AN OPTIMIZATION PROBLEM FOR THE FIRST STEKLOV EIGENVALUE OF A NONLINEAR PROBLEM

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Abstract. In this paper we study the first (nonlinear) Steklov eigenvalue, λ, of the following problem: $-\Delta_p u + |u|^{p-2}u + \alpha\phi|u|^{p-2}u = 0$ in a bounded smooth domain Ω with $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda|u|^{p-2}u$ on the boundary $\partial\Omega$. We analyze the dependence of this first eigenvalue with respect to the weight ϕ and with respect to the parameter α. We prove that for fixed α there exists an optimal ϕ_α that minimizes λ in the class of uniformly bounded measurable functions with fixed integral. Next, we study the limit of these minima as the parameter α goes to infinity and we find that the limit is the first Steklov eigenvalue in the domain with a hole where the eigenfunctions vanish.

1. Introduction

Given a domain $\Omega \subset \mathbb{R}^N$ (bounded, connected, with smooth boundary), $\alpha > 0$ and $E \subset \Omega$ a measurable set, we want to study the eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \alpha\chi_E|u|^{p-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda|u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual $p-$laplacian, $\frac{\partial}{\partial\nu}$ is the outer normal derivative, λ stands for the eigenvalue and α is a positive parameter. Remark that in this problem the eigenvalue appears in the boundary condition. These type of problems are known as *Steklov eigenvalue problems*, see [18]. Observe that when $p = 2$ the problem becomes linear. All our results are new even for the linear case.

We denote the first eigenvalue by $\lambda(\alpha, E)$. The existence of this first eigenvalue and a positive associated eigenfunction follows easily from the variational characterization

$$\lambda(\alpha, E) := \inf_{\substack{v\in W^{1,p}(\Omega),\\ \|v\|_{L^p(\partial\Omega)}=1}} \int_\Omega |\nabla v|^p + |v|^p \,\mathrm{d}x + \alpha\int_E |v|^p \,\mathrm{d}x, \tag{1.2}$$

and the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, see [7].

Once the set E is fixed, it is not difficult to check that when $\alpha \to \infty$ the eigenvalues converge to the first eigenvalue of the problem with E as a hole (the eigenfunctions vanish on E). That is,

$$\lim_{\alpha\to\infty} \lambda(\alpha, E) = \lambda(\infty, E),$$

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where

$$\lambda(\infty, E) := \inf_{\substack{v\in W^{1,p}(\Omega),\, v|_E\equiv 0,\\ \|v\|_{L^p(\partial\Omega)}=1}} \int_\Omega |\nabla v|^p + |v|^p \,\mathrm{d}x.$$

The aim of this paper is to study the following optimization problem: for a fixed α we want to optimize $\lambda(\alpha, E)$ with respect to E, that is, we want to look at the infimum,

$$\inf_{\substack{E\subset\Omega,\\ |E|=A}} \lambda(\alpha, E) \tag{1.3}$$

for a fixed volume $A \in [0, |\Omega|]$ (with $|\cdot|$ denoting volume). Moreover, we want to study the limit as $\alpha \to \infty$ in the above infimum. The natural limit problem for these infima is

$$\lambda(\infty, A) := \inf_{\substack{E\subset\Omega,\\ |E|=A}} \lambda(\infty, E). \tag{1.4}$$

These kind of problems appear naturally in optimal design problems. They are usually formulated as problems of minimization of the energy, stored in the design under a prescribed loading. Solutions of these problems are unstable to perturbations of the loading. The stable optimal design problem is formulated as minimization of the stored energy of the project under the most unfavorable loading. This most dangerous loading is one that maximizes the stored energy over the class of admissible functions. The problem is reduced to minimization of Steklov eigenvalues. See [3].

Also this limit problem (1.4) can be regarded as the study of the best Sobolev trace constant for functions that vanish in a subset of prescribed measure. The study of optimal constants in Sobolev embeddings is a very classical subject (see [5]). Related problems for the best Sobolev trace constant can be found in [6, 9]. In our case the limit problem was studied in [10] where an optimal configuration is shown to exists and some properties of this optimal configuration are obtained. Among them it is proved that $\lambda(\infty, A)$ is strictly increasing with respect to A. In a companion paper [11] the interior regularity of the optimal hole is analyzed.

To begin the study of our optimization problem (1.2) we prove that there exists an optimal configuration. To this end, it is better to relax the problem and consider $\phi \in L^\infty(\Omega)$, such that $0 \le \phi \le 1$ and $\int_\Omega \phi(x)\,\mathrm{d}x = A$ instead of χ_E. Hence we consider the problem,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \alpha\phi|u|^{p-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

This relaxation is natural in the use of the direct method in the calculus of variations since $\{\phi \in L^\infty(\Omega),\, 0 \le \phi \le 1$ and $\int_\Omega \phi(x)\,\mathrm{d}x = A\}$ is closed in the weak$*$ topology in $L^\infty(\Omega)$. In fact, this set is the closure in this topology of the set of characteristic functions $\{\chi_E,\, |E| = A\}$.

We denote by $\lambda(\alpha, \phi)$ the lowest eigenvalue of (1.5). This eigenvalue has the following variational characterization

$$\lambda(\alpha, \phi) := \inf_{\substack{v\in W^{1,p}(\Omega),\\ \|v\|_{L^p(\partial\Omega)}=1}} \int_\Omega |\nabla v|^p + |v|^p \,\mathrm{d}x + \alpha \int_\Omega \phi |v|^p \,\mathrm{d}x. \tag{1.6}$$

As an immediate consequence of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, the above infimum is in fact a minimum. There exists $u = u_{\alpha,\phi} \in W^{1,p}(\Omega)$ such that $\|u\|_{L^p(\partial\Omega)} = 1$ and

$$\lambda(\alpha, \phi) = \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x + \alpha \int_\Omega \phi |u|^p \, \mathrm{d}x.$$

Moreover, u is a weak solution of (1.5), does not changes sign (see [7, 8, 16]) and hence, by Harnack's inequality (see [20]), it can be assumed that u is strictly positive in $\bar{\Omega}$.

Define

$$\Lambda(\alpha, A) = \inf_{\substack{\phi : \int \phi = A, \\ 0 \le \phi \le 1}} \lambda(\alpha, \phi). \tag{1.7}$$

Any minimizer ϕ in (1.7) will be called an optimal configuration for the data (α, A). If ϕ is an optimal configuration and u satisfies (1.5) then (u, ϕ) will be called an optimal pair (or solution).

By the direct method of the calculus of variations it is not difficult to see that there exits an optimal pair. The main point of the following result is to show that we can recover a *classical* solution of our original problem (1.3). In fact, if (u, ϕ) is an optimal pair, then $\phi = \chi_D$ for some measurable set $D \subset \Omega$. Moreover, the set D is shown to be a sublevel set of u.

Theorem 1.1. *For any $\alpha > 0$ and $A \in [0, |\Omega|]$ there exists an optimal pair. Moreover, any optimal pair (u, ϕ) has de following properties:*

(1) *$u \in C^{1,\delta}_{\mathrm{loc}}(\Omega) \cap W^{2,q}(\Omega) \cap C^\gamma(\overline{\Omega})$ for some $\gamma > 0$, $\delta < 1$ and $q = \min\{p, 2\}$.*
(2) *There exists and optimal configuration $\phi = \chi_D$, where D is a sublevel set of u, i.e. there is a number $t \ge 0$ such that $D = \{u \le t\}$.*
(3) *Every level set $\{u = s\}$, has Lebesgue measure zero.*

For the proof we use ideas from [1, 2] where a similar linear problem with homogeneous Dirichlet boundary conditions was studied.

We can compute the derivative from the right of $\lambda(\alpha, \phi)$ with respect to ϕ in an admissible direction. Let us denote by F the set of admissible directions,

$$F = \left\{ f : \, f \le 0 \text{ in } \{\phi = 1\}, \, f \ge 0 \text{ in } \{\phi = 0\}, \, \int_\Omega f = 0 \right\}. \tag{1.8}$$

Then, the right derivative of $\lambda(\alpha, \phi)$ with respect to ϕ in direction of $f \in F$ is given by

$$\lambda'(\alpha, \phi)(f) = \lim_{t \searrow 0^+} \frac{\lambda(\alpha, \phi + tf) - \lambda(\alpha, \phi)}{t} = \alpha \int_\Omega f |u|^p \, dx,$$

where u is an eigenfuction associated to $\lambda(\alpha, \phi)$, see Proposition 2.1.

Next, we analyze the limit as $\alpha \to \infty$ of the optimal configurations found in Theorem 1.1. We give a rigorous proof of the convergence of these optimal configurations to those of (1.4).

Theorem 1.2. *For any sequence $\alpha_j \to \infty$ and optimal pairs (D_j, u_j) of (1.3) there exists a subsequence, that we still call α_j, and an optimal pair (D, u) of (1.4) such* *that*

$$\lim_{j\to\infty} \chi_{D_j} = \chi_D, \qquad \textit{weakly* in } L^\infty(\Omega),$$
$$\lim_{j\to\infty} u_j = u, \qquad \textit{strongly in } W^{1,p}(\Omega).$$

Moreover, $u > 0$ *in* $\Omega \setminus D$.

Finally, we study symmetry properties of the optimal configuration when Ω is the unit ball. For the definition of a spherically symmetric function see [14, 17] and Section 4.

Theorem 1.3. *Fix* $\alpha > 0$ *and* $0 < A < |B(0,1)|$, *there exists an optimal pair of* (1.5), (u, χ_D), *such that* u *and* D *are spherically symmetric. Moreover, when* $p = 2$, *every optimal pair* (u, χ_D) *is spherically symmetric.*

The rest of the paper is organized as follows: in Section 2 we prove that there exists an optimal configuration; in Section 3 we analyze the limit $\alpha \to \infty$ and finally in Section 4 we study the symmetry properties of the optimal pairs in a ball.

2. Existence of an optimal configuration

In this section we prove that there exists an optimal configuration for the relaxed problem and find some properties of it.

Proof of Theorem 1.1. To prove existence, fix α and A, and write $\Lambda = \Lambda(\alpha, A)$, $\lambda(\phi) = \lambda(\alpha, \phi)$ to simplify the notation. Let ϕ_j be a minimizing sequence, i.e., $0 \le \phi_j \le 1$, $\int_\Omega \phi_j \, dx = A$ and $\lambda(\phi_j) \to \Lambda$ as $j \to \infty$.

Let $u_j \in W^{1,p}(\Omega)$, be a normalized eigenfunction associated to $\lambda(\phi_j)$, that is, u_j verifies $\|u_j\|_{L^p(\partial\Omega)} = 1$ and

$$\begin{aligned} \lambda(\phi_j) &= \int_\Omega |\nabla u_j|^p + |u_j|^p \, dx + \alpha \int_\Omega \phi_j |u_j|^p \, dx \\ &= \inf_{\substack{v \in W^{1,p}(\Omega), \\ \|v\|_{L^p(\partial\Omega)}=1}} \int_\Omega |\nabla v|^p + |v|^p \, dx + \alpha \int_\Omega \phi_j |v|^p \, dx. \end{aligned} \tag{2.9}$$

Then, u_j is a positive weak solution of

$$\begin{cases} -\Delta_p u_j + |u_j|^{p-2} u_j + \alpha\phi |u_j|^{p-2} u_j = 0 & \text{in } \Omega, \\ |\nabla u_j|^{p-2} \frac{\partial u_j}{\partial \nu} = \lambda(\phi_j) |u_j|^{p-2} u_j & \text{on } \partial\Omega, \\ \|u_j\|_{L^p(\partial\Omega)} = 1. \end{cases}$$

Since $\lambda(\phi_j)$ is bounded, the sequence u_j is bounded in $W^{1,p}(\Omega)$. Also $\{\phi_j\}$ is bounded in $L^\infty(\Omega)$. Therefore, we may choose a subsequence (again denoted u_j, ϕ_j) and $u \in W^{1,p}(\Omega)$, $\phi \in L^\infty(\Omega)$ such that

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega) \tag{2.10}$$
$$u_j \to u \quad \text{strongly in } L^p(\Omega) \tag{2.11}$$
$$u_j \to u \quad \text{strongly in } L^p(\partial\Omega) \tag{2.12}$$
$$\phi_j \overset{*}{\rightharpoonup} \phi \quad \text{weakly* in } L^\infty(\Omega) \tag{2.13}$$

By (2.11), $\|u\|_{L^p(\partial\Omega)} = 1$ and by (2.13) $0 \le \phi \le 1$ and $\int_\Omega \phi \, dx = A$.

Now taking limits in (2.9), we get

$$\begin{aligned}\Lambda = \lim_{j\to\infty} \lambda(\phi_j) &\geq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^p + |u_j|^p \,\mathrm{d}x + \alpha \int_\Omega \phi_j |u_j|^p \,\mathrm{d}x \\ &\geq \int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x + \alpha \int_\Omega \phi |u|^p \,\mathrm{d}x\end{aligned} \tag{2.14}$$

Therefore, (u, ϕ) is an optimal pair and so u is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \alpha\phi|u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \Lambda |u|^{p-2}u & \text{on } \partial\Omega. \end{cases}$$

That (1) holds is a consequence of the regularity theory for quasilinear elliptic equations with bounded coefficients developed, for instance, in [19].

To prove (2), observe that the minimization problem

$$\inf_{\substack{\phi:\int\phi=A\\ 0\leq\phi\leq 1}} \int_\Omega \phi|u|^p \,\mathrm{d}x$$

has a solution $\phi = \chi_D$ where D is any set with $|D| = A$ and

$$\{u < t\} \subset D \subset \{u \leq t\}, \quad t := \sup\{s : |\{u < s\}| < A\}$$

(compare with the *Bathtub Principle*, see [15]). Therefore, we get from (2.14)

$$\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x + \alpha \int_\Omega \chi_D |u|^p \,\mathrm{d}x \leq \Lambda.$$

By definition of Λ as a minimum, this must actually be an equality, and (u, χ_D) is an optimal pair.

Finally, if (u, χ_D) is any solution and $\mathfrak{N}_s = \{u = s\}$ for any $s > 0$. Using Lemma 7.7 from [12] twice, we see that $\Delta_p u = 0$ a.e. on $\mathfrak{N}_s$. Then

$$|u|^{p-2}u + \alpha\chi_D|u|^{p-2}u = 0 \quad \text{a.e. on } \mathfrak{N}_s. \tag{2.15}$$

As $u = s > 0$ on $\mathfrak{N}_s$, we have

$$|u|^{p-2}u + \alpha\chi_D|u|^{p-2}u > 0 \quad \text{on } \mathfrak{N}_s.$$

Then $|\mathfrak{N}_s| = 0$, we get (3). Taking $s = t$ we get (2). $\square$

Now, we find the derivative of $\lambda(\alpha, \phi)$ in an admissible direction $f \in F$, given by (1.8).

Proposition 2.1. *Let $f \in F$, then the derivative from the right of $\lambda(\alpha, \phi)$ in the direction of $f \in F$ is given by*

$$\lambda'(\alpha, \phi)(f) = \lim_{t\searrow 0} \frac{\lambda(\alpha, \phi + tf) - \lambda(\alpha, \phi)}{t} = \alpha \int_\Omega f|u|^p \,dx, \tag{2.16}$$

where u is an eigenfunction of $\lambda(\alpha, \phi)$.

Proof. Let us consider the curve

$$\phi_t = \phi + tf.$$

Note that since $f \in F$ and ϕ is admissible then ϕ_t is admissible for every $t \geq 0$ small enough. Therefore, we may compute $\lambda(\alpha, \phi_t)$.

Using an eigenfuction u_t of $\lambda(\alpha,\phi_t)$ in the variational formulation of $\lambda(\alpha,\phi)$ we get

$$\frac{\lambda(\alpha,\phi_t)-\lambda(\alpha,\phi)}{t}\le \alpha\int_\Omega f|u_t|^p\,dx. \tag{2.17}$$

On the other hand, using u in the variational formulation of $\lambda(\alpha,\phi_t)$ we get

$$\frac{\lambda(\alpha,\phi_t)-\lambda(\alpha,\phi)}{t}\ge \alpha\int_\Omega f|u|^p\,dx. \tag{2.18}$$

As before, using $v=1$ as a test function in the definition of $\lambda(\alpha,\phi_t)$ we obtain that the family $\{u_t\}_{0\le t\le t_0}$ is bounded in $W^{1,p}(\Omega)$. Then, by our previous arguments we have that

$$u_t\to u \qquad \text{strongly in } L^p(\Omega) \text{ when } t\to 0.$$

Hence, taking limits in (2.17) and (2.18) we conclude (2.16). □

Using this Proposition we can easily prove again that the optimal set must be a sublevel set of u.

Corollary 2.2. *The optimal set D satisfies*

$$D=\{u\le t\}.$$

Proof. As χ_D realizes the minimum of $\lambda(\alpha,\phi)$ we have for all $f\in F$,

$$\lambda'(\alpha,\chi_D)(f)=\alpha\int_\Omega f|u|^p\,dx\ge 0, \tag{2.19}$$

Given two points $x_0\in D$ of positive density (i.e., for every $\varepsilon>0$, $|B(x_0,\varepsilon)\cap D|>0$) and $x_1\in(\Omega\setminus D)$ also with positive density we can take a function $f\in F$ of the form $f=M\chi_{T_0}-M\chi_{T_1}$ with $T_0\subset B(x_0,\varepsilon)\cap D$, $T_1\subset B(x_1,\varepsilon)\cap(\Omega\setminus D)$ and $M^{-1}=|T_0|=|T_1|$. It is clear that $f\in F$. From our expression for the right derivative (2.16) and using that D is a minimizer, taking the limit as $\varepsilon\to 0$ and using the continuity of u we get $u(x_0)\le u(x_1)$. We conclude that $D=\{u\le t\}$. □

3. Limit as $\alpha\to\infty$.

In this section we analyze the limit as $\alpha\to\infty$ in problem (1.7).

Proof of Theorem 1.2. Let $(u_\alpha,\chi_{D_\alpha})$ be a solution to our minimization problem

$$\Lambda(\alpha,A)=\inf_{\substack{u\in W^{1,p}(\Omega),\\ \phi:\int\phi=A,\,0\le\phi\le 1}}\frac{\displaystyle\int_\Omega|\nabla u|^p\,\mathrm{d}x+\int_\Omega|u|^p\,\mathrm{d}x+\alpha\int_\Omega\phi|u|^p\,\mathrm{d}x}{\displaystyle\int_{\partial\Omega}|u|^p\,\mathrm{d}S}.$$

Recall that u_α is a positive weak solution of

$$\begin{cases}-\Delta_p u+|u|^{p-2}u+\alpha\phi|u|^{p-2}u=0 & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}=\Lambda(\alpha,A)|u|^{p-2}u & \text{on } \partial\Omega,\\ \|u\|_{L^p(\partial\Omega)}=1.\end{cases}$$

Let $u_0 \in W^{1,p}(\Omega)$ and $D_0 \subset \Omega$ be such that $|D_0| = A$ and $u_0\chi_{D_0} = 0$. Then, we have that

$$\begin{aligned}\Lambda(\alpha, A) &\leq \frac{\displaystyle\int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega |u_0|^p \,\mathrm{d}x + \alpha \int_\Omega \chi_{D_0}|u_0|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u_0|^p \,\mathrm{d}S} \\ &= \frac{\displaystyle\int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega |u_0|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u_0|^p \,\mathrm{d}S} = K\end{aligned}$$

with K independent of α.

Thus $\{\Lambda(\alpha, A)\}$ is a bounded sequence in $\mathbb{R}$ and it is clearly increasing. As a consequence, $\{u_\alpha\}$ is bounded in $W^{1,p}(\Omega)$. Moreover $\{\chi_{D_\alpha}\}$ is bounded in $L^\infty(\Omega)$. Therefore, we may choose a sequence α_j and $u_\infty \in W^{1,p}(\Omega)$, $\phi_\infty \in L^\infty(\Omega)$ such that

$$u_{\alpha_j} \rightharpoonup u_\infty \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.20}$$
$$u_{\alpha_j} \to u_\infty \quad \text{strongly in } L^p(\Omega), \tag{3.21}$$
$$u_{\alpha_j} \to u_\infty \quad \text{strongly in } L^p(\partial\Omega), \tag{3.22}$$
$$\chi_{D_{\alpha_j}} \overset{*}{\rightharpoonup} \phi_\infty \quad \text{weakly} * \text{ in } L^\infty(\Omega), \tag{3.23}$$

By (3.22) $\|u_\infty\|_{L^p(\Omega)} = 1$ and by (3.23) $0 \leq \phi_\infty \leq 1$ with $\int_\Omega \phi_\infty \,\mathrm{d}x = A$. Also, by (3.21) and (3.23) it holds

$$\int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \to \int_\Omega \phi_\infty |u_\infty|^p \,\mathrm{d}x.$$

As

$$0 \leq \alpha_j \int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \leq \Lambda_{\alpha_j} \leq K \quad \text{for all } j,$$

we have

$$0 \leq \int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \leq \frac{K}{\alpha_j} \quad \text{for all } j,$$

then

$$\int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \to 0.$$

Therefore

$$\int_\Omega \phi_\infty |u_\infty|^p \,\mathrm{d}x = 0$$

and we conclude that

$$\phi_\infty u_\infty = 0 \quad \text{a.e. } \Omega.$$

Since $\{\Lambda(\alpha_j, A)\}$ is bounded and increasing there exists the limit

$$\lim_{j\to\infty} \Lambda(\alpha_j, A) = \Lambda_\infty < +\infty.$$

Then

$$\begin{aligned}
\Lambda_\infty &= \lim_{j\to\infty} \int_\Omega |\nabla u_{\alpha_j}|^p \,\mathrm{d}x + \int_\Omega |u_{\alpha_j}|^p \,\mathrm{d}x + \alpha_j \int_\Omega \chi_{\alpha_j} |u_{\alpha_j}|^p \,\mathrm{d}x \\
&\geq \liminf_{j\to\infty} \int_\Omega |\nabla u_{\alpha_j}|^p \,\mathrm{d}x + \int_\Omega |u_{\alpha_j}|^p \,\mathrm{d}x \\
&\geq \int_\Omega |\nabla u_\infty|^p \,\mathrm{d}x + \int_\Omega |u_\infty|^p \,\mathrm{d}x.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\Lambda_\infty &\geq \int_\Omega |\nabla u_\infty|^p \,\mathrm{d}x + \int_\Omega |u_\infty|^p \,\mathrm{d}x \\
&\geq \inf_{\substack{u\in W^{1,p}(\Omega),\ \|u\|_{L^p(\partial\Omega)}=1, \\ \phi: \int \phi = A,\ \phi u = 0}} \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x \\
&\geq \Lambda(\alpha_j, A) \qquad \text{for all } j.
\end{aligned}$$

Therefore

$$\Lambda_\infty = \inf_{\substack{u\in W^{1,p}(\Omega),\ \|u\|_{L^p(\partial\Omega)}=1, \\ \phi: \int \phi = A,\ \phi u = 0}} \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x = \int_\Omega |\nabla u_\infty|^p + |u_\infty|^p \,\mathrm{d}x,$$

and so the infimum in the above equation is achieved by (u_∞, ϕ_∞).

Now, if we take $D_\infty = \{\phi_\infty > 0\}$ we get that $|D_\infty| = B \geq A$. Hence

$$\lambda(\infty, B) \leq \lambda(\infty, D_\infty) = \Lambda_\infty \leq \lambda(\infty, A).$$

This implies that $|D_\infty| = A$ (otherwise we have a contradiction with the strict monotonicity of $\lambda(\infty, A)$ in A proved in [10]). So, $\phi_\infty = \chi_{D_\infty}$.

We observe that $D_\infty \subset \{u_\infty = 0\}$ and again, by the strict monotonicity of $\lambda(\infty, A)$ in A, see [10], $D_\infty = \{u_\infty = 0\}$. □

4. Symmetry properties.

In this section, we consider the case where Ω is the unit ball, $\Omega = B(0,1)$.

Spherical Symmetrization. Given a measurable set $A \subset \mathbb{R}^N$, the spherical symmetrization A^* of A is constructed as follows: for each r, take $A \cup \partial B(0,r)$ and replace it by the spherical cap of the same area and center re_N. This can be done for almost every r. The union of these caps is A^*. Now, the spherical symmetrization u^* of measurable function $u \geq 0$ is constructed by symmetrizing the super-level sets so that, for all t, $\{u^* \geq t\} = \{u \geq t\}^*$. See [14, 17].

The following theorem is proved in [14] (see also [17]).

Theorem 4.1. *Let $u \in W^{1,p}(\Omega)$ and let u^* be its spherical symmetrization. Then $u^* \in W^{1,p}(B(0,1))$ and*

$$\begin{aligned}
&\int_{B(0,1)} |\nabla u^*|^p \,\mathrm{d}x \leq \int_{B(0,1)} |\nabla u|^p \,\mathrm{d}x,\\
&\int_{B(0,1)} |u^*|^p \,\mathrm{d}x = \int_{B(0,1)} |\nabla u|^p \,\mathrm{d}x,\\
&\int_{\partial B(0,1)} |u^*|^p \,\mathrm{d}S = \int_{\partial B(0,1)} |\nabla u|^p \,\mathrm{d}S,\\
&\int_{B(0,1)} (\alpha\chi_D)_* |u^*|^p \,\mathrm{d}x \leq \int_{B(0,1)} \alpha\chi_D |u|^p \,\mathrm{d}x,
\end{aligned} \tag{4.24}$$

where $D \subset B(0,1)$ and $(\alpha\chi_D)_* = -(-\alpha\chi_D)^*$.

Now we prove Theorem 1.3.

Proof of Theorem 1.3. Fix $\alpha > 0$ and A and assume (u, χ_D) is and optimal pair. Let u^* the spherical symmetrization of u. Define the set D^* by $\chi_{D^*} = (\chi_D)_*$. By Theorem 4.1 we get

$$\begin{aligned}
\lambda(\alpha, D^*) \quad &\leq \quad \frac{\displaystyle\int_\Omega |\nabla u^*|^p \,\mathrm{d}x + \int_\Omega |u^*|^p \,\mathrm{d}x + \int_\Omega (\alpha\chi_D)_* |u^*|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u^*|^p \,\mathrm{d}S}\\
&\leq \quad \frac{\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x + \alpha\int_\Omega \chi_D |u|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u|^p \,\mathrm{d}S} = \lambda(\alpha, D^*).
\end{aligned}$$

Since we have $|D^*| = |D| = A$, optimality of (u, χ_D) implies that (u^*, χ_{D^*}) is also a minimizer.

Now consider $p = 2$. In this case, it is proved in [4] that if equality holds in (4.24) then for each $0 < r \leq 1$ there exists a rotation R_r such that

$$u\mid_{\partial B(0,r)} = (u^* \circ R_r)\mid_{\partial B(0,r)}. \tag{4.25}$$

We can assume that the axis of symmetry e_N was taken so that $R_1 = Id$. Therefore u and u^* coincide on the boundary of $B(0,1)$. Therefore, the optimal sets D, D^* are sublevel sets of u and u^* with the same level, t. As u and u^* are solutions of a second order elliptic equation with bounded measurable coefficients they are C^1. Hence $\{u > t\} \cap \{u^* > t\}$ is an open neighborhood of $\partial\Omega \cap \{u > t\}$. In that neighborhood both functions are solutions of the same equation, $\Delta v = v$ (which has a unique continuation property), and along $\partial\Omega \cap \{u > t\}$ both coincide together with their normal derivatives. Thus they coincide in the whole neighborhood.

Now we observe that the set $\{u > t\}$ is connected, because every connected component of $\{u > t\}$ touches the boundary (since solutions of $\Delta v = v$ cannot have a positive interior maximum) and $\{u > t\} \cap \partial\Omega$ is connected.

We conclude that $\{u > t\} = \{u^* > t\}$ and $u = u^*$ in that set. In the complement of this set both u and u^* satisfy the same equation with the same Dirichlet data, therefore they coincide. $\square$

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