Final Exam – Math 4430 – Fall 2010

Solutions

Problem	Max. points	Points
1	40	
2	40	
3	20	
4	20	
5	20	
6	20	
7	40	
Total	200	

1. Find the general solutions of the differential equations

   a)

$$y' = y \sin t.$$

Answer: $\boxed{y = Ce^{-\cos t}}$

   b)

$$t\frac{dy}{dt} + (t+1)y = 3t^2e^{-t}, \; t > 0.$$

Answer: $\boxed{y = t^2e^{-t} + C\frac{e^{-t}}{t}}$

2. Consider the differential equation

$$y'' + y' + y\,(2 - y) = 0.$$

   a) (15 pt) Write down a first order system equivalent to this equation.

   SOLUTION:

   Set $y' = x$. Then $y'' = x'$ and $x' + x + y\,(2 - y) = 0$.

   Hence the system is

$$\begin{cases} x' = -x - 2y + y^2 \\ y' = x \end{cases}$$

Answer: $\boxed{\begin{cases} x' = -x - 2y + y^2 \\ y' = x \end{cases}}$.

   b) Find the equilibrium solutions of the system you found in the previous part.

   SOLUTION:

   We need to solve the system

$$\begin{cases} -x - 2y + y^2 = 0 \\ x = 0 \end{cases}$$

   We obtain: $x = 0$, $y^2 = 2y$, hence $y = 0$ or $y = 2$

Answer: $\boxed{(0,0) \text{ and } (0,2)}$.

c) For each of the equilibrium solutions determine if it is stable.

SOLUTION:

The matrix of partial derivatives:

$$\begin{bmatrix} -1 & -2+2y \\ 1 & 0 \end{bmatrix}$$

At $(0,0)$ we have the matrix $\begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$ with the eigenvalues

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i.$$

Since the real part of both eigenvalues is negative, this is an (asymptotically) stable equilibrium.

At $(0,1)$ we have the matrix $\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ with the eigenvalues 1 and -2. Since $1 > 0$ this equilibrium is unstable.

Answer: $\boxed{(0,0) \text{ is a stable equilibrium and } (0,2) \text{ is unstable}}$

3. Solve the following initial value problem:

$$t^2 y' = t^2 + ty + y^2, \; t > 0, \; y(1) = 1.$$

SOLUTION: $t^2 y' = t^2 + ty + y^2$, $y' = 1 + y/t + (y/t)^2$, the equation is homogeneous. $y = tv$, $v(1) = 1$, $y' = tv' + v$.

$$tv' + v = 1 + v + v^2, \; v(1) = 1$$

$$tv' = 1 + v^2$$

$$\frac{dv}{1+v^2} = \frac{dt}{t}$$

$$\tan^{-1}(v) = \ln(t) + C$$

From the initial condition $C = \pi/4$. Hence $v = \tan(\ln(t) + \pi/4)$, $y = t\tan(\ln(t) + \pi/4)$. **Answer:** $\boxed{y = t\tan(\ln(t) + \pi/4)}$

4. Solve the following initial value problem by **USING LAPLACE TRANSFORM**:

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{x} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

SOLUTION:

$$s\mathbf{X} = \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{X} + \frac{1}{s-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s+3 & -2 \\ 2 & s-2 \end{bmatrix} \mathbf{X} = \frac{1}{s-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{X} = \frac{1}{(s+3)(s-2)+4} \begin{bmatrix} s-2 & 2 \\ -2 & s+3 \end{bmatrix} \frac{1}{s-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s-1)^2 (s+2)} \begin{bmatrix} s-2 \\ -2 \end{bmatrix}$$

$$\frac{1}{(s-1)^2 (s+2)} \begin{bmatrix} s-2 \\ -2 \end{bmatrix} = \frac{1}{(s-1)^2} \begin{bmatrix} -1/3 \\ -2/3 \end{bmatrix} + \frac{1}{(s-1)} \begin{bmatrix} 4/9 \\ 2/9 \end{bmatrix} + \frac{1}{(s+2)} \begin{bmatrix} -4/9 \\ -2/9 \end{bmatrix}$$

Therefore

$$\mathbf{x}(t) = te^t \begin{bmatrix} -1/3 \\ -2/3 \end{bmatrix} + e^t \begin{bmatrix} 4/9 \\ 2/9 \end{bmatrix} + e^{-2t} \begin{bmatrix} -4/9 \\ -2/9 \end{bmatrix}$$

Answer: $\boxed{te^t \begin{bmatrix} -1/3 \\ -2/3 \end{bmatrix} + e^t \begin{bmatrix} 4/9 \\ 2/9 \end{bmatrix} + e^{-2t} \begin{bmatrix} -4/9 \\ -2/9 \end{bmatrix}}$.

5. Find the solution of the initial value problem

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{x}, \ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda = 1$ of multiplicity 2 and $\lambda = 2$.

Answer: $\boxed{\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - e^t \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) + e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$.

6. Find e^{tA} for

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Answer: $$\boxed{e^{tA} = \begin{bmatrix} (e^t + e^{-t})/2 & 0 & (e^t - e^{-t})/2 \\ 0 & 1 & 0 \\ (e^t - e^{-t})/2 & 0 & (e^t + e^{-t})/2 \end{bmatrix}}.$$

7. a) Given that $y = e^t$ is a solution of the differential equation

$$ty'' - (3t + 1)\,y' + (2t + 1)\,y = 0, \ t > 0$$

find the general solution of this equation.

SOLUTION:

$$y_1(t) = e^t$$

Rewrite:

$$y'' - \left(3 + \frac{1}{t}\right) y' + \frac{2t + 1}{t} y = 0$$

$$p(t) = -\left(3 + \frac{1}{t}\right)$$

$$\int p(t)\,dt = -\int \left(3 + \frac{1}{t}\right) dt = -3t - \ln t$$

$$\frac{e^{-\int p(t)dt}}{y_1^2} = \frac{te^{3t}}{e^{2t}} = te^t$$

$$\frac{y_2}{y_1} = \int te^t dt = te^t - e^t$$

$$y_2 = te^{2t} - e^{2t}$$

Answer: The general solution is $\boxed{y = C_1 e^t + C_2(te^{2t} - e^{2t})}$.

b) Find the general solution of the equation

$$ty'' - (3t + 1)\,y' + (2t + 1)\,y = t^2 e^t.$$

SOLUTION:

Rewrite:

$$y'' - \left(3 + \frac{1}{t}\right) y' + \frac{2t + 1}{t} y = te^t$$

$g(t) = te^t$. $y_1 = e^t$, $y_2 = te^{2t} - e^{2t}$. Also

$$W[y_1, y_2] = \begin{vmatrix} e^t & te^{2t} - e^{2t} \\ e^t & 2te^{2t} - e^{2t} \end{vmatrix} = te^{3t}$$

$u_1' = -\frac{(te^{2t} - e^{2t})te^t}{te^{3t}} = -(t-1)$. $u_1 = -\frac{1}{2}t^2 + t$.
$u_2' = \frac{e^t te^t}{te^{3t}} = e^{-t}$, $u_2 = -e^{-t}$. Therefore we have a particular solution

$$u_1y_1 + u_2y_2 = -\frac{t^2e^t}{2} + e^t$$

Answer: The general solution is $\boxed{y = -\frac{t^2e^t}{2} + e^t + C_1e^t + C_2(te^{2t} - e^{2t})}$.