# SUFFICIENT CONDITIONS FOR OSCILLATIONS OF A FIRST-ORDER SYSTEM WITH DELAY. 

Alberto Déboli ${ }^{\dagger}$, Pablo Amster ${ }^{\ddagger}$ and Paula Kuna ${ }^{\dagger \ddagger}$<br>${ }^{\dagger}$ Universidad Nacional de General Sarmiento. Instituto de Ciencias. Área Matemática Aplicada, afdeboli@ungs.edu.ar<br>${ }^{\ddagger}$ Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires. Departamento de Matemática Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina. IMAS-CONICET pamster@dm.uba.ar mpkuna@dm.uba.ar


#### Abstract

In this work, we consider a system of first order differential equations with delays which appears in many population biological models ([3] [1]). More specifically we prove, using the topological degree theory, existence of


 periodic solutions of following system$$
\begin{equation*}
u_{i}^{\prime}(t)=a_{i} u_{i}(t)+b_{i} u_{i}\left(t-\tau_{i}\right)+g_{i}\left(u_{1}\left(t-\tau_{1}\right), u_{2}\left(t-\tau_{2}\right)\right)+p_{i}(t), \quad i=1,2 . \tag{1}
\end{equation*}
$$

## 1 Introduction

Let us consider the system of delay differential equations (1) where $a_{i}, b_{i} \in \mathbb{R}, \tau_{i}>0$ and $g_{i}: \mathbb{R}^{2} \rightarrow \mathbb{R}$ are continuous and bounded for $i=1,2$. We shall assume that

$$
\begin{equation*}
\left|a_{1}\right|<\left|b_{1}\right| \tag{2}
\end{equation*}
$$

and $p_{i} \in C(\mathbb{R}, \mathbb{R})$ are $T:=2 \pi / \omega$-periodic functions with $\omega:=\sqrt{b_{1}^{2}-a_{1}^{2}}$. For convenience, we shall denote $u=\left(u_{1}, u_{2}\right), g=\left(g_{1}, g_{2}\right)$ and $p=\left(p_{1}, p_{2}\right)$ and write the problem as a functional equation in the following way.

Consider

$$
C_{T}:=\left\{u \in C\left(\mathbb{R}, \mathbb{R}^{2}\right): u(t)=u(t+T)\right\}, C_{T}^{1}:=C_{T} \cap C^{1}\left(\mathbb{R}, \mathbb{R}^{2}\right)
$$

with $L: C_{T}^{1} \rightarrow C\left(\mathbb{R}, \mathbb{R}^{2}\right)$ given by

$$
L\left(u_{1}, u_{2}\right)(t)=\left(L_{1}\left(u_{1}\right)(t), L_{2}\left(u_{2}\right)(t)\right)
$$

where

$$
L_{i}\left(u_{i}\right)(t)=u_{i}^{\prime}(t)-a_{i} u_{i}(t)-b_{i} u_{i}\left(t-\tau_{i}\right), i=1,2
$$

and

$$
\begin{gathered}
N: C_{T} \rightarrow C\left(\mathbb{R}, \mathbb{R}^{2}\right) \\
N(u)(t):=g\left(u_{1}\left(t-\tau_{1}\right), u_{2}\left(t-\tau_{2}\right)\right)+p(t)
\end{gathered}
$$

Then (1) is equivalent to the problem

$$
L u=N u, \quad u \in C_{T}^{1}
$$

We are interested in the resonant case, that is, when the operator $L$ has nontrivial kernel. Specifically, we shall assume that the resonance is produced in the first equation. In more precise terms, consider the characteristic equation of $L_{1}\left(u_{i}\right)(t)=0$ given by

$$
h(\lambda)=\lambda-a_{1}-b_{1} e^{-\lambda \tau_{1}}=0
$$

Let

$$
z=\lambda \tau_{1}, \quad \alpha_{1}=a_{1} \tau_{1} \quad \text { and } \beta_{1}=b_{1} \tau_{1}
$$

and the function $F\left(z, \alpha_{1}, \beta_{1}\right):=z-\alpha_{1}-\beta_{1} e^{-z}$, whose zeros are related to the roots of $h$ via the previous change or variables. We know (see [3]) that $z=i y, y>0$, is a purely imaginary root of $F$ if there we take $\tau_{1}$ such that $\left(\alpha_{1}, \beta_{1}\right) \in C_{k}$, where

$$
C_{k}=\left\{(\alpha(y), \beta(y)) / \alpha(y):=\frac{y \cos y}{\sin y}, \beta(y):=\frac{-y}{\sin y}, y \in(k \pi,(k+1) \pi)\right\}
$$

for some $k \in \mathbb{N}$. In this case, $\lambda=i \omega$ is a root of $h$. Next, consider $\tau_{2}$, such that $\left(\alpha_{2}, \beta_{2}\right) \notin C_{k}$, for any $k \in \mathbb{N}$. It is easy to see that, in this situation, that

$$
\operatorname{Ker}(L)=\operatorname{span}\{\cos (\omega t), \sin (\omega t)\} \times\{0\} .
$$

Let us consider $\mathcal{P}: C_{T}^{1} \rightarrow \operatorname{Ker} L$ the orthogonal projection, given by

$$
\mathcal{P}(u)=\left(\alpha_{u_{1}} \cos (\omega t)+\beta_{u_{1}} \sin (\omega t), 0\right)
$$

where $\alpha_{v}$ and $\beta_{v}$ are the Fourier coefficients given bt

$$
\alpha_{v}:=\frac{2}{T} \int_{0}^{T} \cos (\omega t) v(t) d t
$$

and

$$
\beta_{v}:=\frac{2}{T} \int_{0}^{T} \sin (\omega t) v(t) d t
$$

A straightforward computation shows that the adjoint operator of $L^{1}$ is given by

$$
L_{1}^{*} v(t)=-v^{\prime}(t)-a_{1} v(t)-b_{1} v\left(t+\tau_{1}\right)
$$

where $v \in C^{1}$. Hence, the characteristic equation of $L_{1}^{*} v(t)=0$ reads $h^{*}(\lambda)=\lambda+a_{1}+b_{1} e^{\lambda \tau_{1}}=0$. Observe that $\lambda=i \omega$ is a solution of $h(\lambda)=0$ if and only if $\bar{\lambda}$ is a solution of $h^{*}(\lambda)=0$; thus, $\operatorname{Ker} L_{1}=$ $\operatorname{Ker} L_{1}^{*}$. It follows that $R(L)=\left(\operatorname{Ker} L^{*}\right)^{\perp}=\left(\operatorname{Ker} L_{1}\right)^{\perp} \times C$, where

$$
\left(\operatorname{Ker} L_{1}\right)^{\perp}=\left\{\varphi \in L^{2}([0, T], \mathbb{R}): \int_{0}^{T} \cos (\omega t) \varphi(t) d t=\int_{0}^{T} \sin (\omega t) \varphi(t) d t=0\right\}
$$

Then we may define a right inverse $\mathcal{K}: R(L) \rightarrow C_{T}^{1}$ of the operator $L$, given by $\mathcal{K} \psi=u$, where $u$ is the unique solution of the problem

$$
\left\{\begin{array}{l}
L u=\psi \\
\mathcal{P}(u)=0
\end{array}\right.
$$

Moreover, due to the Open Mapping Theorem, the following standard estimate is obtained

$$
\|u-\mathcal{P} u\|_{H^{1}} \leq c\|L u\|_{L^{2}}, \quad u \in C_{T}^{1}
$$

Hence, from the compactness of the embedding $H^{1}(0, T) \hookrightarrow C_{T}$, we conclude that $\mathcal{K}$ is compact.

## 2 MAIN RESULT

In the sequel, we shall assume that the limits

$$
g_{1}^{i n f}(-\infty):=\liminf _{v \rightarrow-\infty} g_{1}\left(v, u_{2}\right)
$$

and

$$
g_{1}^{s u p}(+\infty):=\limsup _{v \rightarrow+\infty} g_{1}\left(v, u_{2}\right)
$$

exist uniformly for $u_{2} \in B_{r}(0)$, where $r:=c\left(\|g\|_{\infty}+\|p\|_{\infty}\right)$.
Theorem 2.1 Fix $\tau_{1}$ as before such that $\left(\alpha_{1}, \beta_{1}\right) \in C_{k}$ for some $k$ and assume that (2) and one of the following conditions

$$
\begin{equation*}
\frac{4}{T}\left(g_{1}^{\text {inf }}(-\infty)-g_{1}^{\text {sup }}(+\infty)\right)>\sqrt{\alpha_{p_{1}}^{2}+\beta_{p_{1}}^{2}} \tag{3}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{4}{T}\left(g_{1}^{i n f}(+\infty)-g_{1}^{s u p}(-\infty)\right)>\sqrt{\alpha_{p_{1}}^{2}+\beta_{p_{1}}^{2}} \tag{4}
\end{equation*}
$$

hold. Then, for almost all $\tau_{2}>0$, system (1) has at least one nontrivial T-periodic solution.

Proof. Choose $\tau_{2}$ such that $\left(\alpha_{2}, \beta_{2}\right) \notin C_{k}$ for any $k \in \mathbb{N}$. For each $\lambda \in[0,1]$, define the Fredholm operator $F_{\lambda}: C_{T} \rightarrow C_{T}$ given by $F_{\lambda} u=u-T_{\lambda} u$, where the operator $T_{\lambda}$ is defined by

$$
T_{\lambda} u(t)=\mathcal{P} u+\mathcal{P}(N u)+\lambda \mathcal{K}(N u-\mathcal{P}(N u))
$$

We claim that, for $\lambda \in(0,1], F_{\lambda} u=0$ if and only if $u \in C_{T}^{1}$ and $L u=\lambda N u$.
Indeed, if $u$ is a zero of $F_{\lambda}$, then $u=T_{\lambda} u \in C_{T}^{1}$, since $\mathcal{K}: R(L) \rightarrow C_{T}^{1}$. Apply $\mathcal{P}$ at both sides, then $\mathcal{P}(N u)=0$. So, we have that $u=\mathcal{P} u+\lambda \mathcal{K}(N u)$. Since $L \mathcal{P} \equiv 0$ it is deduced that $L u=\lambda(N u)$. Conversely, if $L u=\lambda(N u)$ and $u \in C_{T}^{1}$, then $N u \in R(L)$, hence $\mathcal{P}(N u)=0$ and $\lambda \mathcal{K}(N u)=u-\mathcal{P} u$.

In order to verify that $F_{1}$ has a zero, we shall firstly prove the existence of $R \gg 0$ such that $F_{\lambda} u \neq 0$ for $u \in \partial B_{R}(0)$. Next, by the homotopy invariance of the Leray-Schauder degree, it will suffice to verify that $\operatorname{deg}_{L S}\left(F_{0}, B_{R}(0), 0\right) \neq 0$.

To this end, suppose firstly there exists a sequence $\left(u^{n}\right)_{n \in \mathbb{N}} \subset C_{T}^{1}$ and $\lambda_{n} \in(0,1]$ such that $F_{\lambda_{n}} u^{n}=0$ and $\left\|u^{n}\right\|_{\infty} \rightarrow \infty$. Then

$$
\left(u_{i}^{n}\right)^{\prime}(t)=a_{i} u_{i}^{n}(t)+b_{i} u_{i}^{n}\left(t-\tau_{i}\right)+\lambda_{n}\left(g_{i}\left(u_{1}^{n}\left(t-\tau_{1}\right), u_{2}^{n}\left(t-\tau_{2}\right)\right)+p_{i}(t)\right)
$$

which, in turn, implies that $\mathcal{P}\left(N u^{n}\right)=0$ for all $n \in \mathbb{N}$ and $\left\|u^{n}-\mathcal{P} u^{n}\right\|_{\infty} \leq c\left(\|g\|_{\infty}+\|p\|_{\infty}\right)$. Hence, $\left\|\mathcal{P} u^{n}\right\|_{\infty} \rightarrow+\infty$. Let us write $\mathcal{P} u^{n}=\left(\rho_{n} \cos \left(\omega t-\theta_{n}\right), 0\right)$, where $\rho_{n} \rightarrow \infty, \theta_{n} \in[0,2 \pi]$ and

$$
u^{n}(t)=\mathcal{P} u^{n}(t)+u^{n}(t)-\mathcal{P} u^{n}(t)=\left(\rho_{n} \cos \left(\omega t-\theta_{n}\right)+\tilde{u}_{1}^{n}(t), u_{2}^{n}(t)\right)
$$

where $\tilde{u}_{1}^{n}$ is bounded. Passing to a subsequence if needed, we may assume that $\theta_{n}$ converges to some $\theta \in[0,2 \pi]$. Since $\mathcal{P}(N u)=0$, by substitution we obtain:

$$
\begin{align*}
& -\mathcal{P}\left(p_{1}\right)=\frac{2}{T} \int_{0}^{T} g_{1}\left(\rho_{n} \cos \left(\omega\left(t-\tau_{1}\right)-\theta_{n}\right)+\tilde{u}_{1}^{n}\left(t-\tau_{1}\right), u_{2}^{n}\left(t-\tau_{2}\right)\right) e^{i \omega t} d t= \\
& =\frac{2}{T} e^{i\left(\theta_{n}+\omega \tau_{1}\right)} \int_{0}^{T} g_{1}\left(\rho_{n} \cos \omega s+\tilde{u}_{1}\left(s-\frac{\theta_{n}}{\omega}\right), u_{2}\left(s+\tau_{1}+\frac{\theta_{n}}{\omega}-\tau_{2}\right) e^{i \omega s} d s\right. \tag{5}
\end{align*}
$$

Let us consider the sets $I^{+}=\{t \in[0, T]: \cos \omega t>0\}$ and $I^{-}=\{t \in[0, T]: \cos \omega t<0\}$, then by the Dominated Convergence Theorem and the fact that $\int_{I^{+}} \sin \omega s d s=\int_{I^{-}} \sin \omega s d s=0$, we deduce

$$
\begin{array}{r}
\int_{I^{+}} g_{1}\left(\rho_{n} \cos \omega s+u_{1}\left(s-\frac{\theta_{n}}{\omega}\right), u_{2}\left(s+\tau_{1}+\frac{\theta_{n}}{\omega}-\tau_{2}\right)\right) e^{i \omega s} d s \\
\rightarrow g_{1}^{\text {sup }}(+\infty) \int_{I^{+}} \cos \omega s d s=2 g_{1}^{\text {sup }}(+\infty)
\end{array}
$$

and

$$
\begin{aligned}
& \int_{I^{-}} g_{1}\left(\rho_{n} \cos \omega s+\right.\left.u_{1}\left(s-\frac{\theta_{n}}{\omega}\right), u_{2}\left(s+\tau_{1}+\frac{\theta_{n}}{\omega}-\tau_{2}\right)\right) e^{i \omega s} d s \\
& \rightarrow g_{1}^{i n f}(-\infty) \int_{I^{-}} \cos \omega s d s=-2 g_{1}^{i n f}(-\infty)
\end{aligned}
$$

as $n \rightarrow+\infty$. From (5), we conclude that

$$
\sqrt{\alpha_{p_{1}}^{2}+\beta_{p_{1}}^{2}}=\frac{4}{T}\left(g_{1}^{i n f}(-\infty)-g_{1}^{\text {sup }}(+\infty)\right)
$$

which contradicts condition (3).
Finally, if $F_{0}(u)=0$, then $u=\mathcal{P}(u)$ and a similar argument with $\tilde{u}_{1}=u_{2}=0$ shows that $u \notin \partial B_{R}(0)$ when $R$ is large enough.

This implies that the Leray-Schauder degree of $F_{\lambda}$ at 0 is well defined on $B_{R}(0)$ and that $\operatorname{deg}_{L S}\left(F_{1}, B_{R}(0), 0\right)=$ $d e g_{L S}\left(F_{0}, B_{R}(0), 0\right)$. Moreover, by definition of the degree and the fact that

$$
F_{0} u=u-\mathcal{P}(u+N u)
$$

we conclude that $\operatorname{deg}_{L S}\left(F_{0}, B_{R}(0), 0\right)=\operatorname{deg}_{B}\left(\left.F_{0}\right|_{\operatorname{Ker} L}, B_{R}(0) \cap \operatorname{Ker} L, 0\right)$. Notice that if $u \in \operatorname{Ker} L$, then $F_{0} u=-\mathcal{P}(N u)$; thus, by the product property of the Brouwer degree,

$$
\begin{gathered}
\operatorname{deg}_{B}\left(\left.F_{0}\right|_{\operatorname{Ker} L}, B_{R}(0) \cap \operatorname{Ker} L, 0\right)= \\
=\operatorname{deg}_{B}\left(\pi_{1}(-\mathcal{P} N u), \pi_{1}\left(B_{R}(0)\right) \cap \operatorname{Ker} L_{1}, 0\right) \operatorname{deg}_{B}\left(\pi_{2}(-\mathcal{P}(N u)), \pi_{2}\left(B_{R}(0)\right) \cap \operatorname{Ker} L_{2}, 0\right)= \\
=\operatorname{deg}_{B}\left(\pi_{1}(-\mathcal{P}(N u)), \pi_{1}\left(B_{R}(0)\right) \cap \operatorname{Ker} L_{1}, 0\right)
\end{gathered}
$$

where $\pi_{i}: \mathbb{R} \rightarrow \mathbb{R}, \pi_{i}\left(x_{1}, x_{2}\right)=x_{i}$. Hence, the proof is reduced to see that $\operatorname{de} g_{B}\left(\pi_{1}(-\mathcal{P}(N u)), \pi_{1}\left(B_{R}(0)\right) \cap\right.$ $\left.\operatorname{Ker} L_{1}, 0\right) \neq 0$.

Let $u \in B_{R}(0) \cap \operatorname{Ker} L$, then $u(t)=(\rho \cos (\omega t-\theta), 0)$, with $\theta \in[0,2 \pi]$ and $|\rho| \leq R$ for $R \gg 0$. Via substitution and due to the periodicity of $u$, we have that

$$
\begin{aligned}
\pi_{1}(\mathcal{P}(N u)) & =\frac{2}{T} \int_{0}^{T} g_{1}\left(\rho \cos \left(\omega\left(t-\tau_{1}\right)-\theta\right), 0\right) e^{i \omega t} d t+\alpha_{p_{1}}+i \beta_{p_{1}}= \\
& =e^{i \theta}\left(\frac{2}{T} e^{i \omega \tau_{1}} \int_{0}^{T} g_{1}(\rho \cos \omega s, 0) e^{i \omega s} d s\right)+\alpha_{p_{1}}+i \beta_{p_{1}}
\end{aligned}
$$

Therefore, the degree of the function $\left.F_{0} u\right|_{\operatorname{Ker} L}$ coincides with the index of the curve $\gamma:[0,2 \pi] \rightarrow \mathbb{C}$ defined by

$$
\gamma(t)=e^{i t}\left(\frac{2}{T} e^{i \omega \tau_{1}} \int_{0}^{T} g_{1}(\rho \cos \omega s, 0) e^{i \omega s} d s\right)
$$

around the poing $z_{0}:=-\left(\alpha_{p_{1}}+i \beta_{p_{1}}\right)$.
Via Dominated Convergence Theorem and taking $I^{+}$and $I^{-}$as before, it is seen that

$$
\int_{0}^{T} g_{1}(\rho \cos \omega s, 0) e^{i \omega s} d s \rightarrow_{\rho \rightarrow+\infty} 2\left(g_{1}^{\text {sup }}(+\infty)-g_{1}^{i n f}(-\infty)\right)
$$

Hence, for $\rho$ large enough, $|\gamma(t)| \geq \frac{4}{T}\left(g_{1}^{\text {sup }}(+\infty)-g_{1}^{\text {inf }}(-\infty)\right)>\sqrt{\alpha_{p_{1}}^{2}+\beta_{p_{1}}^{2}}$, by condition (3).
We conclude that

$$
\operatorname{deg}_{L S}\left(F_{1}, B_{R}(0), 0\right)=I\left(\gamma, z_{0}\right)= \pm 1
$$

for $R$ large enough, which proves the existence of a $T$-periodic solution of problem (1).

## ACKNOWLEDGEMENT

This work was partially supported by project CONICET PIO 144-20140100027-CO and UBACyT 20020120100029BA.

## References

[1] R.D. Driver. Ordinary and Delay Differential Equations. Springer-Verlang. New York. 1977.
[2] J. Mawhin and R. Gaines. Coincidence Degree, and Nonlinear Differential Equations. Springer-Verlang. New York. 1977.
[3] H. Smith, An Introduction to Delay Differential Equations with Applications to the Life Sciences, Springer-Verlag, New York, 2011.

