From algebraic actions to C*-algebras and back again

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Let S be a left-cancellative monoid (e.g., S a submonoid of a group).

Definition.

An algebraic S-action is an action of S on a group G by injective group endomorphisms, i.e., a monoid homomorphism

$$\sigma\colon S \to \mathrm{End}\,(G), \quad s \mapsto \sigma_s,$$

such that σ_s is injective for all $s \in S$.[a]

[a] We shall usually assume our action is faithful and that there exists $s \in S$ with $\sigma_s G \lneq G$.

Remark.

If $\sigma\colon S \curvearrowright G$ is an algebraic S-action with G abelian, then we get a dual action $\hat{\sigma}\colon S \curvearrowright \widehat{G}$ by continuous, surjective endomorphisms of the compact group $\widehat{G}$.

Algebraic actions of groups on abelian groups have been studied by Kitchens, Schmidt, Lind, etc. Algebraic actions of semigroups have not received much attention.

Example (Full shifts).

If Σ is any non-trivial group, then the canonical action

$$\sigma\colon S \curvearrowright \bigoplus_S \Sigma, \quad \sigma_s(x)_t = \begin{cases} x_{s^{-1}t} & \text{if } t \in sS, \\ e_\Sigma & \text{if } t \notin sS, \end{cases}$$

for $x = (x_t)_t \in \bigoplus_S \Sigma$ is an algebraic action called the full S-shift over Σ.

Example.

Let R be an integral domain. The multiplicative monoid $R^\times := R \setminus \{0\}$ acts on the (additive group of) R by multiplication, and $\sigma\colon R^\times \curvearrowright R$ is an algebraic action.

Each algebraic S-action $\sigma\colon S \curvearrowright G$ gives rise to a C*-algebra as follows: There is an isometric representation

$$\kappa\colon S \to \mathsf{Isom}(\ell^2 G), \quad \kappa(s)\delta_x = \delta_{\sigma_s(x)},$$

where $\{\delta_x : x \in G\}$ is the canonical orthonormal basis for $\ell^2 G$.

Definition.

We let

$$\mathfrak{A}_\sigma := C^*(\{\kappa(s) : s \in S\} \cup \{\lambda(g) : g \in G\}),$$

where $\lambda\colon G \to \mathcal{U}(\ell^2 G)$ *is the left regular representation of* G*.*

Such C*-algebras from algebraic actions have been considered, e.g., for

- algebraic $\mathbb{N}$-actions (Hirshberg, Cuntz–Vershik, and Vieira).
- examples from rings (Cuntz, Cuntz–Li, and Li);
- special actions of right LCM monoids (Stammeier and Brownlowe–Larsen–Stammeier);
- actions of congruence monoids on rings of algebraic integers (B. and B.–Li).

Example.

- *For the shift* $\sigma\colon \mathbb{N} \curvearrowright \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, *we have* $\mathfrak{A}_\sigma \cong \mathcal{O}_n$.
- *For* $\sigma\colon R^\times \curvearrowright R$, *we see that* $\mathfrak{A}_\sigma = \mathfrak{A}[R]$ *is the (reduced) ring C*-algebra of* R.

Our motivation is to provide a unified framework for studying C*-algebras from algebraic actions of semigroups, so that we can systematically analyze new example classes, e.g., shifts.

Recall that a *partial bijection* of G is a bijective map

$$f\colon \mathrm{dom}(f) \to \mathrm{im}(f),$$

where $\mathrm{dom}(f), \mathrm{im}(f) \subseteq G$.The set $\mathcal{I}_G$ of all partial bijections on G is an inverse semigroup with respect to composition and inversion.

Definition (B.–Li).

Let I_σ be the inverse semigroup generated by the partial bijections

$$\sigma_s\colon G \to \sigma_s G, \quad x \mapsto \sigma_s(x) \quad (\textit{for } s \in S)$$

and

$$\mathbb{t}_g\colon G \to G, \quad x \mapsto \mathbb{t}_g(x) := gx \quad (\textit{for } g \in G).$$

There is a faithful representation

$$\Lambda\colon \mathcal{I}_G \to \mathsf{PIsom}(\ell^2 G), \quad \Lambda_\phi \delta_x = \begin{cases} \delta_{\phi(x)} & \text{if } x \in \mathsf{dom}(\phi), \\ 0 & \text{if } x \notin \mathsf{dom}(\phi). \end{cases}$$

We view Λ as a representation of I_σ. We have:

- $\Lambda_{\sigma_s} = \kappa(s)$ in $\mathsf{Isom}(\ell^2 G)$ for all $s \in S$;
- $\Lambda_{\mathbb{t}_g} = \lambda(g)$ in $\mathcal{U}(\ell^2 G)$ for all $g \in G$;
- $\mathfrak{A}_\sigma = \overline{\mathsf{span}}(\{\Lambda_\phi : \phi \in I_\sigma\})$.

Remark.

From this, we see that there should be a close relationship between the C-algebras associated with I_σ and the concrete C*-algebra $\mathfrak{A}_\sigma$.*

Definition (B.–Li).

We let

$$\mathcal{C}_\sigma := \{\sigma_{s_1}^{-1}\sigma_{t_1}\cdots\sigma_{s_m}^{-1}\sigma_{t_m}G : s_i, t_i \in S, m \in \mathbb{Z}_{>0}\}.$$

Members of $\mathcal{C}_\sigma$ are called S-constructible subgroups. The idempotent semilattice of I_σ is then $\mathcal{E}_\sigma = \{gC : g \in G, C \in \mathcal{C}_\sigma\} \cup \{\emptyset\}$; its members are called S-constructible cosets.

Remark.

Each non-trivial element of I_σ is a composition of the form

$$gC \xrightarrow{\mathbb{t}_{g^{-1}}} C \xrightarrow{\varphi} D \xrightarrow{\mathbb{t}_h} hD,$$

where $\varphi = \sigma_{s_1}^{-1}\sigma_{t_1}\cdots\sigma_{s_m}^{-1}\sigma_{t_m}$ for some $s_i, t_i \in S$ and $C, D \in \mathcal{C}_\sigma$, $g, h \in G$.

Example.

For the full S-shift $S \curvearrowright \bigoplus_S \Sigma$, we have

$$\mathcal{C}_\sigma = \{\bigoplus_X \Sigma : X \in \mathcal{J}_S\},$$

where $\mathcal{J}_S$ is the semilattice of constructible right ideals of S.

Example.

For $\sigma \colon R^\times \curvearrowright R$, $\mathcal{C}_\sigma$ is the semilattice of constructible ring-theoretic ideals of R. E.g., if $R = \mathcal{O}_K$ is the ring of integers in a number fields K, then

$$\mathcal{C}_\sigma = \{I : (0) \neq I \trianglelefteq \mathcal{O}_K\}$$

consists of all non-zero ideals of $\mathcal{O}_K$.

The canonical commutative subalgebra of $\mathfrak{A}_\sigma$ is given by

$$\mathfrak{D}_\sigma := \overline{\mathsf{span}}(\{\Lambda_\phi : \phi \in \mathcal{E}_\sigma\}) = \overline{\mathsf{span}}(\{1_{gC} : g \in G, C \in \mathcal{C}_\sigma\}).$$

Proposition (B.–Li).

The spectrum $\partial\widehat{\mathcal{E}}_\sigma := \mathrm{Spec}\,(\mathfrak{D}_\sigma)$ *can be identified with the space of non-zero characters* $\chi\colon \mathcal{E}_\sigma \to \{0,1\}$ *such that for* $gC, g_iC_i \in \mathcal{E}_\sigma^\times$*, where* $i \in F$ *and* $\#F < \infty$*, we have*

$$\chi(gC) = 1 \text{ and } gC = \textstyle\bigcup_{i\in F} g_iC_i \implies \chi(g_iC_i) = 1 \text{ for some } i \in F.$$

Remark.

In general, $\partial\widehat{\mathcal{E}}_\sigma$ *is a completion of (a quotient of)* G*.*

- ➔ The inverse semigroup I_σ acts canonically on $\partial\widehat{\mathcal{E}}_\sigma$ by partial homeomorphisms, and we form the associated transformation groupoid $\mathcal{G}_\sigma := I_\sigma \ltimes \partial\widehat{\mathcal{E}}_\sigma$.
- ➔ There is a canonical, surjective *-homomorphism $\rho\colon C^*(\mathcal{G}_\sigma) \to \mathfrak{A}_\sigma$.

Remark.

The groupoid $\mathcal{G}_\sigma$ is the tight groupoid of I_σ in the sense of Exel.

Example.

For the $\mathbb{N}$-action $\sigma\colon \mathbb{N} \curvearrowright \mathbb{Z}$ given by $\sigma(m) = 2m$, we have

$$\mathcal{C}_\sigma = \{2^k\mathbb{Z} : k \in \mathbb{N}\} \quad \textit{and} \quad \partial\widehat{\mathcal{E}}_\sigma \cong \mathbb{Z}_2 := \varprojlim_n \mathbb{Z}/2^n\mathbb{Z}.$$

In this case, $\mathcal{G}_\sigma \cong (\mathbb{Z}[1/2] \rtimes \langle 2\rangle) \ltimes \mathbb{Z}_2$.

Definition (B.–Li).

The algebraic action $\sigma \colon S \curvearrowright G$ is said to be exact if

$$\bigcap_{C \in \mathcal{C}_\sigma} C = \{e\}.$$

For example, $\sigma \colon S \curvearrowright G$ is exact whenever we have $\bigcap_{s \in S} \sigma_s G = \{e\}$.

Remark.

Our notion of exactness is a generalization of Rohlin's notion of exactness for a single endomorphism.

Proposition (B.–Li).

The groupoid $\mathcal{G}_\sigma$ is topologically free if $\sigma \colon S \curvearrowright G$ is exact.

Example (Toral endomorphisms (Krzyżewski, Handelman)).

Let $\sigma\colon \mathbb{N} \curvearrowright \mathbb{Z}^d$ *be the algebraic* $\mathbb{N}$*-action defined by a matrix* $a \in \mathsf{M}_d(\mathbb{Z}) \cap \mathsf{GL}(\mathbb{Q})$*. Then*

$$\mathcal{C}_\sigma = \{a^n\mathbb{Z}^d : n \in \mathbb{N}\},$$

and $\bigcap_{n=0}^{\infty} a^n\mathbb{Z}^d = \{0\}$ *if and only if the minimal polynomial of* a *has no prime factors in* $\mathbb{Z}[x]$ *whose constant term lies in* $\{\pm 1\}$*.*

Example.

- $S \curvearrowright \bigoplus_S \Sigma$ *is exact if and only if* S *contains a non-invertible elements.*
- *The action* $\sigma\colon R^\times \curvearrowright R$ *is exact if and only if* R *is not a field.*

We have technical (but checkable!) conditions (H) and (M) on our action $\sigma\colon S \curvearrowright G$ that characterize when $\mathcal{G}_\sigma$ is Hausdorff and minimal, respectively.

Example.

- *If G is a finite rank, torsion-free abelian group, then $\sigma\colon S \curvearrowright G$ satisfies (H) and (M).*
- *The shift $S \curvearrowright \bigoplus_S \Sigma$ satisfies (M) if and only if S is left reversible and satisfies (H) whenever S is right reversible.*
- *$\sigma\colon R^\times \curvearrowright R$ satisfies (M) if and only if R is not a field and satisfies (H) if and only if R is not a field.*

It turns out that pure infiniteness is automatic in the minimal setting:

Theorem (B.–Li).

If $\mathcal{G}_\sigma$ is minimal (i.e., if $\sigma \colon S \curvearrowright G$ satisfies (M)), then $\mathcal{G}_\sigma$ is purely infinite.

The main consequence for our C*-algebras is the following:

Corollary (B.–Li).

Assume S and G are countable and that $\sigma \colon S \curvearrowright G$ is exact and satisfies (M). Then, $C^*_{\mathrm{ess}}(\mathcal{G}_\sigma)$ is simple and purely infinite.

Let χ_e be the character on $\mathcal{E}_\sigma$ given by $\chi_e(C) = 1$ for all $C \in \mathcal{C}$.

Theorem (B.–Li).

*If $\sigma\colon S \curvearrowright G$ is exact and $(\mathcal{G}_\sigma)^{\chi_e}_{\chi_e}$ is amenable, then there exists a canonical *-isomorphism $C^*_{\mathrm{ess}}(\mathcal{G}_\sigma) \cong \mathfrak{A}_\sigma$.*

Remark.

- *In nice cases, $(\mathcal{G}_\sigma)^{\chi_e}_{\chi_e}$ is an enveloping group of S and amenability of $\mathcal{G}_\sigma$ is equivalent to amenability of the group $(\mathcal{G}_\sigma)^{\chi_e}_{\chi_e}$.*
- *We also obtain exotic examples, e.g., for the shift $\sigma\colon \mathbb{F}_2^+ \curvearrowright \bigoplus_{\mathbb{F}_2^+} \Sigma$ over any non-amenable group Σ, we have surjective, non-invertible *-homomorphisms*

$$C^*(\mathcal{G}_\sigma) \to \mathfrak{A}_\sigma \to C^*_r(\mathcal{G}_\sigma)$$

whose composition is the identity on $C_c(\mathcal{G}_\sigma)$.

Our theorems apply to several large example classes, including:

- The C*-algebra of the shift when S is left reversible and embeds in a group.
- C*-algebras associated with exact actions on torsion-free, finite rank abelian groups.
- The ring C*-algebra of an integral domain that is not a field (known by Cuntz–Li).
- Many C*-algebras for actions of the form $S \curvearrowright R$, where $S \subseteq R^\times$ is a submonoid (e.g., the case where S is a congruence monoid in a ring of integers).
- Ring C*-algebras of many non-commutative rings.

Theorem (B.–Li, 2023).

Let K and L be number fields with rings of integers $\mathcal{O}_K$ and $\mathcal{O}_L$. The following are equivalent:

(a) $\mathcal{O}_K \cong \mathcal{O}_L$ *as rings (equivalently, $K \cong L$);*

(b) *the actions $\sigma \colon \mathcal{O}_K^\times \curvearrowright \mathcal{O}_K$ and $\tau \colon \mathcal{O}_L^\times \curvearrowright \mathcal{O}_L$ are isomorphic;*

(c) *there is a *-isomorphism $\alpha \colon \mathfrak{A}_\sigma \xrightarrow{\cong} \mathfrak{A}_\tau$ such that $\alpha(\mathfrak{D}_\sigma) = \mathfrak{D}_\tau$;*

(d) *the groupoids $\mathcal{G}_\sigma$ and $\mathcal{G}_\tau$ are isomorphic.*

- This is in stark contrast with a result by Li and Lück which says that $\mathfrak{A}[\mathcal{O}_K] \cong \mathfrak{A}[\mathcal{O}_L]$ independent of K and L.
- We have an isomorphism $\mathfrak{D}_\sigma \cong \mathfrak{D}_\tau$ independent of K and L.
- Thank you for your attention!