# THE UNIQUE CONTINUATION PROPERTY FOR A NONLINEAR EQUATION ON TREES

LEANDRO M. DEL PEZZO, CAROLINA A. MOSQUERA AND JULIO D. ROSSI

Abstract. In this paper we study the game $p-$Laplacian on a tree, that is,

$$u(x)=\frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)}u(y)+\min_{y\in\mathcal{S}(x)}u(y)\right\}+\frac{\beta}{m}\sum_{y\in\mathcal{S}(x)}u(y),$$

here $x$ is a vertex of the tree and $S(x)$ is the set of successors of $x$. We show, among other properties of the solutions, a characterization of the subsets of the tree that enjoy the unique continuation property, that is, subsets $U$ such that $u\mid_U=0$ implies $u\equiv 0$.

## 1. Introduction

Our main goal in this paper is to analyze for which sets the unique continuation property is valid for the nonlinear equation known as the game $p-$Laplacian on a tree. This nonlinear equation reads as follows

$$u(x)=\frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)}u(y)+\min_{y\in\mathcal{S}(x)}u(y)\right\}+\frac{\beta}{m}\sum_{y\in\mathcal{S}(x)}u(y),\qquad\forall x\in\mathbb{T}_m. \tag{1.1}$$

Here $x$ is a vertex of the $m$-branches directed tree $\mathbb{T}_m$ and $\mathcal{S}(x)$ is the set of successors of that vertex (see Section 2 for details).

Equation (1.1) arises naturally when one considers Tug-of-War games. In fact, let us describe the game that gives rise to (1.1). This is a zero sum game with two players in which the earnings of one of them are the losses of the other. Starting with a token at a vertex $x_0\in\mathbb{T}_m$, the players flip a biased coin with probabilities $\alpha$ and $\beta$, $\alpha+\beta=1$. If the result is a head (probability $\alpha$), they toss a fair coin to decide who move the token. If the outcome of the second toss is heads, then Player I moves the token to any $x_1\in\mathcal{S}(x_0)$, while in case of tails, Player II moves the token to any $x_1\in\mathcal{S}(x_0)$. In the other case, that is, if they get tails in the first coin toss (probability $\beta$), the game state moves according to the uniform probability density to a random vertex $x_1\in\mathcal{S}(x_0)$. They continue playing and given a continuous function $F:[0,1]\to\mathbb{R}$, the final payoff is given by $\lim_{k\to+\infty}u(x_k)=F(\pi)$. This game has a value $u$ that verifies a Dynamic Programming Principle formula, that for this game is given by (1.1). This can be intuitively explained as follows: the expected value of the game is the sum among all possibilities of the expected value in the successors. Note that Player I tries to maximize the expected value while Player II tries to minimize it. Hence, there is $\alpha/2$ probability of each player to win (and hence $\alpha/2$ probability to move to the vertex where the maximum is located and $\alpha/2$ to the minimum) and $\beta$ probability of the random choice of the next point. Formula (1.1) encodes all these possibilities. See Section 3 for more details concerning the game.

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Also equation (1.1) can be viewed as a combination (with coefficients $\alpha$ and $\beta$) of the discrete infinity Laplacian, studied in [22], that is given by

$$\frac{1}{2}\left\{\max_{y\in\mathcal{S}(x)} u(y) + \min_{y\in\mathcal{S}(x)} u(y)\right\} - u(x)$$

and the discrete Laplacian that in this case is given by

$$\frac{1}{m}\sum_{y\in\mathcal{S}(x)} u(y) - u(x).$$

The study of the unique continuation property for solutions of differential equalities and inequalities of second order elliptic operators with smooth and non-smooth real coefficients has a large history and is essentially complete. Let us state a classical strong unique continuation result for the divergence-form linear equation

$$\operatorname{div}(A(x)Du) + \langle b(x), Du\rangle + c(x)u = 0.$$

*Classical unique continuation property.* Let $\Omega \subset \mathbb{R}^N$ be a connected domain. Under adequate assumptions on the coefficients $A$, $b$ and $c$, if $u$ is a weak solution that vanishes in an open subset of $\Omega$, then $u \equiv 0$ in $\Omega$.

A general version of this statement is proved by Hormander [8] using Carleman estimates. See also [4] that contains a proof of the result via monotonicity formulas. This result was recently generalized to fully nonlinear equations (under some assumptions on regularity of the equation) in [1]. For more details and references concerning unique continuation we refer the reader to [7, 9, 12, 13, 20, 23].

Concerning unique continuation for quasilinear problems like the $p-$Laplacian, $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, in [16], the author proves the strong unique continuation property in the plane for all $1 < p < +\infty$, see also [3]. In the higher dimensions, as far as we know, the problem remains open for $p \neq 2$. Recently, in [5] , the authors deal with this problem by studying a certain generalization of Almgren's frequency function for the $p-$Laplacian. Using this approach the authors have obtained some partial results. See also the reference [6].

In the case of connected finite graph this problem can be stated as follows: Let $E$ be a connected finite graph. We assign to every edge of $E$ length one and we define $d(x,y) = \inf_{x\sim y} |x \sim y|$, where $x \sim y$ is the path connecting vertex $x$ to the vertex $y$ and $|x \sim y|$ is the number of edges in this path. Assume that $u$ is a solution to (1.1) on $E$ (these functions are also called $p-$harmonious functions, see [22]) and that $u = 0$ on $B_R(x)$ where $B_R(x)$ is the ball of radius $R > 0$ centered at a node $x$ of $E$ contained within this graph. Does it imply that $u \equiv 0$ on $E$? The answer to this problem is negative, see examples in Section 3.6 of [22]. Also, in [22], the author proves the existence and uniqueness and a comparison principle for the Dirichlet problem for (1.1) in the case of a connected finite graph and in the case where the graph is $\mathbb{T}_3$.

1.1. **Main results.** Our results can be summarized as follows: first, for a general $m$-branches directed tree, we prove existence, uniqueness and a comparison principle for the Dirichlet problem for (1.1). In addition we present an approximation scheme that can be used to approximate numerically the solution when the boundary data is a Lipschitz function. Next, we prove our main result, that is a characterization of the sets $U \subset \mathbb{T}_m$ for which the unique continuation property holds. As we have mentioned, this means that any bounded solution to (1.1) that vanishes on $U$ vanishes everywhere in $\mathbb{T}_m$.

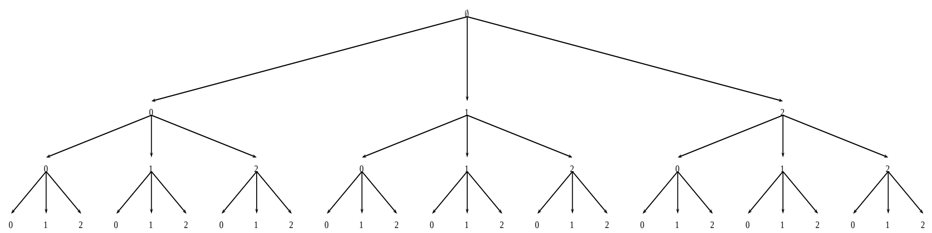
**Organization of the paper.** In Section 2 we collect some preliminary facts concerning trees and solutions to (1.1); in Section 3 we describe with some details the associated Tug-of-War game and use it to prove existence and uniqueness for the Dirichlet problem and a comparison principle for solutions to (1.1); in Section 4 we present a numerical scheme that approximates solutions to (1.1) and, finally, in Section 5 we prove our main result characterizing the sets for which unique continuation hold.

## 2. Preliminaries

2.1. **Directed Tree.** Let $m \in \mathbb{N}_{>2}$. In this work we consider a directed tree $\mathbb{T}_m$ with regular $m-$branching, that is, $\mathbb{T}_m$ consists of the empty set $\emptyset$ and all finite sequences $(a_1, a_2, \dots, a_k)$ with $k \in \mathbb{N}$, whose coordinates $a_i$ are chosen from $\{0, 1, \dots, m-1\}$. The elements in $\mathbb{T}_m$ are called vertices. Each vertex $x$ has $m$ successors, obtained by adding another coordinate. As we mentioned in the introduction, we will denote by $\mathcal{S}(x)$ the set of successors of the vertex $x$. A vertex $x \in \mathbb{T}_m$ is called a $n-$level vertex ($n \in \mathbb{N}$) if $x = (a_1, a_2, \dots, a_n)$. The set of all $n-$level vertices is denoted by $\mathbb{T}_m^n$.

*Example* 2.1. Let $\kappa \in \mathbb{N}_{\geq 3}$. The $1/\kappa-$Cantor set, that we denote by $C_{1/\kappa}$, is the set of all $x \in [0,1]$ that have a base $\kappa$ expansion without the digit 1, that is $x = \sum a_j \kappa^{-j}$ with $a_j \in \{0, 1, \dots, \kappa - 1\}$ with $a_j \neq 1$. Thus $C_{1/\kappa}$ is obtained from $[0,1]$ by removing the second $\kappa-$th part of the line segment $[0,1]$, and then removing the second interval of length $1/\kappa$ from the remaining intervals, and so on. This set can be thought of as a directed tree with regular $m-$branching with $m = \kappa - 1$.

For example, if $\kappa = 3$, we identify $[0,1]$ with $\emptyset$, the sequence $(\emptyset, 0)$ with the first interval right $[0, 1/3]$, the sequence $(\emptyset, 1)$ with the central interval $[1/3, 2/3]$ (that is removed), the sequence $(\emptyset, 2)$ with the left interval $[2/3, 1]$, the sequence $(\emptyset, 0, 0)$ with the interval $[0, 1/9]$ and so on.



A branch of $\mathbb{T}_m$ is an infinite sequence of vertices, each followed by its immediate successor. The collection of all branches forms the boundary $\partial\mathbb{T}_m$ of $\mathbb{T}_m$.

We now define a metric on $\mathbb{T}_m \cup \partial\mathbb{T}_m$. The distance between two sequences (finite or infinite) $\pi = (a_1, \dots, a_k, \dots)$ and $\pi' = (a_1', \dots, a_k', \dots)$ is $m^{-K+1}$ when $K$ is the first index $k$ such that $a_k \neq a_k'$; but when $\pi = (a_1, \dots, a_K)$ and $\pi' = (a_1, \dots, a_K, a_{K+1}', \dots)$, the distance is $m^{-K}$. Hausdorff measure and Hausdorff dimension are defined using this metric. We can observe that $\mathbb{T}_m$ and $\partial\mathbb{T}_m$ have diameter one and $\partial\mathbb{T}_m$ has Hausdorff dimension one. Now, we observe that the mapping $\psi : \partial\mathbb{T}_m \to [0,1]$ defined as

$$\psi(\pi) := \sum_{k=1}^{+\infty} \frac{a_k}{m^k}$$

is surjective, where $\pi = (a_1, \dots, a_k, \dots) \in \partial\mathbb{T}_m$ and $a_k \in \{0, 1, \dots, m-1\}$ for all $k \in \mathbb{N}$. Whenever $x = (a_1, a_2, \dots, a_k)$ is a vertex, we set

$$\psi(x) := \psi(a_1, a_2, \dots, a_k, 0, \dots, 0, \dots).$$

We can also associate to a vertex $x = (a_1, a_2, \dots, a_k)$ an interval $I_x$ of length $\frac{1}{m^k}$ as follows

$$I_x = \left[\psi(x), \psi(x) + \frac{1}{m^k}\right].$$

Observe that for all $x \in \mathbb{T}_m$, $I_x \cap \partial\mathbb{T}_m$ is the subset of $\partial\mathbb{T}_m$ consisting of all branches that start at $x$.

2.2. $p-$**harmonious functions.** Inspired in [22] and [19] we give the definition of the $p-$harmonious function that we will consider throughout this paper.

**Definition 2.2.** Let $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. A function $u : \mathbb{T}_m \to \mathbb{R}$ is called $p-$subharmonious if

$$u(x) \leq \frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)} u(y) + \min_{y\in\mathcal{S}(x)} u(y)\right\} + \frac{\beta}{m}\sum_{y\in\mathcal{S}(x)} u(y) \quad \forall x \in \mathbb{T}_m,$$

and $p-$superharmonious if the opposite inequality holds for all $x \in \mathbb{T}_m$. We say that $u$ is $p-$harmonious if $u$ is both $p-$subharmonious and $p-$superharmonious.

*Remark* 2.3. If $u$ is a $p-$harmonious function on $\mathbb{T}_m$, then $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ are $p-$subharmonious functions on $\mathbb{T}_m$.

Next, we collect some properties of $p-$harmonious functions.

**Lemma 2.4.** *If $u$ is a $p-$subharmonious function bounded above on $\mathbb{T}_m$ and there exists $x \in \mathbb{T}_m$ such that $u(x) = \max_{y\in\mathbb{T}_m} u(y)$ then $u(y) = u(x)$ for any $y \in \mathbb{T}_m$ such that $I_y \subset I_x$.*

*Proof.* Throughout this proof let $M = u(x) = \max_{y\in\mathbb{T}_m} u(y)$. We first observe that it is sufficient to show that $u(y) = M$ for all $y \in \mathcal{S}(x)$. Since $u$ is $p-$subharmonious on $\mathbb{T}_m$, we have that

$$\begin{aligned} M = u(x) &\leq \frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)} u(y) + \min_{y\in\mathcal{S}(x)} u(y)\right\} + \frac{\beta}{m}\sum_{y\in\mathcal{S}(x)} u(y) \\ &\leq \left(\frac{\alpha}{2} + \frac{(m-1)\beta}{m}\right) M + \left(\frac{\alpha}{2} + \frac{\beta}{m}\right)\min_{y\in\mathcal{S}(x)} u(y). \end{aligned}$$

Then

$$\left(\frac{\alpha}{2} + \frac{\beta}{m}\right) M \leq \left(\frac{\alpha}{2} + \frac{\beta}{m}\right)\min_{y\in\mathcal{S}(x)} u(y).$$

Therefore $u(y) = u(x)$ for all $y \in \mathcal{S}(x)$. □

In the same manner, we can prove the following lemma

**Lemma 2.5.** *If $u$ is a $p-$superharmonious function bounded below on $\mathbb{T}_m$ and there exists $x \in \mathbb{T}_m$ such that $u(x) = \min_{y\in\mathbb{T}_m} u(y)$, then $u(y) = u(x)$ for any $y \in \mathbb{T}_m$ such that $I_y \subset I_x$.*

Now we show that $p-$harmonious functions are well behaved with respect to uniform convergence.

**Lemma 2.6.** *The uniform limit of a sequence of $p-$harmonious functions is a $p-$harmonious function.*

*Proof.* Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of $p-$harmonious functions which converges uniformly to $u$. We will show that $u$ is a $p-$harmonious function. Given $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$,

$$|u(x) - u_n(x)| \leq \varepsilon \quad \forall x \in \mathbb{T}_m. \tag{2.2}$$

Then, for all $x \in \mathbb{T}_m$ and $n \geq n_0$ we have that

$$u_n(y) - \varepsilon \leq u(y) \leq u_n(y) + \varepsilon \quad \forall y \in \mathcal{S}(x).$$

Thus, for all $x \in \mathbb{T}_m$ and $n \geq n_0$,

$$\begin{aligned}
u_n(x) - \varepsilon &= \frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)} u_n(y) + \min_{y\in\mathcal{S}(x)} u_n(y)\right\} + \frac{\beta}{m}\sum_{y\in\mathcal{S}(x)} u_n(y) - \varepsilon \\
&\leq \frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)} u(y) + \min_{y\in\mathcal{S}(x)} u(y)\right\} + \frac{\beta}{m}\sum_{y\in\mathcal{S}(x)} u(y) \\
&\leq \frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)} u_n(y) + \min_{y\in\mathcal{S}(x)} u_n(y)\right\} + \frac{\beta}{m}\sum_{y\in\mathcal{S}(x)} u_n(y) + \varepsilon \\
&= u_n(x) + \varepsilon.
\end{aligned}$$

Taking limit as $n \to +\infty$, we get that

$$u(x) - \varepsilon \leq \frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)} u(y) + \min_{y\in\mathcal{S}(x)} u(y)\right\} + \frac{\beta}{m}\sum_{y\in\mathcal{S}(x)} u(y) \leq u(x) + \varepsilon \quad \forall x \in \mathbb{T}_m.$$

Then, since $\varepsilon$ is arbitrary, we have that

$$u(x) = \frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x)} u(y) + \min_{y\in\mathcal{S}(x)} u(y)\right\} + \frac{\beta}{m}\sum_{y\in\mathcal{S}(x)} u(y) \quad \forall x \in \mathbb{T}_m,$$

that is, $u$ is a $p-$harmonious function. □

The Fatou set $\mathcal{F}(u)$ of a function $u$ is the set of the branches $\pi = (x_1, \dots, x_k, \dots)$ on which

$$\lim_{k\to+\infty} u(x_k)$$

exists and is finite, and $BV(u)$ is the set of the branches $\pi = (x_1, \dots, x_k, \dots)$ on which $u$ has finite variation

$$\sum_{k=1}^{\infty} |u(x_{k+1}) - u(x_k)|.$$

Clearly $BV(u) \subseteq \mathcal{F}(u)$.

Now we use the results of [10] to show that the infimum of Hausdorff dimension of $BV(u)$ and $\mathcal{F}(u)$ are equal over all bounded $p-$harmonious functions on $\mathbb{T}_m$.

**Theorem 2.7.** *Let $\mathcal{H}^m$ be the set of bounded $p-$harmonious functions on $\mathbb{T}_m$. Then*

$$\min_{\mathcal{H}^m} \dim \mathcal{F}(u) = \min_{\mathcal{H}^m} \dim BV(u) = \frac{\log\left(\gamma^{-\frac{m\alpha+2(m-1)\beta}{2m}} + (m-1)\gamma^{\frac{m\alpha+2\beta}{2m}}\right)}{\log m}, \tag{2.3}$$

*where*

$$\gamma = \frac{m\alpha + 2(m-1)\beta}{(m-1)(m\alpha+2\beta)}$$

*and* dim *denotes the usual Hausdorff dimension.*

*Proof.* By Theorem A in [10] we have that

$$\min_{\mathcal{H}} \ \dim\ \mathcal{F}(u) = \min_{\mathcal{H}} \ \dim\ BV(u) = \frac{\log f(m)}{\log m},$$

where

$$f(m) = \min\left\{\sum_{j=1}^{m} e^{x_j} : x \in \mathbb{R}^m \text{ s. t.} \frac{\alpha}{2}\left(\max_{1\leq j\leq m} x_j + \min_{1\leq j\leq m} x_j\right) + \frac{\beta}{m}\sum_{j=1}^{m} x_j = 0\right\}.$$

We obseve that the minimum $f(m)$ is attained at

$$x_1 = -\frac{\alpha m + 2(m-1)\beta}{2m}\log\gamma, \qquad x_j = \frac{m\alpha + 2\beta}{2m}\log\gamma, \qquad 2 \le j \le m,$$

with value

$$\gamma^{-\frac{m\alpha+2(m-1)\beta}{2m}} + (m-1)\gamma^{\frac{m\alpha+2\beta}{2m}},$$

which completes the proof. $\square$

*Remark* 2.8. In [11], for the classical discretization of the $p-$Laplacian on trees,

$$\sum_{y\in S(x)} |u(x)-u(y)|^{p-2}(u(x)-u(y)) = 0,$$

the authors prove that

$$\lim_{m\to+\infty} \min_{\mathcal{H}^m} \dim \mathcal{F}(u) = \lim_{m\to+\infty} \min_{\mathcal{H}^m} \dim BV(u) = 1$$

for all $p > 1$. In our case, we can observe that, when $\alpha = 0$, we have that $\gamma = 1$ and therefore

$$\min_{\mathcal{H}^m} \dim \mathcal{F}(u) = \min_{\mathcal{H}^m} \dim BV(u) = 1$$

for all $m \in \mathbb{N}_{\ge 2}$. On the other hand, when $\alpha \neq 0$, if we rewrite (2.3) as

$$\left(\frac{1}{2} + \frac{(m-2)\beta}{2m}\right)\left(1 + \frac{\log\left(\frac{(1-\frac{1}{m})(\alpha+\frac{2\beta}{m})}{(\alpha+\frac{2(m-1)\beta}{m})}\right)}{\log(m)}\right) + \frac{\log\left(\frac{2}{(\alpha+\frac{2\beta}{m})}\right)}{\log(m)}$$

and take limit as $m \to +\infty$, we obtain that

$$\lim_{m\to+\infty} \min_{\mathcal{H}^m} \dim \mathcal{F}(u) = \lim_{m\to+\infty} \min_{\mathcal{H}^m} \dim BV(u) = \frac{1}{2} + \frac{\beta}{2}.$$

## 3. The Dirichlet Problem and a Tug-of-War Game

First, let us introduce what we understand by the Dirichlet problem for $p-$harmonious functions.

**Dirichlet Problem** $(DP)$. Given $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and a continuous function $F : [0,1] \to \mathbb{R}$, find a $p-$harmonious function $u$ such that

$$\lim_{k\to+\infty} u(x_k) = F(\pi) \quad \forall \pi = (x_1, x_2, \dots, x_k, \dots) \in \partial\mathbb{T}_m.$$

We say that $v$ is a supersolution of $(DP)$ if $v$ is $p-$superharmonious and

$$\lim_{k\to+\infty} v(x_k) \ge F(\pi) \quad \forall \pi = (x_1, x_2, \dots, x_k, \dots) \in \partial\mathbb{T}_m.$$

We say that $v$ is a subsolution of $(DP)$ if $v$ is $p-$subharmonious and

$$\lim_{k\to+\infty} v(x_k) \le F(\pi) \quad \forall \pi = (x_1, x_2, \dots, x_k, \dots) \in \partial\mathbb{T}_m.$$

First, we want to show that the $(DP)$ has a unique solution. To this end we use the Tug-of-War game introduced in [21], see also [19]. Now we describe the game and refer to [14] for more details and references. It is a two player zero sum game. Starting with a token at a vertex $x_0 \in \mathbb{T}_m$, the players flip a biased coin with probabilities $\alpha$ of getting a head and $\beta$ of a tail, $\alpha + \beta = 1$. If they get a head (probability $\alpha$), they toss a second coin (a fair coin this time with probabilities $1/2$ and $1/2$) to decide who move the token. If the outcome of the second toss is heads, then Player I moves the token to any $x_1 \in \mathcal{S}(x_0)$. In the case of tails, Player II gets to move the token to any $x_1 \in \mathcal{S}(x_0)$. In the other case, that is, if they get tails in the first coin toss (probability $\beta$), the game state moves according to the uniform probability density to a random vertex $x_1 \in \mathcal{S}(x_0)$. They continue playing the game forever, generating an infinite sequence $\pi = (x_0, x_1, \dots, x_k, \dots)$ where $x_k \in \mathcal{S}(x_{k-1})$ for any $k \in \mathbb{N}$, therefore $\pi \in \partial\mathbb{T}_m$. Then Player I receive from Player II the amount $F(\pi)$, where $F$ is a continuous function from $[0,1]$ to $\mathbb{R}$. This is the reason why we will refer to $F$ as the final payoff function. Now we define the expected payoff for an individual game. First, a strategy $S_I$ for Player I is a collection of measurable mappings $S_I = \{S_I^k\}_{k\in\mathbb{N}}$ such that the next game position is given by

$$S_I^{k+1}(x_0, x_1, \dots, x_k) = x_{k+1} \in \mathcal{S}(x_k)$$

if Player I wins the toss given a partial history $(x_0, x_1, \dots, x_k)$. Similarly, Player II plays according to a strategy $S_{II}$. We can observe that the next game position $x_{k+1} \in S(x_k)$, given a partial history $(x_0, \dots, x_k)$, is distributed according to the probability

$$q_{S_I,S_{II}}(x_0, \dots, x_k, A) = \frac{\alpha}{2}\delta_{S_I^k(x_0,x_1,\dots,x_k)}(A) + \frac{\alpha}{2}\delta_{S_{II}^k(x_0,x_1,\dots,x_k)}(A) + \frac{\beta}{m}\#(A \cap S(x_k)),$$

where $A$ is a subset of $\mathbb{T}_m$ and $\#(A \cap S(x_k))$ denotes the cardinal of the set $A \cap S(x_k)$. Strategies $S_I$ and $S_{II}$ together with an initial state $x_0$ determine a unique probability measure $\mathbb{P}^{x_0}_{S_I,S_{II}}$ in $[0,1]$. For the precise definition of $\mathbb{P}^{x_0}_{S_I,S_{II}}$ we refer to [18]. We define the expected payoff of an individual game as

$$\mathbb{E}^{x_0}_{S_I,S_{II}}[F] = \int_0^1 F(y)\,\mathbb{P}^{x_0}_{S_I,S_{II}}(dy).$$

We also define the value of the game for Player I as

$$u_I(x_0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I,S_{II}}[F]$$

and the value of the game for Player II as

$$u_{II}(x_0) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}}[F].$$

The value $u_I(x_0)$ and $u_{II}(x_0)$ are in a sense the best expected outcomes each player can almost guarantee when the game starts at $x_0$. For more details on values of games, we refer to [15, 22].

The following theorem states that the game has a value, i.e. $u_I = u_{II}$, and this value is a solution of $(DP)$. For a detailed proof of the existence of a value see [14] and, by an argument completely similar to the proof of Theorem 3.4 in [18], we have that the game value is a solution of $(DP)$.

**Theorem 3.1.** *Let $F : [0,1] \to \mathbb{R}$ be a continuous function. Then the game with payoff function $F$ has a value $u$. Furthermore, $u$ is a solution of $(DP)$ with boundary data $F$.*

To see the form of game values $u$ (solutions to $(DP)$) let us mention that in [22], an explicit formulae for $\mathbb{P}^{x_0}_{S_I,S_{II}}$ is given when $F$ is monotone, and therefore we have an explicit formulae for $u$. In the next section, we will show how to approximate $u$ in the general case.

From now on, we assume that $F : [0,1] \to \mathbb{R}$ is a continuous function. Next we show a comparison principle.

**Theorem 3.2.** *Let $G : [0,1] \to \mathbb{R}$ be a continuous function and $v$ be a bounded supersolution of $(DP)$ with boundary data $G$ such that $G \geq F$ in $[0,1]$, then*

$$v(x) \geq u(x)$$

*for any $x \in \mathbb{T}_m$, where $u$ is the value of game with final payoff function $F$.*

*Proof.* First, we show that by choosing a strategy according to the minimal values of $v$, Player II can make the process a supermartingale. More precisely, Player I follows any strategy and Player II follows the following strategy, that we will call $S^0_{II}$ : at $x_{k-1} \in \mathbb{T}_m$ he chooses to step to a vertex that minimizes $v$, i.e. a vertex $x_k \in \mathcal{S}(x_{k-1})$ such that

$$v(x_k) = \min_{y \in \mathcal{S}(x_{k-1})} v(y).$$

We start from a vertex $x_0$. Using that $v$ is a supersolution of $(DP)$ and the estimated the strategy of Player I by the supremum, we have that

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_I, S^0_{II}}[v(X_k)|x_0, \dots, x_{k-1}] \\
&\leq \frac{\alpha}{2}\left\{\min_{y \in \mathcal{S}(x_{k-1})} v(y) + \max_{y \in \mathcal{S}(x_{k-1})} v(y)\right\} + \frac{\beta}{m} \sum_{y \in \mathcal{S}(x_{k-1})} v(y) \\
&\leq v(x_{k-1}).
\end{aligned}$$

Thus $M_k = v(X_k)$ is a supermartingale. From this fact, using Theorem 4.2.2 in [14], the Optional Stopping Theorem, and that $G \geq F$ in $[0,1]$, we get the desired result. □

Moreover, we have an analogous result for bounded subsolutions of $(DP)$.

**Theorem 3.3.** *Let $G : [0,1] \to \mathbb{R}$ be a bounded function and $v$ be a bounded subsolution of $(DP)$ with boundary data $G$ such that $G \leq F$ in $[0,1]$, then*

$$v(x) \leq u(x)$$

*for any $x \in \mathbb{T}_m$, where $u$ is the value of the game with final payoff function $F$.*

*Proof.* The proof is similar to the previous one. □

Then, we arrive to the main result of this section.

**Theorem 3.4.** *There exists a unique bounded solution of $(DP)$ with given boundary data $F$. Moreover, it coincides with the value of the game.*

*Proof.* Theorem 3.1 gives that the value of the game is a solution of $(DP)$. This proves existence. Theorems 3.2 and 3.3 imply uniqueness. □

The above theorem, together with Theorems 3.2 and 3.3, give the Comparison Principle for solutions of $(DP)$.

**Theorem 3.5** (Comparison Principle)**.** *Let $F, G : [0,1] \to \mathbb{R}$ be bounded functions. If $v$ is a bounded supersolution (subsolution) of $(DP)$ with boundary data $G$, $u$ is the solution of $(DP)$ with boundary data $F$ and $F \leq G$ ($F \geq G$) in $[0,1]$, we have that $u \leq v$ ($u \geq v$) in $\mathbb{T}_m$.*

## 4. A Numerical Approximation

In this section we give a numerical approximation for the solutions of $(DP)$ when the boundary datum $F$ is a continuous function.

Let $F$ be a real-valued function on $[0,1]$ and $n \in \mathbb{N}$, we define $F_n : [0,1] \to \mathbb{R}$ as

$$F_n(t) = \sum_{j=0}^{m^n-1} F(t_{nj}) \chi_{(t_{nj}, t_{n(j+1)})}(t)$$

where $t_{nj} = j/m^n$ for all $j \in \{0, \dots, m^n\}$. Note that this function is piecewise constant.

Our next goal is to construct a solution $u_n$ of $(DP)$ with boundary data $F_n$. We first observe that, for all $j \in \{0, \dots, m^n - 1\}$ there exists $x_{nj} \in \mathbb{T}_m^n$ such that $I_{x_{nj}} = (t_{nj}, t_{n(j+1)})$. Then, for all $k \in \{1, \dots, n\}$, we take $\{x_{(n-k)j}\}_{j=0}^{m^{n-k}-1} \subset \mathbb{T}_m$ such that

$$\mathcal{S}(x_{(n-k)j}) = \{x_{(n-k+1)\tau} \colon 1 + (j-1)m \leq \tau \leq jm\} \quad \forall j \in \{0, \dots, m^{n-k} - 1\}.$$

Let $u_n : \mathbb{T}_m \to \mathbb{R}$ such that

$$u_n(y) = F(t_{nj}) \quad \forall y \in \mathbb{T}_m \text{ such that } I_y \subset I_{x_{nj}} \text{ for some } j \in \{1, \dots, m^n - 1\},$$

and for any $k \in \{1, \dots, n\}$

$$u_n(x_{(n-k)j}) = \frac{\alpha}{2}\left\{\max_{y \in \mathcal{S}(x_{(n-k)j})} u(y) + \min_{y \in \mathcal{S}(x_{(n-k)j})} u(y)\right\} + \frac{\beta}{m} \sum_{y \in \mathcal{S}(x_{(n-k)j})} u(y)$$

for all $j \in \{0, \dots, m^{n-k} - 1\}$. It is easy to check that $u_n$ is a solution of $(DP)$ with boundary data $F_n$. Moreover, if $F$ is bounded then $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded on $\mathbb{T}_m$.

*Remark* 4.1. Let $F$ be a continuous function on $[0, 1]$. Then, given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|F(x) - F(y)| \leq \frac{\varepsilon}{2} + \frac{2\|F\|_\infty}{\delta}|x - y|$$

for all $x, y \in [0, 1]$.

We are now ready to state the main result of this section.

**Theorem 4.2.** *Let $F : [0, 1] \to \mathbb{R}$ be a continuous function. Then the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges uniformly to the solution $u$ of $(DP)$ with boundary data $F$. Moreover, if $F$ is a Lipschitz function we have a bound for the error, it holds that*

$$|u_n(x) - u(x)| \leq \frac{L}{m^n}$$

*for all $x \in \mathbb{T}_m$, where $L$ is the Lipschitz constant of $F$.*

*Proof.* We present two proofs of this result. The first proof only uses game theory to show uniqueness and can be viewed as an alternative way to prove existence of a solution.

This first proof we will be divided into 4 steps.

**Step 1.** Since $F$ is a continuous function on $[0, 1]$, by Remark 4.1, given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|F(x) - F(y)| \leq \frac{\varepsilon}{2} + \frac{2\|F\|_\infty}{\delta}|x - y|$$

for all $x, y \in [0, 1]$. Therefore

$$|F_n(x) - F(y)| \leq \frac{\varepsilon}{2} + \frac{2\|F\|_\infty}{\delta m^n} \quad \forall x, y \in \mathbb{T}_m \quad \forall n \in \mathbb{N}.$$

Then $\{F_n\}_{n \in \mathbb{N}}$ converges uniformly to $F$.

**Step 2.** We will prove that $\{u_n\}_{n \in \mathbb{N}}$ is an uniformly Cauchy sequence.

Let $h, k, n \in \mathbb{N}$ and $x \in \mathbb{T}_m^h$. If $n \leq k \leq h$, there exist $i \in \{0, \dots, m^n - 1\}$ and $j \in \{0, \dots, m^k - 1\}$ such that $u_n(x) = F(t_{ni})$ and $u_k(x) = F(t_{kj})$. Moreover $I_x \subset I_{x_{kj}} \subset I_{x_{ni}}$. Then, given $\varepsilon > 0$, using Remark 4.1, we have that

$$|u_n(x) - u_k(x)| \leq |F(t_{ni}) - F(t_{kj})| \leq \frac{\varepsilon}{2} + \frac{2\|F\|_\infty}{\delta m^n} \quad \forall x \in \mathbb{T}_m^h.$$

Thus, there exists $n_0$ such that if $n \geq n_0$,

$$|u_n(x) - u_k(x)| \leq \varepsilon \quad \forall x \in \mathbb{T}_m^h. \tag{4.4}$$

For all $x \in \mathbb{T}_m^{k-1}$, by (4.4), we have that

$$u_k(y) - \varepsilon \leq u_n(y) \leq u_k(y) + \varepsilon \quad \forall y \in \mathcal{S}(x).$$

Then

$$u_k(x) - \varepsilon \leq u_n(x) \leq u_k(x) + \varepsilon \quad \forall x \in \mathbb{T}_m^{k-1},$$

i.e.,

$$|u_n(x) - u_k(x)| \leq \varepsilon \quad \forall x \in \mathbb{T}_m^{k-1}.$$

In the same manner, in $k-1-$steps, we can see that

$$|u_n(x) - u_k(x)| \leq \varepsilon \quad \forall x \in \mathbb{T}_m.$$

Therefore $\{u_n\}_{n\in\mathbb{N}}$ is an uniformly Cauchy sequence.

**Step 3.** Now, we will show that

$$u(x) = \lim_{n\to+\infty} u_n(x) \quad \forall x \in \mathbb{T}_m$$

is the solution of $(DP)$ with boundary data $F$.

By step 2, $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to $u$. Therefore, by Lemma 2.6, $u$ is a $p-$harmonious function. Then we only need to show that

$$\lim_{k\to+\infty} u(x_k) = F(\pi) \quad \forall \pi = (x_1, x_2, \dots, x_k, \dots) \in \partial\mathbb{T}_m.$$

Let $\varepsilon > 0$ and $\pi = (x_1, x_2, \dots, x_k, \dots) \in \partial\mathbb{T}_m$. Since $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to $u$, there exists $n_0 = n_0(\varepsilon)$ such that

$$|u_n(x_j) - u(x_j)| < \frac{\varepsilon}{3} \quad \forall j \in \mathbb{N}, \tag{4.5}$$

for any $n \geq n_0$. On the other hand, we can observe that there exists $n_1 = n_1(\varepsilon)$ such that

$$|F_n(\pi) - F(\pi)| < \frac{\varepsilon}{3} \quad \forall n \geq n_1. \tag{4.6}$$

Finally, taking $n \geq \max\{n_0, n_1\}$, since $u_n$ is the solution of $(DP)$ with boundary data $F_n$, there exists $j_0 = j_0(n, \pi, \varepsilon)$ such that

$$|u_n(x_j) - F_n(\pi)| < \frac{\varepsilon}{3} \quad \forall j \geq j_0. \tag{4.7}$$

Then, if we take $j \geq j_0$, by (4.5), (4.7) and (4.6) we have that

$$|u(x_j) - F(\pi)| \leq |u(x_j) - u_n(x_j)| + |u_n(x_j) - F_n(\pi)| + |F_n(\pi) - F(\pi)| < \varepsilon \quad \forall j \geq j_0.$$

Therefore,

$$\lim_{k\to+\infty} u(x_k) = F(\pi) \quad \forall \pi = (x_1, x_2, \dots, x_k, \dots) \in \partial\mathbb{T}_m.$$

**Step 4.** We observe that if $F$ is a Lipschitz function, in the same manner as in step 2, we obtain that, if $k, n \in \mathbb{N}$,

$$|u_n(x) - u_k(x)| \leq \frac{L}{m^n} \quad \forall x \in \mathbb{T}_m.$$

Therefore,

$$|u_n(x) - u(x)| \leq \frac{L}{m^n} \quad \forall x \in \mathbb{T}_m,$$

where $L$ is the Lipschitz constant of $F$. This completes the first proof.

Now we proceed with the second proof of this result. This proof is shorter but we use here the existence and comparison results proved in the previous section using game theory.

We first observe that, since $F$ is a continuous function on $[0,1]$, using Remark 4.1, we have that given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|F(x) - F(y)| \leq \frac{\varepsilon}{2} + \frac{2\|F\|_\infty}{\delta}|x - y|$$

for all $x, y \in [0,1]$. Then, using that $F_n$ is a continuous function on $I_{x_{n,j}}$ for all $j \in \{0, \dots, m^n - 1\}$, (4.1) and Theorem 3.5, we have that for any $n \in \mathbb{N}$

$$u_n(x) - \varepsilon \leq u(x) \leq u_n(x) + \varepsilon$$

for all $x \in \mathbb{T}_m$ such that $I_x \subset I_{x_{n,j}}$ for some $j \in \{0, \dots, m^n - 1\}$, where $u$ is the solution of $(DP)$ with boundary data $F$. By the above inequality and using that $u_n$ and $u$ are $p-$harmonious functions, we have that

$$u_n(x) - \varepsilon \leq u(x) \leq u_n(x) + \varepsilon \quad \forall x \in \mathbb{T}_m, \quad \forall n \in \mathbb{N}.$$

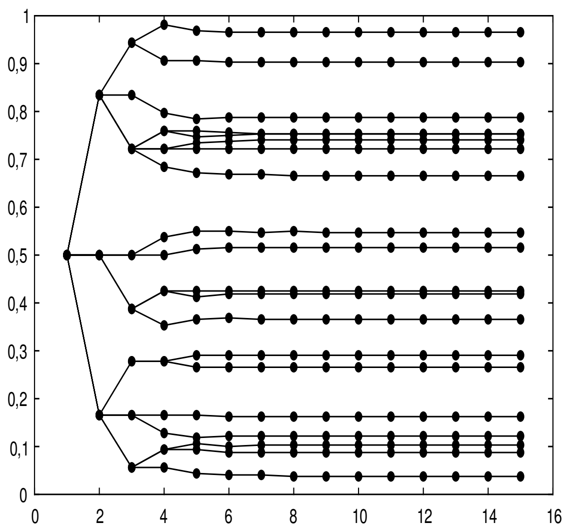
Therefore the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to $u$. □

*Example* 4.3. Case $p = \infty$. Let $m = 3$, $\alpha = 1$, $\beta = 0$ and $F : [0,1] \to \mathbb{R}$ given by $F(t) = t$. In [22], the author proves that the solution of $(DP)$ with boundary data $F$ is

$$u(x) = \int_{I_x} t \, d\mathcal{C}^x(t) \qquad \forall x \in \mathbb{T}_3,$$

where $\mathcal{C}^x$ is the Cantor measure on the interval $I_x$ with $\mathcal{C}^x(I_x) = 1$.

We first observe that $u(x)$ is the middle point of $I_x$ for all $x \in \mathbb{T}_3$. Now, we make a comparison between the real solution $u$ and our approximate solution $u_n$.



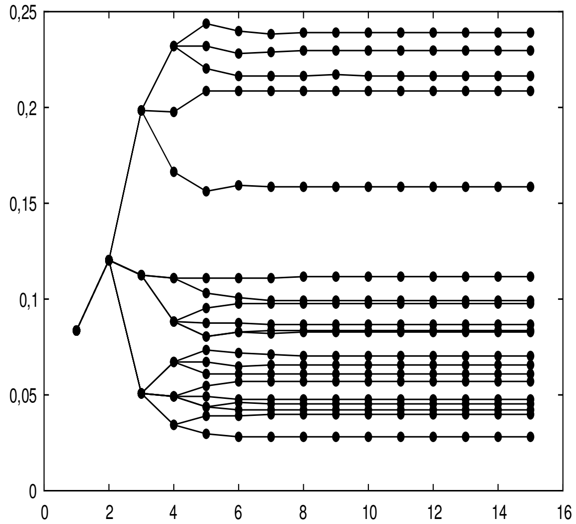

$n = 14$, number of branches $= 20$, • $u$, — $u_n$.

*Example* 4.4. Case $p = 2$. Let $m = 3$, $\alpha = 0$, $\beta = 1$ and $F : [0,1] \to \mathbb{R}$ given by $F(t) = (t - 1/2)^2$. In this case, the solution $u$ of $(DP)$ is

$$u(x) = \frac{1}{|I_x|} \int_{I_x} \left(t - \frac{1}{2}\right)^2 dt \quad \forall x \in \mathbb{T}_3,$$

where $|I_x|$ is the measure of $I_x$.

In the next figure, taking only the branch $\pi$ such that $\pi_1 = 2$, we make a comparison between the real solution $u$ and our approximate solution $u_n$.




$n = 14$, number of branches $= 20$, $\bullet$ $u$, — $u_n$.

## 5. Unique continuation property

In this section we prove our main result, that is a characterization for the subsets of $\mathbb{T}_m$ that have the unique continuation property.

**Definition 5.1.** We say that a subset $U$ of $\mathbb{T}_m$ satisfies the unique continuation property ($UCP$) if for any bounded $p-$harmonious function $u$ such that $u = 0$ in $U$, we have that $u \equiv 0$ in $\mathbb{T}_m$.

Let us first prove that the density of the set $\psi(U)$ in $[0, 1]$ is a necessary condition for $UCP$.

**Theorem 5.2.** *If $U \subset \mathbb{T}_m$ satisfies UCP then $\psi(U)$ is dense in $[0, 1]$.*

*Proof.* We will show that if $\psi(U)$ is not dense in $[0, 1]$, then there exists a $p-$harmonious function $u$ such that $u \neq 0$ in $\mathbb{T}_m$ and $u = 0$ in $U$.

Since $\psi(U)$ is not dense in $[0, 1]$ there exist $\tau > 0$ and $r \in [0, 1]$ such that

$$(r - \tau, r + \tau) \cap \psi(U) = \emptyset. \tag{5.8}$$

Then there exist $k \in \mathbb{N}$ and $x = (a_1, \dots, a_k) \in \mathbb{T}_m$ such that $1/m^k < \tau$ and $I_x \subset (r - \tau, r + \tau)$. Therefore, using (5.8) and the fact that $I_x$ is the subset of $\partial\mathbb{T}_m$ consisting of all branches that start at $x$, we have that $(x, b_1, \dots, b_s) \notin U$ for all $s \in \mathbb{N}$. Now, we construct $u$ as follows

$$u(y) = \begin{cases} 1 & \forall y \in \mathbb{T}_m \text{ such that } I_y \subset I_{(x,0)}, \\ -1 & \forall y \in \mathbb{T}_m \text{ such that } I_y \subset I_{(x,m-1)}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $u$ is a bounded $p-$harmonious function such that $u = 0$ in $U$ and $u \neq 0$. This finishes the proof. $\square$

**Proposition 5.3.** *Let $U$ be a subset of $\mathbb{T}_m$. If $U$ satisfies the following property*

(PA) *There exists $n \in \mathbb{N}$ such that for all $x \in \mathbb{T}_m$ there exist $l \in \{1, \dots, n\}$ and at least one branch starting at $x$ such that its $l-$th node belongs to $U$,* *then $U$ satisfies UCP.*

*Remark* 5.4. Let $U$ be a subset of $\mathbb{T}_m$. It is easy to see that if $U$ satisfies PA, then $\psi(U)$ is dense in $[0,1]$.

*Proof of Proposition 5.3.* Let $u$ be a bounded $p-$harmonious function such that $u=0$ in $U$. Set $M=\sup\{u(x)\colon x\in\mathbb{T}_m\}$ and $\delta=({}^{\alpha}/_{2}+{}^{\beta}/_{m})$. Given $\varepsilon>0$ there exists $x_0\in\mathbb{T}_m$ such that $u(x_0)\geq M-\varepsilon$. Thus, since $u$ is a $p-$harmonious function, we have that

$$\begin{aligned}M-\varepsilon\leq u(x_0)&=\frac{\alpha}{2}\left\{\max_{y\in\mathcal{S}(x_0)}u(y)+\min_{y\in\mathcal{S}(x_0)}u(y)\right\}+\frac{\beta}{m}\sum_{y\in\mathcal{S}(x_0)}u(y)\\&\leq\left(\frac{\alpha}{2}+\frac{m-1}{m}\beta\right)M+\left(\frac{\alpha}{2}+\frac{\beta}{m}\right)\min_{y\in\mathcal{S}(x_0)}u(y).\end{aligned}$$

Then

$$M-\frac{\varepsilon}{\delta}\leq\min_{y\in\mathcal{S}(x_0)}u(y)\leq u(y)$$

for all $y\in\mathcal{S}(x)$.

On the other hand, since $U$ satisfies PA, there exist $l\in\{1,\dots,n\}$ and $(x_0,a_1,\dots,a_l)\in U$ where $a_k\in\{0,\dots,m-1\}$ for all $1\leq k\leq l$. Then, using that $x_1=(x_0,a_1)\in\mathcal{S}(x_0)$ and the above inequality, we get

$$M-\frac{\varepsilon}{\delta}\leq u(x_1).$$

Similarly, we have

$$M-\frac{\varepsilon}{\delta^k}\leq u(x_k)\quad\forall k\in\{2,\dots,l\}$$

where $x_k=(x_{k-1},a_k)$, $2\leq k\leq l$. Then, using that $x_l=(x_{l-1},a_l)=(x_0,a_1,\dots,a_l)\in U$, we have that

$$M\delta^l\leq\varepsilon.$$

Let us now suppose that $M\geq 0$. Using that $l\leq n$, $0<\delta<1$ and the above inequality, we have that

$$M\delta^n\leq\varepsilon\quad\forall\varepsilon>0,$$

then $M=0$. Thus we have that $M\leq 0$.

In the same manner we can show that $N=\inf\{u(x)\colon x\in\mathbb{T}_m\}\geq 0$. Therefore, $M=N=0$, which proves the theorem. □

**Definition 5.5.** Let $U$ be a subset of $\mathbb{T}_m$ such that $\mathbb{T}_m^n\setminus U\neq\emptyset$ for all $n\in\mathbb{N}$. We define the sequence $\{\rho_k(U)\}_{k\in\mathbb{N}}\subset\mathbb{N}$ as follows:

$$\rho_1(U)\colon=\min\{n\in\mathbb{N}\colon\exists x\in\mathbb{T}_m^n\cap U\},$$

and for all $k\in\mathbb{N}_{\geq 2}$,

$$\rho_k(U)\colon=\min\{n\in\mathbb{N}\colon\exists y\in\mathbb{T}_m^{\eta_{k-1}(U)}\setminus U\text{ and }x\in\mathbb{T}_m^{\eta_{k-1}(U)+n}\cap U\text{ s. t. }I_x\subset I_y\},$$

where

$$\eta_{k-1}(U)=\sum_{j=1}^{k-1}\rho_j(U).$$

In addition, for all $k\in\mathbb{N}_{\geq 2}$, we define the sets

$$\mathcal{A}_k(U)\colon=\left\{y\in\mathbb{T}_m^{\eta_{k-1}(U)}\setminus U\colon I_y\cap I_{x_j}=\emptyset\ \forall j\in\{1,\dots,k-1\}\right\}.$$

We will write simply $\rho_k$, $\eta_{k-1}$ and $\mathcal{A}_k$ when no confusion arises.

We can now formulate our main result.

**Theorem 5.6.** *Let $U$ be a subset of $\mathbb{T}_m$ such that $\psi(U)$ is dense in $[0,1]$, $\mathbb{T}_m^n \setminus U \neq \emptyset$ for all $n \in \mathbb{N}$ and $U$ satisfies the following properties*

- (P1) *There exists a unique $x_1 \in U \cap \mathbb{T}_m^{\rho_1}$.*
- (P2) *For all $k \in \mathbb{N}_{\geq 2}$ and for all $y \in \mathcal{A}_{k-1}$ there exists a unique $x \in \mathbb{T}_m^{\eta_{k-1}+\rho_k} \cap U$ such that $I_x \subset I_y$.*

*Then $U$ satisfies UCP if only if*

$$\sum_{k=1}^{\infty} \delta^{\rho_k} = +\infty$$

*where $\delta = 1-\theta$, $\theta = \frac{\alpha}{2} + \frac{m-1}{m}\beta$.*

*Proof.* We will proceed in two steps.

**Step 1.** First we will prove that if $U$ satisfies UCP, then

$$\sum_{k=1}^{\infty} \delta^{\rho_k} = +\infty.$$

Arguing by contradiction, we suppose that $\sum_{k=1}^{\infty} \delta^{\rho_k} < +\infty$. By (P1), there exists a unique $x_1 = (a_1, \dots, a_{\rho_1}) \in U$ such that $\tau_{1i} = (a_1, \dots, a_i) \notin U$ for any $1 \leq i < \rho_1$. We now construct a $p-$harmonious function $u$ such that $u = 0$ in $U$ as follows:

$$\begin{aligned} &u(\emptyset) = 1, \\ &u(a_1) = m_{11} = \min_{y \in S(\emptyset)} u(y) \\ &u(b_1, \dots, b_j) = M_{11} = \max_{y \in \mathcal{S}(\emptyset)} u(y) \text{ if } b_1 \neq a_1 \quad \forall 1 \leq j \leq \rho_1, \end{aligned}$$

and for any $2 \leq i < \rho_1$

$$\begin{aligned} &u(\tau_{1i}) = m_{1i} = \min_{y \in S(\tau_{1i})} u(y) \\ &u(\tau_{1(i-1)}, b_i, \dots, b_j) = M_{1i} = \max_{y \in \mathcal{S}(\tau_{1(i-1)})} u(y) \text{ if } b_i \neq a_i \quad \forall i \leq j \leq \rho_1. \end{aligned}$$

Since $x_1 \in U$ and we need that $u = 0$ in $U$, we define

$$u(x_1) = 0 = m_{1,\rho_1} = \min_{y \in S(\tau_{1(\rho_1-1)})} u(y).$$

We also take $u(y) = 0$ for all $y \in \mathbb{T}_m$ such that $I_y \subset I_{x_1}$. Thus, in order for $u$ to be a $p-$harmonious function, we need to take $M_{11}, \dots, M_{1,\rho_1}$ and $m_{11}, \dots, m_{1,\rho_1-1}$ such that

$$\begin{aligned} 1 &= \frac{\alpha}{2}(M_{11} + m_{11}) + \frac{\beta}{m}\left((m-1)M_{11} + m_{11}\right), \\ m_{1i} &= \frac{\alpha}{2}(M_{1(i+1)} + m_{1(i+1)}) + \frac{\beta}{m}\left((m-1)M_{1(i+1)} + m_{1(i+1)}\right) \quad \forall 1 \leq i < \rho_1. \end{aligned}$$

Then, we can observe that

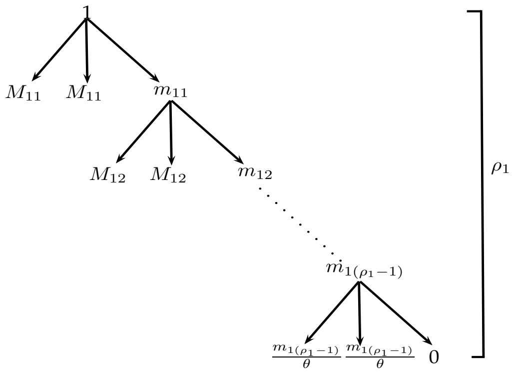
$$\begin{aligned} 1 &= \frac{\alpha}{2}(M_{11} + m_{11}) + \frac{\beta}{m}\left((m-1)M_{11} + m_{11}\right) \\ &= \left(\frac{\alpha}{2} + \frac{m-1}{m}\beta\right) M_{11} + \left(\frac{\alpha}{2} + \frac{\beta}{m}\right) m_{11} \\ &= \theta M_{11} + (1-\theta) m_{11} \end{aligned} \tag{5.9}$$

and in the same manner, we can show that

$$m_{1i} = \theta M_{1(i+1)} + (1-\theta) m_{1(i+1)} \quad \forall 1 \leq i < \rho_1. \tag{5.10}$$

Now, using that $m_{1\rho_1} = 0$, we have that

$$M_{1\rho_1} = \frac{m_{1(\rho_1-1)}}{\theta}.$$




If we take

$$M_{1i} = M_{1\rho_1} = \frac{m_{1(\rho_1-1)}}{\theta} = M_1, \quad \forall 1 \leq i \leq \rho_1, \tag{5.11}$$

by (5.10), we obtain

$$m_{1i} = m_{1(\rho_1-1)} + (1-\theta)m_{1(i+1)} \quad \forall 1 \leq i < \rho_1.$$

Using the above equality, we have

$$m_{1(\rho_1-2)} = m_{1(\rho_1-1)} + (1-\theta)m_{1(\rho_1-1)} = (2-\theta)m_{1(\rho_1-1)},$$

and, for any $2 < j \leq \rho_1 - 1$,

$$m_{1(\rho_1-j)} = \left(\sum_{k=0}^{j-3}(1-\theta)^k + (1-\theta)^{j-2}(2-\theta)\right) m_{1(\rho_1-1)}. \tag{5.12}$$

Thus, by (5.9) and (5.12), we have that

$$m_{1(\rho_1-1)} = \frac{1}{\displaystyle\sum_{k=0}^{\rho_1-3}(1-\theta)^k + (1-\theta)^{\rho_1-2}(2-\theta)}.$$

In addition, since $M_1 = \frac{m_{\rho_1-1}}{\theta}$, we obtain

$$M_1 = \frac{1}{\theta\left(\displaystyle\sum_{k=0}^{\rho_1-3}(1-\theta)^k + (1-\theta)^{\rho_1-2}(2-\theta)\right)}.$$

Then, taking $\delta = 1-\theta$, we get

$$\theta\left(\sum_{k=0}^{\rho_1-3}(1-\theta)^k + (1-\theta)^{\rho_1-2}(2-\theta)\right) = (1-\delta)\sum_{k=0}^{\rho_1-1}\delta^k = 1-\delta^{\rho_1}.$$

Therefore,

$$M_1 = \frac{1}{1-\delta^{\rho_1}}.$$

On the other hand,

$$\mathcal{A}_2 = \{y_j\}_{j=1}^{m^{\rho_1}-1} \text{ and } u(y_j) = M_{11} \quad \forall j \in \{1,\dots,m^{\rho_1-1}\}.$$

Furthermore, by (P2), for all $j \in \{1,\dots,m^{\rho_1}-1\}$ there exists a unique

$$x_2^j = (y_j, a^j_{\rho_1+1}, \dots, a^j_{\rho_1+\rho_2}) \in \mathbb{T}_m^{\rho_1+\rho_2} \cap U$$

with $\tau^j_{2i} = (y_j, a^j_{\rho_1+1}, \dots, a^j_{\rho_1+i}) \notin U$ for any $i \in \{1, \dots, \rho_2\}$.

Let $j \in \{1, \dots, m^{\rho_1} - 1\}$. We define $u$ as follows

$$u(y_j, a^j_{\rho_1+1}) = m_{21} = \min_{y \in S(y_j)} u(y)$$

$$u(y_1, b_{\rho_1+1}, \dots, b_{\rho_1+l}) = M_{21} = \max_{y \in \mathcal{S}(y_j)} u(y) \text{ if } b_{\rho_1+1} \neq a^j_{\rho_1+1} \quad \forall l \in \{1, \dots \rho_2\},$$

and for any $2 \leq i < \rho_2$,

$$u(\tau^j_{2i}) = m_{2i} = \min_{y \in S(\tau^j_{2(i-1)})} u(y),$$

$$u(\tau^j_{2(i-1)}, b_{\rho_1+i}, \dots, b_{\rho_1+j}) = M_{2i} = \max_{y \in \mathcal{S}(\tau_{2(i-1)})} u(y) \text{ if } b_{\rho_1+l} \neq a^j_{\rho_1+i} \quad \forall l \in \{i, \dots, \rho_2\}.$$

Since $x^j_2 \in U$ and we need that $u = 0$ in $U$, we define

$$u(x^j_2) = 0 = m_{2,\rho_2} = \min_{y \in S(\tau^j_{2(\rho_2-1)})} u(y).$$

We also take $u(y) = 0$ for all $y \in \mathbb{T}_m$ such that $I_y \subset I_{x^j_2}$.

Arguing as before, taking

$$M_{2i} = M_{2\rho_2} = \frac{m_{2(\rho_2-1)}}{\theta} = M_2, \quad \forall 1 \leq i \leq \rho_2,$$

we get

$$m_{2(\rho_2-1)} = \frac{M_1}{\displaystyle\sum_{k=0}^{\rho_2-3} (1-\theta)^k + (1-\theta)^{\rho_2-2}(2-\theta)},$$

$$m_{2(\rho_2-l)} = \left( \sum_{k=0}^{l-3} (1-\theta)^k + (1-\theta)^{j-2}(2-\theta) \right) m_{2(\rho_2-1)} \ \forall l \in \{2, \dots, \rho_2 - 1\},$$

and

$$M_2 = \frac{M_1}{1-\delta^{\rho_2}} = \frac{1}{(1-\delta^{\rho_1})(1-\delta^{\rho_2})}.$$

By induction in $k$, we construct $u$ so that $u$ is $p-$harmonious in $\mathbb{T}_m$ such that $u = 0$ in $U$, $u \neq 0$ in $\mathbb{T}_m$ and

$$M_k = \prod_{i=1}^{k} \frac{1}{1-\delta^{\rho_i}} \quad \forall k \in \mathbb{N}.$$

Since

$$\sum_{k=1}^{\infty} \delta^{\rho_k} < +\infty,$$

we have that

$$\sum_{i=1}^{\infty} \frac{\delta^{\rho_k}}{1-\delta^{\rho_k}} < +\infty \Leftrightarrow \sum_{k=1}^{\infty} \log\left(1 + \frac{\delta^{\rho_k}}{1-\delta^{\rho_k}}\right) = \sum_{k=1}^{\infty} \log\left(\frac{1}{1-\delta^{\rho_k}}\right) < +\infty.$$

Thus,

$$\prod_{i=1}^{\infty} \frac{1}{1-\delta^{\rho_k}} < +\infty.$$

Therefore $u$ is a bounded $p-$harmonious function such that $u = 0$ in $U$ and $u \neq 0$ in $\mathbb{T}_m$. This is a contradiction.

**Step 2.** We assume that

$$\sum_{i=1}^{\infty} \delta^{\rho_i} = +\infty$$

and we will prove that $U$ satisfies the $UCP$.

Suppose that there exists a $p-$harmonious function $v \neq 0$ such that $v = 0$ in $U$. We will prove that $v$ is unbounded. Multiplying $v$ by a suitable constant, we can assume that $v(\emptyset) = 1$. Let $u$ be defined as in the above step. First, we need to show that

$$M_k \leq \max\{v(y) \colon y \in \mathbb{T}_m^{\rho_k}\} \qquad \forall k \in \mathbb{N}. \tag{5.13}$$

To this end, we observe that

$$\theta M_1 + (1-\theta) m_{11} = u(\emptyset) = 1 = v(\emptyset) \leq \theta \max_{y \in \mathcal{S}(\emptyset)} v(y) + (1-\theta) \min_{y \in \mathcal{S}(\emptyset)} v(y),$$

then

$$M_1 \leq \max_{y \in \mathcal{S}(\emptyset)} v(y) \quad \text{ or } \quad m_{11} \leq \min_{y \in \mathcal{S}(\emptyset)} v(y).$$

If $M_1 \leq \max_{y \in \mathcal{S}(\emptyset)} v(y)$ then $M_1 \leq \max\{v(y) \colon y \in \mathbb{T}_m^k \text{ with } k \in \{1, \dots, \rho_1\}\}$, and therefore $M_1 \leq \max\{v(y) \colon y \in \mathbb{T}_m^{\rho_1}\}$.

Now we consider the case $M_1 > \max_{y \in \mathcal{S}(\emptyset)} v(y)$ and $m_{11} \leq \min_{y \in \mathcal{S}(\emptyset)} v(y)$.

By (P1), there exists a unique $x_1 = (a_1, \dots, a_{\rho_1}) \in U$ such that $\tau_{1i} = (a_1, \dots, a_i) \notin U$ for any $1 \leq i < \rho_1$. Then, since $m_{11} \leq \min_{y \in \mathcal{S}(\emptyset)} v(y) \leq v(a_1)$, we have that

$$\theta M_1 + (1-\theta) m_{12} = m_{11} \leq v(y_1) \leq \theta \max_{y \in \mathcal{S}(a_1)} v(y) + (1-\theta) \min_{y \in \mathcal{S}(a_1)} v(y),$$

and then

$$M_1 \leq \max_{y \in \mathcal{S}(a_1)} v(y) \quad \text{ or } \quad m_{12} \leq \min_{y \in \mathcal{S}(a_1)} v(y).$$

Again, if $M_1 \leq \max_{y \in \mathcal{S}(a_1)} v(y)$, then we have that $M_1 \leq \max\{v(y) \colon y \in \mathbb{T}_m^{\rho_1}\}$. If $m_{12} \leq \min_{y \in \mathcal{S}(a_1)} v(y) \leq v(\tau_{12})$, then we can prove as before that

$$M_1 \leq \max_{y \in \mathcal{S}(\tau_{12})} v(y) \quad \text{ or } \quad m_{13} \leq \min_{y \in \mathcal{S}(\tau_{12})} v(y).$$

In the same manner, using $\rho_1 - 1$ steps, we show that

$$M_1 \leq \max\{v(y) \colon y \in \mathbb{T}_m^{\rho_1}\} \text{ or } m_{1(\rho-1)} \leq \min_{y \in \mathcal{S}(\tau_{1(\rho-2)})} v(y).$$

If $m_{1(\rho-1)} \leq \min_{y \in \mathcal{S}(\tau_{1(\rho-2)})} v(y) \leq v(\tau_{1(\rho-1)})$, then

$$\theta M_1 = m_{1\rho_1} \leq v(\tau_{1(\rho-1)}) = \theta \max_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y) + (1-\theta) \min_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y).$$

Since $x_1 = (\tau_{1(\rho-1)}, a_{\rho_1}) \in U$ and $v = 0$ in $U$, $\min_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y) \leq 0$ and then

$$M_1 \leq \max_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y).$$

Therefore

$$M_1 \leq \max\{v(y) \colon y \in \mathbb{T}_m^{\rho_1}\}.$$

Then, by induction on $k$, using (P2), we have that (5.13) holds.

Since

$$\sum_{j=1}^{\infty} \delta^{b_j} = +\infty,$$

we have that

$$\lim_{k\to+\infty} M_k = \lim_{k\to+\infty} \prod_{i=1}^{k} \frac{1}{1-\delta^{\rho_i}} = +\infty.$$

Therefore, by (5.13), $v$ is an unbounded. The proof is complete. □

5.1. **Examples.** Below we give some examples of sets verifying (or not) the $UCP$.

*Example* 5.7. Let $U$ be given by

$$U = \bigcup_{k\in\mathbb{N}} \mathbb{T}_m^{2^k}.$$

Then it is clear that $U$ has the $UCP$.

*Example* 5.8. Let $m = 3$ and $U$ be given by

$$U = \{x \in \mathbb{T}_3 :\ x = (a_1, a_2, \ldots, a_n),\ a_i \neq 1, \forall 1 \leq i \leq n\}.$$

It is easy to see that $\psi(U)$ is a Cantor set and therefore $U$ does not have the $UCP$.

*Example* 5.9. Let $U$ be given by

$$U = \{x \in \mathbb{T}_m \colon x = (a_1, a_2, \ldots, a_n),\ a_n = 0\}.$$

Then, since $U$ satisfies (PA) with $n = 1$, $U$ has the $UCP$.

*Example* 5.10. Let $U_1 := \{(0)\}$, $\rho_1 := 1$

$$U_{2n} := \{x \in \mathbb{T}_m^{\mu_{2n-1}+2^{n+1}} \colon x = (y, a_1, \ldots, a_{2^{n+1}}) \colon y \in \mathbb{T}_m^{\mu_{2n-1}} \setminus U_{2n-1}\}, \rho_{2n} := 2^{n+1}$$

$$U_{2n+1} := \{x \in \mathbb{T}_m^{\mu_{2n}+1} \colon x = (y, 0) \colon y \in \mathbb{T}_m^{\mu_{2n}} \setminus U_{2n}\} \text{ and } \rho_{2n+1} := 1$$

for all $n \in \mathbb{N}$, where $\mu_n := \sum_{j=1}^{n} \rho_j$ for all $n \in \mathbb{N}$.

Then $U = \bigcup_{n\in\mathbb{N}} U_n$ is dense and satisfies (P1) and (P2). Since $\sum_{j=1}^{+\infty} \rho_j = \infty$, by Theorem 5.6, $U$ satisfies the $UCP$.

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Leandro M. Del Pezzo and Carolina A. Mosquera
CONICET and Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pabellon I, Ciudad Universitaria (1428), Buenos Aires, Argentina.

*E-mail address*: `ldpezzo@dm.uba.ar, mosquera@dm.uba.ar`

Julio D. Rossi
Departamento de Análisis Matemático, Universidad de Alicante, Ap. correo 99, 03080, Alicante, SPAIN.

*E-mail address*: `julio.rossi@ua.es`