Topological Invariants for G-kernels and Group Actions

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G-kernels

Let G be a countable discrete group and let A be a unital C^*-algebra. An action of G on A is a homomorphism

$$G \to \mathrm{Aut}(A) \ .$$

Consider the following related notion:

Definition

A *G-kernel* (or *anomalous action*) is a group homomorphism

$$\bar{\alpha} \colon G \to \mathrm{Out}(A) \ ,$$

where $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ is the group of outer automorphisms.

Cocycle actions

Definition

A *cocycle action* (α, u) of G on A consists of two maps

$$\alpha\colon G \to \mathrm{Aut}(A) \quad , \quad u\colon G \times G \to U(A) \ ,$$

such that for all $g, h, k \in G$

$$\begin{aligned} \alpha_g \circ \alpha_h &= \mathrm{Ad}(u_{g,h}) \circ \alpha_{gh} \ , \\ \alpha_g(u_{h,k}) u_{g,hk} &= u_{g,h} u_{gh,k} \ . \end{aligned}$$

- Second equation can be derived from associativity, compare

$$(\alpha_g \circ \alpha_h) \circ \alpha_k \quad \text{and} \quad \alpha_g \circ (\alpha_h \circ \alpha_k) \ .$$

The lifting obstruction

Let $\bar{\alpha}\colon G \to \mathrm{Out}(A)$ be a G-kernel. Try lifting to a cocycle action …

- Lift $\bar{\alpha}$ to a map $\alpha\colon G \to \mathrm{Aut}(A)$.
- Since $\bar{\alpha}$ is a group homomorphism, for each pair $g, h \in G$ we have unitaries $u_{g,h} \in U(A)$ such that

$$\alpha_g \circ \alpha_h = \mathrm{Ad}(u_{g,h}) \circ \alpha_{gh} \ .$$

- The associativity constraint for $u_{g,h}$ is in general not satisfied.
- However, for every $g, h, k \in G$ we have $\omega_{g,h,k} \in Z(U(A))$ such that

$$\alpha_g(u_{h,k})u_{g,hk} = \omega_{g,h,k}\, u_{g,h}u_{gh,k} \ .$$

The lifting obstruction (continued)

- The map $\omega\colon G \times G \times G \to Z(U(A))$ is a cocycle representing an element

$$\mathrm{ob}(\bar{\alpha}) = [\omega] \in H^3(G, Z(U(A))) \ .$$

Lemma

The class $\mathrm{ob}(\bar{\alpha})$ *does not depend on the choice of unitaries.*
It vanishes if and only if $\bar{\alpha}$ *lifts to a cocycle action* (α, u)*.*

G-kernels on the hyperfinite II_1-factor

Theorem (Connes, Jones, Ocneanu)

Let $A = R$ be the hyperfinite II_1-factor and let G be a discrete amenable group.

1. *The obstruction*
$$\mathrm{ob}(\bar{\alpha}) \in H^3(G, \mathbb{T})$$
is a complete invariant for injective G-kernels up to conjugacy.
2. *For every class $[\omega] \in H^3(G, \mathbb{T})$ there is an injective G-kernel $\bar{\alpha}\colon G \to Out(R)$ such that $\mathrm{ob}(\bar{\alpha}) = [\omega]$.*

G-kernels on strongly self-absorbing C^*-algebras

Theorem (Evington,Girón Pacheco)

Suppose there exists a G-kernel $\bar{\alpha}\colon G \to Out(\mathcal{Z})$ on the Jiang-Su algebra $\mathcal{Z}$. Then

$$\mathrm{ob}(\bar{\alpha}) = 0 \in H^3(G, \mathbb{T}) \; .$$

Theorem (Evington,Girón Pacheco)

Let G be a finite group. Let

$$D = \bigotimes_{k \in \mathbb{N}} M_{n_k}(\mathbb{C}) \; .$$

Let r be the order of $\mathrm{ob}(\bar{\alpha}) \in H^3(G, \mathbb{T})$. If $\bar{\alpha}\colon G \to Out(D)$ is a G-kernel, then r^∞ divides the supernatural number $\prod_{k \in \mathbb{N}} n_k$.

Let A be a unital C^*-algebra such that

- $U(A)$ is path-connected and $Z(U(A)) = \mathbb{T}$,
- $\pi_1(U(A)) \to K_0(A)$ is an isomorphism.

Examples: $\mathcal{Z}$, $\mathcal{O}_n$ (including $\mathcal{O}_\infty$), UHF-algebras, UHF $\otimes \mathcal{O}_\infty$, ...

Define

$$K_0^{\#}(A) = \{\gamma \colon [0,1] \to U(A) \ : \ f(0) = 1_A, \ f(1) \in \mathbb{T}\} / U(SA)_0 \ .$$

This group fits into the short exact sequence

$$1 \longrightarrow K_0(A) \longrightarrow K_0^{\#}(A) \xrightarrow{ev_1} \mathbb{T} \longrightarrow 1 \, .$$

Izumi's invariant (continued)

Define

$$K_0^{\#}(A) = \{\gamma\colon [0,1] \to U(A) \ : \ f(0) = 1_A,\ f(1) \in \mathbb{T}\} \,/U(SA)_0 \ .$$

Let $\bar{\alpha}\colon G \to \mathrm{Out}(A)$ be a G-kernel.

- Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $u\colon G \times G \to U(A)$ be as before.
- For each pair $g, h \in G$ pick a path $\widetilde{u}_{g,h}\colon [0,1] \to U(A)$ with

$$\widetilde{u}_{g,h}(0) = 1_A \quad \text{and} \quad \widetilde{u}_{g,h}(1) = u_{g,h} \ .$$

- The paths (obtained by point-wise multiplication)

$$\alpha_g(\widetilde{u}_{h,k})\widetilde{u}_{g,hk}\widetilde{u}^*_{gh,k}\widetilde{u}^*_{g,h}$$

  define a cocycle representing an element $\widetilde{\mathrm{ob}}(\bar{\alpha}) \in H^3(G, K_0^{\#}(A))$.

Topological crossed modules

Definition

A *(topological) crossed module* $\mathcal{G} = (\Gamma_0, \Gamma_1, \partial)$ is a morphism

$$\partial \colon \Gamma_1 \to \Gamma_0$$

of topological groups, and a continuous left action of Γ_0 on Γ_1 (denoted by $\alpha(u)$ for $\alpha \in \Gamma_0$ and $u \in \Gamma_1$), which satisfy

1. $\partial(\alpha(u)) = \alpha\,\partial(u)\,\alpha^{-1}$ for all $\alpha \in \Gamma_0$ and $u \in \Gamma_1$,
2. $\partial(u)(v) = u\,v\,u^{-1}$ for all $u, v \in \Gamma_1$.

First examples

1. $\Gamma_0 = \{1\}$, Γ_1 abelian,
2. Γ_1 a normal subgroup of Γ_0.

Cohomology with coefficients in $\mathcal{G}$

Definition

A *1-cocycle on G with values in $\mathcal{G}$* consists of a pair (α, u), where $\alpha\colon G \to \Gamma_0$ and $u\colon G \times G \to \Gamma_1$ satisfying the following conditions

$$\alpha_g\, \alpha_h = \partial(u_{g,h})\, \alpha_{gh}\ ,$$
$$\alpha_g(u_{h,k})\, u_{g,hk} = u_{g,h}\, u_{gh,k}\ .$$

Two cocycles $(\alpha^{(0)}, u^{(0)})$ and $(\alpha^{(1)}, u^{(1)})$ are *cohomologous* if $\exists \gamma \in \Gamma_0$ and $w\colon G \to \Gamma_1$ with:

$$\alpha_g^{(1)} = \partial w_g\, \gamma\, \alpha_g^{(0)}\, \gamma^{-1}$$
$$u_{g,h}^{(1)} = w_g \gamma(\alpha_g^{(0)}(\gamma^{-1}(w_h)) u_{g,h}^{(0)}) w_{gh}^{-1}$$

Define $H^1(G, \mathcal{G}) = Z^1(G, \mathcal{G})/\sim$. In general this is not a group!

Crossed modules associated to C^*-algebras

Let A be a unital C^*-algebra.

Examples

1. $\mathcal{G}_A = (\mathrm{Aut}(A), U(A), \mathrm{Ad})$ with $\mathrm{Ad}(u)(a) = uau^*$ and the canonical action of $\mathrm{Aut}(A)$ on $U(A)$.

$$H^1(G, \mathcal{G}_A) \xleftrightarrow{1:1} \{\text{cocycle actions on } A\}/\sim_{cc}$$

2. $P\mathcal{G}_A = (\mathrm{Aut}(A), PU(A), \mathrm{Ad})$ with $PU(A) = U(A)/Z(U(A))$.

$$H^1(G, P\mathcal{G}_A) \xleftrightarrow{1:1} \{G\text{-kernels on } A\}/\sim_{c}$$

3. If $U(A)$ is connected, then $\widetilde{\mathcal{G}}_A = (\mathrm{Aut}(A), \widetilde{U(A)}, \widetilde{\mathrm{Ad}})$ where $\widetilde{U(A)}$ is the universal cover of $U(A)$.

Revisiting the lifting obstructions

Let A be a unital C^*-algebra and let $\mathcal{Z}_A = (\{1\}, Z(U(A)), \text{triv})$.

The short exact sequence of crossed modules

$$\begin{array}{ccccc} Z(U(A)) & \longrightarrow & U(A) & \longrightarrow & PU(A) \\ \downarrow & & \downarrow{\scriptstyle \text{Ad}} & & \downarrow{\scriptstyle \text{Ad}} \\ 1 & \longrightarrow & \text{Aut}(A) & = & \text{Aut}(A) \end{array}$$

gives rise to a long(-ish) exact sequence of pointed sets

$$\ldots \longrightarrow H^1(G, \mathcal{Z}_A) \longrightarrow H^1(G, \mathcal{G}_A) \longrightarrow H^1(G, P\mathcal{G}_A) \xrightarrow{\text{ob}} H^2(G, \mathcal{Z}_A)$$

Note that $H^2(G, \mathcal{Z}_A)$ only exists, because $Z(U(A))$ is abelian. In fact,

$$H^2(G, \mathcal{Z}_A) \cong H^3(G, Z(U(A))) \,.$$

Revisiting the lifting obstructions (continued)

Let A be a unital C^*-algebra such that

- $U(A)$ is connected and $Z(U(A)) = \mathbb{T}$,
- $\pi_1(U(A)) \to K_0(A)$ is an isomorphism.

It turns out that the group $K_0^{\#}(A)$ fits into an exact sequence of crossed modules of the form

$$\begin{array}{ccccc} K_0^{\#}(A) & \longrightarrow & \widetilde{U(A)} & \xrightarrow{\pi} & PU(A) \\ \downarrow & & \downarrow \widetilde{\mathrm{Ad}} & & \downarrow \mathrm{Ad} \\ 1 & \longrightarrow & \mathrm{Aut}(A) & = & \mathrm{Aut}(A) \end{array}$$

which gives rise to an exact sequence of pointed sets

$$\ldots \longrightarrow H^2(G, K_0^{\#}(A)) \longrightarrow H^1(G, \widetilde{\mathcal{G}}_A) \longrightarrow H^1(G, P\mathcal{G}_A) \xrightarrow{\widetilde{\mathrm{ob}}} H^3(G, K_0^{\#}(A))$$

The special unitary group $SU_\tau(A)$

Definition

Let $\tau\colon A \to \mathbb{C}$ be a tracial state on the unital C^*-algebra A. Define

$$SU_\tau(A) = \{u = e^{ih_1}\cdots e^{ih_n} \; : \; \text{for some } n \in \mathbb{N}, h_i \in A^{sa} \cap \ker(\tau)\}$$

Theorem (Girón Pacheco, Izumi, P.)

Suppose that $\pi_1(U(A)) \to K_0(A)$ is surjective, then

$$SU_\tau(A) = \ker(\Delta_\tau) \cap U(A)_0$$

where $\Delta_\tau\colon U(A)_0 \to \mathbb{R}/\tau_*(K_0(A))$ is the de la Harpe-Skandalis determinant.

The special unitary group $SU_\tau(A)$ (continued)

Let A be a unital C^*-algebra with tracial state τ such that

- $U(A)$ is connected and $Z(U(A)) = \mathbb{T}$,
- $\pi_1(U(A)) \to K_0(A)$ is an isomorphism,
- $\tau_* \colon K_0(A) \to \mathbb{R}$ is injective.

Theorem (Girón Pacheco, Izumi, P.)

Let A be as above. Then

1. *$SU_\tau(A)$ is simply-connected,*
2. *$\widetilde{U(A)} \cong SU_\tau(A) \times \mathbb{R}$,*
3. *we have a short exact sequence*

$$1 \longrightarrow e^{2\pi i \tau_*(K_0(A))} \longrightarrow SU_\tau(A) \longrightarrow PU(A) \longrightarrow 1$$

The tracial lifting obstruction

Let $S\mathcal{G}_A = (\mathrm{Aut}(A), SU_\tau(A), \mathsf{Ad})$. The extension of crossed modules

$$\begin{array}{ccccc} e^{2\pi i \tau_*(K_0(A))} & \longrightarrow & SU_\tau(A) & \xrightarrow{\pi} & PU(A) \\ \downarrow & & \downarrow & & \downarrow \mathsf{Ad} \\ 1 & \longrightarrow & \mathrm{Aut}(A) & = & \mathrm{Aut}(A) \end{array}$$

gives rise to the tracial lifting obstruction

$$H^1(G, S\mathcal{G}_A) \longrightarrow H^1(G, P\mathcal{G}_A) \xrightarrow{\mathsf{ob}_\tau} H^3(G, e^{2\pi i \tau_*(K_0(A))})$$

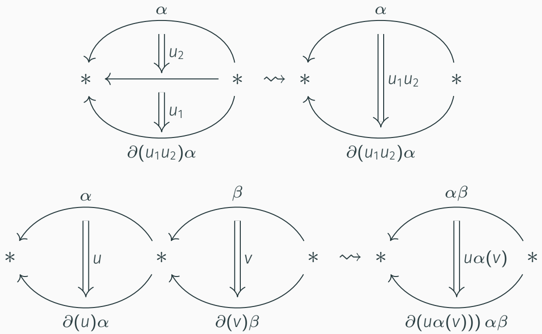
Let $S_{\mathbb{R}}\mathcal{G} = (\mathrm{Aut}(A), SU_\tau(A) \times \mathbb{R}, \mathsf{Ad})$. The following diagram commutes

$$\begin{array}{ccccc} H^1(G, S_{\mathbb{R}}\mathcal{G}_A) & \longrightarrow & H^1(G, P\mathcal{G}_A) & \xrightarrow{(\mathsf{ob}_\tau, 0)} & H^3(G, e^{2\pi i \tau_*(K_0(A))} \times \mathbb{R}) \\ \downarrow \cong & & \| & & \downarrow \cong \\ H^1(G, \widetilde{\mathcal{G}}_A) & \longrightarrow & H^1(G, P\mathcal{G}_A) & \xrightarrow{\widetilde{\mathsf{ob}}} & H^3(G, K_0^{\#}(A)) \end{array}$$

Crossed modules and 2-categories

(top.) crossed module $\mathcal{G} = (\Gamma_0, \Gamma_1, \partial)$ $\rightsquigarrow$ (top.) 2-category

- a single object (denoted $*$),
- 1-morphisms given by Γ_0,
- 2-morphisms given by $\Gamma_0 \rtimes \Gamma_1$,
  where $(\alpha, u) \in \Gamma_0 \rtimes \Gamma_1$ gives a 2-morphism $\alpha \Rightarrow \partial(u)\alpha$.



Classifying space of $\mathcal{G}$

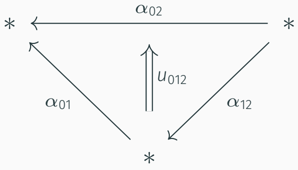
Let $\mathcal{G} = (\Gamma_0, \Gamma_1, \partial)$ be a crossed module. The spaces

$$N_n^D(\mathcal{G}) = \mathrm{Fun}_{\mathrm{su}}([n], \mathcal{G}) \ .$$

given by strictly unital pseudofunctors $[n] \to \mathcal{G}$ form a simplicial space called the Duskin nerve of $\mathcal{G}$. We have

$$N_0^D(\mathcal{G}) = * \quad \text{and} \quad N_1^D(\mathcal{G}) = \Gamma_0 \ .$$

The space $N_2^D(\mathcal{G})$ consists of triangles of the form



(in particular: $\partial(u_{012})\,\alpha_{02} = \alpha_{01}\,\alpha_{12}$)

Classifying space of $\mathcal{G}$ (continued)

Definition

The *classifying space* $B^D\mathcal{G}$ is the geometric realisation of $N_*^D(\mathcal{G})$, i.e.

$$B^D\mathcal{G} = |N_*^D(\mathcal{G})| \ .$$

Each 1-cocycle $(\alpha, u) \in Z^1(G, \mathcal{G})$ induces a pseudofunctor $G \to \mathcal{G}$. Applying the Duskin nerve and geometric realisation gives a map

$$BG \to B^D\mathcal{G} \ .$$

Lemma

This provides a well-defined natural transformation of pointed sets

$$H^1(G, \mathcal{G}) \to [BG, B^D\mathcal{G}]$$

Classifying space of $\mathcal{G}$ (continued)

This allows us to study the lifting obstructions by mapping them to homotopy sets. For example,

$$\begin{array}{ccccc} H^1(G,\widetilde{\mathcal{G}}_A) & \longrightarrow & H^1(G,P\mathcal{G}_A) & \xrightarrow{\widetilde{\text{ob}}} & H^3(G,K_0^{\#}(A)) \\ \downarrow & & \downarrow & & \downarrow \\ [BG,B^D\widetilde{\mathcal{G}}_A] & \longrightarrow & [BG,B^DP\widetilde{\mathcal{G}}_A] & \longrightarrow & [BG,B^D(K_0^{\#}(A)\to\{1\})] \end{array}$$

Caveats:

- The upper sequence is an exact sequence of pointed sets. There is no reason why this should be true for the lower sequence.
- For integer coefficients $H^1(G,\mathbb{Z})\cong H^1(BG,\mathbb{Z})\cong[BG,B\mathbb{Z}]$. This is no longer true for our crossed module coefficients.

Principal bundles and stabilisation

Let D be a strongly self-absorbing C^*-algebra with $K_1(D) = 0$ and let (α, u) be a cocycle action of G on D.

For an appropriate choice of $V_g \in U(\ell^2(G))$ (depending on u_g and α_g)

$$g \mapsto \hat{\alpha}_g = \mathrm{Ad}(V_g) \circ (\alpha_g \otimes \mathrm{id})$$

is an action of G on $D \otimes \mathbb{K} = D \otimes \mathbb{K}(\ell^2(G))$. Let

$$P_{\hat{\alpha}} = EG \times_{\hat{\alpha}} \mathrm{Aut}_0(D \otimes \mathbb{K}) \to BG$$

Theorem (Izumi, Meyer, Gabe-Szabo)

Let D be a strongly self-absorbing Kirchberg algebra, let G be countable discrete amenable, such that BG has a finite CW-complex model. Then

$$\{\textit{outer cocycle actions of } G \textit{ on } D\}/\sim_{cc} \;\xleftrightarrow{1:1}\; [BG, B\mathrm{Aut}_0(D \otimes \mathbb{K})]$$

Classification via cohomology theories

Let D be a strongly self-absorbing C^*-algebra as above.

Theorem (Dadarlat-P.)

We have that $[X, BAut_0(D \otimes \mathbb{K})]$ is a group with respect to $\otimes_X$. If $D \neq \mathbb{C}$, then

$$[X, BAut_0(D \otimes \mathbb{K})] \cong [X, BSL_1(KU^D)] \ .$$

Conjecture

There is a weak homotopy equivalence

$$B^D\mathcal{G}_D \simeq B\mathrm{Aut}_0(D \otimes \mathbb{K})$$

and $H^1(G, \mathcal{G}_D) \to [BG, B^D\mathcal{G}_D] \cong [BG, B\mathrm{Aut}_0(D \otimes \mathbb{K})]$ is the map

$$[(\alpha, u)] \mapsto [P_{\hat{\alpha}}] \ .$$

Moreover, $B^D\mathcal{G}_D$, $B^DP\mathcal{G}_D$ and $B^D\widetilde{\mathcal{G}}_D$ are infinite loop spaces.

Group actions and equivariant bundles

Let G be a finite group and let H be a countable group. An action

$$\theta\colon G \times H \to \mathrm{Aut}(D \otimes \mathbb{K})$$

gives rise to an equivariant principal bundle

$$E(G \times H) \times_{\theta|_H} \mathrm{Aut}(D \otimes \mathbb{K}) \to B_G H$$

where $B_G H = E(G \times H)/H$ is the equivariant classifying space of H. These are classified by

$$[B_G H, B\mathrm{Aut}(D \otimes \mathbb{K})]^G$$

Let $G = \mathbb{Z}/p\mathbb{Z}$ for a prime p. Let V be a finite-dimensional unitary G-representation and

$$D = \mathrm{End}(V)^{\otimes\infty}$$

Let $\mathbb{K}$ be the compact operators on a full G-universe.

Theorem (Bianchi,P.)

The group $Aut(D \otimes \mathbb{K})$ is a G-equivariant infinite loop space. The associated G-equivariant cohomology theory $E^*_D(X)$ satisfies

$$E^*_D(X) \cong gl_1(KU^D)^*_+(X)$$

and $E^1_{D,G}(X) \cong [X, BAut(D \otimes \mathbb{K})]^G$.

Thank you!