INTERPOLATION ERROR ESTIMATES FOR EDGE ELEMENTS ON ANISOTROPIC MESHES

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Abstract. The classical error analysis for the Nédélec edge interpolation requires the so-called regularity assumption on the elements. However, in [18], optimal error estimates were obtained for the lowest order case, under the weaker hypothesis of the maximum angle condition. This assumption allows for anisotropic meshes that become useful, for example, for the approximation of solutions with edge singularities.

In this paper, we prove optimal error estimates for the edge interpolation of any order under the maximum angle condition. We also obtain sharp stability results for that interpolation on appropriate families of elements.

Key words. mixed finite elements, edge elements, anisotropic finite elements.

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1. Introduction

The first family of Nédélec's edge elements, introduced in [17], is a conforming family of finite elements in $H(\mathrm{curl})$. After publication of [17] these finite elements become broadly used in the approximation of elliptic partial differential equations in mixed form, such as Maxwell equations, elasticity equations and their associated eigenproblems [17, 14, 11, 9].

The error estimates for the numerical solutions obtained using these elements depend on the approximation properties of the associated edge interpolation operator. The error analysis for this operator developed in [17] is based on the so-called regularity assumption [10], and therefore the constants involved in the estimates depend on the ratio between the outer and inner diameters. In this way, narrow or anisotropic elements are excluded from that analysis.

Anisotropic meshes appear naturally in applications where the solution presents edge singularities or boundary layers. As described in [8], such a situation is present when considering the time-harmonic Maxwell equations in a Lipschitz polyhedron with non-convex edges or corners. In this case, the poor regularity of the solution causes some obstructions to optimal convergent approximations. One strategy to overcome this difficulty is to use non-quasiuniform finite element meshes that are more refined near some edges or corners. The possibility of using anisotropic elements can make the design of such meshes easier, reduce the number of elements and take advantage of the best regularity properties of the solution: In fact, in

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many problems, the solutions have more regularity in the direction of the edges than transversally to them.

The literature on anisotropic interpolation is nowadays rich. For the standard Lagrange interpolation it is known, since the pioneering works [6, 15] and many generalizations of them (see [5] and its references), that the regularity assumption can be relaxed to a *maximum angle condition* in many cases. In [2], anisotropic estimates are obtained for a generalized Lagrange interpolation with arbitrarily high polynomial degree. These estimates hold uniformly for elements satisfying a maximum angle condition.

We say that a tetrahedron satisfies the maximum angle condition with constant $\bar{\psi} < \pi$ (or shortly MAC($\bar{\psi}$)) if the angles between the faces and inside faces are less than $\bar{\psi}$. For a vector-valued function $\mathbf{u}$ regular enough, denote by $\Pi_l \mathbf{u}$ the edge interpolation of $\mathbf{u}$ of order l on the tetrahedron K (see section 2 for definitions). Let us briefly describe the kind of estimates for Π_l in which we are interested. In Corollary 6.2 (we refer to section 6 for a complete statement) we prove that there exists a constant C depending only on $\bar{\psi}$ and l such that if K is a tetrahedron satisfying a maximum angle condition with constant $\bar{\psi}$ and if $1 \le m \le l$, for all $\mathbf{u} \in W^{m+1,p}(K)$ (see the restrictions on the values of p in the statement of the Corollary) we have

$$\|\mathbf{u} - \Pi_l \mathbf{u}\|_{L^p(K)} \le C\, h^{m+1} \left\| D^{m+1}\mathbf{u} \right\|_{L^p(K)}, \tag{1}$$

with h the diameter of K. We say that the estimate is uniform for elements satisfying a maximum angle condition, because the constant C does not blows up if the maximum angle of the element remains bounded above away from π. We remark that the aspect ratio of the element may degenerate while the maximum angle remains controlled.

Our estimates are also of anisotropic type. Indeed, in Theorem 6.1 we prove that if an element K satisfies MAC($\bar{\psi}$), then it is possible to choose three edges of K, ℓ_1, ℓ_2, and ℓ_3, with lengths h_1, h_2, and h_3, such that if $1 \le m \le l$ and $\mathbf{u} \in W^{m+1,p}(K)$ (see the restrictions on the values of p in the statement of the Theorem) we have the estimate

$$\|\mathbf{u} - \Pi_l \mathbf{u}\|_{L^p(K)} \le C \left\{ \sum_{i+j+k=m+1} h_1^i h_2^j h_3^k \left\| \frac{\partial^{m+1}\mathbf{u}}{\partial\xi_1^i \partial\xi_2^j \partial\xi_3^k} \right\|_{L^p(K)} + h^{m+1} \|D^m \mathbf{curl}\,\mathbf{u}\|_{L^p(K)} \right\} \tag{2}$$

where $\xi_i = \ell_i / \|\ell_i\|$, h is the diameter of K, and where the constant C depends again only on the maximum angle of the element K and on l (we refer again to section 6 for a complete statement of the result). It is important to note that the matrix made up of ξ_i, $i = 1, 2, 3$, as columns, as well as its inverse, have norms bounded only in terms of $\bar{\psi}$, so, the directions are "uniformly" independent. This estimate is not affected by the relative order of the lengths h_1, h_2 and h_3, allowing for elements that are arbitrarily narrow in some directions. Note also that in front of each derivative in the right hand side we have the lengths in the directions corresponding to that derivative. Then, this is an appropriate estimate when approximating anisotropic solutions, that is, solutions with different behaviors along different directions. We observe that we have the diameter h only in front $\mathbf{curl}\,\mathbf{u}$, but in applications, $\mathbf{curl}\,\mathbf{u}$ may be better suited than the solution $\mathbf{u}$ itself.

Our work generalizes a result of [18], where the author proves anisotropic error estimates for the edge interpolation of lowest order under the maximum angle condition. Related results can be founded also in [8], but there the authors consider elements satisfying a stronger condition on the elements (that, as showed in [3], is equivalent to the *regular vertex property* of [1], which we describe below). Moreover their estimates (for tetrahedral meshes) are uniform and anisotropic, in the above sense, only for the lowest order of the interpolation and for functions having constant $\mathbf{curl}$.

We remark that our technique differs significatively from the methods followed in [18] and [8]: firstly, we obtain anisotropic stability estimates for the edge interpolation operator. Then, we combine them with known polynomial approximation results [12], in a classical way, obtaining the desired interpolation error estimates. We need appropriate estimates for Π_l in reference elements in order to obtain, through standard scaling arguments, stability estimates with constants that do not degenerate for narrow elements. In particular, as can be easily checked, by rescaling the inequality $\|\Pi_l \mathbf{u}\|_{L^p(\widehat{K})} \leq C\|\mathbf{u}\|_{W^{1,p}(\widehat{K})}$ $(p > 2)$ for a reference element $\widehat{K}$, we obtain estimates with constants that go to infinity when the element becomes narrower (see section 3).

Let us finally mention, only to make a comparison with results for related operators, that in [3], uniform error estimates under the maximum angle condition for the Raviart-Thomas interpolation are obtained, but such estimates are anisotropic only for elements satisfying the regular vertex property. We say that an element satisfies a regular vertex property with constant $\bar{c}$, if it has a vertex, such that the matrix $M \in \mathbb{R}^{3\times 3}$ that has as columns the unitary vectors with the directions of the edges sharing that vertex, verifies $|\det M| > \bar{c}$. This is a stronger property than the maximum angle condition. In fact, an example in [3] shows that uniform anisotropic error estimates for the Raviart-Thomas interpolation can not hold uniformly under a maximum angle condition.

The plan of the paper is as follows: in the next section we introduce the edge interpolation and some basic facts that we use later. In section 3 we collect analytical aspects of the maximum angle condition. In sections 4 and 5 we obtain sharp stability results for the edge interpolation on suitable families of elements, which allow us to prove, in section 6, the main theorem concerning the anisotropic interpolation error estimates.

2. Preliminaries

Throughout the paper, we use the standard notation for Lebesgue and Sobolev spaces, $L^p(D)$ and $W^{k,p}(D)$, on a domain $D \subset \mathbb{R}^d$ (d=1,2,3), for their norms $\|\cdot\|_{L^p(D)}$ and $\|\cdot\|_{W^{k,p}(D)}$, and for the seminorm $|\cdot|_{W^{k,p}(D)}$. We will use, without explicit mention, the estimates for the traces of functions in $W^{1,p}(D)$ on faces (for arbitrary $p \geq 1$) or edges (for $p > 2$) when $D \subset \mathbb{R}^3$ is a Lipschitz domain.

Bold characters, such as $\mathbf{u}, \mathbf{v}$, denote vector-valued functions in $\mathbb{R}^3$, with components u_i, v_i, $i = 1, 2, 3$, and $\mathbf{x} = (x_1, x_2, x_3)$ denotes the variable in $\mathbb{R}^3$. We will use the standard operator $\mathbf{curl}\,(\cdot) = \nabla \times \cdot$ for vector-valued functions. When necessary, we will denote functions defined in elements $\widetilde{K}$ or $\widehat{K}$, by $\widetilde{\mathbf{u}}$ or $\widehat{\mathbf{u}}$, the variables in those elements by $\widetilde{\mathbf{x}}$ or $\widehat{\mathbf{x}}$ and the differential operators with respect to these variables by, for instance, $\widetilde{\mathbf{curl}}$ or $\widehat{\mathbf{curl}}$. For the norm of vector-valued functions in $[L^p]^3$ or $[W^{k,p}]^3$ we use the same notation, $\|\cdot\|_{L^p(D)}$ and $\|\cdot\|_{W^{k,p}(D)}$, respectively, as for scalar functions.

The space of polynomials of degree less than or equal k on a domain $D \subset \mathbb{R}^d$ is denoted by $P_k(D)$ (the explicit dependence on D indicates, in particular, the number of variables of the polynomials). By $\widetilde{P}_k$ we denote the space of homogeneous polynomials of degree k.

Let $l \geq 1$ be a natural number, and let $K \subset \mathbb{R}^3$ be a tetrahedron. Now, we introduce the first Nédélec family of edge elements [17] on K, $\mathcal{N}_l(K)$, of degree l. It is the subspace of $[P_l(K)]^3$ given by (see for example [14])

$$\mathcal{N}_l(K) = [P_{l-1}(K)]^3 \oplus S_l(K)$$

where

$$S_l(K) = \left\{ \mathbf{p} \in [\widetilde{P}_l(K)]^3 : \mathbf{p}(\mathbf{x}) \cdot \mathbf{x} \equiv 0 \right\}.$$

Remark 2.1. *It is not difficult to check that*

$$\mathcal{N}_l(K) = [P_{l-1}(K)]^3 \oplus [\widetilde{P}_{l-1}(K)]^3 \times \mathbf{x}.$$

If we set $Q_l(K) = [\widetilde{P}_{l-1}(K)]^3 \times \mathbf{x}$ *then we have*

$$S_l(K) = Q_l(K). \tag{3}$$

In fact, consider the maps $\Phi : [\widetilde{P}_l(K)]^3 \to \widetilde{P}_{l+1}(K)$, *with* $\Phi(\mathbf{p}) = \mathbf{p} \cdot \mathbf{x}$ *and* $\Psi : [\widetilde{P}_{l-1}(K)]^3 \to Q_l(K)$, *with* $\Psi(\mathbf{q}) = \mathbf{q} \times \mathbf{x}$. *Then* $S_l(K) = \ker \Phi$ *and since that* Φ *is surjective we have*

$$\dim S_l(K) = \dim\{[\widetilde{P}_l(K)]^3\} - \dim \widetilde{P}_{l+1} = l(l+2).$$

But, Ψ *is also surjective, and, as we can easily check,*

$$\ker \Psi = \{\mathbf{x}\, q : q \in \widetilde{P}_{l-2}(K)\},$$

hence $\dim \ker \Psi = \dim \widetilde{P}_{l-2}(K)$. *Therefore*

$$\dim Q_l(K) = \dim[\widetilde{P}_{l-1}(K)]^3 - \dim \widetilde{P}_{l-2}(K) = l(l+2).$$

Then $S_l(K)$ *and* $Q_l(K)$ *have the same dimension and, since* $Q_l(K) \subseteq S_l(K)$, *we have (3).*

Now we define the associated edge interpolation operator

$$\Pi_l : [W^{1,p}(K)]^3 \to \mathcal{N}_l(K),$$

with $p > 2$. Let $\mathbf{v} \in [W^{1,p}(K)]^3$. For each edge e of K and $q \in P_{l-1}(e)$ we set

$$F_e(\mathbf{v}, q) = \int_e \mathbf{v} \cdot \mathbf{t}\, q$$

($\mathbf{t}$ denotes a unitary tangent field on e). For each face f of K and $\mathbf{q} \in [P_{l-2}(f)]^2$ we set

$$F_f(\mathbf{v}, \mathbf{q}) = \int_f \mathbf{v} \times \mathbf{n} \cdot \mathbf{q}$$

($\mathbf{n}$ denotes the exterior normal field to K on f). Finally, for all $\mathbf{q} \in [P_{l-3}(K)]^3$ we set

$$F_K(\mathbf{v}, \mathbf{q}) = \int_K \mathbf{v} \cdot \mathbf{q}.$$

The edge interpolation of $\mathbf{v}$, $\Pi_l \mathbf{v} \in \mathcal{N}_l(K)$, is defined by

$$\int_e \Pi_l \mathbf{v} \cdot \mathbf{t}\, q = F_e(\mathbf{v}, q), \qquad \forall q \in P_{l-1}(e), \forall e \text{ edge of } K, \tag{4}$$

$$\int_f \Pi_l \mathbf{v} \times \mathbf{n} \cdot \mathbf{q} = F_f(\mathbf{v}, \mathbf{q}), \qquad \forall \mathbf{q} \in [P_{l-2}(f)]^2, \forall f \text{ face of } K, \tag{5}$$

$$\int_K \Pi_l \mathbf{v} \cdot \mathbf{q} = F_K(\mathbf{v}, \mathbf{q}), \qquad \forall \mathbf{q} \in [P_{l-3}(K)]^3. \tag{6}$$

It is well known [14, 17] that the degrees of freedom (4)-(6) define a unique element $\Pi_l \mathbf{v} \in \mathcal{N}_l(K)$.

The next result will be used in sections 4 and 5 in order to obtain stability estimates for the operator Π_l on different families of elements.

Proposition 2.1. *Let K be a tetrahedron. Suppose that $\mathbf{u}$ and $\mathbf{w}$ are functions in $[W^{1,p}(K)]^3$, with $p > 2$, satisfying*

$$\begin{aligned} |F_e(\mathbf{w}, q)| &\leq C^e(q) N(\mathbf{u}) \qquad \forall q \in P_{l-1}(e), \forall e \text{ edge of } K, \\ |F_f(\mathbf{w}, \mathbf{q})| &\leq C^f(\mathbf{q}) N(\mathbf{u}) \qquad \forall \mathbf{q} \in [P_{l-2}(f)]^2, \forall f \text{ face of } K, \\ |F_K(\mathbf{w}, \mathbf{q})| &\leq C^K(\mathbf{q}) N(\mathbf{u}) \qquad \forall \mathbf{q} \in [P_{l-3}(K)]^3 \end{aligned}$$

where $N(\mathbf{u})$ is some norm or seminorm of $\mathbf{u}$, and C^e, C^f and C^K are positive functionals defined on $P_{l-1}(e)$, $[P_{l-2}(f)]^2$ and $[P_{l-3}(K)]^3$, respectively. Then we have the estimate

$$\|\Pi_l \mathbf{w}\|_{L^\infty(K)} \leq C N(\mathbf{u})$$

with C depending on l and K, but being independent of $\mathbf{u}$ and $\mathbf{w}$.

Proof. Let

$$\{\mathbf{q}_i^K : i = 1, \ldots, \dim\{[P_{l-3}(K)]^3\}\}$$

be a basis of $P_{l-3}(K)^3$. Similarly, for each edge e and each face f of K, let

$$\{q_i^e : i = 1, \ldots, \dim P_{l-1}(e)\}$$

and

$$\{\mathbf{q}_i^f : i = 1, \ldots, \dim\{[P_{l-2}(f)]^2\}\}$$

be bases of $P_{l-1}(e)$ and $[P_{l-2}(f)]^2$, respectively. Then, for $[\mathbf{v} \in W^{1,p}(K)]^3$, $\Pi_l \mathbf{v}$ is defined by

$$\begin{aligned} \int_e \Pi_l \mathbf{v} \cdot \mathbf{t}\, q_i^e &= F_e(\mathbf{v}, q_i^e), \qquad i = 1, \ldots, \dim P_{l-1}(e), \forall e \text{ edge of } K, \\ \int_f \Pi_l \mathbf{v} \times \mathbf{n} \cdot \mathbf{q}_i^f &= F_f(\mathbf{v}, \mathbf{q}_i^f), \qquad i = 1, \ldots, \dim\{[P_{l-2}(f)]^2\}, \forall f \text{ face of } K, \\ \int_K \Pi_l \mathbf{v} \cdot \mathbf{q}_i^K &= F_K(\mathbf{v}, \mathbf{q}_i^K), \qquad i = 1, \ldots, \dim\{[P_{l-3}(K)]^3\}. \end{aligned}$$

Now, we can consider the dual basis associated with these equations. We denote this basis by

$$\cup_e \{\mathbf{v}_i^e : i = 1, \ldots, \dim P_{l-1}(e)\} \bigcup \cup_f \left\{\mathbf{v}_i^f : i = 1, \ldots, \dim\{[P_{l-2}(f)]^2\}\right\} \bigcup \left\{\mathbf{v}_i^K : i = 1, \ldots, \dim\{[P_{l-3}(K)]^3\}\right\}.$$

Thus, for example, for each $1 \leq i \leq \dim P_{l-3}(K)^3$, the function $\mathbf{v}_i^K \in \mathcal{N}_l(K)$ is defined by the conditions

$$\begin{aligned} F_e(\mathbf{v}_i^K, q_j^K) &= 0 \qquad j = 1, \dots, \dim P_{l-1}(e), \forall e \text{ edge of } K, \\ F_f(\mathbf{v}_i^K, \mathbf{q}_j^f) &= 0, \qquad j = 1, \dots, \dim\{[P_{l-2}(f)]^2\}, \forall f \text{ face of } K, \\ F_K(\mathbf{v}_i^K, \mathbf{q}_j^K) &= \delta_{ij}, \qquad j = 1, \dots, \dim\{[P_{l-3}(K)]^3\}, \end{aligned}$$

and the functions $\mathbf{v}_i^e$ and $\mathbf{v}_i^f$ are defined similarly.

Then, $\Pi_l \mathbf{w}$ can be written as

$$\Pi_l \mathbf{w} = \sum_e \left(\sum_{i=1}^{\dim P_{l-1}(e)} F(\mathbf{w}, q_i^e) \mathbf{v}_i^e \right) + \sum_f \left(\sum_{i=1}^{\dim\{[P_{l-2}(f)]^2\}} F(\mathbf{w}, \mathbf{q}_i^f) \mathbf{v}_i^f \right) + \sum_{i=1}^{\dim\{[P_{l-3}(K)]^3\}} F(\mathbf{w}, q_i^K) \mathbf{v}_i^K.$$

So, using the assumption of the Proposition we have

$$\begin{aligned} \|\Pi_l \mathbf{w}\|_{L^\infty(K)} &\leq \left[\sum_e \left(\sum_{i=1}^{\dim P_{l-1}(e)} C^e(q_i^e) \|\mathbf{v}_i^e\|_{L^\infty(K)} \right) + \sum_f \left(\sum_{i=1}^{\dim\{[P_{l-2}(f)]^2\}} C^f(\mathbf{q}_i^f) \|\mathbf{v}_i^f\|_{L^\infty(K)} \right) + \sum_{i=1}^{\dim\{[P_{l-3}(K)]^3\}} C^K(q_i^K) \|\mathbf{v}_i^K\|_{L^\infty(K)} \right] N(\mathbf{u}) \\ &=: C(l, K) N(\mathbf{u}), \end{aligned}$$

as we wanted to prove. □

3. The maximum angle condition

The maximum angle condition for tetrahedral meshes was first introduced in [16], as a generalization of the Synge's condition for triangles. We introduce now the definition of this condition and then we present a related result that becomes useful for the rest of the paper.

Definition 3.1. *A tetrahedron K satisfies the "maximum angle condition" with a constant $\bar{\psi} < \pi$ (or shortly $MAC(\bar{\psi})$) if the maximum angle between faces and the maximum angle inside the faces are less than or equal $\bar{\psi}$.*

In order to obtain an analytical equivalent condition, we introduce the following families of tetrahedra. In what follows, $\mathbf{e}_i, i = 1, 2, 3$, will denote the canonical vectors in $\mathbb{R}^3$.

Definition 3.2. *A tetrahedron K belongs to the family $\mathcal{F}_1$ if its vertices are at $\mathbf{0}$, $h_1\mathbf{e}_1$, $h_2\mathbf{e}_2$ and $h_3\mathbf{e}_3$, where $h_i > 0$ are arbitrary lengths (see Figure 1).*

Definition 3.3. *A tetrahedron K belongs to the family $\mathcal{F}_2$ if its vertices are at $\mathbf{0}$, $h_1\mathbf{e}_1 + h_2\mathbf{e}_2$, $h_2\mathbf{e}_2$ and $h_3\mathbf{e}_3$, where $h_i > 0$ are arbitrary lengths (see Figure 1).*

In what follows, for a vector $\xi \in \mathbb{R}^3$, $\|\xi\|$ denotes its euclidean norm, and for a matrix $M \in \mathbb{R}^{3\times 3}$, $\|M\|$ denotes the corresponding matrix norm.

The next theorem is proved in [3](see also [16]).

Theorem 3.1. *For each $0 < \bar{\psi} < \pi$ there exists a constant $C = C(\bar{\psi})$ with the following property: if K is a tetrahedron satisfying $MAC(\bar{\psi})$, then there exists an element $\widetilde{K}_0 \in \mathcal{F}_1 \cup \mathcal{F}_2$ that can be mapped onto K through an affine transformation $F_0(\widetilde{\mathbf{x}}) = M_0\widetilde{\mathbf{x}} + \mathbf{p}_0$ with $\|M_0\|, \|M_0^{-1}\| \le C$.*

We will present a statement of the previous Theorem that is more appropriated for our purposes.

Theorem 3.2. *For each $0 < \bar{\psi} < \pi$, there exists a constant $C = C(\bar{\psi})$ with the following property: for a tetrahedron K satisfying $MAC(\bar{\psi})$, it is possible to choose a vertex $\mathbf{p}_0$ and three of its edges, ℓ_i, $i = 1, 2, 3$, such that, if ξ_i and h_i, $i = 1, 2, 3$ denote the unitary vectors associated to ℓ_i and the lengths of ℓ_i, respectively, and if M is the matrix made up of ξ_1, ξ_2 and ξ_3 as its columns, then the map $\widetilde{\mathbf{x}} \to M\widetilde{\mathbf{x}} + \mathbf{p}_0$ applies $\widetilde{K}$ onto K and $\|M\|, \|M^{-1}\| \le C$, where $\widetilde{K}$ is the tetrahedron with vertices either $\{\mathbf{0}, h_1\mathbf{e}_1, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}$ or $\{\mathbf{0}, h_1\mathbf{e}_1 + h_2\mathbf{e}_2, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}$.*

We shall obtain Theorem 3.2 from Theorem 3.1, but we want to observe that it could be proved directly following the same ideas of [3].

Proof. Suppose K verifies $MAC(\bar{\psi})$ and let $\widetilde{K}_0$, F_0, $\mathbf{p}_0$ and M_0 be as in the statement of the Theorem 3.1, with $\|M_0\|, \|M_0^{-1}\| \le C(\bar{\psi})$.

If $\widetilde{K}_0 \in \mathcal{F}_1$, we suppose that its vertices are at $\mathbf{0}, h_{0,1}\mathbf{e}_1, h_{0,2}\mathbf{e}_2$ and $h_{0,3}\mathbf{e}_3$, while if $\widetilde{K}_0 \in \mathcal{F}_2$ suppose that they are at $\mathbf{0}, h_{0,1}\mathbf{e}_1 + h_{0,2}\mathbf{e}_2, h_{0,2}\mathbf{e}_2$ and $h_{0,3}\mathbf{e}_3$. In any case we define $\ell_i = h_{i,0}M_0\mathbf{e}_i$. Then, we have

$$\begin{aligned}
\ell_1 &= F_0(h_{1,0}\mathbf{e}_1) - F_0(\mathbf{0}) = F_0(h_{1,0}\mathbf{e}_1 + h_{2,0}\mathbf{e}_2) - F_0(h_{2,0}\mathbf{e}_2),\\
\ell_2 &= F_0(h_{2,0}\mathbf{e}_2) - F_0(\mathbf{0}),\\
\ell_3 &= F_0(h_{3,0}\mathbf{e}_3) - F_0(\mathbf{0}),
\end{aligned}$$

that is, the vectors ℓ_i represent three edges of $\widetilde{K}_0$. Let ξ_i be unitary vectors associated with ℓ_i, $i = 1, 2, 3$ (that is, $\xi_i = \ell_i/\|\ell_i\|$). By setting $h_i = \|\ell_i\|, i = 1, 2, 3$, we consider the tetrahedron $\widetilde{K}$ defined as follows: if $\widetilde{K}_0 \in \mathcal{F}_1$, we take $\widetilde{K} \in \mathcal{F}_1$ as the tetrahedron with vertices at $\mathbf{0}, h_1\mathbf{e}_1, h_2\mathbf{e}_2$ and $h_3\mathbf{e}_3$, while if $\widetilde{K}_0 \in \mathcal{F}_2$ we take $\widetilde{K} \in \mathcal{F}_2$ with vertices at $\mathbf{0}, h_1\mathbf{e}_1 + h_2\mathbf{e}_2, h_2\mathbf{e}_2$ and $h_3\mathbf{e}_3$. Finally we define the

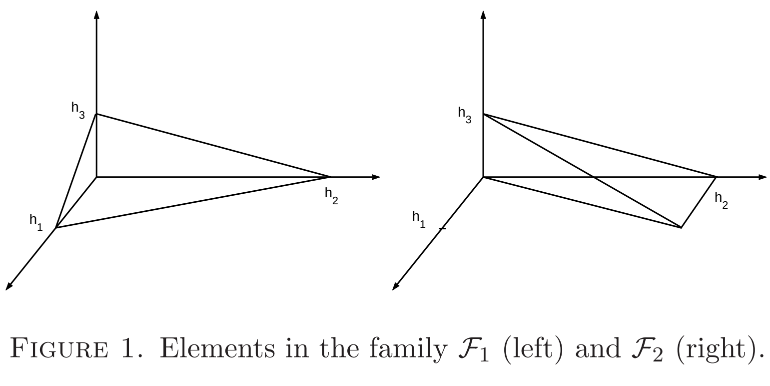

FIGURE 1. Elements in the family $\mathcal{F}_1$ (left) and $\mathcal{F}_2$ (right).

matrix M made up of the vectors ξ_1, ξ_2 and ξ_3 as its columns, and consider the map $F(\widetilde{\mathbf{x}}) = M\widetilde{\mathbf{x}} + \mathbf{p}_0$.

It is clear that $F(h_i\mathbf{e}_i) = F_0(h_{i,0}\mathbf{e}_i)$, $i = 1, 2, 3$, and so $F(\widetilde{K}) = K$. Furthermore, if $\Lambda = diag\left(\frac{h_{1,0}}{h_1}, \frac{h_{2,0}}{h_2}, \frac{h_{3,0}}{h_3}\right)$, we have $M = M_0\Lambda$. Since

$$\frac{h_{i,0}}{h_i} = \|M_0^{-1}\xi_i\|, \quad \frac{h_i}{h_{i,0}} = \|M_0\mathbf{e}_i\|, \qquad i = 1, 2, 3$$

we have $\|\Lambda\| \le C(\bar{\psi})$ and $\|\Lambda^{-1}\| \le C(\bar{\psi})$, and therefore

$$\|M\| \le C(\bar{\psi})^2, \quad \|M^{-1}\| \le C(\bar{\psi})^2.$$

Therefore we have the assertion with the constant C given by $C(\bar{\psi})^2$ (where $C(\bar{\psi})$ comes from Theorem 3.1). ☐

Remark 3.1. *We observe that the constant $C(\bar{\psi})$ of Theorem 3.2 blows up when $\bar{\psi} \to \pi$. In fact, for $h_3 > 0$ consider the tetrahedron $K(h_3)$ with vertices at $\mathbf{0}, (1, 0, 0), (0, 1, 0)$ and $(\frac{1}{2}, \frac{1}{2}, h_3)$. One can easily check that $\bar{\psi}(h_3) \to \pi$ if $h_3 \to 0$, where $\bar{\psi}(h_3)$ denotes the maximum angle of $K(h_3)$. Let $\widetilde{K}(h_3) \in \mathcal{F}_1 \cup \mathcal{F}_2$ be mapped onto $K(h_3)$ by an affine transformation $F_{h_3}(\widetilde{\mathbf{x}}) = M(h_3)\widetilde{\mathbf{x}} + \mathbf{p}_0(h_3)$, as stated in Theorem 3.2. From $\|M(h_3)^{-1}\| \le C(\bar{\psi}(h_3))$ we obtain $|det(M(h_3)^{-1})| \le 6C(\bar{\psi})^3$ or*

$$|\det M(h_3)| \ge \frac{1}{6C(\bar{\psi})^3}.$$

Then we have

$$\frac{1}{6C(\psi(h_3))^3} \le |\det M(h_3)| = \frac{|K(h_3)|}{|\widetilde{K}(h_3)|},$$

where $|K|$ denotes the volume of the tetrahedron K. But, since $|K(h_3)| = \frac{h_3}{6}$ and $|\widetilde{K}(h_3)|$ remains bounded away from 0 because the lengths of the edges of $K(h_3)$ are greater than $\frac{\sqrt{2}}{2}$ (see Theorem 3.2), then we have

$$\frac{1}{6C(\psi(h_3))^3} \to 0 \qquad \textit{when } h_3 \to 0,$$

that proves our assertion.

To obtain uniform error estimates for the edge interpolation on elements satisfying a maximum angle condition, we use, in section 6, uniform stability estimates for the operator Π_l on elements in $\mathcal{F}_1 \cup \mathcal{F}_2$. These stability estimates can be obtained by rescaling the corresponding inequality for reference elements in $\mathcal{F}_1$ and in $\mathcal{F}_2$. Let $\widehat{K}_1$ be the element obtained by taking $h_1 = h_2 = h_3 = 1$ in Definition 3.2, and $\widehat{K}_2$ the one corresponding to Definition 3.3, that we take as reference elements in $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

Clearly, we have the following inequalities ($p > 2$)

$$\|\Pi_l\mathbf{u}\|_{L^p(\widehat{K}_i)} \le \hat{C}\|\mathbf{u}\|_{W^{1,p}(\widehat{K}_i)} \qquad \forall\, \mathbf{u} \in [W^{1,p}(\widehat{K}_i)]^3, i = 1, 2, \tag{7}$$

where if $i = 1$ (resp. $i = 2$) then Π_l is the edge interpolation on $\widehat{K}_1$ (resp. $\widehat{K}_2$). Let K be an element in $\mathcal{F}_1 \cup \mathcal{F}_2$, and let now Π_l be the edge interpolation on K .

Then, it is easy to check that by rescaling inequalities (7) we obtain, for example for the first component $\Pi_{l,1}\mathbf{u}$ of $\Pi_l\mathbf{u}$, with $\mathbf{u} \in [W^{1,p}(K)]^3$, the estimate

$$\|\Pi_{l,1}\mathbf{u}\|_{L^p(K_i)} \le \hat{C}\frac{1}{h_1}\left(\sum_{i=1}^{3} h_i\|u_i\|_{L^p(K)} + \sum_{i,j=1}^{3} h_ih_j\left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^p(K)}\right) \tag{8}$$

(see the proofs of Theorems 4.3 or 5.3) where h_i are as in Definitions 3.2 or 3.3. The presence of, for example, the factor h_2h_3/h_1 may turn useless the estimate when narrow elements are considered, where h_1, h_2 and h_3 are of different orders of magnitude.

To avoid the appearance of problems like the one described, we obtain, in the next two sections, stability estimates on the reference elements sharper than the ones given in (7).

4. Stability of Π_l on elements in the family $\mathcal{F}_1$

Let $\widehat{K}_1$ be the tetrahedron $\{(x_1, x_2, x_3) \in [0,1]^3 : x_1 + x_2 + x_3 \le 1\}$. For $i = 1, \dots, 4$, we denote by f_i the face having normal $\mathbf{n}_i$, where $\mathbf{n}_1 = (-1,0,0)$, $\mathbf{n}_2 = (0,-1,0)$, $\mathbf{n}_3 = (0,0,-1)$ and $\mathbf{n}_4 = \frac{1}{\sqrt{3}}(1,1,1)$. Also, for $i = 1, \dots, 6$, denote by e_i the edge tangent to $\mathbf{t}_i$, where $\mathbf{t}_1 = (1,0,0)$, $\mathbf{t}_2 = (0,1,0)$, $\mathbf{t}_3 = (0,0,1)$, $\mathbf{t}_4 = \frac{1}{\sqrt{2}}(0,-1,1)$, $\mathbf{t}_5 = \frac{1}{\sqrt{2}}(1,0,-1)$ and $\mathbf{t}_6 = \frac{1}{\sqrt{2}}(-1,1,0)$. Let $\widehat{\Pi}_l$ be the edge interpolation operator on the element $\widehat{K}_1$.

Lemma 4.1. *Let* $\mathbf{u} = (0, u_2(x_2,x_3), u_3(x_2,x_3)), \mathbf{v} = (v_1(x_1,x_3), 0, v_3(x_1,x_3))$ *and* $\mathbf{w} = (w_1(x_1,x_2), w_2(x_1,x_2), 0)$ *be functions in* $[W^{1,p}(\widehat{K}_1)]^3$ *with* $p > 2$. *Then we have*

$$\begin{aligned}
\widehat{\Pi}_l\mathbf{u} &= (0, p_2(x_2,x_3) - s_1(x_2,x_3)x_3, p_3(x_2,x_3) + s_1(x_2,x_3)x_2),\\
\widehat{\Pi}_l\mathbf{v} &= (q_1(x_1,x_3) + s_2(x_1,x_3)x_3, 0, q_3(x_1,x_3) - s_2(x_1,x_3)x_1),\\
\widehat{\Pi}_l\mathbf{w} &= (r_1(x_1,x_2) - s_3(x_1,x_2)x_2, r_2(x_1,x_2) + s_3(x_1,x_2)x_1, 0),
\end{aligned}$$

where $p_2, p_3, s_1 \in P_{l-1}(f_1)$, $q_1, q_3, s_2 \in P_{l-1}(f_2)$, *and* $r_1, r_2, s_3 \in P_{l-1}(f_3)$.

Proof. We prove only the first equality, since the other equations follow similarly. Let $\bar{p}_2(x_2)$ be the L^2-projection of $u_2(x_2,0)$ on the space $P_{l-1}(e_2)$, and let $\bar{p}_3(x_3)$ be the L^2-projection of $u_3(0,x_3)$ on the space $P_{l-1}(e_3)$. Furthermore, let $\hat{p}_2, \hat{p}_3 \in P_{l-2}(f_1)$ and $s_1 \in P_{l-1}(f_1)$ such that

$$\begin{aligned}
\int_{e_4} [-x_3(\hat{p}_2 - s_1) + x_2(\hat{p}_3 + q_1)]\, q &= \int_{e_4} [-(u_2 - \bar{p}_2) + (u_3 - \bar{p}_3)]\, q, \quad \forall q \in P_{l-1}(e_4),\\
\int_{f_1} x_3(\hat{p}_2 - s_1)q &= \int_{f_1} (u_2 - \bar{p}_2)\, q \qquad \forall q \in P_{l-2}(f_1) \qquad (9)\\
\int_{f_1} x_2(\hat{p}_3 + s_1)q &= \int_{f_1} (u_3 - \bar{p}_3)\, q \qquad \forall q \in P_{l-2}(f_1).
\end{aligned}$$

Assuming, for the moment, that there exist such $\hat{p}_2, \hat{p}_3$ and s_1, we define

$$p_2 = \bar{p}_2 + x_3\,\hat{p}_2, \qquad p_3 = \bar{p}_3 + x_2\,\hat{p}_3.$$

Then, after some computations, we can check that p_2, p_3 and s_1 satisfy the first equation in the statement of the lemma.

So, it remains only to prove that the system (9) has a solution $\hat{p}_2, \hat{p}_3, s_1$ as required. We will prove that there exist a unique such solution with $\hat{p}_2, \hat{p}_3 \in P_{l-2}(f_1)$ and $s_1 \in \widetilde{P}_{l-1}(f_1)$. For that purpose, being $\dim \widetilde{P}_{l-1}(f_1) = \dim P_{l-1}(e_4)$, it suffices to prove that if

$$\int_{e_4} [-x_3(\hat{p}_2 - s_1) + x_2(\hat{p}_3 + s_1)]\, q = 0 \quad \forall q \in P_{l-1}(e_4), \tag{10}$$

$$\int_{f_1} x_3(\hat{p}_2 - s_1) q = 0 \qquad \forall q \in P_{l-2}(f_1) \tag{11}$$

$$\int_{f_1} x_2(\hat{p}_3 + s_1) q = 0 \qquad \forall q \in P_{l-2}(f_1), \tag{12}$$

then $\hat{p}_2, \hat{p}_3$ and s_1 vanish. Let $\mathbf{z} = (x_3(\hat{p}_2 - s_1), x_2(\hat{p}_3 + s_1))$. Then $\mathbf{z} \in [P_{l-1}(f_1)]^2 + (-x_3, x_2)\widetilde{P}_{l-1}(f_1)$, that is, $\mathbf{z}$ is in the space of edge elements in a 2-dimensional space (see [14]). Using the Green formula on f_1 we have for all $q \in P_{l-1}(f_1)$

$$\int_{f_1} \operatorname{curl} \mathbf{z}\, q = -\int_{f_1} \mathbf{z} \cdot \mathbf{curl}\, q + \int_{\partial f_1} \mathbf{z} \cdot \mathbf{t} q.$$

Note that $\mathbf{z} \cdot \mathbf{t} \equiv 0$ on e_2 and e_3, while on e_4 we have $\mathbf{z} \cdot \mathbf{t} = \frac{1}{\sqrt{2}}[-x_3(\hat{p}_2 - s_1) + x_2(\hat{p}_3 + s_1)$. Since $q|_{e_4} \in P_{l-1}(e_4)$, there follows from equation (10) that $\int_{e_4} \mathbf{z} \cdot \mathbf{t} q = 0$. So

$$\int_{\partial f_1} \mathbf{z} \cdot \mathbf{t} q = 0.$$

Also, since $\mathbf{curl}\, q = (-\frac{\partial q}{\partial x_3}, \frac{\partial q}{\partial x_2}) \in [P_{l-2}(f_1)]^2$, it follows from equations (11) and (12) that

$$\int_{f_1} \mathbf{z} \cdot \mathbf{curl}\, q = 0.$$

Hence

$$\int_{f_1} \operatorname{curl} \mathbf{z}\, q = 0 \qquad \forall q \in P_{l-1}(f_1),$$

and since $\operatorname{curl} \mathbf{z} \in P_{l-1}(f_1)$ we conclude that $\operatorname{curl} \mathbf{z} \equiv 0$. It follows that $\mathbf{z} = \nabla p$ with $p \in P_l(f_1)$, so $\mathbf{z} \in [P_{l-1}(f_1)]^2$ (see, for example, [14], p. 263). But then, $s_1 \in \widetilde{P}_{l-1}(f_1)$ must vanish. Therefore, equations (11) and (12) imply that $\hat{p}_2$ and $\hat{p}_3$ also vanish, as we wanted. $\square$

In what follows, for a function $\mathbf{v} \in [W^{1,p}(\widehat{K}_1)]^3$ we denote the i-component of $\widehat{\Pi}_l \mathbf{v}$ by $\widehat{\Pi}_{l,i}\mathbf{v}$. So $\widehat{\Pi}_l \mathbf{v} = (\widehat{\Pi}_{l,1}\mathbf{v}, \widehat{\Pi}_{l,2}\mathbf{v}, \widehat{\Pi}_{l,3}\mathbf{v})$.

Theorem 4.2. *Let $\mathbf{u} \in [W^{1,p}(\widehat{K}_1)]^3$, with $p > 2$, such that $\mathbf{curl}\,\mathbf{u} \in [W^{1,1}(\widehat{K}_1)]^3$. Denote $\mathbf{curl}\,\mathbf{u}$ by $\theta = (\theta_1, \theta_2, \theta_3)$. Then we have*

$$\begin{aligned}
\left\|\widehat{\Pi}_{l,1}\mathbf{u}\right\|_{L^\infty(\widehat{K}_1)} &\le C\left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_1)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_1)}\right)\\
\left\|\widehat{\Pi}_{l,2}\mathbf{u}\right\|_{L^\infty(\widehat{K}_1)} &\le C\left(\|u_2\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_1\|_{W^{1,1}(\widehat{K}_1)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_1)}\right)\\
\left\|\widehat{\Pi}_{l,3}\mathbf{u}\right\|_{L^\infty(\widehat{K}_1)} &\le C\left(\|u_3\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_1\|_{W^{1,1}(\widehat{K}_1)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_1)}\right)
\end{aligned}$$

where the constant C is independent on $\mathbf{u}$.

Proof. We prove the first inequality, the others follow analogously. Due to the density of $[\mathcal{C}^\infty(\overline{\widehat{K}_1})]^3$ in the space of functions in $[W^{1,p}(\widehat{K}_1)]^3$ with **curl** in $[W^{1,1}(\widehat{K}_1)]^3$, with the norm $\|\cdot\|_{W^{1,p}(\widehat{K}_1)} + \|\mathbf{curl}\,(\cdot)\|_{W^{1,1}(\widehat{K}_1)}$, that can be proved following, for instance, the techniques developed in chapter 3 of [4], we can assume that the function $\mathbf{u}$ is smooth on $\overline{\widehat{K}_1}$. Let $\mathbf{w} = (u_1, u_2 - u_2(0,x_2,x_3), u_3 - u_3(0,x_2,x_3))$.

By Lemma 4.1 we have $\widehat{\Pi}_{l,1}\mathbf{u} = \widehat{\Pi}_{l,1}\mathbf{w}$. But, $\widehat{\Pi}_l\mathbf{w}$ is defined by conditions (4)-(6) on $\widehat{K}_1$. Define

$$N(\mathbf{u}) = \|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_1)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_1)}\,.$$

Then, by Proposition 2.1 it is sufficient to prove that

$$|F_{e_i}(\mathbf{w},q)| \;\leq\; C^{e_i}(q)N(\mathbf{u}) \qquad \forall q \in P_{l-1}(e_i),\ i = 1\ldots,6, \tag{13}$$

$$|F_{f_i}(\mathbf{w},\mathbf{q})| \;\leq\; C^{f_i}(\mathbf{q})N(\mathbf{u}) \qquad \forall q \in [P_{l-2}(f_i)]^2,\ i = 1\ldots,4, \tag{14}$$

$$|F_{\widehat{K}_1}(\mathbf{w},\mathbf{q})| \;\leq\; C^{\widehat{K}_1}(\mathbf{q})N(\mathbf{u}) \qquad \forall \mathbf{q} \in [P_{l-3}(\widehat{K}_1)]^3. \tag{15}$$

For simplicity, we will denote the constants $C^{e_i}(q), C^{f_i}(\mathbf{q})$ or $C^{\widehat{K}_1}(\mathbf{q})$ always by $C(q)$ or $C(\mathbf{q})$ without explicit reference to the edges, faces or element.

Note that $w_1 = u_1$ and $w_2|_{f_1} = w_3|_{f_1} \equiv 0$, and the second and third components of $\operatorname{curl}\mathbf{w}$ coincide with θ_2 and θ_3, respectively.

We begin proving inequalities (13). We have

$$\begin{aligned}
F_{e_1}(\mathbf{w},q) &= \int_{e_1} u_1\,q \qquad \forall q \in P_{l-1}(e_1),\\
F_{e_2}(\mathbf{w},q) &= 0 \qquad \forall q \in P_{l-1}(e_2),\\
F_{e_3}(\mathbf{w},q) &= 0 \qquad \forall q \in P_{l-1}(e_3),\\
F_{e_4}(\mathbf{w},q) &= 0 \qquad \forall q \in P_{l-1}(e_4),\\
F_{e_5}(\mathbf{w},q) &= \frac{1}{\sqrt{2}}\int_{e_5}(w_1 - w_3)\,q \qquad \forall q \in P_{l-1}(e_5),\\
F_{e_6}(\mathbf{w},q) &= \frac{1}{\sqrt{2}}\int_{e_6}(-w_1 + w_2)\,q \qquad \forall q \in P_{l-1}(e_6)
\end{aligned}$$

Clearly, we have estimate (13) for $i = 1,\ldots,4$. It remains to consider the cases $i = 5$ and $i = 6$.

A polynomial $q \in P_{l-1}(e_5)$ can be written as $q = q(x_3)$, and we can see it as a polynomial in $P_{l-1}(f_2)$. For such a polynomial, using the Green formula in f_2, we have (we denote by (n_1,n_3) the unitary outward normal in the plane x_1x_3 to ∂f_2)

$$\begin{aligned}
\int_{f_2}\theta_2\,q &= -\int_{f_2}\left(w_1\frac{\partial q}{\partial x_3} - w_3\frac{\partial q}{\partial x_1}\right) + \int_{\partial f_2}(w_1n_3 - w_3n_1)q\\
&= -\int_{f_2} w_1\frac{\partial q}{\partial x_3} - \int_{e_1} w_1\,q + \frac{1}{\sqrt{2}}\int_{e_5}(w_1 - w_3)\,q,
\end{aligned}$$

so

$$\frac{1}{\sqrt{2}}\int_{e_5}(w_1 - w_3)\,q = \int_{f_2}\theta_2\,q + \int_{e_1} u_1\,q + \int_{f_2} u_1\frac{\partial q}{\partial x_3}.$$

Then, we see that

$$|F_{e_5}(\mathbf{w},q)| \leq C(q)\left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_1)}\right).$$

Similarly, we can prove that

$$|F_{e_6}(\mathbf{w},q)| \le C(q)\left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_1)}\right).$$

So we have proved inequalities (13).

Now consider the estimates (14). We have

$$\begin{aligned}
F_{f_1}(\mathbf{w},\mathbf{q}) &= 0 \qquad \forall \mathbf{q}=(q_1,q_2)\in [P_{l-2}(f_1)]^2,\\
F_{f_2}(\mathbf{w},\mathbf{q}) &= \int_{f_2} w_1\,q_1 + w_3\,q_2 \qquad \forall \mathbf{q}=(q_1,q_2)\in [P_{l-2}(f_2)]^2,\\
F_{f_3}(\mathbf{w},\mathbf{q}) &= \int_{f_2} w_1\,q_1 + w_2\,q_2 \qquad \forall \mathbf{q}=(q_1,q_2)\in [P_{l-2}(f_3)]^2,\\
F_{f_4}(\mathbf{w},\mathbf{q}) &= \int_{f_4} (w_1-w_3)\,q_1 + (w_1-w_2)\,q_2 \qquad \forall \mathbf{q}=(q_1,q_2)\in [P_{l-2}(f_4)]^2
\end{aligned}$$

(we remark that we are taking suitable basis on each face in order to have these formulas). We estimate $F_{f_2}(\mathbf{w},\mathbf{q})$. Given $q_2 \in P_{l-2}(f_2)$ we define $\bar{q}_2 \in P_{l-1}(f_2)$ by

$$\bar{q}_2(x_1,x_3) = -\int_{x_1}^{1-x_3} q_2(t,x_3)\,dt.$$

Then, using again the Green Formula and keeping in mind that $w_3 = 0$ on e_3, $\frac{\partial \bar{q}_2}{\partial x_1} = q_2$ and $\bar{q}_2|_{e_5} = 0$, we obtain

$$\int_{f_2} \theta_2\,\bar{q}_2 = -\int_{f_2}\left(w_1\frac{\partial \bar{q}_2}{\partial x_3} - w_3\,q_2\right) - \int_{e_1} w_1\,\bar{q}_2.$$

Therefore,

$$\int_{f_2} w_3\,q_2 = \int_{f_2}\left(\theta_2\bar{q}_2 + u_1\frac{\partial \bar{q}_2}{\partial x_3}\right) - \int_{e_1} u_1\bar{q}_2.$$

So,

$$|F_{f_2}(\mathbf{w},\mathbf{q})| \le C(\mathbf{q})\left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_1)}\right),$$

as we wanted. In a similar way, we obtain

$$|F_{f_3}(\mathbf{w},\mathbf{q})| \le C(\mathbf{q})\left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_1)}\right).$$

On the other hand, since $w_3|_{f_1} \equiv 0$, we have for each $q_1 \in P_{l-2}(f_4)$

$$\begin{aligned}
\int_{f_4} w_3\,q &= \int_{f_1} w_3(1-x_2-x_3,x_2,x_3)\,q_1(x_2,x_3)\\
&= \int_{f_1}\int_0^{1-x_2-x_3} \frac{\partial w_3}{\partial x_1}(t,x_2,x_3)\,q_1(x_2,x_3)\,dt\,dx_2\,dx_3\\
&= \int_{\widehat{K}_1} \frac{\partial w_3}{\partial x_1}\,q_1\\
&= -\int_{\widehat{K}_1} \theta_2\,q_1 + \int_{\widehat{K}_1}\frac{\partial w_1}{\partial x_3}\,q_1.
\end{aligned}$$

where we have extended q_1 from f_1 to $\widehat{K}_1$ in the natural way. Hence

$$\int_{f_4} (w_1-w_3)\,q_1 \le C(q_1)\left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{L^1(\widehat{K}_1)}\right).$$

Similarly, we have

$$\int_{f_4} (w_1 - w_2)\, q_2 \le C(q_2) \left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_3\|_{L^1(\widehat{K}_1)} \right)$$

and then we obtain

$$|F_{f_4}(\mathbf{w},\mathbf{q})| \le C(q) \left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_1)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_1)} \right).$$

Therefore we obtained (14).

Finally, we prove inequalities (15). We have for $\mathbf{q} = (q_1, q_2, q_3) \in [P_{l-3}(\widehat{K}_1)]^3$

$$F_{\widehat{K}_1}(\mathbf{w},\mathbf{q}) = \int_{\widehat{K}_1} (w_1 q_1 + w_2 q_2 + w_3 q_3)$$

Using again that $w_2|_{f_1} \equiv 0$ we have

$$\begin{aligned}
\int_{\widehat{K}_1} w_2\, q_2 &= \int_{\widehat{K}_1} \int_0^{x_1} \frac{\partial w_2}{\partial x_1}(t, x_2, x_3)\, q_2(\mathbf{x})\, dt\, d\mathbf{x} \\
&= \int_0^1 \int_0^{1-x_2} \int_0^{1-x_2-x_3} \frac{\partial w_2}{\partial x_1}(t, x_2, x_3) \int_t^{1-x_2-x_3} q_2(\mathbf{x})\, dx_1\, dt\, dx_3\, dx_2 \\
&= \int_{\widehat{K}_1} \frac{\partial w_2}{\partial x_1}\, \bar{q}_2 \\
&= \int_{\widehat{K}_1} \theta_3 \bar{q}_2 + \int_{\widehat{K}_1} \frac{\partial u_1}{\partial x_2}\, \bar{q}_2
\end{aligned}$$

where

$$\bar{q}_2(t, x_2, x_3) = \int_t^{1-x_2-x_3} q_2(x_1, x_2, x_3)\, dx_1.$$

So, we see that

$$\left| \int_{\widehat{K}_1} w_2\, q_2 \right| \le C(q_2) \left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_3\|_{L^1(\widehat{K}_1)} \right).$$

Analogously, we have

$$\left| \int_{\widehat{K}_1} w_3\, q_3 \right| \le C(q_3) \left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{L^1(\widehat{K}_1)} \right).$$

Clearly, we have arrived at

$$|F_{\widehat{K}_1}(\mathbf{w},\mathbf{q})| \le C(\mathbf{q}) \left(\|u_1\|_{W^{1,p}(\widehat{K}_1)} + \|\theta_2\|_{L^1(\widehat{K}_1)} + \|\theta_3\|_{L^1(\widehat{K}_1)} \right).$$

Hence, we have proved the first estimate of the assertion. □

Now we can state the main result of this section, concerning the stability estimate for the edge interpolation operator on elements in the family $\mathcal{F}_1$.

Theorem 4.3. *Let $\widetilde{K} \in \mathcal{F}_1$ be the tetrahedron generated by $\{\mathbf{0}, h_1\mathbf{e}_1, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}$. Denote by $\widetilde{\Pi}_l$ the edge interpolation operator of order l in $\widetilde{K}$. Then, there exists a constant C independent of h_1, h_2 and h_3 such that for all $\widetilde{\mathbf{v}} \in [W^{1,p}(\widetilde{K})]^3$, $p > 2$,* *with* $\boldsymbol{curl}\widetilde{\mathbf{v}} \in [W^{1,1}(\widetilde{K})]^3$*, we have*

$$\begin{aligned}\|\widetilde{\Pi}_l\widetilde{\mathbf{v}}\|_{L^\infty(\widetilde{K})} &\leq C\Bigg\{|\widetilde{K}|^{-\frac{1}{p}}\left(\|\widetilde{\mathbf{v}}\|_{L^p(\widetilde{K})} + \sum_{i=1}^3 h_i\left\|\frac{\partial\widetilde{\mathbf{v}}}{\partial\tilde{x}_i}\right\|_{L^p(\widetilde{K})}\right)\\ &\qquad + h\,|\widetilde{K}|^{-1}\left(\left\|\widetilde{\boldsymbol{curl}\mathbf{v}}\right\|_{L^1(\widetilde{K})} + \sum_{i=1}^3 h_i\left\|\frac{\partial\widetilde{\boldsymbol{curl}\mathbf{v}}}{\partial\tilde{x}_i}\right\|_{L^1(\widetilde{K})}\right)\Bigg\}\end{aligned}$$

where h *is the diameter of* $\widetilde{K}$*.*

Proof. Consider the map $\mathbf{x} \to \widetilde{\mathbf{x}} = B\mathbf{x}$ with $B = diag(h_1, h_2, h_3)$ that maps $\widehat{K}_1$ onto $\widetilde{K}$. Let $\widetilde{\mathbf{v}} \in [W^{1,p}(\widetilde{K})]^3$, $p > 2$, with $\mathbf{curl}\,\widetilde{\mathbf{v}} \in [W^{1,1}(\widetilde{K})]^3$ and define $\widehat{\mathbf{v}}$ by setting $\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) = B^t\widetilde{\mathbf{v}}(\widetilde{\mathbf{x}})$. We also set $\widetilde{\theta}(\widetilde{\mathbf{x}}) = \widetilde{\mathbf{curl}\,\mathbf{v}}(\widetilde{\mathbf{x}})$ and $\widehat{\theta}(\widehat{\mathbf{x}}) = \widehat{\mathbf{curl}\,\mathbf{v}}(\widehat{\mathbf{x}})$. It is known that, if $\widehat{\Pi}_l$ denotes the edge interpolation on $\widehat{K}_1$, then $\widehat{\Pi}_l\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) = B^t\widetilde{\Pi}_l\widetilde{\mathbf{v}}(\widetilde{\mathbf{x}})$. So, using the first estimate in Theorem 4.2 we have

$$\begin{aligned}\|\widetilde{\Pi}_{l,1}\widetilde{\mathbf{v}}\|_{L^\infty(\widetilde{K})} &= \frac{1}{h_1}\|\widehat{\Pi}_{l,1}\widehat{\mathbf{u}}\|_{L^\infty(\widehat{K})}\\ &\leq C\frac{1}{h_1}\left[\|\hat{v}_1\|_{W^{1,p}(\widehat{K}_1)} + \|\hat{\theta}_2\|_{W^{1,1}(\widehat{K}_1)} + \|\hat{\theta}_3\|_{W^{1,1}(\widehat{K}_1)}\right].\end{aligned}$$

Now, from this, taking into account that $\hat{\theta}_2(\widehat{\mathbf{x}}) = h_1h_3\tilde{\theta}_2(\widetilde{\mathbf{x}})$ and $\hat{\theta}_3(\widehat{\mathbf{x}}) = h_1h_2\tilde{\theta}_3(\widetilde{\mathbf{x}})$, we obtain

$$\begin{aligned}\|\widetilde{\Pi}_{l,1}\widetilde{\mathbf{v}}\|_{L^\infty(\widetilde{K})} &\leq C|\widetilde{K}|^{-\frac{1}{p}}\left(\|\tilde{v}_1\|_{L^p(\widetilde{K})} + \sum_{i=1}^3 h_i\left\|\frac{\partial\tilde{v}_1}{\partial\tilde{x}_i}\right\|_{L^p(\widetilde{K})}\right) +\\ &\qquad C|\widetilde{K}|^{-1}h_3\left(\|\tilde{\theta}_2\|_{L^1(\widetilde{K})} + \sum_{i=1}^3 h_i\left\|\frac{\partial\tilde{\theta}_2}{\partial\tilde{x}_i}\right\|_{L^1(\widetilde{K})}\right) +\\ &\qquad C|\widetilde{K}|^{-1}h_2\left(\|\tilde{\theta}_3\|_{L^1(\widetilde{K})} + \sum_{i=1}^3 h_i\left\|\frac{\partial\tilde{\theta}_3}{\partial\tilde{x}_i}\right\|_{L^1(\widetilde{K})}\right).\end{aligned}$$

The corresponding estimates for $\widetilde{\Pi}_{l,2}\widehat{\mathbf{v}}$ and $\widetilde{\Pi}_{l,2}\widehat{\mathbf{v}}$ can be analogously proved, thus obtaining the assertion. □

Remark 4.1. *By a simple application of Hölder's inequality we obtain from the previous lemma that for all* $\widetilde{\mathbf{v}} \in [W^{1,p}(\widetilde{K})]^3$*,* $p > 2$*, with* $\boldsymbol{curl}\widetilde{\mathbf{v}} \in [W^{1,p}(\widetilde{K})]^3$*, we have*

$$\begin{aligned}\|\widetilde{\Pi}_l\widetilde{\mathbf{v}}\|_{L^p(\widetilde{K})} &\leq C\Bigg\{\|\widetilde{\mathbf{v}}\|_{L^p(\widetilde{K})} + \sum_{i=1}^3 h_i\left\|\frac{\partial\widetilde{\mathbf{v}}}{\partial\tilde{x}_i}\right\|_{L^p(\widetilde{K})}\\ &\qquad + h\left\|\widetilde{\boldsymbol{curl}\mathbf{v}}\right\|_{L^p(\widetilde{K})} + h\sum_{i=1}^3 h_i\left\|\frac{\partial\widetilde{\boldsymbol{curl}\mathbf{v}}}{\partial\tilde{x}_i}\right\|_{L^p(\widetilde{K})}\Bigg\}.\end{aligned}$$

This is an anisotropic inequality, that should be confronted with (8) at the end of section 3.

5. Stability of Π_l on elements in the family $\mathcal{F}_2$

Now we consider the reference element $\widehat{K}_2$ with vertices at $(0,0,0),(1,1,0),(0,1,0)$ and $(0,0,1)$. For $i=1,\dots,4$, let now f_i, be the faces of $\widehat{K}_2$ with normal n_i, where $n_1=(-1,0,0), n_2=\frac{1}{\sqrt{2}}(1,-1,0), n_3=(0,0,-1)$ and $n_4=\frac{1}{\sqrt{2}}(0,1,1)$, and for $i=1,\dots,6$, let $e_i, i=1,\dots,6$, be the edges with tangential vectors $\mathbf{t}_i$, with $\mathbf{t}_1=\frac{1}{\sqrt{2}}(1,1,0), \mathbf{t}_2=(0,1,0), \mathbf{t}_3=(0,0,1), \mathbf{t}_4=\frac{1}{\sqrt{2}}(0,-1,1), \mathbf{t}_5=\frac{1}{\sqrt{3}}(1,1,-1)$ and $\mathbf{t}_6=(1,0,0)$. We alert the reader that we are redefining some notation used in the previous section.

In this section, we denote by Π_l the edge interpolation operator on $\widehat{K}_2$, and as before, $\widehat{\Pi}_l$ the one operator on $\widehat{K}_1$.

We can use the estimates proved in the previous section to obtain results concerning the stability of Π_l on $\widehat{K}_2$. In fact, the reference element $\widehat{K}_2$ is the image of the element $\widehat{K}_1$, considered in the previous section, by the mapping $\widehat{\mathbf{x}}\to\mathbf{x}=A\widehat{\mathbf{x}}$, where

$$A=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}.$$

Let $\mathbf{u}\in[W^{1,p}(\widehat{K}_2)]^3$ $(p>2)$ with $\mathbf{curl}\,\mathbf{u}\in[W^{1,1}(\widehat{K}_2)]^3$, and define $\widehat{\mathbf{u}}\in[W^{1,p}(\widehat{K}_1)]^3$ by setting $\widehat{\mathbf{u}}(\widehat{\mathbf{x}})=A^t\mathbf{u}(\mathbf{x})$. Then, we have $\widehat{\Pi}_l\widehat{\mathbf{u}}(\widehat{\mathbf{x}})=A^t\Pi_l\mathbf{u}(\mathbf{x})$ or

$$\begin{aligned}\Pi_{l,1}\mathbf{u}(\mathbf{x})&=\widehat{\Pi}_{l,1}\widehat{\mathbf{u}}(\widehat{\mathbf{x}})-\widehat{\Pi}_{l,2}\widehat{\mathbf{u}}(\widehat{\mathbf{x}})\\\Pi_{l,2}\mathbf{u}(\mathbf{x})&=\widehat{\Pi}_{l,2}\widehat{\mathbf{u}}(\widehat{\mathbf{x}})\\\Pi_{l,3}\mathbf{u}(\mathbf{x})&=\widehat{\Pi}_{l,3}\widehat{\mathbf{u}}(\widehat{\mathbf{x}}).\end{aligned}$$

We set $\theta(\mathbf{x})=\mathbf{curl}\,\mathbf{u}(\mathbf{x})$ and $\widehat{\theta}(\widehat{\mathbf{x}})=\widehat{\mathbf{curl}}\,\widehat{\mathbf{u}}(\widehat{\mathbf{x}})$. Then, some easy computations show that

$$\begin{aligned}\widehat{\theta}_1(\widehat{\mathbf{x}})&=\theta_1(\mathbf{x})\\\widehat{\theta}_2(\widehat{\mathbf{x}})&=\theta_2(\mathbf{x})-\theta_1(\mathbf{x})\\\widehat{\theta}_3(\widehat{\mathbf{x}})&=\theta_3(\mathbf{x}),\end{aligned}$$

so, we have

$$\begin{aligned}\|\widehat{\theta}_1\|_{L^1(\widehat{K}_1)}&=\|\theta_1\|_{L^1(\widehat{K}_2)}\\\|\widehat{\theta}_2\|_{L^1(\widehat{K}_1)}&\le\|\theta_1\|_{L^1(\widehat{K}_2)}+\|\theta_2\|_{L^1(\widehat{K}_2)}\\\|\widehat{\theta}_3\|_{L^1(\widehat{K}_1)}&=\|\theta_3\|_{L^1(\widehat{K}_2)}\\\sum_{k=1}^3\left(\left\|\frac{\partial\widehat{\theta}_1}{\partial\hat{x}_k}\right\|_{L^1(\widehat{K}_1)}+\left\|\frac{\partial\widehat{\theta}_2}{\partial\hat{x}_k}\right\|_{L^1(\widehat{K}_1)}\right)&\le 4\sum_{k=1}^3\left(\left\|\frac{\partial\theta_1}{\partial x_k}\right\|_{L^1(\widehat{K}_2)}+\left\|\frac{\partial\theta_2}{\partial x_k}\right\|_{L^1(\widehat{K}_2)}\right)\\\sum_{k=1}^3\left(\left\|\frac{\partial\widehat{\theta}_1}{\partial\hat{x}_k}\right\|_{L^1(\widehat{K}_1)}+\left\|\frac{\partial\widehat{\theta}_3}{\partial\hat{x}_k}\right\|_{L^1(\widehat{K}_1)}\right)&\le 2\sum_{k=1}^3\left(\left\|\frac{\partial\theta_1}{\partial x_k}\right\|_{L^1(\widehat{K}_2)}+\left\|\frac{\partial\theta_3}{\partial x_k}\right\|_{L^1(\widehat{K}_2)}\right).\end{aligned}$$

Then, for instance, from the third inequality of Theorem 4.2, we have

$$\begin{aligned}\|\Pi_{l,3}\mathbf{u}\|_{L^\infty(\widehat{K}_2)} &= \left\|\widehat{\Pi}_{l,3}\widehat{\mathbf{u}}\right\|_{L^\infty(\widehat{K}_1)}\\ &\leq C\left(\|\hat{u}_3\|_{W^{1,p}(\widehat{K}_1)} + \left\|\hat{\theta}_1\right\|_{W^{1,1}(\widehat{K}_1)} + \left\|\hat{\theta}_2\right\|_{W^{1,1}(\widehat{K}_1)}\right)\\ &\leq C\left(\|u_3\|_{W^{1,p}(\widehat{K}_2)} + \|\theta_1\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_2)}\right).\end{aligned}$$

Analogously, we have

$$\|\Pi_{l,2}\mathbf{u}\|_{L^\infty(\widehat{K}_2)} \leq C\left(\|u_2\|_{W^{1,p}(\widehat{K}_2)} + \|\theta_1\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)}\right).$$

We can not obtain an analogous (and suitable for our purposes) estimate for the first component of $\Pi_l\mathbf{u}$ by this simple change of variables. For that, we need the next lemma.

Lemma 5.1. *Let v_2 and v_3 be regular functions defined on the triangle $\{(x,y) \in [0,1]^2 : x+y \leq 1\}$. If $\mathbf{v} : \widehat{K}_2 \to \mathbb{R}^3$ is given by $\mathbf{v} = (0, v_2(x_2,x_3), v_3(x_2,x_3))$ then $\Pi_{l,1}\mathbf{v} = 0$.*

Proof. Consider the map $\widehat{\mathbf{x}} \to \mathbf{x} = A\widehat{\mathbf{x}}$ used previously, which applies $\widehat{K}_1$ onto $\widehat{K}_2$. Let $\mathbf{v}$ be as in the statement of the Lemma, and let $\widehat{\mathbf{v}}$ defined by $\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) = A^t\mathbf{v}(\mathbf{x})$. So, we have, $\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) = (v_2(\hat{x}_1+\hat{x}_2,\hat{x}_3), v_2(\hat{x}_1+\hat{x}_2,\hat{x}_3), v_3(\hat{x}_1+\hat{x}_2,\hat{x}_3))$.

Now, let f be the triangle in the plane x_1x_3 with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, and define $\mathbf{r}(x_1,x_3) = (v_2(x_1,x_3), v_3(x_1,x_3))$. Let $\Pi_l^2\mathbf{r}$ be the 2-dimensional edge interpolation of $\mathbf{r}$ of order l, that is, $\Pi_l^2\mathbf{r}$ is the function in $[P_{l-1}]^2 + P_{l-1}(-x_3,x_1)$, verifying

$$\begin{aligned}\int_0^1 \Pi_{l,1}^2\mathbf{r}(x_1,0)\,q(x_1) &= \int_0^1 v_2(x_1,0)\,q(x_1) \qquad \forall q \in P_{l-1}([0,1]),\\ \int_0^1 \Pi_{l,2}^2\mathbf{r}(0,x_3)\,q(x_3) &= \int_0^1 v_3(0,x_3)\,q(x_3) \qquad \forall q \in P_{l-1}([0,1]),\\ \int_0^1 (\Pi_{l,1}^2\mathbf{r} - \Pi_{l,2}^2\mathbf{r})|_{(x_1,1-x_1)}\,q(x_1) &= \int_0^1 (v_2 - v_3)|_{(x_1,1-x_1)}\,q(x_1) \qquad \forall q \in P_{l-1}([0,1]),\\ \int_f \Pi_{l,1}^2\mathbf{r}\,q &= \int_f v_2\,q \qquad \forall q \in P_{l-2}(f)\\ \int_f \Pi_{l,2}^2\mathbf{r}\,q &= \int_f v_3\,q \qquad \forall q \in P_{l-2}(f).\end{aligned}$$

Then, it is easy to check that

$$\widehat{\Pi}_l\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) = (\Pi_{l,1}^2\mathbf{r}(\hat{x}_1+\hat{x}_2,\hat{x}_3), \Pi_{l,1}^2\mathbf{r}(\hat{x}_1+\hat{x}_2,\hat{x}_3), \Pi_{l,2}^2\mathbf{r}(\hat{x}_1+\hat{x}_2,\hat{x}_3)).$$

So, since $\Pi_l\mathbf{v}(\mathbf{x}) = A^{-t}\widehat{\Pi}_l\widehat{\mathbf{v}}(\widehat{\mathbf{x}})$, we obtain $\Pi_{l,1}\mathbf{v} = 0$ as we wanted. □

Now we are ready to estimate $\Pi_{l,1}\mathbf{u}$. As in the proof of Theorem 4.2 we assume the function $\mathbf{u}$ is smooth on $\overline{\widehat{K}_2}$, and obtain the final result by standard density arguments. Define the vector-valued function $\mathbf{w}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}) - u_2(x_2,x_2,x_3), u_2(\mathbf{x}) - u_2(x_2,x_2,x_3))$. Then $w_2 = w_3 = 0$ on f_2. Also the second and third components of $\mathbf{curl}\,\mathbf{u}$ and $\mathbf{curl}\,\mathbf{w}$ coincide.

From the last lemma, we have $\Pi_{l,1}\mathbf{u} = \Pi_{l,1}\mathbf{w}$. Then, we have to obtain estimates for $\Pi_l \mathbf{w}$ in terms of $\mathbf{u}$. As in the previous section, we will use Proposition 2.1. We will prove that

$$|F_{e_i}(\mathbf{w}, q)| \leq C(q)N(\mathbf{u}) \qquad i = 1 \ldots, 6, \ \forall q \in P_{l-1}(e_i), \tag{16}$$

$$|F_{f_i}(\mathbf{w}, \mathbf{q})| \leq C(\mathbf{q})N(\mathbf{u}) \qquad i = 1 \ldots, 4, \ \forall q \in [P_{l-2}(f_i)]^2, \tag{17}$$

$$|F_{\widehat{K}_2}(\mathbf{w}, \mathbf{q})| \leq C(\mathbf{q})N(\mathbf{u}) \qquad \forall \mathbf{q} \in [P_{l-3}(\widehat{K}_2)]^3 \tag{18}$$

with

$$N(\mathbf{u}) = \|u_1\|_{W^{1,p}(\widehat{K}_2)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)}.$$

We begin by proving (16). We have

$$\begin{aligned}
F_{e_1}(\mathbf{w}, q) &= \frac{1}{\sqrt{2}} \int_{e_1} u_1 q \qquad \forall q \in P_{l-1}(e_1) \\
F_{e_2}(\mathbf{w}, q) &= \int_{e_2} (u_2(0, x_2, 0) - u_2(x_2, x_2, 0))q(x_2) \qquad \forall q \in P_{l-1}(e_2) \\
F_{e_3}(\mathbf{w}, q) &= 0 \qquad \forall q \in P_{l-1}(e_3) \\
F_{e_4}(\mathbf{w}, q) &= \frac{1}{\sqrt{2}} \int_{e_4} [(u_2(0, x_2, x_3) - u_2(x_2, x_2, x_3)) - \\
&\qquad (u_3(0, x_2, x_3) - u_3(x_2, x_2, x_3))] \, q(x_2) \qquad \forall q \in P_{l-1}(e_4) \\
F_{e_5}(\mathbf{w}, q) &= \frac{1}{\sqrt{3}} \int_{e_5} u_1(x_2, x_2, x_3) \, q \qquad \forall q \in P_{l-1}(e_5) \\
F_{e_6}(\mathbf{w}, q) &= \int_{e_6} u_1(x_1, 1, 0) \, q \qquad \forall q \in P_{l-1}(e_6).
\end{aligned}$$

For $i = 1, 3, 5, 6$ inequalities (16) are trivially obtained. Now consider $i = 2$. We have

$$\begin{aligned}
\sqrt{2}\, F_{e_2}(\mathbf{w}, q) &= -\int_0^1 \int_0^{x_2} \frac{\partial u_2}{\partial x_1}(t, x_2, 0)q(x_2) \, dt \, dx_2 \\
&= -\int_0^1 \int_0^{x_2} \left[\frac{\partial u_2}{\partial x_1}(t, x_2, 0) - \frac{\partial u_1}{\partial x_2}(t, x_2, 0) \right] q(x_2) \, dt \, dx_2 - \\
&\qquad \int_0^1 \int_0^{x_2} \frac{\partial u_1}{\partial x_2}(t, x_2, 0)q(x_2) \, dt \, dx_2
\end{aligned}$$

For the second term in the last line, we have

$$\begin{aligned}
\int_0^1 \int_0^{x_2} \frac{\partial u_1}{\partial x_2}(t, x_2, 0)q(x_2) \, dt \, dx_2 &= \int_0^1 \int_0^{x_2} \frac{\partial u_1}{\partial x_2}(x_1, x_2, 0)q(x_2) \, dx_1 \, dx_2 \\
&= \int_0^1 \int_{x_1}^1 \frac{\partial u_1}{\partial x_2}(x_1, x_2, 0)q(x_2) \, dx_2 \, dx_1 \\
&= \int_0^1 [u_1(x_1, 1, 0)q(1) - u_1(x_1, x_1, 0)q(x_1)] \, dx_1 - \\
&\qquad \int_0^1 \int_{x_1}^1 u_1(x_1, x_2, 0)q'(x_2) \, dx_2 \, dx_1.
\end{aligned}$$

Then

$$\left|\int_0^1\int_0^{x_2}\frac{\partial u_1}{\partial x_2}(t,x_2,0)q(x_2)\,dt\,dx_2\right| \leq C(q)\left(\|u_1\|_{L^p(f_3)}+\|u_1\|_{L^p(e_1)}+\|u_1\|_{L^p(e_6)}\right) \leq C(q)\|u_1\|_{W^{1,p}(\widehat{K}_2)}.$$

So, we obtain

$$|F_{e_2}(\mathbf{w},q)| \leq C(q)\left(\|\theta_3\|_{W^{1,1}(\widehat{K}_2)}+\|u_1\|_{W^{1,p}(\widehat{K}_2)}\right).$$

Now consider the case $i=4$. We write

$$\begin{aligned}\sqrt{2}\,F_{e_4}(\mathbf{w},q) &= \int_{e_4}(u_2(0,x_2,x_3)-u_2(x_2,x_2,x_3))\,q(x_2) - \\ &\qquad \int_{e_4}(u_3(0,x_2,x_3)-u_3(x_2,x_2,x_3))\,q(x_2) \\ &=: I_{41}^e - I_{42}^e.\end{aligned}$$

We have

$$\begin{aligned} I_{41}^e &= -\int_0^1\int_0^{x_2}\frac{\partial u_2}{\partial x_1}(t,x_2,1-x_2)q(x_2)\,dt\,dx_2 \\ &= -\int_0^1\int_0^{x_2}\left[\frac{\partial u_2}{\partial x_1}(t,x_2,1-x_2)-\frac{\partial u_1}{\partial x_2}(t,x_2,1-x_2)\right]q(x_2)\,dt\,dx_2 - \\ &\quad -\int_0^1\int_0^{x_2}\frac{\partial u_1}{\partial x_2}(t,x_2,1-x_2)q(x_2)\,dt\,dx_2 \\ &= -\int_0^1\int_0^{x_2}\left[\frac{\partial u_2}{\partial x_1}(t,x_2,1-x_2)-\frac{\partial u_1}{\partial x_2}(t,x_2,1-x_2)\right]q(x_2)\,dt\,dx_2 - \\ &\quad -\int_0^1\int_{x_1}^1\frac{\partial u_1}{\partial x_2}(t,x_2,1-x_2)q(x_2)\,dx_2\,dt \end{aligned}$$

Analogously for I_{42}^e we have

$$\begin{aligned} I_{42}^e = &= -\int_0^1\int_0^{x_2}\frac{\partial u_3}{\partial x_1}(t,x_2,1-x_2)q(x_2)\,dt\,dx_2 \\ &= -\int_0^1\int_0^{x_2}\left[\frac{\partial u_3}{\partial x_1}(t,x_2,1-x_2)-\frac{\partial u_1}{\partial x_3}(t,x_2,1-x_2)\right]q(x_2)\,dt\,dx_2 - \\ &\quad -\int_0^1\int_{x_1}^1\frac{\partial u_1}{\partial x_3}(t,x_2,1-x_2)q(x_2)\,dx_2\,dt \end{aligned}$$

So, we arrive at

$$\begin{aligned} F_{e_4}(\mathbf{w},q) &= -\int_{f_4}(\theta_3+\theta_2)\,q(x_2) - \\ &\qquad \int_{f_4}\left[\frac{\partial u_1}{\partial x_2}(x_1,x_2,1-x_2)-\frac{\partial u_1}{\partial x_3}(x_1,x_2,1-x_2)\right]q(x_2) \\ &=: -\int_{f_4}(\theta_3+\theta_2)\,q(x_2) - I_{43}^e. \end{aligned}$$

But for I^e_{43} we have

$$\begin{aligned} I^e_{43} &= \int_0^1 \int_{x_1}^1 \frac{d}{dx_2}[u_1(x_1,x_2,1-x_2)]\, q(x_2)\, dx_2\, dx_1 \\ &= \int_0^1 [u_1(x_1,1,0)q(1) - u_1(x_1,x_1,1-x_1)q(x_1)]dx_1 - \\ &\quad - \int_0^1 \int_{x_1}^1 u_1(x_1,x_2,1-x_2)q'(x_2)\, dx_2\, dx_1. \end{aligned}$$

So

$$F_{e_4}(\mathbf{w},q) = -\int_{f_4} (\theta_2+\theta_3)q(x_2) + \int_{f_4} u_1\, q'(x_2) - q(1)\int_{e_6} u_1 + \int_{e_5} u_1\, q,$$

and therefore

$$|F_{e_4}(\mathbf{w},q)| \le C(q)\left(\|\theta_2\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)}\right).$$

We proved (16).

Now we consider the face conditions. We have for $\mathbf{q} = (q_1,q_2) \in [P_{l-2}(f_1)]^2$

$$F_{f_1}(\mathbf{w},\mathbf{q}) = I^f_{11} + I^f_{12},$$

where

$$\begin{aligned} I^f_{11} &:= \int_{f_1} [u_3(0,x_2,x_3) - u_3(x_2,x_2,x_3)]\, q_1(x_2,x_3)\, dx_2 dx_3 \\ I^f_{12} &:= \int_{f_1} [u_2(0,x_2,x_3) - u_2(x_2,x_2,x_3)]\, q_2(x_2,x_3)\, dx_2 dx_3 \end{aligned}$$

It follows that

$$\begin{aligned} I^f_{11} &= \int_{f_1} [u_3(0,x_2,x_3) - u_3(x_2,x_2,x_3)]\, q_1(x_2,x_3)\, dx_2 dx_3 \\ &= -\int_{f_1}\int_0^{x_2} \frac{\partial u_3}{\partial x_1}(x_1,x_2,x_3) q_1(x_2,x_3)\, dx_1\, dx_2 dx_3 \\ &= -\int_{\widehat{K}_2} \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) q_1(x_1,x_3) d\mathbf{x} - \int_{\widehat{K}_2} \frac{\partial u_1}{\partial x_3} q_1(x_1,x_3) d\mathbf{x} \end{aligned}$$

So we obtain

$$|I^f_{11}| \le C(\mathbf{q})\left(\|\theta_2\|_{L^1(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)}\right).$$

Analogously, we have

$$|I^f_{12}| \le C(\mathbf{q})\left(\|\theta_3\|_{L^1(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)}\right),$$

and so

$$|F_{f_1}(\mathbf{w},\mathbf{q})| \le C(\mathbf{q})\left(\|\theta_2\|_{L^1(\widehat{K}_2)} + \|\theta_3\|_{L^1(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)}\right).$$

Taking into account that w_2 and w_3 vanish on f_2, we have for $\mathbf{q} = (q_1,q_2) \in [P_{l-2}(f_2)]^2$

$$F_{f_2}(\mathbf{w},\mathbf{q}) = \frac{1}{\sqrt{2}} \int_{f_2} u_1\, q_2.$$

We see that the right side can be bounded in terms of $\|u_1\|_{W^{1,p}(\widehat{K}_2)}$.

On f_3 we have for $\mathbf{q} = (q_1, q_2) \in [P_{l-2}(f_3)]^2$

$$\begin{aligned} F_{f_3}(\mathbf{w}, \mathbf{q}) &= -\int_{f_3} (u_2(x_1, x_2, 0) - u_2(x_2, x_2, 0))\; q_1 + \int_{f_3} u_1(x_1, x_2, 0)\, q_2 \\ &=: -I_{31}^f + I_{32}^f. \end{aligned}$$

We have

$$\begin{aligned} I_{31}^f &:= -\int_0^1 \int_0^{x_2} \int_{x_1}^{x_2} \frac{\partial u_2}{\partial x_1}(t, x_2, 0)\, q_1(x_1, x_2)\, dt\, dx_1 dx_2 \\ &= -\int_0^1 \int_0^{x_2} \int_{x_1}^{x_2} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)\Bigg|_{(t,x_2,0)} q_1(x_1, x_2)\, dt dx_1 dx_2 - \\ &\qquad \int_0^1 \int_0^{x_2} \int_{x_1}^{x_2} \frac{\partial u_1}{\partial x_2}(t, x_2, 0) q_1(x_1, x_2)\, dt dx_1 dx_2 \\ &= -\int_0^1 \int_0^{x_2} \int_0^{t} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)\Bigg|_{(t,x_2,0)} q_1(x_1, x_2)\, dx_1 dt dx_2 - \\ &\qquad \int_0^1 \int_0^{x_2} \int_0^{t} \frac{\partial u_1}{\partial x_2}(t, x_2, 0) q_1(x_1, x_2)\, dx_1 dt dx_2 \\ &= -\int_{f_3} \theta_3\, \widetilde{q}_1 - \int_{f_3} \frac{\partial u_1}{\partial x_2} \widetilde{q}_1 \end{aligned}$$

where

$$\widetilde{q}_1(t, x_2) = \int_0^t q_1(x_1, x_2)\, dx_1. \tag{19}$$

On the other hand we have

$$\begin{aligned} \int_{f_3} \frac{\partial u_1}{\partial x_2} \widetilde{q}_1 &= \int_0^1 \int_{x_1}^1 \frac{\partial u_1}{\partial x_2}(x_1, x_2, 0)\, \widetilde{q}_1(x_1, x_2)\, dx_2 dx_1 \\ &= \int_0^1 \Big[\, u_1(x_1, x_2, 0) \widetilde{q}_1(x_1, x_2)\big|_{x_2 = x_1}^1\, dx_1 - \\ &\qquad \int_{x_1}^1 u_1(x_1, x_2, 0) \frac{d}{dx_2} \widetilde{q}_1(x_1, x_2)\, dx_2 \Big] dx_1 \\ &= \int_0^1 [u_1(x_1, 1, 0) \widetilde{q}_1(x_1, 1) - u_1(x_1, x_1, 0) \widetilde{q}_1(x_1, x_1)] dx_1 - \\ &\qquad \int_0^1 \int_{x_1}^1 u_1(x_1, x_2, 0) \frac{\partial \widetilde{q}_1}{\partial x_2}(x_1, x_2)\, dx_1 dx_1. \end{aligned}$$

Therefore we arrive at

$$|I_{31}^f| \le C(\mathbf{q}) \left(\|\theta_3\|_{W^{1,1}(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)} \right).$$

Also, we clearly have

$$|I_{32}^f| \le C(\mathbf{q}) \|u_1\|_{W^{1,p}(\widehat{K}_2)},$$

and by collecting the previous inequalities we obtain

$$|F_{f_3}(\mathbf{w}, \mathbf{q})| \le C(\mathbf{q}) \left(\|\theta_3\|_{W^{1,1}(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)} \right).$$

Finally, we have for $\mathbf{q} \in [P_{l-2}(f_4)]^2$

$$F_{f_4}(\mathbf{w},\mathbf{q}) = \int_{f_4}(w_2 - w_3)q_1 + \int_{f_4} u_1 q_2$$

We set

$$\begin{aligned} I^f_{41} &:= \int_{f_4}(w_2 - w_3)q_1 \\ I^f_{42} &:= \int_{f_4} u_1 q_2 \end{aligned}$$

Clearly

$$|I^f_{42}| \le C(\mathbf{q})\|u_1\|_{W^{1,p}(\widehat{K}_2)}.$$

Now we deal with I^f_{41}. We have

$$\begin{aligned} \int_{f_4} w_2 q_1 &= \int_0^1\int_0^{x_2}[u_2(x_1,x_2,1-x_2) - u_2(x_2,x_2,1-x_2)]q_1(x_1,x_2)\,dx_1dx_2 \\ &= -\int_0^1\int_0^{x_2}\int_{x_1}^{x_2}\frac{\partial u_2}{\partial x_1}(t,x_2,1-x_2)q_1(x_1,x_2)\,dt\,dx_1dx_2 \\ &= -\int_0^1\int_0^{x_2}\int_{x_1}^{x_2}\theta_3(t,x_2,1-x_2)q_1(x_1,x_2)\,dt\,dx_1dx_2 - \\ &\qquad \int_0^1\int_0^{x_2}\int_{x_1}^{x_2}\frac{\partial u_1}{\partial x_2}(t,x_2,1-x_2)q_1(x_1,x_2)\,dt\,dx_1dx_2 \\ &= -\int_{f_4}\theta_3\widetilde{q}_1 - \int_{f_4}\frac{\partial u_1}{\partial x_2}\widetilde{q}_1 \end{aligned}$$

with $\widetilde{q}_1$ defined by (19). Analogously we have

$$\int_{f_4} w_3\,q_1 = \int_{f_4}\theta_2\widetilde{q}_1 - \int_{f_4}\frac{\partial u_1}{\partial x_3}\widetilde{q}_1$$

So,

$$I^f_{41} = -\int_{f_4}(\theta_2+\theta_3)\widetilde{q}_1 - \int_{f_4}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_1}{\partial x_3}\right)\widetilde{q}_1.$$

But

$$\begin{aligned} \int_{f_4}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_1}{\partial x_3}\right)\widetilde{q}_1 &= \int_0^1\int_{x_1}^1\frac{d}{dx_2}[u_1(x_1,x_2,1-x_2)]\widetilde{q}(x_1,x_2)\,dx_2dx_1 \\ &= \int_0^1[u_1(x_1,1,0)\widetilde{q}_1(x_1,1) - u_1(x_1,x_1,1-x_1)]\widetilde{q}_1(x_1,x_1)dx_1 - \\ &\qquad \int_{f_4} u_1\frac{\partial\widetilde{q}_1}{\partial x_2} \end{aligned}$$

and so,

$$\left|\int_{f_4}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_1}{\partial x_3}\right)\widetilde{q}_1\right| \le C(\mathbf{q})\,\|u_1\|_{W^{1,p}(\widehat{K}_2)}$$

and then we can conclude that

$$|I^f_{41}| \le C(\mathbf{q})\left(\|\theta_2\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)}\right),$$

obtaining

$$F_{f_4}(w,\mathbf{q}) \le C(\mathbf{q}) \left(\|\theta_2\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)} \right).$$

Thus we proved (17).

Finally, we consider the volume conditions. We have for $\mathbf{q} = (q_1, q_2, q_3) \in [P_{l-3}(\widehat{K}_2)]^3$

$$\begin{aligned} F_{\widehat{K}_2}(\mathbf{w},\mathbf{q}) &= \int_{\widehat{K}_2} u_1\, q_1 + \int_{\widehat{K}_2} w_2\, q_2 + \int_{\widehat{K}_2} w_3\, q_3 \\ &=: I_1^V + I_2^V + I_3^V. \end{aligned}$$

Clearly

$$|I_1^V| \le C(\mathbf{q}) \|u_1\|_{L^p(\widehat{K}_2)}.$$

On the other hand we have

$$\begin{aligned} I_2^V &= \int_{\widehat{K}_2} (u_2(x_1,x_2,x_3) - u_2(x_2,x_2,x_3)) q_2 \\ &= -\int_{\widehat{K}_2} \int_{x_1}^{x_2} \frac{\partial u_2}{\partial x_1}(t,x_2,x_3) q_2(x_1,x_2,x_3)\, dt\, dx_1\, dx_2\, dx_3 \\ &= -\int_0^1 \int_0^{1-x_2} \int_0^{x_2} \int_{x_1}^{x_2} \frac{\partial u_2}{\partial x_1}(t,x_2,x_3) q_2(x_1,x_2,x_3)\, dt\, dx_1\, dx_3\, dx_2 \\ &= -\int_0^1 \int_0^{1-x_2} \int_0^{x_2} \int_0^{t} \frac{\partial u_2}{\partial x_1}(t,x_2,x_3) q_2(x_1,x_2,x_3)\, dx_1\, dt\, dx_3\, dx_2 \\ &= \int_{\widehat{K}_2} \frac{\partial u_2}{\partial x_1}(t,x_2,x_3) \bar{q}_2(t,x_2,x_3)\, dt\, dx_3\, dx_2 \\ &= \int_{\widehat{K}_2} \theta_3 \bar{q}_2 + \int_{\widehat{K}_2} \frac{\partial u_1}{\partial x_2} \bar{q}_2 \end{aligned}$$

where

$$\bar{q}_2(t,x_2,x_3) = \int_0^t q_2(x_1,x_2,x_3)\, dx_1.$$

So, we see that

$$|I_2^V| \le C(\mathbf{q}) \left(\|\theta_3\|_{L^1(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)} \right).$$

Similarly we can obtain

$$|I_3^V| \le C(\mathbf{q}) \left(\|\theta_2\|_{L^1(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)} \right).$$

Then, we obtained

$$F_{\widehat{K}_2}(\mathbf{w},\mathbf{q}) \le C(\mathbf{q}) \left(\|\theta_2\|_{L^1(\widehat{K}_2)} + \|\theta_3\|_{L^1(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)} \right)$$

proving (18).

By Proposition 2.1 and inequalities (16)-(18), we have

$$\begin{aligned} \|\Pi_{l,1}\mathbf{u}\|_{L^\infty(\widehat{K}_2)} &\le \|\Pi_l \mathbf{w}\|_{L^\infty(\widehat{K}_2)} \\ &\le C \left(\|\theta_2\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)} + \|u_1\|_{W^{1,p}(\widehat{K}_2)} \right) \end{aligned}$$

Then the following theorem is proved.

Theorem 5.2. *Let $\mathbf{u} \in [W^{1,p}(\widehat{K}_2)]^3$, with $p > 2$, such that $\boldsymbol{curl}\,\mathbf{u} \in [W^{1,1}(\widehat{K}_2)]^3$. Denote $\boldsymbol{curl}\,\mathbf{u}$ by $\theta = (\theta_1, \theta_2, \theta_3)$. Then we have*

$$\begin{aligned}
\|\Pi_{l,1}\mathbf{u}\|_{L^\infty(\widehat{K}_2)} &\leq C\left(\|u_1\|_{W^{1,p}(\widehat{K}_2)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)}\right)\\
\|\Pi_{l,2}\mathbf{u}\|_{L^\infty(\widehat{K}_2)} &\leq C\left(\|u_2\|_{W^{1,p}(\widehat{K}_2)} + \|\theta_1\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_3\|_{W^{1,1}(\widehat{K}_2)}\right)\\
\|\Pi_{l,3}\mathbf{u}\|_{L^\infty(\widehat{K}_2)} &\leq C\left(\|u_3\|_{W^{1,p}(\widehat{K}_2)} + \|\theta_1\|_{W^{1,1}(\widehat{K}_2)} + \|\theta_2\|_{W^{1,1}(\widehat{K}_2)}\right)
\end{aligned}$$

Now we state the main theorem of this section.

Theorem 5.3. *Let $\widetilde{K} \in \mathcal{F}_2$ be the tetrahedron generated by $\{\mathbf{0}, h_1\mathbf{e}_1+h_2\mathbf{e}_2, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}$. Denote by $\widetilde{\Pi}_l$ the edge interpolation operator of order l in $\widetilde{K}$. Then, there exists a constant C independent of h_1, h_2 and h_3 such that for all $\tilde{\mathbf{v}} \in [W^{1,p}(\widetilde{K})]^3$, $p > 2$, with $\boldsymbol{curl}\,\tilde{\mathbf{v}} \in [W^{1,1}(\widetilde{K})]^3$, we have*

$$\begin{aligned}
\|\widetilde{\Pi}_l\tilde{\mathbf{v}}\|_{L^\infty(\widetilde{K})} \leq\; & C\Bigg\{|\widetilde{K}|^{-\frac{1}{p}}\left(\|\tilde{\mathbf{v}}\|_{L^p(\widetilde{K})} + \sum_{i=1}^{3} h_i \left\|\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{x}_i}\right\|_{L^p(\widetilde{K})}\right)\\
&+ h\,|\widetilde{K}|^{-1}\left(\left\|\widetilde{\boldsymbol{curl}}\,\tilde{\mathbf{v}}\right\|_{L^1(\widetilde{K})} + \sum_{i=1}^{3} h_i \left\|\frac{\partial\widetilde{\boldsymbol{curl}}\,\tilde{\mathbf{v}}}{\partial\tilde{x}_i}\right\|_{L^1(\widetilde{K})}\right)\Bigg\}
\end{aligned}$$

where h is the diameter of $\widetilde{K}$.

Proof. The proof follows by rescaling arguments like the ones used in Theorem 4.3, considering the map $\mathbf{x} \to \tilde{\mathbf{x}} = B\mathbf{x}$ with $B = diag(h_1, h_2, h_3)$ that applies $\widehat{K}_2$ onto $\widetilde{K}$. So, we omit the details here. ☐

6. Interpolation error estimates

In this section we give optimal error estimates for edge interpolation of any order. These estimates are derived from the stability results obtained in the previous sections combined with polynomial approximation results.

Let us recall some well known results concerning the approximation of functions in Sobolev spaces by averaged Taylor polynomials, that have been obtained in [12] (see also [7, 13]). For a convex domain $U \subset \mathbb{R}^3$ and any non-negative integer m, given $g \in W^{m+1,p}(U)$ we consider the averaged Taylor polynomial

$$Q_m g(\mathbf{x}) = \frac{1}{|U|}\int_U T_m g(\mathbf{y}, \mathbf{x})\, d\mathbf{y}\,,$$

where

$$T_m g(\mathbf{y}, \mathbf{x}) = \sum_{|\alpha| \leq m} D^\alpha g(\mathbf{y}) \frac{(\mathbf{x} - \mathbf{y})^\alpha}{\alpha!},$$

where, for a function $g = g(\mathbf{x})$ and a multi-index α, $D^\alpha g$ indicates the derivative $\frac{\partial^{|\alpha|} g}{\partial \mathbf{x}^\alpha}$. We remark that in [12], the average is taken with a regular compactly supported weight function ϕ, while for simplicity, here we have taken $\phi = \frac{\chi_U}{|U|}$ (χ_U is the characteristic function of the convex set U). The proofs of the next results given in [12] are not affected by this change.

The following equality holds: if $|\beta| \leq m$ then

$$D^\beta(Q_m g) = Q_{m-|\beta|}\left(D^\beta g\right). \tag{20}$$

The following result is contained in Theorem 3.2 of [12]: let $m \geq 0$ and $p, \bar{p} \in [1, \infty]$. Suppose that

$$\frac{1}{\bar{p}} - \frac{1}{p} + \frac{m+1}{3} \geq 0 \tag{21}$$

and that there exists σ with

$$0 < \sigma \leq \max\left\{ \left\lceil \frac{m+1}{3} \right\rceil, \frac{1}{\bar{p}} - \frac{1}{p} + \frac{m+1}{3}, \min\left\{1 - \frac{1}{p}, \frac{1}{\bar{p}}\right\}\right\}, \tag{22}$$

where $\lceil a \rceil$ denotes the largest integer less than or equal a. Then, there exists a constant C depending on m, σ and U such that for all $g \in W^{m+1,p}(U)$ we have

$$\|(g - Q_m g)\|_{L^{\bar{p}}(U)} \leq C |g|_{W^{m+1,p}(U)}. \tag{23}$$

Now we collect some properties of the averaged polynomials on elements in $\mathcal{F}_1 \cup \mathcal{F}_2$. Of course, some of these properties hold also on more general domains, but we will use them only on those elements.

Let $\widetilde{K} \in \mathcal{F}_1 \cup \mathcal{F}_2$, and consider a function $\widetilde{\mathbf{u}} \in [W^{m+1,p}(\widetilde{T})]^3$. Define $\widetilde{\mathbf{q}} = \widetilde{\mathbf{Q}}_m(\widetilde{\mathbf{u}})$ where

$$\widetilde{\mathbf{Q}}_m(\widetilde{\mathbf{u}}) = (\tilde{Q}_m \tilde{u}_1, \tilde{Q}_m \tilde{u}_2, \tilde{Q}_m \tilde{u}_3) \in [P_m(\widetilde{T})]^3,$$

where $\tilde{Q}_m(g)$ denotes the averaged Taylor polynomial of $g \in W^{m+1,p}(\widetilde{K})$ on $\widetilde{K}$ of degree m. Then, by rescaling inequalities (23) on $U = \widehat{K}_1$ or $U = \widehat{K}_2$, if $m, p, \bar{p}$ and σ satisfy (21)-(22), we obtain

$$\|\widetilde{\mathbf{u}} - \widetilde{\mathbf{q}}\|_{L^{\bar{p}}(\widetilde{K})} \leq C\, |\widetilde{K}|^{\frac{1}{\bar{p}} - \frac{1}{p}} \sum_{i_1+i_2+i_3=m+1} h_1^{i_1} h_2^{i_2} h_3^{i_3} \left\| \frac{\partial^{m+1} \widetilde{\mathbf{u}}}{\partial \tilde{x}_1^{i_1} \partial \tilde{x}_2^{i_2} \partial \tilde{x}_3^{i_3}} \right\|_{L^p(\widetilde{K})} \tag{24}$$

with C depending only on m, σ and $\widehat{K}_1$ or $\widehat{K}_2$. Similarly, using also property (20), if $m \geq 1$, $p, \bar{p}$ and σ satisfy

$$\frac{1}{\bar{p}} - \frac{1}{p} + \frac{m}{3} \geq 0 \tag{25}$$

and

$$0 < \sigma \leq \max\left\{ \left\lceil \frac{m}{3} \right\rceil, \frac{1}{\bar{p}} - \frac{1}{p} + \frac{m}{3}, \min\left\{1 - \frac{1}{p}, \frac{1}{\bar{p}}\right\}\right\}, \tag{26}$$

then there exist C as before, such that

$$\left\| \frac{\partial(\widetilde{\mathbf{u}} - \widetilde{\mathbf{q}})}{\partial \tilde{x}_1} \right\|_{L^{\bar{p}}(\widetilde{K})} \leq C |\widetilde{K}|^{\frac{1}{\bar{p}} - \frac{1}{p}} \sum_{i_1+i_2+i_3=m} h_1^{i_1} h_2^{i_2} h_3^{i_3} \left\| \frac{\partial^{m+1} \widetilde{\mathbf{u}}}{\partial \tilde{x}_1^{i_1+1} \partial \tilde{x}_2^{i_2} \partial \tilde{x}_3^{i_3}} \right\|_{L^p(\widetilde{K})} \tag{27}$$

and the corresponding inequalities for the derivatives with respect to $\tilde{x}_2$ and $\tilde{x}_3$.

Additionally, since

$$\widetilde{\mathbf{curl}}\,(\widetilde{\mathbf{Q}}_m \widetilde{\mathbf{u}}) = \widetilde{\mathbf{Q}}_{m-1}(\widetilde{\mathbf{curl}}\,\widetilde{\mathbf{u}})$$

if $m \geq 1$ and $p \geq 1$ the following inequality holds:

$$\|\widetilde{\mathbf{curl}}\,(\widetilde{\mathbf{u}} - \widetilde{\mathbf{q}})\|_{L^1(\widetilde{K})} \leq C h^m |\widetilde{K}|^{1-\frac{1}{p}} \|\tilde{D}^m \widetilde{\mathbf{curl}}\,\widetilde{\mathbf{u}}\|_{L^p(\widetilde{K})}, \tag{28}$$

with h the diameter of $\widetilde{K}$, and where for a natural number m and a function g, we are denoting by $D^m g$ the sum of the absolute values of all the derivatives of order m of g ($D^1 = D$).

Finally, if $m \geq 1$ and $p \geq 1$ we have

$$\left\| \frac{\partial}{\partial \tilde{x}_i} \widetilde{\mathbf{curl}}\,(\tilde{\mathbf{u}} - \tilde{\mathbf{q}}) \right\|_{L^1(\widetilde{K})} \leq C h^{m-1} |\widetilde{K}|^{1-\frac{1}{p}} \|\tilde{D}^m \widetilde{\mathbf{curl}}\,\tilde{\mathbf{u}}\|_{L^p(\widetilde{K})} \tag{29}$$

with C depending only on m, σ and on the references elements. This inequality follows by applying the estimates just presented, when $m \geq 2$, or from Hölder's inequality, when $m = 1$.

Now, we state and prove the main theorem of this article.

Theorem 6.1. *Let $l \geq 1$. Let K be a tetrahedron satisfying MAC($\bar{\psi}$). There exist three edges of K, $\ell_i, i = 1, 2, 3$, and a constant C, such that for each integer $0 \leq m \leq l - 1$ we have*

1. *if m and p satisfy either $m \geq 2$ and $p \geq 1$ or $m = 1$ and $p > \frac{6}{5}$, then for all $\mathbf{u} \in [W^{m+1,p}(K)]^3$ we have*

$$\|\mathbf{u} - \Pi_l \mathbf{u}\|_{L^p(K)} \leq C \left\{ \sum_{i+j+k=m+1} h_1^i h_2^j h_3^k \left\| \frac{\partial^{m+1} \mathbf{u}}{\partial \xi_1^i \partial \xi_2^j \partial \xi_3^k} \right\|_{L^p(K)} + h^{m+1} \|D^m \boldsymbol{curl}\,\mathbf{u}\|_{L^p(K)} \right\}, \tag{30}$$

2. *if $m = 0$ and $p > 2$, then for all $\mathbf{u} \in [W^{1,p}(K)]^3$ with $D\boldsymbol{curl}\,\mathbf{u} \in [L^s(K)]^3$ for some $s \geq 1$, we have*

$$\|\mathbf{u} - \Pi_l \mathbf{u}\|_{L^p(K)} \leq C \left\{ \sum_{i=1}^{3} h_i \left\| \frac{\partial \mathbf{u}}{\partial \xi_i} \right\|_{L^p(K)} + h \|\boldsymbol{curl}\,\mathbf{u}\|_{L^p(K)} + h^2 |K|^{\frac{1}{p} - \frac{1}{s}} \|D\boldsymbol{curl}\,\mathbf{u}\|_{L^s(K)} \right\}, \tag{31}$$

where h_i denotes the lengths of ℓ_i, $\xi_i = \ell_i / \|\ell_i\|$, $i = 1, 2, 3$, and h is the diameter of K. The constant C depends only on $\bar{\psi}$, l and p, and it is independent of the function $\mathbf{u}$. Furthermore, C can be chosen such that, in addition, if $M \in \mathbb{R}^{3\times3}$ is the matrix made up of ξ_i as columns, then $\|M\|, \|M^{-1}\| \leq C$.

Remark 6.1. *The last sentence in the Theorem implies that* $\det M$ *is bounded bellow in terms of the constant C. It follows that the directions ξ_1, ξ_2 and ξ_3 are "uniformly" linearly independent.*

Proof. From Theorem 3.2 we know that there exists an element $\widetilde{K} \in \mathcal{F}_1 \cup \mathcal{F}_2$ that can be mapped onto K by an affine transformation $\tilde{\mathbf{x}} \to \mathbf{x} = M\tilde{\mathbf{x}} + \mathbf{p_0}$ with $\|M\|, \|M^{-1}\| \leq C$, where C depends only on $\bar{\psi}$. The matrix M is made up of vectors ξ_i, $i = 1, 2, 3$, as it columns, where ξ_i are unitary vectors in the directions of three edges of K, ℓ_i, of lengths h_i, $i = 1, 2, 3$. Also we can assume that $\widetilde{K}$ is the tetrahedron with vertices at either $\{\mathbf{0}, h_1\mathbf{e}_1, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}$ or $\{\mathbf{0}, h_1\mathbf{e}_1 + h_2\mathbf{e}_2, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}$.

Assume $m \geq 1$. From the conditions on m and p we have $\frac{1}{p} - \frac{m}{3} < \frac{1}{2}$, therefore we can always choose $\bar{p} > 2$ and $\sigma > 0$ such that conditions (21), (22), (25) and (26) are verified. From the Sobolev's embedding theorems we know that $W^{m+1,p}(K) \hookrightarrow W^{1,\bar{p}}(K)$. In particular, since $\bar{p} > 2$, $\Pi_l \mathbf{u}$ is well defined for all $\mathbf{u} \in [W^{m+1,p}(K)]^3$.

Define on $\widetilde{K}$ the function $\tilde{\mathbf{u}}$ by $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = M^t \mathbf{u}(\mathbf{x})$. Let $\tilde{\mathbf{q}} = (\widetilde{Q}_m(\tilde{u}_1), \widetilde{Q}_m(\tilde{u}_2), \widetilde{Q}_m(\tilde{u}_3)) \in [P_m(\widetilde{K})]^3$. Finally we set $\mathbf{q}(\mathbf{x}) = M^{-t}\tilde{\mathbf{q}}(\tilde{\mathbf{x}})$, $\mathbf{x} \in K$.

We have

$$\|\mathbf{u} - \Pi_l \mathbf{u}\|_{L^p(K)} \leq \|\mathbf{u} - \mathbf{q}\|_{L^p(K)} + \|\Pi_l(\mathbf{u} - \mathbf{q})\|_{L^p(K)}.$$

Now

$$\begin{aligned}\|\Pi_l(\mathbf{u}-\mathbf{q})\|_{L^p(K)} &= |M|^{\frac{1}{p}}\left(\int_{\widetilde{K}}\left|M^{-t}\widetilde{\Pi}_l(\widetilde{\mathbf{u}}-\widetilde{\mathbf{q}})|_{\widetilde{\mathbf{x}}}\right|^p d\widetilde{\mathbf{x}}\right)^{\frac{1}{p}}\\ &\le |M|^{\frac{1}{p}}\|M^{-1}\|\left\|\widetilde{\Pi}_l(\widetilde{\mathbf{u}}-\widetilde{\mathbf{q}})\right\|_{L^p(\widetilde{K})}\\ &\le C|M|^{\frac{1}{p}}\|M^{-1}\||\widetilde{K}|^{\frac{1}{p}}\left\|\widetilde{\Pi}_l(\widetilde{\mathbf{u}}-\widetilde{\mathbf{q}})\right\|_{L^\infty(\widetilde{K})}.\end{aligned}$$

where we have used an inverse inequality. Here, $\widetilde{\Pi}_l$ is the edge interpolation operator on $\widetilde{K}$ of order l. Since $\widetilde{\mathbf{u}} \in [W^{1,\bar{p}}(\widetilde{K})]^3$ ($\bar{p} > 2$) with $\widetilde{\mathbf{curl}}\,\widetilde{\mathbf{u}} \in [W^{1,1}(\widetilde{K})]^3$ we use Theorem 4.3 if $\widetilde{K} \in \mathcal{F}_1$ or Theorem 5.3 if $\widetilde{K} \in \mathcal{F}_2$, to obtain

$$\begin{aligned}\|\mathbf{u}-\Pi_l\mathbf{u}\|_{L^p(K)} \le\; & C|M|^{\frac{1}{p}}\|M^{-1}\||\widetilde{K}|^{\frac{1}{p}}\\ &\times\Bigg\{|\widetilde{K}|^{-\frac{1}{\bar{p}}}\left(\|\widetilde{u}-\widetilde{\mathbf{q}}\|_{L^{\bar{p}}(\widetilde{K})}+\sum_{i=1}^{3}h_i\left\|\frac{\partial(\widetilde{\mathbf{u}}-\widetilde{\mathbf{q}})}{\partial\tilde{x}_i}\right\|_{L^{\bar{p}}(\widetilde{K})}\right)\\ &+h|\widetilde{K}|^{-1}\left(\left\|\widetilde{\mathbf{curl}}\,(\widetilde{\mathbf{u}}-\widetilde{\mathbf{q}})\right\|_{L^1(\widetilde{K})}+\sum_{i=1}^{3}h_i\left\|\frac{\partial\widetilde{\mathbf{curl}}\,(\widetilde{\mathbf{u}}-\widetilde{\mathbf{q}})}{\partial\tilde{x}_i}\right\|_{L^1(\widetilde{K})}\right)\Bigg\}.\end{aligned}$$

Due to our choice of $\bar{p}$, inequalities (24) and (27) hold true. Using those inequalities together with (28) and (29) we arrive at

$$\begin{aligned}\|\mathbf{u}-\Pi_l\mathbf{u}\|_{L^p(K)} \le\; & C|M|^{\frac{1}{p}}\|M^{-1}\|\Bigg(\sum_{i_1+i_2+i_3=m+1}h_1^{i_1}h_2^{i_2}h_3^{i_3}\left\|\frac{\partial^{m+1}\widetilde{\mathbf{u}}}{\partial\tilde{x}_1^{i_1}\partial\tilde{x}_2^{i_2}\partial\tilde{x}_3^{i_3}}\right\|_{L^p(\widetilde{K})}\\ &+h^{m+1}\left\|\widetilde{D}^m(\widetilde{\operatorname{curl}}\,\widetilde{\mathbf{u}})\right\|_{L^p(\widetilde{K})}\Bigg).\end{aligned}\tag{32}$$

But,

$$\frac{\partial^{m+1}\widetilde{\mathbf{u}}}{\partial\tilde{x}_1^{i_1}\partial\tilde{x}_2^{i_2}\partial\tilde{x}_3^{i_3}}(\widetilde{\mathbf{x}}) = M^t\frac{\partial^{m+1}\mathbf{u}}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\partial\xi_3^{i_3}}(\mathbf{x}),$$

so

$$\left\|\frac{\partial^{m+1}\widetilde{\mathbf{u}}}{\partial\tilde{x}_1^{i_1}\partial\tilde{x}_2^{i_2}\partial\tilde{x}_3^{i_3}}\right\|_{L^p(\widetilde{K})} \le \frac{\|M\|}{|M|^{\frac{1}{p}}}\left\|\frac{\partial^{m+1}\mathbf{u}}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\partial\xi_3^{i_3}}\right\|_{L^p(K)}.\tag{33}$$

On the other hand, if we set $\theta = \mathbf{curl}\,\mathbf{u}$ and $\widetilde{\theta} = \widetilde{\mathbf{curl}}\,\widetilde{\mathbf{u}}$, and define

$$Curl\,\mathbf{u}(\mathbf{x}) = \begin{bmatrix} 0 & -\theta_3(\mathbf{x}) & \theta_2(\mathbf{x})\\ \theta_3(\mathbf{x}) & 0 & -\theta_1(\mathbf{x})\\ -\theta_2(\mathbf{x}) & \theta_1(\mathbf{x}) & 0\end{bmatrix}$$

and

$$\widetilde{Curl}\,\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}) = \begin{bmatrix} 0 & -\widetilde{\theta}_3(\widetilde{\mathbf{x}}) & \widetilde{\theta}_2(\widetilde{\mathbf{x}})\\ \widetilde{\theta}_3(\widetilde{\mathbf{x}}) & 0 & -\widetilde{\theta}_1(\widetilde{\mathbf{x}})\\ -\widetilde{\theta}_2(\widetilde{\mathbf{x}}) & \widetilde{\theta}_1(\widetilde{\mathbf{x}}) & 0\end{bmatrix},$$

then it is known [17] that

$$\widetilde{Curl}\,\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}) = M^tCurl\,\mathbf{u}(\mathbf{x})M.$$

Hence,

$$\begin{aligned}\left\|\frac{\partial^m}{\partial\tilde{x}_1^{i_1}\partial\tilde{x}_2^{i_2}\partial\tilde{x}_3^{i_3}}\widetilde{\mathbf{curl}\,\tilde{\mathbf{u}}}\right\|^p_{L^p(\tilde{K})} &\leq C\int_{\tilde{K}}\left\|\frac{\partial^m}{\partial\tilde{x}_1^{i_1}\partial\tilde{x}_2^{i_2}\partial\tilde{x}_3^{i_3}}\widetilde{Curl}\tilde{\mathbf{u}}(\tilde{\mathbf{x}})\right\|^p d\tilde{\mathbf{x}}\\ &= C\frac{1}{|M|}\int_K\left\|\frac{\partial^m}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\partial\xi_3^{i_3}}M^tCurl\mathbf{u}(\mathbf{x})\,M\right\|^p d\mathbf{x}\\ &\leq C\frac{1}{|M|}\|M\|^{2p}\int_K\left\|\frac{\partial^m}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\partial\xi_3^{i_3}}Curl\mathbf{u}(\mathbf{x})\right\|^p d\mathbf{x}\\ &\leq C\frac{1}{|M|}\|M\|^{2p}\int_K\left\|\frac{\partial^m}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\partial\xi_3^{i_3}}\mathbf{curl}\,\mathbf{u}(\mathbf{x})\right\|^p d\mathbf{x}\\ &\leq C\frac{1}{|M|}\|M\|^{(2+m)p}\int_K\|D^m\mathbf{curl}\,\mathbf{u}(\mathbf{x})\|^p\,d\mathbf{x}.\end{aligned}$$

So, we obtain

$$\left\|\tilde{D}^m(\widetilde{\mathrm{curl}}\,\tilde{\mathbf{u}})\right\|_{L^p(\tilde{K})}\leq C\frac{1}{|M|^{\frac{1}{p}}}\|M\|^{2+m}\|D^m\mathbf{curl}\,\mathbf{u}\|_{L^p(K)}. \tag{34}$$

Then, inserting (34) and (33) in (32) we obtain (30).

Estimate (31) for the simpler case $m = 0$ is clearly proved analogously, by using (24) with $\bar{p} = p$ and Hölder's inequality. □

Taking into account that $h_i \leq h$, $i = 1, 2, 3$, and that, since $\|\xi_i\| = 1$, if $i_1 + i_2 + i_3 = m$ then

$$\left|\frac{\partial^{m+1}\mathbf{u}}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\partial\xi_3^{i_3}}(\mathbf{x})\right|\leq D^m\mathbf{u}(\mathbf{x}),$$

we easily obtain the following simple corollary, containing a uniform interpolation error estimate on elements satisfying $MAC(\bar{\psi})$.

Corollary 6.2. *Let $l \geq 1$. Let K be a tetrahedron satisfying $MAC(\bar{\psi})$. There exists a constant C, such that for each $0 \leq m \leq l-1$ we have*

(1) *if m and p satisfy either $m \geq 2$ and $p \geq 1$ or $m = 1$ and $p > \frac{6}{5}$, for all $\mathbf{u} \in [W^{m+1,p}(K)]^3$ we have*

$$\|\mathbf{u}-\Pi_l\mathbf{u}\|_{L^p(K)}\leq C\,h^{m+1}\left\|D^{m+1}\mathbf{u}\right\|_{L^p(K)},$$

(2) *if $m = 0$ and $p > 2$, for all $\mathbf{u} \in [W^{1,p}(K)]^3$ with $D\boldsymbol{curl}\mathbf{u} \in [L^s(K)]^3$ we have*

$$\|\mathbf{u}-\Pi_l\mathbf{u}\|_{L^p(K)}\leq Ch\left(\|D\mathbf{u}\|_{L^p(K)}+h|K|^{\frac{1}{p}-\frac{1}{s}}\|D\boldsymbol{curl}\mathbf{u}\|_{L^s(K)}\right),$$

where h denotes the diameter of K. The constant C depends only on $\bar{\psi}$, l and p, and it is independent of the function $\mathbf{u}$.

7. Conclusions

The maximum angle condition introduced in section 3 allows for meshes that, for example, appear naturally in the approximation of edge singularities in elliptic problems or layers in singularly perturbed problems, where the classical shape regularity property becomes too restrictive. We refer to Chapter 4 of [5] where the author deals with the construction of families of meshes satisfying the maximum angle condition to obtain adequate approximations for elliptic problems in domains with edges. For such a family of meshes, each element satisfy a maximum angle condition with a constant $\bar{\psi} < \pi$, with $\bar{\psi}$ independent on the element and the mesh.

We have obtained in section 6 error estimates for the edge (Nédélec) interpolation. These estimates are valid uniformly for elements satisfying a maximum angle condition, that means that the constants in the estimates does not degenerate if the maximum angle of the elements remains bounded above away from π (see Corollary 6.2). In this way, by adding the estimates on the individual elements, one can obtain global error estimates.

Our results are also of anisotropic type as showed in Theorem 6.1. We mention that interpolation error estimates of anisotropic type are necessary when one wishes to exploit the independent element sizes h_1, h_2 and h_3 to treat edge singularities or layers: if it is known that the gradient of the solution is large in some direction, it is possible to take a mesh more refined in that direction. Indeed, in many cases, this can be performed controlling the maximum angle of the elements.

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