Morita equivalence for graded rings

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This is joint work with

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Some background and history

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Equivalence of $\mathrm{Mod}\text{-}A$ and $\mathrm{Mod}\text{-}B$.

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Important / guiding example: For any ring R and positive integer n,

$$\text{Mod-}R \quad \text{and} \quad \text{Mod-}\mathrm{M}_n(R)$$

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For instance: $\text{Mod-}R$ and $\text{Mod-}\mathrm{M}_2(R)$ are "the same" ...

$$M \quad \mapsto \quad \begin{pmatrix} M & M \\ 0 & 0 \end{pmatrix}$$

AND, it's not hard to show that

$$\mathrm{Hom}_R(M, N) \;\cong\; \mathrm{Hom}_{\mathrm{M}_2(R)}\Big(\begin{pmatrix} M & M \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} N & N \\ 0 & 0 \end{pmatrix} \Big).$$

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But, e.g., $\text{Mod-}\mathbb{R}$ and $\text{Mod-}\mathbb{C}$ are not equivalent.

Some background and history

Definition: S any ring, $e = e^2 \in S$.

e is *full* in S in case $SeS = S$.

Note: SeS denotes *sums of* elements of the form ses' for $s, s' \in S$.

Example: $S = \mathrm{M}_n(R)$, $e = e_{1,1}$. Then e is full in S.

(So is any $e_{i,i}$.)

Some background and history

Verbiage:

"the rings R and S are *Morita equivalent*"

means

the categories $\mathrm{Mod}\text{-}R$ and $\mathrm{Mod}\text{-}S$ are equivalent categories.

Notation: $R \sim_{ME} S$.

Some background and history

The Original Morita Theorem: These are equivalent.

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(M1) R and S are Morita equivalent (i.e., the categories $\mathrm{Mod}\text{-}R$ and $\mathrm{Mod}\text{-}S$ are equivalent).

(M2) There exist $n \in \mathbb{N}$ and an idempotent $e \in \mathrm{M}_n(S)$ that is full in $\mathrm{M}_n(S)$ and for which the rings R and $e\mathrm{M}_n(S)e$ are isomorphic.

(M3) There exist an R-S-bimodule P and an S-R-bimodule Q and appropriate surjective bimodule homomorphisms $P \otimes_S Q \to R$ and $Q \otimes_R P \to S$.

K. Morita, *Duality for Modules and Its Applications to the Theory of Rings with Minimum Conditions*, Sci. Reports Tokyo Kyoiku Daigaku **6**A, 1958, 83 – 142.

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BUT, there are “local identities” in $\mathrm{FM}_\infty(T)$.

Note: $e_{1,1}$ is full in $\mathrm{FM}_\infty(T)$.

Some background and history

A fourth condition equivalent to those in The Original Morita Theorem:

(M4) The rings $\mathrm{FM}_\infty(R)$ and $\mathrm{FM}_\infty(S)$ are isomorphic.

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Statements (M1) through (M4):

"**The Extended Morita Theorem**"

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Comments on (M4): $\mathrm{FM}_\infty(R) \cong \mathrm{FM}_\infty(S)$ as rings.

1. Stephenson's proof that (M4) implies $R \sim_{ME} S$ invoked some of his own work on isomorphisms between lattices of submodules of various modules.

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1. Stephenson's proof that (M4) implies $R \sim_{ME} S$ invoked some of his own work on isomorphisms between lattices of submodules of various modules.

2. Using a now-well-understood notion of module categories over (nice) nonunital rings, it's not hard to get

$$R \sim_{ME} \mathrm{FM}_\infty(R)$$

for any unital R. From this, the result that (M4) implies $R \sim_{ME} S$ is immediate.

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3. Stephenson's proof that $R \sim_{ME} S$ implies (M4) consists of two steps.

First, show that $R \sim_{ME} S$ yields an (explicitly constructed) isomorphism

$$\Phi : \mathrm{RFM}_\infty(R) \to \mathrm{RFM}_\infty(S).$$

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$$\Phi : \mathrm{RFM}_\infty(R) \to \mathrm{RFM}_\infty(S).$$

Then, show that Φ restricts to an isomorphism between $\mathrm{FM}_\infty(R)$ and $\mathrm{FM}_\infty(S)$.

View the isomorphism in (M4) as "top down".

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(M5) $\mathrm{RFM}_\infty(R)$ and $\mathrm{RFM}_\infty(S)$ are isomorphic as rings.

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4. A (really beautiful!) result of Camillo gives an additional equivalent condition:

   (M5) $\mathrm{RFM}_\infty(R)$ and $\mathrm{RFM}_\infty(S)$ are isomorphic as rings.

5. Many ring theorists were not so impressed by (M4) ...

   after all, $\mathrm{FM}_\infty(R)$ and $\mathrm{FM}_\infty(S)$ are NON-unital rings.

Some background and history

$\mathbb{Z}$-graded rings.

A ring R is $\mathbb{Z}$-graded in case:

1. $(R, +) = \oplus_{t \in \mathbb{Z}} R_t$ as abelian groups, and
2. $R_t \cdot R_u \subseteq R_{t+u}$ for all $t, u \in \mathbb{Z}$.

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(So every element of R is a *finite* sum of homogeneous elements.)

Familiar examples: $k[x]$, $k[x, x^{-1}]$.

Silly (but important?) example: EVERY ring S admits a $\mathbb{Z}$-grading.

$$S_0 := S; \quad S_t := \{0\} \text{ for all } t \neq 0.$$

Some background and history

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Example: Let $R = k[x, x^{-1}]$. For $t \in \mathbb{Z}$, define

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Graded homomorphisms and isomorphisms between $\mathbb{Z}$-graded rings: defined as expected.

Some background and history

Lemma: $n \in \mathbb{N}$. If R is $\mathbb{Z}$-graded, then $\mathrm{M}_n(R)$ is $\mathbb{Z}$-graded.

For each $t \in \mathbb{Z}$,

$$(\mathrm{M}_n(R))_t \ := \ \mathrm{M}_n(R_t).$$

In the same way, $\mathrm{FM}_\infty(R)$ is $\mathbb{Z}$-graded as well.

The *standard* $\mathbb{Z}$-grading on $\mathrm{M}_n(R)$ or $\mathrm{FM}_\infty(R)$.

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Notation for this talk: "graded" means $\mathbb{Z}$-graded.

Some background and history

There are other "natural" gradings on $\mathrm{M}_n(R)$. For example, take *any* R (not necessarily graded). Then e.g., on $\mathrm{M}_3(R)$,

$$(\mathrm{M}_3(R))_0 := \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}, \ (\mathrm{M}_3(R))_1 := \begin{pmatrix} 0 & R & 0 \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_2 := \begin{pmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(\mathrm{M}_3(R))_{-1} := \begin{pmatrix} 0 & 0 & 0 \\ R & 0 & 0 \\ 0 & R & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_{-2} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & 0 & 0 \end{pmatrix},$$

and $(\mathrm{M}_3(R))_i := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all $i \neq -2, -1, 0, 1, 2$.

Some background and history

Suppose S is graded, and $e = e^2 \in S_0$. Then the *corner ring* eSe inherits a grading from S: $(eSe)_t := eS_te \ \forall t \in \mathbb{Z}$.

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Key observation: Suppose S is graded, and $e = e^2 \in S_0$.

Then, even if e is full in S, e need NOT be full in S_0.

Example: $e_{1,1}$ is full in $S = \mathrm{M}_3(R)$. And in the previous example,

$$S_0 := \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix},$$

so $e_{1,1} \in S_0$. But clearly $e_{1,1}$ is NOT full in S_0.

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so $e_{1,1} \in S_0$. But clearly $e_{1,1}$ is NOT full in S_0.

But there are plenty of examples where $e \in S_0$ being full in S does imply that e is full in S_0.

“The Algebraic Stabilization Theorem”

Recall (M2) and (M4) from the Extended Morita Theorem:

(M2) There exist $n \in \mathbb{N}$ and an idempotent $e \in \mathrm{M}_n(S)$ that is full in $\mathrm{M}_n(S)$ and for which R and $e\mathrm{M}_n(S)e$ are isomorphic.

and

(M4) The rings $\mathrm{FM}_\infty(R)$ and $\mathrm{FM}_\infty(S)$ are isomorphic.

Question: is there a graded version of this result?

"The Algebraic Stabilization Theorem"

Algebraic Stabilization Theorem: (A-, Ruiz, Tomforde)
Let R and S be unital graded rings. Assume all gradings on matrix rings and corner rings are standard. Then these two statements are equivalent.

(HG2) There exist $n \in \mathbb{N}$ and an idempotent $e \in \mathrm{M}_n(S)_0$ that is full in $\mathrm{M}_n(S)_0$ and for which the rings R and $e\mathrm{M}_n(S)e$ are graded isomorphic.

and

(HG4) The rings $\mathrm{FM}_\infty(R)$ and $\mathrm{FM}_\infty(S)$ are graded isomorphic.

“The Algebraic Stabilization Theorem”

Strategy of the proof: The key situation is where $n = 1$ and $S = eRe$ for $e \in R_0$ that is full in R_0.

Construct idempotents $\{Q_n, P_n \mid n \in \mathbb{N}\}$ in $\mathrm{RFM}_\infty(R_0)$, together with graded homomorphisms

$$\phi_n\colon\ Q_n\mathrm{FM}_\infty(R)Q_n \ \to\ P_n\mathrm{FM}_\infty(R)P_n \qquad \text{and}$$

$$\psi_n\colon\ P_n\mathrm{FM}_\infty(R)P_n \ \to\ Q_{n+1}\mathrm{FM}_\infty(R)Q_{n+1}$$

such that for all $n \in \mathbb{N}$,

$$Q_nQ_{n+1} = Q_n = Q_{n+1}Q_n, \quad P_nP_{n+1} = P_n = P_{n+1}P_n,$$

$$\bigcup_n P_n\mathrm{FM}_\infty(R)P_n = \mathrm{FM}_\infty(eRe), \ \ \bigcup_n Q_n\mathrm{FM}_\infty(R)Q_n = \mathrm{FM}_\infty(R),$$

"The Algebraic Stabilization Theorem"

and for which the diagram

$$\begin{array}{ccc} Q_n \mathrm{FM}_\infty(R) Q_n & \xrightarrow{i_n} & Q_{n+1} \mathrm{FM}_\infty(R) Q_{n+1} \\ \downarrow \phi_n & \nearrow \psi_n & \downarrow \phi_{n+1} \\ P_n \mathrm{FM}_\infty(R) P_n & \xrightarrow{j_n} & P_{n+1} \mathrm{FM}_\infty(R) P_{n+1} \end{array}$$

commutes. ("Intertwining" homomorphisms")

“The Algebraic Stabilization Theorem”

and for which the diagram

$$\begin{array}{ccc}
Q_n \mathrm{FM}_\infty(R) Q_n & \overset{i_n}{\hookrightarrow} & Q_{n+1} \mathrm{FM}_\infty(R) Q_{n+1} \\
\downarrow \phi_n & \overset{\psi_n}{\nearrow} & \downarrow \phi_{n+1} \\
P_n \mathrm{FM}_\infty(R) P_n & \overset{j_n}{\hookrightarrow} & P_{n+1} \mathrm{FM}_\infty(R) P_{n+1}
\end{array}$$

commutes. (“Intertwining” homomorphisms”)

Note: fullness of e in R_0 is needed to get that the idempotents $\{Q_n, P_n \mid n \in \mathbb{N}\}$ can be chosen in $\mathrm{RFM}_\infty(R_0)$ (as opposed to in $\mathrm{RFM}_\infty(R)$).

“The Algebraic Stabilization Theorem”

The four conditions imply

$$\mathrm{FM}_\infty(R) \cong \varinjlim (Q_n \mathrm{FM}_\infty(R) Q_n, i_n)$$

and

$$\mathrm{FM}_\infty(eRe) \cong \varinjlim (P_n \mathrm{FM}_\infty(R) P_n, j_n),$$

and the commutativity of the diagram implies that these direct limits are not only isomorphic, but in fact graded isomorphic.

“The Algebraic Stabilization Theorem”

Remarks:

1) The construction is motivated by work done in the context of C^*-algebras.

L. G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math, **71**(2), 1977, pp. 335–348.

2) By imposing the trivial grading on non-graded rings, the Algebraic Stabilization Theorem yields the equivalence of (M2) and (M4) in the Extended Morita Theorem.

3) This is a “bottom-up” approach to the isomorphism between $\mathrm{FM}_\infty(R)$ and $\mathrm{FM}_\infty(S)$.

“The Algebraic Stabilization Theorem”

4) So the “naive” extension of (M4) to graded rings

(i.e., to use the standard grading on the $\mathrm{FM}_{\infty}(-)$ rings)

is NOT equivalent to the “naive” extension of (M2).

The additional condition that e be full in $\mathrm{M}_n(S)_0$ is required.

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Question: What, then, is the appropriate extension of (M1) to the graded setting?

Graded modules, the category $\mathrm{Gr}\text{-}R$

Graded modules and graded homomorphisms.

R a graded ring, M_R a right R-module.

M is *graded* in case:

$$M = \oplus_{t \in \mathbb{Z}} M_t, \quad \text{and} \quad M_u R_t \subseteq M_{t+u} \text{ for all } t, u \in \mathbb{Z}.$$

If M, N are graded right R-modules, an R-homomorphism $f : M \to N$ is called *graded* in case $f(M_t) \subseteq N_t$ for all $t \in \mathbb{Z}$.

$$\mathrm{Gr}\text{-}R$$

denotes the category of graded right R-modules with graded homomorphisms.

Graded modules, the category $\mathrm{Gr}\text{-}R$

Here's statement (M1) from the Extended Morita Theorem:

(M1) R and S are Morita equivalent (i.e., the categories $\mathrm{Mod}\text{-}R$ and $\mathrm{Mod}\text{-}S$ are equivalent).

Question, recast: Is there some appropriate statement analogous to (M1) about $\mathrm{Gr}\text{-}R$ and $\mathrm{Gr}\text{-}S$ which would be equivalent to (HG2) and (HG4) ?

Graded modules: some terminology

For a graded right A-module M and $i \in \mathbb{Z}$, the *i-suspension of M*, denoted $M(i)$, is the graded right A-module having $M(i) = M$, with grading given by $M(i)_j = M_{i+j}$.

For $i \in \mathbb{Z}$, $\mathcal{T}_i$ denotes the *i-suspension functor*

$$\mathcal{T}_i : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}A$$

given by $M \mapsto M(i)$ on objects, and the identity on morphisms.

A functor $\phi : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}B$ is called *graded* when

$$\phi \circ \mathcal{T}_\alpha = \mathcal{T}_\alpha \circ \phi$$

for each $\alpha \in \mathbb{Z}$.

Graded modules: some terminology

A graded functor $\phi : \mathrm{Gr}\text{-}A \rightarrow \mathrm{Gr}\text{-}B$ is a *graded equivalence* if there is a graded functor $\psi : \mathrm{Gr}\text{-}B \rightarrow \mathrm{Gr}\text{-}A$ such that ϕ and ψ compose appropriately to the identity functors on each category.

If there is a graded equivalence between $\mathrm{Gr}\text{-}A$ and $\mathrm{Gr}\text{-}B$, we say A and B are *graded equivalent* or, more formally, *graded Morita equivalent*.

Graded modules: some terminology

For any graded ring A, we let U_A (or simply by U) denote the *forgetful functor*

$$U_A : \mathrm{Gr}\text{-}A \to \mathrm{Mod}\text{-}A.$$

A functor $\phi' : \mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B$ is called a *graded functor* if there is a graded functor $\phi : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}B$ such that

$$U_B \circ \phi = \phi' \circ U_A$$

as functors from $\mathrm{Gr}\text{-}A$ to $\mathrm{Mod}\text{-}B$. In this situation the functor ϕ is called an *associated graded functor* of ϕ'.

A functor $\phi' : \mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B$ is called a *graded equivalence* if it is both graded and an equivalence.

Graded modules: some terminology

Let S be a graded ring.

If M is any right S_0-module, then $M \otimes_{S_0} S$ is a graded right S-module, where

$$(M \otimes_{S_0} S)_i = M \otimes_{S_0} S_i$$

for each $i \in \mathbb{Z}$.

This gives a functor

$$- \otimes_{S_0} S : \mathrm{Mod}\text{-}S_0 \to \mathrm{Gr}\text{-}S.$$

Graded modules: some terminology

Definition. We call the graded rings A and B

homogeneously graded equivalent

in case there exists a graded equivalence $\psi : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}B$ for which there is an equivalence of categories

$$\eta : \mathrm{Mod}\text{-}A_0 \to \mathrm{Mod}\text{-}B_0$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Mod}\text{-}A_0 & \xrightarrow{\ \eta\ } & \mathrm{Mod}\text{-}B_0 \\ {\scriptstyle -\otimes_{A_0} A}\downarrow & & \downarrow{\scriptstyle -\otimes_{B_0} B} \\ \mathrm{Gr}\text{-}A & \xrightarrow{\ \psi\ } & \mathrm{Gr}\text{-}B \end{array}$$

commutes on objects of $\mathrm{Mod}\text{-}A_0$ (up to isomorphism).

Graded modules: some terminology

Rephrased:

A and B are called *homogeneously graded equivalent* in case there is a category equivalence

$$\eta : \mathrm{Mod}\text{-}A_0 \to \mathrm{Mod}\text{-}B_0$$

and a graded equivalence

$$\psi : \mathrm{Gr}\text{-}A \to \mathrm{Gr}\text{-}B$$

for which, for each object M of $\mathrm{Mod}\text{-}A_0$, there is an isomorphism

$$\psi(M \otimes_{A_0} A) \cong_{gr} (\eta(M)) \otimes_{B_0} B$$

as objects of $\mathrm{Gr}\text{-}B$.

The connection between these ideas

(Recall the Extended Morita Theorem ...)

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Theorem. Let R and S be unital graded rings. These are equivalent:

(HG1) R is homogeneously graded equivalent to S.

(HG2) There exist $n \in \mathbb{N}$ and an idempotent $e \in \mathrm{M}_n(S)_0$ that is full in $\mathrm{M}_n(S)_0$ and for which the rings R and $e\mathrm{M}_n(S)e$ are graded isomorphic.

(HG4) $\mathrm{FM}_\infty(R)$ is graded isomorphic to $\mathrm{FM}_\infty(S)$ in the standard grading.

The Homogeneously Graded Version of the Extended Morita Theorem

Proof that (HG1) is equivalent to (HG2): Omitted here.

The proof uses a number of known results about graded rings.

Here is a great resource:
R. Hazrat, Graded rings and graded Grothendieck groups. London Mathematical Society Lecture Note Series, **435**. Cambridge University Press, Cambridge, 2016. vii+235 pp.

The Homogeneously Graded Version of the Extended Morita Theorem

There is an appropriate "tensor product of graded bimodules" statement, which is the analog of (M3) in the Extended Morita Theorem, which is equivalent to (HG1), (HG2), (HG4). Omitted today.

This completes the picture corresponding to the existence of a graded isomorphism between $\mathrm{FM}_\infty(R)$ and $\mathrm{FM}_\infty(S)$ (where the standard grading is used to grade the infinite matrix rings).

More gradings on matrix rings

Recall this example. ("Grading #1")
(Here R need NOT be graded.) On $\mathrm{M}_3(R)$,

$$(\mathrm{M}_3(R))_0 := \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}, \ (\mathrm{M}_3(R))_1 := \begin{pmatrix} 0 & R & 0 \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_2 := \begin{pmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(\mathrm{M}_3(R))_{-1} := \begin{pmatrix} 0 & 0 & 0 \\ R & 0 & 0 \\ 0 & R & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_{-2} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & 0 & 0 \end{pmatrix},$$

and $(\mathrm{M}_3(R))_i := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all $i \neq -2, -1, 0, 1, 2$.

More gradings on matrix rings

Here's another $\mathbb{Z}$-grading on $\mathrm{M}_3(R)$. ("Grading #2")
(Again, *any* R.)

$$(\mathrm{M}_3(R))_0 := \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix},$$

$$(\mathrm{M}_3(R))_5 := \begin{pmatrix} 0 & R & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_8 := \begin{pmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(\mathrm{M}_3(R))_{-5} := \begin{pmatrix} 0 & 0 & 0 \\ R & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_{-3} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{pmatrix}, \ (\mathrm{M}_3(R))_{-8} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & 0 & 0 \end{pmatrix}$$

and $(\mathrm{M}_3(R))_i := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all other values of i.

More gradings on matrix rings

So if R is graded, we can grade $\mathrm{M}_n(R)$ using the standard grading.

And for any R, we have gradings on $\mathrm{M}_n(R)$ coming from the matrix structure.

We combine these two ways to grade matrix rings over graded rings.

Definition. If R is graded, we can define a grading on $\mathrm{M}_n(R)$ as follows.

Pick any sequence $\delta = (z_1, z_2, \ldots, z_n)$ in $\mathbb{Z}^n$. For $t \in \mathbb{Z}$,

$$((\mathrm{M}_n(R))_t)_{i,j} \ := \ R_{t+z_j-z_i}.$$

More gradings on matrix rings

Example. $\delta = (12, 7, 4) = (z_1, z_2, z_3)$. Let R be any graded ring.

We grade $\mathrm{M}_3(R)$ by setting, for each $t \in \mathbb{Z}$,

$$(\mathrm{M}_3(R))_t := \begin{pmatrix} R_{t+12-12} & R_{t+7-12} & R_{t+4-12} \\ R_{t+12-7} & R_{t+7-7} & R_{t+4-7} \\ R_{t+12-4} & R_{t+7-4} & R_{t+4-4} \end{pmatrix}$$

$$= \begin{pmatrix} R_t & R_{t-5} & R_{t-8} \\ R_{t+5} & R_t & R_{t-3} \\ R_{t+8} & R_{t+3} & R_t \end{pmatrix}$$

More gradings on matrix rings

So, if R is not graded, then by trivially grading R (i.e., $R_0 = R$, $R_t = 0$ for all $t \neq 0$):

we recover Grading #1 on $\mathrm{M}_3(R)$ using $\delta = (2, 1, 0)$, and

we recover Grading #2 on $\mathrm{M}_3(R)$ using $\delta = (12, 7, 4)$.

More gradings on matrix rings

For R a graded ring, and $\delta = (z_1, z_2, \dots, z_n)$ in $\mathbb{Z}^n$, denote by

$$\mathrm{M}_n(R)[(\delta)]$$

the ring $\mathrm{M}_n(R)$ with the above grading.

More gradings on matrix rings

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the ring $\mathrm{M}_n(R)$ with the above grading.

It's not hard to see: if $a \in \mathbb{Z}$, and $\delta = (z_1, z_2, \ldots, z_n)$ in $\mathbb{Z}^n$, if we define

$$\delta' := (z_1 - a, z_2 - a, \ldots, z_n - a),$$

then $\mathrm{M}_n(R)[(\delta)] = \mathrm{M}_n(R)[(\delta')]$.

More gradings on matrix rings

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then $\mathrm{M}_n(R)[(\delta)] = \mathrm{M}_n(R)[(\delta')]$.

Also, if $\kappa = (z, z, \ldots, z)$ is constant, then $\mathrm{M}_n(R)[(\kappa)]$ gives the standard grading on $\mathrm{M}_n(R)$.

More gradings on matrix rings

AND ... all of these ideas work in the same way to give gradings on $\mathrm{FM}_\infty(R)$:

Given a graded ring R, and sequence $\delta = (z_1, z_2, z_3, \dots)$ in $\mathbb{Z}^{\mathbb{N}}$, define a grading on $\mathrm{FM}_\infty(R)$ by setting, for each $t \in \mathbb{Z}$,

$$((\mathrm{FM}_\infty(R))_t)_{i,j} := R_{t+z_j-z_i}.$$

Denote this by $\mathrm{FM}_\infty(R)[(\delta)]$.

The Graded Version of The Original Morita Theorem

The Graded Version of The Original Morita Theorem. (Hazrat)

For graded unital rings R and S these are equivalent.

(GM1) The categories $\mathrm{Mod}\text{-}R$ and $\mathrm{Mod}\text{-}S$ are graded equivalent.

(GM2) There exist $n \in \mathbb{N}$ and an idempotent $e \in \mathrm{M}_n(S)$ that is full in $\mathrm{M}_n(S)$ and a sequence (δ) in $\mathbb{Z}^n$ for which the rings R and $e\mathrm{M}_n(S)[(\delta)]e$ are graded isomorphic.

(GM3) There exist a graded R-S-bimodule P and a graded S-R-bimodule Q and appropriate surjective graded bimodule homomorphisms $P \otimes_S Q \to R$ and $Q \otimes_R P \to S$.

The Graded Version of The Extended Morita Theorem

Question. Is there an appropriate (GM4) statement about isomorphisms between infinite matrix rings analogous to (M4) or (HG4) which can be added to the Graded Version of the Original Morita Theorem?

The Graded Version of The Extended Morita Theorem

Question. Is there an appropriate (GM4) statement about isomorphisms between infinite matrix rings analogous to (M4) or (HG4) which can be added to the Graded Version of the Original Morita Theorem?

Recall that if $\kappa := (z, z, z, \dots)$ is any constant sequence in $\mathbb{Z}^{\mathbb{N}}$, then $\mathrm{FM}_\infty(R)[(\kappa)]$ is just the standard grading on $\mathrm{FM}_\infty(R)$.

Theorem. (A-, Ruiz, Tomforde) The equivalent statements (GM1), (GM2), and (GM3) are equivalent to:

(GM4) There exists a sequence (δ) in $\mathbb{Z}^{\mathbb{N}}$ such that

$$\mathrm{FM}_\infty(R)[(\kappa)] \text{ is graded isomorphic to } \mathrm{FM}_\infty(S)[(\delta)].$$

Connections

A graded ring R is *strongly* graded in case $R_t R_u = R_{t+u}$ for all $t \in \mathbb{Z}$.

Dade's Theorem: If R is strongly graded, then

$$- \otimes_{R_0} R : \ \text{Mod-}R_0 \to \text{Gr-}R$$

is an equivalence of categories.

So for strongly graded rings, graded equivalence and homogeneous graded equivalence reduce to the same idea.

Connections: C^*-algebras

Connections: C*-algebras

The notion of Morita equivalence is well known to C*-algebraists.

Morita equivalence of the C*-algebras A and B is defined by the existence of an imprimitivity Hilbert bimodule ${}_AX_B$.

Let $\mathcal{K}$ denote the algebra of compact operators on a separable infinite-dimensional Hilbert space.

Theorem: (Brown-Green-Rieffel) The σ-unital C^*-algebras A and B are Morita equivalent if and only if A and B are stably isomorphic (i.e., $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$).

Connections: C*-algebras

Since $\mathcal{K} = \overline{\mathrm{FM}_\infty(\mathbb{C})}$ and $A \otimes \mathcal{K} \cong \overline{\mathrm{FM}_\infty(A)}$, we see that $A \otimes \mathcal{K}$ is the analytic analogue of $\mathrm{FM}_\infty(A)$.

So having A stably isomorphic to B (i.e., $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$) is the analytic analogue of having $\mathrm{FM}_\infty(A) \cong \mathrm{FM}_\infty(B)$.

So for operator algebraists inquiring about corresponding ring-theoretic results, (M4) is a quite natural condition.

More C*-algebra Connections: graph C*-algebras

Let E be a graph and let $C^*(E)$ be the graph C^*-algebra.

Then there is an action γ^E of the circle $\mathbb{T}$ on $C^*(E)$. Specifically, on the generators of $C^*(E)$, γ^E is given by

$$\gamma_z^E(p_v) = p_v \quad \text{and} \quad \gamma_z^E(s_e) = zs_e.$$

for $z \in \mathbb{T}$. This " gauge action" induces a $\mathbb{Z}$-grading on $C^*(E)$ via

$$C^*(E)_n = \{a \in C^*(E) \mid \gamma_z^E(a) = z^n a\}.$$

and then taking the closure.

More C*-algebra Connections: graph C*-algebras

Theorem. Let E and F be finite graphs. Then

there exists a $*$-isomorphism

$$\varphi\colon C^*(E) \to C^*(F) \ \text{ having } \ \gamma_z^F \circ \varphi = \varphi \circ \gamma_z^E$$

if and only if

there exists a graded $*$-isomorphism from $C^*(E)$ to $C^*(F)$.

More C*-algebra Connections: graph C*-algebras

We define

$$\gamma_z^{E,s} := \gamma_z^E \otimes \iota : C^*(E) \otimes \mathcal{K} \to C^*(E) \otimes \mathcal{K}.$$

Call $\gamma_z^{E,s}$ the *stabilized action*.

Then $\gamma_z^{E,s}$ is an action of $\mathbb{T}$ on $C^*(E) \otimes \mathcal{K}$ which induces a $\mathbb{Z}$-grading on $C^*(E) \otimes \mathcal{K}$ (after closing) via

$$(C^*(E) \otimes \mathcal{K})_n = \{x \in C^*(E) \otimes \mathcal{K} \mid \gamma_z^{E,s}(x) = z^n x\}.$$

This grading is the "standard" grading of $C^*(E) \otimes \mathcal{K}$. In fact,

$$(C^*(E) \otimes \mathcal{K})_n = \overline{\bigcup_{k=1}^{\infty} \mathrm{M}_k(C^*(E)_n)}$$

More C*-algebra Connections: graph C*-algebras

Theorem. Let E and F be graphs. Then there exists a $*$-isomorphism

$$\varphi\colon C^*(E)\otimes\mathcal{K}\to C^*(F)\otimes\mathcal{K} \text{ such that } \gamma_z^{F,s}\circ\varphi=\varphi\circ\gamma_z^{E,s}$$

if and only if

there exists a graded $*$-isomorphism

$$\psi : C^*(E)\otimes\mathcal{K}\to C^*(F)\otimes\mathcal{K},$$

(where the stabilizations are given the standard grading).

This is the C*-analog to condition (HG4), for graph C*-algebras.

More C^*-algebra Connections

There is a C^*-algebra analog to the (HG1) condition, in situations more general than the one described above for graph C^*-algebras.

(It has been worked out by Efren Ruiz; still work in progress.)

More C*-algebra Connections

Theorem. (Ruiz) Let G be a locally compact group. Let A and B be unital C^*-algebras and let α and β be actions of G on the C^*-algebras A and B respectively. TFAE:

1. There exists a $*$-isomorphism

$$\varphi\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$$

such that $\beta_g^s \circ \varphi(x) = \varphi \circ \alpha_g^s(x)$ for all $x \in A \otimes \mathcal{K}$ and for all $g \in G$, where α_g^s and β_g^s are the stabilized actions.

2. The systems (A, α) and (B, β) are Morita equivalent via an imprimitivity $A - B$-bimodule (M, γ) such that

$$M^\gamma = \{x \in M \mid \gamma_g(x) = x \text{ for all } g \in G\}$$

is an imprimitivity $A^\alpha - B^\beta$-bimodule.

Leavitt path algebras

Leavitt path algebras

Any Leavitt path algebra is $\mathbb{Z}$-graded, with grading given by setting

$$pq^* \in L_K(E)_n \text{ in case } \ell(p) - \ell(q) = n$$

for paths p, q in E, and $n \in \mathbb{Z}$.

Connections: Leavitt path algebras

For Leavitt path algebras:

Theorem. (Hazrat) Suppose E is a finite graph, and K any field. Then $L_K(E)$ is strongly graded (in the natural $\mathbb{Z}$-grading) if and only if E has no sinks.

So for finite graphs with no sinks, $L_K(E)$ and $L_K(F)$ are homogeneously graded equivalent if and only if they are graded equivalent.

Connections: Leavitt path algebras

For case where the graphs have sinks, the situation is not so clear.

Here's the flavor of one result.

For a finite graph E, let E^n denote the paths of length n. Let $\mathrm{Path}(E)$ denote the set of all paths in E; so

$$\mathrm{Path}(E) = \bigcup_{n \in \mathbb{Z}^+} E^n.$$

Connections: Leavitt path algebras

Proposition. Let E and F be finite acyclic graphs. Suppose E has exactly one sink v and F has exactly one sink w. Then $L_K(E)$ is homogeneously graded equivalent to $L_K(F)$ if and only if

$$\begin{aligned} &\max\{\mathrm{length}(\mu) : \mu \in \mathrm{Path}(E), r(\mu) = v\} \\ = \ &\max\{\mathrm{length}(\nu) : \nu \in \mathrm{Path}(F), r(\nu) = w\}. \end{aligned}$$

Connections: Leavitt path algebras

Proposition. Let E and F be finite acyclic graphs. Suppose E has exactly one sink v and F has exactly one sink w. Then $L_K(E)$ is homogeneously graded equivalent to $L_K(F)$ if and only if

$$\begin{aligned} &\max\{\text{length}(\mu) : \mu \in \text{Path}(E), r(\mu) = v\} \\ = \ &\max\{\text{length}(\nu) : \nu \in \text{Path}(F), r(\nu) = w\}. \end{aligned}$$

Consequently, for example, the Leavitt path algebras of these graphs are not homogeneously graded equivalent.

$$E := \bullet \qquad \text{and} \qquad F := \bullet \longrightarrow \bullet$$

Connections: Leavitt path algebras

$$E := \bullet \quad \text{and} \quad F := \bullet \longrightarrow \bullet$$

Well known: $L_K(E) \cong K$ and $L_K(F) \cong \mathrm{M}_2(K)$.

So $L_K(E)$ and $L_K(F)$ are Morita equivalent.

The natural $\mathbb{Z}$-grading on these Leavitt path algebras: easy to describe.

Connections: Leavitt path algebras

Clearly

$$\mathrm{FM}_\infty(K) \;\cong\; \mathrm{FM}_\infty(\mathrm{M}_2(K)).$$

This isomorphism is not a graded isomorphism in standard grading. (It can't be, by the previous proposition.)

But this isomorphism becomes a graded isomorphism

$$\mathrm{FM}_\infty(K)[(0,-1,0,-1,0,-1,...)] \;\cong_{gr}\; \mathrm{FM}_\infty(\mathrm{M}_2(K)).$$

So $L_K(E)$ and $L_K(F)$ are in fact graded Morita equivalent.

Graded finitely generated projective modules

If R is graded then

$$\mathcal{V}^{gr}(R)$$

denotes the graded-isomorphism classes of graded finitely generated projective right R-modules.

$\mathcal{V}^{gr}(R)$ is an abelian monoid under $\oplus$.

Graded finitely generated projective modules

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$$\mathcal{V}^{gr}(R)$$

denotes the graded-isomorphism classes of graded finitely generated projective right R-modules.

$\mathcal{V}^{gr}(R)$ is an abelian monoid under $\oplus$.

There is a natural "action" of $\mathbb{Z}[x, x^{-1}]$ on $\mathcal{V}^{gr}(R)$, via the suspension functor.

We can then view $\mathcal{V}^{gr}(R)$ as a $\mathbb{Z}[x, x^{-1}]$-module.

Hazrat's Talented Monoid Conjecture

One of the two most-discussed currently-open questions in the subject of Leavitt path algebras is

Hazrat's "Talented Monoid Conjecture"

Let E and F be finite graphs.

Suppose there is a monoid isomorphism between $\mathcal{V}^{gr}(L_K(E))$ and $\mathcal{V}^{gr}(L_K(F))$ which is compatible with the suspension functors.

That is, suppose there is an isomorphism $\mathcal{V}^{gr}(L_K(E)) \to \mathcal{V}^{gr}(L_K(F))$ as $\mathbb{Z}[x, x^{-1}]$-modules.

Question: Are $L_K(E)$ and $L_K(F)$ graded Morita equivalent?

Hazrat's Conjecture

Hazrat conjectures that the answer is YES.

(Our current work is therefore at least tangentially related to Hazrat's Conjecture ...)

Thank you
for your time.