① IVP

$$\dot{X} = \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A} X \qquad X(0) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

$$\chi_A(\lambda) = \begin{vmatrix} \lambda-1 & 1 & -1 \\ -1 & \lambda-1 & 1 \\ 0 & 1 & \lambda-2 \end{vmatrix} = \begin{vmatrix} 0 & 1+(\lambda-1)^2 & \lambda-2 \\ -1 & \lambda-1 & 1 \\ 0 & 1 & \lambda-2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+(\lambda-1)^2 & \lambda-2 \\ 1 & (\lambda-2) \end{vmatrix} = (\lambda-2)(\lambda-1)^2.$$

$$A - 2I = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow[R_2-R_3]{R_1-R_3} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{-R_3 \;\mid\; R_1+R_2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

nullspace: $\lambda(1\ 0\ 1) \longleftarrow$

$$A - I = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

nullspace: $\lambda(1,1,1)$.

$$(A-I)^2 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Nullspace: $\lambda(1,1,1) + \mu(0,2,1)$

Take: $v_1 = (1,0,1)$, $v_2 = (1,1,1)$, $v_3 = (0,2,1)$.

Then: $Av_1 = 2v_1 \Rightarrow w_1 = e^{At}v_1 = e^{2t}v_1 = (e^{2t}, 0, e^{2t})$

$Av_2 = v_2 \Rightarrow w_2 = e^{At}v_2 = e^{t}v_2 = (e^t, e^t, e^t)$.

$(A-I)^2 v_3 = 0 \Rightarrow w_3 = e^{At}v_3 = e^{tI} e^{(A-I)t} v_3$

$\parallel$

$(*)$.

$$\circledast = e^t\left(I + (A-I)t\right)v_3 = e^t\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&-t&t\\t&0&-t\\0&-t&t\end{bmatrix}\right)\begin{bmatrix}0\\2\\1\end{bmatrix}$$

$$= e^t\left(\begin{bmatrix}0\\2\\1\end{bmatrix} + \begin{bmatrix}-t\\-t\\-t\end{bmatrix}\right) = \begin{bmatrix}-te^t\\(2-t)e^t\\(1-t)e^t\end{bmatrix}$$

So general solution is

$$X = c_1\,(e^{2t}, 0, e^{2t}) + c_2\,(e^t, e^t, e^t) + c_3\,(-te^t, (2-t)e^t, (1-t)e^t)$$

IVC:

$$(1, 2, 2) = X(0) = c_1(1,0,1) + c_2(1,1,1) + c_3(0,2,1).$$

Equivalent to

$$\begin{bmatrix}1&1&0\\0&1&2\\1&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix}.$$

Augmented matrix triangulation

$$\left[\begin{array}{ccc:c}1&1&0&1\\0&1&2&2\\1&1&1&2\end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{ccc:c}1&1&0&1\\0&1&2&2\\0&0&1&1\end{array}\right] \xrightarrow{R_2-2R_3}$$

$$\rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\Rightarrow$ Solution $(c_1, c_2, c_3) = (1, 0, 1)$.

Thus

$$X = (e^{2t}, 0, e^{2t}) + (-te^{t}, (2-t)e^{t}, (1-t)e^{t})$$

$$\boxed{X = \left((e^{2t} - te^{t}), (2-t)e^{t}, e^{2t} + (1-t)e^{t}\right)}.$$

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(2) Find e^{tA} for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\chi_A(r) = \begin{vmatrix} r & 0 & -1 \\ 0 & r & 0 \\ -1 & 0 & r \end{vmatrix} = r \begin{vmatrix} r & -1 \\ -1 & r \end{vmatrix}$$

$$= r(r^2 - 1).$$

$$= r(r-1)(r+1).$$

Because A has three distinct eigenvalues and is a 3×3 matrix, the dimension of each of its eigenspaces is 1. So we just need <u>one</u> (nonzero) eigenvector for each eigenvalue.

<u>Eigenvalue 0</u>: The second column of A is zero.

Thus $A\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow (0,1,0)$ is eigenvector of eigenvalue zero.

<u>Eigenvalue 1</u>: $A-I=\begin{bmatrix}-1&0&1\\0&-1&0\\1&0&-1\end{bmatrix} \xrightarrow[-R_2]{R_1+R_3} \begin{bmatrix}0&0&0\\0&1&0\\1&0&-1\end{bmatrix}$

Thus $(1,0,1)$ is eigenvector (says $x_1=x_3$, $x_2=0$).

<u>Eigenvalue -1</u> $A+I=\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}$

Says $\begin{cases} x_1=-x_3\\ x_2=0\end{cases}$

$(1,0,-1)$ is eigenvector.

<u>Basis of eigenvectors</u>: $\{(0,1,0),\ (1,0,1),\ (1,0,-1)\}$.

We have:

$$(*) \quad e^{At} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{=\,V} = \boxed{\begin{matrix} 0 & e^{t} & e^{-t} \\ 1 & 0 & 0 \\ 0 & e^{t} & -e^{-t} \end{matrix}}$$

<u>Finding V^{-1}</u>

$$\left[\begin{array}{ccc:ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_3 - R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & -1/2 \end{array}\right] \xleftarrow{-\frac{1}{2} R_3} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 - R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & +1/2 \\ 0 & 0 & 1 & +1/2 & 0 & -1/2 \end{array}\right] \Rightarrow V^{-1} = \boxed{\begin{matrix} 0 & 1 & 0 \\ 1/2 & 0 & +1/2 \\ +1/2 & 0 & -1/2 \end{matrix}} \quad (**)$$

From (*) and (**), we get:

$$e^{At} = \begin{bmatrix} 0 & e^{t} & e^{-t} \\ 1 & 0 & 0 \\ 0 & e^{t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \frac{e^{t}+e^{-t}}{2}. & 0 & \frac{e^{t}-e^{-t}}{2}. \\ 0 & 1 & 0 \\ \frac{e^{t}-e^{-t}}{2} & 0 & \frac{e^{t}+e^{-t}}{2}. \end{bmatrix} = e^{At}}$$

3 (a) Transform the third order linear equation

$$y''' - y'' + 4y' - 4y = 0$$

into a system of linear differential equations of order 1. Write in matricial form.

$x_1 = y$, $x_2 = y'$, $x_3 = y''$, gives:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 4x_1 - 4x_2 + x_3. \end{cases} \iff \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1. \end{matrix}}\ X = \dot{X}$$

$X = (x_1, x_2, x_3)$.

(b) Find the general solution of the following linear system:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{bmatrix}}_{A} X = \mathring{X}$$

$$\chi_A(r) = \begin{vmatrix} r & -1 & 0 \\ 0 & r & -1 \\ -4 & 4 & r-1 \end{vmatrix} = \begin{vmatrix} r & -1 & 0 \\ 0 & r & -1 \\ -4 & 4+r(r-1) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} r & -1 \\ -4 & 4+r^2-r \end{vmatrix} = r^3 - r^2 + 4r - 4$$

$$= r^2(r-1) + 4(r-1)$$

$$= (r^2+4)(r-1) = \boxed{(r+2i)(r-2i)(r-1) = \chi_A}$$

<u>Eigenvalue 1</u>: Since exponent of $r-1$ in χ_A is 1, it suffices to find 1 eigenvector of this eigenvalue.

$$A - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix} \xrightarrow{R_3 + 4R_1} \boxed{\begin{matrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{matrix}} \xrightarrow{R_1 + R_2}$$

$$\to \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

equations are : $x_1 = x_3$, $x_2 = x_3$.
So $V_1 = (1, 1, 1)$ is eigenvector of eigenvalue 1. It gives the following solution to the SODE:

$$\boxed{W_1 = (e^t, e^t, e^t)}$$

<u>Eigenvalue $2i$</u> Again it suffices to find one eigenvector, which this time will be <u>complex</u>.

$$A - 2i\underline{I} = \begin{bmatrix} -2i & 1 & 0 \\ 0 & -2i & 1 \\ 4 & -4 & 1-2i \end{bmatrix} \xrightarrow[\frac{i}{2}R_2]{R_3 - 2iR_1} \begin{bmatrix} -2i & 1 & 0 \\ 0 & 1 & i/2 \\ 0 & -(4+2i) & 1-2i \end{bmatrix}$$

$$\Big\downarrow R_3 + (4+2i)R_2 \mid R_1 - R_2$$

$$\left(\frac{-1}{4}, -i/2, 1\right) \Longleftarrow \begin{bmatrix} -2i & 0 & -i/2 \\ 0 & 1 & i/2 \\ 0 & 0 & 0 \end{bmatrix}$$

is eigenvector, as is

$$-4\left(-\frac{1}{4}, -\frac{i}{2}, 1\right) = (1, +2i, -4) = \underline{Z}$$

$\underline{Z}$ gives the <u>complex</u> valued solution $e^{2it}\underline{Z}$.
(i.e. $\mathbb{C}^3$-valued)

To find 2 $\mathbb{R}^3$-valued solutions, compute the real and imaginary parts of $e^{2it}z$:

$$e^{2it}z = (\cos 2t + i\sin 2t)(1, 2i, -4).$$

$$= (\cos 2t, -2\sin 2t, -4\cos 2t) \leftarrow \text{Real part}$$

$$+ i\,\boxed{(\sin 2t, 2\cos 2t, -4\sin 2t)} \leftarrow \text{Imaginary part.}$$

thus we get 2 solutions, namely:

$$W_2 = (\cos 2t, -2\sin 2t, -4\cos 2t)$$

$$W_3 = (\sin 2t, 2\cos 2t, -4\sin 2t).$$

General Solution

$$\boxed{x = c_1(e^t, e^t, e^t) + c_2(\cos 2t, -2\sin 2t, -4\cos 2t) + c_3(\sin 2t, 2\cos 2t, -4\sin 2t).}$$

4) A 3×3 SODE $AX=\mathring{X}$ has general solution

$$(*)\quad c_1(e^t,0,0)+c_2(e^{2t},e^{2t},0)+c_3(te^{2t},te^{2t},e^{2t}).$$

(a) Find e^{At}: From $(*)$ we get that

$$W(t)=\begin{bmatrix} e^t & e^{2t} & te^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

satisfies

$$AW(t)=\mathring{W}(t). \quad \circledcirc$$

Hence, by what we have seen in class,

$$W(t)=e^{At}\,W(0) \quad (**).$$

Put $V=W(0)$. Then

$$V=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is invertible (its columns are l.i.),

(e.g. because it is $W(0)$ and $(*)$ is a general solution)

Thus from (**) we get

$$(***) \quad e^{At} = W(t)V^{-1}$$

Compute V^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \quad V^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Plugging this into (***) we get

$$e^{At} = \begin{bmatrix} e^{t} & e^{2t} & te^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} e^{t} & e^{2t}-e^{t} & te^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}}$$

(b) Find A: From identity ⓪ of part (a), we get

$$A V = A W(0) = \mathring{W}(0).$$

$$\Rightarrow A = \mathring{W}(0) V^{-1} \quad (⓪⓪)$$

Where V^{-1} was computed in part (a):

$$V^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now

$$\mathring{W}(t) = \begin{bmatrix} e^t & 2e^{2t} & (1+2t)e^{2t} \\ 0 & 2e^{2t} & (1+2t)e^{2t} \\ 0 & 0 & 2e^{2t} \end{bmatrix}$$

So $$\mathring{W}(0) = \boxed{\begin{matrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}}$$

Plugging everything into (⓪⓪), we get

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}}$$