Deformation quantization and Morita equivalence

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Lecture 2 - Outline:

1. Deformation quantization (lecture 1 reminder)
2. Morita equivalence reminder
3. Morita equivalence of star products

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where $C_r : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ bidifferential

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Moduli of star products: $\mathrm{Def}(M) = \{\star\}/\sim$

Formal Poisson structures: $\pi_\hbar \in \hbar\mathcal{X}^2(M)[[\hbar]]$,

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Problem: When are two star products Morita equivalent ?

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When are two star products $\star$, $\star'$ Morita equivalent?

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Theorem

Star products $\star$ and $\star'$ are Morita equivalent iff $[\star]$, $[\star']$ lie in the same $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$-orbit:

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works also purely algebraically...

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$$\check{H}^2(M,\mathbb{Z}) \circlearrowright \quad \mathrm{FPois}(M) \xrightarrow{\mathcal{K}_*} \mathrm{Def}(M) \quad \circlearrowleft \mathrm{Pic}(M) = \check{H}^2(M,\mathbb{Z})$$

Description of ME star products via characteristic classes?

Poisson-geometric counterpart of algebraic ME of quantum algebras?

Some references:

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◇ B.: *Semiclassical geometry of quantum line bundles and Morita equivalence of star products*, Intern. Math. Res. Notices , 16 (2002) 821–846.

◇ B., Waldmann: *Bimodule deformations, Picard groups and contravariant connections*, K-Theory 31 (2004), 1-37.