GLOBAL BIFURCATION FOR FRACTIONAL p-LAPLACIAN AND APPLICATION

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ABSTRACT. We prove the existence of an unbounded branch of solutions to the non-linear non-local equation

$$(-\Delta)_p^s u = \lambda |u|^{p-2} u + f(x, u, \lambda) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega,$$

bifurcating from the first eigenvalue. Here $(-\Delta)_p^s$ denotes the fractional p-Laplacian and $\Omega \subset \mathbb{R}^n$ is a bounded regular domain. The proof of the bifurcation results relies in computing the Leray–Schauder degree by making an homotopy respect to s (the order of the fractional p-Laplacian) and then to use results of local case (that is $s = 1$) found in [15]. Finally, we give some application to an existence result.

1. INTRODUCTION

In this paper, we study Rabinowitz's global bifurcation type result form the first eigenvalue in a bounded domain of the non-linear non-local operator called the fractional p-Laplacian operator, that is

$$(1.1) \qquad (-\Delta)_p^s u = \mathcal{K}(1-s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy,$$

where $\mathcal{K}$ is a constant depending on the dimension and p. Observe that, this operator extends the fractional Laplacian ($p \equiv 2$).

More precisely , we prove the existence of an unbounded branch of solutions to the non-linear non-local equation

$$(1.2) \qquad (-\Delta)_p^s u = \lambda |u|^{p-2} u + f(x, u, \lambda) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega,$$

bifurcating from the first eigenvalue of the fractional p-Laplacian assuming that f is $o(|u|^{p-2}u)$ near zero and $\Omega \subset \mathbb{R}^n$ is a bounded regular domain.

Bifurcation and global bifurcation are basic principles in mathematical analysis that can be established using, for example, implicitly function theorem or degree theory and, in some simple situation, sub and super solution method, i.e Perron's method. In particular, bifurcation is used as a starting point to prove existence of solution to ODE's and PDE's, see for example [28, 35]. Some of the pioneer works related with our method can be found in [13, 33, 34]. Then many others generalization are established in different context of local operator, see for instance [4, 5, 10, 14, 15, 19, 20, 23, 27] and the reference therein.

Fraction equations are nowadays classical in analysis, see for example [39]. Fractional Laplacian have attracted much interest since they are connected with different applications and sometimes from the mathematical point of view the non-local

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character introduce difficulties that need some new approaches, see for instance [18, 38] and the reference therein.

In [11], the fractional p-Laplacian is studied through energy and test function methods and it is used to obtain Hölder extensions. See also [6, 7], where the authors consider a non-local "Tug-of-War" game, and [26].

Recently, existence and simplicity of the first eigenvalue in a bounded domain for the fractional p-laplacian are obtained and also some regularity result are established in [17, 22, 25, 29]. Some results of these works extend the results of [3] to the non-local case.

In the process of writing this article, appearing the following work [25] where the authors, using barrier arguments, prove C^α-regularity up to the boundary for the weak solutions of a non-local non-linear problem driven by the fractional p-Laplacian operator. This result generalises the main result in [36], where the case $p = 2$ is studied.

Thus, there is natural to ask if bifurcation occurs, this is even not known, as far as we know, for the case $p = 2$, except for some related very recent results that can be found in [21, 32, 37].

In our approach, to establish the Rabinowitz's type of global bifurcation result, we use Leray-Schauder degree that can be computed by making an homotopy respect to s (the order of the fractional p-Laplacian operator) and then use the homotopy invariance of the Leray–Schauder degree to deduce that the degree is the same as in the local case ($s = 1$), i.e. the p-Laplacian, which is already computed in [15]. Notice that in [15] similar ideas are used, where the homotopy was done with respect to p and the result were deduced from the (by now) classical case of the Laplacian. To do this homotopy with respect to s, we need as a starting point different properties of the first eigenvalue in terms of s up to $s = 1$, analogous properties to the ones that were obtained in [15], but now with respect to s not respect to p.

Notice that one of our limiting procedures s to 1 are obtained in the weak formulation with the help of some limiting properties of the fractional Sobolev spaces already studied in [8]. Moreover, in [26] this limiting procedure is done by viscosity solution techniques for a very close related operator.

Before stated our main theorem we will give the precisely assumption of the function $f \colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

(1) f satisfies a Carathéodory condition in the first two variables;
(2) $f(x, t, \lambda) = o(|t|^{p-1})$ near $t = 0$, uniformly a.e. with respect to x and uniformly with respect to λ on bounded sets;
(3) There exists $q \in (1, p_s^\star)$ such that

$$\lim_{|t| \to +\infty} \frac{|f(x, t, \lambda)|}{|t|^{q-1}} = 0$$

uniformly a.e. with respect to x and uniformly with respect to λ on bonded sets. Where $p_s^\star$ is the fractional critical Sobolev exponent, that is

$$p_s^\star := \begin{cases} \dfrac{np}{n - sp}, & \text{if } sp < n, \\ +\infty, & \text{if } sp \geq n. \end{cases}$$

Our main result,

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, and $p \in (1, +\infty)$ The pair $(\lambda_1(s,p), 0)$ is a bifurcation point of (5.41). Moreover, there is a connected component of the set of non-trivial weak solutions of (5.41) in $\mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ whose closure contains $(\lambda_1(s,p), 0)$ and it is either unbounded or contains a pair $(\lambda, 0)$ for some λ, eigenvalue of (4.21) with $\lambda > \lambda_1(s,p)$.*

Notice that the ideas of the proof can be used for other problems. As for example, a very close related problem such as bifurcation from infinity by the change of variable $v = u/\|u\|^2_{\widetilde{W}^{s,p}(\Omega)}$, for details see for example [20].

Then, we use the above theorem for some application, more precisely, we prove existence of a non-trivial weak solution of the following non-linear non-local problem

$$(1.3) \qquad \begin{cases} (-\Delta_p)^s u = g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $g(s)/|s|^{p-2}s$ is bounded an crosses the first eigenvalue, for the precisely assumption see Section 6. For the prove of this existence result we need some extra qualitative properties of the branch of solutions in the above theorem. Some of these properties come in some cases from the study of the first eigenvalue of the fractional p-Laplacian with weights, see Section 4.

The paper is organized as follows. In Section 2, we review some results of fractional Sobolev space and some properties of the Leray-Schauder degree; in Section 3 we study the Dirichlet problem with special interest in proving continuity in terms of s (see Lemma 3.1 below); in Section 4 we study the eigenvalue problem with weights. In addition, we establish the continuity of the eigenvalue respect to s that will help us to make the homotopy and then to compute the degree. In Section 5 we prove our main theorem. Finally, in Section 6 we prove our existence results.

2. Preliminaries

2.1. **Fractional Sobolev spaces.** First, we briefly recall the definitions and some elementary properties of the fractional Sobolev spaces. We refer the reader to [1, 16, 18, 24] for further reference and for some of the proofs of the results in this subsection.

Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1, +\infty)$. We define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \colon \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\, dxdy < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\Omega)} := \int_\Omega |u(x)|^p\, dx \quad \text{and} \quad |u|^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\, dxdy.$$

A proof of the following proposition can be found in [1, 16].

Proposition 2.1. *Let Ω be an open set in $\mathbb{R}^n$, $s \in (0, 1)$ and $p \in [1, +\infty)$. We have that*

- *$W^{s,p}(\Omega)$ is a separable Banach space;*
- *If $1 < p < +\infty$ then $W^{s,p}(\Omega)$ is reflexive.*

We denote by $W_0^{s,p}(\Omega)$ the closure of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in $W^{s,p}(\Omega)$. We denote by $\widetilde{W}^{s,p}(\Omega)$ the space of all $u \in W^{s,p}(\Omega)$ such that $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$, where $\tilde{u}$ is the extension by zero of u.

The proofs of the next theorem is given in [1, Theorem 7.38].

Theorem 2.2. *For any $s \in (0, 1)$ and $p \in [1, +\infty)$, the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$, that is $W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$.*

In the next result, we show the explicit dependence of the constant of [18, Proposition 2.1] on s, that is needed for our propose.

Lemma 2.3. *Let Ω be an open set in $\mathbb{R}^n$, $p \in [1, +\infty)$ and $0 < s \le s' < 1$. Then*

$$(2.4) \qquad |u|_{W^{s,p}(\Omega)}^p \le |u|_{W^{s',p}(\Omega)}^p + C(n,p)\left(\frac{1}{sp} - \frac{1}{s'p}\right)\|u\|_{L^p(\Omega)}^p$$

for any $u \in W^{s',p}(\Omega)$.

Proof. Let $u \in W^{s',p}(\Omega)$, then

$$
\begin{aligned}
|u|_{W^{s,p}(\Omega)}^p &= \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,dxdy \\
&= \int_\Omega \int_A \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,dxdy + \int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,dxdy
\end{aligned}
$$
(2.5)

where $A = \Omega \cap \{|x - y| < 1\}$.

Using that $s' \ge s$, we have that

$$(2.6) \qquad \int_\Omega \int_A \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,dxdy \le \int_\Omega \int_A \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps'}}\,dxdy.$$

On the other hand, we have that

$$\int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,dxdy = \int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps'}}|x-y|^{(s'-s)p}\,dxdy$$

$$= \int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps'}}\left(|x-y|^{(s'-s)p} - 1\right)dxdy + \int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps'}}\,dxdy$$

$$= 2^p \int_\Omega \int_{\Omega \setminus A} \frac{|u(x)|^p}{|x-y|^{n+ps'}}\left(|x-y|^{(s'-s)p} - 1\right)dxdy + \int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps'}}\,dxdy$$

$$\le 2^p \|u\|_{L^p(\Omega)}^p \int_{\{|z| \ge 1\}} \frac{|z|^{(s'-s)p} - 1}{|z|^{n+ps'}}\,dz + \int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps'}}\,dxdy.$$

Then

$$
\begin{aligned}
\int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,dxdy \le\; & C(n,p)\left(\frac{1}{ps} - \frac{1}{ps'}\right)\|u\|_{L^p(\Omega)}^p \\
& + \int_\Omega \int_{\Omega \setminus A} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps'}}\,dxdy.
\end{aligned}
$$
(2.7)

Therefore, combining (2.5), (2.6) and (2.7), we get

$$|u|^p_{W^{s,p}(\Omega)} \le |u|^p_{W^{s',p}(\Omega)} + C(n,p)\left(\frac{1}{sp} - \frac{1}{s'p}\right)\|u\|^p_{L^p(\Omega)}.$$

The proof is now complete. □

Remark 2.4. The space $W^{s',p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$ for any $0 < s \le s' < 1$ and $1 \le p < +\infty$.

Lemma 2.5. *Let Ω be an bounded open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1,+\infty)$. Then*

$$\|u\|^p_{L^p(\Omega)} \le \frac{sp|\Omega|^{\frac{sp}{n}}}{2\omega_n^{\frac{sp}{n}+1}}|u|^p_{W^{s,p}(\mathbb{R}^n)}$$

for any $u \in \widetilde{W}^{s,p}(\Omega)$. Here ω_n denotes n-dimensional measure of the unit sphere S^n.

Proof. Let $u \in \widetilde{W}^{s,p}(\Omega)$. Then

$$|u|^p_{W^{s,p}(\mathbb{R}^n)} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,dxdy + 2\int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{n+ps}}\,dxdy$$

$$\ge 2\int_\Omega |u(x)|^p \int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+ps}}\,dydx.$$

By Lemma 6.1 in [18],

$$|u|^p_{W^{s,p}(\mathbb{R}^n)} \ge 2\int_\Omega |u(x)|^p \int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+ps}}\,dydx$$

$$\ge 2\frac{\omega_n^{\frac{sp}{n}+1}}{|\Omega|^{\frac{sp}{n}}}\|u\|^p_{L^p(\Omega)}$$

which proves the lemma. □

The proofs of the next two theorems are given in [18, Proposition 2.2], and [16, Proposition 4.43], respectively.

Theorem 2.6. *Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary $s \in (0,1)$, $p \in [1,+\infty)$, and $u \in W^{1,p}(\Omega)$. Then, there exists a positive constant $C = C(n,s,p)$ such that*

$$\|u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

In particular, $W^{1,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$.

Theorem 2.7. *Let $p \in [1,+\infty)$, $s \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Then $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,p}(\mathbb{R}^n)$.*

The proofs of the following embedding theorems can be found in [16, Theorems 4.47 and 4.54].

Theorem 2.8. *Let $s \in (0,1)$ and $p \in (1,+\infty)$. Then we have the following continuous embeddings:*

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \qquad \text{for all } 1 \le q \le p_s^\star \qquad \text{if } sp \le n;$$

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow C_b^{0,\beta}(\mathbb{R}^n) \quad \text{where } \beta = s - n/p, \qquad \text{if } sp > n.$$

Where $p_s^\star$ is the fractional critical Sobolev exponent, that is

$$p_s^\star := \begin{cases} \dfrac{np}{n - sp}, & \text{if } sp < n, \\ +\infty, & \text{if } sp \geq n, \end{cases}$$

Remark 2.9. Note that $p_s^\star$, as a function of s, is continuous in $(0, 1]$ where $p_1^\star$ is the critical Sobolev exponent, i.e.

$$p_1^\star := \begin{cases} \dfrac{np}{n - p}, & \text{if } p < n, \\ +\infty, & \text{if } p \geq n, \end{cases}$$

A proof of the next theorem can be found in [30, Theorem 1].

Theorem 2.10. *Let $s \in (0, 1)$, $p \in [1, +\infty)$ and $sp < n$. Then there exists a constant $C = C(n, p)$ such that*

$$\|u\|_{L^{p_s^\star}(\mathbb{R}^n)} \leq C \frac{s(1 - s)}{(n - sp)^{p-1}} |u|_{W^{s,p}(\mathbb{R}^n)}^p$$

for all $u \in W^{s,p}(\mathbb{R}^n)$.

By Theorem 2.7 and Theorem 2.8, we have the next result.

Corollary 2.11. *Let $s \in (0, 1)$, $p \in (1, +\infty)$ and $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. The conclusions of Theorem 2.8 remain true if $\mathbb{R}^n$ is replaced by Ω.*

The following embedding theorem is established in [16, Theorem 4.58]. See also [1].

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$ a bounded open set with Lipschitz boundary, $s \in (0, 1)$ and $p \in (1, +\infty)$. Then we have the following compact embeddings:*

$$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for all } q \in [1, p_s^\star), \qquad\qquad \text{if } sp \leq n;$$
$$W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) \quad \text{for all } \lambda < s - {}^n/_p, \qquad\qquad \text{if } sp > n.$$

Remark 2.13. Let $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary. By the above theorem, we have that the embedding of $W^{s,p}(\Omega)$ into $L^p(\Omega)$ is compact for every $s \in (0, 1)$ and for every $p \in (1, +\infty)$.

The next results are proven in [8, Corollaries 2 and 7].

Theorem 2.14. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p \in (1, +\infty)$. Assume $u \in L^p(\Omega)$, then*

$$\lim_{s \to 1^-} \mathcal{K}(1 - s)|u|_{W^{s,p}(\Omega)}^p = |u|_{W^{1,p}(\Omega)}^p$$

with

$$|u|_{W^{1,p}(\Omega)}^p = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, dx, & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{if } u \notin W^{1,p}(\Omega). \end{cases}$$

Here $\mathcal{K}$ depends only the p and n.

Remark 2.15. Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $p \in (1, +\infty)$ and $\phi \in C_0^\infty(\Omega)$. Then

$$|\phi|_{W^{s,p}(\Omega)}^p \leq |\phi|_{W^{s,p}(\mathbb{R}^n)}^p = |\phi|_{W^{s,p}(\Omega)}^p + 2 \int_\Omega \int_\Omega \frac{|\phi(x)|^p}{|x-y|^{n+sp}} \, dx$$

$$\leq |\phi|_{W^{s,p}(\Omega)}^p + \frac{C}{sp} \frac{\|\phi\|_{L^p(\Omega)}^p}{\operatorname{dist}(K, \partial\Omega)^{sp}},$$

where K is the support of ϕ and C depends only of n. Then by Theorem 2.14 we have

$$\lim_{s \to 1^-} \mathcal{K}(1-s)|\phi|_{W^{s,p}(\mathbb{R}^n)}^p = |\phi|_{W^{1,p}(\Omega)}^p.$$

Theorem 2.16. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $p \in (1, +\infty)$ and $u_s \in W^{s,p}(\Omega)$. Assume that*

$$(1-s)|u_s|_{W^{s,p}(\Omega)} \leq C.$$

Then, there exists $u \in W^{1,p}(\Omega)$ and a subsequence $\{u_{s_k}\}_{k \in \mathbb{N}}$ such that

$$u_{s_k} \to u \quad \text{strongly in } L^p(\Omega)$$

$$u_{s_k} \rightharpoonup u \quad \text{weakly in } W^{1-\varepsilon,p}(\Omega)$$

for all $\varepsilon > 0$.

Remark 2.17. Let $0 < s < s' < 1$, and $1 < p < \infty$. From the proof of the Lemma 2 and Corollary 7 in [8], it follows that

$$(2.8) \qquad (1-s)|u|_{W^{s,p}(\Omega)}^p \leq p(1-s)|u|_{W^{s,p}(\Omega)}^p \leq 2^{(1-s)p}(1-s')|u|_{W^{s',p}(\Omega)}^p$$

for all $u \in W^{s',p}(\Omega)$. See also [8, Remark 6].

Observe also that for any $u \in W^{1,p}(\Omega)$, passing to the limit in (2.8) as $s' \to 1$ and using Theorem 2.14, we get

$$(1-s)|u|_{W^{s,p}(\Omega)}^p \leq \frac{1}{\mathcal{K}}|u|_{W^{1,p}(\Omega)}^p,$$

that is

$$\mathcal{K}(1-s)|u|_{W^{s,p}(\Omega)}^p \leq |u|_{W^{1,p}(\Omega)}^p.$$

Remark 2.18. Let $s_0 \in (0, \min\{n/p, s\})$, by Theorem 2.10 and Remark 2.17, there exists a constant $C = C(n, p)$ such that

$$\|u\|_{L^{p_{s_0}^\star}(\mathbb{R}^n)} \leq C \frac{s_0(1-s_0)}{(n-s_0 p)^{p-1}} |u|_{W^{s_0,p}(\mathbb{R}^n)}^p \leq C 2^{(1-s_0)p} \frac{s_0(1-s)}{(n-s_0 p)^{p-1}} |u|_{W^{s,p}(\mathbb{R}^n)}^p$$

for all $u \in W^{s,p}(\mathbb{R}^n)$.

Our last result gives a characterization of $W_0^{1,p}(\Omega)$. For the proof we refer the reader to [24, Corollary 1.4.4.5].

Theorem 2.19. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set with Lipschitz boundary, $s \in (0, 1]$ and $p \in (1, +\infty)$. If $s \neq \frac{1}{p}$ then*

$$W_0^{s,p}(\Omega) = \widetilde{W}^{s,p}(\Omega),$$

Furthermore, when $0 < s < 1/p$ we have

$$W_0^{s,p}(\Omega) = W^{s,p}(\Omega).$$

Remark 2.20. $\widetilde{W}^{s,p}(\Omega)$ is a Banach space for the norm induced by $W^{s,p}(\mathbb{R}^n)$. More-over, if $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in (0,1]$ and $p \in (1,+\infty)$, then $C_0^\infty(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ and $\widetilde{W}^{s,p}(\Omega) \subset W_0^{s,p}(\Omega)$. See [24, Theorem 1.4.2.2 and Corollary 1.4.4.10].

For $s \in (0,1)$ and $p \in (1,\infty)$, we define the space $W^{-s,p'}(\Omega)$ $(\widetilde{W}^{-s,p'}(\Omega))$ as the dual space of $W_0^{s,p}(\Omega)$ $(\widetilde{W}^{s,p}(\Omega))$ where $1/p' + 1/p = 1$.

2.2. **Leray-Schauder degree.** For the definition and some properties of Leray-Schauder degree, for instance, see [12, 35].

The proof of the next Leray-Schauder degree property is given in [15, Lemma 2.4].

Lemma 2.21. *Let X, Y be Banach spaces with respective norms $\| \cdot \|_X$ and $\| \cdot \|_Y$. Assume that $Y \subset X$ and that the inclusion $i \colon Y \to X$ is continuous. Let Ω_X, Ω_Y be bounded open sets in X and Y, respectively, both containing 0, let $T \colon X \to Y$ be a completely continuous operator such that*

$$x - Tx \neq 0 \quad \forall x \in X \setminus \{0\}.$$

Then

$$\deg_X(I - i \circ T, \Omega_X, 0) = \deg_Y(I - T \circ i, \Omega_Y, 0).$$

3. The Dirichlet problem

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p \in (1,+\infty)$. We consider the operator

$$(3.9) \qquad \mathcal{L}_{s,p}u := \begin{cases} -\Delta_p u & \text{if } s = 1, \\ (-\Delta)_p^s u, & \text{if } 0 < s < 1, \end{cases}$$

where Δ_p is the $p-$Laplace operator, that is

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

and $(-\Delta)_p^s$ is the fractional $p-$Laplace operator, that is

$$(3.10) \qquad (-\Delta)_p^s u = 2\mathcal{K}(1-s)\text{P.V.}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+sp}}\, dy.$$

with $\mathcal{K}$ is the constant of Theorem 2.14.

For further details on the fractional $p-$Laplace operator, we refer to [22, 29] and references therein.

It is well known that the Dirichlet problem

$$(3.11) \qquad \begin{cases} -\Delta_p u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution for each $h \in W^{-1,p}(\Omega)$, i.e. there exists an uniqe $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla\phi(x)\, dx = \langle h, \phi\rangle \quad \forall \phi \in C_0^\infty(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

We also recall that the weak solution is the critical point of the functional $J_{1,p}$: $W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_{1,p}(v) = \frac{1}{p}|v|_{W_0^{1,p}(\Omega)}^p - \langle h, v \rangle.$$

See, for instance, [40] and references therein.

Now, we study the Dirichlet problem for fractional p-Laplace equation.

Let $s \in (0,1)$, $p \in (1, +\infty)$ and $h \in W^{-s,p'}(\Omega)$. We say that $u \in W_0^{s,p}(\Omega)$ is a weak solution of the Dirichlet problem

(3.12)
$$\begin{cases} (-\Delta)_p^s u = h & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

if

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u, \phi) = \langle h, \phi \rangle \quad \forall \phi \in C_0^\infty(\Omega),$$

where

(3.13) $\quad \mathcal{H}_{s,p}(u, \phi) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} (\phi(x) - \phi(y)) \, dy dx,$

and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{s,p}(\Omega)$ and $W^{-s,p'}(\Omega)$.

It is clear that, the weak solutions are critical points of the functional $J_{s,p}$: $\widetilde{W}^{s,p}(\Omega) \to \mathbb{R}$ given by

$$J_{s,p}(v) = \frac{1}{p}\mathcal{K}(1-s)|v|_{W^{s,p}(\mathbb{R}^n)}^p - \langle h, v \rangle.$$

Now, it is easy to see that $J_{s,p}$ is bounded below, coercive, strictly convex and sequentially weakly lower semi continuous. Then it has a unique critical point which is a global minimum. Therefore the Dirichlet problem (3.12) has a unique weak solution.

Thus, given $s \in (0,1]$ and $h \in \widetilde{W}^{-s,p'}(\Omega)$, the Dirichlet problem

(3.14)
$$\begin{cases} \mathcal{L}_{s,p} u = h & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

has a unique weak solution $u_{s,p,h} \in \widetilde{W}^{s,p}(\Omega)$. Moreover, the operator

$$\mathcal{R}_{s,p} \colon \widetilde{W}^{-s,p'}(\Omega) \to \widetilde{W}^{s,p}(\Omega)$$
$$h \to u_{s,p,h},$$

is continuous. By the Rellich-Kondrachov theorem (case $s = 1$) and Theorem 2.12 (case $s \in (0,1)$), the restriction of $\mathcal{R}_{s,p}$ to $L^{q'}(\Omega)$ with $q \in (1, p_s^\star)$ is a completely continuous operator, that is for every weakly convergent sequence $\{h_k\}_{k \in \mathbb{N}}$ from $L^{q'}(\Omega)$, the sequence $\{\mathcal{R}_{s,p}(h_k)\}_{k \in \mathbb{N}}$ is norm-convergent in $\widetilde{W}^{s,p}(\Omega)$.

Our next result show that the operator $\mathcal{R}_{s,p}$ is continuous with respect to s and h.

Lemma 3.1. *Let $s_0 \in (0,1)$, $p \in (1,+\infty)$ and $1 < q < p_{s_0}^\star$. Then the operator*

$$\mathcal{R}_p \colon [s_0, 1] \times L^{q'}(\Omega) \to L^q(\Omega)$$
$$(s, h) \to \mathcal{R}_{s,p}(h),$$

is completely continuous.

Proof. We start by proving that $\mathcal{R}_p$ is compact.

Let $\{(s_k, h_k)\}_{k \in \mathbb{N}}$ be a bounded sequence in $[s_0, 1] \times L^{q'}(\Omega)$. We want to prove that $u_k = \mathcal{R}_p(s_k, h_k)$ has a strongly convergent subsequence in $L^q(\Omega)$.

For all $k \in \mathbb{N}$, u_k satisfies

$$|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)} = \int_\Omega h_k(x) u_k(x)\, dx.$$

Then, by Hölder inequality and using $q \le p_{s_0}^\star$, we have

$$(3.15) \qquad |u_k|^p_{W^{s_k,p}(\mathbb{R}^n)} \le \|h_k\|_{L^{q'}(\Omega)} \|u_k\|_{L^q(\Omega)} \le C \|u_k\|_{W^{s_0,p}_0(\Omega)}$$

where C is a constant independent of k. Thus, by Lemma 2.3, Lemma 2.5 and (3.15), we get

$$\|u_k\|_{W^{s_0,p}(\Omega)} \le C$$

for some constant C independent of k. Hence $\{u_k\}_{k \in \mathbb{N}}$ has a strongly convergent subsequence in $L^q(\Omega)$ due to $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $W^{s_0,p}(\Omega)$ and $1 < q < p_{s_0}^\star$.

Finally, we show that $\mathcal{R}_p$ is continuous.

Let $(s_k, h_k) \to (s, h)$ in $[s_0, 1] \times L^{q'}(\Omega)$ as $k \to +\infty$, $u_k = \mathcal{R}_p(s_k, h_k)$ $k \in \mathbb{N}$, and $u = \mathcal{R}_p(s, h)$. We want to show that $u_k \to u$ strongly in $L^q(\Omega)$. In fact, we only need to show that u is the only accumulation point of $\{u_k\}_{k \in \mathbb{N}}$ due to $\mathcal{R}_p$ is compact.

Let $\{u_j\}_{j \in \mathbb{N}}$ be a subsequence of $\{u_k\}_{k \in \mathbb{N}}$ converging to v in $L^q(\Omega)$. We have to prove that $v = u$. It is enough to prove that

$$(3.16) \qquad \frac{1}{p} |\tilde{v}|^p_{s,p} - \int_\Omega v(x) h(x)\, dx \le \frac{1}{p} |w|^p_{s,p} - \int_\Omega w(x) h(x)\, dx \quad \forall w \in \widetilde{W}^{s,p}(\Omega)$$

where

$$|w|^p_{s,p} = \begin{cases} |w|^p_{W^{1,p}(\Omega)} & \text{if } s = 1, \\ \mathcal{K}(1-s)|w|^p_{W^{s,p}(\mathbb{R}^n)} & \text{if } s \in (0,1), \end{cases}$$

and $\tilde{v}$ is the continuation of v by zero outside Ω.

On the other hand, we know that

$$(3.17) \quad \frac{1}{p} |u_j|^p_{s_j,p} - \int_\Omega u_j(x) h(x)\, dx \le \frac{1}{p} |w|^p_{s_j,p} - \int_\Omega w(x) h(x)\, dx \quad \forall w \in \widetilde{W}^{s_j,p}(\Omega).$$

Now we need consider the following two cases.

Case $s \ne 1$. Since $u_j \to v$ strongly in $L^q(\Omega)$, we have that $u_j \to \tilde{v}$ a.e. in $\mathbb{R}^n$. Then, using that $h_j \to h$ strongly in $L^{q'}(\Omega)$ and by Fatou's lemma, we have

$$(3.18) \qquad \frac{1}{p} |\tilde{v}|^p_{s,p} - \int_\Omega v(x) h(x)\, dx \le \liminf_{j \to +\infty} \frac{1}{p} |u_j|^p_{s_j,p} - \int_\Omega u_j(x) h_j(x)\, dx$$

Thus, for any $\phi \in C_0^\infty(\Omega)$, by (3.18), (3.17) and dominate convergence theorem, we get

$$\frac{1}{p} |\tilde{v}|^p_{s,p} - \int_\Omega v(x) h(x)\, dx \le \frac{1}{p} |\phi|^p_{s,p} - \int_\Omega \phi(x) h(x)\, dx.$$

Therefore, $v \in \widetilde{W}^{s,p}(\Omega)$ and by density, (3.16) holds.

Case $s = 1$. Let $\phi \in C_0^\infty(\Omega)$. By (3.17) and Remark 2.15, we have

$$\limsup_{j \to +\infty} \frac{1}{p} |u_j|_{s_j,p}^p - \int_\Omega v(x) h(x)\, dx = \limsup_{j \to +\infty} \frac{1}{p} |u_j|_{s_j,p}^p - \int_\Omega u_j(x) h_j(x)\, dx$$

$$\leq \frac{1}{p} |\phi|_{1,p}^p - \int_\Omega \phi(x) h(x)\, dx,$$

due to $u_j \to v$ strongly in $L^q(\Omega)$ and $h_j \to h$ strongly in $L^{q'}(\Omega)$. Then

$$(3.19) \qquad \limsup_{j \to +\infty} \frac{1}{p} |u_j|_{s_j,p}^p \leq \frac{1}{p} |\varphi|_{1,p}^p - \int_\Omega \varphi(x) h(x)\, dx + \int_\Omega v(x) h(x)\, dx.$$

Therefore

$$|u_j|_{s_j,p} \leq C$$

for some constant C independent of j.

Thus, by Theorem 2.16, there exist $w \in W_0^{1,p}(\Omega)$ and a subsequence of $\{u_j\}_{j \in \mathbb{N}}$, still denoted by $\{u_j\}_{j \in \mathbb{N}}$, such that

$$u_j \to w \quad \text{strongly in } L^p(\Omega)$$

$$u_j \rightharpoonup w \quad \text{weakly in } W^{1-\varepsilon,p}(\Omega)$$

for all $\varepsilon > 0$. Then $v = w$, and $v \in W_0^{1,p}(\Omega)$.

On the other hand, given $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that $1 - \varepsilon < s_j$ for all $j \geq j_0$ due to $s_j \to 1$. Then, by Remark 2.17

$$(3.20) \qquad \mathcal{K}\varepsilon |u_j|_{W^{1-\varepsilon,p}(\Omega)}^p \leq 2^{\varepsilon p} \mathcal{K}(1 - s_j) |u_j|_{W^{s_j,p}(\Omega)}^p \leq 2^{\varepsilon p} |u_j|_{s_j,p}^p \quad \forall j \geq j_0.$$

Thus, using $u_j \rightharpoonup v$ weakly in $W^{1-\varepsilon,p}(\Omega)$ and by (3.20) and (3.19),

$$2^{-\varepsilon p} \frac{1}{p} \mathcal{K}\varepsilon |v|_{W^{1-\varepsilon,p}(\Omega)}^p \leq \liminf_{j \to +\infty} |u_j|_{s_j,p}^p \leq \frac{1}{p} |\varphi|_{1,p}^p - \int_\Omega \varphi(x) h(x)\, dx + \int_\Omega v(x) h(x)\, dx.$$

Now, by Theorem 2.14, letting $\varepsilon \to 0^+$ we get

$$\frac{1}{p} |v|_{W^{1,p}(\Omega)}^p \leq \frac{1}{p} |\varphi|_{1,p}^p - \int_\Omega \varphi(x) h(x)\, dx + \int_\Omega v(x) h(x)\, dx.$$

Thus, since φ is arbitrary, we have that

$$\frac{1}{p} |v|_{W^{1,p}(\Omega)}^p - \int_\Omega v(x) h(x)\, dx \leq \frac{1}{p} |\varphi|_{1,p}^p - \int_\Omega \varphi(x) h(x)\, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Hence, by density, (3.16) holds. This completes the proof. $\qquad\square$

Remark 3.2. Let $p \in (1, +\infty)$. Then the operator

$$\mathcal{R}_p \colon (0,1] \times L^{p'}(\Omega) \to L^p(\Omega)$$

$$(s, h) \to \mathcal{R}_{s,p}(h),$$

is completely continuous.

4. The eigenvalue problem with weight

In this section we show some results concerning the the following eigenvalue problems

$$
(4.21) \qquad \begin{cases} \mathcal{L}_{s,p}(u) = \lambda h(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}
$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $s \in (0,1]$, $p \in (1,+\infty)$ and $h \in \mathcal{A} = \{f \in L^\infty(\Omega) \colon |\{x \in \Omega \colon f(x) > 0\}| > 0\}$.

4.1. The case $s = 1$, the first $p-$eigenvalue.
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $p \in (1,+\infty)$ and $h \in \mathcal{A}$.

The first eigenvalue $\lambda_1(1,p,h)$ can be characterized as

$$
\lambda_1(1,p,h) = \inf\left\{ |u|_{W^{1,p}(\Omega)}^p \colon u \in W_0^{1,p}(\Omega), \int_\Omega h(x)|u(x)|^p \, dx = 1 \right\},
$$

and it is simple and isolated, see [3]. For simplicity, we omit mention of h when $h \equiv 1$, and thus we write $\lambda_1(1,p)$ in place of $\lambda_1(1,p,1)$.

4.2. Case $s \in (0,1)$, the first fractional $p-$eigenvalue.
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,+\infty)$ and $h \in \mathcal{A}$. In this section, we analyse the (non-linear non-local) eigenvalue problem

$$
(4.22) \qquad \begin{cases} (-\Delta)_p^s u = \lambda h(x)|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}
$$

A function $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak solution of (4.22) if it satisfies

$$
\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,\phi) = \lambda \int_\Omega h(x)|u(x)|^{p-2}u(x)\phi(x) \, dx, \quad \forall \phi \in C_0^\infty(\Omega).
$$

We say that $\lambda \in \mathbb{R}$ is a fractional $p-$eigenvalue provided there exists a non-trivial weak solution $u \in \widetilde{W}^{s,p}(\Omega)$ of (4.22). The function u is a corresponding eigenfunction.

The first fractional p-eigenvalue is

$$
(4.23) \quad \lambda_1(s,p,h) := \mathcal{K}(1-s)\inf\left\{ |u|_{W^{s,p}(\mathbb{R}^n)}^p \colon u \in \widetilde{W}^{s,p}(\Omega), \int_\Omega h(x)|u(x)|^p dx = 1 \right\}.
$$

As before, in the case $h \equiv 1$, for simplicity, we write $\lambda_1(s,p)$ in place of $\lambda_1(s,p,1)$.

First we want to mention that $\{u \in \widetilde{W}^{s,p}(\Omega) \colon \int_\Omega h(x)|u(x)|^p \, dx = 1\} \neq \emptyset$ due to $|\{x \in \Omega \colon h(x) > 0\}| > 0$. Therefore $\lambda_1(s,p,h)$ is well defined and is non-negative.

We also know that $\lambda_1(s,p) > 0$ and there exists a non-negaive function $u \in W_0^{s,p}(\Omega)$ such that

- $u > 0$ in Ω, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$;
- u is a minimizer of (4.23) with $h \equiv 1$;
- u is a weak solution of (4.22) with $\lambda = \lambda_1(s,p)$ and $h \equiv 1$, that is u is an eigenfunction of (3.9) with eigenvalue $\lambda_1(s,p)$.

Moreover $\lambda_1(s,p)$ is simple, and if $sp > n$ then $\lambda_1(s,p)$ is isolated. See [29, Theorem 5, Theorem 14 and Theorem 19], [9, Theorems A1] and [22, Theorem 4.2].

The rest of this section is devoted to generalize these results for the first eigenvalue of (4.22).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,+\infty)$, and $h \in \mathcal{A}$ There exists a non-negative function $u \in \widetilde{W}^{s,p}(\Omega)$, such that*

- *$u \neq 0$ in Ω;*
- *u is a minimizer of (4.23);*
- *u is a weak solution of (4.22) with $\lambda = \lambda_1(s,p,h)$, that is u is an eigenfunction of (3.9) with eigenvalue $\lambda_1(s,p,h)$.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence, that is $u_j \in \widetilde{W}^{s,p}(\Omega)$,

$$\int_\Omega h(x)|u_j(x)|^p\,dx = 1 \text{ and } \lim_{j\to+\infty}|u_j|_{W^{s,p}(\Omega)} = \lambda(s,p,h).$$

Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $\widetilde{W}^{s,p}(\Omega)$. Therefore, there exits a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, and $u \in \widetilde{W}^{s,p}(\Omega)$ such that

$$u_j \rightharpoonup u \text{ weakly in } \widetilde{W}^{s,p}(\Omega),$$
$$u_j \to u \text{ strongly in } L^p(\Omega).$$

Thus

$$\int_\Omega h(x)u(x)\,dx = 1$$

and

$$|u|^p_{\widetilde{W}^{s,p}(\Omega)} \leq \lim_{j\to+\infty}|u_j|_{\widetilde{W}^{s,p}(\Omega)} = \lambda(s,p,h).$$

Then $|u|^p_{\widetilde{W}^{s,p}(\Omega)} = \lambda(s,p,h)$, that is u is a minimizer of (4.23). It is easy to see that $|u|$ is also a minimizer of (4.23), this shows that there exists a non-negative minimizer of (4.23).

Finally, by the Lagrange multiplier rule (see [31, Theorem 2.2.10]) there exists $a, b \in \mathbb{R}$ such that $a + b \neq 0$, and

$$a\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) + b\int_\Omega h(x)|u(x)|^{p-2}u(x)v(x)\,dx = 0 \quad \forall v \in \widetilde{W}^{s,p}(\Omega).$$

If $a = 0$, then $b \neq 0$ and taking $v = u$, we get $\int_\Omega h(x)|u(x)|^p\,dx = 0$ a contradiction because $\int_\Omega h(x)|u(x)|^p\,dx = 1$. Hence $a \neq 0$, and without any loss of generality, we can assume that $a = 1$. Then

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) + b\int_\Omega h(x)|u(x)|^{p-2}u(x)v(x)\,dx = 0 \quad \forall v \in \widetilde{W}^{s,p}(\Omega).$$

Again, taking $v = u$ and using that

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) = \mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} = \lambda_1(s,p,h)$$

and

$$\int_\Omega h(x)|u(x)|^p\,dx = 1,$$

we have $b = -\lambda_1(s,p,h)$. $\square$

Our next aim is to show that a non-negative eigenfunction associated to $\lambda_1(s, p, h)$ is in really positive. For this we will need a strong minimum principle.

We star by a definitions. Let $p \in (1, +\infty)$, $s \in (0, 1)$, $h \in L^\infty(\Omega)$, and $\lambda \in \mathbb{R}$. We say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak super-solution of (4.22) if

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u, \phi) \geq \lambda \int_\Omega h(x)|u(x)|^{p-1}u(x)\phi(x)\,dx$$

for all $\phi \in C_0^\infty(\Omega), \phi \geq 0$.

Following the proof of the DKP logarithmic lemma (see [17, Lemma 1.3]) we have the following result.

Lemma 4.2. *Let Ω be a bounded domain, $s \in (0, 1), p \in (1, +\infty)$, $h \in \mathcal{A}$, $\lambda > 0$ and u be a weak super-solution of (4.22) such that $u \geq 0$ in $B_R(x_0) \subset\subset \Omega$. Then for any $B_r = B_r(x_0) \subset B_{R/2}(x_0)$ and $0 < \delta < 0$*

$$\int_{B_r}\int_{B_r}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \frac{dxdy}{|x-y|^{n+sp}} \leq Cr^{n-sp}\left\{\delta^{1-p}r^{sp}\int_{\mathbb{R}^n \setminus B_{2r}}\frac{u^-(y)^{p-1}}{|y-x_0|}\,dy + 1\right\}$$
$$+ |\lambda|\|h\|_{L^1(B_{2r})},$$

where $u^- = \max\{-u, 0\}$ and C depends only on $n, s,$ and p.

Proceeding as in the proof of Theorem A.1 in [9] using Lemma 4.2 in place of DKP logarithmic lemma, we get the following strong minimum principle.

Theorem 4.3. *Let Ω be a bounded domain, $s \in (0, 1), p \in (1, +\infty)$, $h \in \mathcal{A}$, $\lambda > 0$, and u be a weak super-solution of (4.22) such that $u \geq 0$ in Ω. If $u \neq 0$ in Ω then $u > 0$ a.e. in Ω.*

Corollary 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $s \in (0, 1)$, $p \in (1, +\infty)$, $h \in \mathcal{A}$, and $u \in W_0^{s,p}(\Omega)$ be a non-negative eigenfunction corresponding to $\lambda_1(s, p, h)$. Then $u > 0$ almost everywhere in Ω.*

The proof of the result given below follows from a careful reading of [22, proof of Theorem 3.2].

Theorem 4.5. *Let Ω be a bounded domain with Lipschitz, $s \in (0, 1), p \in (1, +\infty)$, $h \in \mathcal{A}$, $\lambda > 0$, and $u \in W_0^{s,p}(\Omega)$ be a weak solution to (4.22). Then $u \in L^\infty(\mathbb{R}^n)$.*

Now, we prove that $\lambda_1(s, p, h)$ is also simple when $h \neq 1$. For this we need the following lemma. For the proof see Lemma 6.2 in [2].

Lemma 4.6. *Let $p \in (1, +\infty)$. For $v > 0$ and $u \geq 0$, we have*

$$L(u, v) \geq 0 \quad in \; \mathbb{R}^n \times \mathbb{R}^n$$

where

$$L(u, v)(x, y) = |u(y)-u(x)|^p - |v(y)-v(x)|^{p-2}(v(y)-v(x))\left(\frac{u(y)^p}{v(y)^{p-1}} - \frac{u(x)^p}{v(x)^{p-1}}\right).$$

The equality holds if and only if $u = kv$ a.e. for some constant k.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0, 1)$, $p \in (1, +\infty)$, $h \in \mathcal{A}$, and u be a positive eigenfunction corresponding to $\lambda_1(s, p, h)$. If $\lambda > 0$ is such that there exists a non-negative eigenfunction v of (3.9) with eigenvalue λ, then $\lambda = \lambda_1(s, p, h)$ and there exists $k \in \mathbb{R}$ such that $v = ku$ a.e. in Ω. Therefore the first eigenvalue $\lambda_1(s, p, h)$ is simple.*

Proof. Since $\lambda_1(s,p,h)$ is the first eigenvalue we have that $\lambda_1(s,p,h) \leq \lambda$. Let $m \in \mathbb{N}$ and $v_m := v + \dfrac{1}{m}$.

We begin by proving that $w_m := \dfrac{u^p}{v_m^{p-1}} \in \widetilde{W}^{s,p}(\Omega)$. It is immediate that $w_m = 0$ in $\mathbb{R}^n \setminus \Omega$ and $w_m \in L^p(\Omega)$, due to $u \in L^\infty(\Omega)$, see Theorem 4.5.

On the other hand

$$
\begin{aligned}
|w_m(y) - w_m(x)| &= \left| \frac{u(y)^p - u(x)^p}{v_m(y)^{p-1}} + \frac{u(x)^p \left(v_m(x)^{p-1} - v_m(x)^{p-1}\right)}{v_m(y)^{p-1} v_m(x)^{p-1}} \right| \\
&\leq m^{p-1} |u(y)^p - u(x)^p| + \|u\|_{L^\infty(\Omega)}^p \frac{\left|v_m(y)^{p-1} - v_m(x)^{p-1}\right|}{v_m(y)^{p-1} v_m(x)^{p-1}} \\
&\leq m^{p-1} p (u(y)^{p-1} + u(x)^{p-1}) |u(y) - u(x)| \\
&\quad + \|u\|_{L^\infty(\Omega)}^p (p-1) \frac{|v_m(y)^{p-2} + v_m(x)^{p-2}|}{v_m(y)^{p-1} v_m(x)^{p-1}} |v_m(y) - v_m(x)| \\
&\leq 2\|u\|_{L^\infty(\Omega)}^{p-1} m^{p-1} p |u(y) - u(x)| \\
&\quad + \|u\|_{L^\infty(\Omega)}^p (p-1) \left(\frac{1}{v_m(y)} + \frac{1}{v_m(x)} \right) |v(y) - v(x)| \\
&\leq C(m,p,\|u\|_{L^\infty(\Omega)}) \left(|u(y) - u(x)| + |v(y) - v(x)| \right)
\end{aligned}
$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Hence $w_m \in \widetilde{W}^{s,p}(\Omega)$ for all $m \in \mathbb{N}$ due to $u,v \in \widetilde{W}^{s,p}(\Omega)$.

Then, by Lemma 4.6 and since $u,v \in W_0^{s,p}(\Omega)$ are two positive eigenfunctions of problem (3.9) with eigenvalue $\lambda_1(s,p,h)$ and λ respectively, we have

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{L(u,v_n)(x,y)}{|x-y|^{n+sp}} \, dx\, dy \\
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|x-y|^{n+sp}} \, dx\, dy \\
&\quad - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(y) - v(x)|^{p-2}(v(y) - v(x))}{|x-y|^{n+sp}} \left(\frac{u(y)^p}{v_m(y)^{p-1}} - \frac{u(x)^p}{v_m(x)^{p-1}} \right) dx\, dy \\
&\leq \frac{1}{\mathcal{K}(1-s)} \left\{ \lambda_1(s,p,h) \int_\Omega h(x) u(x)^p \, dx - \lambda \int_\Omega h(x) v(x)^{p-1} \frac{u(x)^p}{v_m(x)^{p-1}} \, dx \right\}.
\end{aligned}
$$

By the Fatou's lemma and the dominated convergence theorem

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{L(u,v)(x,y)}{|x-y|^{n+sp}} \, dx\, dy = 0
$$

due to $\lambda_1(s,p,h) \leq \lambda$. Then $L(u,v)(x,y) = 0$ a.e. Hence, by Lemma 4.6, $u = kv$ for some constant $k > 0$. $\qquad\square$

By Corollary 4.4 and Theorem 4.7, we have that $\lambda_1(s,p,h)$ is simple.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,+\infty)$, and $h \in \mathcal{A}$. Then $\lambda_1(s,p)$ is simple.*

Now, we get a lower bound for the measure of the nodal sets

Lemma 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $s \in (0,1)$, $p \in (1,+\infty,)$, $s_0 \in (0, \min\{n/p, s\})$, and $h \in \mathcal{A}$. If $u \in \widetilde{W}^{s,p}(\Omega)$ is an*

eigenfunctions of (3.9) with eigenvalue $\lambda > \lambda(s,p,h)$, then there exists a constant C (independent of λ and u) such that

$$\left(\frac{\mathcal{K}(n-s_0 p)^{p-1}}{C 2^{(1-s_0)p} s_0 \lambda \|h\|_{L^\infty(\Omega)}} \right)^{\frac{p_{s_0}^\star}{p_{s_0}^\star - p}} \leq |\Omega^\pm|$$

Here $\Omega^+ = \{x \in \Omega \colon u(x) > 0\}$, $\Omega^- = \{x \in \Omega \colon u(x) < 0\}$, and C is a constant depending both on n and p.

Proof. Let $u^+(x) = \max\{0, u(x)\}$. By Theorem 4.7, u changes sign then $u^+ \neq 0$. In addition, $u^+ \in W_0^{s,p}(\Omega)$ and

$$|u^+(x) - u^+(y)|^p \leq |u(x) - u(y)|^{p-2}(u(x)-u(y))(u^+(x)-u^+(y))$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Therefore,

$$\begin{aligned}
(4.24) \quad |u^+|_{W^{s,p}(\mathbb{R}^n)}^p &\leq \mathcal{H}_{s,p}(u, u^+) \leq \frac{\lambda}{\mathcal{K}(1-s)} \int_{\Omega^+} h(x)|u^+(x)|^p dx \\
&\leq \frac{\lambda \|h\|_{L^\infty(\Omega)}}{\mathcal{K}(1-s)} \int_{\Omega^+} |u^+(x)|^p dx.
\end{aligned}$$

On the other hand, by Remark 2.18, there exists a constant $C = C(n,p)$ such that

$$\|u\|_{L^{p_{s_0}^\star}} \leq C 2^{(1-s_0)p} \frac{s_0(1-s)}{(n-s_0 p)^{p-1}} |u^+|_{W^{s,p}(\mathbb{R}^n)}^p$$

Then, by (4.24) and Hölder's inequality, we get

$$\|u^+\|_{L^{p_{s_0}^\star}(\mathbb{R}^n)} \leq C 2^{(1-s_0)p} \frac{s_0 \lambda \|h\|_{L^\infty(\Omega)}}{\mathcal{K}(n-s_0 p)^{p-1}} \|u^+\|_{L^{p_{s_0}^\star}(\Omega)}^p |\Omega^+|^{\frac{p_{s_0}^\star - p}{p_{s_0}^\star}}.$$

Hence

$$\left(\frac{\mathcal{K}(n-s_0 p)^{p-1}}{C 2^{(1-s_0)p} s_0 \lambda \|h\|_{L^\infty(\Omega)}} \right)^{\frac{p_{s_0}^\star}{p_{s_0}^\star - p}} \leq |\Omega^+|.$$

In order to prove the second inequality, it will suffice to proceed as above, using the function $u^-(x) = \max\{0, -u(x)\}$ instead of u^+. $\square$

Finally, we show that the first eigenvalue is isolated.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1, +\infty)$, and $h \in \mathcal{A}$. Then the first eigenvalue is isolated.*

Proof. By the definition of $\lambda_1(s,p,h)$ we have that $\lambda_1(s,p,h)$ is left–isolated.

To prove that $\lambda_1(s,p,h)$ is right–isolated, we argue by contradiction. We assume that there exists a a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k > \lambda_1(s,p,h)$ and $\lambda_k \searrow \lambda_1(s,p,h)$ as $k \to +\infty$. Let u_k be an eigenfunction associated to λ_k, we can assume that

$$\int_\Omega h(x)|u_k(x)|^p dx = 1.$$

Then $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\widetilde{W}^{s,p}(\Omega)$ and therefore we can extract a subsequence (that we still denoted by $\{u_k\}_{k\in\mathbb{N}}$) such that

$$u_k \rightharpoonup u \quad \text{weakly in } \widetilde{W}^{s,p}(\Omega),$$
$$u_k \to u \quad \text{strongly in } L^p(\Omega).$$

Then

$$\int_\Omega h(x)|u(x)|^p\,dx = 1$$

and

$$\mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} \leq \mathcal{K}(1-s)\liminf_{k\to+\infty}|u_k|^p_{W^{s,p}(\mathbb{R}^n)}$$

$$= \lim_{k\to+\infty}\lambda_k\int_\Omega h(x)|u_k(x)|^p\,dx$$

$$= \lambda_1(s,p,h)\int_\Omega h(x)|u(x)|^p\,dx$$

.

Hence, u is an eigenvalue of (3.9) with eigenvalue $\lambda_1(s,p,h)$. By Corollary 4.4, we can assume that $u > 0$.

On the other hand, by the Egorov's theorem, for any $\varepsilon > 0$ there exists a subset A_ε of Ω such that $|A_\varepsilon| < \varepsilon$ and $u_k \to u > 0$ uniformly in $\Omega \setminus A_\varepsilon$. This contradicts the fact that, by Lemma 4.9,

$$\left(\frac{\mathcal{K}(n-s_0 p)^{p-1}}{C2^{(1-s_0)p}s_0\lambda_k\|h\|_{L^\infty(\Omega)}}\right)^{\frac{p^\star_{s_0}}{p^\star_{s_0}-p}} \leq |\{x \in \Omega\colon u_k(x) < 0\}|.$$

where $s_0 \in (0, \min\{s, {}^n\!/_p\})$. This proves the theorem. $\qquad\square$

4.3. **Global properties.** In the rest of this section, for simplicity, we will take $h \equiv 1$.

Lemma 4.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and $p \in (1,+\infty)$. The first eigenvalue function $\lambda_1(\cdot,p)\colon (0,1] \to \mathbb{R}$ is continuous.*

Proof. Let $\{s_j\}_{j\in\mathbb{N}}$ be a sequence in $(0,1]$ convergent to $s \in (0,1]$. We will show that

$$(4.25) \qquad\qquad \lim_{j\to+\infty}\lambda_1(s_j,p) = \lambda_1(s,p).$$

We need to consider two cases: $s \in (0,1)$ and $s = 1$.

Case $s \in (0,1)$. Let $\varphi \in C_0^\infty(\Omega)$. Then

$$\lambda_1(s_j,p) \leq \mathcal{K}(1-s_j)\frac{\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+s_j p}}\,dxdy}{\displaystyle\int_\Omega|\varphi(x)|^p\,dx}$$

for all $j \in \mathbb{N}$. Therefore, by dominated convergence theorem,

$$\limsup_{j\to+\infty}\lambda_1(s_j,p) \leq \mathcal{K}(1-s)\frac{\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+sp}}\,dxdy}{\displaystyle\int_\Omega|\varphi(x)|^p\,dx}.$$

As φ is arbitrary

$$\limsup_{j\to+\infty}\lambda_1(s_j,p) \leq \lambda_1(s,p)$$

due to (4.23).

Thus, to prove (4.25), we need to show that

$$\liminf_{j \to +\infty} \lambda_1(s_j, p) \geq \lambda_1(s, p).$$

Let $\{s_k\}_{k \in \mathbb{N}}$ be a subsequence of $\{s_j\}_{j \in \mathbb{N}}$ such that

$$(4.26) \qquad \lim_{k \to +\infty} \lambda(s_k, p) = \liminf_{j \to +\infty} \lambda_1(s_j, p).$$

Let u_k be an eigenfunction of (3.9) with eigenvalue $\lambda_1(s_k, p)$ such that

$$|u_k|_{W^{s_k, p}(\mathbb{R}^n)} = 1.$$

Then, for any $k \in \mathbb{N}$ we have that $u_k \in \widetilde{W}^{s_k, p}(\Omega)$, $u_k \neq 0$ in Ω, $u_k = 0$ in $\mathbb{R}^n \setminus \Omega$, and

$$(4.27) \qquad \lambda_1(s_k, p)\|u_k\|^p_{L^p(\Omega)} = \mathcal{K}(1 - s_k)|u_k|_{W^{s_k, p}(\mathbb{R}^n)} = K(1 - s_k).$$

On the other hand, given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $s - \varepsilon < s_k$ for all $k \geq k_0$ and, by Lemma 2.3, we have

$$(4.28) \qquad |u_k|_{W^{s-\varepsilon, p}(\mathbb{R}^n)} \leq |u_k|^p_{W^{s_k, p}(\mathbb{R}^n)} + C(n, p)\left(\frac{1}{(s - \varepsilon)p} - \frac{1}{s_k p}\right)\|u_k\|^p_{L^p(\Omega)}.$$

for all $k \geq k_0$. Thus, by Lemma 2.5, (4.28) and using that $|u_k|_{W^{s_k, p}(\mathbb{R}^n)} = 1$, we have that $\{u_k\}_{k \geq k_0}$ is bounded in $\widetilde{W}^{s-\varepsilon, p}(\Omega)$. Then there exists $u \in \widetilde{W}^{s-\varepsilon, p}(\Omega)$ such that, for a subsequence that still call $\{u_k\}_{k \geq k_0}$,

$$(4.29) \qquad \begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } \widetilde{W}^{s-\varepsilon, p}(\mathbb{R}^n), \\ u_k &\to u \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

By (4.27), (4.26) and (4.29),

$$\liminf_{j \to +\infty} \lambda_1(s_j, p)\|u\|^p_{L^p(\Omega)} = K(1 - s).$$

Thus $u \neq 0$ in Ω.

On the other hand, by (4.29), (4.28) and (4.27), we get

$$\begin{aligned} \mathcal{K}(1 - s)|u|^p_{W^{s-\varepsilon, p}(\mathbb{R}^n)} &\leq \liminf_{k \to +\infty} \mathcal{K}(1 - s_k)|u_k|^p_{W^{s-\varepsilon, p}(\mathbb{R}^n)} \\ &\leq \liminf_{k \to +\infty} \left\{ \mathcal{K}(1 - s_k)|u_k|^p_{W^{s_k, p}(\Omega)} \right. \\ &\qquad \left. + C(n, p)\mathcal{K}(1 - s_k)\left(\frac{1}{(s - \varepsilon)p} - \frac{1}{s_k p}\right)\|u_k\|^p_{L^p(\Omega)} \right\} \\ &= \liminf_{k \to +\infty} \left\{ \lambda_1(s_k, p)\|u_k\|^p_{L^p(\Omega)} \right. \\ &\qquad \left. + C(n, p)\mathcal{K}(1 - s_k)\left(\frac{1}{(s - \varepsilon)p} - \frac{1}{s_k p}\right)\|u_k\|^p_{L^p(\Omega)} \right\} \\ &= \liminf_{j \to +\infty} \lambda_1(s_j, p)\|u\|^p_{L^p(\Omega)} \\ &\qquad + C(n, p)\mathcal{K}(1 - s)\left(\frac{1}{(s - \varepsilon)p} - \frac{1}{sp}\right)\|u\|^p_{L^p(\Omega)}. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, by Fatou Lemma, we have

(4.30)
$$\mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} \leq \mathcal{K}(1-s)\liminf_{\varepsilon\to 0^+}|u|^p_{W^{s-\varepsilon,p}(\mathbb{R}^n)}$$
$$\leq \liminf_{j\to+\infty}\lambda_1(s_j,p)\|u\|^p_{L^p(\Omega)}.$$

Finally, by (4.23) and (4.30), we get

$$\lambda(s,p) \leq \mathcal{K}(1-s)\frac{|u|^p_{W^{s,p}(\mathbb{R}^n)}}{\|u\|^p_{L^p(\Omega)}} \leq \liminf_{j\to+\infty}\lambda_1(s_j,p).$$

Case $s = 1$. Let $\varphi \in C_0^\infty(\Omega)$. Then, for any $j \in \mathbb{N}$

$$\lambda_1(s_j,p) \leq \mathcal{K}(1-s_j)\frac{\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+s_jp}}\,dxdy}{\displaystyle\int_{\Omega}|\varphi(x)|^p\,dx}.$$

Thus, by Remark 2.15 and the above inequality, we get

$$\liminf_{j\to+\infty}\lambda_1(s_j,p) \leq \frac{\displaystyle\int_{\Omega}|\nabla\varphi(x)|^p\,dx}{\displaystyle\int_{\Omega}|\varphi(x)|^p\,dx}.$$

As φ is arbitrary

$$\limsup_{j\to+\infty}\lambda_1(s_j,p) \leq \lambda_1(1,p).$$

As in the previous case, to prove (4.25), we need to show that

$$\liminf_{j\to+\infty}\lambda_1(s_j,p) \geq \lambda_1(s,p).$$

Let $\{s_k\}_{k\in\mathbb{N}}$ be a subsequence of $\{s_j\}_{j\in\mathbb{N}}$ such that

(4.31)
$$\lim_{k\to+\infty}\lambda(s_k,p) = \liminf_{j\to+\infty}\lambda_1(s_j,p).$$

Let u_k be an eigenfunction of (3.9) with eigenvalue $\lambda_1(s_k,p)$ such that

$$\mathcal{K}(1-s_k)|u_k|_{W^{s_k,p}(\mathbb{R}^n)} = 1.$$

Then, for any $k \in \mathbb{N}$ we have that $u_k \in \widetilde{W}^{s_k,p}(\Omega)$, $u_k \neq 0$ in Ω, $u_k = 0$ in $\mathbb{R}^n \setminus \Omega$, and

(4.32)
$$\lambda_1(s_k,p)\|u_k\|^p_{L^p(\Omega)} = \mathcal{K}(1-s_k)|u_k|_{W^{s_k,p}(\mathbb{R}^n)} = 1.$$

On the other hand, by Theorem 2.16, we can extract a subsequence (that we still denote by $\{u_k\}_{k\in\mathbb{N}}$) such that

(4.33) $$u_k \rightharpoonup u \quad \text{weakly in } W^{1-\varepsilon,p}(\Omega),$$
(4.34) $$u_k \to u \quad \text{strongly in } L^p(\Omega),$$

for all $\varepsilon > 0$, to some $u \in W_0^{1,p}(\Omega)$. Thus, by (4.31), (4.32) and (4.33), we have

$$\liminf_{j\to+\infty}\lambda_1(s_j,p)\|u\|^p_{L^p(\Omega)} = 1.$$

Then

$$(4.35) \qquad \lambda(1,p) \leq \frac{\displaystyle\int_{\Omega} |\nabla u|^p \, dx}{\displaystyle\int_{\Omega} |u|^p \, dx} = \liminf_{j \to +\infty} \lambda_1(s_j,p) \int_{\Omega} |\nabla u(x)|^p \, dx.$$

On the other hand, given $\varepsilon > 0$ there exists k_0 such that $1 - \varepsilon < s_k$ for all $k \geq k_0$. Then, by Remark 2.17 and (4.32), we get

$$\mathcal{K}\varepsilon|u_k|_{W^{1-\varepsilon,p}(\Omega)} \leq 2^{\varepsilon p} \mathcal{K}(1-s_k)|u_k|_{W^{s_k,p}(\Omega)} \leq 2^{\varepsilon p} \quad \forall k \geq k_0.$$

Thus, by (4.33),

$$(4.36) \qquad \mathcal{K}\varepsilon|u|_{W^{1-\varepsilon,p}(\Omega)} \leq \liminf_{k \to +\infty} \mathcal{K}\varepsilon|u_k|_{W^{1-\varepsilon,p}(\Omega)} \leq 2^{\varepsilon p}.$$

As $\varepsilon > 0$ is arbitrary, by (4.36) and Theorem 2.14, we have that

$$(4.37) \qquad \int_{\Omega} |\nabla u(x)|^p = \lim_{\varepsilon \to 0^+} \mathcal{K}\varepsilon|u|_{W^{1-\varepsilon,p}(\Omega)} \leq 1.$$

Finally, by (4.35) and (4.35), we get

$$\lambda(1,p) \leq \liminf_{j \to +\infty} \lambda_1(s_j,p).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the proof of the following lemma we borrow ideas from [15, Lemma 2.3].

Lemma 4.12. *For every interval $[a,b] \subset (0,1)$ there is a $\delta > 0$ such that for all $s \in [a,b]$ there is not eigenvalue of (4.21) in $(\lambda_1(s,p), \lambda_1(s,p) + \delta]$.*

Proof. Suppose the lemma were false. Then we could find sequences $\{s_k\}_{k \in \mathbb{N}}$ in $(0,1]$, $\{\lambda_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}_+$ and $\{u_k\}_{k \in \mathbb{N}}$ in $\widetilde{W}^{s,p}(\Omega) \setminus \{0\}$ such that

$$\lim_{k \to \infty} s_k = s \in (0,1], \quad \lambda_k > \lambda_1(s_k,p) \quad \forall k \in \mathbb{N}, \quad \lim_{k \to \infty} \lambda_k - \lambda_1(s_k,p) = 0,$$

and for all $k \in \mathbb{N}$ $\|u_k\|_{L^p(\Omega)} = 1$ and

$$(4.38) \qquad u_k = \mathcal{R}_{s_k,p}(\lambda_k |u_k|^{p-2} u_k).$$

By Lemma 4.11,

$$(4.39) \qquad \lim_{k \to \infty} \lambda_k = \lambda_1(s,p)$$

On the other hand $\{|u_k|^{p-2} u_k\}_{k \in \mathbb{N}}$ is bounded in $L^{p'}(\Omega)$ due to $\|u_k\|_{L^p(\Omega)} = 1$ for all $k \in \mathbb{N}$. Then, by Lemma 3.1, there exist $u \in L^p(\Omega)$ and a subsequence of $\{u_k\}_{k \in \mathbb{N}}$, still denoted $\{u_k\}_{k \in \mathbb{N}}$, such that $u_k \to u$ in $L^p(\Omega)$ as $k \to +\infty$. Thus

$$(4.40) \qquad |u_k|^{p-2} u_k \to |u|^{p-2} u \quad \text{strongly in } L^{p'}(\Omega).$$

Then, passing to the limit in (4.38), using (4.39), (4.40) and Lemma 3.1, we get

$$u = \mathcal{R}_{s,p}(\lambda_1(s,p)|u|^{p-2} u).$$

Therefore u is an eigenfunction associated to $\lambda_1(s,p)$. Then, by Corollary 4.4 and Theorem 4.7, we may assume without loss of generality, that $u > 0$.

On the other hand, given $s_0 \in (0, \min\{s, n/p\})$ there exists $k_0 \in N$ such that $s_k \geq s$ for all $k \geq k_0$ due $s_k \to s$ as $k \to +\infty$. Thus, by Lemma 4.9, we get

$$\left(\frac{\mathcal{K}(n - s_0 p)^{p-1}}{C 2^{(1-s_0)p} s_0 \lambda_k} \right)^{\frac{p_{s_0}^\star}{p_{s_0}^\star - p}} \leq |\{x \in \Omega \colon u_k(x) < 0\}| \quad \forall k \geq k_0.$$

Then, since $u_k \to u$ in $L^p(\Omega)$, u must change its sign in Ω, contrary to the fact that $u > 0$. $\qquad\square$

5. BIFURCATION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1]$, and $p \in (1, +\infty)$. In this section we consider the following non-linear problem:

$$(5.41) \qquad \begin{cases} \mathcal{L}_{s,p}(u) = \lambda |u|^{p-2} u + f(x, u, \lambda) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

where $f \colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that
  (1) f satisfies a Carathéodory condition in the first two variables;
  (2) $f(x, t, \lambda) = o(|t|^{p-1})$ near $t = 0$, uniformly a.e. with respect to x and uniformly with respect to λ on bonded sets;
  (3) There exists $q \in (1, p_s^\star)$ such that

$$\lim_{|t| \to +\infty} \frac{|f(x, t, \lambda)|}{|t|^{q-1}} = 0$$

  uniformly a.e. with respect to x and uniformly with respect to λ on bonded sets.

A pair $(\lambda, u) \in \mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ is a weak solution of (5.41) if

$$\mathfrak{H}_{s,p}(u, \phi) = \int_\Omega \left(\lambda |u(x)|^{p-2} u(x) \phi(x) + f(x, u, \lambda) \right) \phi(x) \, dx,$$

for all $\phi \in C_0^\infty(\Omega)$. Where

$$\mathfrak{H}_{s,p}(u, \phi) = \begin{cases} \mathcal{K}(1-s) \mathcal{H}_{s,p}(u, \phi), & \text{if } 0 < s < 1, \\ \displaystyle\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \phi(x) \, dx & \text{if } s = 1. \end{cases}$$

Remark 5.1. The pair (λ, u) is weak solution of (5.41) iff (u, λ) satisfies

$$u = \mathcal{R}_{s,p}(\lambda |u|^{p-2} u + F(u, \lambda))$$

where $F(\cdot, \lambda)$ is the Nemitsky operator associated with f.

We say that $(\lambda, 0) \in \mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ is a bifurcation point of (5.41) if in any neighbourhood of $(\lambda, 0)$ in $\mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ there exists a nontrivial solution of (5.41).

The proof of the following result is analogous to that of Proposition 2.1 in [15]

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1]$, and $p \in (1, +\infty)$. If $(\lambda, 0)$ is a bifurcation point of (5.41) then λ is an eigenvalue of (4.21).*

Let, $s \in (0,1)$, and $p \in (1,+\infty)$. For $\lambda < \lambda_1(s,p)$ or $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ the function $u \equiv 0$ is the unique solution of

$$u = \mathcal{R}_{s,p}(\lambda |u|^{p-2} u),$$

where

$$\lambda_2(s,p) := \inf \left\{ \lambda > \lambda_1(s,p) \colon \lambda \text{ is an eigenvalue of } (4.21) \right\}.$$

Then for $\lambda < \lambda_1(s,p)$ or $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ we define the completely continuous operator $\mathcal{T}_{s,p}^\lambda \colon \widetilde{W}^{s,p}(\Omega) \to \widetilde{W}^{s,p}(\Omega)$

$$\mathcal{T}_{s,p}^\lambda(u) := \mathcal{R}_{s,p}(\lambda |u|^{p-2} u).$$

Thus

$$\deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathcal{T}_{s,p}^\lambda, B(0,r), 0)$$

is well defined for any $\lambda < \lambda_1(s,p)$ or $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ and $r > 0$.

Theorem 5.3. *Let* $t \in (0,1]$, $p \in (1,+\infty)$, *and* $r > 0$. *then*

$$\deg_{\widetilde{W}^{t,p}(\Omega)}(I - \mathcal{T}_{t,p}^\lambda, B(0,r), 0) = \begin{cases} 1, & \text{if } \lambda < \lambda_1(t,p), \\ -1, & \text{if } \lambda_1(t,p) < \lambda < \lambda_2(t,p). \end{cases}$$

This result is a generalization of Proposition 2.2 in [15], where the authors show that

$$(5.42) \quad \deg_{W_0^{1,p}(\Omega)}(I - \mathcal{T}_{1,p}^\lambda(u), B(0,r), 0) = \begin{cases} 1, & \text{if } \lambda < \lambda_1(1,p), \\ -1, & \text{if } \lambda_1(1,p) < \lambda < \lambda_2(1,p). \end{cases}$$

Proof. We begin by the case $\lambda_1(t,p) < \lambda < \lambda_1(t,p)$. By Lemmas 4.11 and 4.12, there exists a continuous function $\rho \colon (0,1] \to \mathbb{R}$ such that

$$\lambda_1(s,p) < \rho(s) < \lambda_1(s,p) \quad \forall s \in (0,1],$$

and $\rho(t) = \lambda$. Then it is sufficient to prove that the function $d \colon (0,1] \to \mathbb{R}$

$$d(s) := \deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathcal{T}_{\tilde{s},p}^{\rho(\tilde{s})}, B(0,r), 0)$$

is constant due to $d(1) = -1$

Let $s \in (0,1]$. We define the operator $\mathcal{P}_s \colon L^p(\Omega) \to \widetilde{W}^{s,p}(\Omega)$ as

$$\mathcal{P}_s(u) := \mathcal{R}_{s,p}(\rho(s)|u|^{p-2} u)$$

Then $\mathcal{P}_s$ is completely continuous and

$$\mathcal{T}_{s,p}^{\rho(s)} = \mathcal{P}_s \circ i$$

where $i \colon \widetilde{W}^{s,p}(\Omega) \to L^p(\Omega)$ is the usual inclusion. Thus, by Lemma 2.21, we get

$$(5.43) \qquad d(s) = \deg_{L^p(\Omega)}(I - i \circ \mathcal{P}_s, O, 0) \quad \forall s \in (0,1]$$

where O is any open bounded set in $L^p(\Omega)$ such that $0 \in O$.

On the other hand, since ρ is continuous and by Lemma 4.11, we get that the homotopy

$$(0,1] \times L^p(\Omega) \to L^p(\Omega)$$

$$(s,u) \to \mathcal{R}_{s,p}(\rho(s)|u|^{p-2} u) = (i \circ \mathcal{P}_s)(u)$$

is completely continuous. Then $d(s)$ is constant in $(0,1]$ due to the invariance of the Leray-Schauder degree under compact homotopy and (5.43).

Finally, we consider the case $\lambda < \lambda_1(t,p)$. Given $a \in [0,1]$, the degree

$$\deg_{\widetilde{W}^{t,p}(\Omega)}(I - \mathcal{R}_{t,p}(a\lambda\Psi_p(\cdot)), B(0,r), 0)$$

is well defined. Here $\Psi_p(u) = |u|^{p-2}u$. Then, from the invariance of the degree under homotopies, we get

$$\deg_{\widetilde{W}^{t,p}(\Omega)}(I - \mathcal{R}_{\tilde{s},p}(t\lambda\Psi_p(\cdot)), B(0,r), 0) = \deg_{\widetilde{W}^{t,p}(\Omega)}(I, B(0,r), 0) = 1$$

for all $a \in [0,1]$. $\qquad\square$

Finally, proceeding as in the proof of Theorem 1.1 in [15], we can prove Theorem 1.1.

6. Existence of constant-sign solution

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,+\infty)$, and $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(0) = 0$. In this section, we will apply Theorem 1.1 to show that the following non-linear non-local problem

$$(6.44) \qquad \begin{cases} (-\Delta_p)^s u = g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

has a non-trivial weak solution. Observe that $u \equiv 0$ is a solution of (6.44).

We will keep the following assumptions about g, throughout this section:

A1. $\dfrac{g(s)}{|s|^{p-2}s}$ is bounded;

A2. $\underline{\lambda} := \lim\limits_{s \to 0} \dfrac{g(s)}{|s|^{p-2}s} < \lambda_1(s,p) < \liminf\limits_{|s|\to+\infty} \dfrac{g(s)}{|s|^{p-2}s}$.

Note that, if g satisfies A1 and A2 then

$$g(s) = \underline{\lambda}|s|^{p-2}s + f(s),$$

where $f(s) = o(|s|^{p-1})$ near $s = 0$. Then, our problem is related to the next bifurcation problem

$$(6.45) \qquad \begin{cases} (-\Delta_p)^s u = \lambda|u|^{p-2}u + f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

By Theorem 1.1 there exists a connected component $\mathscr{C}$ of the set of non-trivial solution of (6.45) in $\mathbb{R} \times W_0^{s,p}(\Omega)$ whose closure contains $(\lambda_1(s,p),0)$ and it is either unbounded or contains a pair $(\lambda,0)$ for some λ, eigenvalue of (4.21) with $\lambda > \lambda_1(s,p)$.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, and $p \in (1,+\infty)$. Then $\mathscr{C}$ is unbounded and*

$$\overline{\mathscr{C}} \subset \mathscr{H} := \{(\lambda_1(s,p),0)\} \cup (\mathbb{R} \times \mathscr{P}),$$

where $\mathscr{P} := \{v \in \widetilde{W}^{s,p}(\Omega)\colon v \text{ has constant-sign in } \Omega\}$.

Proof. We split the proof in 3 steps.

Step 1. There exist a neighbourhood U of $(\lambda_1(s,p),0)$ in $\mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ such that $\mathscr{C} \cap U \setminus \{(\lambda_1(s,p),0)\} \subset \mathbb{R} \times \mathscr{P}$.

Let us assume by contradiction the existence of a sequence $\{(\lambda_k, u_k)\}_{k \in \mathbb{N}}$ of non-trivial solution of (6.45) such that u_k changes sign in Ω for all $k \in \mathbb{N}$ and $(\lambda_k, u_k) \to (\lambda_1(s,p),0)$ in $\mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ as $k \to +\infty$.

For any $k \in \mathbb{N}$, since $h_k = \lambda_k + {f(u_k)}/{|u_k|^{p-2}u_k}$ is uniformly bounded in Ω and u_k changes sign in Ω, by Corollary 4.4 we have that 1 is an eigenvalue (4.22) with $h = h_n$ and $1 > \lambda_1(s,p,h_k)$. Thus, by Lemma 4.9 and using that h_k is uniformly bounded in Ω, there exist a constant C independent of k such that

$$(6.46) \qquad |\{x \in \Omega \colon u_k(x) > 0\}| \geq C \text{ and } |\{x \in \Omega \colon u_k(x) < 0\}| \geq C \quad \forall k \in \mathbb{N}.$$

On the other hand, taking $\hat{u}_k := {u_k}/{\|u_k\|_{\widetilde{W}^{s,p}(\Omega)}}$, it follows that the sequence $\{\hat{u}_k\}_{k \in \mathbb{N}}$ is bounded in $\widetilde{W}^{s,p}(\Omega)$ then, via a subsequence if necessary, we have that there exists $u \in \widetilde{W}^{s,p}(\Omega)$ such that

$$\hat{u}_k \rightharpoonup u \text{ weakly in } \widetilde{W}^{s,p}(\Omega),$$
$$\hat{u}_k \to u \text{ strongly in } L^p(\Omega),$$
$$\hat{u}_k \to u \text{ a.e. in } \Omega$$

By (6.46),

$$(6.47) \qquad\qquad\qquad u \neq 0 \text{ and } u \text{ changes sign }.$$

Moreover

$$\mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} \leq \lim_{k \to +\infty} \mathcal{K}(1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\hat{u}_k(x) - \hat{u}_k(y)|^p}{|x-y|^{n+sp}} dxdy$$
$$= \lim_{k \to +\infty} \int_\Omega h_k(x)|\hat{u}_k|^p\, dx$$
$$= \lambda_1(s,p) \int_\Omega |u|^p dx,$$

due to h_k is uniformly bounded in Ω, $h_k(x) \to \lambda_1(s,p)$ a.e. in Ω and $\hat{u}_k \to u$ strongly in $L^p(\Omega)$. Then

$$\mathcal{K}(1-s) \frac{\displaystyle\int_{\mathbb{R}^n}\int_{R^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy}{\displaystyle\int_\Omega |u|^p dx} \leq \lambda_1(s,p).$$

Thus, by definition of $\lambda_1(s,p)$, we have that u is an eigenfunction associated to $\lambda_1(s,p)$. Therefore, by Corollary 4.4, $|u| > 0$, that is u has constant sign, this yield a contradiction with (6.47). Hence the claim follows.

Step 2. $\overline{\mathscr{C}} \subset \mathscr{H}$.

Again we proceed by contradiction. Suppose that there exist $(\lambda_0, u_0) \in \overline{\mathscr{C}}$ such that can be approximated by elements of $\mathscr{C}$ from inside and from without $\mathscr{H}$. By step 1, $(\lambda_0, u_0) \neq (\lambda_1(s,p),0)$.

Case $u_0 \equiv 0$. Thus $\lambda_0 \neq \lambda_1(s,p)$. Proceeding in a similar manner as in the previous step, we can see that λ_0 is an eigenvalue of (4.22) with $h \equiv 1$ different to $\lambda_1(s,p)$ and arrive to a contradiction.

Case $u_0 \neq 0$. We know that there exist $\{(\lambda_k, u_k)\}_{k \in \mathbb{N}} \subset \mathscr{C} \cap \mathscr{H}$ such that $(\lambda_k, u_k) \to (\lambda_0, u_0)$ in $\mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$. Therefore u_0 is either non-negative or non-positive and u_0 is a weak solution of

$$\begin{cases} (-\Delta)_p^s u = \left(\lambda_0 + \dfrac{f(u_0)}{|u_0|^{p-2}u_0}\right)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Without loss of generality, we can assume that $u_0 \geq 0$ a.e. in Ω. Since $\lambda_0 + f(u_0)/|u_0|^{p-2}u_0$ is bounded, by Theorem 4.3, we have that $u_0 > 0$ a.e. in Ω. Thus $(\lambda_0, u_0) \in \mathscr{H}$. Argument in similar manner that in step 1, we can show that (λ_0, u_0) can not be approximated by elements of $\mathscr{C}$ from without $\mathscr{H}$, contradicting the fact that (λ_0, u_0) can be approximated by elements of $\mathscr{C}$ from without $\mathscr{H}$.

Step 3. $\mathscr{C}$ is unbounded.

Since $\overline{\mathscr{C}} \subset \mathscr{H}$, $\mathscr{C}$ does not contain a pair $(\lambda, 0)$ for some λ, eigenvalue of (4.21) with $\lambda > \lambda_1(s,p)$. Then by Theorem 1.1, $\mathscr{C}$ is unbounded. $\qquad \square$

Our next aim is to show that $\mathscr{C} \cap \left([\underline{\lambda}, +\infty) \times \widetilde{W}^{s,p}(\Omega)\right)$ is bounded. For this, we will need the following result. The proof is identical to the proof of [15, Lemma 3.2].

Lemma 6.2. *There exists a positive constant C such that if $(\lambda, u) \in \mathscr{C}$ then $\lambda \leq \mathscr{C}$.*

Then for showing that $\mathscr{C} \cap \left([\underline{\lambda}, +\infty) \times \widetilde{W}^{s,p}(\Omega)\right)$ is bounded, it is enough to prove the result given below.

Lemma 6.3. *There exists a positive constant M such that for any $(\lambda, u) \in \mathscr{C} \cap \left([\underline{\lambda}, C] \times \widetilde{W}^{s,p}(\Omega)\right)$ we have that $\|u\|_{\widetilde{W}^{s,p}(\Omega)} \leq M$. Here C is the constant of Lemma 6.2.*

Proof. Suppose by contradiction that there exists a sequence $\{(\lambda_k, u_k)\}_{k \in \mathbb{N}}$ of elements of $\mathscr{C} \cap \left([\underline{\lambda}, C] \times \widetilde{W}^{s,p}(\Omega)\right)$ such that $\lambda_k \to \lambda_0$ and $\|u_k\|_{\widetilde{W}^{s,p}(\Omega)} \to +\infty$ as $k \to +\infty$. Without loss of generality we can assume that $u_k > 0$ for all $k \in \mathbb{N}$.

Taking $\hat{u}_k = u_k/\|u_k\|_{\widetilde{W}^{s,p}(\Omega)}$ and $h_k = f(u_k)/|u_k|^{p-2}u_k$, for any $k \in \mathbb{N}$ we have that

$$\hat{u}_k = \mathcal{R}_{s,p}\left(\lambda_k|\hat{u}_k|^{p-2}\hat{u}_k + \frac{f(u_k)}{|u_k|^{p-2}u_k}|\hat{u}_k|^{p-2}\hat{u}_k\right).$$

On the other hand, $\{f(u_k)/|u_k|^{p-2}u_k\}_{k \in \mathbb{N}}$ is uniformly bounded due to g satisfies A1, then there exists $h \in L^\infty(\Omega)$ such that

$$\frac{f(u_k)}{|u_k|^{p-2}u_k} \rightharpoonup h \text{ weakly in } L^q(\Omega) \quad \forall q > 1.$$

Since $\mathcal{R}_{s,p}$ to $L^{q'}(\Omega)$ with $q \in (1, p_s^\star)$ is a completely continuous operator, we have that there exists $u_0 \in \widetilde{W}^{s,p}(\Omega)$ such that $u_k \to u_0$ strongly in $\widetilde{W}^{s,p}(\Omega)$ and

$$u_0 = \mathcal{R}_{s,p}\left(\lambda_0|u_0|^{p-2}u_0 + h|u_0|^{p-2}u_0\right),$$

that is u_0 is a weak solution of

$$\begin{cases} (-\Delta)_p^s u = (\lambda_0 + h(x))|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Observe that $u_0 \neq 0$ and $u \geq 0$ due to $\|\hat{u}_k\|_{\widetilde{W}^{s,p}(\Omega)} = 1$ and $\hat{u}_k > 0$ in Ω. The $\mu = 1$ is the first eigenvalue of

$$(6.48) \qquad \begin{cases} (-\Delta)_p^s u = \mu \left(\lambda_0 + h(x) \right) |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

and u_0 is an eigenfunction associated to 1. Then, by Corollary 4.4, we have that $u_0 > 0$ in Ω.

Claim. $h \geq \overline{\lambda} - \underline{\lambda}$ a.e. in Ω where $\lambda_1(s,p) < \overline{\lambda} < \liminf\limits_{|s| \to +\infty} \dfrac{g(s)}{|s|^{p-2}s}$.

Suppose the contrary, that is the set $A = \{x \in \Omega \colon h(x) < \overline{\lambda} - \underline{\lambda}\}$ has positive measure. Since $\hat{u}_k \to u_0 > 0$ a.e. in Ω, by the Egorov's theorem, there exists a set $U \subset \Omega$ such that $|\Omega \setminus U| < |A|$ and $u_k \to +\infty$ uniformly in U. Then there exists $k_0 \in \mathbb{N}$ such that $f(u_k)/|u_k|^{p-2}u_k \geq \overline{\lambda} - \underline{\lambda}$ for all $k \geq k_0$ because

$$\lambda_1(s,p) < \overline{\lambda} < \liminf_{|s| \to +\infty} \frac{g(s)}{|s|^{p-2}s} = \underline{\lambda} + \liminf_{|s| \to +\infty} \frac{f(s)}{|s|^{p-2}s}$$

and therefore $h(x) \geq \overline{\lambda} - \underline{\lambda}$ a.e. in U. Thus $A \subset \Omega \setminus U$, then $|A| \leq |\Omega \setminus U| < |A|$, which is a contradiction. Hence, the claim follows.

Since $h(x) \geq \overline{\lambda} - \underline{\lambda}$ a.e. in Ω, $\lambda_0 - \underline{\lambda} \geq 0$ and $\overline{\lambda} > \lambda_1(s,p)$, we get $\lambda_0 + h(x) \geq \lambda_0 + \overline{\lambda} - \underline{\lambda} > \lambda_1(s,p)$.

On the other hand, since μ is the first eigenvalue of 6.48, we have that

$$1 \leq \mathcal{K}(1-s) \frac{|\phi|_{W^{s,p}(\mathbb{R}^n)}^p}{\displaystyle\int_\Omega (\lambda_0 + h(x))|\phi(x)|^p \, dx} \qquad \forall \phi \in C_0^\infty(\Omega).$$

Then for any $\phi \in C_0^\infty(\Omega)$

$$(\lambda_0 + \overline{\lambda} - \underline{\lambda})\|\phi\|_{L^p(\Omega)}^p \leq \int_\Omega (\lambda_0 + h(x))|\phi(x)|^p \, dx \leq \mathcal{K}(1-s)|\phi|_{W^{s,p}(\mathbb{R}^n)}^p$$

due to our claim. Then

$$\lambda_0 + \overline{\lambda} - \underline{\lambda} \leq \lambda_1(s,p) < \lambda_0 + \overline{\lambda} - \underline{\lambda},$$

getting a contradiction. Thus the lemma is true.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Finally we establish the main result of this section.

Theorem 6.4. *Let $g\colon \Omega \to \mathbb{R}$ continuous such that $g(0) = 0$ and g satisfies A1 and A2. Then there exists a non-trivial weak solution u of* (6.44) *such that u has constant-sign in Ω.*

Proof. By Lemma 6.2 and Lemma 6.3, $\mathscr{C} \cap \left([\underline{\lambda}, +\infty) \times \widetilde{W}^{s,p}(\Omega) \right)$ is bounded. On other hand, by Lemma 6.1, $\mathscr{C}$ is unbounded. Then there exists $(\underline{\lambda}, \underline{u}) \in \mathscr{C}$, due to $\mathscr{C}$ is connected. By A2 $\underline{\lambda} < \lambda_1(s,p)$ and Lemma 6.1, $\underline{u}$ has constant-sign in Ω. Therefore u is a non-trivial weak solution of (6.44). $\qquad\qquad \square$

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