A construction of real numbers in the category of categories

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Abstract. We present a construction of the poset of real numbers as an object in the theory of the category of categories. We follow an axiomatization derived from the work of McLarty and give the object in question the algebraic structure of a complete ordered field.

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1. Introduction

In his pioneering work [1], Lawvere proposed to base the foundations of mathematics upon a first order theory of the category of categories. Through subsequent proposals such as [5] and [3] some technical difficulties (see [4]) have been overcome and the ideas refined, setting a categorical framework within which most of usual mathematics, including elementary set theory, could be developed. Such a theory was strong enough to formulate the concept of natural numbers and to develop from there real numbers and calculus. Usual constructions of classical real numbers in categorical set theory involve Cauchy sequences and Dedekind cuts; these can be formulated in any topos and generally give rise to distinct real number objects (for an account of these constructions see, for instance, [6] and [7]). Another construction is presented in [8], also formulated for an arbitrary topos and that reduces to the classical construction in the topos of sets. The aim of this article is to propose an alternate construction for the classical case, mimicking Dedekind cuts but working now in the setting of the category of categories. The construction makes no use of sets and provides thus a categorical framework, different from usual set/topoi-theoretic settings, in which calculus can be developed.

2. The axioms

We will work in a slightly stronger variant of the axiomatisation proposed by Mclarty in [3], adding to those axioms the existence of a natural number object (which amounts to the set theoretical axiom of infinity). These axioms are:

CC_0: The Eilenberg-MacLane axioms for a category, presented either as a two-sorted first order theory with objects and arrows as variables or as a one-sorted theory with only arrows as variables (see [1]).

CC_1: The axioms of finite roots as well as the property of cartesian closedness, that is, the existence of non isomorphic initial and terminal objects, $\mathbf{0}$ and $\mathbf{1}$, products, coproducts, equalizers, coequalizers and exponentials.

CC_2: The characterization of the category $\mathbf{2}$. Two different functors $0 : \mathbf{1} \to \mathbf{2}$ and $1 : \mathbf{1} \to \mathbf{2}$ are postulated, as well as exactly three different endofunctors $\mathbf{2} \to \mathbf{2}$ ($Id_{\mathbf{2}}$, $0 \circ !_{\mathbf{2}}$ and $1 \circ !_{\mathbf{2}}$, where $!_{\mathbf{2}}$ is the unique functor $\mathbf{2} \to \mathbf{1}$), and the property of arrow extensionality (two functors from $\mathbf{A}$ to $\mathbf{B}$ are equal if and only if their compositions with every functor $\mathbf{2} \to \mathbf{A}$ are equal), which means that $\mathbf{2}$ is a generator.

CC_3: The functor $\mathbf{1} + \mathbf{1} \to \mathbf{2}$ obtained through the cocone $0 : \mathbf{1} \to \mathbf{2}$ and $1 : \mathbf{1} \to \mathbf{2}$ is not an epimorphism. This is equivalent to stating that the two arrows from $\mathbf{2}$ to the pushout $\mathbf{E}$ of $\mathbf{1} + \mathbf{1} \to \mathbf{2}$ along itself are different. In addition we shall postulate that these are the only non-identity arrows of $\mathbf{E}$ (although this latter assumption could be actually provable from the other axioms).

CC_4: The pushout $\mathbf{3}$ of $0 : \mathbf{1} \to \mathbf{2}$ and $1 : \mathbf{1} \to \mathbf{2}$ has exactly three arrows; two of these are α, β satisfying $\alpha \circ 1 = \beta \circ 0$ (commutativity of the pushout) and a third $\gamma : \mathbf{2} \to \mathbf{3}$ satisfies $\gamma \circ 0 = \alpha \circ 0$ and $\gamma \circ 1 = \beta \circ 1$.

CC_5: Functorial comprehension. That is, any first order formula R relating arrows from a category $\mathbf{A}$ (i.e., functors from $\mathbf{2}$ to $\mathbf{A}$) to those of category $\mathbf{B}$ and satisfying functorial relations defines an actual functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ such that $R(f, g) \iff \mathbf{F}f = g$.

CC_6: There exists a category with a non identity isomorphism arrow.

CC_7: There is an unary predicate ^{op} that preserves identity functors, domain, codomain and composites of functors, such that $(\mathbf{A}^{op})^{op} = \mathbf{A}$ and $(\mathbf{F}^{op})^{op} = F$ for every category $\mathbf{A}$ and every functor $\mathbf{F}$.

CC_8: $0^{op} = 1 : \mathbf{1} \to \mathbf{2}$ (this prevents the operator ^{op} from being the identity).

CC_9: There exists a natural number object $\mathbf{N}$. That is, there are functors $0 : \mathbf{1} \to \mathbf{N}$ and $\mathbf{s} : \mathbf{N} \to \mathbf{N}$ such that for every $\mathbf{M}$ and every pair of functors $a : \mathbf{1} \to \mathbf{M}$ and $\mathbf{t} : \mathbf{M} \to \mathbf{M}$ there is a unique $\mathbf{f} : \mathbf{N} \to \mathbf{M}$ such that $\mathbf{f} \circ 0 = a$ and $\mathbf{fs} = \mathbf{tf}$.

The axioms above are all provably independent of each other (with the possible exception of the extra assumption in CC_3) and serve as an alternative to Lawvere's theory.

3. The real number object

The existence of a natural number object allows to develop the integers $\mathbf{Z}$ and the rational numbers $\mathbf{Q_d}$ through a series of applications of the axioms above, as well as to define the usual operations of sum and product making $\mathbf{Q_d}$ have the properties of a field. The construction is the same as the one carried out in any topos (see [7]) and proceeds as follows: consider the functor $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ and take its pullback $\mathbf{P}$ along itself, which gives two functors $\mathbf{A}, \mathbf{B} : \mathbf{P} \to \mathbf{N} \times \mathbf{N}$. Then $\mathbf{Z}$ can be defined as the codomain of the coequalizer T of the two functors $(\pi_1\mathbf{A}, \pi_2\mathbf{B}), (\pi_2\mathbf{A}, \pi_1\mathbf{B}) : \mathbf{P} \to \mathbf{N}\times\mathbf{N}$. This construction allows to extend the operations on $\mathbf{N}$ to $\mathbf{Z}$ through the usual definitions by noting that $\mathbf{N}$ can be regarded as a subobject of $\mathbf{Z}$ through the monomorphism $T\circ(Id_{\mathbf{N}}\times 0) : \mathbf{N}\times\mathbf{1} \to \mathbf{Z}$; also, a total ordering can be defined making $\mathbf{Z}$ an internal poset. To construct $\mathbf{Q_d}$ we apply the same method as above, considering the functor $\times \circ (1 \times s) : \mathbf{Z} \times \mathbf{N} \to \mathbf{Z}$, taking its pullback $\mathbf{P}'$ along itself (which provides functors $\mathbf{A}', \mathbf{B}' : \mathbf{P}' \to \mathbf{Z} \times \mathbf{N}$) and finally taking the coequalizer of the two functors $(\pi_1\mathbf{A}', \pi_2\mathbf{B}'), (\pi_2\mathbf{A}', \pi_1\mathbf{B}') : \mathbf{P}' \to \mathbf{Z} \times \mathbf{N}$. Again, operations on $\mathbf{Z}$ can be extended to $\mathbf{Q_d}$ conveniently and a total ordering can be defined making $\mathbf{Q_d}$ an archimedean field.

The construction above gives a discrete category $\mathbf{Q_d}$ (a category $\mathbf{A}$ is said to be discrete if every functor $\mathbf{F} : \mathbf{2} \to \mathbf{A}$ factors through $\mathbf{1}$). For the construction of the real numbers we will nevertheless need to work with the archimedean poset $\mathbf{Q}$ of rational numbers, which intuitively should be the category whose objects are those of $\mathbf{Q_d}$ and whose arrows correspond to the ordering relation in $\mathbf{Q_d} \times \mathbf{Q_d}$. To see that such a category is definable within our axiom system, note that it is possible to define the corresponding internal small category in the sense of [3], which is a pair of categories $\mathbf{Ar}, \mathbf{Ob}$ with functors $\mathbf{Comp} : \mathbf{Ar} \times_{\mathbf{Ob}} \mathbf{Ar} \to \mathbf{Ob}$, $\Delta_0, \Delta_1 : \mathbf{Ar} \to \mathbf{Ob}$ and $\mathbf{Id} : \mathbf{Ob} \to \mathbf{Ar}$ satisfying the usual relations of the poset $\mathbf{Q}$. It is a consequence of axioms CC_3 and CC_6 that whenever an internal small category is such that every internal endomorphism is an (internal) identity arrow and each internal isomorphism class of objects and internal classes of parallel arrows admit a choice definable within our axioms, there exists an actual category corresponding to the internal one (see Theorem 27 of [3], pp. 1252). In our case there is evidently one object and one arrow in each nonempty class, which allows us to deduce the existence of $\mathbf{Q}$.

We can now state the following:

Definition 3.1. The poset of real numbers $\mathbf{R}$ is the full subcategory of non constant cocontinuous functors in $\mathbf{2}^{\mathbf{Q}^{op}}$.

Of course, it needs to be checked that such a category is definable within our system. This can be done through axiom CC_5 and CC_6, noting that axiom CC_6 implies that there is a full subcategory classifier $\mathbf{Cl}$ (see [3]) and then the definition of such a full subcategory depends on the definition of a functor $S : \mathbf{2}^{\mathbf{Q}^{op}} \to \mathbf{Cl}$, which can be easily proven to exist through a convenient application of CC_5.

Cocontinuous functors in $\mathbf{A}^{\mathbf{B}}$ can be defined, following [2], as those functors such that for every $\mathbf{D}$ the following diagram:

$$\begin{tikzcd} \mathbf{A}^{\mathbf{D}} \arrow[r, "F^{\mathbf{D}}"] \arrow[d, "\lim_{\to}"'] & \mathbf{B}^{\mathbf{D}} \arrow[d, "\lim_{\to}"'] \\ \mathbf{A} \arrow[r, "F"] & \mathbf{B} \end{tikzcd}$$

commutes (here the functor $\lim_{\to}$ is the left adjoint of the diagonal functor $\Delta : \mathbf{A} \to \mathbf{A}^{\mathbf{D}}$).

Slice categories and representable functors in $\mathbf{2}^{\mathbf{A}^{op}}$ (where $\mathbf{A}$ is a poset) can be defined in our theory by applying CC_5 together with axioms CC_7 and CC_8, and the usual calculations yield Yoneda's lemma and its common corollaries. In particular, there is a monic $\mathbf{I} : \mathbf{A} \to \mathbf{2}^{\mathbf{A}^{op}}$ such that for every functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$, where $\mathbf{B}$ is cocomplete, there exists a unique functor $\overline{\mathbf{F}} : \mathbf{2}^{\mathbf{A}^{op}} \to \mathbf{B}$ such that $\overline{\mathbf{F}}\mathbf{I} = \mathbf{F}$. We prove now the following:

Lemma 3.2. *The functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ between posets is left adjoint of some functor $\mathbf{G} : \mathbf{B} \to \mathbf{A}$ if and only if the cocontinuous extension $\overline{\mathbf{F}} : \mathbf{2}^{\mathbf{A}^{op}} \to \mathbf{2}^{\mathbf{B}^{op}}$ is the transpose $\mathbf{G}^*$ of some $\mathbf{G} : \mathbf{B} \to \mathbf{A}$.*

Proof. We have $\mathbf{F} \dashv \mathbf{G}$ if and only if $[b, \mathbf{F}(a)] = [\mathbf{G}(b), a]$ with naturality conditions on a, b. This means that $\overline{\mathbf{F}}([-, a])(b) = \mathbf{G}^*([-, a])(b)$ and therefore $\overline{\mathbf{F}}$ and $\mathbf{G}^*$ are equal on the full subcategory of representable functors. Since they are both cocontinuous, this is equivalent to stating that they are equal on all $\mathbf{2}^{\mathbf{A}^{op}}$. This completes the proof. □

Lemma 3.2 allows to extend the operations of $\mathbf{Q}$ to $\mathbf{R}$ by considering the embedding $\mathbf{Q} \to \mathbf{2}^{\mathbf{Q}^{op}}$. Note, however, that representable functors are not cocontinuous and therefore they do not provide the standard copy of $\mathbf{Q}$ inside $\mathbf{R}$. To find the right copy we first define the usual operations on $\mathbf{R}$. Given a real number a and a rational number r, consider the functor $\mathbf{S}_r : \mathbf{Q} \to \mathbf{Q}$ defined on objects by summing r and consider the cocontinuous extension $\overline{\mathbf{S}_r} : \mathbf{2}^{\mathbf{Q}^{op}} \to \mathbf{2}^{\mathbf{Q}^{op}}$. Since $\mathbf{S}_r$ has an inverse $\mathbf{H} = \mathbf{S}_{-r}$, it is also a right adjoint of $\mathbf{S}_r$ and, by lemma 3.2, $\overline{\mathbf{S}_r} = \mathbf{H}^*$. But $\mathbf{H}^*$ applies cocontinuous functors in $\mathbf{2}^{\mathbf{Q}^{op}}$ into cocontinuous functors in $\mathbf{2}^{\mathbf{Q}^{op}}$ (since for a cocontinuous F, $\mathbf{H}^*(F)(b) = F(\mathbf{H}(b))$ and $F \circ \mathbf{H}$ is cocontinuous since both F and $\mathbf{H}$ are), and therefore applies real numbers into real numbers. This defines the functor $\overline{\mathbf{S}_r} : \mathbf{R} \to \mathbf{R}$ which intuitively means "to sum the rational number r".

$$
\begin{tikzcd}
 & & \mathbf{R} \arrow[dl, hook'] \arrow[dd, "(\overline{\mathbf{S}_r})|_{\mathbf{R}}"] \\
\mathbf{Q} \arrow[r, tail] \arrow[dd, "\mathbf{S}_r"] & 2^{\mathbf{Q}^{op}} \arrow[dd, "\overline{\mathbf{S}_r}"] & \\
 & & \mathbf{R} \arrow[dl, hook'] \\
\mathbf{Q} \arrow[r, tail] & 2^{\mathbf{Q}^{op}} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathbf{R} \arrow[dl, hook'] \arrow[ddl, "(\overline{\mathbf{S}_a})|_{\mathbf{R}}"] \\
\mathbf{Q} \arrow[r, tail] \arrow[dr, "\mathbf{S}_a"'] & 2^{\mathbf{Q}^{op}} \arrow[d, "\overline{\mathbf{S}_a}"'] & \\
 & \mathbf{R} &
\end{tikzcd}
$$

If we define now the functor $\mathbf{S}_a : \mathbf{Q} \to \mathbf{R}$ by applying the object r into $a + r = \overline{\mathbf{S}_r}(a)$, it is now possible to define $b + a$ as $\overline{\mathbf{S}_a}(b)$ (note that we are making use of CC_5 here), where $\overline{\mathbf{S}_a}$ is the unique cocontinuous extension, that exists because $\mathbf{R}$ is cocomplete.

Remark 3.3. Cocompleteness of $\mathbf{R}$, which follows in turn from cocompleteness of $\mathbf{2}$ and the fact that a colimit of cocontinuous functors is cocontinuous, expresses as a first order sentence that every bounded subcategory of real numbers has a supremum (which is its colimit).

The arrows of $\mathbf{R}$ induce a total ordering that satisfies the trichotomy law. To see this, note that $a \leq b$ if the canonical expression of a as colimit of representable functors is a subcategory of that of b, that is, if for every r such that there is an arrow $[-, r] \to a$, there is also an arrow $[-, r] \to b$. Now, if $a \not\leq b$, there is some s such that there is an arrow $[-, s] \to a$ but there is no arrow $[-, s] \to b$, which implies that whenever there is an arrow $[-, t] \to b$, then there is also an arrow $[-, t] \to [-, s]$. Hence there exists an arrow $b \to [-, s]$ and trichotomy holds.

There is also a copy of $\mathbf{Q}$ inside $\mathbf{R}$, explicited as follows: consider a functor $\mathbf{i} : \mathbf{Q} \to \mathbf{R}$ defined on objects by $\mathbf{i}(r) = \lim_{(s<r)}[-, s]$ (we invoke again CC_5 and use the full subcategory classifier to define $(s < r)$ as the full subcategory of those rationals s in $\mathbf{Q}$ such that $s < r$). It can be seen to be well defined, monic and to preserve the field operations; for example, we have:

$$\mathbf{i}(r) + \mathbf{i}(r') = \lim_{(s<r)} [-, s] + \lim_{(s'<r')} [-, s'] = \lim_{(s<r)} \left([-, s] + \lim_{(s'<r')} [-, s'] \right)$$

$$= \lim_{(s<r)} \left(\lim_{(s'<r')} [-, s + s'] \right) = \lim_{(s+s'<r+r')} [-, s + s'] = \mathbf{i}(r + r')$$

and similarly with the product.

An argument analogous to that of the sum can be used to define the product of real numbers, although in this case we define first the functor

$\mathbf{P}_r : \mathbf{Q} \to \mathbf{Q}$ for positive r, and then the functor $\mathbf{P}_a : \mathbf{Q}_{\geq 0} \to \mathbf{R}$ for positive a is extended to all $\mathbf{Q}$ through the usual definitions. This allows to define ba for positive a and again the general definitions are extended conveniently.

Uniqueness of functors extensions to $\mathbf{2}^{\mathbf{Q}^{op}}$ can be used to prove the associative, commutative and distributive laws. For example, for a fixed a the functors $a+x$ and $x+a$ are equal for rational x, and thus their cocontinuous extensions are equal, that is, $a+b=b+a$ for real numbers a, b.

It is now clear that $a+0 = a1 = a$. For example, if we define $([-,s] \to a)$ (through the use of the full subcategory classifier) as the full subcategory of representable functors $[-,s]$ in $\mathbf{R}$ such that there is an arrow $-,s] \to a$, we have the following calculation:

$$a+0 = \lim_{([-,s]\to a)} [-,s] + \lim_{(r<0)} [-,r] = \lim_{(r<0)} \left([-,r] + \lim_{([-,s]\to a)} [-,s] \right)$$

$$= \lim_{(r<0)} \left(\lim_{([-,s]\to a)} [-,r+s] \right) = \lim_{([-,r+s]\to a)} [-,r+s] = a$$

where the last equality is justified by noting that given an arrow $[-,d] \to a$ there is some rational s such that there are non identity arrows $[-,d] \to [-,s] \to a$, and defining $r = d - s$ yields $[-,d] = [-,r+s]$. This implies that the double limit over the categories $([-,s] \to a)$ and $(r < 0)$ can be expressed as a single limit over the category $([-,r+s] \to a)$, from which the equality follows.

Define now the additive inverse of a as the colimit:

$$-a = \lim_{(a\to[-,r])^{op}} [-,-r] = \lim_{(a\to[-,r])^{op}} \mathbf{i}(-r)$$

where $(a \to [-,r])$ is the full subcategory (we use again the full subcategory classifier) of those representable functors $[-,r]$ such that there is an arrow $a \to [-,r]$ (note that a itself cannot be representable). Since each $\mathbf{i}(-r)$ is cocontinuous, it follows that $-a$ is cocontinuous as well, and since it is non constant, it is therefore a real number. To prove that $a+(-a) = 0$ note that, by definition, we have:

$$a + (-a) = \lim_{([-,s]\to a)} [-,s] + \lim_{(a\to[-,-r])^{op}} [-,r]$$

$$= \lim_{(a\to[-,-r])^{op}} \left([-,r] + \lim_{([-,s]\to a)} [-,s] \right)$$

$$= \lim_{(a\to[-,-r])^{op}} \left(\lim_{([-,s]\to a)} [-,r+s] \right) = \lim_{(r+s<0)} [-,r+s] = 0$$

Here the last equality can be justified through the following argument: for every pair (r,s) such that there are arrows $a \to [-,-r]$, and $[-,s] \to a$ there is a non identity arrow $[-,s] \to [-,-r]$, and therefore a non identity arrow

$[-, r+s] \to [-, 0]$, which proves $r+s < 0$. Conversely, let d be a rational number, $d < 0$; then $a + d \leq a$ (i.e., there is an arrow $\overline{\mathbf{S}_d}(a) \to a$). Moreover, it can be seen using the archimedean property for $\mathbf{Q}$ that this is not the identity arrow; for if it were, then for every rational r such that there is an arrow $[-, r] \to a$, there should be an arrow $[-, r-d] \to a$ (because $\mathbf{R}$ is totally ordered), and inductively, an arrow $[-, r - nd] \to a$, which contradicts archimedeanity since a is not the constant functor 1. It follows that there must be some rational s such that there is an arrow $[-, s] \to a$ but there is no arrow $[-, s] \to a + d$. If we call $r = -s + d$, (and since $\mathbf{R}$ is totally ordered), it follows that $a < [-, -r]$ (i.e., there is an arrow $a \to [-, -r]$) and $r + s = d < 0$.

It can also be verified that a is positive if and only if $-a$ is negative.

The multiplicative inverse can be defined first for positive a as:

$$a^{-1} = \lim_{(a \to [-,r])^{op}} [-, 1/r] = \lim_{(a \to [-,r])^{op}} \mathbf{i}(1/r)$$

which can be seen to be a real number as before. We have now:

$$\begin{aligned} aa^{-1} &= \lim_{([-,s] \to a)} [-, s] \lim_{(a \to [-,1/r])^{op}} [-, r] \\ &= \lim_{(a \to [-,1/r])^{op}} \left([-, r] \lim_{([-,s] \to a)} [-, s] \right) \\ &= \lim_{(a \to [-,1/r])^{op}} \left(\lim_{([-,s] \to a)} [-, rs] \right) = \lim_{(rs<1)} [-, rs] = 1 \end{aligned}$$

As before, we can justify the last equality through this argument: for every pair (r, s) such that there are arrows $a \to [-, 1/r]$, and $[-, s] \to a$ there is a non identity arrow $[-, s] \to [-, 1/r]$, and therefore a non identity arrow $[-, rs] \to [-, 1]$, which proves $rs < 1$. Conversely, let d be a rational number, $d < 1$, and consider two cases:

a) $d > 0$: Since $ad < a$ (again using archimedeanity), it follows that there must be some rational s such that there is an arrow $[-, s] \to a$ but there is no arrow $[-, s] \to ad$. If we call $r = d/s$, then $a < [-, 1/r]$, $s < a$ and $rs = d < 1$.

b) $d \leq 0$: Since a is not the constant functor 1, there exists a rational q such that $a < q$. Then the choice $r = 1/q$, $s = d/r$ yields $a < [-, 1/r]$, $[-, s] < a$ and $rs = d$.

A similar argument can be given when $a < 0$, using now that a is not the constant functor 0.

Finally, it can be verified that the total ordering on $\mathbf{R}$ is compatible with the field operations. To check it, note that we have $a > 0$ if and only if there is some positive rational r such that there is an arrow $[-, r] \to a$, from which we deduce that positive real numbers are closed under addition and multiplication, as well as compatibility with the sum and product.

This establishes the existence of a category satisfying the properties of a complete ordered field, and allows to develop calculus in a purely categorical base.

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