THE DECOMPOSITION–COORDINATION METHOD FOR THE $p(x)$-LAPLACIAN

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Abstract. In this paper we construct two algorithms to approximate the minimizer of a discrete functional which comes from using a discontinuous Galerkin method for a variational problem related to the $p(x)$-Laplacian. We also make some numerical experiments in dimension two.

1. Introduction

This work is devoted to developing and analysing two algorithms to approximate the minimizer of a discrete functional which comes from using a discontinuous Galerkin method for a nonlinear, nonhomogeneous variational problem. This variational problem is related to an image processing model of Chen, Levin and Rao [8], see also [3].

More precisely, we consider the following nonlinear variational problem:

$$\text{(P)}\qquad \begin{aligned} &\text{Find } u \in \mathcal{A} := \left\{ v \in W^{1,p(\cdot)}(\Omega) \colon v - u_D \in W_0^{1,p(\cdot)}(\Omega) \right\} \text{ such that} \\ &J(u) = \min_{v \in \mathcal{A}} J(v), \end{aligned}$$

where

$$J(v) := \int_\Omega |\nabla v|^{p(x)} + |v - \xi|^2 \, dx,$$

Ω is a bounded connected open set in $\mathbb{R}^N$ with Lipschitz continuous boundary, $p : \overline{\Omega} \to [p_1, p_2]$ is a log-Hölder continuous function with $1 < p_1 \le p_2 \le 2$, $u_D \in W_0^{1,p(\cdot)}(\Omega)$ and $\xi \in L^2(\Omega)$. It is well-known that the functional J admits a unique minimizer $u \in \mathcal{A}$. For the definitions of the log-Hölder continuous function and the variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, see Section 2.

Note that this functional is related to the so-called $p(x)-$Laplacian operator, that is

$$\Delta_{p(x)} = \text{div}(|\nabla u|^{p(x)-2} \nabla u).$$

This operator extends the classical Laplacian ($p(x) \equiv 2$) and the $p-$Laplacian ($p(x) \equiv p$, $1 < p < +\infty$). The interest in this operator was originally motivated by the model for electrorheological fluids, see [27, 28].

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In [12], the so-called discontinuous Galerkin method is considered to approximate the minimizer of (P). More precisely, the authors study the following discrete functional,

$$I_h(v_h) := \int_\Omega |\nabla v_h + R_h(v_h)|^{p(x)} + |v_h - \xi|^{q(x)}\,dx + \int_{\partial\Omega} |v_h - u_D|^{p(x)}\mathbf{h}^{1-p(x)}\,dS + \int_{\Gamma_{int}} |[\![v_h]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\,dS,$$

where $\mathbf{h}$ is the local mesh size, h is the global mesh size, Γ_{int} is the union of the interior edges of the elements, $[\![v_h]\!]$ is the jump of the function between two edges, ∇v_h denotes the elementwise gradient of v_h and R_h is the lifting operator, see Section 2 for a precise definition. Observe that the boundary condition is weakly imposed by the second term of the functional.

With this setting, the discrete problem is to find a minimizer u_h of J_h over the space $S^k(\mathcal{T}_h)$ of all the functions that are polynomials of degree at most k in each element, with $k \geq 1$, see Subsection 2.2 for details.

In [12], the authors prove the following result.

Theorem 1.1. *Let Ω be a polyhedral domain, $u_D \in W^{2,2}(\Omega)$, and $u_h \in S^k(\mathcal{T}_h)$ be the minimizer of J_h for any $h \in (0,1]$. If u is the minimizer of J then*

$$I_h(u_h) \to J(u), \quad R_h(u_h) \to 0, \quad u_h \to u \text{ strongly in } W^{1,p(\cdot)}(\Omega), \text{ and}$$

$$\int_{\partial\Omega} |u_h - u_D|^{p(x)}\mathbf{h}^{1-p(x)}\,dS + \int_{\Gamma_{int}} |[\![u_h]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\,dS \to 0$$

Since we want to implement this method for some examples, the next step is to find a good approximation of the minimizer of the discrete functional.

The methods for finding minimizers of functionals, such as the BFGS Quasi- Newton, work when the dimension of the space is small. However, we observe that these methods are to slow when we refine the mesh. We also observe, in some numerical experiments, that the decomposition– coordination–method (DCM), defined in [23, Chapter VI], is more suitable for our problem.

The DCM is used to approximate the minimizers of functionals that can be written in the form

$$J(v) = F(Bv) + G(v),$$

where $F : H \to \mathbb{R}$, $G : V \to \mathbb{R}$ are convex functions, $B : V \to H$ is a linear operator and V and H are topological vector spaces.

In this context, the problem of finding minimizers of J over V is equivalent to find $(q_0, v_0) \in W := \{(q,v) \in V \times H : Bv - q = 0\}$ such that

$$F(q_0) + G(v_0) = \min_{(q,v)\in W} \{F(q) + G(v)\}. \tag{1.1}$$

In the practical applications, under the following assumptions

(H1) $F : H \to \mathbb{R}$, $G : V \to \mathbb{R}$ are lower semicontinuous functions and

$$\operatorname{dom}(F \circ B) \cap \operatorname{dom}(G) \neq \emptyset;$$

(H2) F is a convex Gateaux–diffentiable functional and

$$\lim_{|q|\to\infty} \frac{F(q)}{|q|} = \infty;$$

(H3) The rank of B is close in H;
(H4) B injective;

In [23] the authors prove that there exists a sequence $\{(u^n, q^n)\} \subset V \times H$ such that u^n solves a linear differential equation, q^n solves a non-linear equation, and

$$\begin{aligned} u^n &\to v_0 = u \quad \text{strongly in } V \\ q^n &\to q_0 = B(u) \quad \text{strongly in } H \end{aligned}$$

where u is the minimizer of J.

If we write the functional I_h in this form, we have that

$$I_h(v) = F(Bv) + G(v)$$

where here $V = S^k(\mathcal{T}_h)$, $H = S^l(\mathcal{T}_h) \times S^l(\mathcal{T}_h)$, $k, l \in \mathbb{N}_0$ with $l \geq k-1$, $Bv = R_h(v) + \nabla v$,

$$\begin{aligned} F(q) &= \int_\Omega |q|^{p(x)}\,dx \quad \text{and} \\ G(v) &= \int_\Omega |v-\xi|^{q(x)}\,dx + \int_{\partial\Omega} |v-u_D|^{p(x)}\mathbf{h}^{1-p(x)}\,dS + \int_{\Gamma_{int}} |[\![v]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\,dS. \end{aligned}$$

We can observe that, (H1), (H2), (H3) hold, but (H4) does not hold, that is B is not injective. Moreover, G' is not linear.

To prove the convergence of the method, it is not necessary the assumption that G' is linear, but it is useful for the implementation. For this reason and since we are interested in the case $p(x) \leq 2$, we define a new discrete functional

$$J_h(v) = F(Bv) + G(v)$$

where now

$$G(v) = \int_\Omega |v_h-\xi|^2\,dx + \int_{\partial\Omega} |v_h-u_D|^2\mathbf{h}^{-2/p'(x)}\,dS + \int_{\Gamma_{int}} |[\![v_h]\!]|^2\mathbf{h}^{-2/p'(x)}\,dS,$$

and F and B are defined as before. In this manner, G' is linear. Observe that, here we have to change the power over the function $\mathbf{h}$.

To overcome the lack of injectivity of the functional B, we will use that our functional G is Gateaux-differentiable and convex.

Now, we are ready to state the main results of this paper.

Since we change the discrete functional we have to prove a result similar to Theorem 1.1. More precisely, we prove the following theorem.

Theorem 1.2. *Let Ω be a polygonal domain in $\mathbb{R}^N$, $p : \overline{\Omega} \to [p_1, 2]$ ($N/2 < p_1 \leq 2$) be a* log-*Hölder continuous and $u_D \in W^{2,2}(\Omega)$. For each $h \in (0,1]$, let $u_h \in S^k(\mathcal{T}_h)$ be the minimizer of*

J_h. If u is the minimizer of J then

$$\begin{aligned} u_h &\to u \text{ strongly in } L^{s(\cdot)}(\Omega) \quad \forall s \in \mathcal{K}, \\ u_h &\to u \text{ strongly in } L^2(\partial\Omega), \\ J_h(u_h) &\to J(u), \\ R_h(u_h) &\to 0, \\ \int_{\partial\Omega} |u_h - u_D|^2 \mathbf{h}^{-2/p'(x)}\, dS + \int_{\Gamma_{int}} |[\![u_h]\!]|^2 \mathbf{h}^{-2/p'(x)}\, dS &\to 0, \\ \nabla u_h &\to \nabla u \text{ strongly in } L^{p(\cdot)}(\Omega), \end{aligned}$$

where $\mathcal{K} = \{s \in L^\infty(\Omega) \colon 1 \le s(x) < p^*(x) - \varepsilon$ for some $\varepsilon > 0\}$.

We define two algorithms that construct a sequence $\{u_h^n\}_{n\in\mathbb{N}}$ that approximates, for each $h \ge 0$ the minimizers of J_h and finally we prove the convergence of both algorithms.

Theorem 1.3. *Let $h \ge 0$ and $(u_h, \eta_h, \lambda_h) \in V \times H \times H$ be a saddle-point of $\mathcal{L}_r$. If*

$$0 < \alpha_0 \le \rho_n \le \alpha_1 < 2r \tag{1.2}$$

and $(u_h^n, \eta_h^n, \lambda_h^n) \in V \times H \times H$ is the solutions given by Algorithm 1 then

$$\begin{aligned} u_h^n &\to u_h \text{ in } V, \\ \eta_h^n &\to \eta_h = Bu_h \text{ in } H, \\ \lambda_h^{n+1} - \lambda_h^n &\to 0 \text{ in } H, \end{aligned}$$

$$\lambda_h^n \text{ is bounded.}$$

Moreover,

$$\begin{aligned} [\![u_h^n - u_h]\!] &\to 0 \text{ in } L^2(\Gamma_{int}), \\ u_h^n &\to u_h \text{ in } L^2(\partial\Omega), \\ R(u_h^n) &\to R(u_h) \text{ in } H, \\ \nabla u_h^n &\to \nabla u_h \text{ in } H. \end{aligned}$$

Theorem 1.4. *Let $h \ge 0$ and $(u_h, \eta_h, \lambda_h) \in V \times H \times H$ be a saddle-point of $\mathcal{L}_r$. If*

$$0 < \rho_n = \rho < r\frac{(1+\sqrt{5})}{2} \tag{1.3}$$

and $(u_h^n, \eta_h^n, \lambda_h^n) \in V \times H \times H$ is the solutions given by Algorithm 2 then

$$\begin{aligned} \eta_h^n &\to \eta_h = Bu_h \text{ in } H, \\ \lambda_h^{n+1} - \lambda_h^n &\to 0 \text{ in } H, \end{aligned}$$

$$\lambda_h^n \text{ is bounded .}$$

Moreover,

$$\begin{aligned} u_h^n &\to u_h \ \text{in } V, \\ [\![u_h^n - u_h]\!] &\to 0 \ \text{in } L^2(\Gamma_{int}), \\ u_h^n &\to u \ \text{in } L^2(\partial\Omega), \\ R(u_h^n) &\to R(u_h) \ \text{in } H, \\ \nabla u_h^n &\to \nabla u_h \ \text{in } H. \end{aligned}$$

Let us end the introduction with a brief comments on previous bibliography. In [4, 10], the convergence of conforming finite element method approximations for the Dirichlet problem of the $p(\cdot)-$Laplacian is studied. Moreover, in [10], we study the convergence rate using the regularity results obtained in [11].

Finally, we want to mention that in [3] and [8] the authors find an approximation of the solutions by using an explicit finite difference scheme for the associated parabolic problem.

Outline of the paper. In Section 2, we state several properties of the variable exponent Sobolev spaces, we give some definitions and properties related to the mesh and to the broken Sobolev spaces; In section 3, we prove Theorem 1.2; In section 4, the decomposition-coordination method is studied and the convergence of the algorithms are proved; Finally, in Section 5, we give some numerical examples.

2. Preliminaries

We begin with a review of the basic results that will be needed in subsequent sections. The results are generally stated without proof, although we attempt to provide good references where the proofs can be found. Also, we introduce some of our notational conventions.

2.1. The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$.

We first introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let $p\colon \Omega \to [p_1, p_2]$ be a measurable bounded function, called a variable exponent on Ω where $p_1 := \operatorname{ess\,inf}\, p(x)$ and $p_2 := \operatorname{ess\,sup}\, p(x)$ with $1 \le p_1 \le p_2 < \infty$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} := \inf\{k > 0\colon \varrho_{p(\cdot)}(u/k) \le 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

The following properties can be obtained directly from the definition of the norm. For the proof see [21, Theorem 1.3 and Theorem 1.4].

Proposition 2.1. *If $u, u_n \in L^{p(\cdot)}(\Omega)$, $\|u\|_{p(\cdot)} = \lambda$, then*

(1) $\lambda < 1$ $(= 1, > 1)$ *iff* $\displaystyle\int_\Omega |u(x)|^{p(x)}\, dx < 1$ $(= 1, > 1)$;

(2) *If* $\lambda \ge 1$, *then* $\displaystyle\lambda^{p_1} \le \int_\Omega |u(x)|^{p(x)}\, dx \le \lambda^{p_2}$;

(3) *If* $\lambda \le 1$, *then* $\lambda^{p_2} \le \int_\Omega |u(x)|^{p(x)}\,dx \le \lambda^{p_1}$;

(4) $\int_\Omega |u_n(x)|^{p(x)}\,dx \to 0$ *iff* $\|u_n\|_{p(\cdot)} \to 0$.

For the proofs of the following two theorems we refer the reader to [26].

Theorem 2.2. *Let* $q(x) \le p(x)$, *then* $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *continuously.*

Theorem 2.3. *Let* $p'(x)$ *such that,* $1/p(x) + 1/p'(x) = 1$. *Then* $L^{p'(\cdot)}(\Omega)$ *is the dual of* $L^{p(\cdot)}(\Omega)$. *Moreover, if* $p_1 > 1$, $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are reflexive.*

Now we give some well known inequalities.

Proposition 2.4. *For any* x *fixed we have the following inequalities*

$$|\eta - \xi|^{p(x)} \le C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta - \xi) \qquad \text{if } p(x) \ge 2,$$

$$|\eta - \xi|^2\Big(|\eta| + |\xi|\Big)^{p(x)-2} \le C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta - \xi) \qquad \text{if } p(x) < 2,$$

$$|\eta|^{p(x)} \le 2^{p(x)-1}(|\eta - \xi|^{p(x)} + |\xi|^{p(x)}) \qquad \text{if } p(x) \ge 1.$$

The following properties will be used throughout the paper.

Proposition 2.5. *Let* $F_n, F \in L^{p(\cdot)}(\Omega)$.

(1) *If*

$$F_n \rightharpoonup F \text{ weakly in } L^{p(\cdot)}(\Omega)$$

then

$$\int_\Omega |F|^{p(x)}\,dx \le \liminf_{n\to\infty} \int_\Omega |F_n|^{p(x)}\,dx.$$

(2) *If*

$$F_n \to F \text{ strongly in } L^{p(\cdot)}(\Omega)$$

then

$$\int_\Omega |F_n|^{p(x)}\,dx \to \int_\Omega |F|^{p(x)}\,dx.$$

(3) *If*

$$F_n \rightharpoonup F \text{ weakly in } L^{p(\cdot)}(\Omega) \quad \text{and} \quad \int_\Omega |F_n|^{p(x)}\,dx \to \int_\Omega |F|^{p(x)}\,dx$$

then

$$F_n \to F \text{ strongly in } L^{p(\cdot)}(\Omega).$$

Proof. For the proof of (1) and (3) see [16, Theorem 3.9 and Lemma 2.4.17]. Finally (2) follows by [18, Proposition 2.3]. □

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that, u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

We now introduce the most important condition on the exponent in the study of variable exponent spaces, the log-Hölder continuity condition.

Definition 2.6. We say that a function $\alpha : \overline{\Omega} \to \mathbb{R}$ is log-Hölder continuous if there exists a constant C_{log} such that

$$|\alpha(x) - \alpha(y)| \leq \frac{C_{log}}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \forall\, x, y \in \overline{\Omega}.$$

For example, it was proved in [14, Theorem 3.7], that if one assumes that $\partial\Omega$ is Lipschitz and $p : \overline{\Omega} \to [1, +\infty)$ is log-Hölder continuous then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$. See also [13, 17, 19, 26, 29].

We now state two Sobolev embedding Theorems. Here, p^* and p_* are the Sobolev critical exponents for these spaces, i.e.

$$p^*(x) := \begin{cases} \dfrac{p(x)N}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \quad \text{and} \quad p_*(x) := \begin{cases} \dfrac{p(x)(N-1)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

For the proofs of the following theorems see [15] and [20, Corollary 2.4], respectively.

Theorem 2.7. *Let Ω be a Lipschitz domain. Let $p : \overline{\Omega} \to [1, \infty)$ be a* log-*Hölder continuous function. Then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous.*

Theorem 2.8. *Let Ω be a bounded Lipschitz domain. Suppose that $p \in C^0(\overline{\Omega})$ with $p_1 > 1$. If $r \in C^0(\partial\Omega)$ satisfies the condition $1 \leq r(x) < p_*(x)$ for all $x \in \partial\Omega$, then there is a compact boundary trace embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.*

2.2. **The mesh $\mathcal{T}_h$ and properties of $W^{1,p(\cdot)}(\mathcal{T}_h)$.** We now give some definitions and properties related to the mesh and to the broken Sobolev space.

Hypothesis 2.9. *Let Ω be a bounded polygonal domain and $(\mathcal{T}_h)_{h\in(0,1]}$ be a family of partitions of $\overline{\Omega}$ into polyhedral elements. We assume that there exists a finite number of reference polyhedral $\hat{\kappa}_1, \ldots, \hat{\kappa}_r$ such that for all $\kappa \in \mathcal{T}_h$ there exists an invertible affine map F_κ such that, $\kappa = F_\kappa(\hat{\kappa}_i)$. We assume that each $\kappa \in \mathcal{T}_h$ is closed and that $diam(\kappa) \leq h$ for all $\kappa \in \mathcal{T}_h$.*

Now we give some notation,

$$\mathcal{E}_h := \{\kappa \cap \kappa' \colon \dim_H(\kappa \cap \kappa') = N - 1\} \cup \{\kappa \cap \partial\Omega \colon \dim_H(\kappa \cap \partial\Omega) = N - 1\},$$
$$\Gamma_{int} := \bigcup \{e \in \mathcal{E}_h := \dim_H(e \cap \partial\Omega) < N - 1\},$$

where $\dim_H$ is the Hausdorff dimension.

We also assume that the mesh satisfies the following hypotheses.

Hypothesis 2.10. *The family of partitions $(\mathcal{T}_h)_{h\in(0,1]}$ satisfies the Hypothesis 2.9 and*

(a) *There exist positive constants C_1 and C_2, independent of h, such that for each element $\kappa \in \mathcal{T}_h$*
$$C_1 h_\kappa^N \leq |\kappa| \leq C_2 h_\kappa^N.$$
(b) *There exists a constant $C_1 > 0$ such that for all $h \in (0,1]$ and for all face $e \in \mathcal{E}_h$ there exists a point $x_e \in e$ and a radius $\rho_e \geq C_1 \operatorname{diam}(e)$ such that $B_{\rho_e}(x_e) \cap A_e \subset e$, where A_e is the affine hyperplane spanned by e. Moreover, there are positive constants such that*
$$ch_\kappa \leq h_e \leq Ch_\kappa, \quad ch_{\kappa'} \leq h_e \leq Ch_{\kappa'}$$
where $e = \kappa \cap \kappa'$.

Now, we introduce the finite element spaces associated with $\mathcal{T}_h$. We define the variable broken Sobolev space as

$$W^{1,p(\cdot)}(\mathcal{T}_h) := \{u \in L^1(\Omega) \colon u|_\kappa \in W^{1,p(\cdot)}(\kappa) \text{ for all } \kappa \in \mathcal{T}_h\},$$

and the subspaces

$$\begin{aligned} U^k(\mathcal{T}_h) :=& \{u \in C(\Omega) \colon u|_\kappa \in P^k \text{ for all } \kappa \in \mathcal{T}_h\}, \\ S^k(\mathcal{T}_h) :=& \{u \in L^1(\Omega) \colon u|_\kappa \in P^k \text{ for all } \kappa \in \mathcal{T}_h\}, \end{aligned}$$

where P^k is the space of polynomials functions of degree at most k.

For each face $e \in \mathcal{E}_h$, $e \subset \Gamma_{int}$ we denote by κ^+ and κ^- its neighboring elements. We write ν^+, ν^- to denote the outward normal unit vectors to the boundaries $\partial\kappa^\pm$, respectively. The jump of a function $u \in W^{1,p(\cdot)}(\mathcal{T}_h)$ and the average of a vector-valued function $\phi \in (W^{1,p(\cdot)}(\mathcal{T}_h))^N$, with traces $u^\pm$, $\phi^\pm$ from $k^\pm$ are, respectively, defined as the vectors

$$[\![u]\!] := u^+\nu^+ + u^-\nu^- \quad \text{ and } \quad \{\phi\} := \frac{\phi^+ + \phi^-}{2}.$$

Let $\mathbf{h} \colon \partial\Omega \cup \Gamma_{int} \to \mathbb{R}$ a piecewise constant function define by

$$\mathbf{h}(x) := \operatorname{diam}(e) \quad \text{ if } x \in e,$$

where $e \in \mathcal{E}_h$.

We consider the following seminorm on $W^{1,p(\cdot)}(\mathcal{T}_h)$,

$$|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} := \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|[\![u]\!]\mathbf{h}^{\frac{-1}{p'(x)}}\|_{L^{p(\cdot)}(\Gamma_{int})}.$$

2.3. **The lifting operator.** Finally we define, as in [6] (see also [1]), the lifting operator.

Definition 2.11. Let $l \geq 0$ and $R_h \colon W^{1,p(\cdot)}(\mathcal{T}_h) \to S^l(\mathcal{T}_h)^N$ defined as,

$$\int_\Omega \langle R_h(u), \phi\rangle \, dx := -\int_{\Gamma_{int}} \langle [\![u]\!], \{\phi\}\rangle \, dS$$

for all $\phi \in S^l(\mathcal{T}_h)^N$.

This operator appears in the first term of the discretized functional J_h. As we can see from the definition, this operator represents the contribution of the jumps to the distributional gradient. This is the reason why it is crucial to add this term in order to have the consistency of the method.

We point out that this lifting operator was first used in [2] in order to describe the contributions of the jumps across the interelements of the computed solution on the (computed) gradient of the solution in a mixed formulation of Navier-Stokes equations. It was also used in [5] where a solid mathematical background for the method introduced in [2] was proposed.

Now, we state a bound of the $L^{p(\cdot)}(\Omega)$-norm of $R_h(u)$ in terms of the jumps of u in Γ_{int}. For the proof see [12].

Lemma 2.12. *Let $p : \overline{\Omega} \to\colon [1, \infty)$ be a* log- *Hölder continuous in Ω. Then, there exists a constant C such that,*

$$\|R_h(u)\|_{L^{p(\cdot)}(\Omega)} \leq C\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h) \quad \forall h \in (0, 1].$$

3. Convergence of the discontinuous Galerkin FEM

In this section we prove the convergence of the discontinuous Galerkin FEM.

From now on, we make the following assumption: Let Ω be a bounded polygonal domain in $\mathbb{R}^N$ and $p : \overline{\Omega} \to [p_1, 2]$ $(1 < p_1 \leq 2)$ be a log-Hölder continuous function.

Our next result follows by using Lemma 2.12 and the fact that $L^2(\Gamma_{int}) \subset L^{p(\cdot)}(\Gamma_{int})$.

Lemma 3.1. *There exists a constant C such that*

$$\|R_h(v)\|_{L^{p(\cdot)}(\Omega)} \leq C\|\mathbf{h}^{-1/p'(x)}[\![v]\!]\|_{L^2(\Gamma_{int})}$$

for all $v \in W^{1,p(\cdot)}(\mathcal{T}_h)$ and for all $h \in (0, 1]$.

Now, we prove the coercivity of the functional.

Theorem 3.2. *For each $h \in (0, 1]$, let $v_h \in W^{1,p(\cdot)}(\mathcal{T}_h)$. If there exists a constant C independent of h such that $J_h(v_h) \leq C$ for all $h \in (0, 1]$, then*

$$\sup_{h\in(0,1]} \Big(\|v_h\|_{L^1(\Omega)} + |v_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}\Big) < \infty.$$

Moreover,

$$\sup_{h\in(0,1]} \int_{\partial\Omega} |v_h - u_D|^{p(x)}\mathbf{h}^{1-p(x)}\, dS < \infty.$$

Proof. Since $J_h(v_h) \leq C$, we have that

$$\int_{\Gamma_{int}} |[\![v_h]\!]|^2\mathbf{h}^{-2/p'(x)}\, dS \leq C$$

then, by Lemma 3.1, $\|R_h(v_h)\|_{L^{p(\cdot)}(\Omega)}$ is bounded. Therefore, using Proposition 2.4, we have

$$J_h(v_h) + C \geq C \int_{\Omega} |\nabla v_h|^{p(x)}\, dx + \int_{\partial\Omega} |v_h - u_D|^2\mathbf{h}^{-2/p'(x)}\, dS + \int_{\Gamma_{int}} |[\![v_h]\!]|^2\mathbf{h}^{-2/p'(x)}\, dS.$$

By the above inequality and the fact that $L^2 \subset L^{p(\cdot)}$ we get

$$\int_{\Omega} |\nabla v_h|^{p(x)}\, dx \leq C,$$

$$\int_{\partial\Omega} |v_h - u_D|^{p(x)}\mathbf{h}^{1-p(x)}\, dS \leq C,$$

$$\int_{\Gamma_{int}} |[\![v_h]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\, dS \leq C.$$

Finally, the proof follows as in the end of the proof of Theorem 6.2 in [12]. □

The following theorem was proved in [12].

Theorem 3.3. *Let $u_h \in S^k(\mathcal{T}_h)$ be such that*

$$\sup_{h\in(0,1]} \Big(\|u_h\|_{L^1(\Omega)} + |u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}\Big) < \infty \quad \text{and} \quad \sup_{h\in(0,1]} \int_{\partial\Omega} |u_h - u_D|^{p(x)}\mathbf{h}^{1-p(x)}\, dS < \infty.$$

Then, there exist a sequence $h_j \to 0$ and a function $u \in W^{1,p(\cdot)}(\Omega)$ such that

$$\begin{aligned} u_{h_j} &\overset{*}{\rightharpoonup} u \quad \text{weakly* in } BV(\Omega) \\ \nabla u_{h_j} + R_h(u_{h_j}) &\rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega), \\ u_{h_j} &\to u \quad \text{strongly in } L^{p(\cdot)}(\partial\Omega), \\ u_{h_j} &\to u \quad \text{strongly in } L^{s(\cdot)}(\Omega) \quad \forall s \in \mathcal{K}, \end{aligned}$$

where $\mathcal{K} = \{s \in L^\infty(\Omega) \colon 1 \le s(x) < p^*(x) - \varepsilon$ for some $\varepsilon > 0\}$.

Before proving the convergence of the sequence of minimizers, we need an auxiliary lemma.

Lemma 3.4. *Let $h \in (0,1]$ and $u_D \in W^{2,2}(\Omega)$. If $v \in W^{2,2}(\Omega) \cap \mathcal{A}$, then there exists $v_h \in U^1(\mathcal{T}_h)$ such that*

$$\|v_h - v\|_{H^1(\Omega)} \to 0 \quad \text{as } h \to 0,$$

and

$$J_h(v_h) \to J(v) \quad \text{as } h \to 0.$$

Proof. Given $v \in W^{2,2}(\Omega) \cap \mathcal{A}$, by standard approximation theory (see [9, Theorem 3.1.5]), we have that there exists $v_h \in U^1(\mathcal{T}_h)$ such that

$$\|v_h - v\|_{H^1(\Omega)} \to 0$$

as $h \to 0$, and

$$\|v - v_h\|_{L^2(\partial\Omega)} \le Ch\|D^2 v\|_{L^2(\Omega)}.$$

Since $1 < p_1 \le p(x) \le 2$, we have

$$\int_{\partial\Omega} |v - v_h|^2 \mathbf{h}^{-2/p'(x)}\, dS \le Ch^{-2/p_1} \int_{\partial\Omega} |v - v_h|^2\, dS \le Ch^{-2/p_1} h^2 \|D^2 v\|^2_{L^2(\Omega)} \to 0$$

as $h \to 0$.

Finally, since $v_h \in W^{1,p(\cdot)}(\Omega)$, we have that $[\![v_h]\!] = 0$ and $R_h(v_h) = 0$. Then, using Proposition 2.5, we have

$$J_h(v_h) = \int_\Omega |\nabla v_h|^{p(x)} + |v_h - \xi|^2\, dx + \int_{\partial\Omega} |v_h - u_D|^2 \mathbf{h}^{-2/p'(x)}\, dS \to \int_\Omega (|\nabla v|^{p(x)} + |v - \xi|^2)\, dx$$

as $h \to 0$. The proof is complete. ☐

Now we prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 3.4, there exists $w_h \in U^1(\mathcal{T}_h)$ such that $J_h(w_h) \to J(u_D)$. Then

$$J_h(u_h) \le J_h(w_h) \le C \quad \forall h > 0. \tag{3.4}$$

By Theorem 3.2 and Theorem 3.3, we obtain that there exists a subsequence u_{h_j} of u_h such that

$$\begin{aligned} u_{h_j} &\to u \text{ strongly in } L^{s(\cdot)}(\Omega) \quad \forall s \in \mathcal{K} \\ u_{h_j} &\to u \text{ strongly in } L^{p(\cdot)}(\partial\Omega), \\ \nabla u_{h_j} + R_h(u_{h_j}) &\rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega). \end{aligned}$$

On the other hand, by (3.4)

$$\int_{\partial\Omega} |u_{h_j} - u_D|^2 \mathbf{h_j}^{-2/p'(x)}\, dS \le C$$

then

$$\int_{\partial\Omega} |u_{h_j} - u_D|^2 dS \to 0 \quad \text{as } j \to +\infty,$$

that is $u_{h_j} \to u_D$ strongly in $L^2(\partial\Omega)$. Therefore, since $u_{h_j} \to u$ strongly in $L^{p(\cdot)}(\partial\Omega)$, we have that $u = u_D$ on $\partial\Omega$, which proves that $u \in \mathcal{A}$.

Now, since $u_{h_j} \to u$ strongly in $L^2(\Omega)$ (due to $s(x) \equiv 2 \in \mathcal{K}$) and $\nabla u_{h_j} + R_h(u_{h_j}) \rightharpoonup \nabla u$ weakly in $L^{p(\cdot)}(\Omega)$ we have

$$J(u) \le \liminf_{j\to+\infty} \int_\Omega |\nabla u_{h_j} + R(u_{h_j})|^{p(x)} + |u_{h_j} - \xi|^2 \, dx \le \liminf_{j\to+\infty} J_{h_j}(u_{h_j}) \le \limsup_{j\to+\infty} J_{h_j}(u_{h_j}).$$

Let $v \in \mathcal{A} \cap W^{2,2}(\Omega)$ and v_{h_j} as in Lemma 3.4, we obtain

$$J(u) \le \liminf_{j\to+\infty} J_{h_j}(u_{h_j}) \le \limsup_{j\to+\infty} J_{h_j}(u_{h_j}) \le \lim_{j+\infty} J_{h_j}(v_{h_j}) = J(v).$$

By a density argument, we also have that

$$J(u) \le \liminf_{j\to+\infty} J_{h_j}(u_{h_j}) \le \limsup_{j\to+\infty} J_{h_j}(u_{h_j}) \le J(w)$$

for any $w \in \mathcal{A}$. Therefore u is a minimizer of J. Moreover, if we take $w = u$, we have that $J_{h_j}(u_{h_j}) \to J(u)$ as $j \to +\infty$. Thus,

$$\int_{\Gamma_{int}} |[\![u_{h_j}]\!]|^2 \mathbf{h_j}^{-2/p'(x)} \, dS. \to 0 \quad \text{as } j \to +\infty,$$

Then, by Lemma 3.1, we have that $R_{h_j}(u_{h_j}) \to 0$ as $j \to +\infty$ and

$$\nabla u_{h_j} \rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega).$$

On the other hand, since

$$\nabla u_{h_j} + R_{h_j}(u_h) \rightharpoonup \nabla u \text{ weakly in } L^{p(\cdot)}(\Omega) \text{ and } \int_\Omega |\nabla u_{h_j} + R_h(u_{h_j})|^{p(x)} \, dx \to \int_\Omega |\nabla u|^{p(x)} \, dx,$$

by Proposition 2.5, $\nabla u_{h_j} + R_{h_j}(u_{h_j}) \to \nabla u$ strongly in $L^{p(\cdot)}(\Omega)$. Therefore, since $R_{h_j}(u_{h_j}) \to 0$ strongly in $L^{p(\cdot)}(\Omega)$, we get that $\nabla u_{h_j} \to \nabla u$ strongly in $L^{p(\cdot)}(\Omega)$.

Finally, since u is the unique minimizer, the hole sequence converges. □

Remark 3.5. In the above theorem, we ask that $p_1 > N/2$ since in those cases we obtain $p^*(x) > 2$. Observe that, when $N \in \{1, 2\}$, we are not adding any new assumption on p_1.

4. The decomposition–coordination method

In this section we will study the decomposition–coordination method to approximate, for each h, the minimizer of J_h.

Throughout this section, to simplify notation, we omit the subindex h, and $]\!| \cdot |\![$, $\| \cdot \|$, and $\langle \cdot, \cdot \rangle$ denote the L^2-norm, $L^2 \times L^2$-norm and the inner product associated to the $L^2 \times L^2$-norm, respectively.

Let $V = S^k(\mathcal{T}_h)$ and $H = S^l(\mathcal{T}_h) \times S^l(\mathcal{T}_h)$ where $k, l \in \mathbb{N}_0$ with $l \ge k - 1$ we consider the following functionals

- $F\colon H \to \mathbb{R}$, $F(q) := \int_\Omega |q|^{p(x)}\,dx$;
- $G\colon V \to \mathbb{R}$, $G(v) := \int_\Omega |v-\xi|^2\,dx + \int_{\Gamma_{int}} |[\![v]\!]|^2 \mathbf{h}^{-2/p'(x)}\,dS + \int_{\partial\Omega} |v-u_D|^2 \mathbf{h}^{-2/p'(x)}\,dS$;
- $B\colon V \to H$, $Bv := R(v) + \nabla v$.

Observe that

$$J(v) = F(Bv) + G(v),$$

F and G are convex and Gateaux–diffentiable functionals, B is a linear operator, and

$$\operatorname{dom}(F\circ B) \cap \operatorname{dom}(G) \neq \emptyset.$$

In [23, Chapter VI], in a more general context, the author show that the problem

$$\min_{v\in V} J(v) = \min_{v\in V}\{F(Bv) + G(v)\} \tag{4.5}$$

is equivalent to

$$\min_{(v,q)\in W}\{F(q) + G(v)\}, \tag{4.6}$$

where

$$W = \{(v,q) \in V\times H : Bv = q\}.$$

We then define for $r \geq 0$ an augmented Lagrangian $\mathcal{L}_r$ associated with (4.6), by

$$\mathcal{L}_r\colon V\times H\times H \to \mathbb{R}$$
$$\mathcal{L}_r(v,q,\lambda) := F(q) + G(v) + \langle \lambda, Bv - q\rangle + \frac{r}{2}]\!|Bv-q|\![^2,$$

and we will say that $(u,\eta,\lambda) \in V\times H\times H$ is a saddle point of $\mathcal{L}_r$ on $V\times H\times H$ if

$$\mathcal{L}_r(u,\eta,\mu) \leq \mathcal{L}_r(u,\eta,\lambda) \leq \mathcal{L}_r(v,q,\lambda) \quad \forall (v,q,\mu)\in V\times H\times H. \tag{4.7}$$

The following lemma establishes a fundamental relationship between the saddle points of $\mathcal{L}_r$ and the solution of (4.5). For the proof see [23, Theorem 2.1– Chapter VI].

Lemma 4.1. *Let (u,η,λ) be a saddle point of $\mathcal{L}_r$, on $V\times H\times H$ then u is the solution of* (4.5) *and $Bu = \eta$.*

Then, a method for solving (4.5) is to solve the saddle point problem (4.7).

Remark 4.2. Let (u,η,λ) be a saddle point of $\mathcal{L}_r$, then

$$\mathcal{L}_r(u,\eta,\lambda) \leq \mathcal{L}_r(v,q,\lambda) \quad \forall (v,q,\mu)\in V\times H\times H, \quad (u,\eta)\in V\times H.$$

Therefore (u,η) is characterized by

$$G(v) - G(u) + \langle \lambda, B(v-u)\rangle + r\langle Bu-\eta, B(v-u)\rangle \geq 0, \quad \forall v\in V,\, u\in V,$$
$$F(q) - F(\eta) - \langle \lambda, q-\eta\rangle + r\langle \eta - Bu, q-\eta\rangle \geq 0, \quad \forall q\in H,\, \eta\in H.$$

Moreover, since F and G are convex and Gateaux–diffentiable, (u,η) is also characterized by

$$\begin{aligned} &G'(u)(v-u) + \langle \lambda, B(v-u)\rangle + r\langle Bu-\eta, B(v-u)\rangle \geq 0, \quad \forall v\in V,\, u\in V,\\ &F'(\eta)(q-\eta) - \langle \lambda, q-\eta\rangle + r\langle \eta-Bu, q-\eta\rangle \geq 0, \quad \forall q\in H,\, \eta\in H, \end{aligned} \tag{4.8}$$

where G' and F' are the Gateaux-derivative of G and F, respectively.

For both characterizations, see [23, Chapter I and VI].

4.1. **Algorithms.** To solve the saddle point problem (4.7) we will use an Uzawa type algorithm and a variant of it. See [7, 22, 24].

Algorithm 1. Let $\lambda^0 \in H$: then λ^n is known , we define $(u^n, \eta^n, \lambda^{n+1}) \in V \times H \times H$ by

$$\mathcal{L}_r(u^n, \eta^n, \lambda^n) \leq \mathcal{L}_r(v, q, \lambda^n) \quad \forall (v, q) \in V \times H,$$
$$\lambda^{n+1} = \lambda^n + \rho_n (Bu^n - \eta^n), \quad \rho_n > 0.$$

Remark 4.3. Observe that the first inequality of this algorithm is equivalent to the following system of two coupled variational inequalities,

$$\begin{aligned} G(v) - G(u^n) + \langle \lambda^n, B(v - u^n) \rangle + r\langle Bu^n - \eta^n, B(v - u^n) \rangle \geq 0, \quad \forall v \in V, u^n \in V, \\ F(q) - F(\eta^n) - \langle \lambda^n, q - \eta^n \rangle + r\langle \eta^n - Bu^n, q - \eta^n \rangle \geq 0, \quad \forall q \in H, \eta^n \in H, \end{aligned} \tag{4.9}$$

see [23, Chapter VI–Section 3].

The main difficulty of Algorithm 1 is that it requires the solution of the coupled system of equations at each iteration. To overcome this difficulty, in [23] the authors propose the following algorithm.

Algorithm 2. Let $(\eta^0, \lambda^1) \in H \times H$; then (η^{n-1}, λ^n) known, we define $(u^n, \eta^n, \lambda^{n+1}) \in V \times H \times H$ by

$$\begin{aligned} G(v) - G(u^n) + \langle \lambda^n, B(v - u^n) \rangle + r\langle Bu^n - \eta^{n-1}, B(v - u^n) \rangle \geq 0, \quad \forall v \in V, u^n \in V, \\ F(q) - F(\eta^n) - \langle \lambda^n, q - \eta^n \rangle + r\langle \eta^n - Bu^n, q - \eta^n \rangle \geq 0, \quad \forall q \in H, \eta^n \in H, \\ \lambda^{n+1} = \lambda^n + \rho_n (Bu^n - \eta^n), \quad \rho_n > 0. \end{aligned} \tag{4.10}$$

Observe that now the two first equations are uncoupled.

Remark 4.4. In [23], in a more general context, the convergence of both algorithms are proved. More precisely, if F, G and B satiefy the assumptions (H1), (H2) and (H3), then

$$\begin{aligned} \lim_{n \to +\infty} \|Bu^{-n} - \eta^{-n}\| = 0, \\ \lim_{n \to +\infty} \langle F'(\eta^n) - F'(\eta), \eta^n - \eta \rangle = 0, \\ \lim_{n \to +\infty} \|Bu^n - \eta^n\| = 0, \\ \lim_{n \to +\infty} \|\eta^n - \eta\| = 0, \\ Bu^n \to \eta = Bu \text{ in } H, \\ \lambda^{n+1} - \lambda^n \to 0 \text{ in } H, \\ \lambda^n \text{ is bounded}, \end{aligned} \tag{4.11}$$

see Theroem 4.1 and Theorem 5.1 in [23, Chapter VI]. The assumption (H4) is only used to concluded that $u^n \to u$ in V.

In our case, (H4) does not hold, that is B is not injective. To overcome the lack of this assumption, we use that our functional G is Gateaux-differentiable and convex.

Proof of Theorem 1.3. By Remark 4.3 and using the same argument of Remark 4.2, u^n can be characterized as

$$G'(u^n)(v - u^n) + \langle \lambda^n, B(v - u^n) \rangle + r\langle Bu^n - \eta^n, B(v - u^n) \rangle \geq 0, \quad \forall v \in V, \ u^n \in V, \tag{4.12}$$

Let us denote $u^{-n} = u^n - u$ and $\eta^{-n} = \eta^n - \eta$. By Remark 4.4, we have that (4.11) holds.

On the other hand, taking $v = u^n$ in (4.8), $v = u$ in (4.12) and summing we obtain

$$(G'(u^n) - G'(u))(u - u^n) + \langle \lambda^n - \lambda, B(u - u^n)\rangle + r\langle B(u^n - u) + \eta - \eta^n, B(u - u^n)\rangle \geq 0,$$

then

$$(G'(u^n) - G'(u))(u^n - u) + \langle \lambda^n - \lambda, B(u^n - u)\rangle + r\langle B(u^n - u) - (\eta^n - \eta), B(u^n - u)\rangle \leq 0.$$

Since $(G'(u^n) - G'(u))(u^n - u) \geq 0$, by (4.11), we get

$$(G'(u^n) - G'(u))(u^n - u) = \int_\Omega |u^n - u|^2 dx + \int_{\Gamma_{int}} |[\![u^n - u]\!]|^2 \mathbf{h}^{-2/p'(x)} dS + \int_{\partial\Omega} |u^n - u|^2 \mathbf{h}^{-2/p'(x)} dS \to 0 \quad \text{as } n \to +\infty,$$

then

$$\begin{aligned} u^n &\to u \text{ in } V, \\ [\![u^n - u]\!] &\to 0 \text{ in } L^2(\Gamma_{int}), \\ u^n &\to u \text{ in } L^2(\partial\Omega). \end{aligned} \tag{4.13}$$

Finally,

$$\nabla u^{-n} \to 0 \quad \text{in } H. \tag{4.14}$$

due to

$$\begin{aligned} \|R(u^{-n})\|_{L^2(\Omega)} &\leq C\|\mathbf{h}^{-1/2}[\![u^{-n}]\!]\|_{L^2(\Gamma_{int})} \to 0, \\ B(u^{-n}) &\to 0 \quad \text{in } H. \end{aligned}$$

The proof is now completed. □

Finally, we prove the convergence of Algorithm 2.

Proof of Theorem 1.3. We began by observing that, as in the proof of Theorem 1.3, by Remark 4.4, we obtain that (4.11) holds.

On the other hand, for this algorithm, we get that u^n satisfies that

$$G'(u^n)(v - u^n) + \langle \lambda^n, B(v - u^n)\rangle + r\langle Bu^n - \eta^{n-1}, B(v - u^n)\rangle \geq 0, \quad \forall v \in V \tag{4.15}$$

Therefore, taking $v = u^n$ in (4.8), $v = u$ in (4.15) and fallowing the lines of the proof of Theorem 1.3, we get (4.13) and (4.14). The proof is now completed. □

5. Numerical Results

In this section, we only implement the uncoupled Algorithm 2. For any h, we obtain a sequence $\{u_h^n\}$ such that $u_h^n \to u^h$ as $n \to +\infty$, where u_h is the minimizer of J_h.

For simplicity, we take

$$J_h(v) = F(Bv) + G(v),$$

where

$$\begin{aligned} F(q) &= \int_\Omega \frac{|q|^{p(x)}}{p(x)}\, dx, \\ G(v) &= \frac{1}{2}\left(\int_\Omega |v - \xi|^2\, dx + \int_{\partial\Omega} |v - u_D|^2 \mathbf{h}^{-2/p'(x)}\, dS + \int_{\Gamma_{int}} |[\![v]\!]|^2 \mathbf{h}^{-2/p'(x)}\, dS\right). \end{aligned}$$

Observe that, this new definition of F does not change any of the results that we prove in the preceding sections.

If we take $\rho_n = r$ then the algorithm is:
Given

$$(\eta^0, \lambda^1) \in H \times H,$$

then, $(\eta_h^{n-1}, \lambda_h^n)$ known, we define $(u_h^n, \eta_h^n, \lambda_h^{n+1}) \in V \times H \times H$ by

$$\begin{aligned}\int_\Omega (u_h^n - \xi)v\,dx + r\int_\Omega \left(Bu_h^n - \eta_h^{n-1}\right)Bv\,dx + \int_\Omega \lambda_h^n Bv\,dx \\ + \int_{\Gamma_{int}} [\![u_h^n]\!][\![v]\!]\mathbf{h}^{-2/p'(x)}dS + \int_{\partial\Omega}(u_h^n - u_D)v\mathbf{h}^{-2/p'(x)}dS = 0,\end{aligned} \tag{5.16}$$

$$\int_\Omega |\eta_h^n|^{p(x)-2}\eta_h^n \Phi\,dx + r\int_\Omega (\eta_h^n - Bu_h^n)\Phi\,dx = \int_\Omega \lambda_h^n \Phi\,dx, \tag{5.17}$$

$$\lambda_h^{n+1} = \lambda_h^n + r(Bu_h^n - \eta_h^n), \tag{5.18}$$

for all $v \in S^k(\mathcal{T}_h)$ and for all $\Phi \in S^l(\mathcal{T}_h) \times S^l(\mathcal{T}_h)$.

Remark 5.1. Since V, H, F, G, B, ρ_n and r satisfy the assumptions of Theorem 1.4, then the conclusions of Theorem 1.4 hold, that is, $u_h^n \to u_h$ and $\nabla u_h^n \to \nabla u_h$, as $n \to +\infty$.

Observe that (5.16) can be replace by,

$$MU^n = F^n,$$

where

$$M_{ij} = \int_\Omega \varphi_i\varphi_j\,dx + r\int_\Omega B\varphi_i B\varphi_j\,dx + \int_{\Gamma_{int}} [\![\varphi_i]\!][\![\varphi_i]\!]\mathbf{h}^{-2/p'(x)}\,dS + \int_{\partial\Omega}\varphi_i\varphi_j\mathbf{h}^{-2/p'(x)}\,dS,$$

$$F_j^n = \int_\Omega \varphi_j\xi\,dx + \int_{\partial\Omega}\varphi_j u_D\mathbf{h}^{-2/p'(x)}\,dS + \int_\Omega (r\eta_h^{n-1} - \lambda_h^n)B\varphi_j\,dx,$$

and $\{\varphi_j\}_{j\le m}$ is a basis of V with $m = \dim(V)$. Thus

$$u_h^n = \sum_{j=1}^m U_j^n \varphi_j.$$

On the other hand, we define $\eta_{n,\kappa} = \eta_h^n|_\kappa$, in the same way we define $\lambda_{n,\kappa}$ and $B_\kappa u_h^n$. We can see from (5.17) that $\eta_{n,\kappa}$ satisfies

$$\left(\frac{1}{|\kappa|}\int_\kappa |\eta_{n,\kappa}|^{p(x)-2}\,dx + r\right)\eta_{n,\kappa} = \lambda_{n,\kappa} + B_\kappa u_h^n.$$

Let $\bar{p}_\kappa = p(\bar{x}_\kappa)$, where $\bar{x}_\kappa$ is the varicenter of κ. Then using a quadrature rule for the first term, we can approximate $\eta_{n,\kappa}$ by the equation,

$$(|\eta_{n,\kappa}|^{\bar{p}_\kappa - 2} + r)\eta_{n,\kappa} = \lambda_{n,\kappa} + B_\kappa u_h^n,$$

thus $|\eta_{n,\kappa}|$ solves

$$(|\eta_{n,\kappa}|^{\bar{p}_\kappa - 2} + r)|\eta_{n,\kappa}| = |\lambda_{n,\kappa} + B_\kappa u_h^n|,$$

and therefore

$$\eta_{n,\kappa} = \frac{\lambda_{n,\kappa} + B_\kappa u_h^n}{|\eta_{n,\kappa}|^{\bar{p}_\kappa - 2} + r}.$$

Summarizing, each iteration of the algorithm can be reduced to the following:

Find $(u_h^n, \eta_h^n, \lambda_h^{n+1}) \in V \times H \times H$ such that

$$\begin{aligned} u_h^n &= \sum_{j=1}^m U_j^n \varphi_j, \\ \eta_{n,\kappa} &= \frac{\lambda_{n,\kappa} + B_\kappa u_h^n}{x^{\bar{p}_\kappa - 2} + r}, \\ \lambda_h^{n+1} &= \lambda_h^n + r(Bu_h^n - \eta_h^n). \end{aligned}$$

where U^n solves,

$$MU^n = F^n \tag{5.19}$$

and $x \in \mathbb{R}_{\geq 0}$ solves

$$x^{\bar{p}_\kappa - 1} + rx = |\lambda_{n,\kappa} + B_\kappa u_h^n|, \tag{5.20}$$

Observe that each step of the algorithm consists in solving the linear equation (5.19) and then the one dimensional nonlinear equation (5.20).

We now apply the algorithm to a family of examples. For each h, we approximate u_h by u_h^n, and finally we compute $\|u_h^n - u\|_{L^2(\Omega)}$.

Motivate by [25], where the authors analyse a P_0 discontinuous Galerkin formulation for image denosing, we test this algorithm in the following example; we have considered a rectangular domain $\Omega = [-1\ 1] \times [-1\ 1]$ and a uniform rectangular mesh, with constant finite elements in all the rectangles. We denote by m the number of degrees of freedom in the finite element approximation. We take $r = 1$.

We take the following function $p(x)$,

$$p(x) = \begin{cases} 1 + \left(\frac{b}{2}(x_1 + x_2) + 1 + b \right)^{-1} & \text{if } b \neq 0, \\ 2 & \text{if } b = 0. \end{cases}$$

Observe that $p_2 = 2$ and $p_1 = 1 + {}^1\!/_{1+2b}$, then p_1 is close to one when $b >> 0$.

It is easy to see that the solution of (P) is

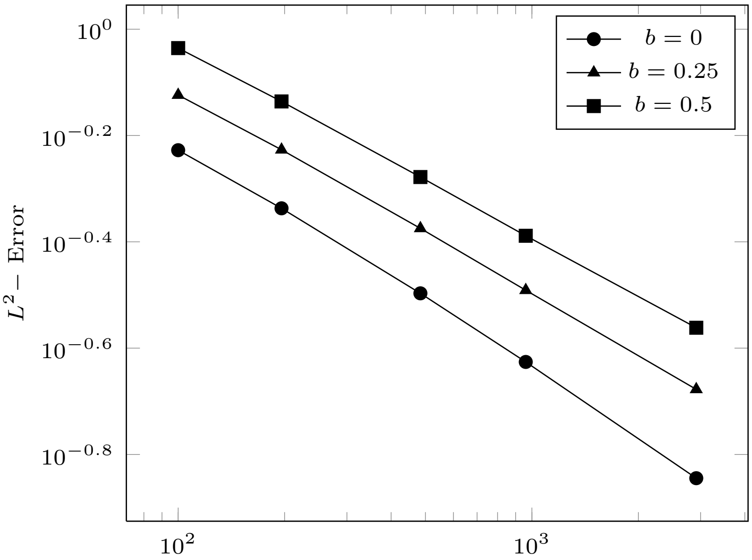
$$u(x) = \begin{cases} \frac{\sqrt{2}e^{b+1}}{b} \left(e^{\frac{b}{2}(x_1+x_2)} - 1 \right) & \text{if } b \neq 0, \\ \frac{\sqrt{2}e}{2}(x_1 + x_2) & \text{if } b = 0. \end{cases}$$

with $\xi = u$.

The experimental results for different values of b and m are shown in the following table.

b	0		0.25		0.5	
m	L^2-Error	Iter.	L^2-Error	Iter.	L^2-Error	Iter.
100	0.5921	3	0.7519	46	0.9214	50
196	0.4603	3	0.5932	47	0.7313	52
484	0.3185	3	0.4220	48	0.5271	56
961	0.2366	3	0.3228	49	0.4087	59
2916	0.1430	3	0.2101	50	0.2744	63

Where Iter. is the number of iterations required in the algorithm in order to satisfy our stopping criteria. Observe that as b grows, the number of iterations increases and the rate of convergence decreases.



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