H^2 REGULARITY FOR THE $p(x)-$LAPLACIAN IN TWO-DIMENSIONAL CONVEX DOMAINS

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ABSTRACT. In this paper we study the H^2 global regularity for solutions of the $p(x)-$Laplacian in two dimensional convex domains with Dirichlet boundary conditions. Here $p : \Omega \to [p_1, \infty)$ with $p \in \mathrm{Lip}(\overline{\Omega})$ and $p_1 > 1$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ and let $p : \Omega \to (1, +\infty)$ be a measurable function. In this work, we study the H^2 global regularity of the weak solution of the following problem

$$\begin{cases} -\Delta_{p(x)} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$Laplacian. The hypothesis over p, f and g will be specified later.

Note that, the $p(x)-$Laplacian extends the classical Laplacian ($p(x) \equiv 2$) and the $p-$Laplacian ($p(x) \equiv p$ with $1 < p < +\infty$). This operator has been recently used in image processing and in the modeling of electrorheological fluids, see [3, 5, 24].

Motivate by the applications to image processing problem, in [8], the authors study two numerical methods to approximate solutions of the type of (1.1). In Theorem 5.1, the authors prove the convergence in $W^{1,p(\cdot)}(\Omega)$ of the conformal Galerking finite element method. It is of our interest to study, in a future work, the rate of this convergence. In general, all the error bounds depend on the global regularity of the second derivatives of the solutions, see for example [6, 22]. However, there appear to be no existing regularity results in the literature that can be applied here, since all the results have either a first order or local character.

The H^2 global regularity for solutions of the $p-$Laplacian is studied in [22]. There the authors prove the following: Let $1 < p \leq 2$, $g \in H^2(\Omega)$, $f \in L^q(\Omega)$ ($q > 2$) and u be the unique weak solution of (1.1). Then

- If $\partial\Omega \in C^2$ then $u \in H^2(\Omega)$;
- If Ω is convex and $g = 0$ then $u \in H^2(\Omega)$;

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- If Ω is convex with a polygonal boundary and $g \equiv 0$ then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

Regarding the regularity of the weak solution of (1.1) when $f = 0$, in [1, 7], the authors prove the $C^{1,\alpha}_{loc}$ regularity (in the scalar case and also in the vectorial case). Then, in the paper [15] the authors study the case where the functional has the so called $(p, q)-$ growth conditions. Following these ideas, in [17], the author proves that the solutions of (1.1) are in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ if Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with $C^{1,\gamma}$ boundary, $p(x)$ is a Hölder function, $f \in L^\infty(\Omega)$ and $g \in C^{1,\gamma}(\overline{\Omega})$. While in [4], the authors prove that the solutions are in $H^2_{loc}(\{x \in \Omega\colon p(x) \leq 2\})$ if $p(x)$ is uniformly Lipschitz ($\mathrm{Lip}(\Omega)$) and $f \in W^{1,q(\cdot)}_{loc}(\Omega) \cap L^\infty(\Omega)$.

Our aim, it is to generalized the results of [22] in the case where $p(x)$ is a measurable function. To this end, we will need some hypothesis over the regularity of $p(x)$. Moreover, in all our result we can avoid the restriction $g = 0$, assuming some regularity of $g(x)$.

On the other hand, to prove our results, we can assume weaker conditions over the function f than the ones on [4]. Since, we only assume that $f \in L^{q(\cdot)}(\Omega)$, we do not have a priori that the solutions are in $C^{1,\alpha}(\Omega)$. Then we can not use it to prove the H^2 global regularity. Nevertheless, we can prove that the solutions are in $C^{1,\alpha}(\overline{\Omega})$, after proving the H^2 global regularity.

The main results of this paper are:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geq p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of* (1.1)*. If*

(F1) *$f \in L^{q(x)}(\Omega)$ with $q(x) \geq q_1 > 2$ in the set $\{x \in \Omega\colon p(x) \leq 2\}$;*
(F2) *$f \equiv 0$ in the set $\{x \in \Omega\colon p(x) > 2\}$.*

then $u \in H^2(\Omega)$.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with convex boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geq p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of* (1.1)*. If f satisfies (F1) and (F2) then $u \in H^2(\Omega)$.*

Using the above theorem we can prove the following,

Corollary 1.3. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ with polygonal boundary, p and f as in the previous theorem, $g \in W^{2,q(x)}(\Omega)$ and u be the weak solution of* (1.1) *then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Observe that this result extends the one in [17] in the case where Ω is a polygonal domain in $\mathbb{R}^2$.

Organization of the paper. The rest of the paper is organized as follows. After a short Section 2 where we collect some preliminaries results, in Section 3, we study the H^2- regularity for the non-degenerated problem. In Section 4 we prove Theorem 1.1. Then, in Section 5, we study the regularity of the solution u of (1.1) if Ω is convex. In Section 6, we make some comments on the dependence of the H^2-norm of u on p_1. Lastly, in Appendices A and B we give some results related to elliptic linear equation with bounded coefficients and Lipschitz functions, respectively.

2. Preliminaries

We now introduce the space $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let Ω be a bounded open set of $\mathbb{R}^n$ and $p\colon \Omega \to [1, +\infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_1 := essinf\, p(x)$ and $p_2 := esssup\, p(x)$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\{k > 0\colon \varrho_{p(\cdot)}(u/k) \le 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

For the proofs of the following theorems, we refer the reader to [12].

Theorem 2.1 (Hölder's inequality)**.** *Let $p, q, s : \Omega \to [1, +\infty]$ be a measurable functions such that*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{s(x)} \quad in\ \Omega.$$

Then the inequality

$$\|fg\|_{L^{s(\cdot)}(\Omega)} \le 2\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{q(\cdot)}(\Omega)}$$

for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that, u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

Theorem 2.2. *Let $p'(x)$ such that, $1/p(x) + 1/p'(x) = 1$. Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_1 > 1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.*

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have the following version of Poincare's inequity (see Theorem 3.10 in [21]).

Lemma 2.3 (Poincare's inequity)**.** *If $p : \Omega \to [1, +\infty)$ is continuous in $\overline{\Omega}$, there exists a constant C such that for every $u \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In order to have better properties of these spaces, we need more hypotheses on the regularity of $p(x)$.

We say that p is log-*Hölder continuous* in Ω if there exists a constant C_{log} such that

$$|p(x) - p(y)| \leq \frac{C_{log}}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \forall\, x, y \in \Omega.$$

It was proved in [10], Theorem 3.7, that if one assumes that p is log-Hölder continuous then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ (see also [9, 12, 13, 21, 25]).

We now state the Sobolev embedding Theorem (for the proofs see [12]). Let,

$$p^*(x) := \begin{cases} \frac{p(x)N}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

be the Sobolev critical exponent. Then we have the following,

Theorem 2.4. *Let Ω be a Lipschitz domain. Let $p : \Omega \to [1, \infty)$ and p log-Hölder continuous. Then the imbedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous.*

3. H^2-Regularity for the non-degenerated problem for any dimension

In this section we assume that Ω is a bounded domain in $\mathbb{R}^N$, with $N \geq 2$.

We want to study higher regularity of the weak solution of the regularized equation,

$$\begin{cases} -\operatorname{div}\left((\varepsilon + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u\right) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $0 < \varepsilon \leq 1$, and $f \in \mathrm{Lip}(\Omega)$ and $g \in W^{1,p(\cdot)}(\Omega)$.

The existence of a weak solution of (3.2) holds by Theorem 13.3.3 in [12].

Remark 3.1. Given $\varepsilon \geq 0$, $p \in C^{\alpha_0}(\overline{\Omega})$ for some $\alpha_0 > 0$, and $g \in L^\infty(\Omega)$ we have the following results,

(1) Since $f, g \in L^\infty(\Omega)$, by Theorem 4.1 in [18], we have that $u \in L^\infty(\Omega)$.
(2) By Theorem 1.1 in [17], $u \in C^{1,\alpha}_{loc}(\Omega)$ for some α depending on p_1, p_2, $\|u\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}$. Moreover, given $\Omega_0 \subset\subset \Omega$, $\|u\|_{C^{1,\alpha}(\Omega_0)}$ depends on the same constants and $dist(\Omega_0, \partial\Omega)$.
(3) Finally, by Theorem 1.2 in [17], if $\partial\Omega \in C^{1,\gamma}$ and $g \in C^{1,\gamma}(\partial\Omega)$ for some $\gamma > 0$ then $u \in C^{1,\alpha}(\overline{\Omega})$, where α and $\|u\|_{C^{1,\alpha}(\Omega)}$ depend on $p_1, p_2, N, \|u\|_{L^\infty(\Omega)}, \|p\|_{C^{\alpha_0}(\Omega)}, \alpha_0, \gamma$.

We will first prove the H^2-local regularity assuming only that $p(x)$ is Lipschitz. Then, we will prove the global regularity under the stronger condition that $\nabla p(x)$ is Hölder.

3.1. **H^2-Local regularity.** While we where finishing this paper, we found the work [4], where the authors give a different proof of the H^2-local regularity of the solutions of (3.2). Anyhow, we leave the proof for the completeness of this paper.

Theorem 3.2. *Let $p, f \in \mathrm{Lip}(\Omega)$ with $p_1 > 1$ and u a weak solution of (3.2), then $u \in H^2_{loc}(\Omega)$.*

Proof. First, let us define for any function F and $h > 0$,

$$\Delta^h F(x) = \frac{F(x+\mathbf{h}) - F(x)}{h},$$

where $\mathbf{h} = he_k$ where e_k is a vector of the canonical base of $\mathbb{R}^N$.

Let $\eta(x) = \xi(x)^2 \Delta^h u(x)$ where ξ is a regular function with compact support. Therefore, if we take $v_\varepsilon = (|\nabla u|^2 + \varepsilon)^{1/2}$ and $h < \mathrm{dist}(\mathrm{supp}(\xi), \partial\Omega)$, we have

$$\int_\Omega \langle v_\varepsilon(x)^{p(x)-2} \nabla u(x), \nabla \eta(x)\rangle \, dx = \int_\Omega f(x)\eta(x)\, dx$$

$$\int_\Omega \langle v_\varepsilon(x+\mathbf{h})^{p(x+\mathbf{h})-2} \nabla u(x+\mathbf{h}), \nabla \eta(x)\rangle \, dx = \int_\Omega f(x+\mathbf{h})\eta(x)\, dx.$$

Subtracting, using that $\nabla\eta = 2\xi\nabla\xi\Delta^h u + \xi^2 \Delta^h(\nabla u)$ and dividing by h we obtain,

$$\begin{aligned}
I =& \int_\Omega \langle \Delta^h (v_\varepsilon(x)^{p(x)-2}\nabla u), \Delta^h(\nabla u)\rangle \xi^2 \, dx \\
=& -2\int_\Omega \langle \Delta^h (v_\varepsilon(x)^{p(x)-2}\nabla u), \xi\nabla\xi\Delta^h u\rangle \, dx + \int_\Omega \xi^2 \Delta^h f \Delta^h u \, dx \\
=& 2\int_\Omega \left(\int_0^1 (v_\varepsilon(x+\mathbf{h}t)^{p(x+\mathbf{h}t)-2} \nabla u(x+\mathbf{h}t)\, dt \right) \frac{\partial}{\partial x_k}(\xi\nabla\xi\Delta^h u) dx \\
&+ \int_\Omega \xi^2 \Delta^h f \Delta^h u \, dx \\
=& II + III.
\end{aligned}$$

Now, let as fix a ball B_R such that $B_{3R} \subset\subset \Omega$ and take $\xi \in C_0^\infty(\Omega)$ supported in B_{2R} such that $0 \le \xi \le 1$, $\xi = 1$ in B_R, $|\nabla\xi| \le 1/R$ and $|D^2\xi| \le CR^{-2}$.

By Remark 3.1, there exist a constant $C_1 > 0$ such that $|\nabla u| \le C_1$ in B_{3R}, therefore we get

$$\begin{aligned}
II &\le 2\int_{B_{2R}} \frac{C}{R} |\Delta^h u_{x_k}| \xi \, dx + 2\int_{B_{2R}} \frac{C}{R^2} |\Delta^h u| \, dx \\
&\le \frac{C}{R} \int_{B_{2R}} |\Delta^h(\nabla u)| \xi \, dx + CR^{N-2}.
\end{aligned}$$

On the other hand, since f is Lipschitz we have that,

$$|f(x+\mathbf{h}) - f(x)| \le C_2 h$$

for some constant $C_2 > 0$. This implies that,

$$III \le C_2 R^N.$$

Therefore, summing II and III, and using Young's inequality, we have that for any $\delta > 0$

$$I \leq \delta \int_{B_{2R}} |\Delta^h(\nabla u)|^2 \xi^2 \, dx + C, \tag{3.3}$$

for some constant C depending on R and δ.

On the other hand observe that $I = I_1 + I_2$ where,

$$I_1 = \frac{1}{h} \int_{B_{2R}} \langle (v_\varepsilon(x+\mathbf{h})^{p(x+\mathbf{h})-2} \nabla u(x+\mathbf{h}) - v_\varepsilon(x)^{p(x+\mathbf{h})-2} \nabla u(x)), \Delta^h(\nabla u) \rangle \xi^2 \, dx,$$

and

$$I_2 = \frac{1}{h} \int_{B_{2R}} \langle \Big(v_\varepsilon(x)^{p(x+\mathbf{h})} - v_\varepsilon(x)^{p(x)} \Big) \frac{\nabla u(x)}{v_\varepsilon(x)^2}, \Delta^h(\nabla u) \rangle \xi^2 \, dx.$$

Using that $p(x)$ is Lipschitz and the fact that $|\nabla u(x)| \leq C_1$ we have that, for some b between $p(x+h)$ and $p(x)$,

$$\frac{1}{h} \left| v_\varepsilon(x)^{p(x+\mathbf{h})} - v_\varepsilon(x)^{p(x)} \right| = \left| v_\varepsilon(x)^b \log(v_\varepsilon(x)) \frac{p(x+\mathbf{h}) - p(x)}{h} \right| \leq C,$$

for some constant $C > 0$ depending on $p_1, p_2, \varepsilon, C_1$ and the Lipschitz constant of $p(x)$.

Therefore, we have that

$$-I_2 \leq C C_1 \varepsilon^{-1} \int_{B_{2R}} |\Delta^h(\nabla u)| \xi^2 \, dx.$$

By (3.3), the last inequality and using again Young's inequality we have that, for any $\delta > 0$

$$I_1 \leq \delta \int_{B_{2R}} |\Delta^h(\nabla u)|^2 \xi^2 \, dx + C, \tag{3.4}$$

for some constant $C > 0$ depending on $p_1, p_2, \varepsilon, C_1$ and the Lipschitz constant of $p(x)$.

To finish the proof, we have to find a lower bound for I_1. By a well known inequality, we have that

$$\langle (v_\varepsilon(x+\mathbf{h})^{p(x+h)-2} \nabla u(x+\mathbf{h}) - v_\varepsilon(x)^{p(x+\mathbf{h})-2} \nabla u(x)), (\nabla u(x+\mathbf{h}) - \nabla u(x)) \rangle \geq C_\varepsilon |\nabla u(x+\mathbf{h}) - \nabla u(x)|^2,$$

where

$$C_\varepsilon = \begin{cases} \varepsilon^{p(x+\mathbf{h})-2/2} & \text{if } p(x+\mathbf{h}) \geq 2, \\ (p(x+\mathbf{h})-1)\varepsilon^{p(x+\mathbf{h})-2/2} & \text{if } p(x+\mathbf{h}) \leq 2. \end{cases}$$

Therefore, using that $p_1 > 1$, we arrive at

$$I_1 \geq \int_{B_{2R}} C h^{-2} |\nabla u(x+\mathbf{h}) - \nabla u(x)|^2 \xi^2 \, dx = C \int_{B_{2R}} |\Delta^h(\nabla u(x))|^2 \xi^2 \, dx.$$

Finally combining the last inequality with (3.4) we have that,

$$\int_{B_R} |\Delta^h(\nabla u(x))|^2 \, dx \leq C(N, p, f, \varepsilon).$$

This proves that $u \in H^2_{loc}(\Omega)$. □

3.2. H^2-**Global Regularity.** Now we want to prove that if $f \in \mathrm{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$, the regularized equation (3.2) has a weak solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for an $\alpha \in (0,1)$. We already know, by Remark 3.1, that $u \in C^{1,\alpha}(\overline{\Omega})$. Then, we only need to prove that $u \in C^2(\Omega)$.

Lemma 3.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega \in C^{1,\gamma}$, $p \in C^{1,\beta}(\Omega) \cap C^{\alpha_0}(\overline{\Omega})$, $f \in \mathrm{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$. Then, the Dirichlet Problem (3.2) has a solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.*

Proof. Observe that by Theorem 3.2, we know that the solution is in $H^2_{loc}(\Omega)$. Then for any $\Omega' \subset\subset \Omega$ we can derive the equation and look the solution of (3.2) as the solution of the following equation,

$$\begin{cases} L_\varepsilon u = a(x) & \text{in } \Omega', \\ u = u & \text{on } \partial\Omega'. \end{cases} \tag{3.5}$$

Here,

$$L_\varepsilon u = a^\varepsilon_{ij}(x) u_{x_i x_j}$$

with

$$\begin{aligned} a^\varepsilon_{ij}(x) &= \delta_{ij} + (p(x)-2)\frac{u_{x_i}u_{x_j}}{v_\varepsilon^2}, \quad v_\varepsilon = \left(\varepsilon + |\nabla u|^2\right)^{\frac{1}{2}}, \\ a_\varepsilon(x) &= \ln(v_\varepsilon)\langle \nabla u, \nabla p\rangle + f v_\varepsilon^{2-p}. \end{aligned} \tag{3.6}$$

The operator L_ε is uniformly elliptic in Ω, since for any $\xi \in \mathbb{R}^N$

$$\min\{(p_1-1),1\}|\xi|^2 \le a^\varepsilon_{ij}\xi_i\xi_j \le \max\{(p_2-1),1\}|\xi|^2. \tag{3.7}$$

On the other hand, by Remark 3.1, $u \in C^{1,\alpha}(\overline{\Omega})$. Then, $a^\varepsilon_{ij} \in C^\alpha(\overline{\Omega})$, since $\varepsilon > 0$. Using that $f \in \mathrm{Lip}(\Omega)$, we have that $a \in C^\rho(\Omega)$ where $\rho = \min(\alpha,\beta)$. If $\partial\Omega' \in C^2$, as u is the unique solution of (3.5), by Theorem 6.13 in [19], we have that $u \in C^{2,\rho}(\Omega')$. This ends the proof. □

Remark 3.4. By the H^2 global estimate for linear elliptic equations with $L^\infty(\Omega)$ coefficients in two variables (see Lemma A.1 and (3.7)) we have that,

$$\|u\|_{H^2(\Omega)} \le C\left(\|a_\varepsilon\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\right)$$

where u is the solution of (3.2) and C is a constant independents of ε.

4. Proof of Theorem 1.1

Before proving the theorem, we will need a global bound for the derivatives of the solutions of (3.2).

Lemma 4.1. *Let $f \in L^{q(x)}(\Omega)$ with $q'(x) \le p^*(x)$, $g \in W^{1,p(\cdot)}(\Omega)$, $\varepsilon > 0$ and u_ε be the weak solution of (3.2) then*

$$\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)} \le C$$

where C is a constant depending on $\|f\|_{L^{q(\cdot)}(\Omega)}, \|g\|_{W^{1,p(\cdot)}(\Omega)}$ but not on ε.

Proof. Let

$$J(v) := \int_\Omega \frac{1}{p(x)} (|\nabla v|^2 + \varepsilon)^{p(x)/2}\, dx.$$

By the convexity of J and using (3.2) we have that,

$$\begin{aligned} J(u_\varepsilon) &\le J(g) - \int_\Omega (|\nabla u_\varepsilon|^2 + \varepsilon)^{(p-2)/2} \nabla u_\varepsilon (\nabla g - \nabla u_\varepsilon)\, dx \\ &\le C\left(1 + \int_\Omega f(u_\varepsilon - g)\, dx\right) \\ &\le C\left(1 + \|f\|_{L^{q(\cdot)}(\Omega)} \|u_\varepsilon - g\|_{L^{q'(\cdot)}(\Omega)}\right) \\ &\le C\left(1 + \|f\|_{L^{q(\cdot)}(\Omega)} \|\nabla u_\varepsilon - \nabla g\|_{L^{p(\cdot)}(\Omega)}\right), \end{aligned}$$

where in the last inequality we are using that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ continuously and Poincare's inequality.

Thus we have that there exist a constant independent of ε such that,

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)}\, dx \le C(1 + \|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}),$$

and using the properties of the $L^{p(\cdot)}(\Omega)-$ norms this means that

$$\|\nabla u_\varepsilon\|^m_{L^{p(\cdot)}(\Omega)} \le C(1 + \|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}),$$

for some $m > 1$. Therefore $\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}$ is bounded independent of ε. □

To prove Theorem 1.1, we will use the results of Section 3. Therefore, we will first need to assume that $p \in C^{1.\beta}(\Omega) \cap C(\overline{\Omega})$.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary, $p \in C^{1.\beta}(\Omega) \cap C^{\alpha_0}(\overline{\Omega})$ with $p(x) \ge p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of* (1.1)*. If f satisfies (F1) and (F2) then $u \in H^2(\Omega)$.*

Proof. Let $f_\varepsilon \in \mathrm{Lip}(\Omega)$ and $g_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ such that

$$\begin{aligned} &f_\varepsilon \to f \text{ strongly in } L^{q(\cdot)}(\Omega), \\ &g_\varepsilon \to g \text{ strongly in } H^2(\Omega), \end{aligned}$$

as $\varepsilon \to 0$. Observe that, since $f(x) = 0$ if $p(x) > 2$, we can take $f_\varepsilon \equiv 0$ in $\{x \in \Omega \colon p(x) > 2\}$.

Now, let us consider the solution of (3.2) as the solution of

$$\begin{cases} a^\varepsilon_{11}(x)\frac{\partial^2 u_\varepsilon}{\partial x_1^2} + 2a^\varepsilon_{12}(x)\frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} + a^\varepsilon_{22}(x)\frac{\partial^2 u_\varepsilon}{\partial x_2^2} = a_\varepsilon(x) & \text{in } \Omega, \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega, \end{cases}$$

where $a^\varepsilon_{11}, a^\varepsilon_{22}, a^\varepsilon_{12}, a_\varepsilon$ are defined as in Lemma 3.3, substituting f and g by f_ε and g_ε respectively. By Lemma 3.3 we know that $u_\varepsilon \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

First we will prove the $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $H^2(\Omega)$. By Remark 3.4, we have that

$$\begin{aligned} \|u_\varepsilon\|_{H^2(\Omega)} &\le C(\|a_\varepsilon(x)\|_{L^2(\Omega)} + \|g_\varepsilon\|_{H^2(\Omega)}) \\ &\le C(\|\ln(v_\varepsilon)\nabla u_\varepsilon \nabla p\|_{L^2(\Omega)} + \|f_\varepsilon v^{2-p}\|_{L^2(\Omega)} + \|g_\varepsilon\|_{H^2(\Omega)}). \end{aligned} \tag{4.8}$$

Taking $\Omega_1 = \{x \in \Omega : |\nabla u_\varepsilon(x)| > 1\}$, using that $p(x)$ is Lipschitz and Hölder's inequality, we have

$$\|\ln(v_\varepsilon)\nabla u_\varepsilon \nabla p\|_{L^2(\Omega)} \leq C\|\ln^2(v_\varepsilon)\nabla u_\varepsilon\|^{1/2}_{L^{p'(\cdot)}(\Omega_1)}\|\nabla u_\varepsilon\|^{1/2}_{L^{p(\cdot)}(\Omega_1)} + C. \tag{4.9}$$

On the other hand, since $q(x) \geq q_1 > 2$, we have that $q'(x) \leq p^*(x)$. Then, as $\|f_\varepsilon\|_{L^{q(\cdot)}(\Omega)}$ and $\|g_\varepsilon\|_{H^2(\Omega)}$ are bounded independent of ε, using Lemma 4.1 we conclude that $\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}$ is uniformly bounded.

Observe that, for all $s > 0$ there exist a constant $C > 0$ such that

$$\ln(v_\varepsilon) \leq C v_\varepsilon^{s/2} < C|\nabla u_\varepsilon|^{s/2} \quad \text{ in } \Omega_1,$$

thus

$$\begin{aligned}
\|\ln^2(v_\varepsilon)|\nabla u_\varepsilon|\|_{L^{p'(\cdot)}(\Omega_1)} &\leq C\||\nabla u_\varepsilon|^{1+s}\|_{L^{p'(\cdot)}(\Omega_1)} \\
&\leq C\|\nabla u_\varepsilon\|^{(1+s)}_{L^{p'(\cdot)(1+s)}(\Omega_1)} \\
&\leq C\|u_\varepsilon\|^{(1+s)}_{H^2(\Omega_1)}.
\end{aligned}$$

In the last line, we are using that $2^* = \infty$, since $N = 2$.

Then, by the last inequality, (4.8) and (4.9), we get

$$\|u_\varepsilon\|_{H^2(\Omega)} \leq C\left(\|u_\varepsilon\|^{(1+s)/2}_{H^2(\Omega)} + \|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(\Omega)} + 1\right). \tag{4.10}$$

Taking

$$A_1 = \{x \in \Omega : p(x) = 2\} \quad \text{ and } \quad A_2 = \{x \in \Omega : p(x) < 2\}$$

and using that $f_\varepsilon \equiv 0$ in $\{x \in \Omega : p(x) > 2\}$, we have that

$$\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(\Omega)} \leq \|f_\varepsilon\|_{L^2(A_1)} + \|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)}.$$

Since $\|f_\varepsilon\|_{L^2(A_1)}$ is bounded, to prove that $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $H^2(\Omega)$, we only have to find a bound of $\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)}$.

Let as define in A_2 the function

$$\widetilde{q}(x) = \begin{cases} \frac{1}{2p(x)-3} + 1 & \text{if} \quad \frac{1}{q(x)} + \frac{3}{2} \leq p(x) < 2, \\ \frac{q(x)}{2} + 1 & \text{if} \quad p(x) < \frac{1}{q(x)} + \frac{3}{2}. \end{cases}$$

It is easy to see that $2 < \widetilde{q}(x) \leq q(x)$ for any $x \in A_2$.

On the other hand, let us denote $\mu(x) = \frac{2\widetilde{q}(x)}{\widetilde{q}(x)-2}$ and $\gamma(x) = \mu(x)(2-p(x))$ then

$$1 < 1 + \frac{2}{q_2} \leq \gamma(x) \leq \max\left\{2, 2 + \frac{8}{q_1 - 2}\right\} \quad \forall x \in A_2.$$

Now, using Hölder's inequality with exponent $\widetilde{q}(x)/2$, we have

$$\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)} \leq C\|f_\varepsilon\|_{L^{\widetilde{q}(\cdot)}(A_2)}\|v_\varepsilon^{2-p}\|_{L^{\mu(\cdot)}(A_2)}. \tag{4.11}$$

Then, if $\|v_\varepsilon\|_{L^{\gamma(\cdot)}(A_2)} \leq 1$ we have $\|v_\varepsilon^{2-p}\|_{L^{\mu(\cdot)}(A_2)} \leq 1$ and since $\widetilde{q}(x) \leq q(x)$ we get

$$\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)} \leq C.$$

If $\|v\|_{L^{\gamma(\cdot)}(A_2)} \geq 1$, we have

$$\|v_\varepsilon^{2-p}\|_{L^{\mu(\cdot)}(A_2)} \leq \|v_\varepsilon\|^{2-p_1}_{L^{\gamma(\cdot)}(A_2)} \leq C(1 + \|\nabla u_\varepsilon\|^{2-p_1}_{L^{\gamma(\cdot)}(A_2)}), \tag{4.12}$$

where in the last inequality we are using that $\varepsilon \le 1$.

Since $2^* = \infty$ and $1 < \gamma_1 \le \gamma(x) \le \gamma_2 < \infty$, by the Sobolev embedding inequality, we have that

$$\|\nabla u_\varepsilon\|^{2-p_1}_{L^{\gamma(\cdot)}(A_2)} \le C\|u_\varepsilon\|^{2-p_1}_{H^2(A_2)} \le C\|u_\varepsilon\|^{2-p_1}_{H^2(\Omega)}.$$

Combining this last inequality with inequalities (4.12), (4.11), (4.10) and the fact that $\widetilde{q}(x) \le q(x)$, we get

$$\|u_\varepsilon\|_{H^2(\Omega)} \le C(\|u_\varepsilon\|^{(1+s)/2}_{H^2(\Omega)} + \|u_\varepsilon\|^{2-p_1}_{H^2(\Omega)} + 1).$$

Finally, we get that for any $0 < s < 1$ there exist a constant $C = C(p, g, f, s)$ such that

$$\|u_\varepsilon\|_{H^2(\Omega)} \le C.$$

Then, there exist a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in H^1(\Omega)$ such that

$$\begin{aligned} &u_\varepsilon \to u \text{ strongly in } H^1(\Omega),\\ &u_\varepsilon \rightharpoonup u \text{ weakly in } H^2(\Omega), \end{aligned}$$

It is clear that u satisfies the boundary condition.

Lastly, by Proposition 3.2 in [2], there exist a constant $M > 0$ independent of ε such that,

$$|(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon - (\varepsilon + |\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u| \le M|\nabla(u_\varepsilon - u)|^{p(x)-1} \tag{4.13}$$

for all $x \in \Omega$. Then, passing to the limit in the weak formulation of (3.2) and using the above inequality, we have that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\, dx = \int_\Omega f\varphi\, dx$$

for any $\varphi \in C^\infty_0(\Omega)$. Therefore $u \in H^2(\Omega)$ and solves (1.1). □

Now, we are able to prove the theorem.

Proof of Theorem 1.1. First, we consider the case $p \in C^1(\overline{\Omega})$. Let $p_\varepsilon \in C^\infty(\overline{\Omega})$ such that $p_\varepsilon \to p$ in $C^1(\Omega)$. Now, we define

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{if } p_\varepsilon(x) \le 2,\\ 0 & \text{if } p_\varepsilon(x) > 2. \end{cases} \tag{4.14}$$

Observe that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$ as $\varepsilon \to 0$.

Then, by Theorem 4.2, the solution u_ε of (1.1) (with p_ε and f_ε instead of p and f) is bounded in $H^2(\Omega)$ by a constant independent of ε. Therefore, there exist a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in H^2(\Omega)$ such that

$$\begin{aligned} &u_\varepsilon \to u \quad \text{in } H^1(\Omega),\\ &u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^2(\Omega). \end{aligned} \tag{4.15}$$

It remains to prove that u is a solution of (1.1). Let $\varphi \in C_0^\infty(\Omega)$, then

$$\begin{aligned}\int_\Omega f_\varepsilon \varphi\, dx &= \int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon(x)-2}\nabla u_\varepsilon \nabla\varphi\, dx \\ &= \int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi\, dx \\ &+ \int_\Omega (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2})\nabla u_\varepsilon \nabla\varphi\, dx.\end{aligned} \tag{4.16}$$

Therefore, using that $H^2(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega)$ compactly, we have that

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi\, dx \to \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\, dx. \tag{4.17}$$

On the other hand, we have

$$|\nabla u_\varepsilon(x)|^{p_\varepsilon(x)-1} - |\nabla u_\varepsilon(x)|^{p(x)-1} = |\nabla u_\varepsilon(x)|^{b_\varepsilon(x)} \log(|\nabla u_\varepsilon(x)|)(p_\varepsilon(x) - p(x)),$$

where $b_\varepsilon(x) = p_\varepsilon(x)\theta + (1-\theta)p(x) - 1$ for some $0 < \theta < 1$. Therefore, using that $2^* = \infty$ and that $p_\varepsilon \to p$ uniformly, we obtain

$$\int_\Omega (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2})\nabla u_\varepsilon \nabla\varphi\, dx \to 0. \tag{4.18}$$

Then, using that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$, (4.16), (4.17) and the (4.18) we conclude that u is a solution of (1.1).

Now, we consider the case $p \in \mathrm{Lip}(\overline{\Omega})$. By Lemmas B.1 and B.2 there exists $p_\varepsilon \in C^1(\overline{\Omega})$ such that $|\Omega \setminus \Omega_0| < \varepsilon$ where

$$\Omega_0 = \{x \in \Omega \colon p_\varepsilon(x) = p(x) \text{ and } \nabla p_\varepsilon(x) = \nabla p(x)\}.$$

We define f_ε as in (4.14). Then, the solution u_ε of (1.1) with p_ε and f_ε instead of p and f is bounded in $H^2(\Omega)$ by a constant independent of ε. Therefore there exist a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in H^2(\Omega)$ satisfying (4.15).

Lastly, we prove that u is a solution of (1.1). Let $\varphi \in C_0^\infty(\Omega)$. By Hölder inequality, since $2^* = \infty$ and by (3) of Lemma B.2 we have

$$\begin{aligned}&\int_{\Omega\setminus\Omega_0} (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2})\nabla u_\varepsilon \nabla\varphi\, dx \\ &\qquad \le C(\|\nabla u_\varepsilon\|_{L^{p_\varepsilon}(\Omega)}\|1\|_{L^{p_\varepsilon}(\Omega\setminus\Omega_0)} + \|\nabla u_\varepsilon\|_{L^{p}(\Omega)}\|1\|_{L^{p}(\Omega\setminus\Omega_0)}) \\ &\qquad \le C\|u_\varepsilon\|_{H^2(\Omega)}(\|1\|_{L^{p_\varepsilon}(\Omega\setminus\Omega_0)} + \|1\|_{L^{p}(\Omega\setminus\Omega_0)}).\end{aligned}$$

Then, since $\|u_\varepsilon\|_{H^2(\Omega)}$ is bounded independent of ε and $|\Omega \setminus \Omega_0| < \varepsilon$ we obtain that

$$\int_{\Omega\setminus\Omega_0} (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2})\nabla u_\varepsilon \nabla\varphi\, dx \to 0.$$

Therefore, since (4.16), (4.17) again hold, using that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$, and the above equation, we conclude that u is a solution of (1.1). □

5. The convex case

Lastly, we want to prove that the solution is in $H^2(\Omega)$ if we only assume that $\partial\Omega$ is convex. We want to remark here that this result generalize the one in Theorem 2.2 in [22] in two ways. In that paper the authors consider the case $p = constant$ and $g = 0$. Instead, we are allowed to cover the case where g is any function in $H^2(\Omega)$ and $p(x) \in \mathrm{Lip}(\overline{\Omega})$.

Remark 5.1. Let Ω be a convex set and $p : \Omega \to [1, \infty)$ be log −continuous in $\overline{\Omega}$. Then, there exists a sequence $\{\Omega_m\}_{m\in\mathbb{N}}$ of convex subset of Ω with C^2 boundary such that $\Omega_m \subset \Omega_{m+1}$ for any $m \in \mathbb{N}$ and $|\Omega \setminus \Omega_m| \to 0$.

(1) Then, there exists a constant C depending on $p(x), |\Omega|$ such that

$$\|v\|_{L^{p(\cdot)}(\Omega_m)} \leq C\|\nabla v\|_{L^{p(\cdot)}(\Omega_m)} \quad \forall v \in W_0^{1,p(\cdot)}(\Omega_m),$$

for any $m \in \mathbb{N}$. This follows by Theorem 3.3 in [21], using that $\Omega_m \subset \Omega_{m+1}$ for any $m \in \mathbb{N}$.

(2) The Lipschitz constants of Ω_m ($m \in \mathbb{N}$) are uniformly bounded (see Remark 2.3 in [22]). Therefore, the extension operators

$$E_{1,m} : W^{1,p(\cdot)}(\Omega_m) \to W^{1,p(\cdot)}(\Omega) \text{ and } E_{2,m} : H^2(\Omega_m) \to H^2(\Omega)$$

define as Theorem 4.2 in [11] satisfy that $\|E_{1,m}\|$ and $\|E_{2,m}\|$ are uniformly bounded.

(3) By (2) and Corollary 8.3.2 in [12], there exists a constant C independent of m such that

$$\|v\|_{L^{p^*(\cdot)}(\Omega_m)} \leq C\|v\|_{W^{1,p(\cdot)}(\Omega_m)} \quad \forall v \in W^{1,p(\cdot)}(\Omega_m),$$

for any $m \in \mathbb{N}$.

We want to remark that all the constants of the above inequalities are independent of p_1 (see Section 6 for the applications).

Proof of Theorem 1.2. We begin taking $\{\Omega_m\}_{m\in\mathbb{N}}$ as in Remark 5.1 and u_m the solution of

$$\begin{cases} -\Delta_{p(x)} u_m = f & \text{in } \Omega_m, \\ u_m = g & \text{on } \partial\Omega_m. \end{cases}$$

By Theorem 1.1, $u_m \in H^2(\Omega_m)$ for any $m \in \mathbb{N}$. Moreover, u_m solves

$$\begin{cases} L^m u_m = a_{ij}^m(x) u_{m,x_i x_j} = a^m(x) & \text{in } \Omega_m, \\ u_m = g & \text{on } \partial\Omega_m, \end{cases}$$

with

$$a_{ij}^m(x) = \delta_{ij} + (p(x) - 2)\frac{u_{m,x_i}(x) u_{m,x_j}(x)}{|\nabla u_m(x)|^2},$$
$$a^m(x) = \ln(|\nabla u_m(x)|)\langle \nabla u_m(x), \nabla p(x)\rangle + f(x)|\nabla u_m(x)|^{2-p(x)}.$$

Then $v_m = u_m - g$ solves

$$\begin{cases} L^m v_m = -L^m g + a^m(x) & \text{in } \Omega_m, \\ v_m = 0 & \text{on } \partial\Omega_m. \end{cases}$$

Thus, using that $v_m \in H^2(\Omega_m) \cap H_0^1(\Omega_m)$ and since the coefficients $a_{ij}^m(x)$ are bounded independent of m, we can argue as in Theorem 2.2 in [22] and obtain,

$$\begin{aligned}
\|v_m\|_{H^2(\Omega_m)} &\leq C\| - L^m g + f|\nabla u_m|^{2-p(\cdot)} + \ln(|\nabla u_m|)|\nabla u_m|\|_{L^2(\Omega_m)} \\
&\leq C\left(\||\nabla u_m|^{2-p(\cdot)}\|_{L^2(\Omega_m)} + \|\ln(|\nabla u_m|)|\nabla u_m|\|_{L^2(\Omega_m)} + 1\right)
\end{aligned} \tag{5.19}$$

where the constant C is independent of m.

As in Lemma 4.1 we can prove, using Remark 5.1 (1) and (3), that the norms $\|\nabla u_m\|_{L^{p(\cdot)}(\Omega_m)}$ are uniformly bounded. Therefore, proceeding as in Theorem 4.2 we obtain

$$\begin{aligned}
&\|\ln(|\nabla u_m|)|\nabla u_m|\|_{L^2(\Omega_m)} + \|f|\nabla u_m|^{2-p}\|_{L^2(\Omega_m)} \\
&\leq C\left(\|\nabla u_m\|^{(1+s)/2}_{L^{p'(\cdot)(1+s)}(\Omega_{1,m})} + \|\nabla u_m\|^{2-p_1}_{L^{\gamma(\cdot)}(A_{2,m})} + 1\right),
\end{aligned} \tag{5.20}$$

with C independent of m, where

$$\Omega_{1,m} = \{x \in \Omega_m : |\nabla u_m(x)| > 1\} \text{ and } A_{2,m} = \{x \in \Omega_m : p(x) < 2\}.$$

Now, using Remark 5.1 (3) and (2), we have that for any $r > 1$ that

$$\begin{aligned}
\|v_m\|_{W^{1,r}(\Omega_m)} &\leq \|E_{2,m}v_m\|_{W^{1,r}(\Omega)} \\
&\leq C\|E_{2,m}v_m\|_{H^2(\Omega)} \\
&\leq C\|v_m\|_{H^2(\Omega_m)}
\end{aligned} \tag{5.21}$$

where C is independent of m.

Therefore, using (5.19), (5.20) and (5.21), we get

$$\begin{aligned}
\|v_m\|_{H^2(\Omega_m)} \leq & C(\|v_m\|^{(1+s)/2}_{H^2(\Omega_m)} + \|v_m\|^{2-p_1}_{H^2(\Omega_m)} + \|g\|^{(1+s)/2}_{H^2(\Omega_m)} + \|g\|^{2-p_1}_{H^2(\Omega_m)} + 1) \\
\leq & C(\|v_m\|^{(1+s)/2}_{H^2(\Omega_m)} + \|v_m\|^{2-p_1}_{H^2(\Omega_m)} + 1),
\end{aligned}$$

where the constant C is independent of m. This proves that $\{\|v_m\|_{H^2(\Omega_m)}\}_{m\in\mathbb{N}}$ is bounded.

Now we have, as in the proof of Theorem 2.2 in [22], that there exist a subsequence still denote $\{v_m\}_{m\in\mathbb{N}}$ and a function $v \in H^2(\Omega) \cap H_0^1(\Omega)$ such that,

$$v_m \to v \quad \text{strongly in } H^1(\Omega')$$

for any $\Omega' \subset\subset \Omega$. Then $u = v + g \in H^2(\Omega)$ and

$$u_m \to u \quad \text{strongly in } H^1(\Omega')$$

for any $\Omega' \subset\subset \Omega$. Thus, using (4.13), we have

$$|\nabla u_m|^{p(x)-2}\nabla u_m \to |\nabla u|^{p(x)-2}\nabla u \quad \text{strongly in } L^{p'(\cdot)}(\Omega') \tag{5.22}$$

for any $\Omega' \subset\subset \Omega$.

On the other hand, for any $\varphi \in C_0^\infty(\Omega)$ there exist m_0 such that for all $m \geq m_0$

$$\int_{\Omega_m} |\nabla u_m|^{p(x)-2}\nabla u_m \nabla\varphi \, dx = \int_{\Omega_m} f\varphi \, dx.$$

Therefore, using (5.22) we have that u is a weak solution of (1.1). □

Proof of Corollary 1.3. By the previous theorem we have that $u \in H^2(\Omega)$, then we can derive the equation (1.1) and obtain

$$\begin{cases} -a_{ij}(x)u_{x_ix_j} = a(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} a_{ij}(x) &= \delta_{ij} + (p(x)-2)\frac{u_{x_i}(x)u_{x_j}(x)}{|\nabla u(x)|^2}, \\ a(x) &= \ln(|\nabla u(x)|)\langle \nabla u(x), \nabla p(x)\rangle + f(x)|\nabla u(x)|^{2-p(x)}. \end{aligned}$$

Using that $f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geq q_1 > 2$ and following the lines in the proof of Theorem 4.2, we have that $a(x) \in L^s(\Omega)$ with $s > 2$. Therefore, by Remark A.3, we have that $u \in C^{1,\alpha}(\overline{\Omega})$. ☐

6. Comments

In the image processing problem it is of interest the case where p_1 is close to 1. By this reason, we are also interested in the dependence of the H^2-norm on p_1.

If $N = 2$, $g \in H^2(\Omega)$ and u_ε is the solution of (3.2), we have by Lemma A.1, (3.6) and (3.7), that there exists a constant C independent of p_1 and ε such that

$$\|u_\varepsilon\|_{H^2(\Omega)} \leq \frac{C}{(p_1-1)^\kappa}\left(\|a_\varepsilon\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\right),$$

where $\kappa = 1$ if Ω is convex and $\kappa = 2$ if $\partial\Omega \in C^2$. Therefore, using that the Poincare's inequality and the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ hold in the case $p_1 = 1$ and following the lines of Theorem 1.1 and Theorem 1.2 we have that

$$\|u\|_{H^2(\Omega)} \leq \frac{C}{(p_1-1)^\kappa},$$

where the constant C is independent of p_1.

Appendix A. Regularity results for elliptic linear equations with coefficients in L^∞

Let Ω be an bounded open subset of $\mathbb{R}^2$ and

$$\mathcal{M}u = a_{ij}(x)u_{x_ix_j},$$

such that $a_{ij} = a_{ji}$ and for any $\xi \in \mathbb{R}^N$

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \tag{A.1}$$

and

$$M_1 \leq a_{11}(x) + a_{22}(x) \leq M_2 \quad \text{in } \Omega \tag{A.2}$$

where λ, Λ, M_1 and M_2 are positive constant.

In the next lemma, we will give a H^2-bound for solutions of

$$\begin{cases} \mathcal{M}u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{A.3}$$

In fact, the following result is proved in Theorem 37,III in [23], but it is not explicit the dependence of the bounds on the ellipticity and the $L^\infty-$norm of $(a_{ij}(x))$. Then, following the proof of the mentioned theorem we can prove

Lemma A.1. *Let Ω be a bounded domain in $\mathbb{R}^2$, $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$. Then, if u is a solution of* (A.3) *and $u \in H^2(\Omega)$ we have that*

$$\|u\|_{H^2(\Omega)} \leq \frac{C}{\lambda^\kappa} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\right),$$

where $\kappa = 1$ if Ω is convex and $\kappa = 2$ if $\partial\Omega \in C^2$ and C is a constant independent of λ.

Proof. In this proof, we denote $u_{ij} = u_{x_i x_j}$ for all $i, j = 1, 2$ and C is a constant independent of λ.

First, we consider the case $g \equiv 0$. Using (A.1), we have that

$$\begin{aligned}(a_{11}(x) + a_{22}(x))(u_{12}^2 - u_{11}u_{22}) = &\sum_{i,j,k=1}^{2} a_{ij}u_{ki}u_{kj} - \Delta u \sum_{ij=1}^{2} a_{ij}u_{ij} \\ \geq &\lambda \sum_{ik=1}^{2} u_{ki}^2 - \Delta u f(x).\end{aligned}$$

Then, using Young's inequality, we get

$$\frac{\lambda}{2(a_{11}(x) + a_{22}(x))} \sum_{ik=1}^{2} u_{ki}^2 \leq \frac{4}{\lambda(a_{11}(x) + a_{22}(x))} f(x)^2 + u_{12}^2 - u_{11}u_{22},$$

and by (A.2), we have that

$$\sum_{ik=1}^{2} u_{ki}^2 \leq \frac{C}{\lambda^2} f(x)^2 + \frac{C}{\lambda}(u_{12}^2 - u_{11}u_{22}), \tag{A.4}$$

Now, using (37.4) and (37.6) in [23], we have that for any $u \in H^2(\Omega)$

$$\int_\Omega (u_{12}^2 - u_{11}u_{22})\, dx = -\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \frac{H}{2} ds \tag{A.5}$$

where H is the curvature of $\partial\Omega$. If Ω is convex, then $H \geq 0$ and therefore, using (A.4) and (A.5) we have that

$$\|D^2 u\|_{L^2(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^2(\Omega)}. \tag{A.6}$$

In the general case, we can use the following inequality

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 ds \leq C\left((1 + \delta^{-1}) \int_\Omega |\nabla u|^2\, dx + \delta \int_\Omega \sum_{ik=1}^{2} u_{ki}^2\, dx\right) \tag{A.7}$$

for any $\delta > 0$. See equation (37.6) of [23].

Then, by (A.4), (A.5), using that H is bounded and (A.7) (choosing δ properly) we arrive to

$$\int_\Omega \sum_{ik=1}^{2} u_{ki}^2\, dx \leq \frac{C}{\lambda^2} \left(\int_\Omega f(x)^2\, dx + \int_\Omega |\nabla u|^2\, dx\right). \tag{A.8}$$

On the other hand, using that $Lu = f$ in Ω, (A.1) and the Poincare's inequality, we have

$$\|\nabla u\|_{L^2(\Omega)} \leq \frac{C}{\lambda}\|f\|_{L^2(\Omega)}. \tag{A.9}$$

Therefore, by (A.8) and (A.9), we get

$$\|D^2 u\|_{L^2(\Omega)} \leq \frac{C}{\lambda^2}\|f\|_{L^2(\Omega)}.$$

Thus, by the last inequality, (A.9) and (A.6) the lemma is proved in the case $g = 0$.

When g is any function in $H^2(\Omega)$ the lemma follows taking $v = u - g$. □

The following theorem is proved in Corollary 8.1.6 in [20].

Theorem A.2. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$, $\mathcal{M}$ satisfying* (A.1) *and $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a solution of* (A.3) *with $g = 0$ and $f \in L^p(\Omega)$ with $p > 2$. Then $\nabla u \in C^\mu(\overline{\Omega})$ for some $0 < \mu < 1$.*

Remark A.3. Observe that the above Theorem holds also if we consider any $g \in W^{2,p}(\Omega)$, since we can take $v = u - g$ in (A.3) and use that $W^{2,p}(\Omega) \hookrightarrow C^{1,1-2/p}(\overline{\Omega})$.

Appendix B. Lipschitz Functions

Using the linear extension operator define in [14], we have the following lemma

Lemma B.1. *Let Ω be a bounded open domain with Lipschitz boundary and $f \in \mathrm{Lip}(\overline{\Omega})$. Then, there exists a function $\overline{f} : \mathbb{R}^N \to \mathbb{R}$ such that $\overline{f}$ is a Lipschitz function, $\sup_{\mathbb{R}^N} \overline{f} = \inf_{\overline{\Omega}} f$ and $\inf_{\mathbb{R}^N} \overline{f} = \max_{\overline{\Omega}} f$.*

Lemma B.2. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz function. Then for each $\varepsilon > 0$, there exists a C^1 function $f_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ such that*

(1) *$|\{x \in \mathbb{R}^N : f_\varepsilon(x) \neq f(x)$ or $Df_\varepsilon(x) \neq Df(x)\}| \leq \varepsilon$.*
(2) *There exist a constant C depending only on N such that,*
$$\|Df_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq CLip(f).$$
(3) *If $1 < f_1 \leq f(x) \leq f_2$ in $\mathbb{R}^N$, we have*
$$1 < f_\varepsilon(x) \leq f_2 + C\varepsilon^{\frac{1}{N}} \text{ in } \mathbb{R}^N$$
with C a constant depending only on N.

Proof. Items (1) and (2) follow by Theorem 1, pag. 251 in [16].

To prove (3), let as define

$$\Omega_0 = \{x \in \mathbb{R}^N : f_\varepsilon(x) = f(x) \text{ and } Df_\varepsilon(x) = Df(x)\}$$

and let as suppose that there exist $x \in \mathbb{R}^N \setminus \Omega_0$ such that $f_\varepsilon(x) = f_2 + \delta$ with $\delta > 0$. If $x_0 \in \Omega_0$, by (2), we have

$$CLip(f)|x - x_0| \geq f_\varepsilon(x) - f_\varepsilon(x_0) = f_2 + \delta - f(x_0) \geq \delta.$$

Then $B_\rho(x) \subset \mathbb{R}^N \setminus \Omega_0$ where $\rho = \delta(CLip(f))^{-1}$ and using (1) we get $\delta \leq C\varepsilon^{1/N}$, for some constant C independent of ε.

Analogously we can prove the other inequality. □

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