Lectures on dg-categories

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Contents

1 Lecture 1: Dg-categories and localization

The purpose of this first lecture is to explain one motivation for working with dg-categories concerned with the localization construction in category theory (in the sense of Gabriel-Zisman, see below). I will start by presenting some very concrete problems often encountered when using the localization construction. In a second part I will introduce the homotopy category of dg-categories, and propose it as a setting in order to define a better behaved localization construction. This homotopy category of dg-categories will be further studied in the next lectures.

1.1 Bad behaviour of the Gabriel-Zisman localization

Let C be a category and S be a subset of the set of morphisms in C[1]. A *localization of C with respect to S* is the data of a category $S^{-1}C$ and a functor

$$l : C \longrightarrow S^{-1}C$$

satisfying the following property: for any category D the functor induced by composition with l

$$l^* : \underline{Hom}(S^{-1}C, D) \longrightarrow \underline{Hom}(C, D)$$

is fully faithful and its essential image consists of all functors $f : C \longrightarrow D$ such that $f(s)$ is an isomorphism in D for any $s \in S$ (here $\underline{Hom}(A, B)$ denotes the category of functors from a category A to another category B).

Using the definition it is not difficult to show that if a localization exists then it is unique, up to an equivalence of categories, which is itself unique up to a unique isomorphism. It can also be proved that a localization always exists. However, in general localization are extremely difficult to describe in a useful manner, and the existence of localizations does not say much in practice (thought it is sometimes useful to know that hey exist). The following short list of examples show that localized categories are often encountered and provide interesting categories in general.

Examples:

1. If all morphisms in S are isomorphisms then the identity functor $C \to C$ is a localization.

2. If S consists of all morphisms in C, then $S^{-1}C$ is the groupoid completion of C. When C has a unique objet with a monoid M of endomorphisms, then $S^{-1}C$ has unique object with the group M^+ as automorphisms (M^+ is the group completion of the monoid M).

3. Let R be a ring and $C(R)$ be the category of (unbounded) complexes over R. Its objects are families of R-modules $\{E^n\}_{n\in\mathbb{Z}}$ together with maps $d^n : E^n \to E^{n+1}$ such that $d^{n+1}d^n = 0$. Morphisms are simply families of morphisms commuting with the d's. Finally, for $E^* \in C(R)$, we can define its n-th cohomology by $H^n(E^*) := Ker(d^n)/Im(d^{n-1})$, which is an R-module. The construction $E^* \mapsto H^n(E^*)$ provides a functor H^n from $C(R)$ to R-modules.

   A morphism $f : E^* \longrightarrow F^*$ in $C(R)$ is called a *quasi-isomorphism* if for all $i \in \mathbb{Z}$ the induced map
   $$H^i(f) : H^i(E^*) \longrightarrow H^i(F^*)$$

[1] In these lectures I will not take into account set theory problems, and will do as if all categories are small. These set theory problems can be solved for instance by fixing Grothendieck universes.

is an isomorphism. We let S be the set of quasi-isomorphisms in $C(R)$. Then $S^{-1}C(R)$ is the *derived category* of R and is denoted by $D(R)$. Understanding the hidden structures of derived categories is one of the main objectives of dg-category theory.

4. Let Cat be the category of categories: its objects are categories and its morphisms are functors (note that Cat is not a 2-category here). We let S be the set of categorical equivalences. The localization category $S^{-1}Cat$ is called the *homotopy category of categories*. It can be shown quiete easily that $S^{-1}Cat$ is equivalent to the category whose objetcs are categories and whose morphismes are isomorphism classes of functors.

5. Let Top be the category of topological spaces and continuous maps. A morphism $f : X \longrightarrow Y$ is called a *weak equivalence* if it induces isomorphisms on all homotopy groups (with respect to all base points). If S denotes the set of weak equivalences then $S^{-1}Top$ is called the *homotopy category of spaces*. It can be shown that $S^{-1}Top$ is equivalent to the category whose objects are CW-complexes and whose morphisms are homotopy classes of continuous maps.

In these lectures we will be mainly interested in localized categories of the type $D(R)$ for some ring R (or some more general object, see lecture 2). I will therefore explain the bad behaviour of the localization using examples of derived categories. However, this bad behaviour is a general fact and also apply to other examples of localized categories.

Thought the localization construction is useful to construct interesting new categories, the resulting localized categories are in general badly behaved. Here is a list of problems often encountered in practice.

1. The derived category $D(R)$ lacks the standard categorical constructions of limits and colimits. There exists a non-zero morphism $e : \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2[1]$ in $D(\mathbb{Z})$, corresponding to the non-zero element in $Ext^1(\mathbb{Z}/2, \mathbb{Z}/2)$ (recall that $Ext^i(M,N) \simeq [M, N[i]]$, where $N[i]$ is the complex whose only non-zero part is N in degree $-i$, and $[-,-]$ denotes the morphisms in $D(R)$). Suppose that the morphism e has a kernel, i.e. that a fiber product

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow e \\ 0 & \longrightarrow & \mathbb{Z}/2[1] \end{array}$$

exists in $D(\mathbb{Z})$. Then, for any integer i, we have a short exact sequence

$$0 \longrightarrow [\mathbb{Z}, X[i]] \longrightarrow [\mathbb{Z}, \mathbb{Z}/2[i]] \longrightarrow [\mathbb{Z}, \mathbb{Z}/2[i+1]],$$

or in other words

$$0 \longrightarrow H^i(X) \longrightarrow H^i(\mathbb{Z}/2) \longrightarrow H^{i+1}(\mathbb{Z}/2).$$

This implies that $X \longrightarrow \mathbb{Z}/2$ is a quasi-isomorphism, and thus an isomorphism in $D(\mathbb{Z})$. In particular $e = 0$, which is a contradiction.

2. The fact that $D(R)$ has no limits and colimits might not be a problem by itself, as it is possible to think of interesting categories which does not have limits and colimits (e.g. any non-trivial groupoid has no final object). However, the case of $D(R)$ is very frustating as it seems that $D(R)$ is very close to have limits and colimits. For instance it is possbile to show that $D(R)$ admits *homotopy limits and homotopy colimits* in the following sense. For a category I, let $C(R)^I$ be the category of functors from I to $C(R)$. A morphism $f : F \longrightarrow G$ (i.e. a natural transformation between two functors $F, G : I \longrightarrow C(R)$) is called a *levelwise quasi-isomorphism* if for any $i \in I$ the induced morphism $f(i) : F(i) \longrightarrow G(i)$ is a quasi-isomorphism in $C(R)$. We denote by $D(R, I)$ the category $C(R)$ localized along levelwise quasi-isomorphisms. The constant diagram functor $C(R) \longrightarrow C(R)^I$ is compatibe with localizations on both sides and provides a functor

   $$c : D(R) \longrightarrow D(R, I).$$

   It can then by shown that the functor c has a left and a right adjoint denoted by

   $$Hocolim_I : D(R, I) \longrightarrow D(R) \qquad D(R) \longleftarrow D(R, I) : Holim_I,$$

   called the *homotopy colimit* and the *homotopy limit* functor. Homotopy limits are colimits are very good replacement of the notions of limits and colimits, as they are the best possible approximation of the colimit and limit functors that are compatible with the notion of quasi-isomorphisms. However, this is quite unsatisfactory as the category $D(R, I)$ depends on more than the category $D(R)$ alone (note that $D(R, I)$ is not equivalent to $D(R)^I$), and in general it is impossible to recontruct $D(R, I)$ from $D(R)$.

3. To the ring R is associated several invariants such as its K-theory spectrum, its Hochschild (resp. cyclic) homology .... It is tempting to think that these invariants can be directly defined on the level of derived categories, but this is not the case (see [Sch]). However, it has been noticed that these invariants only depends on R up to some notion of equivalence that is much weaker than the notion of isomorphism. For instance, any functor $D(R) \longrightarrow D(R')$ which is induced by a complex of (R, R')-bi-modules induces a map on K-theory, Hochschild homology and cyclic homology. However, it is not clear that any functor $D(R) \longrightarrow D(R')$ comes from a complex of (R, R')-bi-modules (there are counter examples when R and R' are dg-algebras, see [Du-Sh, 2.5,6.8]). Definitely, the derived category of complexes of (R, R')-bi-modules is not equivalent to the category of functors $D(R) \longrightarrow D(R')$. This is again an unsatisfactory situation and it is then quite difficult (if not impossible) to understand the true nature of these invariants (i.e. of which mathematical structures are they trully invariants ?).

4. Another important problem with the categories $D(R)$ is its non local nature. To explain this let $\mathbb{P}^1$ be the projective line (e.g. over $\mathbb{Z}$). As a scheme $\mathbb{P}^1$ is the push-out

   $$\begin{array}{ccc} Spec\,\mathbb{Z}[X, X^{-1}] & \longrightarrow & Spec\,\mathbb{Z}[T] \\ \downarrow & & \downarrow \\ Spec\,\mathbb{Z}[U] & \longrightarrow & \mathbb{P}^1, \end{array}$$

where T is sent to X and U is sent to X^{-1}. According to the push-out square, the category of quasi-coherent sheaves on $\mathbb{P}^1$ can be described as the (categorical) pull-back

$$\begin{array}{ccc} QCoh(\mathbb{P}^1) & \longrightarrow & Mod(\mathbb{Z}[T]) \\ \downarrow & & \downarrow \\ Mod(\mathbb{Z}[U]) & \longrightarrow & Mod(\mathbb{Z}[X, X^{-1}]). \end{array}$$

In other words, a quasi-coherent module on $\mathbb{P}^1$ is the same thing as a triple (M, N, u), where M (resp. N) is a $\mathbb{Z}[T]$-module (resp. $\mathbb{Z}[U]$-module), and u is an isomorphism

$$u : M \otimes_{\mathbb{Z}[T]} \mathbb{Z}[X, X^{-1}] \simeq N \otimes_{\mathbb{Z}[U]} \mathbb{Z}[X, X^{-1}]$$

of $\mathbb{Z}[X, X^{-1}]$-modules. This property is extremely useful in order to reduce problems of quasi-coherent sheaves on schemes to problems of modules over rings. Unfortunately, this property is lost when passing to the derived categories. The square

$$\begin{array}{ccc} D_{qcoh}(\mathbb{P}^1) & \longrightarrow & D(\mathbb{Z}[T]) \\ \downarrow & & \downarrow \\ D(\mathbb{Z}[U]) & \longrightarrow & D(\mathbb{Z}[X, X^{-1}]), \end{array}$$

is not cartesian anymore (e.g. there exists non zero morphisms $\mathcal{O} \longrightarrow \mathcal{O}(-2)[1]$ that go to zero as a morphism in $D(\mathbb{Z}[U]) \times_{D(\mathbb{Z}[X,X^{-1}])} D(\mathbb{Z}[T])$). The derived categories of the affine pieces of $\mathbb{P}^1$ does not determine the derived category of quasi-coherent sheaves on $\mathbb{P}^1$.

The list of problems above suggests the existence of a some sort of categorical structure lying in between the category of complexes $C(R)$ and its derived category $D(R)$, which is rather close to $D(R)$ (i.e. in which the quasi-isomorphisms are inverted in some sense), but for which $(1)-(4)$ above are no longer a problem. There exist several possible approaches, and my purpose is to present an appraoch using dg-categories.

1.2 Localization as a dg-category

We now fix a base commutative ring k. Unless specified, all the modules and tensor products will be over k.

We start by recalling that a *dg-category* T consists of the following data.

- A set of objects $Ob(T)$, also sometimes denoted by T itself.
- For any pair of objects $(x, y) \in Ob(T)^2$ a complex $T(x, y) \in C(k)$.
- For any triple $(x, y, z) \in Ob(T)^3$ a composition morphism $\mu_{x,y,z} : T(x, y) \otimes T(y, z) \longrightarrow T(x, z)$.
- For any object $x \in Ob(T)$, a morphism $e_x : k \longrightarrow T(x, x)$.

These data are required to satisfy the following associativity and unit conditions.

1. (Associativity) For any four objects (x, y, z, t) in T, the following diagram

$$\begin{array}{ccc} T(x,y) \otimes T(y,z) \otimes T(z,t) & \xrightarrow{id \otimes \mu_{y,z,t}} & T(x,y) \otimes T(y,t) \\ \downarrow{\scriptstyle \mu_{x,y,z} \otimes id} & & \downarrow{\scriptstyle \mu_{x,y,t}} \\ T(x,z) \otimes T(z,t) & \xrightarrow[\mu_{x,z,t}]{} & T(x,t) \end{array}$$

   commutes.

2. (Unit) For any $(x, y) \in Ob(T)^2$ the two morphisms

$$T(x,y) \simeq k \otimes T(x,y) \xrightarrow{e_x \otimes id} T(x,x) \otimes T(x,y) \xrightarrow{\mu_{x,x,y}} T(x,y)$$

$$T(x,y) \simeq T(x,y) \otimes k \xrightarrow{id \otimes e_y} T(x,y) \otimes T(y,y) \xrightarrow{\mu_{x,y,y}} T(x,y)$$

   are equal to the identities.

For two dg-categories T and T', a *morphism of dg-categories* (or simply a *dg-functor*) $f : T \longrightarrow T'$ consists of the following data.

- A map of sets $f : Ob(T) \longrightarrow Ob(T')$.
- For any pair of objects $(x, y) \in Ob(T)^2$, a morphism in $C(k)$

$$f_{x,y} : T(x,y) \longrightarrow T'(f(x), f(y)).$$

These data are required to satisfy the following associativity and unit conditions.

1. For any $(x, y, t) \in Ob(T)^3$ the following diagram

$$\begin{array}{ccc} T(x,y) \otimes T(y,z) & \xrightarrow{\mu_{x,y,z}} & T(x,z) \\ \downarrow{\scriptstyle f_{x,y} \otimes f_{y,z}} & & \downarrow{\scriptstyle f_{x,z}} \\ T'(f(x),f(y)) \otimes T'(f(y),f(z)) & \xrightarrow[\mu'_{f(x),f(y),f(z)}]{} & T'(f(x),f(z)) \end{array}$$

   commutes.

2. For any $x \in Ob(T)$, the following diagram

$$\begin{array}{ccc} k & \xrightarrow{e_x} & T(x,x) \\ & \searrow^{e_{f(x)}} & \downarrow{\scriptstyle f_{x,x}} \\ & & T'(f(x),f(x)) \end{array}$$

   commutes.

Dg-functors can be composed in an obvious manner, and dg-categories together with dg-functors form a category denoted by $dg - cat_k$ (or $dg - cat$ if the base ring k is clear).

For a dg-category T, we define a category $[T]$ in the following way. The set of objects of $[T]$ is the same as the set of objects of T. For two objects x and y the set of morphisms in $[T]$ is defined by

$$[T](x, y) := H^0(T(x, y)).$$

Finally, the composition of morphisms in $[T]$ is defined using the natural morphisms

$$H^0(T(x, y)) \otimes H^0(T(y, z)) \longrightarrow H^0(T(x, y) \otimes T(y, z))$$

composed with the morphism

$$H^0(\mu_{x,y,z}) : H^0(T(x, y) \otimes T(y, z)) \longrightarrow H^0(T(x, z)).$$

Definition 1.1 *The category $[T]$ is called the* homotopy category *of T.*

One of the most important notion in dg-category theory is the notion of quasi-equivalences, a mixture in between quasi-isomorphisms and categorical equivalences.

Definition 1.2 *Let $f : T \longrightarrow T'$ be a dg-functor between dg-categories*

1. *The morphism f is* quasi-fully faithful *if for any two objects x and y in T the morphism $f_{x,y} : T(x, y) \longrightarrow T'(f(x), f(y))$ is a quasi-isomorphism of complexes.*
2. *The morphism f is* quasi-essentially surjective *if the induced functor $[f] : [T] \longrightarrow [T']$ is essentially surjective.*
3. *The morphism f is a* quasi-equivalence *if it is quasi-fully faithful and quasi-essentially surjective.*

We will be mainly interested in dg-categories up to quasi-equivalences. We therefore introduce the following category.

Definition 1.3 *The* homotopy category of dg-categories *is the category $dg - cat$ localized along quasi-equivalences. It is denoted by $Ho(dg - cat)$.*

Remark 1.4 In the last section we have seen that the localization construction is not well behaved, but in the definition above we consider $Ho(dg - cat)$ which is obtained by localization. Therefore, the category $Ho(dg - cat)$ will not be well behaved itself. In order to get the most powerful approach the category $dg - cat$ should have been itself localized in a more refined maner (e.g. as a higher category, see [To2, §2]). We will not need such a evolved approach, and the category $Ho(dg - cat)$ will be enough for most of our purpose.

Note that the construction $T \mapsto [T]$ provides a functor $[-] : dg-cat \longrightarrow Cat$, which descends as a functor on homotoy categories

$$[-] : Ho(dg-cat) \longrightarrow Ho(Cat).$$

The derived category $D(R)$ is defined as a localization of the category $C(R)$, and thus has a universal property in $Ho(Cat)$. The purpose of this series of lectures is to show that $C(R)$ can also be localized as a dg-category in order to get an object $L(R)$ satisfying a universal property in $Ho(dg-ca)$. The two objects $L(R)$ and $D(R)$ will be related by the formula

$$[L(R)] \simeq D(R),$$

and we will see that the extra informations encoded in $L(R)$ is enough in order to solve all the problems mentioned in §1.1.

2 Lecture 2: Model categories and dg-categories

The purpose of this second lecture is to study in more details the category $Ho(dg-cat)$. Localizations of categories are in general very difficult to describe in general. Things get much more simpler in the presence of a model category structure. In this lecture, I will start by some brief reminders on model categories. I will then explain how model category structures appear in the context of dg-categories by describing the model category of dg-categories (due to G. Tabuada, [Tab]) and the model category of dg-modules. We will also see how model categories can be used in order to construct interesting dg-categories. In the next lecture these model categories will be used in order to understand maps in $Ho(dg-cat)$, and to prove the existence of several important constructions such as localization and internal Homs.

2.1 Reminders on model categories

We let M be a category with arbitrary limits and colimits. Recall that a (closed) model category structure on M is the data of three classes of morphisms in M, the fibration Fib, the cofibration Cof and the equivalences W, satisfying the following axioms (see [Ho1]).

1. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in M, then f, g and gf are all in W if and only if two of them are in W.

2. The fibrations, cofibrations and equivalences are all stable by retracts.

3. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

be a commutative square in M with $i \in Cof$ and $p \in Fib$. If either i or p is also in W then there exists a morphism $h : B \longrightarrow X$ such that $ph = g$ and $hi = f$.

4. Any morphism $f : X \longrightarrow Y$ can be factorized in two ways as $f = pi$ and $f = qj$, with $p \in Fib$, $i \in Cof \cap W$, $q \in Fib \cap W$ and $j \in Cof$. Moreover, the existence of these factorizations are required to be functorial in f.

A model category structure is a rather simple notion, but in practice it is never easy to check that three given classes Fib, Cof and W satisfy the four axioms above. This can be explained by the fact that the existence of a model category structure on M has a very important consequence on the localized category $W^{-1}M$. For this, we introduce the notion of homotopy between morphisms in M in the following way. Two morphisms $f, g : X \longrightarrow Y$ are called *homotopic* if there exists a commutative diagram in M

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle i} & \searrow^{f} & \\ C(X) & \xrightarrow{h} & Y \\ \uparrow{\scriptstyle j} & \nearrow_{g} & \\ X & & \end{array}$$

satisfying the following two properties:

1. There exists a morphism $p : C(X) \longrightarrow X$, which belongs to $Fib \cap W$, such that $pi = pj = id$.

2. The induced morphism
$$i \coprod j : X \coprod X \longrightarrow C(X)$$
is a cofibration.

When X is cofibrant and Y is fibrant in M (i.e. $\emptyset \longrightarrow X$ is a cofibration and $Y \longrightarrow *$ is a fibration), it can be shown that being homotopic as defined above is an equivalence relation on the set of morphisms from X to Y. Moreover, the localized category $W^{-1}M$ is naturally equivalent to the category whose objects are fibrant and cofibrant objects in M and whose morphisms are homotopy classes of morphisms (see [Ho1]).

Our first main example of a model category will be $C(k)$, the category of complexes over some base commutative ring k. The fibrations are taken to be the surjective morphisms, and the equivalences are taken to be the quasi-isomorphisms. This determines the class of cofibrations as the morphisms having the correct lifting property. It is an important theorem that this defines a model category structure on $C(k)$ (see [Ho1]). The homotopy category of this model category is of course $D(k)$ the derived category of k. Therefore, maps in $D(k)$ can be described as homotopy classes of morphisms between fibrant and cofibrant complexes. As the cofibration objects in $C(k)$ are essentially the complexes of projective modules (see [Ho1]) and that every object is fibrant, this gives back essentially the usual way of describing maps in derived categories.

Before going back to dg-catgeories we will need a more structured notion of a model category structure, the notion of a $C(k)$-*model category structure*. Suppose that M is a model category. A $C(k)$-model category structure on M is the data of a functor

$$- \otimes - : C(k) \times M \longrightarrow M$$

satisfying the following two conditions.

1. The functor $\otimes$ above defines a closed $C(k)$-module structure on M (see [Ho1, §4]). In other words, we are given functorial isomorphisms in M

$$E \otimes (E' \otimes X) \simeq (E \otimes E') \otimes X \qquad k \otimes X \simeq X$$

   for any $E, E' \in C(k)$ and $X \in M$ (satisfying the usual associativity and unit conditions). We are also given for two objects X and Y in M a complex $\underline{Hom}(X,Y) \in C(k)$, together with functorial isomorphisms of complexes

$$Hom(E, \underline{Hom}(X,Y)) \simeq Hom(E \otimes X, Y)$$

   for $E \in C(k)$, and $X, Y \in M$.

2. For any cofibration $i : E \longrightarrow E'$ in $C(k)$, and any cofibration $j : A \longrightarrow B$ in M, the induced morphism

$$E \otimes B \coprod_{E \otimes A} E' \otimes A \longrightarrow E' \otimes B$$

   is a cofibration in M, which is moreover an equivalence if i or j is so.

Condition (1) above is a purely categorical stucture, and simply asserts the existence of an enrichement of M into $C(k)$ in a rather strong sense. The second condition is a compatibility condition between this enrichement and the model structures on $C(k)$ and M (which is the non trivial part to check in practice).

2.2 Model categories and dg-categories

There a several places in the theory of dg-categories where model categories appear.

We start by the model category of dg-categories itself. The equivalences for this model structure are the quasi-equivalences. The fibrations are the morphisms $f : T \longrightarrow T'$ satisfying the following two properties.

1. For any two objects x and y in T, the induced morphism

$$f_{x,y} : T(x,y) \longrightarrow T'((f(x), f(y))$$

   is a fibration in $C(k)$ (i.e. is surjective).

2. For any isomorphism $u' : x' \to y'$ in $[T']$, and any $y \in [T]$ such that $f(y) = y'$, there exists an isomorphism $u : x \to y$ in $[T]$ such that $[f](u) = u'$.

Theoreme 2.1 *(see [Tab]) The above definitions define a model category structure on $dg - cat$.*

This is a key statement in the homotopy theory of dg-categories, and many results in the sequel will depend in an essential way from the existence of this model structure.

Let now T be a dg-category. A T-dg-module is the data of a dg-functor $F : T \longrightarrow C(k)$. In other words a T-dg-module F consists of the data of complexes $F_x \in C(k)$ for each object x of T, together with morphisms

$$F_x \otimes T(x, y) \longrightarrow F_y$$

for each objects x and y, satisfying the usual associativity and unit conditions. A morphism of T-dg-module consists of a natural transformation between dg-functors (i.e. families of morphisms $F_x \longrightarrow F'_x$ commuting with the maps $F_x \otimes T(x, y) \longrightarrow F_y$ and $F'_x \otimes T(x, y) \longrightarrow F'_y$).

We let $T - Mod$ be the category of T-dg-modules. We define a model category structure on $T - Mod$ by defining equivalences (resp. fibrations) to be the morphisms $f : F \longrightarrow F'$ such that for all $x \in T$ the induced morphism $f_x : F_x \longrightarrow F'_x$ is an equivalence (resp. a fibration) in $C(k)$. It is known that this defines a model category structure (see [To1]). This model category is in a natural way a $C(k)$-model category, for which the $C(k)$-enrichement is defined by the formula $(E \otimes F)_x := E \otimes F_x$.

Definition 2.2 *The* derived category *of a dg-catgeory T is*

$$D(T) := Ho(T - Mod).$$

The previous definition generalizes the derived categories of rings. Indeed, if R is a k-algebra it can also be considered as a dg-category, sometimes denoted by BR, with a unique object and R as endomorphism of this object (considered as a complex of k-modules concentrated in degree 0). Then $D(BR) \simeq D(R)$. Indeed, a BR-dg-module is simply a complex of R-modules.

Any morphism of dg-categories $f : T \longrightarrow T'$ induces an adjunction on the corresponding model categories of dg-modules

$$f_! : T - Mod \longrightarrow T' - Mod \qquad T - Mod \longleftarrow T' - Mod : f^*,$$

for which the functor f^* is defined by composition with f, and $f_!$ is its left adjoint. This adjunction is a *Quillen adjunction*, i.e. f^* preserves fibrations and trivial fibrations, and therefore can be derived into an adjunction on the level of homotopy categories

$$\mathbb{L}f_! : D(T) \longrightarrow D(T') \qquad D(T) \longleftarrow D(T') : f^* = \mathbb{R}f^*.$$

It is also not difficult to see that when f is a quasi-equivalence then f^* and $\mathbb{L}f_!$ are equivalences of categories inverse to each others (see [To1]).

For a $C(k)$-model category M we can also define a notion of *T-dg-modules with coefficients in M* as being dg-functors $T \longrightarrow M$ (where M is considered as a dg-category using its $C(k)$-enrichement). This category is denoted by M^T (so that $T - Mod = C(k)^T$). When M satisfies some mild assumptions (that of being *cofibrantly generated*, see [Ho1]) we can endow M^T with a model category structure similar to $T - Mod$, for which equivalences and fibrations are defined levelwise in M. The existence of model categories as M^T will be used in the sequel to describe morphisms in $Ho - (dg - cat)$.

We finish this second lecture by describing a way to construct many examples of dg-categories using model categories. For this, let M be a $C(k)$-enriched model category. Using the $C(k)$-enrichement M can also be considered as a dg-category whose set of objects is the same as the set of objects of M and whose complexes of morphisms are $\underline{Hom}(x, y)$. We let $Int(M)$ be the full sub-dg-category of M consisting of fibrant and cofibrant objects in M. From the general theory of model categories it can be easily seen that the category $[Int(M)]$ is naturally isomorphic to the category of fibrant and cofibrant objects in M and homotopy classes of morphisms between them. In particular there exists a natural equivalence of categories

$$[Int(M)] \simeq Ho(M).$$

The dg-category $Int(M)$ is therefore a dg-enhancement of the homotopy category $Ho(M)$. Of course, not every dg-category is of form $Int(M)$. However, we will see that any dg-catgeory can be, up to a quasi-equivalence, fully embedded into some dg-category of the form $Int(M)$. This explains the importance of $C(k)$-model categories in the study of dg-categories.

Remark 2.3 The construction $M \mapsto Int(M)$ is an ad-hoc construction, and does not seem very intrinsic (e.g. as it is defined it depends on the choice of fibrations and cofibrations in M, and not only on equivalences). However, we will see in the next lecture that $Int(M)$ can also be characterized by as the localization of M along the equivalences in M, showing that it only depends on the dg-category M and the subset W.

3 Lecture 3: Structure of the homotopy category of dg-categories

In this lecture I will present some results describing the homotopy category $Ho(dg - cat)$. I will start by a theorem describing the set of maps between two objects in $Ho(dg - cat)$. This fundamental result has two important consequences: the existence of localizations of dg-categories, and the existence of dg-categories of morphisms between two dg-categories, both characterized by universal properties in $Ho(dg - cat)$. I will also introduce the notion of triangulated dg-categories and Morita equivalences between dg-categories.

3.1 Maps in the homotopy category of dg-categories

We start by computing the set of maps in $Ho(dg - cat)$ from a dg-category T to a dg-category of the form $Int(M)$. As any dg-category full embedds into some $Int(M)$ this will be enough to compute maps in $Ho(dg - cat)$ between any two objects.

Let M be a $C(k)$-model category. We assume that M satisfies the following two conditions.

1. M is cofibrantly generated.
2. For any cofibrant object X in M, and any quasi-isomorphism $E \longrightarrow E'$ in $C(k)$, the induced morphism $E \otimes X \longrightarrow E' \otimes X$ is an equivalence.

Condition (1) this is a very mild condition, as almost all model categories encountered in real life are cofibrantly generated. Condition (2) is more serious, as it states that cofibrant objects of M are flat in some sense, which is not always the case. For example, to be sure that the model category $T - Mod$ satisfies (2) we need to impose the condition that all the complexes $T(x, y)$ are flat (e.g. cofibrant in $C(k)$). The following proposition is wrong if (2) is not satisfied (it is unclear to me that condition (1) is really needed).

Proposition 3.1 *Let T be any dg-category and M be a $C(k)$-model category satisfying conditions* (1) *and* (2) *above. Then, there exists a natural bijection*

$$[T, Int(M)] \simeq Iso(Ho(M^T))$$

between the set of morphisms from T to $Int(M)$ in $Ho(dg - cat)$ and the set of isomorphism classes of objects in $Ho(M^T)$.

Ideas of proof (see [To1] for details): Let $Q(T) \longrightarrow T$ be a cofibrant model for T. The pull-back functor on dg-modules with coefficients in M induces a functor

$$Ho(M^T) \longrightarrow Ho(M^{Q(T)}).$$

Condition (2) on M insures that this is an equivalence of categories. This implies that we can assume that T is cofibrant. As all objects in $dg-cat$ are fibrant $[T, Int(M)]$ is then the quotient of the set of morphisms in $dg - cat$ by the homotopy relations. In particular, the natural map $[T, Int(M)] \longrightarrow Iso(Ho(M^T))$ is surjective (this uses that a cofibrant and fibrant object in M^T factors as $T \to Int(M) \to M$, i.e. is levelwise fibrant and cofibrant). To prove injectivity, we start with two morphisms $u, v : T \longrightarrow Int(M)$ in $dg-cat$, and we assume that the corresponding objects F_u and F_v in M^T are equivalent. Using that any equivalences can be factorized as a composition of trivial cofibrations and trivial fibrations, we easily reduce the problem to the case where there exists a trivial fibration $F_u \longrightarrow F_v$ (the case of cofibration is somehow dual). This morphism can be considered as an object in $Int(Mor(M)^T)$, where $Mor(M)$ is the model category of morphisms in M (note that fibrant objects in $Mor(M)$ are fibrations between fibrant objects in M). Moreover, this object belongs to $T' \subset Int(Mor(M)^T)$, the full sub-dg-category corresponding to equivalences in M (i.e. the dg-functor $T \longrightarrow Mor(M)$ whose image of any object of T is an equivalence in M). We therefore have a commutative diagram in $dg - cat$

$$\begin{array}{ccc} & & Int(M) \\ & \overset{u}{\nearrow} & \uparrow \\ T & \longrightarrow & T' \\ & \underset{v}{\searrow} & \downarrow \\ & & Int(M). \end{array}$$

The two morphisms $T' \longrightarrow Int(M)$ are easily seen to be quasi-equivalences, and to possess a common section $Int(M) \longrightarrow T'$ correponding to the identity morphisms in M. Projecting this diagram in $Ho(dg-ca)$, we see that $[u]=[v]$ in $Ho(dg-cat)$. $\Box$

We will now deduce from proposition 3.1 a description of the set of maps $[T,T']$ between two objects in $Ho(dg-cat)$. For this we use the $C(k)$-enriched Yoneda embedding

$$\underline{h} : T' \longrightarrow Int((T')^{op} - Mod),$$

sending an object $x \in T'$ to the $(T')^{op}$-dg-module defined by

$$\begin{array}{cccc} \underline{h}_x : & (T')^{op} & \longrightarrow & C(k) \\ & y & \mapsto & T'(y,x). \end{array}$$

The dg-module $\underline{h}$ is easily seen to be cofibrant and fibrant in $(T')^{op} - Mod$, and thus we have $\underline{h}_x \in Int((T')^{op} - Mod)$ as required. The enriched version of the Yoneda lemma implies that $\underline{h}$ is a quasi-fully faithful dg-functor. More precisely, we can show that the induced morphism of complexes

$$T'(x,y) \longrightarrow \underline{Hom}(\underline{h}_x, \underline{h}_y) = Int((T')^{op} - Mod)((\underline{h}_x, \underline{h}_y)$$

is an isomorphims of complexes.

Using the description of maps in $Ho(dg-cat)$ as being homotopy classes of morphisms between cofibrant objects, we see that the morphism $\underline{h}$ induces a injective map

$$[T,T'] \hookrightarrow [T, Int((T')^{op} - Mod)]$$

whose image consists of morphisms $T \longrightarrow Int((T')^{op} - Mod)$ factorizing in $Ho(dg-cat)$ throught the quasi-essential image of $\underline{h}$. We easily get this way the following corollary (see §3.2 for the definition of the tensor product of two dg-categories).

Corolary 3.2 *Let T and T' be two dg-categories, one of them having cofibrant complexes of morphisms. Then, there exists a natural bijection between $[T,T']$ and the subset of $Iso(Ho(T \otimes (T')^{op} - Mod))$ consisting of $T \otimes (T')^{op}$-dg-modules F such that for any $x \in T$, there exists $y \in T'$ such that $F_{x,-}$ and $\underline{h}_y$ are isomorphic in $Ho((T')^{op} - Mod)$.*

3.2 Existence of internal Homs

For two dg-catgeories T and T' we can construct their tensor product $T \otimes T'$ in the following way. The set of objects of $T \otimes T'$ is the product $Ob(T) \times Ob(T')$. For $(x,y) \in Ob(T)^2$ and $(x',y') \in Ob(T')^2$ we set

$$(T \otimes T')((x,x'),(y,y')) := T(x,y) \otimes T(x',y'),$$

with the obvious compositions and units. When k is not a field the functor $\otimes$ does not preserves quasi-equivalences. However, it can be derived by the following formula

$$T \otimes^{\mathbb{L}} T' := Q(T) \otimes Q(T'),$$

where Q is a cofibrant replacement functor on $dg-cat$. This defines a symmetric monoidal structure

$$-\otimes^{\mathbb{L}}-: Ho(dg-cat)\times Ho(dg-cat)\longrightarrow Ho(dg-cat).$$

Proposition 3.3 *The monoidal structure* $-\otimes^{\mathbb{L}}-$ *is closed. In other words, for two dg-categories* T *and* T' *there exists* $\mathbb{R}\underline{Hom}(T,T')\in Ho(dg-cat)$*, such that for any third dg-category* U *there exists a bijection*

$$[U,\mathbb{R}\underline{Hom}(T,T')]\simeq[U\otimes^{\mathbb{L}}T,T'],$$

functorial in $U\in Ho(dg-cat)$*.*

Idea of proof: As for the poposition 3.1 we can reduce the problem of showing that $\mathbb{R}\underline{Hom}(T,Int(M))$ exists for a $C(k)$-model category M. Under the same hypothesis than corollary 3.2 it can be checked (using proposition 3.1) that $\mathbb{R}\underline{Hom}(T,Int(M))$ exists and is given by $Int(M^T)$. $\Box$

For two dg-categories T and T', one of them having cofibrant complexes of morphisms it is possible to show that $\mathbb{R}\underline{Hom}(T,T')$ is given by the full sub-dg-category of $Int(T\otimes(T')^{op}-Mod)$ consisting of dg-modules satifying the condition of corollary 3.2.

Finally, note that when $M=C(k)$ we have

$$\mathbb{R}\underline{Hom}(T,Int(C(k)))\simeq Int(T-Mod).$$

In particular, we find a natural equivalence of categories

$$D(T)\simeq[\mathbb{R}\underline{Hom}(T,Int(C(k)))],$$

which is an important formula.

3.3 Existence of localizations

Let T be a dg-category and let S be subset of morphisms in $[T]$ we would like to invert in $Ho(dg-cat)$. For this, we will say that a morphism $l:T\longrightarrow L_ST$ in $Ho(dg-cat)$ is a *localization of* T *along* S if for any $T'\in Ho(dg-cat)$ the induced morphism

$$l^*:[L_ST,T']\longrightarrow[T,T']$$

is injective and its image consists of all morphisms $T\longrightarrow T'$ in $Ho(dg-cat)$ whose induced functor $[T]\longrightarrow[T']$ sends all morphisms in S to isomorphisms in $[T]$. Note that the functor $[T]\longrightarrow[T']$ is only well defined in $Ho(Cat)$ (i.e. up to isomorphism), but this is enough for the definition to makes sense as the condition of sending S to isomorphisms is stable by isomorphism between functors.

Proposition 3.4 *For any dg-category* T *and any set of maps* S *in* $[T]$*, a localization* $T\longrightarrow L_ST$ *exists in* $Ho(dg-cat)$*.*

Idea of proof (see [To1] for details): We start by the most simple example of a localization. We first suppose that $T := \Delta^1_k$ is the dg-category freely generated by two objects, 0 and 1, and a unique morphism $u : 0 \to 1$. More concretely, $T(0,1) = T(0,0) = T(1,1) = k$ and $T(1,0) = 0$, together with the obvious compositions and units. We let $\mathbf{1}$ be the dg-category with a unique object $*$ and $\mathbf{1}(*,*) = k$ (with the obvious composition). We consider the dg-foncteur $T \longrightarrow \mathbf{1}$ sending the non trivial morphism of T to the identity of $*$ (i.e. $k = T(0,1) \to \mathbf{1}(*,*) = k$ is the identity). We claim that this morphism $T \longrightarrow \mathbf{1}$ is a localization of T along S consisting of the morphism $u : 0 \to 1$ of $T = [T]$. This in fact follows easily from our proposition 3.1. Indeed, for a $C(k)$-model category M the model category M^T is the model category of morphisms in M. It is then easy to check that the functor $Ho(M) \longrightarrow Ho(M^T)$ sending an object of M to the identity morphism in M is fully faithful and that its essential image consists of all equivalences in M.

In the general case, let S be a subset of morphisms in $[T]$ for some dg-category T. We can represent the morphisms S by a dg-functor

$$\coprod_S \Delta^1_k \longrightarrow T,$$

sending the non trivial morphism of the component s to a representative of s in T. We define $L_S T$ as being the homotopy push-out (see [Ho1] for this notion)

$$L_S T := (\coprod_S \mathbf{1}) \coprod^{\mathbb{L}}_{\coprod_S \Delta^1_k} T.$$

The fact that each morphism $\Delta^1_k \longrightarrow \mathbf{1}$ is a localization and the universal properties of homotopy push-outs imply that the induced morphism $T \longrightarrow L_S T$ defined as above is a localization of T along S. □

The following proposition describes $Int(M)$ as a dg-localization of M.

Proposition 3.5 *Let M be a cofibrantly generated $C(k)$-model category. There exists a natural isomorphism in $Ho(dg-cat)$*

$$Int(M) \simeq L_W M,$$

where M is a considered as a dg-category using its $C(k)$-enrichement.

Idea of proof: We consider the natural inclusion dg-functor $i : Int(M) \longrightarrow M$. This inclusion factors as

$$Int(M) \xrightarrow{j} M^f \xrightarrow{k} M,$$

where M^f is the full sub-dg-category of M consisting of fibrant objects. Using that M is cofibrantly generated we can construct dg-functors

$$r : M \longrightarrow M^f \qquad q : M^f \longrightarrow Int(M)$$

together with morphisms

$$jq \to id \qquad qj \to id \qquad id \to ri \qquad id \to ir.$$

Moreover, these morphisms between dg-functors are levelwise in W. This can be seen to imply that the induced morphisms on localizations

$$L_W Int(M) \longrightarrow L_W M^f \longrightarrow L_W M$$

are isomorphisms in $Ho(dg-cat)$. Finally, as morphisms in W are already invertible in $[Int(M)] \simeq Ho(M)$, we have $L_W Int(M) \simeq Int(M)$. $\square$

Finally, one possible way to understand localizations of dg-categories is by the following proposition.

Proposition 3.6 *Let T be a dg-category and S be a subset of morphisms in $[T]$. Then, the localization morphism $l : T \longrightarrow L_S T$ induces a fully faithful functor*

$$l^* : D(L_S T) \longrightarrow D(T)$$

whose image consists of all T-dg-modules $F : T \longrightarrow C(k)$ sending all morphisms of S to quasi-isomorphisms in $C(k)$.

Idea of proof: This follows from the existence of internal Homs and localizations, as well as the formula

$$D(T) \simeq [\mathbb{R}\underline{Hom}(T, Int(C(k)))] \qquad D(L_S T) \simeq [\mathbb{R}\underline{Hom}(L_S T, Int(C(k)))].$$

Indeed, the universal properties of localizations and internal Homs implies that $\mathbb{R}\underline{Hom}(L_S T, Int(C(k)))$ can be identified full the full sub-dg-category of $\mathbb{R}\underline{Hom}(T, Int(C(k)))$ consisting of dg-functors sending S to quasi-isomorphisms in $C(k)$. $\square$

3.4 Triangulated dg-categories

In this section we will introduce a class of dg-categories called *triangulated*. The notion of being triangulated is the dg-analog of the notion of being Karoubian for linear categories. We will see that any dg-category has a triangulated hull, and this will allow us to introduce a notion of Morita equivalences which is a dg-analog of the usual notion of Morita equivalences between linear categories.

Let T be a dg-category. We recall the existence of the Yoneda embedding

$$T \longrightarrow Int(T^{op} - Mod).$$

Passing to homotopy categories we get a fully faithful morphism

$$\underline{h} : [T] \longrightarrow D(T^{op}).$$

An object in the essential image of this functor will be called *quasi-representable.*

Recall that an object $x \in D(T^{op})$ is *compact* if the functor

$$[x, -] : D(T^{op}) \longrightarrow k - Mod$$

commutes with arbitrary direct sums. It is easy to see that any quasi-representable object is compact. The converse is not true and we set the following definition.

Definition 3.7 *A dg-category T is* triangulated *if and only if every compact object in $D(T^{op})$ is quasi-representable.*

When T is triangulated we have $[T] \simeq D(T^{op})_c$, where $D(T^{op})_c$ is the full sub-category of $D(T)$ of compact objects. The category $D(T)$ has a natural triangulated structure which restricts to a triangulated structure on compact objects. Therefore, when T is triangulated dg-category its homotopy category $[T]$ comes equiped with a natural triangulated structure. However, it is not necessary to know the theory of triangulated categories in order to understand triangulated dg-categories.

We let $Ho(dg - cat^{tr}) \subset Ho(dg - cat)$ be the full sub-category of triangulated dg-categories.

Proposition 3.8 *The natural inclusion*

$$Ho(dg - cat^{tr}) \longrightarrow Ho(dg - cat)$$

has a left adjoint. In other words, any dg-category has a triangulated hull.

Idea of proof: Let T be a dg-category. We consider the Yoneda embedding

$$\underline{h} : T \longrightarrow Int(T^{op} - Mod).$$

We consider $\widehat{T}_{pe} \subset Int(T^{op} - Mod)$, the full sub-dg-category consisting of compact objects. The Yoneda embedding factors as a full embedding

$$\underline{h} : T \longrightarrow \widehat{T}_{pe}.$$

Let T' be a triangulated dg-category. By definition, the natural morphism

$$T' \longrightarrow \widehat{T'}_{pe}$$

is an isomorphism in $Ho(dg - cat)$. We can then consider the induced morphism

$$[\widehat{T}_{pe}, \widehat{T'}_{pe}] \longrightarrow [T, \widehat{T'}_{pe}].$$

The hard point is to show that this map in bijective and that $\widehat{T}_{pe}$ is triangulated. These two facts follow from the fact that T generates $\widehat{T}_{pe}$ by the operations of taking homotopy push-outs, shifts and retracts (see [To-Va] for details). $\Box$

The proof of the proposition shows that the left ajoint to the inclusion is given by

$$\widehat{(-)}_{pe} : Ho(dg-cat) \longrightarrow Ho(dg-cat^{tr}).$$

For example, if R is a k-algebra, considered as a dg-category with a unique object BR, $\widehat{BR}_{pe}$ is the dg-category of cofibrant and perfect complexes of R-modules. In particular

$$[\widehat{BR}_{pe}] \simeq D_{parf}(R)$$

is the perfect derived category of R. This follows from the fact that compact objects in $D(R)$ are precisely the perfect complexes (see [To-Va]). Therefore, we see that the dg-category of perfect complexes over some ring R is the triangulated hull of R.

Definition 3.9 *A morphism $T \longrightarrow T'$ in $Ho(dg-cat)$ is called a* Morita equivalence *if the induced morphism in the triangulated hull*

$$\widehat{T}_{pe} \longrightarrow \widehat{T'}_{pe}$$

is an isomorphism in $Ho(dg-cat)$.

It follows formally that $Ho(dg-cat^{tr})$ is equivalent to the localized category $W_{mor}^{-1}dg-cat$, where W_{mor} is the subset of Morita equivalences in $dg-cat$ as defined above.

We can characterize the Morita equivalences in the following way.

Proposition 3.10 *Let $f : T \longrightarrow T'$ be a morphism of dg-categories. The following are equivalent.*

1. *The morphism f is a Morita equivalence.*
2. *For any triangulated dg-category T_0, the induced map*
$$[T', T_0] \longrightarrow [T, T_0]$$
is bijective.
3. *The induced functor*
$$f^* : D(T') \longrightarrow D(T)$$
is an equivalence of categories.
4. *The functor*
$$\mathbb{L}f_! : D(T) \longrightarrow D(T')$$
induces an equivalence between the full sub-category of compact objects.

We finish this section by a description of $Ho(dg-cat^{tr})$ in termes of derived categories of bi-dg-modules.

Proposition 3.11 *Let T and T' be two dg-categories. Then, there exists a natural bijection between $[\widehat{T}_{pe}, \widehat{T'}_{pe}]$ and the subset of $Iso(D(T \otimes^{\mathbb{L}} (T')^{op}))$ consisting of $T \otimes^{\mathbb{L}} (T')^{op}$-dg-modules F such that for any $x \in T$, the $(T')^{op}$-dg-module $F_{x,-}$ is compact.*

4 Some applications

In this last lecture I will present some applications of the homotopy theory of dg-categories. We will see in particular how the problems mentioned in §1.1 are solved using dg-categories. The very last section will be some discussions on the notion of saturated dg-categories and their use in the definition of a *secondary K-theory* functor.

4.1 Functorial cones

One of the problem encountered with derived categories is the non existence of functorial cones. In the context of dg-categories this problem can be solved as follows.

Let T be a triangulated dg-category. We let Δ^1_k be the dg-category freely generated by two objects 0 and 1 and one morphism, and $\mathbf{1}$ be the dg-category freely generated on one objects. There is a morphism

$$\Delta^1_k \longrightarrow \widehat{\mathbf{1}}_{pe}$$

sending 0 to 0 and 1 to k. We get an induced morphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(\widehat{\mathbf{1}}_{pe}, T) \longrightarrow \mathbb{R}\underline{Hom}(\Delta^1_k, T).$$

As T is triangulated we have

$$\mathbb{R}\underline{Hom}(\widehat{\mathbf{1}}_{pe}, T) \simeq \mathbb{R}\underline{Hom}(\mathbf{1}, T) \simeq T.$$

Therefore, we have defined a morphism in $Ho(dg-cat)$

$$f : T \longrightarrow \mathbb{R}\underline{Hom}(\widehat{\mathbf{1}}_{pe}, T) =: Mor(T).$$

The dg-category $Mor(T)$ is also the full sub-dg-category of $Int(Mor(T^{op}-Mod))$ corresponding to quasi-representable dg-modules, and is called the dg-category of morphisms in T. The morphism f defined above intuitively sends an object $x \in T$ to $0 \rightarrow z$ in $Mor(T)$ (note that 0 is an object in T because T is triangulated).

Proposition 4.1 *There exists a unique morphism in $Ho(dg-cat)$*

$$c : Mor(T) \longrightarrow T$$

such that the following two $(T \otimes^{\mathbb{L}} Mor(T)^{op})$-dg-modules

$$(z, u) \mapsto Mor(T)(u, f(z)) \qquad (z, u) \mapsto T(c(u), z).$$

In other words, the morphism f admits a left adjoint*.*

Idea of proof: We consider the following explicit models for T, $Mor(T)$ and f. We let T' be the full sub-dg-category of $Int(T^{op}-Mod)$ consisting of quasi-representable objects (or equivalentely of compact objects as T is triangulated). We let $Mor(T)'$ be the full sub-dg-category of $Int(Mor(T^{op}-Mod))$ consisting of morphisms between quasi-representable objects (these are also the compact objects in $Ho(Mor(T^{op}-Mod))$ because T is triangulated). We note that $Mor(T)'$ is the dg-category whose objects are cofibrations between cofibrant and quasi-representable T^{op}-dg-modules. To each compact and cofibrant T^{op}-dg-module z we consider $0 \to z$ as an object in T'. This defines a dg-functor $T' \longrightarrow Mor(T)'$ which is a model for f. We define c as being a $C(k)$-enriched left adjoint to c (in the most naive sense), sending an object $c : x \longrightarrow y$ of $Mor(T)'$ to $c(u)$ defined by the push-out in $T^{op} - Mod$

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & c(u). \end{array}$$

We note that the T^{op}-module $c(u)$ is compact and thus belongs to T'. It is easy to check that c, as a morphism in $Ho(dg - cat)$ satisfies the property of the proposition.

The unicity of c is proved formally, in the same way that one proves the unicity of adjoints in usual category theory. $\square$

The morphism $c : Mor(T) \longrightarrow T$ is a functorial cone construction for the triangulated dg-category T. The important fact here is that there exists a natural functor

$$[Mor(T)] \longrightarrow Mor([T]),$$

which is essentially surjective, full but not faithful in general. The functor

$$[c] : [Mor(T)] \longrightarrow [T]$$

does not factor in general throught $Mor([T])$.

To finish, proposition 4.1 becomes really powerful when combined with the following fact.

Proposition 4.2 *Let T be a triangulated dg-category and T' be any dg-category. Then $\mathbb{R}\underline{Hom}(T', T)$ is triangulated.*

One important feature of triangulated dg-categories is that any dg-functor $f : T \longrightarrow T'$ between triangulated dg-categories commutes with cones. In other words, the diagram

$$\begin{array}{ccc} Mor(T) & \xrightarrow{c} & T \\ {\scriptstyle c(f)}\downarrow & & \downarrow{\scriptstyle f} \\ Mor(T') & \xrightarrow[c]{} & T' \end{array}$$

commutes in $Ho(dg - cat)$. This has to be understood as a generalization of the fact that any linear functor between additive categories commutes with finite direct sums. This property of triangulated dg-categories is very useful in practice, as any dg-functor $T \longrightarrow T'$ automatically induces a triangulated functor $[T] \longrightarrow [T']$.

4.2 Some invariants

Another problem mentioned in §1.1 is the fact that the usual invariants, (K-theory, Hochschild homology and cohomology ...), are not invariants of derived categories. We will see here that these invariants can be defined on the level of dg-categories. We will treat the examples of K-theory and Hochschild cohomology.

1. Let T be a dg-category. We consider $T^{op} - Mod^{cc}$ the full sub-category of compact and cofibrant T^{op}-dg-modules. We can endow $T^{op} - Mod^{cc}$ with a structure of a Waldhausen category whose equivalences are quasi-isomorphisms and cofibrations are the cofibrations of the model category structure on $T^{op} - Mod$. This Waldhausen category defines a K-theory spectrum $K(T)$. We note that if T is triangulated we have

   $$K_0(T) := \pi_0(K(T)) \simeq K_0([T]),$$

   where the last K-group is the Grothendieck group of the triangulated category $[T]$.

   Now, let $f : T \longrightarrow T'$ be a morphism between dg-categories. It induces a functor

   $$f_! : T^{op} - Mod \longrightarrow (T')^{op} - Mod.$$

   This functor preserves cofibrations, compact cofibrant objects and push-outs (when they exists). Therefore, it induces a functor between Waldhausen categories

   $$f_! : T^{op} - Mod^{cc} \longrightarrow (T')^{op} - Mod^{cc}$$

   and a morphism on the corresponding spectra

   $$f_! : K(T) \longrightarrow K(T').$$

   This defines a functor

   $$K : dg - cat \longrightarrow Sp$$

   from dg-categories to spectra. It is possible to show that this functor sends Morita equivalences to stable equivalences, and thus defines a functor

   $$K : Ho(dg - cat^{tr}) \longrightarrow Ho(Sp).$$

   We see it particular that two dg-categories which are Morita equivalent have the same K-theory.

2. Let T be a dg-category. We consider $\mathbb{R}\underline{Hom}(T,T)$, the dg-category of (derived) endomorphisms of T. The identity gives an object $id \in \mathbb{R}\underline{Hom}(T,T)$, and we can set

   $$HH(T) := \mathbb{R}\underline{Hom}(T,T)(id, id),$$

   the Hochschild complex of T. This is a well defined object in $D(k)$, the derived category of complexes of k-modules, and the construction $T \mapsto HH(T)$ provides a functor of groupoids

   $$Ho(dg - cat)^{iso} \longrightarrow D(k)^{iso}.$$

Using the results of §3.2 we can see that

$$HH^*(T) \simeq Ext^*(T,T),$$

where the Ext-group is computed in the derived category of $T \otimes^{\mathbb{L}} T^{op}$-dg-modules. In particular, when T is given by a usual non-dg flat k-algebra R we find

$$HH^*(T) \simeq Ext^*_{R\otimes R}(R,R),$$

which is usual Hochschild cohomology. The Yoneda embedding $T \longrightarrow \widehat{T}_{pe}$, provides an isomorphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(\widehat{T},\widehat{T}) \simeq \mathbb{R}\underline{Hom}(T,\widehat{T}),$$

and a quasi-fully faithful morphism

$$\mathbb{R}\underline{Hom}(T,T) \longrightarrow \mathbb{R}\underline{Hom}(T,\widehat{T}).$$

Therefore, we get a quasi-fully faithful morphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(T,T) \longrightarrow \mathbb{R}\underline{Hom}(\widehat{T},\widehat{T})$$

sending the identity to the indentity. Therefore, we obtain a natural isomorphism

$$HH(T) \simeq HH(\widehat{T}_{pe}).$$

We get that way that Hochschild cohomology is a Morita invariant.

4.3 Descent

In this section we will see how to solve the non-local nature of derived categories explained in §1.1. For this, let X be a scheme. We have the Grothendieck category $C(\mathcal{O}_X)$ of (unbounded) complexes of sheaves of $\mathcal{O}_X$-modules. This category can be endowed with a model category structure for which the equivalences are the quasi-isomorphisms (of complexes of sheaves) and the cofibrations are the monomorphisms (see e.g. [Ho2]). Moreover, when X is a k-scheme then the natural $C(k)$-enrichement of $C(\mathcal{O}_X)$ makes it into a $C(k)$-model category. We let

$$L(\mathcal{O}_X) := Int(C(\mathcal{O}_X)),$$

and we let $L_{pe}(X)$ be the full sub-dg-category consisting of perfect complexes on X. The K-theory of X can be defined as

$$K(X) := K(L_{pe}(X)),$$

using the definition of K-theory of dg-categories we saw in the last section. It can be shown that this coincides, up to an equivalence, with the definition using a Waldhausen category of perfect complexes on X.

When $f : X \longrightarrow Y$ is a morphism of schemes, it is possible to define two morphisms in $Ho(dg-cat)$

$$\mathbb{L}f^* : L(\mathcal{O}_Y) \longrightarrow L(\mathcal{O}_X) \qquad L(\mathcal{O}_Y) \longleftarrow L(\mathcal{O}_X) : \mathbb{R}f_*,$$

which are adjoints (according to the model we chose $\mathbb{L}f^*$ is a bit tricky to define explicitly). The morphism

$$\mathbb{L}f^* : L(\mathcal{O}_Y) \longrightarrow L(\mathcal{O}_X)$$

always preserves perfect complexes are induces a morphism

$$\mathbb{L}f^* : L_{pe}(Y) \longrightarrow L_{pe}(X).$$

Proposition 4.3 *Let $X = U \cup V$, where U and V are two Zariski open subschemes. Then the following square*

$$\begin{array}{ccc} L_{pe}(X) & \longrightarrow & L_{pe}(U) \\ \downarrow & & \downarrow \\ L_{pe}(V) & \longrightarrow & L_{pe}(U \cap V) \end{array}$$

is homotopy cartesian in $dg - cat$.

The previous proposition can be generalized to more complicated descent properties, such as faithfully flat descent (see e.g. [Hir-Si]).

4.4 Saturated dg-categories and secondary K-theory

We arrive at the last section of these lectures. We have seen that dg-categories can be used in order to replace derived categories, and that they can be used in order to define K-theory. In this section we will see that dg-categories can also be considered as *coefficients* that can themselves be used in order to define a secondary version of K-theory. For this I will present an analogy between the categories $Ho(dg - cat^{tr})$ and $k - Mod$. Throught this analogy projective k-modules of finite rank correspond to the notion of *saturated dg-categories*. I will then show how to define secondary K-theory spectrum $K^{(2)}(k)$ using saturated dg-categories, and give some ideas of how to define analogs of the rank and chern character maps in order to see that this secondary K-theory $K^{(2)}(k)$ is non-trivial. I will also mention a relation between $K^{(2)}(k)$ and the Brauer group, analog to the well known relation between K-theory and the Picard group.

We start by the analogies between the categories $k - Mod$ of k-modules and $Ho(dg - cat^{tr})$. The true analogy is really betwen $k - Mod$ and the homotopy theory of triangulated dg-categories, e.g. the simplicial category $Ldg - cat^{tr}$ obtained by simplicial localization (see [To2]). The homotopy category $Ho(dg - cat^{tr})$ is sometimes too coarse to see the analogy. We will however restrict ourselves with $Ho(dg - cat^{tr})$, even thought some of the facts below about $Ho(dg-cat^{tr})$ are not completely intrinsic and requires to lift things to the model category of dg-categories.

1. The category $k-Mod$ is a closed symmetric monoidal category for the usual tensor product. In the same way, $Ho(dg - cat^{tr})$ has a closed symmetric monoidal structure induced from the one of $Ho(dg-cat)$ (see §3.2). Explicitly, if T and T' are two triangulated dg-category we form $T \otimes^{\mathbb{L}} T' \in Ho(dg - cat)$. This is not a triangulated dg-category anymore and we set
$$T \widehat{\otimes}^{\mathbb{L}} T' := (\widehat{T \otimes^{\mathbb{L}} T'})_{pe} \in Ho(dg - cat^{tr}).$$

The unit of this monoidal structure is the triangulated hull of **1**, i.e. the dg-category of cofibrant and perfect complexes of k-modules. The corresponding internal Homs is the one of $Ho(dg-cat)$, as we already saw that $\mathbb{R}\underline{Hom}(T,T')$ is triangulated if T and T' are.

2. The category $k-Mod$ has a zero object and finite sums are also finite products. This is again true in $Ho(dg-cat^{tr})$. The zero dg-category (with one object and 0 as endomorphism ring of this object) is a zero object in $Ho(dg-cat^{tr})$. Also, for two triangulated dg-categories T and T' their sum $T\coprod T'$ as dg-categories is not triangulated anymore. Their direct sum in $Ho(dg-cat^{tr})$ is the triangulated hull of $T\coprod T'$, that is

$$\widehat{T\coprod T'}_{pe} \simeq \widehat{T}_{pe}\times \widehat{T'}_{pe}\simeq T\times T'.$$

   We note that this second remarkable property of $Ho(dg-cat^{tr})$ is not satisfied by $Ho(dg-cat)$ itself. We can say that $Ho(dg-cat^{tr})$ is *semi-additive*, which is justififed by the fact that the Homs in $Ho(dg-cat^{tr})$ are abelian monoids (or abelian semi-groups).

3. The category $k-Mod$ has arbitrary limits and colimits. The corresponding statement is not true for $Ho(dg-cat^{tr})$. However, we have homotopy limits and homotopy colimits in $Ho(dg-cat^{tr})$, whose existence are insured by the model category structure on $dg-cat$.

4. There is a natural notion of short exact sequences in $k-Mod$. In the same way, there is a natural notion of short exact sequences in $Ho(dg-cat^{tr})$. These are the sequences of the form

$$T_0 \xrightarrow{j} T \xrightarrow{p} \widehat{(T/T_0)}_{pe},$$

   where i is a quasi-fully faithful functor between triangulated dg-categories, and $\widehat{(T/T_0)}_{pe}$ is the quotient defined as the triangulated hull of the homotopy push-out of dg-categories

$$\begin{array}{ccc} T_0 & \longrightarrow & T \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & T/T_0. \end{array}$$

   These sequences are natural in terms of the homotopy theory of triangulated dg-categories as it can be shown that quasi-fully faithful dg-functors are precisely the *homotopy monomorphisms* in $dg-cat$, i.e. the morphsms $T\longrightarrow T'$ such that the diagonal map

$$T\longrightarrow T\times^h_{T'}T$$

   is a quasi-equivalence (the right hand side is a homotopy pull-back). This defines a dual notion of homotopy epimorphisms of triangulated dg-categories as being the morphism $T\longrightarrow T'$ such that for any triangulated dg-categories T'' the induced morphism

$$\mathbb{R}\underline{Hom}(T',T'')\longrightarrow \mathbb{R}\underline{Hom}(T,T'')$$

   is a homotopy monomorphisms (i.e. is quasi-fully faithful). In the exact sequences above j is a homotopy monomorphism, p is a homotopy epimorphism, p is the cokernel of j and j is the kernel of p. The situation is therefore really close to the situation in $k-Mod$.

If $k - Mod$ and $Ho(dg - cat^{tr})$ are so analoguous then we should be able to say what is the analog property of being projective of finite rank, and to define a K-group or even a K-theory spectrum of such objects. It turns that this can be done and that the theory can actually be pushed rather far. Also, we will see that this new K-theory migt have some geometric and arithmetic significance.

It is well know that the projective modules of finite rank over k are precisely the dualizable (also called rigid) objects in the closed monoidal category $k - Mod$. Recall that any k-module M has a dual $M^{\vee} := \underline{Hom}(M, k)$, and that there always exists an evaluation map

$$M^{\vee} \otimes M \longrightarrow \underline{Hom}(M, M).$$

The k-module M is dualizable if this evaluation map is an isomorphism, and this is known to be equivalent to the fact that M is projective of finite rank.

We will take this as a definition of *projective triangulated dg-categories of finite rank*. The striking fact is that these dg-categories have already been studied for other reasons under the name of *saturated dg-categories*, or *smooth and proper dg-categories*.

Definition 4.4 *A triangulated dg-category T is* saturated *if it is dualizale in $Ho(dg - cat^{tr})$, i.e. if the evaluation morphism*

$$\mathbb{R}\underline{Hom}(T, \widehat{\mathbf{1}}_{pe}) \widehat{\otimes}^{\mathbb{L}} T \longrightarrow \mathbb{R}\underline{Hom}(T, T)$$

is an isomorphism in $Ho(dg - cat^{tr})$.

The saturated triangulated dg-categories can be characterized nicely using the notion of smooth and proper dg-algebras (see [To3, To-Va, Ko-So]). Recall that a dg-algebra B is smooth if B is a compact object in $D(B \otimes^{\mathbb{L}} B^{op})$. Recall also that such a dg-algebra is proper if its underlying complex if perfect (i.e. if B is compact in $D(k)$). The following proposition can be deduced from the results of [To-Va].

Proposition 4.5 *A triangulated dg-category is saturated if and only if it is Morita equivalent to a smooth and proper dg-algebra.*

This proposition is interesting as it allows us to show that there exists many examples of saturated dg-categories. The two main examples are the following.

1. Let X be a smooth and proper k-scheme. Then $L_{pe}(X)$ is a saturated dg-category (see [To-Va]).
2. For any k-algebra, which is projective of finite rank as a k-module and which is of finite global cohomlogical dimension, the dg-category $\widehat{A}_{pe}$ of perfect complexes of A-modules is saturated.

The symmetric monoidal category $Ho(dg - cat^{sat})$ of saturated dg-categories is rigid. Note that any object T has a dual $T^{\vee} := \mathbb{R}\underline{Hom}(T, \widehat{\mathbf{1}}_{pe})$. Moroever, it can be shown that $T^{\vee} \simeq T^{op}$

is simply the opposite dg-category (this is only true when T is saturated). In particular, for T and T' two saturated dg-categories we have the following important formula

$$T^{op}\widehat{\otimes}^{\mathbb{L}}T' \simeq \mathbb{R}\underline{Hom}(T,T').$$

We can now define the secondary K-group. We start by $\mathbb{Z}[sat]$, the free abelian group on isomorphism classes (in $Ho(dg-cat^{tr})$) of satuared dg-categories. We define $K_0^{(2)}(k)$ to be the quotient of $\mathbb{Z}[sat]$ by the relation

$$[T] = [T_0] + [\widehat{(T/T_0)}_{pe}]$$

for any full saturated sub-dg-category $T_0 \subset T$ with quotient $\widehat{(T/T_0)}_{pe}$.

More generally, we can consider a certain Waldhausen category Sat, whose objects are cofibrant dg-categories T such that $\widehat{T}_{pe}$ is saturated, whose morphisms are morphisms of dg-categories, whose equivalences are Morita equivalences, and whose cofibrations are cofibrations of dg-categories which are also fully faithful. From this we can construct a spectrum, denoted by $K^{(2)}(k)$ by applying Waldhausen's construction, called the *secondary K-theory spectrum of k*. We have

$$\pi_0(K^{(2)}(k)) \simeq K_0^{(2)}(k).$$

We no finish with some arguments that $K^{(2)}(k)$ to show that is non trivial and interesting. First of all, we have the following two basic properties.

1. $k \mapsto K^{(2)}(k)$ defines a functor from the category of commutative rings to the homotopy category of spectra. To a map of rings $k \to k'$ we associate the base change $- \otimes_k^{\mathbb{L}} k'$ from saturated dg-categories over k to saturated dg-categories over k', which induces a functor of Waldhausen categories and thus a morphism on the corresponding K-theory spectra.

2. If $k = colim_i k_i$ is a filtered colimit of commutative rings then we have

$$K^{(2)}(k) \simeq colim_i K^{(2)}(k_i).$$

   This follows from the non trivial statement that the homotopy theory of saturated dg-categories over k is the filtered colimit of the homotopy theories of saturated dg-categories over the k_i (see [To4]).

3. The monoidal structure on $Ho(dg-cat^{tr})$ induces a commutative ring structure on $K_0^{(2)}(k)$. I guess that this monoidal structure also induces a E_∞-multiplication on $K^{(2)}(k)$.

Our next task is to prove that $K^{(2)}(k)$ is non zero. For this we construct a rank map

$$rk_0^{(2)} : K_0^{(2)}(k) \longrightarrow K_0(k)$$

which an analog of the usual rank map (also called the trace map)

$$rk_0 : K_0(k) \longrightarrow HH_0(k) = k.$$

Let T be a saturated dg-category. As T is dualizable in $Ho(dg - cat^{tr})$ there exists a trace map

$$\mathbb{R}\underline{Hom}(T,T) \simeq T^{op}\widehat{\otimes}^{\mathbb{L}}T \longrightarrow \widehat{\mathbf{1}}_{pe},$$

which is the dual of the identity map

$$id : \widehat{\mathbf{1}}_{pe} \longrightarrow T^{op}\widehat{\otimes}^{\mathbb{L}}T.$$

The image of the identity provides a perfect complex of k-modules, and thus an element

$$rk_0^{(2)}(T) \in K_0(k).$$

This defines the map

$$rk_0^{(2)} : K_0^{(2)}(k) \longrightarrow K_0(k).$$

It can be shown that $rk_0^{(2)}(T)$ is in fact $HH_*(T)$, the Hochschild homology complex of T.

Lemma 4.6 *For any saturated dg-category T we have*

$$rk_0^{(2)}(T) = [HH_*(T)] \in K_0(k),$$

where $HH_*(T)$ is the (perfect) complex of Hochschild homology of T.

In particular we see that for X a smooth and proper k-scheme we have

$$rk_0^{(2)}(L_{pe}(X)) = [HH_*(X)] \in K_0(k).$$

When $k = \mathbb{C}$ then $HH_*(X)$ can be identified with Hodge cohomology $H^*(X, \Omega_X^*)$, and thus $rk_0^{(2)}(L_{pe}(X))$ is then the euler characteristic of X. In other words, we can say that the rank of $L_{pe}(X)$ is $\chi(X)$. The map $rk_0^{(2)}$ shows that $K_0^{(2)}(k)$ is non zero.

The usual rank $rk_0 : K_0(k) \longrightarrow HH_0(k) = k$ is only the zero part of a rank map

$$rk_* : K_*(k) \longrightarrow HH_*(k).$$

In the same way, it is possible to define a secondary rank map

$$rk_*^{(2)} : K_*^{(2)}(k) \longrightarrow K_*(S^1 \otimes^{\mathbb{L}} k),$$

where $S^1 \otimes^{\mathbb{L}} k$ is a simplicial ring that can be defined as

$$S^1 \otimes^{\mathbb{L}} k = k \otimes^{\mathbb{L}}_{k\otimes^{\mathbb{L}}_{\mathbb{Z}}k} k.$$

Note that by definition of Hochschild homology we have

$$HH_*(k) \simeq S^1 \otimes^{\mathbb{L}} k,$$

so we can also write

$$rk_*^{(2)} : K_*^{(2)}(k) \longrightarrow K_*(HH_*(k)).$$

Using this map I guess it could be possible to check that the higher K-groups $K_i^{(2)}(k)$ are also non zero in general. Actually, I think it is possible to construct an analog of the Chern character

$$Ch : K_*(k) \longrightarrow HC_*(k)$$

as a map

$$Ch^{(2)} : K_*^{(2)}(k) \longrightarrow HC_*^{(2)}(k) := K_*^{S^1}(S^1 \otimes^{\mathbb{L}} k),$$

where the right hand side is the S^1-equivariant K-theory of $S^1 \otimes^{\mathbb{L}} k$ (note that S^1 acts on $S^1 \otimes^{\mathbb{L}} k$), which we take as a definition of secondary cyclic homology.

To finish we show that $K_0^{(2)}(k)$ has a relation with the Brauer group, analog to the relation between $K_0(k)$ and the Picard group. For this, we define $Br_{dg}(k)$ to be the group of isomorphism classes of invertible objects (for the monoidal structure) in $Ho(dg - cat^{tr})$. As being invertible is stronger than being dualizable we have a natural map

$$Br_{dg}(k) \longrightarrow K_0^{(2)}(k)$$

analog to the natural map

$$Pic(k) \longrightarrow K_0(k).$$

Now, by definition $Br_{dg}(k)$ can also be described as the Morita equivalence classes of *Azumaya's dg-algebras*, that is of dg-algebras B such that the natural map

$$B^{op} \otimes^{\mathbb{L}} B \longrightarrow \mathbb{R}\underline{End}_{C(k)}(B)$$

is a quasi-isomorphism. In particular, a non-dg Azumaya's algebra over k defines an element in $Br_{dg}(k)$, and we thus get a map $Br(k) \longrightarrow Br_{dg}(k)$, from the usual Brauer group of k (see [Mi]) to the dg-Brauer group of k. Composing with the map $Br_{dg}(k) \longrightarrow K_0^{(2)}(k)$ we get a map

$$Br(k) \longrightarrow K_0^{(2)}(k),$$

from the usual Brauer group to the secondary K-group of k. I do not know if this map is injective in general, but I guess it should be possible to prove that it is non zero in some examples by using the Chern character mentioned above.

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