Generalized asymptotic algebras and $\mathbb{E}$-theory for non-separable C^*-algebras

Christopher Wulff

Universidad Georg–August de Gotinga

Seminario de Geometría No Conmutativa del Atlántico Sur
18 de abril de 2023

E-theory as bivariant K-theory

E-theory consists of groups $\mathrm{E}(A, B)$ for all C^*-algebras A, B with the following properties:

- $\mathrm{E}(\mathbb{C}, B) = \mathrm{K}_0(B) = \mathrm{K}(B)$ is the K-theory of B.
- $\mathrm{E}(A, \mathbb{C}) = \mathrm{K}^0(A)$ is the K-homology of A.
- $*$-homomorphisms $\varphi\colon A \to B$ give rise to classes $[\![\varphi]\!] \in \mathrm{E}(A, B)$.
- There are composition products $\mathrm{E}(A, B) \otimes \mathrm{E}(B, C) \to \mathrm{E}(A, C)$ extending the composition of $*$-homomorphisms.
- Many more properties.

Tool for computing K-theory

E.g. composition product $\mathrm{K}(B) \otimes \mathrm{E}(B, C) \to \mathrm{K}(C)$ extends functoriality.

Caveat

Almost all definitions of bivariant K-theory require separability of the C^*-algebras either to construct the composition products or to retrieve long exact sequences.

Asymptotic morphisms

Definition (Guentner–Higson–Trout 2000)

Asymptotic algebra of C^*-algebra B:

$$\mathfrak{A}B := \frac{\mathrm{C}_b([1,\infty);B)}{\mathrm{C}_0([1,\infty);B)}$$

An asymptotic morphism from A to B is a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}B$.

φ is represented by bounded continuous family of maps

$$\varphi_t\colon A \to B$$

such that for all $a, b \in A$, $\lambda \in \mathbb{C}$:

$$\left.\begin{array}{c} \varphi_t(\lambda a + b) - \lambda\varphi_t(a) - \varphi_t(b) \\ \varphi_t(ab) - \varphi_t(a)\varphi_t(b) \\ \varphi_t(a^*) - \varphi_t(a)^* \end{array}\right\} \xrightarrow{t\to\infty} 0$$

E-theory classes of Dirac operators

Example

Let $S \to M$ be a Dirac bundle over a compact manifold and D the associated Dirac operator. Then

$$\varphi_t(f \otimes g) := f(t^{-1}D)g$$

defines an asymptotic morphism

$$\mathrm{C}_0(\mathbb{R}) \otimes \mathrm{C}(M) \to \mathfrak{A}(\mathfrak{K}(L^2(M,S)))$$

is an asymptotic morphism.

It gives rise to a class $[\![D]\!] \in \mathrm{E}(\mathrm{C}(M), \mathbb{C}) = \mathrm{K}_0(M)$ such that

$$\mathrm{K}^0(M) \otimes \mathrm{K}_0(M) = \mathrm{E}(\mathbb{C}, \mathrm{C}(M)) \otimes \mathrm{E}(\mathrm{C}(M), \mathbb{C}) \to \mathrm{E}(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$$

maps $[V] \otimes [\![D]\!]$ to the index of the twisted operator D_V.

Functoriality

Definition (Guentner–Higson–Trout 2000)

Asymptotic algebra of C^*-algebra B:

$$\mathfrak{A}B := \frac{\mathrm{C}_b([1,\infty);B)}{\mathrm{C}_0([1,\infty);B)}$$

An asymptotic morphism from A to B is a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}B$.

Funtoriality of K-theory under asymptotic morphisms

If $P \in A^{n\times n}$ is a projection, then $\varphi_t(P)$ is close to a projection for t large. More formally:

$$\mathrm{K}(A) \xrightarrow{\varphi_*} \mathrm{K}(\mathfrak{A}B) \xleftarrow{\cong} \mathrm{K}(\mathrm{C}_b([1,\infty);B)) \xrightarrow{\text{eval. at } 1} \mathrm{K}(B)$$

Category?

Definition (Guentner–Higson–Trout 2000)

Asymptotic algebra of C^*-algebra B:

$$\mathfrak{A}B := \frac{\mathrm{C}_b([1,\infty);B)}{\mathrm{C}_0([1,\infty);B)}$$

An asymptotic morphism from A to B is a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}B$.

Question

Can we make a category out of asymptotic morphisms?

We can compose $\varphi\colon A \to \mathfrak{A}B$ and $\psi\colon B \to \mathfrak{A}C$ to $A \xrightarrow{\varphi} \mathfrak{A}B \xrightarrow{\mathfrak{A}\psi} \mathfrak{A}\mathfrak{A}C$.

Asymptotic homotopies

Definition (Guentner–Higson–Trout 2000)

Two $*$-homomorphisms

$$\varphi_0, \varphi_1 \colon A \to \mathfrak{A}^n B := \mathfrak{A} \dots \mathfrak{A} B$$

are called *n-homotopic* if there is a $*$-homomorphism

$$\Phi \colon A \to \mathfrak{A}^n(\mathrm{C}([0,1]; B))$$

whose compositions with $\mathfrak{A}^n(\mathrm{ev}_i)$ is φ_i for $i = 0, 1$,
$\mathrm{ev}_t \colon \mathrm{C}([0,1]; B) \to B$ evaluation at t.

Example: 0-homotopic = homotopic

Fact: n-homotopy is an equivalence relation.

Definition (Guentner–Higson–Trout 2000)

$$[\![A, B]\!]_n := \{*\text{-homomorphisms } A \to \mathfrak{A}^n B\}/n\text{-homotopy}$$

The asymptotic category

There are canonical maps $[\![A,B]\!]_n \to [\![A,B]\!]_{n+1}$ induced by $*$-homomorphisms $\mathfrak{A}^n B \to \mathfrak{A}^{n+1} B$. Define

$$[\![A,B]\!] := [\![A,B]\!]_{\mathfrak{A}} := \varinjlim_n [\![A,B]\!]_n \,.$$

The composition products

$$[\![A,B]\!]_m \times [\![B,C]\!]_n \to [\![A,C]\!]_{m+n}\,, \quad ([\![\varphi]\!],[\![\psi]\!]) \to [\![\mathfrak{A}^m(\psi)\circ\varphi]\!]$$

pass to the direct limits.

Proposition (Guentner–Higson–Trout 2000)

The sets $[\![A,B]\!]$ are the morphism sets of a category.
If A is separable, then $[\![A,B]\!]_1 \to [\![A,B]\!]$ is a bijection.

Definition (Guentner–Higson–Trout 2000)

$\mathrm{E}(A,B) := [\![\mathrm{C}_0(0,1)\otimes A\otimes\mathfrak{K}\,,\ \mathrm{C}_0(0,1)\otimes B\otimes\mathfrak{K}]\!]$

Long exact sequences

For separable C^*-algebras A, D and $I \subset A$ ideal:

$$\cdots \to \mathrm{E}(D, A/I \otimes \mathrm{C}_0(0,1)) \xrightarrow{[\![\delta]\!]\circ} \mathrm{E}(D, I) \to \mathrm{E}(D, A) \to \mathrm{E}(D, A/I)$$

$$\mathrm{E}(A/I, D) \to \mathrm{E}(A, D) \to \mathrm{E}(I, D) \xrightarrow{\circ[\![\delta]\!]} \mathrm{E}(A/I \otimes \mathrm{C}_0(0,1), D) \to \dots$$

Choose quasicentral approximate unit $\{u_n\}_{n\in\mathbb{N}}$, i.e.

- $\forall n \in \mathbb{N}\colon u_n \in I, 0 \le u_n \le 1$
- $\forall j \in I\colon \lim_{n\to\infty} u_n j = j$
- $\forall a \in A\colon \lim_{n\to\infty} [u_n, a] = 0$

Extend it to a continuous map $u\colon [1,\infty) \to I$.

Then there is an asymptotic morphism

$$\begin{aligned} \delta\colon A/I \otimes \mathrm{C}_0(0,1) &\to \mathfrak{A}I \\ [a] \otimes f &\mapsto [t \mapsto a f(u_t)] \end{aligned}$$

which defines an element $[\![\delta]\!] \in \mathrm{E}(A/I \otimes \mathrm{C}_0(0,1), I)$.

What about non-separable C^*-algebras?

For A, I non-separable, there are in general only approximate units $\{u_m\}_{m\in\mathfrak{m}}$ index over a directed set $\mathfrak{m}$.

Extending it affine linearily to $u\colon |\Delta^{\mathfrak{m}}| \to I$, we obtain a $*$-homomorphism

$$\delta\colon A/I \otimes \mathrm{C}_0(0,1) \to \mathfrak{S}^{\mathfrak{m}} I := \frac{\mathrm{C}_b(|\Delta^{\mathfrak{m}}|; I)}{\mathrm{C}_0(|\Delta^{\mathfrak{m}}|; I)}$$

$$[a] \otimes f \mapsto [t \mapsto af(u_t)]\,.$$

What is $\mathrm{C}_0(|\Delta^{\mathfrak{m}}|; I)$?

Let $\mathfrak{m} \triangleright m := \{x \in \mathfrak{m} \mid x \geq m\}$. Then

$$\mathrm{C}_0(|\Delta^{\mathfrak{m}}|; I) := \{f \in \mathrm{C}_b(|\Delta^{\mathfrak{m}}|; I) \mid \forall \varepsilon > 0 \exists m \in \mathfrak{m}\colon \|f|_{|\Delta^{\mathfrak{m}\triangleright m}|}\| \leq \varepsilon\}\,.$$

Idea

To obtain an element $[\![\delta]\!] \in \mathrm{E}(A/I \otimes \mathrm{C}_0(0,1), I)$, we need a definition of E-theory based on the simpltotic algebras $\mathfrak{S}^{\mathfrak{m}} B$ instead of the asymptotic algebras $\mathfrak{A}^n B$.

The simpltotic morphism sets

Definition

$$[\![A, B]\!]_{\mathfrak{m}} := \{*\text{-homomorphisms } A \to \mathfrak{S}^{\mathfrak{m}} B\}/\mathfrak{m}\text{-homotopy}$$

Example

For $\mathfrak{m} = \mathbb{N} := \{1, 2, \dots\}$ consider the following two maps:

$$\iota\colon [1,\infty) \to |\Delta^{\mathbb{N}}|, \quad n + t \mapsto (1-t)[n] + t[n+1] \qquad \text{for } n \in \mathbb{N}, t \in [0,1]$$

$$\tau\colon |\Delta^{\mathbb{N}}| \to [1,\infty), \quad \sum_{n\in\mathbb{N}} \lambda_n[n] \mapsto \sum_{n\in\mathbb{N}} \lambda_n \cdot n$$

They induce bijections

$$[\![A, B]\!]_1 \underset{\iota^*}{\overset{\tau^*}{\rightleftarrows}} [\![A, B]\!]_{\mathbb{N}}$$

by composition with the $*$-homomorphisms $\mathfrak{A}B \underset{\iota^*}{\overset{\tau^*}{\rightleftarrows}} \mathfrak{S}^{\mathbb{N}} B$.

The simpltotic morphism sets

If $\alpha\colon \mathfrak{m} \to \mathfrak{n}$ is a cofinal map, then $|\alpha|\colon |\Delta^{\mathfrak{m}}| \to |\Delta^{\mathfrak{n}}|$ induces natural $*$-homomorphisms $\alpha^*\colon \mathfrak{S}^{\mathfrak{n}}B \to \mathfrak{S}^{\mathfrak{m}}B$ whose $\mathfrak{m}$-homotopy class does not depend on the choice of α.

Lemma

If a cofinal map $\mathfrak{m} \to \mathfrak{n}$ exists, then there is a canonical map

$$[\![A, B]\!]_{\mathfrak{n}} \to [\![A, B]\!]_{\mathfrak{m}}\,.$$

Definition

$$[\![A, B]\!]_{\mathfrak{S}} := \varinjlim_{\mathfrak{m}} [\![A, B]\!]_{\mathfrak{m}}$$

The direct limit exists, because $[\![A, B]\!]_{\mathfrak{S}} \cong [\![A, B]\!]_{\mathfrak{m}}$ for $\mathfrak{m}$ large enough.

Proposition

If A is separable: $\qquad [\![A, B]\!]_{\mathfrak{S}} \cong [\![A, B]\!]_{\mathfrak{A}}$

Composition in the simpltotic category

Proposition

- *There are composition maps*

$$\llbracket A, B\rrbracket_{\mathfrak{m}} \times \llbracket B, C\rrbracket_{\mathfrak{n}} \to \llbracket A, C\rrbracket_{\mathfrak{m}\sharp\mathfrak{n}},$$
$$(\llbracket\varphi\rrbracket, \llbracket\psi\rrbracket) \mapsto \llbracket\psi\rrbracket \circ \llbracket\varphi\rrbracket := \llbracket\Theta^* \circ \mathfrak{S}^{\mathfrak{m}}(\psi) \circ \varphi\rrbracket$$

  where $\mathfrak{m}\sharp\mathfrak{n} := \mathfrak{m} \times \mathfrak{n}^{\Delta^{\mathfrak{m}}}$ *and* $\Theta\colon |\Delta^{\mathfrak{m}\sharp\mathfrak{n}}| \to |\Delta^{\mathfrak{m}}| \times |\Delta^{\mathfrak{n}}|$ *is a certain continuous map inducing a natural transformation* $\mathfrak{S}^{\mathfrak{m}}\mathfrak{S}^{\mathfrak{n}}B \to \mathfrak{S}^{\mathfrak{m}\sharp\mathfrak{n}}B$.
- *These compositions pass to the direct limit, turning the latter into the morphism sets of a category.*
- *There are cannonical maps* $\llbracket A, B\rrbracket_{\mathfrak{A}} \to \llbracket A, B\rrbracket_{\mathfrak{S}}$ *which constitute a functor from the asymptotic to the simpltotic category.*

E-theory for non-separable C^*-algebras

Definition (W. '22)

$$\mathrm{E}(A,B) := \varinjlim_{H} [\![\mathrm{C}_0(0,1) \otimes A\,,\ \mathrm{C}_0(0,1) \otimes B \otimes \mathfrak{K}(H)]\!]_{\mathfrak{S}}$$

This model has all products, long exact sequences, etc.

Question

Does this model satisfy the universal characterization of E-theory?

Question

Are there interesting approximation procedures over directed sets which could yield canonical elements in the new model of E-theory?

Elements in $\mathbb{E}$-theory

Recall asymptotic morphism associated to Dirac operator over compact manifold:

Example

$$\mathrm{C}_0(\mathbb{R}) \otimes \mathrm{C}(M) \to \mathfrak{A}(\mathfrak{K}(L^2(M,S)))\,, \quad f \otimes g \mapsto [t \mapsto f(t^{-1}D)g]$$

1. If $\dim(M) = \infty$, we should rescale D with different factors in different directions.
   Example: For M infinite dimensional Euclidean space, a construction of Higson–Kasparov–Trout ('98) yields a canonical simpltotic morphism with $\mathfrak{m} := \{\text{finite dimensional subspaces of } M\}$.
2. If M is a complete Riemannian manifold, then we have

$$\mathrm{C}_0(\mathbb{R}) \otimes \mathrm{C}_b(M;\mathfrak{K}) \to \mathfrak{S}^{\mathfrak{m}}(\mathrm{C}^*(M))\,, \quad f \otimes g \mapsto [\lambda \mapsto f(D^\lambda)g]$$

   where $\mathfrak{m} := \{\lambda \colon M \to [1,\infty) \text{ smooth}\}$ and D^λ is the Dirac operator after conformally changing the metric by λ^2 (cf. W. '18).

¡Muchas gracias por su atención!