EIGENVALUES FOR A NONLOCAL PSEUDO $p-$LAPLACIAN

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ABSTRACT. In this paper we study the eigenvalue problems for a nonlocal operator of order s that is analogous to the local pseudo $p-$Laplacian. We show that there is a sequence of eigenvalues $\lambda_n \to \infty$ and that the first one is positive, simple, isolated and has a positive and bounded associated eigenfunction. For the first eigenvalue we also analyze the limits as $p \to \infty$ (obtaining a limit nonlocal eigenvalue problem analogous to the pseudo infinity Laplacian) and as $s \to 1^-$ (obtaining the first eigenvalue for a local operator of $p-$Laplacian type). To perform this study we have to introduce anisotropic fractional Sobolev spaces and prove some of their properties.

1. INTRODUCTION

Our main goal is to introduce a nonlocal operator that is a nonlocal analogous to the local pseudo $p-$Laplacian, $\Delta_{p,x}u + \Delta_{p,y}u$ (here the subindexes x and y denote differentiation with respect to the $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ variables respectively). The local pseudo $p-$Laplacian appears naturally when one considers critical points of the functional $F(u) = \int_\Omega |\nabla_x u|^p + |\nabla_y u|^p \, dxdy$. See [5, 14, 25, 33, 34]. On the other hand, recently, it was introduced a nonlocal $p-$Laplacian that is given by

$$(-\Delta)^s_p v(x) = 2 \text{ P.V.} \int_{\mathbb{R}^k} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{k+ps}} \, dx,$$

the symbol P.V. stands for the principal value of the integral. We will omit it in what follows. For references involving this kind of operator we refer to [9, 16, 18, 23, 24, 26, 29, 30, 32, 31] and references therein.

Here, we introduce the following nonlocal operator that we will call the nonlocal pseudo $p-$Laplacian,

$$\begin{aligned}\mathcal{L}_{s,p}(u)(x,y) \coloneqq 2 \int_{\mathbb{R}^n} & \frac{|u(x,y) - u(z,y)|^{p-2}(u(x,y) - u(z,y))}{|x-z|^{n+sp}} dz \\ & + 2 \int_{\mathbb{R}^m} \frac{|u(x,y) - u(x,w)|^{p-2}(u(x,y) - u(x,w))}{|y-w|^{m+sp}} dw.\end{aligned}$$

The natural space to consider when one deals with the operator $\mathcal{L}_{s,p}$ is given by

$$\mathcal{W}^{s,p}(\mathbb{R}^{n+m}) \coloneqq \left\{ u \in L^p(\mathbb{R}^{n+m}) \colon [u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} < \infty \right\},$$

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where for $p < +\infty$,

$$[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \coloneqq \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy + \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{|u(x,y)-u(x,w)|^p}{|y-w|^{m+sp}} dwdxdy$$

and for $p = +\infty$,

$$[u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} \coloneqq \max\left\{ \sup\left\{ \frac{|u(x,y)-u(z,y)|}{|x-z|^s} : (x,y) \neq (z,y) \right\} ; \sup\left\{ \frac{|u(x,y)-u(x,w)|}{|y-w|^s} : (x,y) \neq (x,w) \right\} \right\}.$$

In this paper, we deal with the eigenvalue problem for this operator, that is, given a bounded domain Ω we look for pairs (λ, u) such that $\lambda \in \mathbb{R}$ and $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\}$ are such that u is a weak solution of

$$\begin{cases} \mathcal{L}_{s,p} u(x,y) = \lambda |u(x,y)|^{p-2} u(x,y) & \text{in } \Omega, \\ u(x,y) = 0 & \text{in } \Omega^c = \mathbb{R}^{n+m} \setminus \Omega. \end{cases}$$

Here $\widetilde{\mathcal{W}}^{s,p}(\Omega) = \{u \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m}) \colon u \equiv 0 \text{ in } \Omega^c\}$. We will study the Dirichlet problem for this operator in a companion paper.

We impose the following assumptions on the data:

A1. Ω is a bounded Lipschitz domain in $\mathbb{R}^{n+m}$;
A2. $s \in (0,1)$, and $p \in (1, +\infty)$.

Under these conditions we have the following result.

Theorem 1.1. *There exists a sequence of eigenvalues λ_n such that $\lambda_n \to +\infty$ as $n \to +\infty$. Moreover, every eigenfunction is in $L^\infty(\mathbb{R}^{n+m})$. The first eigenvalue (the smallest eigenvalue) is given by*

$$\lambda_1(s,p) \coloneqq \inf\left\{ \frac{[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}}{\|u\|^p_{L^p(\Omega)}} : u \in \widetilde{\mathcal{W}}^{s,p}(\Omega), u \not\equiv 0 \right\}.$$

This eigenvalue $\lambda_1(s,p)$ is simple, isolated and an associated eigenfunction is strictly positive (or negative) in Ω.

Next, we analyze the limit as $s \to 1^-$ of the first eigenvalue obtaining that there is a limit that is the first eigenvalue of a local operator that involve two $p-$Laplacians (one in the x variables and another one in y variables).

Theorem 1.2. *Let Ω is bounded domain in $\mathbb{R}^{n+m}$ with smooth boundary, and fix $p \in (1, \infty)$. Then*

$$\lim_{s \to 1^-} (1-s)\lambda_1(s,p) = \lambda_1(1,p) \coloneqq \inf\left\{ \frac{K_{n,p}\|\nabla_x u\|^p_{L^p(\Omega)} + K_{m,p}\|\nabla_y u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}} : u \in W^{1,p}_0(\Omega), u \not\equiv 0 \right\}, \tag{1.1}$$

where the constant $K_{n,p} > 0$ depends only on n and p, while $K_{m,p} > 0$ depends only on m and p.

Observe that the limit value, $\lambda_1(1,p)$, is the first eigenvalue of the following eigenvalue problem

$$\begin{cases} -K_{n,p}\Delta_{p,x}u - K_{m,p}\Delta_{p,y}u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Concerning the limit as $p \to \infty$ (for a fixed s) for the first eigenvalue we have the following result.

Theorem 1.3. *It holds that*

$$\lim_{p\to\infty} [\lambda_1(s,p)]^{1/p} = \Lambda_\infty(s)$$

where

$$\Lambda_\infty(s) := \inf\left\{ [u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} \colon u \in \mathcal{W}^{s,\infty}(\mathbb{R}^{n+m}), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ in } \Omega^c \right\}.$$

In addition, the eigenfunctions u_p normalized by $\|u_p\|_{L^p(\Omega)} = 1$ converge along subsequences $p_n \to \infty$ uniformly to a continuous limit u_∞, that is a nontrivial viscosity solution to

$$\begin{cases} \max\{A; C\} = \max\{-B; -D; \Lambda_\infty(s)u\} & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

with

$$A = \sup_w \frac{u(x,w) - u(x,y)}{|y-w|^s}, \qquad B = \inf_w \frac{u(x,w) - u(x,y)}{|y-w|^s},$$
$$C = \sup_z \frac{u(z,y) - u(x,y)}{|x-z|^s}, \qquad D = \inf_z \frac{u(z,y) - u(x,y)}{|x-z|^s}.$$

We can give a simple geometric characterization of the limit value $\Lambda_\infty(s)$, this value is related to the maximum distance (measured in a way that involves the exponent s, see below) from one point $(x,y) \in \Omega$ to the boundary. In fact,

$$\Lambda_\infty(s) = \frac{1}{\displaystyle\max_{(x,y)\in\Omega} \min_{(z,w)\in\partial\Omega} (|x-z|^s + |y-w|^s)}.$$

That the limit equation is verified in the viscosity sense and involve quotients of the form $\frac{u(x,w)-u(x,y)}{|y-w|^s}$ is not surprising. In fact, viscosity solutions provide the right framework to deal with limits of $p-$Laplacians as $p \to \infty$, see [4, 6, 27], and quotients like the one mentioned above appeared in other related limits, see [12, 23, 29]. What is remarkable in the limit equation is that it involves the limit value $\Lambda_\infty(s)$ and that the quotients that appear have perfectly identified the two groups of variables that are present in the fractional pseudo $p-$Laplacian that we introduced here.

Our results say that we can take the limits as $s \to 1^-$ and as $p \to \infty$ in the first eigenvalue. With the above notations we have the following commutative diagram

$$\begin{array}{ccc} ((1-s)\lambda_1(s,p))^{1/p} & \xrightarrow[s\to 1^-]{} & (\lambda_1(1,p))^{1/p} \\ {\scriptstyle p\to\infty}\Big\downarrow & & \Big\downarrow{\scriptstyle p\to\infty} \\ \Lambda_\infty(s) & \xrightarrow[s\to 1^-]{} & \Lambda_\infty. \end{array}$$

Here

$$\Lambda_\infty := \frac{1}{\displaystyle\max_{(x,y)\in\Omega}\min_{(z,w)\in\partial\Omega}(|x-z|+|y-w|)}.$$

The limit

$$\lim_{p\to\infty}(\lambda_1(1,p))^{1/p}=\Lambda_\infty$$

can be obtained as in [27] using the variational characterization of $\lambda_1(1,p)$ given in (1.1). We omit the details.

To end this introduction, let us comment on previous results. The limit as $p\to\infty$ of the first eigenvalue λ_p^D of the usual local p-Laplacian with Dirichlet boundary condition was studied in [27, 28], (see also [5] for an anisotropic version). In those papers the authors prove that

$$\lambda_\infty^D := \lim_{p\to+\infty}\left(\lambda_p^D\right)^{1/p} = \inf\left\{\frac{\|\nabla v\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} : v\in W_0^{1,\infty}(\Omega), v\not\equiv 0\right\} = \frac{1}{R},$$

where R is the largest possible radius of a ball contained in Ω. In addition, it was shown the existence of extremals, i.e. functions where the above infimum is attained. These extremals can be constructed taking the limit as $p\to\infty$ in the eigenfunctions of the $p-$Laplacian eigenvalue problems (see [27]) and are viscosity solutions of the following eigenvalue problem (called the infinity eigenvalue problem in the literature)

$$\begin{cases}\min\left\{|Du|-\lambda_\infty^D u,\,\Delta_\infty u\right\}=0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

The limit operator Δ_∞ that appears here is the ∞-Laplacian given by $\Delta_\infty u = -\langle D^2uDu, Du\rangle$. Remark that solutions to $\Delta_p v_p = 0$ with a Dirichlet data $v_p = f$ on $\partial\Omega$ converge as $p\to\infty$ to the viscosity solution to $\Delta_\infty v = 0$ with $v = f$ on $\partial\Omega$, see [4, 6, 13]. This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in Ω of a boundary data f, see [2, 4]. Limits of $p-$Laplacians are also relevant in mass transfer problems, see [7, 19].

On the other hand, the pseudo infinity Laplacian is the second order nonlinear operator given by $\tilde\Delta_\infty u = \sum_{i\in I(\nabla u)} u_{x_ix_i}|u_{x_i}|^2$, where the sum is taken over the indexes in $I(\nabla u) = \{i : |u_{x_i}| = \max_j |u_{x_j}|\}$. This operator, as happens for the usual infinity Laplacian, also appears naturally as a limit of $p-$Laplace type problems. In fact, any possible limit of u_p, solutions to $\tilde\Delta_p u = \sum_{i=1}^N (|u_{x_i}|^{p-2}u_{x_i})_{x_i} = 0$, is a viscosity solution to $\tilde\Delta_\infty u = 0$. A proof of this fact is contained in [5], where are also studied the eigenvalue problem for this operator.

Concerning regularity, we mention [35] where it it proved that infinity harmonic functions, that is, viscosity solutions to $-\Delta_\infty u = 0$, are C^1 in two dimensions and [20, 21] where it is proved differentiability in any dimension. For the pseudo infinity Laplacian, we refer here to solutions to $\tilde\Delta_\infty u = 0$, the optimal regularity is Lipschitz continuity, see [34].

For references concerning nonlocal fractional problems we refer to [18, 26, 29, 30, 32, 31, 17] and references therein. For limits as $p\to+\infty$ in nonlocal $p-$Laplacian problems and its relation with optimal mass transport we refer to [26] (eigenvalue problems were not considered there).

Finally, concerning limits as $p \to \infty$ in fractional eigenvalue problems, we mention [9, 23, 28]. In [28] the limit of the first eigenvalue for the fractional $p-$Laplacian is studied while in [23] higher eigenvalues are considered. We borrow ideas and techniques from these papers. In particular, when we prove the fact that there is a limit problem that is verified in the viscosity sense. For example, the fact that continuous weak solutions to our pseudo fractional $p-$Laplacian are viscosity solutions runs exactly as in [28] and hence we omit the details here.

The paper is organized as follows: In Section 2 we collect some preliminary results; in Section 3 we deal with our eigenvalue problem and prove Theorem 1.1; in Section 4 we analyze the limit as $s \to 1^-$, Theorem 1.2; finally, in Section 5 we study the limit as $p \to \infty$ proving Theorem 1.3.

2. Preliminaries

Throughout this section $s \in (0,1)$, $p \in (1,+\infty]$, Ω is an open set of $\mathbb{R}^{n+m}$. We henceforth use the notation:

- $(x,y) = (x_1,\dots,x_n,x_{n+1},\dots,x_{n+m}) \in \mathbb{R}^{n+m}$ with $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $y = (x_{n+1},\dots,x_{n+m}) \in \mathbb{R}^m$;
- $\Omega^2 = \Omega \times \Omega$;
- $\Omega_x = \{y \in \mathbb{R}^m \colon (x,y) \in \Omega\}$, and $\Omega_y = \{x \in \mathbb{R}^n \colon (x,y) \in \Omega\}$;
- $B^N(x,r)$ denotes the ball of $N-$ball of radius r and center x, and ω_N denotes the $(N-1)-$dimensional Hausdorff measure of the $N-$sphere of radius 1;
- $(a)^{p-1} = |a|^{p-2}a$.

Given a measurable function $u \colon \Omega \to \mathbb{R}$, we set for $p < +\infty$,

$$\|u\|^p_{L^p(\Omega)} := \int_\Omega |u(x,y)|^p \, dxdy,$$

$$|u|^p_{W^{s,p}(\Omega)} = \int_{\Omega^2} \frac{|u(x,y)-u(z,w)|^p}{|(x,y)-(z,w)|^{n+m+sp}} dxdydzdw,$$

$$\begin{aligned}[u]^p_{W^{s,p}(\Omega)} = &\int_\Omega \int_{\Omega_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy \\ &+ \int_\Omega \int_{\Omega_x} \frac{|u(x,y)-u(x,w)|^p}{|y-w|^{m+sp}} dwdxdy\end{aligned}$$

and for $p = +\infty$,

$$|u|_{W^{s,\infty}(\Omega)} = \sup\left\{ \frac{|u(x,y)-u(z,y)|}{|(x,y)-(z,w)|^s} \colon (x,y) \neq (z,w) \in \Omega \right\} = |u|_{C^{0,s}(\Omega)},$$

$$\begin{aligned}[u]_{\mathcal{W}^{s,\infty}(\Omega)} = \max\Bigg\{ &\sup\left\{ \frac{|u(x,y)-u(z,y)|}{|x-z|^s} \colon (x,y) \neq (z,y) \in \Omega \right\}; \\ &\sup\left\{ \frac{|u(x,y)-u(x,w)|}{|y-w|^s} \colon (x,y) \neq (x,w) \in \Omega \right\} \Bigg\}.\end{aligned}$$

We denote by $W^{s,p}(\Omega)$ (here p can be $+\infty$) the usual fractional Sobolev space, that is $W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \colon |u|_{W^{s,p}(\Omega)} < +\infty \right\}$.

We introduce the space $\mathcal{W}^{s,p}(\Omega)$ (again here p can be $+\infty$) as follows:

$$\mathcal{W}^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \colon [u]^p_{\mathcal{W}^{s,p}(\Omega)} < \infty \right\}.$$

This space is a Banach space. We state this as a proposition but we omit its proof that is standard.

Proposition 2.1. *The space $\mathcal{W}^{s,p}(\Omega)$ endowed with the norm*

$$\|u\|_{\mathcal{W}^{s,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{\mathcal{W}^{s,p}(\Omega)} \right)^{1/p}$$

is a Banach space. Moreover $\mathcal{W}^{s,p}(\Omega)$ is separable for $1 \le p < +\infty$ and it is reflexive for $1 < p < \infty$.

For $u \colon \Omega \to \mathbb{R}$ a measurable function, we set

$$u_+(x,y) = \max\{u(x,y), 0\} \quad \text{and} \quad u_-(x,y) = \min\{-u(x,y), 0\}.$$

Observe that

$$|u_\pm(x,y) - u_\pm(z,w)| \le |u(x,y) - u(z,w)|$$

for all $(x,y),(z,w) \in \Omega$. Therefore, we have

Lemma 2.2. *Let $\mathcal{X} = W^{s,p}(\Omega)$ or $\mathcal{W}^{s,p}(\Omega)$. If $u \in \mathcal{X}$ then $u_+, u_- \in \mathcal{X}$.*

For $1 \le p < \infty$, we denote by $\widetilde{\mathcal{W}}^{s,p}(\Omega)$ the space of all $u \in \mathcal{W}^{s,p}(\Omega)$ such that $\tilde{u} \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m})$ where $\tilde{u}$ is the extension by zero of u.

The next result can be found in [1, 15].

Theorem 2.3. *Under the assumptions* A1 *and* A2 *we have that*

- *If $sp < n+m$, then $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \le q < p^\star_s = {(n+m)p}/{(n+m-sp)}$.*
- *If $sp = n+m$, then $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \le q < \infty$.*
- *If $sp > n+m$, then $W^{s,p}(\Omega)$ is compactly embedded in $C^{0,\lambda}(\overline{\Omega})$ with $\lambda < s - {(n+m)}/{p}$.*

Lemma 2.4. *Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If $\Omega = \Omega_1 \times \Omega_2$, and $p \in [1,+\infty)$, then $\mathcal{W}^{s,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$. Moreover, there exists a constant $C = C(n,m)$ such that*

$$|u|^p_{W^{s,p}(\Omega)} \le C[u]_{\mathcal{W}^{s,p}(\Omega)}$$

for all $u \in \mathcal{W}^{s,p}(\Omega)$.

Proof. Let $u \in \mathcal{W}^{s,p}(\Omega)$. We have

$$\begin{aligned}
|u|^p_{W^{s,p}(\Omega)} &= \int_{\Omega^2} \frac{|u(x,y)-u(z,w)|^p}{|(x,y)-(z,w)|^{n+m+sp}}\, dxdydzdw \\
&\le 2^{p-1} \int_{\Omega^2} \frac{|u(x,y)-u(z,y)|^p}{|(x,y)-(z,w)|^{n+m+sp}} dxdydzdw \\
&\qquad + 2^{p-1} \int_{\Omega^2} \frac{|u(z,y)-u(z,w)|^p}{|(x,y)-(z,w)|^{n+m+sp}} dxdydzdw \\
&= 2^{p-1} I_1 + 2^{p-1} I_2.
\end{aligned} \tag{2.1}$$

Now, we observe that

$$\begin{aligned}
I_1 &= \int_{\Omega^2} \frac{|u(x,y)-u(z,y)|^p}{|(x,y)-(z,w)|^{n+m+sp}} dxdydzdw \\
&\le \int_{\Omega}\int_{\Omega_2}\int_{\mathbb{R}^m} \frac{|u(x,y)-u(z,y)|^p}{|(x,y)-(z,w)|^{n+m+sp}} dwdzdxdy \\
&\le \int_{\Omega}\int_{\Omega_2} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} \int_{\mathbb{R}^m} \frac{|x-z|^{n+sp}dw}{(|x-z|^2+|y-w|^2)^{\frac{n+m+sp}{2}}} dzdxdy \\
&= \omega_m \int_{\Omega}\int_{\Omega_2} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy \int_0^{+\infty} \frac{r^{m-1}}{(1+r^2)^{\frac{n+m+sp}{2}}} dr.
\end{aligned}$$

Since

$$\int_0^{+\infty} \frac{r^{m-1}}{(1+r^2)^{\frac{n+m+sp}{2}}} dr \le \int_0^1 r^{m-1} dr + \int_1^{+\infty} \frac{1}{r^{n+sp+1}} dr = \frac{1}{m} + \frac{1}{n+sp}$$

we have that

$$I_1 \le 2\omega_m \int_{\Omega}\int_{\Omega_2} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy. \tag{2.2}$$

One can also, in an analogous way, obtain

$$I_2 \le 2\omega_n \int_{\Omega}\int_{\Omega_1} \frac{|u(x,y)-u(x,w)|^p}{|y-w|^{m+sp}} dwdxdy. \tag{2.3}$$

By (2.1), (2.2) and (2.3), we get

$$|u|_{W^{s,p}(\Omega)} \le C(n,m)[u]_{\mathcal{W}^{s,p}(\Omega)}.$$

This completes the proof. □

Remark 2.5. If $p=\infty$, it is straightforward to show that $W^{s,\infty}(\Omega) \subset \mathcal{W}^{s,\infty}(\Omega)$. Moreover, if $\Omega = \Omega_1 \times \Omega_2$ then $\mathcal{W}^{s,\infty}(\Omega) = W^{s,\infty}(\Omega)$.

Lemma 2.6. *Let Ω be an open subset of $\mathbb{R}^{n+m}$ and $p \in (1,\infty)$. If $0<t<s<1$ then $\mathcal{W}^{s,p}(\Omega) \subset \mathcal{W}^{t,p}(\Omega)$, and the embedding is continuous. Moreover*

$$[u]^p_{\mathcal{W}^{t,p}(\Omega)} \le [u]^p_{\mathcal{W}^{s,p}(\Omega)} + \frac{2^p(\omega_n+\omega_m)}{tp} \|u\|^p_{L^p(\Omega)} \qquad \forall u \in \mathcal{W}^{s,p}(\Omega). \tag{2.4}$$

Proof. Let $u \in \mathcal{W}^{s,p}(\Omega)$. Observe that,

$$\begin{aligned}
\int_{\Omega}\int_{\Omega_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+tp}} dzdxdy \le \int_{\Omega}\int_{A_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+tp}} dzdxdy \\
+ \int_{\Omega}\int_{A_y^c} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+tp}} dzdxdy
\end{aligned}$$

where $A_y = \{z \in \Omega_y \colon |z-x| < 1\}$. Since $t < s$, we have that

$$\begin{aligned}
&\int_\Omega \int_{\Omega_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+tp}} dzdxdy \leq \\
&\leq \int_\Omega \int_{A_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy + 2^{p-1} \int_\Omega \int_{A_y^c} \frac{|u(x,y)|^p + |u(z,y)|^p}{|x-z|^{n+tp}} dzdxdy \\
&\leq \int_\Omega \int_{A_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy + 2^p \int_\Omega \int_{A_y^c} \frac{|u(x,y)|^p}{|x-z|^{n+tp}} dzdxdy \\
&\leq \int_\Omega \int_{A_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy + \frac{2^p \omega_n}{tp} \int_\Omega |u(x,y)|^p dxdy.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega \int_{\Omega_x} \frac{|u(x,y)-u(x,w)|^p}{|y-w|^{n+tp}} dzdxdy \leq \\
&\qquad \leq \int_\Omega \int_{A_x} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy + \frac{2^p \omega_m}{tp} \int_\Omega |u(x,y)|^p dxdy,
\end{aligned}$$

where $A_x = \{w \in \Omega_x \colon |y-w| < 1\}$. Therefore (2.4) holds. □

Finally, we prove a Poincaré type inequality.

Lemma 2.7. *Let Ω be an open bounded subset of $\mathbb{R}^{n+m}$, $s \in (0,1)$ and $p \in (1,\infty)$. Then there is a positive constant C such that*

$$\|u\|_{L^p(\Omega)} \leq C[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \quad \forall u \in \widetilde{\mathcal{W}}^{s,p}(\Omega).$$

Proof. Let $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$ and $d = 2\,\mathrm{diam}(\Omega)$. It holds that

$$[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \geq \int_\Omega |u(x,y)|^p \int_{\mathbb{R}^{n+m}\setminus B^n(x,d)} \frac{dz}{|x-z|^{n+sp}} \geq \frac{\omega_n d^{-sp}}{sp} \|u\|^p_{L^p(\Omega)}.$$

□

3. The first eigenvalue

Under assumptions A1 and A2, a natural definition of an eigenvalue is a real value λ for which there exists $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\}$ such that u is a weak solution of

$$\begin{cases} \mathcal{L}_{s,p} u(x,y) = \lambda (u(x,y))^{p-1} & \text{in } \Omega, \\ u(x,y) = 0 & \text{in } \Omega^c, \end{cases} \tag{3.1}$$

that is

$$\mathcal{H}_{s,p}(u,v) = \lambda \int_\Omega (u(x,y))^{p-1} v(x,y)\, dxdy \qquad \forall v \in \widetilde{\mathcal{W}}^{s,p}(\Omega).$$

The function u is called a corresponding eigenfunction. Here

$$\begin{aligned}
\mathcal{H}_{s,p}(u,v) \coloneqq &\int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{(u(x,y)-u(z,y))^{p-1}(v(x,y)-v(z,y))}{|x-z|^{n+sp}} dzdxdy \\
&+ \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{(u(x,y)-u(x,w))^{p-1}(v(x,y)-v(x,w))}{|y-w|^{m+sp}} dwdxdy.
\end{aligned}$$

Observe that

$$\mathcal{H}_{s,p}(u,u) = [u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \qquad \forall u \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m}),$$

and, by Hölder's inequality,

$$\mathcal{H}_{s,p}(u,v) \leq 2[u]^{p-1}_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}[v]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \qquad \forall u, v \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m}).$$

Observe that, when λ is an eigenvalue, then there is $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\}$ such that

$$\mathcal{H}_{s,p}(u,u) = \lambda \int_\Omega |u(x,y)|^p dxdy.$$

Then, we have that

$$\lambda = \frac{[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}}{\|u\|^p_{L^p(\Omega)}} \geq 0.$$

By a standard compactness argument, we have the following result.

Theorem 3.1. *Under the assumptions* A1 *and* A2*, the first eigenvalue is given by*

$$\lambda_1(s,p) := \inf \left\{ \frac{[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}}{\|u\|^p_{L^p(\Omega)}} : u \in \widetilde{\mathcal{W}}^{s,p}(\Omega), u \not\equiv 0 \right\}.$$

Proof. Consider a minimizing sequence u_n normalized according to $\|u_n\|_{L^p(\Omega)} = 1$. Then, as u_n in bounded in $\widetilde{\mathcal{W}}^{s,p}(\Omega)$, by Lemma 2.4 and Theorem 2.3, there is a subsequence such that $u_{n_j} \rightharpoonup u$ weakly in $\widetilde{\mathcal{W}}^{s,p}(\Omega)$ and $u_{n_j} \to u$ strongly in $L^p(\Omega)$. Therefore, u is a nontrivial minimizer to the variational problem defining $\lambda_1(s,p)$. The fact that this minimizer is a weak solution to (3.1) is straightforward and can be obtained from the arguments in [29].

To finish the proof we just observe that any other eigenfunction associated with an eigenvalue λ verifies

$$\lambda = \frac{[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}}{\|u\|^p_{L^p(\Omega)}} \geq \lambda_1(s,p),$$

and then we get that $\lambda_1(s,p)$ is the first eigenvalue. $\square$

Now we observe that using a topological tool (the genus) we can construct an unbounded sequence of eigenvalues.

Theorem 3.2. *Assume* A1 *and* A2*. There is a sequence of eigenvalues λ_n such that $\lambda_n \to +\infty$ as $n \to +\infty$.*

Proof. We follow ideas from [22] and hence we omit the details. Let us consider

$$M_\alpha = \{u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) \colon [u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} = p\alpha\}$$

and

$$\varphi(u) = \frac{1}{p} \int_\Omega |u(x,y)|^p \, dxdy.$$

We are looking for critical points of φ restricted to the manifold M_α using a minimax technique. We consider the class

$$\Sigma = \{A \subset \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\} \colon A \text{ is closed, } A = -A\}.$$

Over this class we define the genus, $\gamma\colon \Sigma \to \mathbb{N} \cup \{\infty\}$, as

$$\gamma(A) = \min\{k \in \mathbb{N} \colon \text{there exists } \phi \in C(A, \mathbb{R}^k - \{0\}),\ \phi(x) = -\phi(-x)\}.$$

Now, we let $C_k = \{C \subset M_\alpha \colon C \text{ is compact, symmetric and } \gamma(C) \le k\}$ and let

$$\beta_k = \sup_{C \in C_k} \min_{u \in C} \varphi(u).$$

Then $\beta_k > 0$ and there exists $u_k \in M_\alpha$ such that $\varphi(u_k) = \beta_k$ and u_k is a weak eigenfuction with $\lambda_k = \alpha/\beta_k$. □

The following lemma shows that the eigenfunctions are bounded.

Lemma 3.3. *Under assumptions* A1 *and* A2*, if u is an eigenfunction associated to some eigenvalue λ then $u \in L^\infty(\mathbb{R}^{n+m})$.*

Proof. In this proof we follow ideas form [23].

If $ps > n + m$, by Lemma 2.4 and Theorem 2.3, then the assertion holds. From now on, we suppose that $sp \le n + m$.

We will show that if $\|u_+\|_{L^p(\Omega)} \le \delta$ then u_+ is bounded, where $\delta > 0$ is some small constant to be determined. Let $k \in \mathbb{N}_0$, we define the function u_k by

$$u_k(x, y) := (u(x, y) - 1 + 2^{-k})_+.$$

Observe that, $u_0 = u_+$ and for any $k \in \mathbb{N}_0$ we have that $u_k \in \widetilde{W}^{s,p}(\Omega)$ verifies

$$\begin{aligned} &u_{k+1} \le u_k \text{ a.e. } \mathbb{R}^{n+m}, \\ &u < (2^{k+1} - 1)u_k \text{ in } \{u_{k+1} > 0\}, \\ &\{u_{k+1} > 0\} \subset \{u_k > 2^{-(k+1)}\}. \end{aligned} \tag{3.2}$$

Now, for any function $v\colon \mathbb{R}^{n+m} \to \mathbb{R}$, it holds that

$$|v_+(x, y) - v_+(z, w)|^p \le |v(x, y) - v(z, w)|^{p-1}(v_+(x, y) - v_+(x, w))$$

for all $(x, y), (z, w) \in \mathbb{R}^{n+m}$. Then

$$[u_{k+1}]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \le \mathcal{H}_{s,p}(u, u_{k+1}) = \lambda \int_\Omega (u(x, y))^{p-1} u_{k+1}(x, y)\, dxdy$$

for all $k \in \mathbb{N}_0$. Hence, by (3.2) and Hölder's inequality, we get

$$\begin{aligned} [u_{k+1}]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} &\le \lambda \int_\Omega (u(x, y))^{p-1} u_{k+1}(x, y)\, dxdy \\ &\le (2^{k+1} - 1)^{p-1} \lambda \|u_k\|^p_{L^p(\Omega)} \end{aligned} \tag{3.3}$$

for all $k \in \mathbb{N}_0$.

On the other hand, in the case $sp < n+m$, using Hölder's inequality, Lemma 2.4 and Theorem 2.3, the formulas in (3.2), and Chebyshev's inequality, for any $k \in \mathbb{N}_0$

we have that

$$\begin{aligned}\|u_{k+1}\|^p_{L^p(\Omega)} &\le \|u_{k+1}\|^p_{L^{p_s^\star}(\Omega)} |\{u_{k+1}>0\}|^{sp/(n+m)}\\ &\le C[u_{k+1}]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} |\{u_k > 2^{-(k+1)}\}|^{sp/(n+m)} \\ &\le C[u_{k+1}]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \left(2^{(k+1)p}\|u_k\|^p_{L^p(\Omega)}\right)^{sp/(n+m)},\end{aligned} \tag{3.4}$$

where C is a constant independent of k. Then, by (3.3) and (3.4), for any $k\in\mathbb{N}_0$ we obtain

$$\|u_{k+1}\|^p_{L^p(\Omega)} \le C\left(2^{(k+1)p}\|u_k\|^p_{L^p(\Omega)}\right)^{1+\alpha}, \tag{3.5}$$

where C is a constant independent of k and $\alpha = {sp}/{(n+m)} > 0$.

Arguing similarly, in the case $sp = n+m$, taking $r>p$ and proceeding as in the previous case, $sp<n+m$ (with r in place of $p_s^\star$), we obtain that (3.5) holds with $\alpha = 1 - {p}/{r} > 0$.

Therefore, if $sp \le n+m$, there exist $\alpha>0$ and a constant $C>1$ such that

$$\|u_{k+1}\|^p_{L^p(\Omega)} \le C^k\left(\|u_k\|^p_{L^p(\Omega)}\right)^{1+\alpha},$$

for any $k\in\mathbb{N}_0$. Hence, if $\|u_0\|^p_{L^p(\Omega)} = \|u_+\|^p_{L^p(\Omega)} \le C^{-1/\alpha^2} =: \delta^p$ then $u_k\to 0$ strongly in $L^p(\Omega)$. But $u_k \to (u-1)_+$ a.e in $\mathbb{R}^{n+m}$, then we conclude that $(u-1)_+\equiv 0$ in $\mathbb{R}^{n+m}$. Therefore, u_+ is bounded.

Taking $-u$ in place of u we have that u_- is bounded if $\|u_-\|_{L^p(\Omega)} < \delta$.

Hence, as we can multiply an eigenfunction u by a small constant in order to obtain $\|u_+\|_{L^p(\Omega)}$ and $\|u_-\|_{L^p(\Omega)}<\delta$, we conclude that u is bounded. □

Our next goal is to show that if u is a eigenfunction associated with $\lambda_1(s,p)$ then u does not change sign. Before showing this result we need the following two technical lemmas.

Lemma 3.4. *Assume* A1 *and* A2. *If* $u\in\widetilde{\mathcal{W}}^{s,p}(\Omega)$ *is such that*

$$\mathcal{H}_{s,p}(u,v)\ge 0 \quad \forall v\in\widetilde{\mathcal{W}}^{s,p}(\Omega), v\ge 0 \text{ in } \Omega. \tag{3.6}$$

and $u\ge 0$ *in* $B^n(x_0,R)\times B^m(y_0,R)\subset\subset\Omega$ *for some* $R>0$ *then for any* $d>0$ *and* $0<2r<R$ *there holds*

$$\begin{aligned}&\int_{B_r^m}\int_{B_r^n}\int_{B_r^n}\frac{1}{|x-z|^{n+sp}}\left|\log\left(\frac{u(x,y)+d}{u(z,y)+d}\right)\right|^p dzdxdy\\ &+\int_{B_r^n}\int_{B_r^m}\int_{B_r^n}\frac{1}{|y-w|^{m+sp}}\left|\log\left(\frac{u(x,y)+d}{u(x,w)+d}\right)\right|^p dwdxdy\\ &\quad\le Cr^{n+m-sp}\left\{\frac{r^{sp}}{d^{p-1}r^m}\int_{\mathbb{R}^m}\int_{(B_R^n)^c}\frac{u_-(x,y)^p}{|x-x_0|^{n+sp}}dxdy\right.\\ &\qquad\left.+\frac{r^{sp}}{d^{p-1}r^n}\int_{\mathbb{R}^n}\int_{(B_R^m)^c}\frac{u_-(x,y)^p}{|y-y_0|^{m+sp}}dydx+1\right\}\end{aligned} \tag{3.7}$$

where $B_\rho^n = B^n(x_0,\rho)$, $B_\rho^m = B^m(y_0,\rho)$ *and* $C=C(n,m,p,s)>0$ *is a constant.*

Proof. Let $d > 0$, $r \in (0, R/2)$,

$$\phi \in C_0^\infty(B^n_{3r/2}), \quad 0 \le \phi \le 1, \quad \phi \equiv 1 \text{ in } B^n_r, \quad |D_x\phi| < \frac{c}{r} \text{ in } B^n_{3r/2}, \text{ and}$$
$$\psi \in C_0^\infty(B^m_{3r/2}), \quad 0 \le \psi \le 1, \quad \psi \equiv 1 \text{ in } B^m_r, \quad |D_x\psi| < \frac{c}{r} \text{ in } B^m_{3r/2}.$$

Taking $v(x,y) = \phi^p(x)\psi^p(y)(u(x,y)+d)^{1-p}$ as test function in (3.6) and following the proof of Lemma 1.3 in [16], we get (3.7). □

Lemma 3.5. *Assume* A1 *and* A2. *If* Ω *is connected and* $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$ *is such that*

$$\mathcal{H}_{s,p}(u,v) \ge 0 \quad \forall v \in \widetilde{\mathcal{W}}^{s,p}(\Omega), v \ge 0 \text{ in } \Omega,$$

$u \ge 0$ *in* Ω *and* $u \not\equiv 0$ *in* Ω *then* $u > 0$ *in* Ω.

Proof. In this proof we borrow ideas from [8]. Since Ω is a bounded connected open set, it is enough to prove that $u > 0$ in K for any $K \subset\subset \Omega$ a connected compact set such that $u \not\equiv 0$ in K.

Let $K \subset\subset \Omega$ be a connected compact set such that $u \neq 0$ in K. Then there exists $r > 0$ such that

$$K \subset \left\{(x,y) \in \Omega \colon \max_{(z,w)\in\partial\Omega} \{|z-x|, |w-y|\} > 2r\right\}.$$

Since K is compact, there exists $\{(x_j, y_j)\}_{j=1}^k \subset K$ such that

$$K \subset \bigcup_{j=1}^k B^n_j \times B^m_j, \quad \text{and} \quad |(B^n_j \times B^m_j) \cap (B^n_{j+1} \times B^m_{j+1})| > 0 \tag{3.8}$$

for any $j \in \{1, \dots, k-1\}$, where $B^n_j = B^n(x_j, r/2)$ and $B^m_j = B^m(y_j, r/2)$.

To obtain a contradiction, suppose that $|\{(x,y) \colon u(x,y) = 0\} \cap K| > 0$ then there exists $j \in \{1, \dots, k\}$ such that

$$Z = \{(x,y) \colon u(x,y) = 0\} \cap (B^n_j \times B^m_j)$$

has positive measure.

Given $d > 0$, we define

$$F_d \colon B^n_j \times B^m_j \to \mathbb{R} \quad \text{by} \quad F_d(x,y) = \log\left(1 + \frac{u(x,y)}{d}\right).$$

Then, for any $(x,y) \in B^n(x_j, r/2) \times B^m(y_j, r/2)$ and $(z,w) \in Z$ we have

$$\begin{aligned}
F_d(z,w) &= 0 \\
|F_d(x,y)|^p &= |F(x,y) - F(z,w)|^p \\
&\leq 2^{p-1} \frac{|F(x,y) - F(z,y)|^p}{|z-x|^{n+sp}} |z-x|^{n+sp} \\
&\quad + 2^{p-1} \frac{|F(z,y) - F(z,w)|^p}{|w-y|^{m+sp}} |w-y|^{n+sp} \\
&\leq 2^{p-1} r^{n+sp} \frac{|F(x,y) - F(z,y)|^p}{|z-x|^{n+sp}} \\
&\quad + 2^{p-1} r^{m+sp} \frac{|F(z,y) - F(z,w)|^p}{|w-y|^{m+sp}} \\
&= 2^{p-1} r^{n+sp} \left| \log\left(\frac{u(x,y)+d}{u(z,y)+d} \right) \right|^p \frac{1}{|z-x|^{n+sp}} \\
&\quad + 2^{p-1} r^{m+sp} \left| \log\left(\frac{u(z,y)+d}{u(z,w)+d} \right) \right|^p \frac{1}{|w-y|^{m+sp}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|Z||F_d(x,y)|^p &= \iint_Z |F_d(x,y)|^p dwdz \\
\leq c_1 r^{n+m+sp} &\int_{B_j^n} \left| \log\left(\frac{u(x,y)+d}{u(z,y)+d} \right) \right|^p \frac{dz}{|z-x|^{n+sp}} \\
+ 2^{p-1} r^{m+sp} &\int_{B_j^n} \int_{B_j^m} \left| \log\left(\frac{u(z,y)+d}{u(z,w)+d} \right) \right|^p \frac{dwdz}{|w-y|^{m+sp}}
\end{aligned}$$

for any $(x,y) \in B^n(x_j, r/2) \times B^m(y_j, r/2)$. Here $c_1 = c_1(m,p) > 0$ is a constant. Then

$$\begin{aligned}
&\int_{B_j^n} \int_{B_j^m} |F_d(x,y)|^p dxdy \\
&\qquad \leq \frac{c_1 r^{n+m+sp}}{|Z|} \int_{B_j^m} \int_{B_j^n} \int_{B_j^n} \left| \log\left(\frac{u(x,y)+d}{u(z,y)+d} \right) \right|^p \frac{dzdxdy}{|z-x|^{n+sp}} \\
&\qquad + \frac{c_2 r^{n+m+sp}}{|Z|} \int_{B_j^n} \int_{B_j^m} \int_{B_j^m} \left| \log\left(\frac{u(x,y)+d}{u(x,w)+d} \right) \right|^p \frac{dwdxdy}{|w-y|^{m+sp}}.
\end{aligned}$$

Thus, by Lemma 3.4 and since $u \geq 0$ in Ω, we get

$$\int_{B_j^n} \int_{B_j^m} |F_d(x,y)|^p dxdy \leq C \frac{r^{2n+2m}}{|Z|},$$

where $C = C(n,m,s,p) > 0$ is a constant. Taking $d \to 0$ in the last inequality, we get that $u \equiv 0$ in $B_j^n \times B_j^m$.

By (3.8), there exists $i \in \{1, \ldots, k\}$ such that $i \neq j$ and

$$|(B_i^n \times B_i^m) \cap \{(x,y) \colon u(x,y) = 0\}| > 0.$$

Then, we can repeat the previous argument for $B_i^n \times B_i^m$ and obtain $u \equiv 0$ in $B_i^n \times B_i^m$. In this way we conclude that $u \equiv 0$ in K which contradicts the fact that $u \not\equiv 0$ in K. Thus $|\{(x,y) \colon u(x,y) = 0\} \cap K| = 0$. $\square$

Now, we are ready to prove that the eigenfunctions associated to $\lambda_1(s,p)$ do not change sign.

Theorem 3.6. *Assume* A1 *and* A2. *If u is an eigenfunction associated to $\lambda_1(s,p)$ then $|u|>0$ in Ω.*

Proof. We start by showing that if u is an eigenfunction corresponding to $\lambda_1(s,p)$ then $|u|\not\equiv 0$ in all connected components of Ω. Our proof is by contradiction. We therefore assume that there is a connected component A of Ω such that $|u|\equiv 0$. Since u is an eigenfunction corresponding to $\lambda_1(s,p)$ then so is $|u|$. Then

$$\begin{aligned}0&=\lambda_1(s,p)\int_\Omega |u(x,y)|^{p-1}\phi(x,y)\,dxdy=\mathcal{H}_{s,p}(|u|,\phi)\\&=-2\int_{A^c}\int_{A_y}\frac{|u(x,y)|^{p-1}\phi(z,y)}{|x-z|^{n+sp}}dzdxdy-2\int_{A^c}\int_{A_x}\frac{|u(x,y)|^{p-1}\phi(x,w)}{|y-w|^{m+sp}}dwdxdy\end{aligned}$$

for all $\phi\in C_0^\infty(A)$, which is a contradiction.

Therefore, if A connected components C of Ω then $|u|\not\equiv 0$ in A and

$$\mathcal{H}_{s,p}(|u|,v)=\lambda_1(s,p)\int_\Omega |u(x,y)|^{p-1}v(x,y)\,dxdy\geq 0\quad \forall v\in\widetilde{\mathcal{W}}^{s,p}(A).$$

Then, by Lemma 3.5, $|u|>0$ in A. Therefore $|u|>0$ in Ω. ☐

Our next result show that $\lambda_1(s,p)$ is simple.

Theorem 3.7. *Assume* A1 *and* A2. *Let u be a positive eigenfunction corresponding to $\lambda_1(s,p)$. If $\lambda>0$ is such that there exists a non-negative eigenfunction v of* (3.1) *with eigenvalue λ, then $\lambda=\lambda_1(s,p)$ and there exists $k\in\mathbb{R}$ such that $v=ku$ a.e. in Ω.*

Proof. Since $\lambda_1(s,p)$ is the first eigenvalue we have that $\lambda_1(s,p)\leq\lambda$. Let $k\in\mathbb{N}$ and define $v_k:=v+1/k$.

We begin proving that $w_k:=u^p/v_k^{p-1}\in\widetilde{\mathcal{W}}^{s,p}(\Omega)$. It is immediate that $w_k=0$ in Ω^c and $w_k\in L^p(\Omega)$, due to the fact that $u\in L^\infty(\Omega)$, see Lemma 3.3.

On the other hand

$$\begin{aligned}&|w_k(x,y)-w_k(z,w)|\\&\quad=\left|\frac{u(x,y)^p-u(z,w)^p}{v_k(x,y)^{p-1}}+\frac{u(z,w)^p\left(v_k(z,w)^{p-1}-v_k(x,y)^{p-1}\right)}{v_k(x,y)^{p-1}v_k(z,w)^{p-1}}\right|\\&\quad\leq k^{p-1}\left|u(x,y)^p-u(z,w)^p\right|+\|u\|^p_{L^\infty(\Omega)}\frac{\left|v_k(x,y)^{p-1}-v_k(z,w)^{p-1}\right|}{v_k(x,y)^{p-1}v_k(w,z)^{p-1}}\\&\quad\leq 2\|u\|^{p-1}_{L^\infty(\Omega)}k^{p-1}p|u(x,y)-u(z,w)|\\&\qquad+\|u\|^p_{L^\infty(\Omega)}(p-1)\frac{v_k(x,y)^{p-2}+v_k(z,w)^{p-2}}{v_k(x,y)^{p-1}v_k(z,w)^{p-1}}|v_k(x,y)-v_k(z,w)|\\&\quad\leq 2\|u\|^{p-1}_{L^\infty(\Omega)}k^{p-1}p|u(x,y)-u(z,w)|\\&\qquad+\|u\|^p_{L^\infty(\Omega)}(p-1)k^{p-1}\left(\frac{1}{v_k(x,y)}+\frac{1}{v_k(z,w)}\right)|v(y)-v(x)|\\&\quad\leq C(k,p,\|u\|_{L^\infty(\Omega)})\left(|u(x,y)-u(z,w)|+|v(x,y)-v(z,w)|\right)\end{aligned}$$

for all $(x,y),(z,w)\in\mathbb{R}^{n+m}$. Hence, we have that $w_k\in\widetilde{\mathcal{W}}^{s,p}(\Omega)$ for all $k\in\mathbb{N}$ since $u,v\in\widetilde{\mathcal{W}}^{s,p}(\Omega)$.

Set

$$\begin{aligned}L(u,v_k)(x,y,z,w)&=|u(x,y)-u(w,z)|^p\\&\qquad-(v_k(x,y)-v_k(w,z))^{p-1}\left(\frac{u(x,y)^p}{v_k(x,y)^{p-1}}-\frac{u(z,w)^p}{v_k(z,w)^{p-1}}\right).\end{aligned}$$

Then, by [2, Lemma 6.2] and since u,v are two positive eigenfunctions of problem (3.1) with eigenvalues $\lambda_1(s,p)$ and λ respectively, we have

$$\begin{aligned}0\le&\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n}\frac{L(u,v_k)(x,y,z,y)}{|x-z|^{n+sp}}dzdxdy+\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^m}\frac{L(u,v_k)(x,y,x,w)}{|y-w|^{m+sp}}dwdxdy\\\le&\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n}\frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}}dzdxdy+\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^m}\frac{|u(x,y)-u(x,w)|^p}{|y-w|^{n+sp}}dwdxdy\\&-\mathcal{H}_{s,p}(v,w_k)\\\le&\lambda_1(s,p)\int_\Omega u(x,y)^p\,dxdy-\lambda\int_\Omega v(x,y)^{p-1}w_k(x,y)\,dxdy\\=&\lambda_1(s,p)\int_\Omega u(x,y)^p\,dxdy-\lambda\int_\Omega v(x,y)^{p-1}\frac{u(x,y)^p}{v_k(x,y)^{p-1}}\,dxdy.\end{aligned}$$

By Fatou's lemma and the dominated convergence theorem we obtain

$$\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n}\frac{L(u,v)(x,y,z,y)}{|x-z|^{n+sp}}dzdxdy+\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^m}\frac{L(u,v)(x,y,x,w)}{|y-w|^{m+sp}}dwdxdy=0$$

due to $\lambda_1(s,p,h)\le\lambda$. Then $L(u,v)(x,y,z,y)=L(u,v)(x,y,x,w)=0$ a.e. Hence, again by Lemma 6.2 in [2], $u(x,y)=\ell_1(y)v(x,y)$ and $u(x,y)=\ell_2(x)v(x,y)$ for all $(x,y)\in\mathbb{R}^{n+m}$. Then, we conclude that $u=\ell v$ for some constant $\ell>0$. $\square$

Finally we will prove that $\lambda_1(s,p)$ is isolated.

Theorem 3.8. *Assume* A1 *and* A2. *Them* $\lambda_1(s,p)$ *is isolated.*

Proof. We split the proof into two steps.

Step 1. If u is an eigenfunction associated to some eigenvalue $\lambda>\lambda_1(s,p)$ then there is a positive constant C such that

$$\left(\frac{1}{C\lambda}\right)^{r/(r-p)}\le|\Omega_\pm| \tag{3.9}$$

for all $p<r<p_s^\star$. Here $\Omega_\pm=\{(x,y)\colon u_\pm\not\equiv0\}$, and

$$p_s^\star=\begin{cases}\dfrac{(n+m)p}{n+m-sp}, & \text{if } sp<n+m,\\ \infty & \text{if } sp\ge n+m.\end{cases}$$

Let $r\in(p,p_s^\star)$. By Theorem 2.3, Lemmas 2.7 and 2.4 and Hölder inequality, we have

$$\|u_+\|^p_{L^r(\Omega)}\le C\|u_+\|^p_{W^{s,p}(\Omega)}\le C\mathcal{H}_{s,p}(u,u_+)=C\lambda\|u_+\|^p_{L^r(\Omega)}|\Omega_+|^{(r-p)/r}.$$

Then

$$\left(\frac{1}{C\lambda}\right)^{r/(r-p)}\le|\Omega_+|.$$

In order to prove the inequality for $|\Omega_-|$, it suffices to proceed as above, using the function $-u$ instead of u.

Step 2. By definition, $\lambda_1(s,p)$ is left-isolated. To prove that $\lambda_1(s,p)$ is right-isolated, we argue by contradiction. We assume that there is a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k \searrow \lambda_1(s,p)$ as $k\to\infty$. Let u_k be an eigenfunction associated to λ_k such that $\|u_k\|_{L^p(\Omega)} = 1$. Then $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\widetilde{\mathcal{W}}^{s,p}(\Omega)$ and therefore we can extract a subsequence (that we still denoted by $\{u_k\}_{k\in\mathbb{N}}$) such that

$$u_k \rightharpoonup u \text{ weakly in } \widetilde{\mathcal{W}}^{s,p}(\Omega), \qquad u_k \to u \text{ strongly in } L^p(\Omega).$$

Then $\|u\|_{L^p(\Omega)} = 1$ and

$$[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \le \liminf_{k\to\infty} [u_k]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} = \lim_{k\to\infty} \lambda_k = \lambda_1(s,p).$$

Then u is an eigenfunction associated to $\lambda_1(s,p)$. Therefore u has constant sign.

Now, proceeding as in the proof of [3, Theorem 2], we arrive to a contradiction. In fact, by Egoroff's theorem we can find a subset A_δ of Ω such that $|A_\delta| < \delta$ and $u_k \to u$ uniformly in $\Omega\setminus A_\delta$. From (3.9) we get that u and the uniform convergence in $\Omega\setminus A_\delta$ we obtain that $|\{u>0\}| > 0$ and $|\{u>0\}| < 0$. This contradicts the fact that an eigenfunction associated with the first eigenvalue does not change sign. □

4. The limit as $s \to 1^-$

In this section, our goal is to show that

$$\begin{aligned} &\lim_{s\to 1^-} (1-s)\lambda_1(s,p) = \lambda_1(1,p) \\ &= \inf_{u\in W^{1,p}_0(\Omega), u\not\equiv 0} \left\{ \frac{K_{n,p}\int_\Omega |\nabla_x u(x,y)|^p dxdy + K_{m,p}\int_\Omega |\nabla_y u(x,y)|^p dxdy}{\|u\|^p_{L^p(\Omega)}} \right\} \end{aligned} \tag{4.1}$$

where $K_{n,p}$ is a constant that depends only on n and p, and $K_{m,p}$ depends only on m and p. Before proving (4.1), we need some technical results.

Lemma 4.1. *Let Ω be an open subsets of $\mathbb{R}^{n+m}$ with smooth boundary and $p \in (1,\infty)$. For all $s\in(0,1)$ we have that $W^{1,p}(\Omega)$ is continuity embedded in $\mathcal{W}^{s,p}(\Omega)$.*

Proof. In this proof, we follow the ideas of the proof of [11, Theorem 1]. Let $u \in W^{1,p}(\Omega)$. By an extension argument, we can assume that $u\in W^{1,p}(\mathbb{R}^{n+m})$. We have that

$$\begin{aligned} &\int_{\mathbb{R}^{n+m}} |u(x+h,y)-u(x,y)|^p dxdy \le |h|^p \int_{\mathbb{R}^{n+m}} |\nabla_x u(x,y)|^p dxdy, \\ &\int_{\mathbb{R}^{n+m}} |u(x,y+h)-u(x,y)|^p dxdy \le |h|^p \int_{\mathbb{R}^{n+m}} |\nabla_y u(x,y)|^p dxdy. \end{aligned} \tag{4.2}$$

The proof of this fact can be carried out as that of Proposition XI.3 in [10] and is omitted.

Then, by (4.2), we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^{n+m}} & \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}}dxdydz\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n+m}} \frac{|u(x+h,y)-u(x,y)|^p}{|h|^{n+sp}}dxdydh\\
&\le \int_{\{|h|\le 1\}}\frac{dh}{|h|^{(s-1)p+n}}\int_{\mathbb{R}^{n+m}}|\nabla_x u(x,y)|^p dxdy\\
&\quad + 2\int_{\{|h|>1\}}\frac{dh}{|h|^{sp+n}}\int_{\mathbb{R}^{n+m}}|u(x,y)|^p dxdy\\
&\le \frac{\omega_n}{(1-s)p}\int_{\mathbb{R}^{n+m}}|\nabla_x u(x,y)|^p dxdy + \frac{2\omega_n}{sp}\int_{\mathbb{R}^{n+m}}|u(x,y)|^p dxdy.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\mathbb{R}^m}\int_{\mathbb{R}^{n+m}} & \frac{|u(x,y)-u(x,w)|^p}{|y-w|^{m+sp}}dxdydw\\
&\le \frac{\omega_m}{(1-s)p}\int_{\mathbb{R}^{n+m}}|\nabla_y u(x,y)|^p dxdy + \frac{2\omega_m}{sp}\int_{\mathbb{R}^{n+m}}|u(x,y)|^p dxdy,
\end{aligned}$$

which completes the proof. $\square$

Remark 4.2. Proceeding as in the proof of previous lemma and using using the Poincaré inequality, we have that

$$(1-s)[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \le C\left(1+\frac{1}{s}\right)\int_\Omega |\nabla u|^p\, dxdy \qquad \forall u\in W^{1,p}_0(\Omega)$$

where C is a constant independent of s.

Lemma 4.3. *Let Ω be an open subset of $\mathbb{R}^{n+m}$ with smooth boundary and $p\in(1,\infty)$. If $u\in W^{1,p}_0(\Omega)$ then*

$$(1-s)[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \to K_{n,p}\int_\Omega |\nabla_x u|^p\, dxdy + K_{m,p}\int_\Omega |\nabla_y u|^p\, dxdy$$

as $s\to 1^-$.

Proof. We split the proof into two cases.

Case 1. First we prove the lemma for $\phi\in C^\infty_0(\Omega)$. Let B_1 and B_2 be two open balls in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively such that $\Omega\subset B_1\times B_2$.

Given $y\in B_2$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\phi(x,y)-\phi(z,y)|^p}{|x-z|^{n+sp}}dxdz &= \int_{B_1}\int_{B_1}\frac{|\phi(x,y)-\phi(z,y)|^p}{|x-z|^{n+sp}}dxdz\\
&\quad + 2\int_{B_1}\int_{B_1^c}\frac{|\phi(x,y)|^p}{|x-z|^{n+sp}}dxdz.
\end{aligned}\tag{4.3}$$

By [11, Theorem 1], there is a constant $K_{n,p}$ (that depends only the n and p) such that

$$(1-s)\int_{B_1}\int_{B_1}\frac{|\phi(x,y)-\phi(z,y)|^p}{|x-z|^{n+sp}}dxdz \to K_{n,p}\int_{B_1}|\nabla_x\phi(x,y)|^p dx \tag{4.4}$$

as $s \to 1^-$. On the other hand, since $\mathrm{supp}(\varphi) \subset\subset \Omega \subset B_1 \times B_2$, there exists $\delta > 0$ such that $|x - z| > \delta$ for all $z \in B_1^c$ and $x \in \{t \in B_1 \colon (t, y) \in \mathrm{supp}(\varphi)\}$. Thus

$$(1-s)\int_{B_1}\int_{B_1^c} \frac{|\phi(x,y)|^p}{|x-z|^{n+sp}} dxdz \le (1-s)\frac{\omega_n}{sp\delta^{sp}} \|\phi(\cdot,y)\|_{L^p(B_1)}^p \to 0 \tag{4.5}$$

as $s \to 1^-$. Then by (4.3), (4.4), and (4.5) we have that

$$(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(x,y)-\phi(z,y)|^p}{|x-z|^{n+sp}} dxdz \to K_{n,p}\int_{B_1} |\nabla_x \phi(x,y)|^p dx \tag{4.6}$$

as $s \to 1^-$. Proceeding as in the proof of Lemma 4.1, we have that

$$\begin{aligned}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(x,y)-\phi(z,y)|^p}{|x-z|^{n+sp}} dxdz &\le \frac{\omega_n}{p}\int_{\mathbb{R}^n} |\nabla_x \phi(x,y)|^p dxdy \\ &\quad + (1-s)\frac{2\omega_n}{s_0 p}\int_{\mathbb{R}^n} |\phi(x,y)|^p dxdy.\end{aligned}$$

Thus, (4.6) and the dominated convergence theorem imply

$$(1-s)\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n} \frac{|\phi(x,y)-\phi(z,y)|^p}{|x-z|^{n+sp}} dzdxdy \to K_{n,p}\int_{\mathbb{R}^m}\int_{B_1} |\nabla_x \phi(x,y)|^p dxdy,$$

as $s \to 1^-$, that is,

$$(1-s)\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n} \frac{|\phi(x,y)-\phi(z,y)|^p}{|x-z|^{n+sp}} dzdxdy \to K_{n,p}\int_{\Omega} |\nabla_x \phi(x,y)|^p dxdy,$$

as $s \to 1^-$.

In the same manner we can see that there exists a constant $K_{m,p}$ (that depends only the m and p) such that

$$(1-s)\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^m} \frac{|\phi(x,y)-\phi(x,w)|^p}{|y-w|^{m+sp}} dwdxdy \to K_{m,p}\int_{\Omega} |\nabla_y \phi(x,y)|^p dxdy,$$

as $s \to 1^-$.

Then, we have

$$(1-s)[\phi]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \to K_{n,p}\int_{\Omega} |\nabla_x \phi|^p \, dxdy + K_{m,p}\int_{\Omega} |\nabla_y \phi|^p \, dxdy,$$

as $s \to 1^-$.

Case 2. Now we prove the general case. Given $u \in W_0^{1,p}(\Omega)$, we define

$$\begin{aligned} F_s^u(x,y,z) &= (1-s)^{1/p} \frac{|u(x,y)-u(z,y)|}{|x-z|^{n/p+s}}, \\ G_s^u(x,y,z) &= (1-s)^{1/p} \frac{|u(x,y)-u(x,w)|}{|y-w|^{m/p+s}} \end{aligned}$$

and we want to show that

$$\|F_s^u\|_{L^p(\mathbb{R}^{2n+m})} \to K_{n,p}^{1/p}\|\nabla_x u\|_{L^p(\Omega)}, \qquad \|G_s^u\|_{L^p(\mathbb{R}^{n+2m})} \to K_{m,p}^{1/p}\|\nabla_y u\|_{L^p(\Omega)},$$

as $s \to 1^-$.

Given $\varepsilon > 0$ there is $\phi \in C_0^\infty(\Omega)$ such that

$$\|\nabla u - \nabla \phi\|_{L^p(\Omega)} < \varepsilon.$$

Thus

$$|\|\nabla_x u\|_{L^p(\Omega)} - \|\nabla_x \phi\|_{L^p(\Omega)}| < \varepsilon \text{ and } |\|\nabla_x u\|_{L^p(\Omega)} - \|\nabla_x \phi\|_{L^p(\Omega)}| < \varepsilon. \tag{4.7}$$

By case 1, there exists $s_0 \in (0,1)$ such that

$$\begin{aligned} |\|F_s^{\phi}\|_{L^p(\mathbb{R}^{2n+m})} - K_{n,p}^{1/p}\|\nabla_x \phi\|_{L^p(\Omega)}| < \varepsilon, \\ |\|G_s^{\phi}\|_{L^p(\mathbb{R}^{n+2m})} - K_{m,p}^{1/p}\|\nabla_y \phi\|_{L^p(\Omega)}| < \varepsilon, \end{aligned} \tag{4.8}$$

for all $s \in (s_0, 1)$.

On the other hand, using Remark 4.2, we have that

$$\begin{aligned} |\|F_s^{u}\|_{L^p(\mathbb{R}^{2n+m})} - \|F_s^{\phi}\|_{L^p(\mathbb{R}^{2n+m})}| \le C\|\nabla u - \nabla \phi\|_{L^p(\Omega)} < C\varepsilon, \\ |\|G_s^{u}\|_{L^p(\mathbb{R}^{2n+m})} - \|G_s^{\phi}\|_{L^p(\mathbb{R}^{2n+m})}| \le C\|\nabla u - \nabla \phi\|_{L^p(\Omega)}, < C\varepsilon, \end{aligned} \tag{4.9}$$

where C is a constant independent of s.

Finally, by (4.7), (4.8), and (4.9), we obtain that

$$\begin{aligned} |\|F_s^{u}\|_{L^p(\mathbb{R}^{2n+m})} - K_{n,p}^{1/p}\|\nabla_x u\|_{L^p(\Omega)}| < C\varepsilon, \\ |\|G_s^{u}\|_{L^p(\mathbb{R}^{n+2m})} - K_{m,p}^{1/p}\|\nabla_y u\|_{L^p(\Omega)}| < C\varepsilon, \end{aligned}$$

and the proof is complete. □

Corollary 4.4. *Let Ω be an open subset of $\mathbb{R}^{n+m}$ with smooth boundary and $p \in (1,\infty)$. If $u \in W_0^{1,p}(\Omega)$ then*

$$(1-s)[u]^p_{\mathcal{W}^{s,p}(\Omega)} \to K_{n,p}\int_{\Omega} |\nabla_x u|^p \, dxdy + K_{m,p}\int_{\Omega} |\nabla_y u|^p \, dxdy$$

as $s \to 1^-$.

Proof. By Lemma 4.3, we only need to proof that if $u \in W_0^{1,p}(\Omega)$ then

$$(1-s)\left([u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} - [u]^p_{\mathcal{W}^{s,p}(\Omega)}\right) \to 0$$

as $s \to 1^-$. First we prove the result for $\phi \in C_0^\infty(\Omega)$. We have

$$\begin{aligned} \left([\phi]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} - [\phi]^p_{\mathcal{W}^{s,p}(\Omega)}\right) &= 2\int_{\text{supp}(\phi)}\int_{\Omega_y^c} \frac{|\phi(x,y)|}{|x-z|^{n+sp}}^p \, dzdxdy \\ &\quad + 2\int_{\text{supp}(\phi)}\int_{\Omega_x^c} \frac{|\phi(x,y)|}{|y-w|^{m+sp}}^p \, dwdxdy. \end{aligned} \tag{4.10}$$

Since $\text{supp}(\phi) \subset \Omega$ is compact, there exists $\delta > 0$ such that $|x-z| > \delta$ and $|y-w| > \delta$ for all $(x,y) \in \text{supp}(\phi)$, $z \in \Omega_y^c$, $w \in \Omega_x^c$. Then

$$\begin{aligned} \int_{\text{supp}(\phi)}\int_{\Omega_y^c} \frac{|\phi(x,y)|}{|x-z|^{n+sp}}^p \, dzdxdy \le \frac{\omega_n}{sp\delta^{sp}}\int_{\Omega} |\phi(x,y)|^p dxdy, \\ \int_{\text{supp}(\phi)}\int_{\Omega_y^c} \frac{|\phi(x,y)|}{|y-w|^{m+sp}}^p \, dwdxdy \le \frac{\omega_m}{sp\delta^{sp}}\int_{\Omega} |\phi(x,y)|^p dxdy. \end{aligned}$$

Therefore, using (4.10), we have that

$$(1-s)\left([\phi]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} - [\phi]^p_{\mathcal{W}^{s,p}(\Omega)}\right) \to 0$$

as $s \to 1^-$.

The argument for the general case is analogous to the one performed in case 2 in the proof of Lemma 4.3. $\square$

For the proof of the following lemma, see [11, Lemma 2].

Lemma 4.5. *Let $\delta > 0$ and $g, h\colon (0,\delta) \to (0,+\infty)$. Assume that $g(t) \le g(t/2)$ and that h in non-increasing. Then*

$$\int_0^\delta t^{N-1} g(t)h(t)\,dt \ge \frac{N}{(2\delta)^N}\int_0^\delta t^{N-1}g(t)dt \int_0^\delta t^{N-1}h(t)dt$$

for all $N > 0$.

Lemma 4.6. *Let $0 < s_0 < s$ and $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$. Then*

$$\frac{(1-s_0)[u]^p_{\mathcal{W}^{s_0,p}(\Omega)}}{2^{(1-s_0)p}\operatorname{diam}(\Omega)^{(s-s_0)p}} \le (1-s)[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}$$

Proof. Let B_1 and B_2 be two balls in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively such that $\Omega \subset B_1 \times B_2$ and $\operatorname{diam}(B_1) = \operatorname{diam}(B_2) = \operatorname{diam}(\Omega)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy \ge \\
&\ge \int_{\mathbb{R}^m}\int_0^\infty \int_{S^{n-1}}\int_{\mathbb{R}^n} \frac{|u(x+tw,y)-u(x,y)|^p}{t^{1+sp}} dxd\sigma dtdy \\
&\ge \int_{\mathbb{R}^m}\int_0^{\operatorname{diam}(\Omega)} \int_{S^{n-1}} t^{(1-s_0)p-1}\int_{\mathbb{R}^n} \frac{|u(x+t\omega,y)-u(x,y)|^p}{t^p}\frac{dxd\sigma dtdy}{t^{(s-s_0)p}}
\end{aligned}$$

Taking $N = (1-s_0)p$, $\delta = \operatorname{diam}(\Omega)$, we get

$$g(t) = \int_{S^{n-1}}\int_{\mathbb{R}^m} \frac{|u(x+t\omega,y)-u(x,y)|^p}{t^p} dxd\sigma, \qquad \text{and} \qquad h(t) = \frac{1-s}{t^{(s-s_0)p}}.$$

By Lemma 4.5, we have that

$$\begin{aligned}
&(1-s)\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+sp}} dzdxdy \ge \\
&\ge \frac{(1-s_0)p}{2^{(1-s_0)p}\operatorname{diam}(\Omega)^{(1-s_0)p}}\int_{\mathbb{R}^m}\int_0^\delta t^{(1-s_0)p-1}g(t)dt\int_0^\delta t^{(1-s_0)p-1}h(t)dt \\
&\ge \frac{(1-s_0)p}{2^{(1-s_0)p}\operatorname{diam}(\Omega)^{(1-s_0)p}}\int_{\mathbb{R}^m}\int_0^\delta t^{(1-s_0)p-1}g(t)dt\int_0^\delta (1-s)t^{(1-s)p-1}dt \\
&\ge \frac{(1-s_0)}{2^{(1-s_0)p}\operatorname{diam}(\Omega)^{(s-s_0)p}}\int_{\mathbb{R}^m}\int_0^\delta\int_{S^{n-1}}\int_{\mathbb{R}^m} \frac{|u(x+t\omega,y)-u(x,y)|^p}{t^{1+s_0p}} dxd\sigma dtdy \\
&\ge \frac{(1-s_0)}{2^{(1-s_0)p}\operatorname{diam}(\Omega)^{(s-s_0)p}}\int_\Omega\int_{\Omega_y} \frac{|u(x,y)-u(z,y)|^p}{|x-z|^{n+s_0p}} dzdxdy.
\end{aligned}$$

Similarly

$$\begin{aligned}
&(1-s)\int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^n} \frac{|u(x,y)-u(x,w)|^p}{|y-w|^{m+sp}} dzdxdy \ge \\
&\qquad \ge \frac{(1-s_0)}{2^{(1-s_0)p}\operatorname{diam}(\Omega)^{(s-s_0)p}}\int_\Omega\int_{\Omega_x} \frac{|u(x,y)-u(z,y)|^p}{|y-w|^{m+s_0p}} dwdxdy.
\end{aligned}$$

This conclude the proof. $\square$

We can now show the main result of this section.

Theorem 4.7. *Let Ω is bounded domain in $\mathbb{R}^{n+m}$ with smooth boundary, $s \in (0,1)$ and $p \in (1,\infty)$. Then*
$$\lim_{s\to 1^-}(1-s)\lambda_1(s,p) = \lambda_1(1,p).$$

Proof. First, we observe that, from Lemma 4.1, if $u \in W_0^{1,p}(\Omega)$ then $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$. Then
$$(1-s)\lambda_1(s,p) \le \frac{[u]^p_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}}{\|u\|^p_{L^p(\Omega)}}$$
for all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. Therefore, by Lemma 4.3, we have that
$$\limsup_{s\to 1^-}(1-s)\lambda_1(s,p) \le \frac{K_{n,p}\displaystyle\int_\Omega |\nabla_x u(x,y)|^p dxdy + K_{m,p}\int_\Omega |\nabla_y u(x,y)|^p dxdy}{\|u\|^p_{L^p(\Omega)}}$$
for all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. Then
$$\limsup_{s\to 1^-}(1-s)\lambda_1(s,p) \le \lambda_1(1,p). \tag{4.11}$$

To finish the proof, we have to show that
$$\liminf_{s\to 1^-}(1-s)\lambda_1(s,p) \ge \lambda_1(1,p).$$

Let $\{s_k\}_{k\in\mathbb{N}} \subset (0,1)$ be such that $s_k \to 1$ as $k \to \infty$,
$$\lim_{k\to\infty}(1-s_k)\lambda_1(s_k,p) = \liminf_{s\to 1^-}(1-s)\lambda_1(s,p). \tag{4.12}$$
For each $k \in \mathbb{N}$, we let u_k be an eigenfunction corresponding to $\lambda_1(s_k,p)$ such that $\|u_k\|_{L^p(\Omega)} = 1$. By (4.12), there is a positive constant C such that
$$(1-s_k)[u_k]^p_{\mathcal{W}^{s_k,p}(\mathbb{R}^{n+m})} \le C \qquad \forall k \in \mathbb{N}.$$
Then, by Lemma 2.4, there is a positive constant C such that
$$(1-s_k)|u_k|^p_{W^{s_k,p}(\mathbb{R}^{n+m})} \le C \qquad \forall k \in \mathbb{N}.$$
Thus, by [11, Corollary 7], up to a subsequence, $\{u_k\}_{k\in\mathbb{N}}$ converges in $L^p(\Omega)$ to some $u \in W_0^{1,p}(\Omega)$. Moreover, for all $\delta > 0$, $u_k \to u$ strongly in $W^{1-\delta,p}(\Omega)$. Therefore $\|u\|_{L^p(\Omega)} = 1$.

Let $s_0 \in (0,1)$. Since $s_k \to 1$, there exists $k_0 \in \mathbb{N}$ such that $s_0 < s_k$ for all $k \ge k_0$. Then, by Lemma 4.6, we have that
$$\begin{aligned}\frac{(1-s_0)[u_k]^p_{\mathcal{W}^{s_0,p}(\Omega)}}{2^{(1-s_0)p}} &\le \operatorname{diam}(\Omega)^{(s_k-s_0)p}(1-s_k)[u_k]^p_{\mathcal{W}^{s_k,p}(\mathbb{R}^n)} \\ &= \operatorname{diam}(\Omega)^{(s_k-s_0)p}(1-s_k)\lambda_1(s_k,p).\end{aligned}$$
Thus, by (4.12) and Fatou's lemma, we get
$$\frac{(1-s_0)[u]^p_{\mathcal{W}^{s_0,p}(\Omega)}}{2^{(1-s_0)p}} \le \operatorname{diam}(\Omega)^{(1-s_0)p}\liminf_{s\to 1^-}(1-s)\lambda_1(s,p).$$

By Corollary 4.4, it holds that

$$K_{n,p}\int_\Omega |\nabla_x u(x,y)|^p dxdy + K_{m,p}\int_\Omega |\nabla_y u(x,y)|^p dxdy = \lim_{s_0\to 1^-} \frac{(1-s_0)[u]^p_{\mathcal{W}^{s_0,p}(\Omega)}}{2^{(1-s_0)p}}$$
$$\leq \liminf_{s\to 1^-}(1-s)\lambda_1(s,p).$$

Then

$$\lambda_1(1,p) \leq \liminf_{s\to 1^-}(1-s)\lambda_1(s,p).$$

Therefore, by (4.11),

$$\lambda_1(1,p) = \lim_{s\to 1^-}(1-s)\lambda_1(s,p),$$

as we wanted to prove. □

5. The limit as $p\to\infty$

Now we want to pass to the limit as $p\to\infty$ in the first eigenvalue $\lambda_1(s,p)$. Our goal now is to show that

$$[\lambda_1(s,p)]^{1/p} \to \Lambda_\infty(s)$$

where

$$\Lambda_\infty(s) = \inf\left\{[u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} \colon u \in \mathcal{W}^{s,\infty}(\mathbb{R}^{n+m}), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ in } \Omega^c\right\}.$$

Observe that, by Arzela-Ascoli's theorem, the previous infimum is attained.

We first prove a geometric characterization of $\Lambda_\infty(s)$.

Lemma 5.1. *Let* $R_s = \max_{(x,y)\in\Omega}\min_{(z,w)\in\partial\Omega}(|x-z|^s + |y-w|^s)$, *then*

$$\Lambda_\infty(s) = \frac{1}{R_s}.$$

Proof. Let $u\in\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})$, such that $\|u\|_{L^\infty(\Omega)} = 1$, $u = 0$ in Ω^c and $\Lambda_\infty(s) = [u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})}$. Then, let $(x_0,y_0)\in\Omega$ be such that $u(x_0,y_0) = 1$. If $(z,w)\in\partial\Omega$ we have

$$|u(x_0,y_0) - u(z,y_0)| \leq \Lambda_\infty(s)|x_0-z|^s$$

and

$$|u(z,y_0) - u(z,w)| \leq \Lambda_\infty(s)|y_0-w|^s.$$

Then

$$|u(x_0,y_0) - u(z,w)| \leq \Lambda_\infty(s)(|x_0-z|^s + |y_0-w|^s).$$

Therefore,

$$1 \leq \Lambda_\infty(s)\min_{(z,w)\in\partial\Omega}(|x_0-z|^s + |y_0-w|^s),$$

and then, we get

$$\Lambda_\infty(s) \geq \frac{1}{\min_{(z,w)\in\partial\Omega}(|x_0-z|^s + |y_0-w|^s)} \geq \frac{1}{R_s}. \tag{5.1}$$

Now, we choose (x_0,y_0) that solves the geometric maximization problem

$$R_s = \max_{(x,y)\in\Omega}\min_{(z,w)\in\partial\Omega}(|x-z|^s + |y-w|^s),$$

and consider the function

$$u(x,y) = \left(1 - \frac{|x_0 - x|^s + |y_0 - y|^s}{R_s}\right)_+.$$

Observe that, $\|u\|_{L^\infty(\Omega)} = 1$. On the other hand, since for any $s \in (0,1]$

$$|a^s - b^s| \leq |a - b|^s \quad \forall a, b \in [0, \infty),$$

we have that $[u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} \leq {}^1\!/_{R_s}$. Hence, using this functions as a test function in the variational problem defining $\Lambda_\infty(s)$ we get

$$\Lambda_\infty(s) \leq \frac{1}{R_s}. \tag{5.2}$$

From (5.1) and (5.2) we obtain the desired result. $\square$

Lemma 5.2. *Let u_p be a positive eigenfunction for $\lambda_1(s,p)$ normalized according to $\|u_p\|_{L^p(\Omega)} = 1$. Then, there exists a sequence $p_j \to \infty$ such that*

$$u_j \to u$$

uniformly in $\mathbb{R}^N$. This limit function u belongs to the space $\mathcal{W}^{s,\infty}(\Omega)$ and is a solution to the variational problem

$$\Lambda_\infty(s) = \min\left\{[u]_{\mathcal{W}^{s,\infty}(\Omega)} \colon u \in \mathcal{W}^{s,\infty}(\Omega), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ on } \partial\Omega\right\}.$$

In addition, it holds that

$$[\lambda_1(s,p)]^{1/p} \to \Lambda_\infty(s).$$

Proof. Let $\alpha > 1$ and

$$R_{s\alpha} = \max_{(x,y)\in\Omega} \min_{(z,w)\in\partial\Omega} (|x - z|^{s\alpha} + |y - w|^{s\alpha}).$$

We first claim that

$$\frac{(R_s)^\alpha}{2^{\alpha-1}} \leq R_{s\alpha} \tag{5.3}$$

for any $\alpha > 1$. To this end, let $(x_0, y_0) \in \Omega$ such that

$$R_s = \min_{(z,w)\in\partial\Omega} (|x_0 - z|^s + |y_0 - w|^s).$$

Then for all $(z, w) \in \partial\Omega$ we have

$$\begin{aligned}(R_s)^\alpha &\leq (|x_0 - z|^s + |y_0 - w|^s)^\alpha \leq 2^{\alpha-1}\left(|x_0 - z|^{s\alpha} + |y_0 - w|^{s\alpha}\right)\\ &\leq 2^{\alpha-1} R_{s\alpha},\end{aligned}$$

that is, (5.3). On the other hand, it is clear that if $s\alpha < 1$ we have that

$$u_\alpha(x,y) = \left(1 - \frac{|x - x_0|^{\alpha s} + |y - y_0|^{\alpha s}}{R_{s\alpha}}\right)_+$$

belongs to $\widetilde{\mathcal{W}}^{s,p}(\Omega)$ for all $p > 1$. Then

$$(\lambda_1(s,p))^{1/p} \leq \frac{[u_\alpha]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}}{\|u_\alpha\|_{L^p(\Omega)}} \tag{5.4}$$

for all $p > 1$ and $1 < \alpha < {}^1\!/_s$. Therefore

$$\limsup_{p\to\infty} (\lambda_1(s,p))^{1/p} \leq \frac{[u_\alpha]_{\mathcal{W}^{s,\infty}(\Omega)}}{\|u_\alpha\|_{L^\infty(\Omega)}} \quad \forall \alpha \in (1, {}^1\!/_s).$$

Observe that if $\alpha \in (1, {}^1\!/_s)$, by (5.3), we have

$$\frac{|u_\alpha(x,y) - u_\alpha(z,y)|}{|x-z|^s} \le \frac{|x-z|^{(\alpha-1)s}}{R_{s\alpha}} \le 2^{\alpha-1}\frac{\mathrm{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha}$$

for all $(x,y) \neq (z,y) \in \overline{\Omega}$, and

$$\frac{|u_\alpha(x,y) - u_\alpha(x,w)|}{|y-w|^s} \le \frac{|y-w|^{(\alpha-1)s}}{R_{s\alpha}} \le 2^{\alpha-1}\frac{\mathrm{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha},$$

for all $(x,y) \neq (z,y) \in \overline{\Omega}$, that is

$$[u_\alpha]_{\mathcal{W}^{s,\infty}(\Omega)} \le 2^{\alpha-1}\frac{\mathrm{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha}.$$

Then, by (5.4) we get

$$\limsup_{p\to\infty}(\lambda_1(s,p))^{1/p} \le 2^{\alpha-1}\frac{\mathrm{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha} \qquad \alpha \in (1, {}^1\!/_s),$$

since $\|u_\alpha\|_{L^\infty(\Omega)} = 1$. Therefore, passing to the limit as $\alpha \to 1$ in the previous inequality we get

$$\limsup_{p\to\infty}(\lambda_1(s,p))^{1/p} \le \frac{1}{R_s} = \Lambda_\infty(s). \tag{5.5}$$

Our next goal is to show that

$$\Lambda_\infty(s) \le \liminf_{p\to\infty}(\lambda_1(s,p))^{1/p}.$$

Let $p_j > 1$ be such that

$$\liminf_{p\to\infty}(\lambda_1(s,p))^{1/p} = \lim_{j\to\infty}(\lambda_1(s,p_j))^{1/p_j}.$$

By (5.5), without of loss of generality, we can assume

$$(\lambda_1(s,p_j))^{1/p_j} = [u_{p_j}]_{\mathcal{W}^{s,p_j}(\mathbb{R}^{n+m})} \le \Lambda_\infty(s) + \epsilon \qquad \forall j \in \mathbb{N},$$

where u_{p_j} is an eigenfunction for $\lambda_1(s,p_j)$ normalized according to $\|u_{p_j}\|_{L^{p_j}(\Omega)} = 1$ and ϵ is any positive number. Then, by Lemma 2.4, we have that there exists a constant C, independent of j, such that

$$|u_{p_j}|_{W^{s,p_j}(\Omega)} \le C \qquad \forall j \in \mathbb{N}.$$

Therefore, for any $j \in \mathbb{N}$ there exists a constant C independent of j, such that

$$\|u_{p_j}\|_{W^{s,p_j}(\Omega)} \le C. \tag{5.6}$$

On the other hand, given $q > 1$ such that $sq > 2(n+m)$ and taking $t = s - {}^{n+m}\!/_q$, by Hölder's inequality, for any $p_j > q$ we have that

$$\|u_{p_j}\|^q_{L^q(\Omega)} \le |\Omega|^{1-\frac{q}{p_j}}\|u_{p_j}\|^q_{L^p(\Omega)} = |\Omega|^{1-\frac{q}{p_j}},$$

and

$$\begin{aligned}|u_{p_j}|^q_{W^{t,q}(\Omega)} &= \int_{\Omega^2} \frac{|u_{p_j}(x,y)-u_{p_j}(z,w)|^q}{|(x,y)-(z,w)|^{sq}}\,dxdydzdw\\ &\le |\Omega|^{2(1-\frac{q}{p_j})}\left(\int_{\Omega^2} \frac{|u_{p_j}(x,y)-u_{p_j}(z,w)|^{p_j}}{|(x,y)-(z,w)|^{sp_j}}\,dxdydzdw\right)^{\frac{q}{p_j}}\\ &\le |\Omega|^{2(1-\frac{q}{p_j})}\max\left\{1,\operatorname{diam}(\Omega)^{(n+m)\frac{q}{p_j}}\right\}|u_{p_j}|^q_{W^{s,p_j}(\Omega)}.\end{aligned}$$

Hence, by (5.6), for j large there exists a constant C, independent of j, such that

$$\|u_{p_j}\|_{W^{t,q}(\Omega)} \le C\max\left\{|\Omega|^{\frac{1}{q}-\frac{1}{p_j}}, |\Omega|^{2(\frac{1}{q}-\frac{1}{p_j})}, |\Omega|^{2(\frac{1}{q}-\frac{1}{p_j})}\operatorname{diam}(\Omega)^{\frac{n+m}{p_j}}\right\},$$

that is, there exists $j_0 > 1$ such that $\{u_{p_j}\}_{j>j_0}$ is bounded in $W^{t,q}(\Omega)$. Then, since $tq > n+m$, by Theorem 2.3, there exists a subsequence $\{u_k\}_{k\in\mathbb{N}}$ of $\{u_{p_j}\}_{j>j_0}$ and a function $u\in C^{0,\gamma}(\overline{\Omega})$ $(0<\gamma<t-{}^{(n+m)}/_{q})$ such that $u_k \to u$ uniformly in $\overline{\Omega}$.

Thus, if $q>1$ there exists $k_0\in\mathbb{N}$ such that $p_k>q$ if $k>k_0$ and therefore, by Hölder's inequality, for any $k>k_0$ we have

$$\begin{aligned}&\left(\int_\Omega\int_{\Omega_y} \frac{|u_k(x,y)-u_k(z,y)|^q}{|x-z|^{qs}}dzdxdy\right)^q\\ &\le C^{\frac{1}{q}-\frac{1}{p_k}}\max\left\{1,\operatorname{diam}(\Omega)^{\frac{n}{p_k}}\right\}\left(\int_\Omega\int_{\Omega_y} \frac{|u_k(x,y)-u_k(z,y)|^{p_k}}{|x-z|^{p_k s+n}}dzdxdy\right)^{\frac{1}{p_k}}\\ &\le C^{\frac{1}{q}-\frac{q}{p_k}}\max\left\{1,\operatorname{diam}(\Omega)^{\frac{n}{p_k}}\right\}[u_k]_{\mathcal{W}^{s,p_k}(\Omega)},\end{aligned}$$

and similarly

$$\begin{aligned}&\left(\int_\Omega\int_{\Omega_x} \frac{|u_k(x,y)-u_k(x,w)|^q}{|y-w|^{qs}}dwdxdy\right)^q\\ &\qquad\qquad \le C^{\frac{1}{q}-\frac{q}{p_k}}\max\left\{1,\operatorname{diam}(\Omega)^{\frac{m}{p_k}}\right\}[u_k]_{\mathcal{W}^{s,p_k}(\Omega)}.\end{aligned}$$

Here C is a constant independent of k. Then passing to the limit as $k\to\infty$ and using Fatou's lemma we have that

$$\begin{aligned}\left(\int_\Omega\int_{\Omega_y} \frac{|u(x,y)-u(z,y)|^q}{|x-z|^{qs}}dzdxdy\right)^q &\le C^{\frac{1}{q}}\liminf_{k\to\infty}[u_k]_{\mathcal{W}^{s,p_k}(\Omega)}\\ &\le C^{\frac{1}{q}}\liminf_{p\to\infty}(\lambda_1(s,p))^{1/p},\\ \left(\int_\Omega\int_{\Omega_x} \frac{|u(x,y)-u(x,w)|^q}{|y-w|^{qs}}dwdxdy\right)^q &\le C^{\frac{1}{q}}\liminf_{k\to\infty}[u_k]_{\mathcal{W}^{s,p_k}(\Omega)}\\ &\le C^{\frac{1}{q}}\liminf_{p\to\infty}(\lambda_1(s,p))^{1/p}\end{aligned}$$

for all $q>1$. Now passing to the limit as $q\to\infty$ we obtain

$$\sup\left\{\frac{|u(x,y)-u(z,y)|}{|x-z|^s}\colon (x,y)\neq(z,y)\in\Omega\right\} \le \liminf_{p\to\infty}(\lambda_1(s,p))^{1/p},$$

$$\sup\left\{\frac{|u(x,y)-u(x,w)|}{|x-z|^s}\colon (x,y)\neq(x,w)\in\Omega\right\} \le \liminf_{p\to\infty}(\lambda_1(s,p))^{1/p},$$

that is

$$[u]_{\mathcal{W}^{s,\infty}(\Omega)} \leq \liminf_{p\to\infty} (\lambda_1(s,p))^{1/p}. \tag{5.7}$$

To conclude we need to show that $\|u\|_{L^\infty(\Omega)} = 1$. For all $q > 1$ there exists $k_0 \in \mathbb{N}$ such that $p_k > q$ if $k > k_0$ and therefore, by Hölder's inequality, for any $k > k_0$ we get

$$\|u_k\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p_k}} \|u_{p_j}\|^q_{L^p(\Omega)} = |\Omega|^{\frac{1}{q}-\frac{1}{p_j}}.$$

Then passing to the limit as $k \to \infty$ and using that $u_k \to u$ uniformly in $\overline{\Omega}$, $\|u\|_{L^q(\Omega)} \leq 1$ for all $q > 1$. Hence $\|u\|_{L^\infty(\Omega)} \leq 1$. On the other hand, for all k we have $1 = \|u_k\|_{L^{p_k}(\Omega)} \leq |\Omega|^{1/p_k} \|u_k\|_{L^\infty(\Omega)}$. Then, since $u_k \to u$ uniformly in $\overline{\Omega}$, we get $1 \leq \|u\|_{L^\infty(\Omega)}$. Hence $\|u\|_{L^\infty(\Omega)} = 1$. Thus, by (5.7), we get

$$\Lambda_\infty(s) \leq [u]_{\mathcal{W}^{s,\infty}(\Omega)} \leq \liminf_{p\to\infty} (\lambda_1(s,p))^{1/p},$$

and by (5.5) we conclude that

$$\Lambda_\infty(s) = \lim_{p\to\infty} (\lambda_1(s,p))^{1/p}.$$

This ends the proof. □

Using the geometric characterization given in Lemma 5.1 we can compute $\Lambda_\infty(s)$ in some concrete examples.

Example 1. When $\Omega = B_R$ is a ball of radius R we have

$$\Lambda_\infty(s) = \frac{1}{R^s}.$$

Example 2. When $\Omega = (-R, R) \times (-L, L)$ is a rectangle in $\mathbb{R}^2$ we have

$$\Lambda_\infty(s) = \frac{1}{\min\{R^s, L^s\}}.$$

Remark 5.3. One can consider two different powers r and s in the definition of the pseudo $p-$Laplacian. In this case we get that,

$$\Lambda_\infty(r,s) = \max_{(x,y)\in\Omega} \min_{(z,w)\in\partial\Omega} (|x-z|^r + |y-w|^s).$$

Viscosity solutions. To obtain an eigenvalue problem that is satisfied by the limit of the eigenfunctions u_p when $p \to \infty$, we need to introduce the definition of viscosity solutions. This is a notion of solution different from the weak one considered before. We refer to [13] for an introduction to the subject of viscosity solutions. In the theory of viscosity solutions the equation is evaluated for test functions at points where they touch the graph of a solution. Viscosity solutions are assumed to be continuous and the fractional Sobolev space is absent from the definition (no derivatives of a solutions are needed).

Definition 5.4. (Viscosity solutions). Suppose that the function u is continuous in $\mathbb{R}^{n+m}$ and that $u = 0$ in Ω^c. We say that u is a viscosity supersolution of the equation $-\mathcal{L}_{s,p}u + \lambda|u|^{p-2}u = 0$ if the following holds: whenever $x_0 \in \Omega$ and $\varphi \in C^1_0(\mathbb{R}^{n+m})$ (the test function) are such that $\varphi(x_0) = u(x_0)$ and $\varphi(x) \leq u(x)$ for every $x \in \mathbb{R}^{n+m}$, then we have

$$-\mathcal{L}_{s,p}\varphi(x_0) + \lambda|\varphi(x_0)|^{p-2}\varphi(x_0) \leq 0.$$

The requirement for being a viscosity subsolution is symmetric: the test function is touching from above and the inequality is reversed.

Finally, a viscosity solution is defined as being both a viscosity supersolution and a viscosity subsolution.

For our eigenvalue problem, we have that a continuos weak solution is a viscosity solution. For the proof we refer to [29].

Theorem 5.5. *An eigenfunction $u \in C(\overline{\Omega})$ (in the weak sense) is a viscosity solution of the equation $-\mathcal{L}_{s,p}u + \lambda|u|^{p-2}u = 0$ in the sense of Definition 5.4.*

We will also use the following lemmas.

Lemma 5.6. *Assume that*

$$(A_p)^{1/p} \to A, \qquad (B_p)^{1/p} \to -B,$$
$$(C_p)^{1/p} \to C, \qquad (D_p)^{1/p} \to -D,$$

and that

$$\theta_p \to \Theta,$$

as $p \to \infty$. If

$$2^{1/p}(A_p + C_p)^{1/p} \geq (B_p + D_p + \theta_p^{p-1})^{1/p}$$

for every p large enough, then, passing to the limit, it holds that

$$\max\{A; C\} \geq \max\{-B; -D; \Theta\}.$$

Proof. First, assume that $A > C$ and $-B > \max\{-D; \Theta\}$. Then for p large enough we have $A_p \geq C_p$, $-B_p \geq -D_p$ and $-B_p \geq (\theta_p)^p$. Then taking $p \to \infty$ in

$$(A_p)^{1/p} 2^{1/p} \left(1 + \frac{C_p}{A_p}\right)^{1/p} \geq (B_p)^{1/p} \left(1 + \frac{D_p}{B_p} + \frac{\theta_p^{p-1}}{B_p}\right)^{1/p}$$

we get

$$A \geq -B.$$

The rest of the cases ($A = C$, $A < C$, etc) can be handled in an analogous way. □

Lemma 5.7. *For a smooth test function ϕ let*

$$A_p = \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} dz.$$

If $x_p \to x_0$, $y_p \to y_0$ as $p \to \infty$, then

$$(A_p)^{1/p} \to A = \sup_z \frac{\phi(x_0, y_0) - \phi(z, y_0)}{|x_0 - z|^s}.$$

Proof. We just have to observe that

$$(A_p)^{1/p} = \left(\int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} dz\right)^{1/p}.$$

The integrand satisfies

$$\frac{|\phi(x_p,y_p)-\phi(z,y_p)|^{p-2}(\phi(x_p,y_p)-\phi(z,y_p))^+}{|x_p-z|^{n+sp}} \sim \frac{|\phi(x_0,y_0)-\phi(z,y_0)|^{p-2}(\phi(x_0,y_0)-\phi(z,y_0))^+}{|x_0-z|^{n+sp}}$$

and hence the result follows from the fact that $\left(\int f^p\right)^{1/p} \to \|f\|_\infty$. □

Lemma 5.8. *Any uniform limit of u_p a sequence of eigenfunctions for $\lambda_1(s,p)$ normalized according to $\|u_p\|_{L^p(\Omega)}=1$, u is a nontrivial solution to*

$$\begin{cases} \max\{A;C\} = \max\{-B;-D;\Lambda_\infty(s)u\} & \text{in } \Omega,\\ u=0 & \text{in } \Omega^c, \end{cases}$$

in the viscosity sense. Here

$$A=\sup_w \frac{u(x,w)-u(x,y)}{|y-w|^s}, \qquad B=\inf_w \frac{u(x,w)-u(x,y)}{|y-w|^s},$$
$$C=\sup_z \frac{u(z,y)-u(x,y)}{|x-z|^s}, \qquad D=\inf_z \frac{u(z,y)-u(x,y)}{|x-z|^s}.$$

Proof. We call u_p a sequence of solutions to $-\mathcal{L}_{s,p}u+\lambda|u|^{p-2}u=0$ that converges uniformly to u. That $u=0$ in Ω^c follows since $u_p=0$ in Ω^c and we have uniform convergence.

Let $\phi\in C_0^1(\mathbb{R}^{n+m})$ be such that $u-\phi$ has a strict minimum at $(x_0,y_0)\in\Omega$. Since u_p converges uniformly to u we have that there exist $(x_p,y_p)\in\Omega$ such that $u_p-\phi$ has a minimum at (x_p,y_p) and $(x_p,y_p)\to(x_0,y_0)$ as $p\to\infty$. Since u_p is a viscosity solution to $-\mathcal{L}_{s,p}v(x,y)+\lambda_1(s,p)v(x,y)^{p-1}=0$ in Ω, we obtain

$$\begin{aligned} &((\lambda_1(s,p))^{1/(p-1)}u_p(x_p,y_p))^{p-1}\le\\ &\le 2\int_{\mathbb{R}^n}\frac{|\phi(x_p,y_p)-\phi(z,y_p)|^{p-2}(\phi(x_p,y_p)-\phi(z,y_p))}{|x_p-z|^{n+sp}}dz\\ &\quad+2\int_{\mathbb{R}^m}\frac{|\phi(x_p,y_p)-\phi(x_p,w)|^{p-2}(\phi(x_p,y_p)-\phi(x_p,w))}{|y_p-w|^{m+sp}}dw\\ &=2(A_p-B_p+C_p-D_p), \end{aligned} \tag{5.8}$$

where

$$A_p=\int_{\mathbb{R}^n}\frac{|\phi(x_p,y_p)-\phi(z,y_p)|^{p-2}(\phi(x_p,y_p)-\phi(z,y_p))^+}{|x_p-z|^{n+sp}}dz,$$
$$B_p=\int_{\mathbb{R}^n}\frac{|\phi(x_p,y_p)-\phi(z,y_p)|^{p-2}(\phi(x_p,y_p)-\phi(z,y_p))^-}{|x_p-z|^{n+sp}}dz,$$
$$C_p=\int_{\mathbb{R}^m}\frac{|\phi(x_p,y_p)-\phi(x_p,w)|^{p-2}(\phi(x_p,y_p)-\phi(x_p,w))^+}{|y_p-w|^{m+sp}}dw,$$
$$D_p=\int_{\mathbb{R}^m}\frac{|\phi(x_p,y_p)-\phi(x_p,w)|^{p-2}(\phi(x_p,y_p)-\phi(x_p,w))^-}{|y_p-w|^{m+sp}}dw.$$

We observe that

$$(A_p)^{1/p} \to A, \qquad (B_p)^{1/p} \to -B,$$
$$(C_p)^{1/p} \to C, \qquad (D_p)^{1/p} \to -D,$$

and

$$(\lambda_1(s,p))^{1/(p-1)} u_p(x_p, y_p) \to \Lambda_\infty u(x_0, y_0).$$

Hence, taking limit as $p \to \infty$ in (5.8), from Lemma 5.6, we get

$$\max\{-B; -D; \Lambda_\infty(s) u(x_0, y_0)\} \le \max\{A; C\}.$$

Now, if ψ is such that $u - \psi$ has a strict minimum at $(x_0, y_0) \in \Omega$. Since u_p converges uniformly to u we have that there exist $(x_p, y_p) \in \Omega$ such that $u_p - \psi$ has a minimum at (x_p, y_p) and $(x_p, y_p) \to (x_0, y_0)$ as $p \to \infty$. Since u_p is a solution to $-\mathcal{L}_{s,p} v(x,y) + \lambda v(x,y)^{p-1} = 0$ in Ω we obtain

$$\begin{aligned} ((\lambda_{1,p})^{1/(p-1)} u_p(x_p, y_p))^{p-1} &\ge \\ \ge 2 \int_{\mathbb{R}^n} &\frac{|\psi(x_p, y_p) - \psi(z, y_p)|^{p-2} (\psi(x_p, y_p) - \psi(z, y_p))}{|x_p - z|^{n+sp}} dz \\ + 2 \int_{\mathbb{R}^m} &\frac{|\psi(x_p, y_p) - \psi(x_p, w)|^{p-2} (\psi(x_p, y_p) - \psi(x_p, w))}{|y_p - w|^{m+sp}} dw, \end{aligned}$$

and, arguing as before, we obtain

$$\max\{A; C\} \ge \max\{-B; -D; \Lambda_\infty(s) u(x_0, y_0)\}.$$

□

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