is simple, we will show that, in the general case, λ is differentiable from the left and from the right at $\delta = 0$.

Lemma 4.3. *Let Γ be a connected compact metric graph, V_D be a non-empty subset of $\mathrm{V}(\Gamma)$ and $p \in (1, +\infty)$. Then the function $\lambda(\delta)$ is left and right differentiable at $\delta = 0$ and*

$$\lim_{\delta\to 0^+} \frac{\lambda(\delta) - \lambda(0)}{\delta} = \min_{u\in\mathcal{E}} \left\{ -\frac{(p-1)}{\ell_{e_0}} \int_0^{\ell_{e_0}} |u_0'|^p - \frac{\lambda(0)}{\ell_{e_0}} \int_0^{\ell_{e_0}} |u_0|^p \right\},$$

$$\lim_{\delta\to 0^-} \frac{\lambda(\delta) - \lambda(0)}{\delta} = \max_{u\in\mathcal{E}} \left\{ -\frac{(p-1)}{\ell_{e_0}} \int_0^{\ell_{e_0}} |v_0'|^p - \frac{\lambda(0)}{\ell_{e_0}} \int_0^{\ell_{e_0}} |v_0|^p \right\},$$

where $\mathcal{E}$ is the set of eigenfunctions u associated to $\lambda(0)$ normalized with $\|u\|_{L^p(\Gamma)} =$ 1.

Proof. We split the proof in several steps.

Step 1. We start by showing that

$$\limsup_{\delta\to 0^+} \frac{\lambda(\delta) - \lambda(0)}{\delta} \le -\frac{(p-1)}{\ell_{e_0}} \int_0^{\ell_{e_0}} |u'(x)|^p \, dx - \frac{\lambda(0)}{\ell_{e_0}} \int_0^{\ell_{e_0}} |u(x)|^p \, dx$$

for any eigenfunction u associated to $\lambda(0)$ normalized by $\|u\|_{L^p(\Gamma)} = 1$.

Let u be an eigenfunction associated to $\lambda(0)$ normalized by $\|u\|_{L^p(\Gamma)} = 1$. By (4.11), we have

$$\lambda(\delta) - \lambda(0) \le \frac{\left[\left(\frac{\ell_{e_0}}{\ell_{e_0} + \delta} \right)^{p-1} - 1 \right] \int_0^{\ell_{e_0}} |u'(x)|^p \, dx - \lambda(0) \frac{\delta}{\ell_{e_0}} \int_0^{\ell_{e_0}} |u(x)|^p \, dx}{1 + \frac{\delta}{\ell_{e_0}} \int_0^{\ell_{e_0}} |u(x)|^p \, dx}$$

for all δ. Then

$$\frac{\lambda(\delta)-\lambda(0)}{\delta} \leq \frac{\dfrac{\left(\dfrac{\ell_{e_0}}{\ell_{e_0}+\delta}\right)^{p-1}-1}{\delta}\displaystyle\int_0^{\ell_{e_0}} |u'(x)\,|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u(x)|^p \;dx}{1+\dfrac{\delta}{\ell_{e_0}}\displaystyle\int_0^{\ell_{e_0}} |u(x)|^p \;dx}$$

for all $\delta > 0$. Therefore

$$\limsup_{\delta\to 0^+} \frac{\lambda(\delta)-\lambda(0)}{\delta} \leq -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u'(x)\,|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u(x)|^p \;dx.$$

Step 2. With a similar procedure, we obtain

$$\liminf_{\delta\to 0^-} \frac{\lambda(\delta)-\lambda(0)}{\delta} \geq -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u'(x)\,|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u(x)|^p \;dx.$$

for any eigenfunction u associated to $\lambda(0)$ normalized by $\|u\|_{L^p(\Gamma)} = 1$.

Step 3. Now we show that there exists an eigenfunction u_0 associated to $\lambda(0)$ normalized by $\|u_0\|_{L^p(\Gamma)} = 1$ such that

$$\liminf_{\delta\to 0^+} \frac{\lambda(\delta)-\lambda(0)}{\delta} \geq -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0'(x)\,|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0(x)|^p \;dx.$$

Let u_δ be an eigenfunction associated to $\lambda(\delta)$ normalized by $\|u_\delta\|_{L^p(\Gamma_\delta)} = 1$. By (4.16), we have

$$\lambda(\delta)-\lambda(0) \geq -\frac{A(\delta)}{B(\delta)} \qquad \forall\delta$$

where

$$A(\delta) = \frac{\lambda(\delta)\delta}{(\ell_{e_0}+\delta)}\int_0^{\ell_{e_0}+\delta} |u_\delta(x)|^p\,dx - \left[(1+{}^{\delta}/_{\ell_{e_0}})^{p-1}-1\right]\int_0^{\ell_{e_0}+\delta_j} |u_\delta'(x)|^p\,dx$$

$$B(\delta) = 1 - \frac{\delta}{(\ell_{e_0}+\delta)}\int_0^{\ell_{e_0}+\delta} |u_\delta(x)|^p\,dx.$$

Then

$$\frac{\lambda(\delta)-\lambda(0)}{\delta} \geq \frac{\frac{A(\delta)}{\delta}}{B(\delta)} \tag{4.20}$$

for all $\delta > 0$. Let $\{\delta_j\}_{j\in\mathbb{N}}$ such that $\delta_j \to 0^+$ as $j\to\infty$ and

$$\lim_{j\to+\infty} \frac{\lambda(\delta_j)-\lambda(0)}{\delta_j} = \liminf_{\delta\to 0^+} \frac{\lambda(\delta)-\lambda(0)}{\delta}. \tag{4.21}$$

Then, by Corollary 4.2, there exist a subsequence (still denoted δ_j) and an eigenfunction u_0 associated to $\lambda(0)$ such that

$$\|u_0\|_{L^p(\Gamma)} = 1,$$
$$\lim_{j\to\infty}\int_0^{\ell_{e_0}+\delta_j} \left|u_{\delta_j}(x)\right|^p\;dx = \int_0^{\ell_{e_0}} |u_0(x)|^p\,dx,$$
$$\lim_{j\to\infty}\int_0^{\ell_{e_0}+\delta_j} \left|u_j'(x)\right|^p\;dx = \int_0^{\ell_{e_0}} |u_0'(x)|^p\,dx.$$

Therefore

$$\lim_{j\to+\infty}\frac{A(\delta_j)}{\delta_j} = -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0'(x)|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0(x)|^p\,dx,$$
$$\lim_{j\to+\infty} B(\delta_j) = 1.$$

In addition, by (4.20) and (4.21), we get

$$\liminf_{\delta\to 0^+}\frac{\lambda(\delta)-\lambda(0)}{\delta} \geq -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0'(x)|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0(x)|^p\,dx.$$

Hence, by *step 1*, we have that

$$\lim_{\delta\to 0^+}\frac{\lambda(\delta)-\lambda(0)}{\delta} = -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0'(x)|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0(x)|^p\,dx.$$

Step 4. In the same way, we can show that there exists an eigenfunction v_0 associated to $\lambda(0)$ such that

$$\lim_{\delta\to 0^-}\frac{\lambda(\delta)-\lambda(0)}{\delta} = -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |v_0'(x)|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |v_0(x)|^p\,dx.$$

□ □

Thus, if the first eigenvalue is simple then the function $\lambda(\delta)$ is differentiable at $\delta = 0$.

Theorem 4.4. *Let Γ be a connected compact metric graph, V_D be a non-empty subset of $\mathrm{V}(\Gamma)$ and $p \in (1, +\infty)$. If the first eignevalue $\lambda_{1,p}(\Gamma, \mathrm{V}_D)$ is simple, then the function $\lambda(\delta)$ is differentiable at $\delta = 0$ and*

$$\lambda'(0) = -\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0'(x)|^p\,dx - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |u_0(x)|^p\,dx$$

where u_0 is an eigenfunction associated to $\lambda(0)$ normalized by $\|u\|_{L^p(\Gamma)} = 1$.

Remark 4.5. *Note that the result of Theorem 4.4 does not hold if we remove the assumption that the first eigenvalue is simple. For example, let Γ defined as in Remark 3.4 and $e_0 = [\mathrm{v}_2, \mathrm{v}_3]$ we have that*

$$\begin{aligned}\lim_{t\to 0^+}\frac{\lambda(\delta)-\lambda(0)}{\delta} &= \min_{u\in\mathcal{E}}\left\{-\frac{(p-1)}{L}\int_0^{\ell_{e_0}} |u_0'|^p - \frac{\lambda(0)}{L}\int_0^{\ell_{e_0}} |u_0|^p\right\}\\ &= \min_{u\in\mathcal{E}}\left\{-\frac{p\lambda(0)}{L}\int_0^{\ell_{e_0}} |u_0'|^p\right\} = -\frac{p\lambda(0)}{L},\end{aligned}$$

$$\begin{aligned}\lim_{t\to 0^-}\frac{\lambda(\delta)-\lambda(0)}{\delta} &= \max_{u\in\mathcal{E}}\left\{-\frac{(p-1)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |v_0'|^p - \frac{\lambda(0)}{\ell_{e_0}}\int_0^{\ell_{e_0}} |v_0|^p\right\}\\ &= \max_{u\in\mathcal{E}}\left\{-\frac{p\lambda(0)}{L}\int_0^{\ell_{e_0}} |u_0'|^p\right\} = 0.\end{aligned}$$

Hence λ is not differentiable (but Lipschitz) at $\delta = 0$.

5. The limit as $p \to \infty$.

In this section we deal with the limit as $p \to \infty$ of the eigenvalue problem (3.5).

Theorem 5.1. *Let Γ be a connected compact metric graph, V_D be a non-empty subset of $\mathrm{V}(\Gamma)$, and u_p be a minimizer for (3.5) normalized by $\|u_p\|_{L^p(\Gamma)} = 1$. Then, there exists a sequence $p_j \to \infty$ such that*

$$u_{p_j} \to u_\infty$$

uniformly in Γ and weakly in $W^{1,q}(\Gamma)$ for every $q < \infty$.

Moreover, any possible limit u_∞ is a minimizer for

$$\Lambda_\infty(\Gamma, \mathrm{V}_D) = \inf \left\{ \frac{\|v'\|_{L^\infty(\Gamma)}}{\|v\|_{L^\infty(\Gamma)}} : v \in W^{1,\infty}(\Gamma), v = 0 \text{ on } \mathrm{V}_D, v \neq 0 \right\}.$$

This value $\Lambda_\infty(\Gamma, \mathrm{V}_D)$ is the limit of $\lambda_{1,p}(\Gamma, V_D)^{1/p}$ and can be characterized as

$$\Lambda_\infty(\Gamma, \mathrm{V}_D) = \frac{1}{\max\limits_{x \in \Gamma} d(x, \mathrm{V}_D)}.$$

Note that

$$\max_{z \in \Gamma} d(x, \mathrm{V}_D) = \frac{1}{2} \max_{z \in \Gamma} \min_{\mathrm{v} \in \mathrm{V}_D} d(x, \mathrm{v}).$$

Proof. In this proof we use ideas from [17]. Let u_p be an eigenfunction associated with $\lambda_{1,p}(\Gamma, V_D)$ normalized by $\|u_p\|_{L^p(\Gamma)} = 1$. We first prove a uniform bound (independent of p) for the L^p-norm of u_p'.To this end, take v any smooth function that vanishes on V_D. Using that u_p is a minimizer for (3.5) we obtain

$$\frac{\displaystyle\int_\Gamma |u_p'(x)|^p \, dx}{\displaystyle\int_\Gamma |u_p(x)|^p \, dx} \leq \frac{\displaystyle\int_\Gamma |v'(x)|^p \, dx}{\displaystyle\int_\Gamma |v(x)|^p \, dx},$$

hence we get

$$\left(\int_\Gamma |u_p'(x)|^p \, dx \right)^{1/p} \leq \left(\frac{\displaystyle\int_\Gamma |v'(x)|^p \, dx}{\displaystyle\int_\Gamma |v(x)|^p \, dx} \right)^{1/p}.$$

Now we observe that

$$\left(\frac{\displaystyle\int_\Gamma |v'(x)|^p \, dx}{\displaystyle\int_\Gamma |v(x)|^p \, dx} \right)^{1/p} \to \frac{\|v'\|_{L^\infty(\Gamma)}}{\|v\|_{L^\infty(\Gamma)}}$$

as $p \to \infty$. Therefore, we conclude that there exists a constant C independent of p such that

$$\left(\int_\Gamma |u_p'(x)|^p \, dx \right)^{1/p} \leq C.$$

Then, by Hölder inequality, we have

$$\begin{aligned} \left(\int_\Gamma |u_p'(x)|^q \, dx \right)^{1/q} &\leq \mathrm{card}(\mathrm{E}(\Gamma))^{1/q} \left(\int_\Gamma |u_p'(x)|^p \, dx \right)^{1/p} \ell(\Gamma)^{(p-q)/pq} \\ &\leq C \mathrm{card}(\mathrm{E}(\Gamma))^{1/q} \ell(\Gamma)^{(p-q)/pq} \end{aligned}$$

for all $1 \leq q \leq p$. Then we obtain that the family $\{u_p\}_{p\geq q}$ is bounded in $W^{1,q}(\Gamma)$ for any $q < \infty$ and therefore by a diagonal procedure we can extract a sequence $p_j \to \infty$ such that

$$u_{p_j} \to u_\infty$$

uniformly in Γ and weakly in $W^{1,q}(\Gamma)$ for every $q < \infty$.

From our previous computations we obtain

$$\left(\int_\Gamma |u_\infty'(x)|^q \, dx\right)^{1/q} \leq \frac{\|v'\|_{L^\infty(\Gamma)}}{\|v\|_{L^\infty(\Gamma)}} \mathrm{card}(\mathrm{E}(\Gamma))^{1/q} \ell(\Gamma)^{1/q}$$

and then (taking $q \to \infty$) we conclude that

$$\|u_\infty'\|_{L^\infty(\Gamma)} \leq \frac{\|v'\|_{L^\infty(\Gamma)}}{\|v\|_{L^\infty(\Gamma)}},$$

for every v smooth that vanishes on V_D.

Now, using that u_{p_j} converges uniformly to u_∞ we obtain that

$$\|u_\infty\|_{L^\infty(\Gamma)} = 1.$$

In fact, we have

$$\begin{aligned} \left(\int_\Gamma |u_\infty(x)|^p \, dx\right)^{1/p} &\leq \left(\int_\Gamma |u_\infty(x) - u_p(x)|^p \, dx\right)^{1/p} + \left(\int_\Gamma |u_p(x)|^p \, dx\right)^{1/p} \\ &= \left(\int_\Gamma |u_\infty(x) - u_p(x)|^p \, dx\right)^{1/p} + 1. \end{aligned}$$

Now we have that

$$\left(\int_\Gamma |u_\infty(x) - u_p(x)|^p \, dx\right)^{1/p} \leq \|u_\infty - u_p\|_{L^\infty(\Gamma)} \ell(\Gamma)^{1/p} \to 0$$

as $p \to \infty$ and we conclude that $\|u_\infty\|_{L^\infty(\Gamma)} \leq 1$. On the other hand,

$$\begin{aligned} 1 &= \left(\int_\Gamma |u_p(x)|^p \, dx\right)^{1/p} \\ &\leq \left(\int_\Gamma |u_\infty(x) - u_p(x)|^p \, dx\right)^{1/p} + \left(\int_\Gamma |u_\infty(x)|^p \, dx\right)^{1/p} \end{aligned}$$

and then we obtain the reverse inequality, $\|u_\infty\|_{L^\infty(\Gamma)} \geq 1$.

We have proved that u_∞ is a minimizer for

$$\Lambda_\infty(\Gamma, \mathrm{V}_D) = \inf \left\{ \frac{\|v'\|_{L^\infty(\Gamma)}}{\|v\|_{L^\infty(\Gamma)}} : v \in W^{1,\infty}(\Gamma), v = 0 \text{ on } \mathrm{V}_D, v \neq 0 \right\}.$$

and that

$$\lambda_{1,p}(\Gamma, \mathrm{V}_D)^{1/p} \to \Lambda_\infty(\Gamma, \mathrm{V}_D)$$

as $p \to \infty$.

It remains to show that

$$\Lambda_\infty(\Gamma, \mathrm{V}_D) = \frac{1}{\max\limits_{z\in\Gamma} d(z, \mathrm{V}_D)}.$$

To this end, first let us consider a point $z_0 \in \Gamma$ such that

$$\max_{z\in\Gamma} d(z, V_1) = d(z_0, V_1)$$

and the cone

$$v(x) = \left(1 - \frac{1}{d(z_0, \mathrm{V}_D)} d(x, z_0)\right)_+ .$$

This function v is Lipschitz and vanishes on V_D, hence it is a competitor for the infimum for $\Lambda_\infty(\Gamma, \mathrm{V}_D)$ and then we get

$$\Lambda_\infty(\Gamma, \mathrm{V}_D) \leq \frac{1}{d(z_0, \mathrm{V}_D)} = \frac{1}{\max_{z\in\Gamma} d(z, \mathrm{V}_D)}.$$

To see the reverse inequality we argue as follows: let v be a smooth function vanishing on V_D and normalize it according to $\|v\|_{L^\infty(\Gamma)} = 1$. Let $z_1 \in \Gamma$ be such that $v(z_1) = 1$. Since $z_1 \in \Gamma$ it holds that

$$\max_{z\in\Gamma} d(z, \mathrm{V}_D) \geq d(z_1, \mathrm{V}_D).$$

Hence there is a vertex $\mathrm{v} \in V_D$ such that

$$\max_{z\in\Gamma} d(z, \mathrm{V}_D) \geq d(z_1, \mathrm{v}),$$

and we get

$$1 = v(z_1) - v(\mathrm{v}) = v'(\xi) d(z_1, \mathrm{v}) \leq |v'(\xi)| \max_{z\in\Gamma} d(z, \mathrm{V}_D).$$

We conclude that

$$\|v'\|_{L^\infty(\Gamma)} \geq \frac{1}{\max_{z\in\Gamma} d(z, \mathrm{V}_D)}$$

and therefore

$$\Lambda_\infty(\Gamma, \mathrm{V}_D) \geq \frac{1}{\max_{z\in\Gamma} d(z, \mathrm{V}_D)}.$$

This ends the proof. □ □

6. The limit as $p \to 1$.

In this section we study the other limit case, $p = 1$. We will use functions of bounded variation on the graph (that we will denote by $BV(\Gamma)$) and the perimeter of a subset of the graph (denoted by $\mathrm{Per}(D)$). We refer to [1] for precise definitions and properties of functions and sets in this context.

Theorem 6.1. *Let Γ be a connected compact metric graph, V_D be a non-empty subset of $\mathrm{V}(\Gamma)$, and u_p be a minimizer for (3.5) normalized by $\|u_p\|_{L^1(\Gamma)} = 1$. Then, there exists a sequence $p_j \to 1^+$ such that*

$$u_{p_j} \to u_1$$

in $L^1(\Gamma)$.

Moreover, any possible limit u_1 is a minimizer for

$$\Lambda_1(\Gamma, V_D) = \inf\left\{ \frac{\|v'\|_{BV(\Gamma)}}{\|v\|_{L^1(\Gamma)}} : v \in BV(\Gamma), v = 0 \text{ on } \mathrm{V}_D, v \neq 0 \right\}.$$

This value $\Lambda_1(\Gamma, V_D)$ is the limit of $\lambda_{1,p}(\Gamma, V_D)$.

Proof. Without loss of generality, we can assume that $u_p(x) \geq 0$ for all $x \in \Gamma$. Let $v_p = (u_p)^p$. Then $v_p \in W^{1,1}(\Omega)$ and

$$\int_\Gamma |v_p(x)|\, dx = 1$$

$$\begin{aligned}\int_\Gamma |v_p'(x)|\, dx &= p \int_\Gamma u(x)^{p-1} |u'(x)|\, dx \\ &\leq p \left(\int_\Gamma u(x)^p\, dx \right)^{1/p'} \left(\int_\Gamma |u'(x)|^p\, dx \right)^{1/p} \\ &= p \left(\int_\Gamma |u'(x)|^p\, dx \right)^{1/p}.\end{aligned}$$

Hence

$$\begin{aligned}\Lambda_1(\Gamma, V_D) \leq \frac{\|v_p'\|_{BV(\Gamma)}}{\|v_p\|_{L^1(\Gamma)}} &\leq p \frac{\left(\int_\Gamma |u_p'(x)|^p\, dx \right)^{1/p}}{\left(\int_\Gamma |u_p(x)|^p\, dx \right)^{1/p}} \\ &= p \lambda_{1,p}(\Gamma, \mathrm{V}_D)^{1/p}.\end{aligned}$$

From where we get

$$\Lambda_1(\Gamma, \mathrm{V}_D) \leq \liminf_{p \to 1^+} \lambda_1(\Gamma, \mathrm{V}_D)^{1/p}. \tag{6.22}$$

On the other hand, for any smooth function v that vanishes on V_D we have

$$\lambda_1(\Gamma, \mathrm{V}_D)^{1/p} \leq \frac{\left(\int_\Gamma |v'(x)|^p\, dx \right)^{1/p}}{\left(\int_\Gamma |v(x)|^p\, dx \right)^{1/p}}$$

from where it follows

$$\limsup_{p \to 1^+} \lambda_1(\Gamma, \mathrm{V}_D)^{1/p} \leq \frac{\int_\Gamma |v'(x)|\, dx}{\int_\Gamma |v(x)|\, dx}$$

and we conclude that

$$\limsup_{p \to 1^+} \lambda_1(\Gamma, \mathrm{V}_D)^{1/p} \leq \Lambda_1(\Gamma, \mathrm{V}_D). \tag{6.23}$$

Therefore, from (6.22) and (6.23) we obtain

$$\lim_{p \to 1^+} \lambda_1(\Gamma, \mathrm{V}_D) = \Lambda_1(\Gamma, \mathrm{V}_D). \tag{6.24}$$

Moreover, by [10, Theorem 4 Section 5.2.3] we have that there is $u_1 \in BV(\Gamma)$ such that

$$\|u_{p_j} - u_1\|_{L^1(\Gamma)} \to 0$$

for a sequence $p_j \to 1^+$. From the lower semicontinuity of the variation measure (see [10, Theorem 1 Section 5.2.1]), we have

$$\|u_1\|_{BV(\Gamma)} \leq \liminf_{p_j \to 1} \|u_{p_j}\|_{BV(\Gamma)}.$$

From this we conclude that every possible limit of a sequence of u_p as $p \to 1$ is an extramal for $\Lambda_1(\Gamma, \mathrm{V}_D)$. □ □

Theorem 6.2. *It holds that*

$$\Lambda_1(\Gamma, \mathrm{V}_D) = \inf \left\{ \frac{\mathrm{Per}(D)}{|D|} : D \subset \Gamma,\, D \cap \mathrm{V}_D = \emptyset \right\}.$$

Proof. We have

$$\Lambda_1(\Gamma, \mathrm{V}_D) \le \lambda = \inf \left\{ \frac{\mathrm{Per}(D)}{|D|} : D \subset \Gamma,\, D \cap \mathrm{V}_D = \emptyset \right\}.$$

By Theorem 6.1 there exists a function $u \in BV(\Gamma)$, $u \neq 0$, such that

$$\Lambda_1(\Gamma, \mathrm{V}_D) = \frac{\|u'\|_{BV(\Gamma)}}{\|u\|_{L^1(\Gamma)}}.$$

We can consider without loss of generality that $u \ge 0$. Let

$$E_t \coloneqq \{x \in \Gamma \colon u(x) > t\}.$$

We have

$$|u'|(\Gamma) = \int_0^\infty \mathrm{Per}(E_t) dt.$$

Hence, we get using Cavalieri's principle,

$$\begin{aligned} 0 &= \|u'\|_{BV(\Gamma)} - \Lambda_1(\Gamma, \mathrm{V}_D)\|u\|_{L^1(\Gamma)} \\ &= \int_0^\infty (\mathrm{Per}(E_t) - \Lambda_1(\Gamma, \mathrm{V}_D)|E_t|) dt \\ &\ge \int_0^\infty (\mathrm{Per}(E_t) - \lambda |E_t|) dt \ge 0. \end{aligned}$$

Therefore, we conclude that for almost every $t \in \mathbb{R}$ (in the sense of the Lebesgue measure on $\mathbb{R}$),

$$\mathrm{Per}(E_t) = \lambda |E_t|$$

and

$$\lambda = \Lambda_1(\Gamma, \mathrm{V}_D).$$

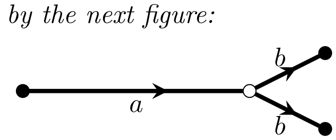
□ □

Sets D^* such that

$$\inf \left\{ \frac{\mathrm{Per}(D)}{|D|} : D \subset \Gamma,\, D \cap \mathrm{V}_D = \emptyset \right\} = \frac{\mathrm{Per}(D^*)}{|D^*|}$$

are called Cheeger sets. See [28] and references therein.

Example 6.3. *To see that the optimal value $\Lambda_1(\Gamma, \mathrm{V}_D)$ depends strongly on the geometric configuration of the graph Γ, let us consider the following example: let Γ be a simple graph with 4 nodes (3 of them, the terminal nodes, are in V_D) and 3 edges as the one described by the next figure:*



Let us compute

$$\Lambda_1(\Gamma, \mathrm{V}_D) = \inf\left\{ \frac{\|v'\|_{BV(\Gamma)}}{\|v\|_{L^1(\Gamma)}} : v \in BV(\Gamma), v = 0 \text{ on } \mathrm{V}_D, v \neq 0 \right\}$$
$$= \inf\left\{ \frac{\mathrm{Per}(D)}{|D|} : D \subset \Gamma,\ D \cap \mathrm{V}_D = \emptyset \right\},$$

in this case. As we will see its value (and the corresponding optimal set D^*) depends on the lengths a and b.

First, let us compute the value of $\frac{\mathrm{Per}(D)}{|D|}$ *for $D = \Gamma$. We have*

$$|\Gamma| = \ell(\Gamma) = a + 2b, \qquad \text{and} \qquad \mathrm{Per}(\Gamma) = 3.$$

Hence

$$\frac{\mathrm{Per}(\Gamma)}{\ell(\Gamma)} = \frac{3}{a+2b}.$$

On the other hand, if we consider D_a the characteristic function of the edge of length a we obtain

$$|D_a| = a, \qquad \text{and} \qquad \mathrm{Per}(D_a) = 2,$$

and then

$$\frac{\mathrm{Per}(D_a)}{|D_a|} = \frac{2}{a}.$$

Now we remark that any other subset D of Γ has a ratio $\frac{\mathrm{Per}(D)}{|D|}$ *bigger or equal than one of the previous two sets. Therefore, we conclude that*

$$\Lambda_1(\Gamma, \mathrm{V}_D) = \begin{cases} \dfrac{3}{a+2b}, & \text{if } a \leq 4b, \\ \dfrac{2}{a} & \text{if } a > 4b. \end{cases}$$

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