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Existence of positive T-periodic solutions of a generalized Nicholson's blowflies model with a nonlinear harvesting term[☆]

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ABSTRACT

We give sufficient and necessary conditions for the existence of at least one positive T-periodic solution for a generalized Nicholson's blowflies model with a nonlinear harvesting term. Our results extend those of the previous work Li and Du (2008) [1].

1. Introduction

In [1], the authors considered the generalized Nicholson's blowflies model

$$x'(t) = -\delta(t)x(t) + \sum_{k=1}^{N} P_k(t)x(t-\tau_k(t))e^{-x(t-\tau_k(t))} \tag{1}$$

where for $k = 1, \ldots, N$ the functions δ_k, P_k and τ_k are positive, continuous and T-periodic. The existence of at least one positive T-periodic solution was proven under the assumption

$$\delta(t) < \sum_{k=1}^{N} P_k(t) \quad \text{for all } t.$$

Also, it was proven that the previous inequality is necessary for some t; furthermore, it was seen that if $\delta(t) \geq \sum_{k=1}^{N} P_k(t)$ for all t, then all positive solutions of (1) tend to 0 as $t \to +\infty$.

In this work we generalize these results by including into the model a nonlinear harvesting term $H(t,x)$ with $H : \mathbb{R} \times [0,+\infty) \to [0,+\infty)$ continuous and T-periodic in t such that $H(t,0) = 0$. Namely, we shall consider the problem

$$x'(t) = -\delta(t)x(t) + \sum_{k=1}^{N} P_k(t)x(t-\tau_k(t))e^{-x(t-\tau_k(t))} - H(t,x(t)). \tag{2}$$

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Our existence result for problem (2) reads as follows:

Theorem 1.1. *Assume that the upper limit* $H_{sup}(t) := \limsup_{x\to 0^+} \frac{H(t,x)}{x}$ *is uniform in* t *and satisfies*

$$\delta(t) + H_{sup}(t) < \sum_{k=1}^{N} P_k(t) \tag{3}$$

for all t*. Then problem* (2) *admits at least one* T*-periodic positive solution.*

Remark 1.2. Condition (3) implies the existence of constants $\gamma, \varepsilon > 0$ such that

$$\delta(t) + \frac{H(t,x)}{x} < \sum_{k=1}^{N} P_k(t) - \gamma$$

for every t and $0 < x < \varepsilon$. In particular, if H is continuously differentiable with respect to x, then (3) can be written as: $\delta(t) + \frac{\partial H}{\partial x}(t,0) < \sum_{k=1}^{N} P_k(t)$ for all t.

Moreover, we shall prove that the condition

$$\sum_{k=1}^{N} P_k(t) > \delta(t) + \frac{H(t,x)}{x} \quad \text{for some } t, x > 0 \tag{4}$$

is necessary for the existence of positive T-periodic solutions. But as in [1], in fact we prove a little more: namely, that if (4) does not hold, then the equilibrium point $\hat{x} = 0$ is a global attractor for the solutions with positive initial data. Indeed, let

$$\tau^* = \max_{1\le k\le m,\ 0\le t\le T} \tau_k(t) - t$$

and consider the initial condition for problem (2):

$$x(t) = \varphi(t) \quad t \in [-\tau^*, 0] \tag{5}$$

for some continuous function φ. Then we have:

Theorem 1.3. *If* $\sum_{k=1}^{N} P_k(t) \le \delta(t) + \frac{H(t,x)}{x}$ *for all* t *and all* $x > 0$ *then all solutions of the initial value problem* (2)–(5) *with* $\varphi > 0$ *are globally defined and tend to* 0 *as* $t \to +\infty$*.*

The paper is organized as follows. In Section 2 we shall prove Theorem 1.1. In Section 3 we give a proof of Theorem 1.3. Finally, in Section 4 we make some final comments and introduce an open problem.

2. Proof of Theorem 1.1

Let us firstly introduce some notation. The set of continuous and T-periodic real functions shall be denoted C_T. For $x \in C_T$, its maximum and minimum values and its average $\frac{1}{T}\int_0^T x(t)\,dt$ shall be denoted respectively by x^*, x_* and $\overline{x}$. For $\varphi \in C_T$ such that $\overline{\varphi} = 0$, let $\mathcal{K}\varphi$ be the unique T-periodic solution with zero average of the problem $x'(t) = \varphi(t)$. For convenience, let us also define the operator $\phi : C_T \to C_T$ by

$$\phi(x)(t) := -\delta(t)x(t) + \sum_{k=1}^{N} P_k(t)x(t-\tau_k(t))e^{-x(t-\tau_k(t))} - H(t, x(t)).$$

We shall apply the standard Leray–Schauder degree techniques (see e. g. [2]). For $\lambda \in [0, 1]$, define the compact operator $F_\lambda : C_T \to C_T$ given by

$$F_\lambda(x) = x - \overline{x} - \overline{\phi(x)} - \lambda\mathcal{K}(\phi(x) - \overline{\phi(x)}).$$

It is easy to verify that if $\lambda > 0$ then $x \in C_T$ is a zero of F_λ if and only if $x' = \lambda\phi(x)$. Thus, we look for a positive zero of F_1.

Let $\Omega := \{x \in C_T : m < x(t) < M\}$ for some constants $M > m > 0$ to be established. By the standard continuation method, it suffices to prove that F_λ does not vanish on $\partial\Omega$ for $\lambda \in [0, 1]$ and that $\deg(F_0, \Omega, 0) \neq 0$. Furthermore, observe that $F_0(x) - x \in \mathbb{R}$ for every $x \in C_T$; thus, its degree over Ω is different from zero if and only if $F_0(m)$ and $F_0(M)$ have opposite signs, i.e. $F_0(m)F_0(M) < 0$.

Indeed, if $x \in \mathbb{R}^+$ then

$$F_0(x) = \overline{\delta}x - \sum_{k=1}^{N} \overline{P_k}xe^{-x} + \frac{1}{T}\int_0^T H(t,x)\,dt.$$

Thus, $F_0(x) > x(\overline{\delta} - \sum_{k=1}^N \overline{P_k}e^{-x})$ and hence $F_0(M) > 0$ for $M \geq \ln \frac{\sum_{k=1}^N \overline{P_k}}{\overline{\delta}}$. On the other hand, if $0 < x < \varepsilon$ with ε as in Remark 1.2, then

$$F_0(x) = \frac{x}{T}\int_0^T \left(\delta(t) + \frac{H(t,x)}{x} - \sum_{k=1}^N P_k(t)e^{-x}\right) dt$$
$$\leq \frac{x}{T}\int_0^T \left(\sum_{k=1}^N P_k(t)(1-e^{-x}) - \gamma\right) dt$$

and we deduce that $F_0(x) < 0$ if x is small enough.

It remains to prove that if m and M are respectively small and large enough then $F_\lambda(x) \neq 0$ for $x \in \partial\Omega$ and $\lambda \in (0, 1]$.

Let $\lambda \in (0, 1]$ and assume for some positive x that $F_\lambda(x) = 0$, that is, $x' = \lambda\phi(x)$. If ξ is an absolute maximizer of x, then

$$\sum_{k=1}^N P_k(\xi)x(\xi - \tau_k(\xi))e^{-x(\xi-\tau_k(\xi))} > \delta(\xi)x(\xi),$$

and from the fact that the function $f(A) := Ae^{-A} \leq f(1) = \frac{1}{e}$, we deduce:

$$x^* \leq \left(\frac{\sum\limits_{k=1}^N P_k}{e\delta}\right)^* .$$

On the other hand, if η is an absolute minimizer of x then

$$\sum_{k=1}^N P_k(\eta)x(\eta - \tau_k(\eta))e^{-x(\eta-\tau_k(\eta))} = \left[\delta(\eta) + \frac{H(\eta, x_*)}{x_*}\right]x_*.$$

As before, if $x_* < \varepsilon$ then we know from the hypothesis that $\delta(\eta) + \frac{H(\eta,x_*)}{x_*} < \sigma\sum_{k=1}^N P_k(\eta)$ for some constant $\sigma < 1$ independent of η.

Suppose that $x_* \ll 1$, then $x(\eta - \tau_k(\eta))e^{-x(\eta-\tau_k(\eta))} \geq x_*e^{-x_*}$ and hence

$$e^{x_*} \geq \frac{\sum\limits_{k=1}^N P_k(\eta)}{\delta(\eta) + \frac{H(\eta,x_*)}{x_*}} \geq \frac{1}{\sigma}.$$

Thus, $x_* > -\ln\sigma > 0$ and the proof follows.

3. Necessary conditions

In this section, we shall prove that condition (4) is necessary for the existence of positive T-periodic solutions. This is actually seen directly as in the proof of Theorem 1.1: if x is a positive T-periodic solution and ξ is a global maximizer then

$$x^*\left(\delta(\xi) + \frac{H(\xi, x^*)}{x^*}\right) = \sum_{k=1}^N P_k(\xi)x(\xi - \tau_k(\xi))e^{-x(\xi-\tau_k(\xi))}.$$

If $x^* \leq 1$, then the right hand-side term is less or equal than $\sum_{k=1}^N P_k(\xi)x^*e^{-x^*}$ and the proof follows; otherwise we obtain that

$$\delta(\xi) + \frac{H(\xi, x^*)}{x^*} \leq \frac{\sum\limits_{k=1}^N P_k(\xi)}{ex^*} < \sum_{k=1}^N P_k(\xi)$$

and so completes the proof. But, as mentioned, we shall prove furthermore that $\hat{x} = 0$ is asymptotically stable over the set of positive solutions.

Proof of Theorem 1.3. We shall proceed in several steps.

1. Assume that x is defined up to t_0 and $x(t) > 0$ for all $t < t_0$. Then $x(t_0) > 0$. Indeed, if $x(t_0) = 0$ then

$$0 \geq x'(t_0) = \sum_{k=1}^N P_k(t_0)x(t_0 - \tau_k(t_0))e^{-x(t_0-\tau_k(t_0))} > 0,$$

a contradiction.

2. If $x'(t_0) \geq 0$, then $x(t_0) \leq \frac{1}{e}$. *Proof*: as $x'(t_0) \geq 0$ and x is positive,

$$\delta(t_0) + \frac{H(t_0, x(t_0))}{x(t_0)} \leq \frac{\sum_{k=1}^{N} P_k(t_0)}{ex(t_0)}$$

   and the proof follows from the assumptions.

   In particular, we deduce from 1 and 2 that x is defined and strictly positive on $[0, +\infty)$.

3. If x is strictly decreasing on $[0, +\infty)$, then $x(t) \to 0$ as $t \to +\infty$. *Proof*: suppose that $x(t) \to \alpha > 0$, then for arbitrary $\beta > 0$ there exists t_0 such that $\sum_{k=1}^{N} P_k(t)x(t-\tau_k(t))e^{-x(t-\tau_k(t))} \leq \sum_{k=1}^{N} P_k(t)\alpha e^{-\alpha} + \beta$ for $t \geq t_0$. From this inequality and the hypotheses we obtain:

$$\begin{aligned} x'(t) &\leq \sum_{k=1}^{N} P_k(t)\alpha e^{-\alpha} + \beta - \delta(t)x(t) - H(t, x(t)) \\ &\leq \delta(t)(\alpha - x(t)) + H(t, \alpha) - H(t, x(t)) + \beta - (\delta(t)\alpha + H(t, \alpha))(1 - e^{-\alpha}). \end{aligned}$$

   Thus, if we fix $\beta < (\delta\alpha + H(\cdot, \alpha))^*(1 - e^{-\alpha})$ it follows that $x'(t) \leq -\kappa$ for some $\kappa > 0$ and t sufficiently large, which contradicts the fact that x is always positive.

As a conclusion from the first three steps, we deduce the existence of $t_1 \geq 0$ such that $0 < x(t) < \frac{1}{e}$ for $t \geq t_1 - \tau^*$. Next, define $x_1 = \frac{1}{e}$ and, as in step 2, we deduce that if $x'(t) \geq 0$ for some $t \geq t_1$ then $x(t) \leq f(x_1) := x_2$, where as before $f(x) = xe^{-x}$. Repeating the procedure we obtain a sequence $t_1 \leq t_2 \leq \cdots$ such that $0 < x(t) < x_n := f(x_{n-1})$ for $t \geq t_n - \tau^*$. As the sequence $\{x_n\}$ is strictly decreasing and positive, it must converge to a fixed point of f and so completes the proof. □

4. Concluding remarks and open problem

In the very recent paper [3], the authors solved a particular case of an open problem posed in [4]: study the original model (i.e. with $m = 1$) with linear harvesting term depending on the delayed estimate of the population. An important (implicitly stated) assumption in [3] was the fact that the delay in the harvesting term was equal to the one in the original equation. Following the ideas in Theorem 1.1, the existence result in [3] can be improved and extended for the generalized model (2) if the harvesting term is replaced by a nonlinear term with a delay $\tau = \tau_{\hat{k}}$ for some $\hat{k}$. More precisely:

Theorem 4.1. *Consider Eq. (2) with the harvesting function $H(t, x(t))$ replaced by $H(t, x(t-\tau_{\hat{k}}(t)))$ for some $\hat{k}$. Assume that (3) is satisfied and that*

$$\frac{H(t, x)}{x} \leq P_{\hat{k}}(t)e^{-x} \quad \text{for all } t \text{ and } 0 < x < \left(\frac{\sum_{k=1}^{N} P_k}{e\delta}\right)^*. \tag{6}$$

Then the problem has at least one positive T-periodic solution.

The proof follows the outline of Section 2, so the details are left to the reader. Simply observe that if $F_\lambda(x) = 0$ for $\lambda \in (0, 1]$ then a bound for x^* is obtained exactly in the same way, and the lower bound is now obtained as follows: suppose $x_* \ll 1$, then

$$\delta x_* \geq \sum_{k \neq \hat{k}} P_k(\eta)x_* e^{-x_*} + x_{\hat{k}}\left(P_{\hat{k}}(\eta)e^{-x_{\hat{k}}} - \frac{H(\eta, x_{\hat{k}})}{x_{\hat{k}}}\right)$$

where $x_{\hat{k}} := x(\eta - \tau_{\hat{k}}(\eta))$. As $x_{\hat{k}} \geq x_*$, the desired bound is obtained using (3) and (6).

Open question. Find sufficient conditions for the existence of positive T-periodic solutions without making the assumption $\tau = \tau_{\hat{k}}$.

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