Deformation quantization and Morita equivalence

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Lecture 3 - Outline:

1. Morita equivalence of star products (lecture 2 reminder)
2. B-field symmetries of Poisson structures
3. Kontsevich's classes of Morita equivalent star products

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Theorem

Star products $\star$ and $\star'$ are Morita equivalent iff $[\star]$, $[\star']$ lie in the same $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$-orbit:

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Description in terms of Kontsevich's classes?

$$\check{H}^2(M, \mathbb{Z}) \circlearrowright \quad \mathrm{FPois}(M) \xrightarrow{\mathcal{K}_*} \mathrm{Def}(M) \quad \circlearrowleft \mathrm{Pic}(M) = \check{H}^2(M, \mathbb{Z})$$

2. B-field symmetries of Poisson structures

B-field or gauge symmetries:

view $\Omega^2_{cl}(M)$ as symmetries of Poisson structures (Severa-Weinstein '01).

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E.g., $$\omega^{-1} \mapsto \omega^{-1}(1 + B\omega^{-1})^{-1} = (\omega + B)^{-1}$$

Things are nicer in the formal world:

$$\pi_\hbar = \hbar\pi_1 + \hbar^2\pi_2 + \ldots \quad \text{in} \quad \hbar\mathcal{X}^2(M)[[\hbar]]$$

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Theorem (B., Dolgushev, Waldmann, '09)
This action descends to

$H^2(M, \mathbb{C}) \times \mathrm{FPois(M)} \longrightarrow \mathrm{FPois(M)}, \quad [\pi_\hbar] \mapsto [\pi_\hbar^B]$

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General description of characteristic classes of Morita equivalent star products.

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Symplectic case: Fedosov-Deligne classes

$$c(\star) - c(\star'_\varphi) \in 2\pi i H^2(M,\mathbb{Z})$$

Some references:

◇ B., Waldmann: *Deformation quantization of Hermitian vector bundles*, Lett. Math. Phys. 53.4 (2000), 349–365

◇ B.: *Semiclassical geometry of quantum line bundles and Morita equivalence of star products*, Intern. Math. Res. Notices , 16 (2002) 821–846.

◇ B., Waldmann: *The characteristic classes of Morita equivalent star products on symplectic manifolds*, Comm. Math. Phys. 228.1 (2002), 103-121.

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◇ B., Waldmann: *Bimodule deformations, Picard groups and contravariant connections*, K-Theory 31 (2004), 1-37.

◇ Dolgushev: *Covariant and Equivariant Formality Theorems*, Adv. Math. 191, (2005) 147-177.

◇ B., Waldmann, Dolgushev: *Morita equivalence and characteristic classes of star products*, arXiv:0909.4259. To appear in Crelle's journal.